Axisymmetry in elasticity
"Old wine in new bottles"

J. BLAAUWENDRAAD
Delft University of Technology
Department of Civil Engineering

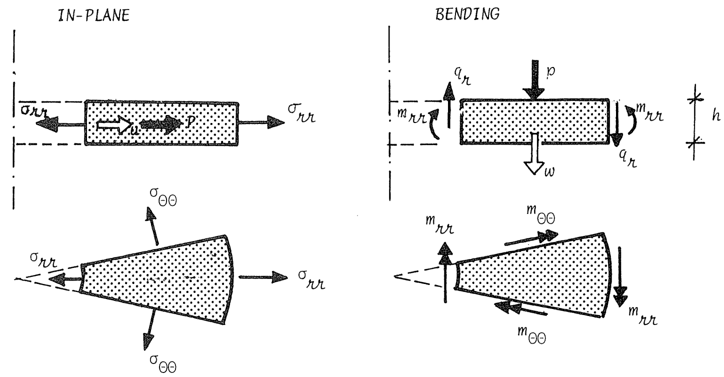
1 Outline

The theory of elasticity has been a well-established domain of engineering science for over a century, and solutions exist for many particular problems. It is not the aim of this contribution to extend the scope of the theory, but instead to draw attention to a specific way to formulate problems. So this contribution touches upon didactics in mechanics or, if you like, systematics; one may regard it as an educational section in a valedictory volume dedicated to a devoted teacher.

Though the method and message is valid generally, we shall here confine ourselves, to solids of revolution which are subjected to axisymmetric loads. The states of stress and strain will also be axisymmetric in this case. We will discuss in more detail plates which are stretched (load in-plane) or loaded in bending (load normal to plane), and which behave linear-elastically. It is quite usual to consider such plates as special cases of general two-dimensional formulations in which the biharmonic differential equation plays its role, either for Airy's stress function or for the deflection. The fourth-order partial differential equation in coordinates x and y is then transformed mathematically into a fourth-order differential equation in the radial coordinate r. This approach has serious disadvantages, at least for plates loaded in-plane. It is more advantageous to define the problem immediately form the very basis of axisymmetry.

The basic quantities and relations in axisymmetric plates have been assembled in Fig. 1. As shown in this diagram, they reflect the conventional representation of the strains (curvatures), stresses (moments), kinematic relations, equilibrium conditions and the constitutive relations [1, 2]. We will demonstrate that this representation must be modified to reach our goal.

In general an elastic state under static loading is determined by a continuous field of displacements $\{u\}$. Each component of this field corresponds to a component of the volume forces $\{P\}$. Stresses $\{\sigma\}$ and strains $\{\varepsilon\}$ occur which are related by Hooke's law. The surface S of the elastic solid consists of a part S_p on which the surface loads $\{p\}$ have been specified and a part S_u on which the displacements $\{u\}$ have been specified. In the volume V of the solid the displacements, strains, stresses and volume forces are interrelated by kinematic equations, constitutive relations and equilibrium conditions. This is indicated schematically in Fig. 2. The symbols $[D]$ and $[D^*]$ denote differential matrix-operators; $[S_\varepsilon]$ is the rigidity matrix in the stiffness relation and $[F_\sigma]$ is the compliance matrix in the flexibility relation. It can be shown that $[D]$ and $[D^*]$ are



$$\begin{cases} \varepsilon_{rr} = \dfrac{du}{dr} \\ \varepsilon_{\theta\theta} = \dfrac{u}{r} \end{cases} \qquad \begin{cases} \kappa_{rr} = -\dfrac{d^2 w}{dr^2} \\ \kappa_{\theta\theta} = -\dfrac{1}{r}\dfrac{dw}{dr} \end{cases}$$

$$\begin{cases} \varepsilon_{rr} = \dfrac{1}{E}\left(\sigma_{rr} - \nu\,\sigma_{\theta\theta}\right) \\ \varepsilon_{\theta\theta} = \dfrac{1}{E}\left(\sigma_{\theta\theta} - \nu\,\sigma_{rr}\right) \end{cases} \qquad \begin{cases} \kappa_{rr} = \dfrac{12}{Eh^3}\left(m_{rr} - \nu\, m_{\theta\theta}\right) \\ \kappa_{\theta\theta} = \dfrac{12}{Eh^3}\left(m_{\theta\theta} - \nu\, m_{rr}\right) \end{cases}$$

$$\frac{d\,\sigma_{rr}}{dr} + \frac{\sigma_{rr}}{r} - \frac{\sigma_{\theta\theta}}{r} + P = 0 \qquad \frac{d^2 m_{rr}}{dr^2} + \frac{2}{r}\frac{d\,m_{rr}}{dr} - \frac{1}{r}\frac{d\,m_{\theta\theta}}{dr} + p = 0$$

Fig. 1. Conventional formulation of axisymmetric plate problems.

"transposed" to each other; in the case of differentiations of even order the transposed terms have the same sign, and in the case of odd order the signs are opposite [3].

We have at our disposal two major strategies to solve elasticity problems, namely, the *stiffness method* and the *flexibility method.* In the stiffness method we start with continuous displacements and we so substitute the equations into each other that we end up with equilibrium equations, Fig. 2. The degrees of freedom are the displacements $\{u\}$. In the flexibility method we select a solution which a priori satisfies equilibrium

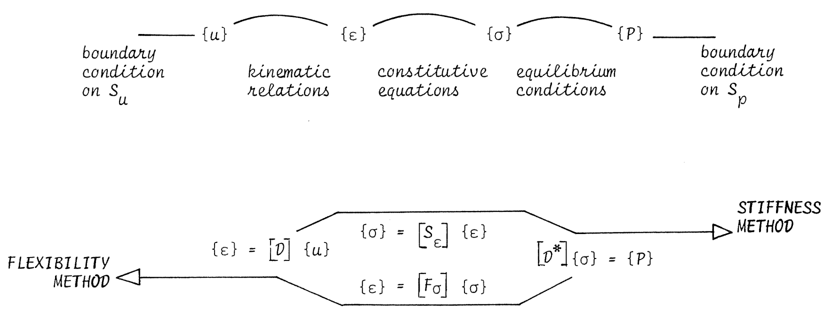


Fig. 2. Schematic representation of quantities and relations in theory of elasticity.

and we so substitute that we end up with compatibility equations, with stress functions as unknowns. It is *the essence of a rigorous and consistent formulation* that the stresses $\{\sigma\}$ and $\{\varepsilon\}$ are so chosen that their inner product determines the strain energy E' stored in a unit volume of the solid. Besides, the differential matrix-operators must be each other's transpose as defined before. These requirements have to be satisfied in the following Chapter 2. After that some applications will be shown in Chapters 3 and 4.

2 Energy-based definition of axisymmetry

In order to avoid any influence of the conventional approach, we choose a truly independent system of definitions. In the case of general axisymmetry a unit volume is a ring element of unit cross-section and a sector angle of one radial. The strain energy stored in it is called E'. To derive this energy density we compute the strain energy E which is stored in a ring with cross-sectional dimensions dr and dz, Fig. 3.

We make use of the rule that the stored strain energy equals the work done by the external load. The ring element considered is loaded by stresses σ_{rr}, σ_{zz}, σ_{zr} and σ_{rz} and by volume forces P_r and P_z. The work done by σ_{rr} on the outer face of the ring element is $\frac{1}{2}(2\pi r \cdot \sigma_{rr} \cdot u_r)dz$. On the inner face a similar energy term applies, but with an opposite sign. The total work of σ_{rr} is the sum of these two terms, which reads

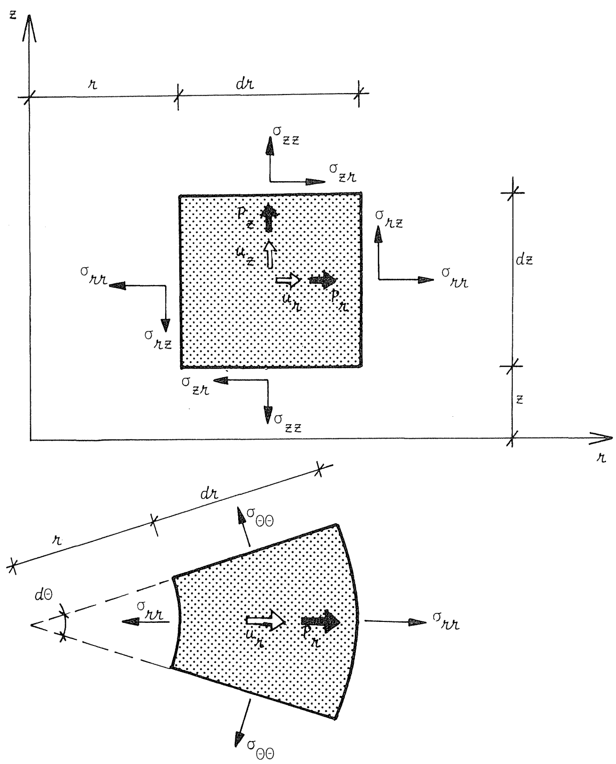


Fig. 3. Loads and stresses acting on a ring element with dimensions dr, dz and $d\theta$.

$\frac{1}{2}\frac{\partial}{\partial r}(r\sigma_{rr}\cdot u_r)dr\cdot dz\cdot 2\pi$. Likewise we can compute the work on all faces by all stresses.

The work done by the volume forces is $\frac{1}{2}(P_r u_r + P_z u_z)2\pi r\cdot dr\cdot dz$. The total work done by all the loads on the ring element is:

$$E=\tfrac{1}{2}\left\{\frac{\partial}{\partial r}(r\sigma_{rr}\cdot u_r)+\frac{\partial}{\partial z}(r\sigma_{zz}\cdot u_z)+\frac{\partial}{\partial r}(r\sigma_{rz}\cdot u_z)+\frac{\partial}{\partial z}(r\sigma_{zr}\cdot u_r)+\right.$$

$$\left.+\,rP_r\cdot u_r+rP_z\cdot u_z\right\}2\pi drdz$$

In this stage it is convenient to introduce generalized stresses and loads as follows

$$\Sigma_{rr}=r\,\sigma_{rr} \qquad \mathcal{P}_r=r\,P_r$$
$$\Sigma_{zz}=r\,\sigma_{zz} \qquad \mathcal{P}_z=r\,P_z$$
$$\Sigma_{rz}=r\,\sigma_{rz}$$
$$\Sigma_{zr}=r\,\sigma_{zr}$$

The energy E' per unit volume now becomes

$$E'=\tfrac{1}{2}\left\{\frac{\partial}{\partial r}(\Sigma_{rr}u_r)+\frac{\partial}{\partial z}(\Sigma_{zz}u_z)+\frac{\partial}{\partial r}(\Sigma_{rz}u_z)+\frac{\partial}{\partial z}(\Sigma_{zr}u_r)+\mathcal{P}_r u_r+\mathcal{P}_z u_z\right\}$$

Here a unit volume is defined by unit dimensions in r-, z- and θ-direction. Performing the differentiations yields

$$E'=\tfrac{1}{2}\left\{\left(\frac{\partial\Sigma_{rr}}{\partial r}+\frac{\partial\Sigma_{zr}}{\partial z}+\mathcal{P}_r\right)u_r+\left(\frac{\partial\Sigma_{rz}}{\partial r}+\frac{\partial\Sigma_{zz}}{\partial z}+\mathcal{P}_z\right)u_z+\right.$$

$$\left.+\,\Sigma_{rr}\frac{\partial u_r}{\partial r}+\Sigma_{zz}\frac{\partial u_z}{\partial z}+\Sigma_{rz}\frac{\partial u_z}{\partial r}+\Sigma_{zr}\frac{\partial u_r}{\partial z}\right\}$$

This specific energy can be rewritten using the three equilibrium conditions of a ring segment with dimensions $rd\theta$, dr, dz. They are

$$\frac{\partial}{\partial r}(\sigma_{rr}\cdot rd\theta\cdot dz)dr+\frac{\partial}{\partial z}(\sigma_{zr}\cdot rd\theta\cdot dr)dz-(\sigma_{\theta\theta}\cdot dr\cdot dz)d\theta+P_r\cdot rd\theta\cdot dr\cdot dz=0$$

$$\frac{\partial}{\partial r}(\sigma_{rz}\cdot rd\theta\cdot dz)dr+\frac{\partial}{\partial z}(\sigma_{zz}\cdot rd\theta\cdot dr)dz+P_z\cdot rd\theta\cdot dr\cdot dz=0$$

$$(\sigma_{rz}\cdot rd\theta\cdot dz)dr=(\sigma_{zr}\cdot rd\theta\cdot dr)dz$$

We can divide by $d\theta\cdot dr\cdot dz$ and introduce the generalized stresses and loads, which transforms the equilibrium equations into

$$\frac{\partial\Sigma_{rr}}{\partial r}+\frac{\partial\Sigma_{zr}}{\partial z}-\sigma_{\theta\theta}+\mathcal{P}_r=0$$

$$\frac{\partial\Sigma_{rz}}{\partial r}+\frac{\partial\Sigma_{zz}}{\partial z}+\mathcal{P}_z=0$$

$$\Sigma_{rz}=\Sigma_{zr}$$

These equations are now introduced into the expression E', which yields

$$E' = \tfrac{1}{2}\left\{\Sigma_{rr}\frac{\partial u_r}{\partial r} + \Sigma_{zz}\frac{\partial u_z}{\partial z} + \Sigma_{rz}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) + \sigma_{\theta\theta}\, u_r\right\}$$

From this inner product we deduce the definition of the generalized stress vector $\{\sigma\}$ and related generalized strain vector $\{\varepsilon\}$:

$$\{\sigma\}^T = \left\{\; \Sigma_{rr} \;\vdots\; \Sigma_{zz} \;\vdots\; \Sigma_{rz} \;\vdots\; \sigma_{\theta\theta} \;\right\}$$

$$\{\varepsilon\}^T = \left\{\; \frac{\partial u_r}{\partial r} \;\vdots\; \frac{\partial u_z}{\partial z} \;\vdots\; \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \;\vdots\; u_r \;\right\}$$

For the generalized strains we introduce the designations ε_{rr}, ε_{zz}, γ_{rz} and $E_{\theta\theta}$. They are related respectively to the generalized stresses Σ_{rr}, Σ_{zz}, Σ_{rz} and $\sigma_{\theta\theta}$. The equilibrium conditions and the kinematic relations now are

$$\begin{bmatrix} -\frac{\partial}{\partial r} & 0 & -\frac{\partial}{\partial z} & 1 \\ 0 & -\frac{\partial}{\partial z} & -\frac{\partial}{\partial r} & 0 \end{bmatrix} \begin{Bmatrix} \Sigma_{rr} \\ \Sigma_{zz} \\ \Sigma_{rz} \\ \sigma_{\theta\theta} \end{Bmatrix} = \begin{Bmatrix} P_r \\ P_z \end{Bmatrix} \qquad [D^*]\{\sigma\} = \{P\}$$

$$\begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \gamma_{rz} \\ E_{\theta\theta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} u_r \\ u_z \end{Bmatrix} \qquad \{\varepsilon\} = [D]\{u\}$$

The corresponding rigidity matrix is, using the conventional stiffnesses from Hooke's law:

$$\begin{Bmatrix} \Sigma_{rr} \\ \Sigma_{zz} \\ \Sigma_{rz} \\ \sigma_{\theta\theta} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} r & \nu r & 0 & \nu \\ \nu r & r & 0 & \nu \\ 0 & 0 & \frac{1-\nu}{2} r & 0 \\ \nu & \nu & 0 & \frac{1}{r} \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \gamma_{rz} \\ E_{\theta\theta} \end{Bmatrix} \qquad \{\sigma\} = [S_\varepsilon]\{\varepsilon\}$$

We conclude that the resulting generalized stresses and strains satisfy the requirements stated earlier. The differential matrix operators in the equilibrium conditions and kinematic relations are transposed to each other in the required manner, and the inner product of $\{\sigma\}$ and $\{\varepsilon\}$ determines the strain energy per unit volume. The derivation did not naturally oblige us to replace $\sigma_{\theta\theta}$ by a generalized stress $\Sigma_{\theta\theta}$ as was done for the other stresses. On the contrary, no need at all exists to do so. As a consequence, we end up with a corresponding generalized strain $E_{\theta\theta}$ which is unconventional, and the rigidity matrix needs proper attention.

If one prefers, however, to introduce $\Sigma_{\theta\theta}$ instead of $\sigma_{\theta\theta}$, one can do so. This brings us back to the conventional strain $\varepsilon_{\theta\theta} = u_r/r$ to define E' properly. The complete set of definitions then becomes

$$\{\sigma\}^T = \left\{ \Sigma_{rr} \;\vdots\; \Sigma_{zz} \;\vdots\; \Sigma_{rz} \;\vdots\; \Sigma_{\theta\theta} \right\}$$

$$\{\varepsilon\}^T = \left\{ \frac{\partial u_r}{\partial r} \;\vdots\; \frac{\partial u_z}{\partial z} \;\vdots\; \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \;\vdots\; \frac{u_r}{r} \right\}$$

$$\begin{bmatrix} -\frac{\partial}{\partial r} & 0 & -\frac{\partial}{\partial z} & \frac{1}{r} \\ 0 & -\frac{\partial}{\partial z} & -\frac{\partial}{\partial r} & 0 \end{bmatrix} \begin{Bmatrix} \Sigma_{rr} \\ \Sigma_{zz} \\ \Sigma_{rz} \\ \Sigma_{\theta\theta} \end{Bmatrix} = \begin{Bmatrix} P_r \\ P_z \end{Bmatrix} \qquad [D^*]\{\sigma\} = \{P\}$$

$$\begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \gamma_{rz} \\ \varepsilon_{\theta\theta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \\ \frac{1}{r} & 0 \end{bmatrix} \begin{Bmatrix} u_r \\ u_z \end{Bmatrix} \qquad \{\varepsilon\} = [D]\{\sigma\}$$

$$\begin{Bmatrix} \Sigma_{rr} \\ \Sigma_{zz} \\ \Sigma_{rz} \\ \Sigma_{\theta\theta} \end{Bmatrix} = \frac{rE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & \nu \\ \nu & 1 & 0 & \nu \\ 0 & 0 & \frac{1-\nu}{2} & 0 \\ \nu & \nu & 0 & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \gamma_{rz} \\ \varepsilon_{\theta\theta} \end{Bmatrix} \qquad \{\sigma\} = [S_\varepsilon]\{\varepsilon\}$$

This definition set is slightly closer to what we are accustomed to. For the rest, however, it offers no advantages. Besides that, it is not possible for plates loaded in bending, Chapter 4. In the following chapters we will use the definition set based on $\sigma_{\theta\theta}$ and $E_{\theta\theta}$.

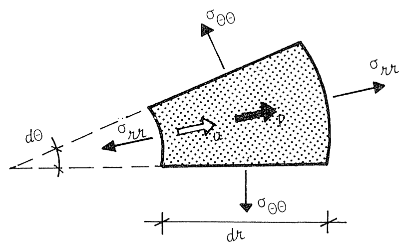


Fig. 4. Axisymmetric plate loaded in-plane.

3 Axisymmetric plate loaded in-plane

Consider the state of plane stress in which σ_{zz} and σ_{rz} do not occur. So we only consider the stresses σ_{rr} and $\sigma_{\theta\theta}$ and a volume force P, see Fig. 4. The definition set is

Strain energy: $E' = \frac{1}{2}\{\Sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}E_{\theta\theta}\}$

Equilibrium: $-\dfrac{d\Sigma_{rr}}{dr} + \sigma_{\theta\theta} = P$

Constitutive equations:

$$\left.\begin{aligned} \varepsilon_{rr} &= \frac{1}{E}\left(\frac{1}{r}\Sigma_{rr} - \nu\sigma_{\theta\theta}\right) \\ E_{\theta\theta} &= \frac{1}{E}(\Sigma_{rr} + r\sigma_{\theta\theta}) \end{aligned}\right\} \text{ flexibility method}$$

$$\left.\begin{aligned} \Sigma_{rr} &= \frac{E}{1-\nu^2}(r\varepsilon_{rr} + \nu E_{\theta\theta}) \\ \sigma_{\theta\theta} &= \frac{E}{1-\nu^2}\left(\nu\varepsilon_{rr} + \frac{1}{r}E_{\theta\theta}\right) \end{aligned}\right\} \text{ stiffness method}$$

Kinematic relations:

$$\varepsilon_{rr} = \frac{du}{dr}$$

$$E_{\theta\theta} = u$$

We will solve the general problem of a plate with edges at r_1 and r_2. Either the radial external edge loads p_1 and p_2 in the r-direction are known, or the displacements u_1 and u_2. A constant volume force P is present ($P = rP$). The flexibility method and the stiffness method will both be applied.

3.1 *Flexibility method*

We introduce a stress function ϕ. A solution for Σ_{rr} and $\sigma_{\theta\theta}$ which satisfies equilibrium is

$$\Sigma_{rr} = \phi$$

$$\sigma_{\theta\theta} = \frac{d\theta}{dr} + rP$$

Using the constitutive equations we find the generalized strains

$$\varepsilon_{rr} = \frac{\phi}{rE} - \frac{\nu}{E}\frac{d\phi}{dr} - \frac{\nu}{E} rP$$

$$E_{\theta\theta} = -\frac{\nu\phi}{E} + \frac{r}{E}\frac{d\phi}{dr} + \frac{r^2}{E} P$$

This result is substituted into the compatibility condition which results from the kinematic relations by elimination of the displacement u:

$$-\varepsilon_{rr} + \frac{dE_{\theta\theta}}{dr} = 0$$

This substitution yields the differential equation

$$\frac{d}{dr} r \frac{d\phi}{dr} - \frac{1}{r}\phi = -(2+\nu)rP$$

Or

$$\boxed{L\phi = -(2+\nu)rP}$$

in which L is the following differential operator

$$L = \frac{d}{dr} r \frac{d}{dr} - \frac{1}{r}$$

The reader easily can check that this operator also can be put in the following form

$$\boxed{L = r\frac{d}{dr}\frac{1}{r}\frac{d}{dr} r}$$

The differential equation is of the second order. The general solution of the homogeneous equation $L\phi = 0$ is

$$\phi = A\frac{1}{r} + Br$$

In the case of given edge loads p_1 at r_1 and p_2 at r_2 and no volume load P we can solve

$$A = \frac{r_1^2 - r_2^2}{r_2^2 - r_1^2}(p_2 - p_1)$$

$$B = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2}$$

The stresses then become

$$\sigma_{rr} = \frac{1}{r}\Sigma_{rr} = \frac{\phi}{r} = \frac{A}{r^2} + B$$

$$\sigma_{\theta\theta} = \frac{d\phi}{dr} = -\frac{A}{r^2} + B$$

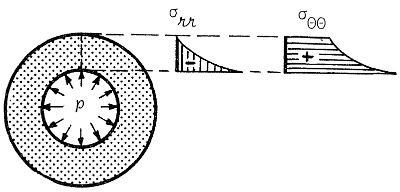


Fig. 5. Results of tick-walled tube under internal pressure.

If we take $p_2 = 0$ we get the solution of a thickwalled tube under inner pressure, see Fig. 5.

Note 1:
If we use Airy's biharmonic equation we get a fourth-order differential equation with a general solution:

$$\phi_{\text{Airy}} = A + Br^2 + C \ln r + Dr^2 \ln r$$

Now we need four boundary conditions, but we have only two. So additional consideration of displacements is needed [1]. It is clear that the direct approach advocated in this paper has considerable advantages. A much simpler differential equation has to be solved, which requires only two boundary conditions.

Note 2:
The derivation given here also applies to states in plane strain. The same operator $\boldsymbol{L}$ then occurs in the homogeneous equation.

3.2 *Stiffness method*

First we substitute the kinematic relations into the constitutive equations:

$$\Sigma_{\text{rr}} = \frac{E}{1-\nu^2}\left(r\frac{du}{dr} + \nu u\right)$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left(\nu\frac{du}{dr} + \frac{1}{r}u\right)$$

and this is fed into the equilibrium equation, yielding:

$$-\frac{E}{1-\nu^2}\boldsymbol{L}u = rP$$

The same operator $\boldsymbol{L}$ appears which was found in the flexibility method! The homogeneous equation is now:

$$\boldsymbol{L}u = 0$$

with a similar general solution for u as was found for ϕ in the flexibility method

$$u = \frac{A}{r} + Br$$

From this we find stresses

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left\{-\frac{1-\nu}{r^2}A + (\nu+1)B\right\} = \frac{A'}{r^2} + B'$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left\{\frac{1-\nu}{r^2}A + (1+\nu)B\right\} = -\frac{A'}{r^2} + B'$$

This corresponds to the solution which was found in the flexibility method. The stiffness method also applies when displacements u_1 and u_2 are specified. So it is more general than the flexibility method. The operator L also appears in cases of plane strain.

4 Axisymmetric plate loaded in bending

Consider a plate as shown in Fig. 5. The load consists of a uniformly distributed load p per unit area. Bending moments m_{rr} and $m_{\theta\theta}$ occur, and an accompanying shear force q_r, all defined per unit length. Similarly to in-plane loading we introduce generalized quantities:

$$M_{rr} = r\,m_{rr}$$
$$Q_r = r\,q_r$$
$$P = rp$$

The strain energy E' per unit area, i.e. the work done by all loads on the plate ring element, is now

$$E' = \tfrac{1}{2}\left\{\frac{d}{dr}(Q_r w) + \frac{d}{dr}(M_{rr}\varphi) + P_w\right\}$$

in which φ denotes the rotation. The definition of φ is

$$\varphi = -\frac{dw}{dr}$$

Performing the derivations in E' we arrive at

$$E' = \tfrac{1}{2}\left\{\left(\frac{dQ_r}{dr} + P\right)w - \left(Q_r - \frac{dM_{rr}}{dr}\right)Q + M_{rr}\frac{d\varphi}{dr}\right\}$$

The vertical and rotational equilibrium conditions become

$$-\frac{d}{dr}Q_r = P$$

$$Q_r = \frac{dM_{rr}}{dr} - m_{\theta\theta}$$

So we can rewrite E' as follows

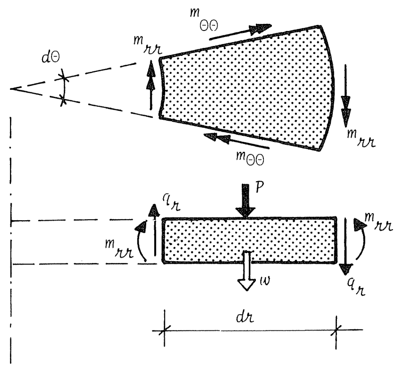


Fig. 6. Axisymmetric plate loaded in bending.

$$E' = \tfrac{1}{2}\left\{M_{\text{rr}}\frac{d\varphi}{dr} + m_{\theta\theta}\,\varphi\right\}$$

which is very similar to the expression for in-plane loading. The rotation φ now takes the place of u there. For the generalized strains $d\varphi/dr$ and φ we introduce the designations $\varkappa_{\text{rr}}$ and $K_{\theta\theta}$.

The resulting definition set is found to be

Equilibrium:
$$-\frac{d^2M_{\text{rr}}}{dr^2} + \frac{dm_{\theta\theta}}{dr} = P$$

Constitutive equations:
$$\left.\begin{aligned}\varkappa_{\text{rr}} &= \frac{1}{K_0}\left(\frac{1}{r}M_{\text{rr}} - \nu m_{\theta\theta}\right)\\ K_{\theta\theta} &= \frac{1}{K_0}\left(-\nu M_{\text{rr}} + r m_{\theta\theta}\right)\end{aligned}\right\}\ \text{flexibility method}\ \left(K_0 = \frac{Eh^3}{12}\right)$$

$$\left.\begin{aligned}M_{\text{rr}} &= K(r\varkappa_{\text{rr}} + \nu K_{\theta\theta})\\ m_{\theta\theta} &= K\left(\nu\varkappa_{\text{rr}} + \frac{1}{r}K_{\theta\theta}\right)\end{aligned}\right\}\ \text{stiffness method}\ \left(K = \frac{Eh^3}{12(1-\nu^2)}\right)$$

Kinematic relations:
$$\varkappa_{\text{rr}} = \left(\frac{d\varphi}{dr} = \right) - \frac{d^2w}{dr^2}$$

$$K_{\theta\theta} = (\varphi =) - \frac{dw}{dr}$$

Here the equilibrium equation was obtained by substitution of Q_r from the rotational equation into the vertical equation. It can easily be checked that this definition set yields a $[D]$ and $[D^*]$ which are "transposed" to each other.

The plate has edges at r_1 and r_2. At an edge in general the normal moment, the transverse shear force, the rotation or the vertical displacement can be specified. In the following we assume p to be constant. Again the flexibility method and stiffness method will both be applied.

4.1 *Flexibility method*

We introduce a stress function ϕ such that

$$M_{rr} = \phi$$

$$m_{\theta\theta} = \frac{d\phi}{dr} + \tfrac{1}{2} pr^2$$

This solution satisfies the equilibrium condition in which $P = pr$. Substitution into the constitutive relations yields:

$$\varkappa_{rr} = \frac{1}{K_0}\left(\frac{\phi}{r} - \nu\frac{d\phi}{dr} - \frac{\nu}{2}pr^2\right)$$

$$K_{\theta\theta} = \frac{1}{K_0}\left(-\nu\phi + r\frac{d\phi}{dr} + \tfrac{1}{2}pr^3\right)$$

This result is fed into the compatibility condition which is found by elimination of w (or φ) from the kinematic relations:

$$-\varkappa_{rr} + \frac{dK_{\theta\theta}}{dr} = 0$$

So we obtain a differential equation for ϕ

$$\boxed{\boldsymbol{L}\phi = -\frac{3+\nu}{2}pr^2}$$

Surprisingly, we again find the operator $\boldsymbol{L}$ which is also valid for in-plane loading. The general solution of ϕ is

$$\phi = \frac{A}{r} + Br - \frac{3+\nu}{16}pr^3$$

Hence the moments are

$$m_{rr} = \frac{\phi}{r} = \frac{A}{r^2} + B - \frac{3+\nu}{16}pr^2$$

$$m_{\theta\theta} = \frac{d\phi}{dr} + \tfrac{1}{2}pr^2 = -\frac{A}{r^2} + B - \frac{1+3\nu}{16}pr^2$$

This yields all well-known solutions which can be found from boundary conditions where moments are specified. For instance, we consider a plate with $r_1 = 0$ and $r_2 = a$ which is simply supported at $r_2 = a$, see Fig. 7. In this case A vanishes and

$$B = \frac{3+\nu}{16}pa^2:$$

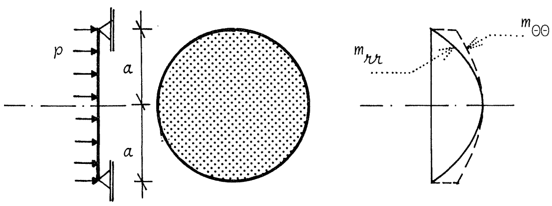


Fig. 7. Simply supported plate and homogeneously distributed load.

$$m_{rr} = \frac{3+\nu}{16}(a^2 - r^2)p$$

$$m_{\theta\theta} = \frac{3+\nu}{16}\left(a^2 - \frac{1+3\nu}{3+\nu}r^2\right)p$$

4.2 *Stiffness method*

Substitution of the kinematic relations into the constitutive equations yields:

$$M_{rr} = K\left(r\frac{d\varphi}{dr} + \nu\varphi\right)$$

$$m_{\theta\theta} = K\left(\nu\frac{d\varphi}{dr} + \frac{\varphi}{r}\right)$$

Next, we substitute this into the equilibrium condition. As we shall see later on, it is advantageous to do this in two steps. First we use the rotational equilibrium condition, which results in:

$$Q_r = K\left(\frac{d}{dr}r\frac{d}{dr} - \frac{1}{r}\right)\varphi$$

Notice herein again the ever repeating operator $\boldsymbol{L}$:

$$\boxed{Q_r = K\boldsymbol{L}\varphi}$$

We introduce this into the vertical equilibrium condition, and simultaneously we replace φ by $-dw/dr$. The differential equation for w becomes:

$$\boxed{K\frac{d}{dr}\boldsymbol{L}\frac{d}{dr}w = P}$$

Note that P equals rp. The differential equation is of the fourth order, and the general solution in the case of constant p is as follows:

$$w = A + Br^2 + C\ln r + Dr^2\ln r + \frac{pr^4}{64K}$$

This approach allows for all possible boundary conditions.

Note 1:
From $Q_r = KL\varphi$ we can derive how to express q_r into w. Using $Q_r = rq_r$, $\varphi = -dw/dr$ and

$$L = r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r$$

we find:

$$q_r = -K\frac{d}{dr}\underline{\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}w}$$

The underlined part of the right hand member can be expanded into

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}$$

This is the sum of the curvatures in the radial and the circumferential direction, apart of the sign. We can designate this sum with the Laplace operator ∇^2. So we find the well-known expression for the shear force:

$$q_r = -K\frac{d}{dr}\nabla^2 w$$

Note 2:
The fourth-order differential equation, written in full and divided by r, is for constant K:

$$K\underline{\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}}\;\underline{\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}}w = p$$

Herein we easily identify a repetition of the Laplace operator ∇^2, so the differential equation is the well-known biharmonic equation:

$$K\nabla^2\nabla^2 w = p$$

5 Summary and final remark

Axisymmetric states of elasticity can be tackled either by the flexibility method or by the stiffness method. The dual approach requires a careful definition of generalized stresses and strains. The choice which has been made is based on an energy concept. Next, the theory has been applied to simple axisymmetric plates loaded in-plane and loaded in bending. In all the resulting differential equations the same second-order differential operator appears

$$L = r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r$$

The respective homogeneous equations are:

In-plane: $\boldsymbol{L}\phi = 0$ (flexibility method)

$\boldsymbol{L}u = 0$ (stiffness method)

Bending: $\boldsymbol{L}\phi = 0$ (flexibility method)

$\frac{d}{dr}\boldsymbol{L}\frac{d}{dr}w = 0$ (stiffness method)

The theory set forth here does not produce any new and hitherto unknown solutions. The aim of this paper is to present a simple and straightforward way to obtain the results. Its objects are, firstly, to demonstrate systematic features and, secondly, to highlight the surprise of the major analogy in the several formulations.

Finally it is remarked that a consistent definition of the generalized stresses and strains is indispensable and essential for correct finite element analyses. This is particularly true for equilibrium models and hybrid formulations.

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