BOOLEAN METRIC SPACES

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KORT OVERZICHT VAN DE INHOUD

Dit proefschrift handelt over Boolese metrische ruimten M; dat zijn ruimten waarbij aan elk tweetal elementen a en b uit M een element d (a,b) van een Boolese algebra B is toegevoegd als afstand.

In het bijzonder worden onderzocht de zg. geassocieerde Boolese metrische ruimten M, ontstaan uit Boolese valuatie-ringen R door middel van de definitie d (a,b) = φ (a — b), waarbij φ de valuatie is van R in de verzameling B van idempotente elementen van R.

Ringen van partitie-afbeeldingen van een gegeven ring R in een gegeven Boolese algebra B worden bestudeerd om aan de hand hiervan complete, separabele, geassocieerde, zwak convexe Boolese metrische ruimten te kunnen construeren. Tevens worden voorwaarden aangegeven waaronder een Boolese valuatie-ring een ring van partitie-afbeeldingen is.

Na het invoeren van een topologie in de Boolese metrische ruimte M worden de begrippen maximale keten, boog en segment gedefinieerd. Verschillende eigenschappen en karakteriseringen worden afgeleid.

PREFACE

This treatise originated from an article by L. M. Blumenthal [5] [1]), Boolean geometry I, to which I will refer by BGI. Blumenthal's manuscript [2]) for the intended continuation of the article BGI, Boolean geometry II (to which I will refer by BGII) was the outset of these underlying investigations. Some of the following material is taken from BGII. Where this occurs it has been indicated.

In his paper BGI the author developes some aspects of the distance geometry of a Boolean metric space B, obtained by attaching to each two elements a and b of a Boolean algebra B the element $d(a,b) = (a \cap b') \cup (a' \cap b)$ of the algebra as distance. The methods and results of that study are entirely algebraic in the sense that no topological notions are involved. The in BGII presented continuation of the program deals with continuity notions based upon the introduction of a topology in the Boolean algebra B.

The underlying thesis "Boolean metric spaces" has been set up on a more general basis, making use of an article by J. L. Zemmer [20], Some remarks on p-rings and their Boolean geometry. This means that the sets giving rise to Boolean metric spaces are not restricted to Boolean algebras. We will also allow so called Boolean valued rings as sets from which to obtain Boolean metric spaces.

It follows from a result obtained by W. Krull [3]) that any Boolean valued ring may be considered as a subdirect sum of integral domains [4]). Since Krull's result is rather deep and since it turned out to be possible to prove several properties of Boolean valued rings without making use of this result, we have tried to refrain from basing our developement on the above mentioned theorem. In fact we have kept our results completely independent of it.

[1]) Numbers between brackets refer to the References at the back of this thesis.
[2]) Not published; but see the abstract [5a].
[3]) [15], references at the bottom of p. 113.
[4]) [15], Theorem 31, p. 123.

For more extensive and detailed information on the field of distance geometry we refer to L. M. Blumenthal [4], Theory and applications of distance geometry, especially to chap. xv. For the lattice-theoretical aspect we refer to H. Hermes [10], Einführung in die Verbandstheorie.

Most of the notation will be developed in the text. A few general remarks may be given here. Ring operations will be indicated by the usual justaposition and $+$. Boolean operations will be denoted by $\cap$ and $\cup$. For the complementation in a Boolean algebra we use the accent $'$, while the order-relation is written $\leq$, $<$ meaning $\leq$ and $\neq$. Since set-operations are Boolean operations we will make no distinction between these unless confusion might occur. In that case the set operations are denoted by $\odot$, $\uplus$, $\leqslant$ and c for complementation.

In general capitals will be used for sets.

CHAPTER I

ASSOCIATE RINGS

1. Boolean algebras, Boolean rings and idempotents.

It is known that Boolean rings with identity may be identified with Boolean algebras [1]; i.e. a Boolean ring with identity can be considered as a Boolean algebra under suitable modifications of the ring operations and conversely. The Boolean operations expressed in terms of the ring operations are

$$\begin{aligned} a \cap b &= ab, \\ a \cup b &= a + b + ab, \\ a' &= 1 + a. \end{aligned}$$

The ring operations expressed in terms of the Boolean operations are

$$\begin{aligned} ab &= a \cap b, \\ a + b &= (a' \cap b) \cup (a \cap b'). \end{aligned}$$

It is also known that the idempotents of a commutative ring R with identity form a Boolean ring B with identity and hence a Boolean algebra [2]. The operations of the set B, considered as a Boolean algebra, expressed in terms of the ring operations are

$$\begin{aligned} a \cap b &= ab, \\ a \cup b &= a + b - ab, \\ a' &= 1 - a. \end{aligned}$$

The operations of the set B, considered as a Boolean ring, expressed in terms of the ring operations are

$$\begin{aligned} a \otimes b &= ab, \\ a \oplus b &= a + b - 2\,ab. \end{aligned}$$

In order to avoid the difficulty and inconvenience of distinguishing between too many kinds of operations we will always consider the set B of idempotents of a commutative ring R with identity as a

[1] [18]; also [10], § 22 and [11], chap. VII.
[2] [8].

Boolean algebra. Thus we only have to distinguish between Boolean operations and ring operations. Since the Boolean multiplication coincides with the ring multiplication, mostly the justaposition will be used. Only when the fact is to be stressed that a Boolean multiplication is meant, we will use the Boolean notation (cap).

2. Boolean valued rings.

2.1 DEFINITION. *A commutative ring* R *with identity is called a* BOOLEAN VALUED RING [1]), *provided there exists a mapping* φ *of* R *into a Boolean algebra* B

$$\varphi: a \rightarrow \varphi(a) \qquad a \in R, \varphi(a) \in B, \textit{ such that}$$

$$\text{(i)} \quad \varphi(a) = 0 \quad \text{if and only if} \quad a = 0,$$
$$\text{(ii)} \quad \varphi(ab) = \varphi(a) \cap \varphi(b),$$
$$\text{(iii)} \quad \varphi(a + b) \leq \varphi(a) \cup \varphi(b).$$

The subset of B consisting of all images of R under the mapping φ will be indicated by Φ (R).

Some properties of Boolean valued rings, immediate consequences of the definition, are:

2.1.1 $a^n = 0$ implies $a = 0$;

or: a Boolean valued ring does not contain proper (i.e. non zero) nilpotent elements.

2.1.2 $\varphi(a) = \varphi(-a)$.

2.1.3 $\varphi(a) \leq \varphi(1)$ for all $a \in R$.

2.1.4 $\varphi(a) = \varphi(1)$ if a is a unity of R (i.e. if $a^{-1} \in R$).

2.1.5 $\varphi(a + b) = \varphi(a) \cup \varphi(b)$ if $ab = 0$.

From 2.1.1 it follows that R is isomorphic to a subdirect sum of integral domains [2]). As pointed out in the Preface we will not make use of this fact in the sequel. Since, however, conversely any subdirect sum with identity of integral domains is a Boolean valued ring, as we will show, one would then have

2.2 THEOREM. *A commutative ring* R *with identity is a Boolean valued ring if and only if* R *does not contain proper nilpotent elements.*

To show that any subdirect sum with identity of integral domains is a Boolean valued ring we remark that such a subdirect sum is a subring of a full direct sum of integral domains. A full direct sum

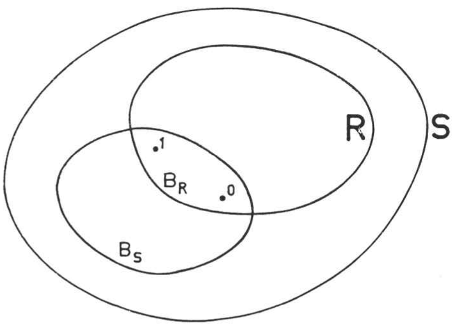
[1]) [20].

[2]) [15], Theorem 31, p. 123.

of integral domains may be considered as the commutative ring S with identity consisting of all functions f of a variable $\omega \in \Omega$, such that $f(\omega) \in I(\omega)$, where $I(\omega)$ is an integral domain for each $\omega \in \Omega$ [1]. Equality, addition and multiplication of elements of the full direct sum are to be defined component-wise, where $f(\omega)$ is the ω-component of the element f of the full direct sum. We will denote the full direct sum by

$$S = \sum_{\omega \in \Omega}^{*} I(\omega),$$

Ω being the cardinality of the set of component integral domains $I(\omega)$. The idempotents of S are those functions of ω that only assume the values zero and one, as $f(\omega) f(\omega) = f(\omega)$ is equivalent to $f(\omega) = 0$ or $f(\omega) = 1$ since $I(\omega)$ is an integral domain. One could say: the idempotents of S are the characteristic functions in S. This Boolean algebra B_S of idempotents of S will be used for the valuation of R. The situation therefore is:



S: full direct sum of integral domains.
R: subdirect sum of S.
B_S: idempotents of S.
B_R: idempotents of R.

It follows easily that if a subdirect sum of integral domains has an identity, this must be the identity of the full direct sum, i.e. the function f such that $f(\omega) = 1 \in I(\omega)$ for all $\omega \in \Omega$.
Using the fact that for the Boolean algebra B_S of idempotents of S $f \leq g$ is equivalent to $f \cap g = f$ (or to $f \cup g = g$) one proves easily: the idempotent f precedes the idempotent g if and only if $f(\omega) = 0$ for all those elements $\omega \in \Omega$ for which $g(\omega) = 0$.

[1]) [16].

Now let $s \in S$ and let σ be the characteristic function (i.e. idempotent) in B_S defined by

$$\sigma(\omega) = 0 \quad \text{if} \quad s(\omega) = 0,$$
$$\sigma(\omega) = 1 \quad \text{if} \quad s(\omega) \neq 0.$$

Then the mapping φ

$$\varphi\colon s \to \varphi(s) = \sigma, \qquad s \in S, \sigma \in B_S$$

is a mapping of the full direct sum S onto the Boolean algebra B_S of idempotents of S. One may easily verify that the conditions for a Boolean valuation are satisfied. Since R is a subring of $S = \sum_{\omega\in\Omega}^{*} I(\omega)$ we thus have constructed a Boolean valuation for R.

Property 2.1.3 implies that one can always assume that $\varphi(1) = 1$. For if $\varphi(1) \neq 1$, consider the subset B^* of B consisting of all elements of B preceding $\varphi(1)$. This is a distributive sublattice with 0 and $\varphi(1)$ as its least and greatest element. Defining $u^* = u' \cap \varphi(1)$ for all $u \in B^*$ one sees readily that

$$u \cap u^* = 0,$$
$$u \cup u^* = \varphi(1).$$

Thus u^* is the complement of u in B^* so that B^* is a Boolean algebra.

Sussman [19] introduced the notion of associate ring. Since we want to refrain from using the fact that a Boolean valued ring may be considered as a subdirect sum of integral domains, we give a different definition.

2.3 DEFINITION. *If* B *denotes the Boolean algebra of all idempotents of a Boolean valued ring* R, *we call* R *an* ASSOCIATE RING *provided*

$$\text{(i)} \quad \Phi(R) \leq B,$$
$$\text{(ii)} \quad \varphi(u) = u \quad \text{for all} \quad u \in B.$$

If, in addition, the Boolean algebra B *is complete, we call* R *a* COMPLETE, ASSOCIATE RING.

Denoting by C the set of all $x \in R$ such that $\varphi(x) = 1$ and by U the set of all unities of R the following properties of associate rings may easily be proved

2.3.1 $\Phi(R) = B$ and $\varphi(1) = 1$.
2.3.2 $\varphi(b_1 - b_2) = 1$ if and only if $b_1 = b_2'$ $(b_1, b_2 \in B)$.
2.3.3 $a\,\varphi(a)' = 0$ for all $a \in R$.
2.3.4 $a\,\varphi(a) = a$ for all $a \in R$.
2.3.5 $U \leq C$.
2.3.6 $a + \varphi(a)' \in C$.

As an example of associate rings we will discuss the commutative regular rings with identity [1]). A ring R is called regular provided for each element $a \in R$ there exists an element $x \in R$ such that $axa = a$.

2.4 THEOREM. *A commutative regular ring* R *with identity is an associate ring* [2]).

Proof. Let $a \in R$ and let $x \in R$ such that $a^2x = a$. Such an element x must exist in R since R is regular. Suppose there was also another element $y \in R$ such that $a^2y = a$. Then we would have $ax = ay$ since $a^2xy = ax$ but also $a^2xy = ay$. This proves that the mapping φ

$$\varphi: a \to \varphi(a) = ax; \qquad a,x \in R \text{ such that } a^2x = a$$

is single valued.

Furthermore ax is an idempotent of R since $axax = ax$. If u is an idempotent of R we have $\varphi(u) = u$. Left to verify whether the properties required for a Boolean valuation are satisfied.

(i) $a = 0$ implies $ax = 0$; also $ax = 0$ implies $a = 0$; for if not, we would have $a^2x = a$ while $(ax)\,a = 0$ and $a \neq 0$.

(ii) if $\varphi(a) = ax$ and $\varphi(b) = by$, where $a^2x = a$ and $b^2y = b$, we see immediately that $\varphi(ab) = abxy = \varphi(a)\,\varphi(b)$ since $a^2b^2xy = ab$.

(iii) $\varphi(a) = ax$; $\varphi(b) = by$; $\varphi(a + b) = (a + b)z$; again holding $a^2x = a$, $b^2y = b$ and $(a + b)^2z = a + b$. We have to show that $\varphi(a + b) \leq \varphi(a) \cup \varphi(b)$, which is equivalent to $\varphi(a + b)\{\varphi(a) \cup \varphi(b)\} = \varphi(a + b)$. Expressed in terms of the ring operations solely this means: $\varphi(a + b)\{\varphi(a) + \varphi(b) - \varphi(a)\,\varphi(b)\} = \varphi(a + b)$.

Straight forward substitution and computation shows that the equality is valid. Examples of regular rings are the p-rings [3]). A p-ring (p is prime) is a ring with more than one element, with the property that for every element a it holds that $a^p = a$ and $pa = 0$. p-Rings are necessarily commutative. Furthermore: a p-ring is regular since for any element a of the ring it holds $aa^{p-2}a = a$, thus satisfying the requirement for regular rings. A special instance of p-rings are 2-rings, the so called Boolean rings.

Thus we can say that p-rings with identity are associate rings.

The valuation in case of p-rings is

[1]) [17].

[2]) see also [19].

[3]) [15], chap. VII.

$$\varphi\colon a \to \varphi(a) = aa^{p-2} = a^{p-1},$$

and more in particular for 2-rings (Boolean rings)

$$\varphi\colon a \to \varphi(a) = a,$$

so that in case of a 2-ring the valuation is the identity mapping of the ring onto itself.
As any Boolean algebra can be converted into a Boolean ring (2-ring), we see that also Boolean algebras are instances of associate rings.

Finally we prove

2.5 THEOREM. *An associate ring* R *is regular if and only if* U = C.
Proof. Suppose U = C. Let $a \in R$; then $\varphi(a) \in B$ and $\varphi(a)' \in B$. Set $a^* = a + \varphi(a)'$, so that $a \in C$ (prop. 2.3.6) and hence $a \in U$ so that $(a^*)^{-1} \in R$. Now we have $aa^* = a(a + \varphi(a)') = a^2$; thus $a^2 (a^*)^{-1} = a$ and R is regular.
Conversely, suppose that R is regular. Let $a \in C$; then $a^2x = a$ for some $x \in R$. $\varphi(a) = 1$ and $\varphi(a) = ax$, from which $ax = 1$, so that $a \in U$. Since we also have $U \leq C$ (prop. 2.3.5) it follows U = C.

3. Boolean metric spaces.

3.1 DEFINITION. *An abstract set* M *is called a* BOOLEAN METRIC SPACE [1], *provided there exists a mapping* d *of* M × M *into a Boolean algebra* B

$$d\colon (a,b) \to d(a,b),\quad (a,b) \in M \times M,\quad d(a,b) \in B$$

such that

$$\text{(i)}\quad d(a,b) = 0 \quad \text{if and only if} \quad a = b,$$
$$\text{(ii)}\quad d(a,b) = d(b,a),$$
$$\text{(iii)}\quad d(a,b) \leq d(a,c) \cup d(c,b).$$

3.2 THEOREM. *Every Boolean valued ring* R *can be made into a Boolean metric space* M *by defining* $d(a,b) = \varphi(a - b)$. R *will be said to be the underlying set of* M *and* M *will be said to be obtained from* R.

Proof. One may easily verify that $\varphi(a - b)$ satisfies the requirements for a Boolean distance.
Denoting the set of all distances of pairs of elements of R by D (R), we have D (R) = Φ (R), so that in case of an associate ring R we have D (R) = B, where B is, again, the Boolean algebra of idempotents of R.

[1] [6], [7], [5], [20].

3.3 DEFINITION. *A Boolean metric space* M *obtained from a (complete) associate ring* R *by defining* d (a,b) $= \varphi$ (a — b) *for* A,b $\in$ R, *will be called a* (COMPLETE) ASSOCIATE BOOLEAN METRIC SPACE.
In the same way: a REGULAR BOOLEAN METRIC SPACE M is a Boolean metric space obtained from a commutative regular ring R with identity.
If the underlying set R is a p-ring with identity the Boolean metric space M, obtained from R, will be called a BOOLEAN METRIC p-SPACE.
In particular a BOOLEAN METRIC 2-SPACE is a Boolean metric space M obtained from a Boolean ring R (2-ring) with identity or a Boolean algebra R; i.e. from a ring R for which every element is idempotent, so that R = B.
It is this class of Boolean metric 2-spaces that Blumenthal deals with in BGI and BGII.

3.4 THEOREM. *If* M *is an associate Boolean metric space and* B *the Boolean algebra of idempotents of* R, *the underlying set of* M, B *is a Boolean metric 2-space with* d (a,b) = a'b $\cup$ b'a *for all* a,b $\in$ B. *We call* B *the Boolean metric 2-space* ASSOCIATED *with* M.

Proof. $a - b = a(1 - b) - b(1 - a)$;
$\varphi(a - b) \leq \varphi(a)\,\varphi(1 - b) \cup \varphi(b)\,\varphi(1 - a)$.

Referring to section 1 of this chapter we have: $1 - a = a'$ and $1 - b = b'$.
Since R is an associate ring and a, b, a', b' are all elements of B, so that $\varphi(a) = a$, $\varphi(b) = b$, $\varphi(a') = a'$ and $\varphi(b') = b'$; we thus have $\varphi(a - b) \leq ab' \cup a'b$ . . . . (i)

Moreover: $ab'(a - b) = ab'$,
from which $\varphi(ab')\,\varphi(a - b) = \varphi(ab')$,
or $\varphi(ab') \leq \varphi(a - b)$, or $ab' \leq \varphi(a - b)$.

Similarly: $\varphi(a'b) \leq \varphi(a - b)$, or $a'b \leq \varphi(a - b)$,
so that $a'b \cup b'a \leq \varphi(a - b)$ . . . . (ii)

(i) and (ii) imply: $\varphi(a - b) = a'b \cup b'a$.

3.5 COROLLARY. *If* M *is a Boolean metric 2-space,* d (a,b) = a'b $\cup$ b'a *for all* a,b $\in$ R [1]).
Proof. R is a Boolean ring (2-ring) so that all elements are idempotent: R = B.
Note: in the sequel Boolean metric p-spaces will be denoted by M_p.

[1]) see also [5].

3.6 DEFINITION. *A distance-preserving correspondence between the elements of two subsets of an associate Boolean metric space* M *is called a* CONGRUENCE, *and such a mapping of the space onto itself is called a* MOTION.

It is obvious that for a fixed element $a \in M$ the mapping $m(x) = x + a$ is a motion. This class of motions will be called TRANSLATIONS. There is a unique translation that takes any assigned element a into any assigned element b, namely the translation $m(x) = x + (b-a)$. If M_2 is a Boolean metric 2-space, we see from section 1 and from COROLLARY 3.5 that the translation $m(x) = a + x$ becomes $m(x) = d(x,a)$ or $m(x) = d(x,m(0))$ since $a = m(0)$. Blumenthal[1]) has proved that any motion of a Boolean metric 2-space M_2 can be written as $m(x) = d(x,m(0))$. Thus we may say that for Boolean metric 2-spaces translations are the only motions.

Blumenthal also showed that any congruent mapping f of M_2 into itself is involutary: $ff(x) = x$, for all $x \in M_2$, from which it follows that f is a motion. Since these results also apply to the Boolean metric 2-space B, associated with any associate Boolean metric space M, we can say that every congruent mapping f of M such that $f(B) \leq B$ can be written as $f(x) = d(x,f(0))$ as far as f applies to B, i.e. for all $x \in B \leq M$; and also that any such congruence f is involutory for B: $ff(x) = x$ for all $x \in B$, from which it follows that f is a motion of B. Zemmer[2]) has described the motions of Boolean metric p-spaces M_p by means of matrices with elements from the Boolean algebra of idempotents of the p-ring R, underlying M_p.

3.7 DEFINITION. *A subset* $\{a_\alpha\}$ *of an associate Boolean metric space* M *is called a* METRIC BASIS *for* M, *provided every element* $x \in M$ *is uniquely determined by its distances* $d(x,a_\alpha)$ *from the elements of the set* $\{a_\alpha\}$.

It follows readily from LEMMA 2.2 chap. III that the set of constants of a homogeneous Boolean valued ring R, underlying a Boolean metric space M, forms a metric basis.

This implies that the identity 1 and its successive summands 2,3,......,p of a Boolean metric p-space M_p form a metric basis for M_p [2]).

[1]) [4], § 133, p. 334.
[2]) [20].

CHAPTER II

RINGS OF PARTITIONAL MAPPINGS

1. The rings KB and KB*.

Let B be a complete Boolean algebra and K a commutative ring with identity. For the sequel it is of importance to remark that in a complete Boolean algebra the distributive law

$$x \cap \bigcup y_\alpha = \bigcup (x \cap y_\alpha) \quad \text{(and dually)}$$

holds [1]), from which

$$\bigcup x_\alpha \cap \bigcup y_\beta = \bigcup (x_\alpha \cap y_\beta) \quad \text{(and dually).}$$

1.1 DEFINITION. *A mapping* ψ

$$\psi \colon a \to \psi(\alpha) \qquad \alpha \in K,\ \psi(\alpha) \in B$$

of a commutative ring K *with identity into a complete Boolean algebra* B *such that*

(i) $\psi(\mu)\,\psi(\nu) = 0$ if $\mu \neq \nu$,
which is also expressed by saying that the elements $\psi(\alpha)$, $\alpha \in K$ are PAIRWISE ORTHOGONAL,

(ii) $\bigcup\limits_{\alpha \in K} \psi(\alpha) = 1$

is called a PARTITIONAL MAPPING.

We say that ψ *assumes the value* α on $\psi(\alpha)$ if $\psi(\alpha) \neq 0$ and that ψ *does not assume* the value α if $\psi(\alpha) = 0$.
A partitional mapping is called a FINITE partitional mapping if ψ only assumes finitely many values.

1.2 THEOREM. *The set of all partitional mappings of a commutative ring* K *with identity into a complete Boolean algebra* B *is a commutative ring* KB* *with identity.*
If F *is a field, the ring* FB* *is regular.*

Proof. We first introduce a multiplication and addition for the elements of KB*.

1) [10], Satz 24.1, p. 130.

Therefore let ψ_1 and ψ_2 be two elements of KB*

$$\psi_1 \colon \alpha \to \psi_1(\alpha); \quad \alpha \in K, \psi_1(\alpha) \in B;$$
$$\psi_2 \colon \alpha \to \psi_2(\alpha); \quad \alpha \in K, \psi_2(\alpha) \in B.$$

Then we define

$$\psi_1 \psi_2 \colon \alpha \to \bigcup_{\mu\nu=\alpha} \psi_1(\mu)\, \psi_2(\nu)$$

$$\psi_1 + \psi_2 \colon \alpha \to \bigcup_{\mu+\nu=\alpha} \psi_1(\mu)\, \psi_2(\nu)$$

It is clear that the mappings $\psi_1 \psi_2$ and $\psi_1 + \psi_2$ both belong to KB*. Furthermore it is clear that the multiplication and the addition are commutative. That the multiplication is associative may be seen as follows

$(\psi_1 \psi_2)\, \psi_3 \colon \alpha \to \bigcup_{\varkappa\lambda=\alpha} \{(\bigcup_{\mu\nu=\varkappa} \psi_1(\mu)\, \psi_2(\nu))\, \psi_3(\lambda)\}$, which is equivalent to $\alpha \to \bigcup_{\varkappa\lambda=\alpha} \bigcup_{\mu\nu=\varkappa} \psi_1(\mu)\, \psi_2(\nu)\, \psi_3(\lambda)$, or to $\alpha \to \bigcup_{\mu\nu\lambda=\alpha} \psi_1(\mu)\, \psi_2(\nu)\, \psi_3(\lambda)$.

Similarly we find $\psi_1(\psi_2 \psi_3) : \alpha \to \bigcup_{\mu\nu\lambda=\alpha} \psi_1(\mu)\, \psi_2(\nu)\, \psi_3(\lambda)$,

from which $(\psi_1 \psi_2)\, \psi_3 = \psi_1(\psi_2 \psi_3)$.

The associativity for the addition is proved similarly.

To prove the distributive law we have

$\psi_1(\psi_2 + \psi_3) : \alpha \to \bigcup_{\lambda\varkappa=\alpha} \psi_1(\lambda)\, \{\bigcup_{\mu+\nu=\varkappa} \psi_2(\mu)\, \psi_3(\nu)\}$, or equivalently

$$\alpha \to \bigcup_{\lambda\varkappa=\alpha} \bigcup_{\mu+\nu=\varkappa} \psi_1(\lambda)\, \psi_2(\mu)\, \psi_3(\nu),$$

or also $\alpha \to \bigcup_{\lambda(\mu+\nu)=\alpha} \psi_1(\lambda)\, \psi_2(\mu)\, \psi_3(\nu)$ . . . . (i)

Furthermore we have

$$\psi_1 \psi_2 : \alpha \to \bigcup_{\mu\nu=\alpha} \psi_1(\mu)\, \psi_2(\nu) \text{ and } \psi_1 \psi_3 : \alpha \to \bigcup_{\mu\nu=\alpha} \psi_1(\mu)\, \psi_3(\nu)$$

from which $\psi_1 \psi_2 + \psi_1 \psi_3 : \alpha \to \bigcup_{\beta+\gamma=\alpha} [\{\bigcup_{\rho\mu=\beta} \psi_1(\rho)\, \psi_2(\mu)\}$

$$\{\bigcup_{\sigma\nu=\gamma} \psi_1(\sigma)\, \psi_2(\nu)\}]$$

which is equal to $\alpha \to \bigcup_{\beta+\gamma=\alpha} \bigcup_{\rho\mu=\beta} \bigcup_{\sigma\nu=\gamma} \psi_1(\rho)\, \psi_2(\mu)\, \psi_1(\sigma)\, \psi_3(\nu)$.

But since $\psi_1(\rho)\, \psi_1(\sigma) = 0$ if $\rho \neq \sigma$ and $\psi_1(\rho)\, \psi_1(\sigma) = \psi_1(\rho) = \psi_1(\sigma)$ if $\rho = \sigma$

we find $\psi_1 \psi_2 + \psi_1 \psi_3 : \alpha \to \bigcup_{\beta+\gamma=\alpha} \bigcup_{\lambda\mu=\beta} \bigcup_{\lambda\nu=\gamma} \psi_1(\lambda)\, \psi_2(\mu)\, \psi_3(\nu)$

which is equivalent to $\alpha \to \bigcup_{\lambda(\mu+\nu)=\alpha} \psi_1(\lambda)\, \psi_2(\mu)\, \psi_3(\nu)$ . . . . . (ii)

From (i) and (ii) it follows $\psi_1(\psi_2 + \psi_3) = \psi_1\psi_2 + \psi_1\psi_3$.

To complete the proof that KB* is a ring with identity we have to indicate the zero and the identity of KB* together with a negative for each $\psi \in$ KB*.
The zero of KB* is the mapping $0 : \alpha \to 0(\alpha)$; $\alpha \in R$, $0(\alpha) \in B$, such that

$$0(0) = 1 \quad \text{and } 0(\alpha) = 0 \quad \text{if } \alpha \neq 0.$$

Apparently $0 \in$ KB*. It holds that $\psi + 0 = \psi$ for every $\psi \in$ KB* as may be seen from $\psi + 0 : \alpha \to \bigcup_{\mu+\nu=\alpha} \psi(\mu)\, 0(\nu) = \psi(\alpha)$

For $\psi \in$ KB*, $\psi : \alpha \to \psi(\alpha)$
we introduce the mapping $-\psi : \alpha \to \psi(-\alpha)$
We then have $\psi + (-\psi) : \alpha \to \bigcup_{\mu+\nu=\alpha} \psi(\mu)\,\psi(-\nu)$;

(i) $\bigcup_{\mu+\nu=0} \psi(\mu)\,\psi(-\nu) = \bigcup_{\mu} \psi(\mu) = 1$,

(ii) $\bigcup_{\mu+\nu=\alpha\neq 0} \psi(\mu)\,\psi(-\nu) = 0$, since $\mu \neq -\nu$.

From (i) and (ii) it follows $\psi + (-\psi) = 0$.

The identity of KB* is the mapping $1 : \alpha \to 1(\alpha)$; $\alpha \in R$, $1(\alpha) \in B$, such that

$$1(1) = 1 \text{ and } 1(\alpha) = 0 \text{ if } \alpha \neq 1.$$

Apparently $1 \in$ KB* It holds that $\psi 1 = 1$ for every $\psi \in$ KB* as may be seen from $\psi 1 : \alpha \to \bigcup_{\mu\nu=\alpha} \psi(\mu)\, 1(\nu) = \psi(\alpha)$.

To show that for a field F the ring FB* is regular we consider for the mapping $\psi \in$ FB* the mapping $\psi^* : \alpha \to \psi^*(\alpha)$; $\alpha \in F$, $\psi^*(\alpha) \in B$, such that

$$\psi^*(\alpha) = \psi\left(\frac{1}{\alpha}\right) \text{ if } \alpha \neq 0 \text{ and } \psi^*(\alpha) = \psi(0) \text{ if } \alpha = 0,$$

and we will show $\psi\,\psi^*\,\psi = \psi$.

$$\psi\,\psi^*\,\psi : \alpha \to \bigcup_{\varkappa\lambda\mu=\alpha} \psi(\varkappa)\,\psi^*(\lambda)\,\psi(\mu);$$

(i) $\alpha \neq 0$; then $\lambda \neq 0$, so that we have $\bigcup_{\varkappa\lambda\mu=\alpha} \psi(\varkappa)\,\psi^*(\lambda)\,\psi(\mu) =$

$$\bigcup_{\varkappa\lambda\mu=\alpha} \psi(\varkappa)\,\psi\left(\frac{1}{\lambda}\right)\psi(\mu) = \bigcup_{\varkappa^2\lambda=\alpha} \psi(\varkappa)\,\psi\left(\frac{1}{\lambda}\right) = \bigcup_{\varkappa=\alpha} \psi(\varkappa) = \psi(\alpha).$$

(ii) $\alpha = 0;\ \bigcup_{\varkappa\lambda\mu=\alpha} \psi(\varkappa)\,\psi^*(\lambda)\,\psi(\mu) =$

$$\{ \bigcup_{\substack{\varkappa\lambda\mu=0\\ \lambda\neq 0}} \psi(\varkappa)\,\psi\left(\frac{1}{\lambda}\right)\psi(\mu) \} \cup \{ \bigcup_{\substack{\varkappa\lambda\mu=0\\ \lambda=0}} \psi(\varkappa)\,\psi(0)\,\psi(\mu) \} =$$

$$\{ \bigcup_{\substack{\varkappa^2\lambda=0\\ \lambda\neq 0}} \psi(\varkappa)\,\psi\left(\frac{1}{\lambda}\right) \} \cup \psi(0) = \{ \bigcup_{\lambda\neq 0} \psi(0)\,\psi\left(\frac{1}{\lambda}\right) \} \cup \psi(0) =$$

$$0 \cup \psi(0) = \psi(0).$$

This completes the proof of THEOREM 2.2.
It follows from the definition of a finite partitional mapping ψ that $\psi(\alpha)$ only differs from zero for finitely many elements $\alpha \in K$. Repeating the preceding construction of KB*, but now only allowing finite partitional mappings to occur, one obtains a commutative ring KB with identity. Again, if F is a field, FB is regular. B need not be complete for this construction. Thus we have

2.3 THEOREM. *The set of all finite partitional mappings of a commutative ring* K *with identity into a Boolean algebra* B *is a commutative ring* KB *with identity.*
If F *is a field, the ring* FB *is regular.*
It may be noted that $KB \leq KB^*$ and that $KB = KB^*$ if K is finite.

2. The sets $< K,B >$, K^* and B^*.

Let KB be the set of all finite partitional mappings of a commutative ring K with identity into a Boolean algebra B. We will consider the subset $< K,B >$ of KB, consisting of all finite partitional mappings $< \xi,b >$

$$< \xi,b >:\ \alpha \rightarrow < \xi,b > (\alpha);\ \alpha \in K,\ 0 \neq \xi \in K,\ 0 \neq b \in B$$

such that (i) $< \xi,b > (\xi) = b$,
(ii) $< \xi,b > (0) = b'$,
(iii) $< \xi,b > (\alpha) = 0$ if $0 \neq \alpha \neq \xi$,

while for $\xi = 0$ or $b = 0$ we define the mapping $< \xi,b >$ to be the zero mapping $0 \in KB$.

2.1 LEMMA. $< \xi_1,b_1 > < \xi_2,b_2 > = < \xi_1\,\xi_2,b_1\,b_2 >$.

Proof. Consider $\bigcup_{\mu\nu=\alpha} < \xi_1,b_1 > (\mu) < \xi_2,b_2 > (\nu)$.

$< \xi_1,b_1 > (\mu)$ is zero for all μ such that $0 \neq \mu \neq \xi_1$ due to (iii) above;

$< \xi_2, b_2 > (\nu)$ is zero for all ν such that $0 \neq \nu \neq \xi_2$ due to (iii) above. Therefore we only have to investigate the following cases:

$\alpha = \mu\nu$	μ	ν	$< \xi_1, b_1 > (\mu) < \xi_2, b_2 > (\nu)$
$\xi_1 \xi_2$	ξ_1	ξ_2	$b_1 b_2$
0	ξ_1	0	$b_1 b_2'$
0	0	ξ_2	$b_1' b_2$
0	0	0	$b_1' b_2'$

Since $b_1 b_2' \cup b_1' b_2 = (b_1 b_2)'$, it follows

$$\bigcup_{\mu\nu=\alpha} < \xi_1, b_1 > (\mu) < \xi_2, b_2 > (\nu) = < \xi_1 \xi_2, b_1 b_2 > (\alpha)$$

or $< \xi_1, b_1 > < \xi_2, b_2 > = < \xi_1 \xi_2, b_1 b_2 >$.

2.2 Corollary. $< \xi, b > = < \xi, 1 > < 1, b >$.

2.3 Lemma. $< \xi_1, b > + < \xi_2, b > = < \xi_1 + \xi_2, b >$.
Proof. Similar as above.

2.4 Corollary. *For any integer* k *it holds that* $k < \xi, b > = < k\xi, b >$.

Proof. It follows from Lemma 2.3 by induction that $n < \xi, b > = < n\xi, b >$, for any natural member n. Since $- < \xi, b > = < -\xi, b >$ by definition, we have $k < \xi, b > = < k\xi, b >$ for any integer k.

2.5 Lemma. $< \xi, b_1 > + < \xi, b_2 > = < \xi, b_1 \cup b_2 >$ if $b_1 b_2 = 0$.

Proof. Consider $\bigcup_{\mu+\nu=\alpha} < \xi, b_1 > (\mu) < \xi, b_2 > (\nu)$. For the same reasons as above we only have to investigate the following cases:

$\alpha = \mu + \nu$	μ	ν	$< \xi, b_1 > (\mu) < \xi, b_2 > (\nu)$
2ξ	ξ	ξ	$b_1 b_2 = 0$
ξ	ξ	0	$b_1 b_2' = b_1$ since $b_1 \leq b_2'$
ξ	0	ξ	$b_1' b_2 = b_2$ since $b_2 \leq b_1'$
0	0	0	$b_1' b_2' = (b_1 \cup b_2)'$

Thus it follows that $\bigcup\limits_{\mu+\nu=\alpha} <\xi,b_1>(\mu)<\xi,b_2>(\nu) = <\xi,b_1 \cup b_2>(\alpha)$ or $<\xi,b_1> + <\xi,b_2> = <\xi,b_1 \cup b_2>$.

2.6 LEMMA. $<\xi_1,b_1> + <\xi_2,b_2> = <\xi_1+\xi_2,b_1 b_2> +$
$<\xi_1,b_1 b_2'> + <\xi_2,b_1' b_2>$.

Proof. $<\xi_1,b_1> = <\xi_1,b_1 b_2 \cup b_1 b_2'> = <\xi_1,b_1 b_2> + <\xi_1,b_1 b_2'>$
$<\xi_2,b_2> = <\xi_2,b_1 b_2 \cup b_1' b_2> = <\xi_2,b_1 b_2> + <\xi_2,b_1' b_2>$

from which the result, using LEMMA 2.3.

Let $< K,b >$ be the subset of $< K,B >$ consisting of all elements $< \xi,b >$ for a fixed element b of B, $b \neq 0$.

2.7 THEOREM. *For any* $b \in B$, *different from zero,* $< K,b > \cong K$.

Proof. We will let $\xi \in K$ correspond with the element $< \xi,b >$ of $< K,b >$. This is a one to one correspondence between $< K,b >$ and K. To prove $< K,b > \cong K$ we have to show for $\xi_1,\xi_2 \in K$

(i) $< \xi_1,b > < \xi_2,b > = < \xi_1 \xi_2,b >$,
(ii) $< \xi_1,b > + < \xi_2,b > = < \xi_1 + \xi_2,b >$.

(i) follows from LEMMA 2.1 and (ii) from LEMMA 2.3.
For the special case $b = 1$ we will denote the element $< \xi,1 >$ by ξ, so that ξ is the mapping $\xi\colon \alpha \to \xi(\alpha)$; $\alpha \in R, \xi(\alpha) \in B$, such that $\xi(\xi) = 1$ and $\xi(\alpha) = 0$ if $\alpha \neq \xi$.

In particular we have $0\colon \alpha \to 0(\alpha)$ such that $0(0) = 1$ and $0(\alpha) = 0$ if $\alpha \neq 0$,
and $1\colon \alpha \to 1(\alpha)$ such that $1(1) = 1$ and $1(\alpha) = 0$ if $\alpha \neq 1$,

as we have already introduced before.
This special subset $< K,1 >$ of all elements $\xi = < \xi,1 >$ will be called the set of constants of KB (and of KB*) and will be denoted by K*.

Resuming we have $K \cong < K,b > \leq KB \leq KB^*$ for any $b \neq 0$, and in particular $K \cong < K,1 > = K^* \leq KB \leq KB^*$.

Let $< 1,B >$ be the subset of $< K,B >$ consisting of all elements $< 1,b >$.

2.8 THEOREM. $< 1,B >$ *is lattice-isomorphic to* B.

Proof. We will let $b \in B$ correspond with $< 1,b > \in < 1,B >$. This is a one to one correspondence between B and $< 1,B >$. In order to establish the lattice-isomorphism between B and $< 1,B >$ we will prove

(i) $< 1,b_1 > < 1,b_2 > = < 1,b_1 b_2 >$
(ii) $< 1,b_1 > + < 1,b_2 > - < 1,b_1 > < 1,b_2 > = < 1,b_1 \cup b_2 >$.

(i) follows from LEMMA 2.1.

Since $< 1,b_1 > + < 1,b_2 > - < 1,b_1 > < 1,b_2 > = < 1,b_1 > + < 1,b_2 > + < -1,b_1 b_2 >$ we will consider

$$\bigcup_{\mu+\nu+\lambda=\alpha} < 1,b_1 > (\mu) < 1,b_2 > (\nu) < -1,b_1 b_2 > (\lambda).$$

$\alpha=\mu+\nu+\lambda$	μ	ν	λ	$<1,b_1>(\mu) <1,b_2>(\nu) <-1,b_1 b_2>(\lambda)$				
1	1	1	−1	b_1	b_2	$b_1 b_2$	=	$b_1 b_2$
0	1	0	−1	b_1	b_2'	$b_1 b_2$	=	0
0	0	1	−1	b_1'	b_2	$b_1 b_2$	=	0
−1	0	0	−1	b_1'	b_2'	$b_1 b_2$	=	0
2	1	1	0	b_1	b_2	$(b_1 b_2)'$	=	0
1	1	0	0	b_1	b_2'	$(b_1 b_2)'$	=	$b_1 b_2'$
1	0	1	0	b_1'	b_2	$(b_1 b_2)'$	=	$b_1' b_2$
0	0	0	0	b_1'	b_2'	$(b_1 b_2)'$	=	$b_1' b_2'$

Since $b_1 b_2 \cup b_1 b_2' \cup b_1' b_2 = b_1 \cup b_2$ and $b_1' b_2' = (b_1 \cup b_2)'$

we have $\bigcup_{\mu+\nu+\lambda=\alpha} < 1,b_1 > (\mu) < 1,b_2 > (\nu) < -1,b_1 b_2 > (\lambda) = < 1,b_1 \cup b_2 > (\alpha)$

and thus $<1,b_1> + <1,b_2> - <1,b_1> <1,b_2> = <1,b_1 \cup b_2>$. This proves that $< 1,B >$ is a Boolean algebra, lattice-isomorphic to B, whose Boolean operations expressed in terms of the ring operations of KB are

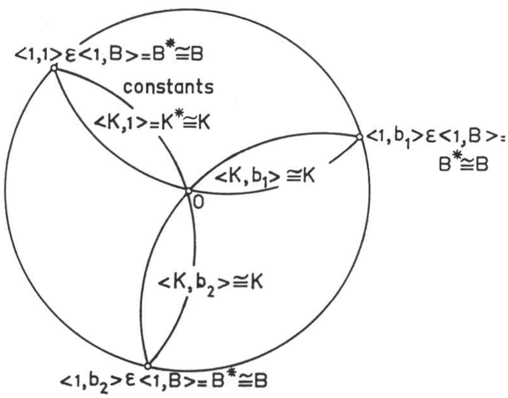
$< 1,b_1 > \cap < 1,b_2 > = < 1,b_1 > < 1,b_2 >$,
$<1,b_1> \cup <1,b_2> = <1,b_1> + <1,b_2> - <1,b_1> <1,b_2>$.

2.9 THEOREM. *If* D *is a commutative integral domain with identity, the set* B* *of idempotents of* DB* *coincides with the set of idempotents of* DB *and* B* *is lattice-isomorphic to* B.

Proof. We first will show that B* $= < 1,B >$. Let $< 1,b > \in <1,B>$. Then $<1,b> <1,b> = <1,b>$ so that $<1,b> \in$ B*.

Now, conversely, let $\psi \in B^*$. This means $\bigcup\limits_{\mu\nu=\alpha} \psi(\mu)\,\psi(\nu) = \psi(\alpha)$, which is equivalent to $\bigcup\limits_{\mu^2=\alpha} \psi(\mu) = \psi(\alpha)$, from which it follows $\bigcup\limits_{\mu^2=\alpha} \psi(\mu)\,\psi(\alpha) = \psi(\alpha)$. If $\psi(\alpha) \neq 0$ there must exist an element $\mu \in D$ such that $\mu = \alpha$ and $\mu^2 = \alpha$. For the integral domain D this is equivalent to $\alpha = 0$ or $\alpha = 1$. Thus we may say that, if $0 \neq \alpha \neq 1$, $\psi(\alpha)$ must be equal to zero. Setting $\psi(1) = b$, $\psi(0)$ must equal b′ in order to satisfy the requirements for a partitional mapping. Since ψ thus turns out to be the element $< 1,b >$, we have proved $\psi \in < 1,B >$. Thus $B^* = < 1,B >$. As $< 1,B > \leq < D,B > \leq DB$ we have proved at the same time that all idempotents of DB* are also idempotents of DB. The converse being trivial it follows that the set of idempotents of DB* and DB coincide and are equal to B*.

Referring to section 1 chap. I and to THEOREM 2.8 it follows that $B^* \cong B$.



circle surface: KB.
circumference with centre: $B^* \cong B$.
centre: zero element of KB.

As will be shown in the next section: the leaf of constants together with the circumference generate KB.

3. KB generated by K* and B*.

3.1 THEOREM. *Any* $\psi \in KB$ *that assumes* n *values* $(n \geq 2)$ α_i $(i = 1 \rightarrow n)$ *on* $\psi(\alpha_i)$ *resp., such that* $\alpha_n = 0$, *can be written as* $\psi = \sum\limits_{i=1}^{n-1} < \alpha_i, \psi(\alpha_i >$.

Proof. The statement is apparently true for $n = 2$, as we then have $\psi = < \alpha_1, \psi(\alpha_1) >$.
Assume the statement to be true for $n = k - 1$: any $\psi \in KB$ that assumes $k - 1$ values α_i ($i = 1 \to k - 1$) on $\psi(\alpha_i)$ resp., such that $\alpha_{k-1} = 0$, can be written as $\psi = \sum_{i=1}^{k-2} < \alpha_i, \psi(\alpha_i) >$. We will prove that then the statement is also true for $n = k$.
Therefore let ψ be an element of KB assuming k values α_i ($i = 1 \to k$) on $\psi(\alpha_i)$ resp., such that $\alpha_k = 0$. Consider the element ψ^* derived from ψ in the following fashion

$$\psi^*(\alpha_i) = \psi(\alpha_i) \qquad i = 1 \to k - 2$$
$$\psi^*(0) = \psi(\alpha_{k-1}) \cup \psi(\alpha_k).$$

Then ψ^* is an element of KB that assumes $k - 1$ values α_i ($i = 1 \to k - 2$), 0 so that ψ^* can be written as $\psi^* = \sum_{i=1}^{k-2} < \alpha_i, \psi(\alpha_i) >$ according to the induction assumption. We assert that $\psi = \psi^* + < \alpha_{k-1}, \psi(\alpha_{k-1}) >$. To prove this consider

$\bigcup_{\mu+\nu=\alpha} \psi^*(\mu) < \alpha_{n-1}, \psi(\alpha_{n-1}) > (\nu)$ with the table ($i = 1 \to k - 2$):

$\alpha = \mu + \nu$	μ	ν	$\psi^*(\mu) < \alpha_{k-1}, \psi(\alpha_{k-1}) > (\nu)$
$\alpha_i + \alpha_{k-1}$	α_i	α_{k-1}	$\psi(\alpha_i) \quad \psi(\alpha_{k-1}) \quad = 0$
α_i	α_i	0	$\psi(\alpha_i) \quad \psi'(\alpha_{k-1}) \quad = \psi(\alpha_i)$
α_{k-1}	0	α_{k-1}	$(\psi(\alpha_{k-1}) \cup \psi(\alpha_k))\ \psi(\alpha_{k-1}) = \psi(\alpha_{k-1})$
0	0	0	$(\psi(\alpha_{k-1}) \cup \psi(\alpha_k))\ \psi'(\alpha_{k-1}) = \psi(\alpha_k) = \psi(0)$

Thus we see $\bigcup_{\mu+\nu=\alpha} \psi^*(\mu) < \alpha_{k-1}, \psi(\alpha_{k-1}) > (\nu) = \psi(\alpha)$, or
$\psi = \psi^* + < \alpha_{k-1}, \psi(\alpha_{k-1}) >$.
Together with the induction assumption this yields
$\psi = \sum_{i=1}^{k-2} < \alpha_i, \psi(\alpha_i) > + < \alpha_{k-1}, \psi(\alpha_{k-1}) >$, or $\psi = \sum_{i=1}^{n-1} < \alpha_i, \psi(\alpha_i) >$.
A slightly changed version of the preceding theorem we have in the following

3.2 Theorem. *Any* $\psi \in KB$ *assuming* n *values* α_i ($i = 1 \to n$) *and not assuming the value zero can be written as* $\psi = \sum_{i=1}^{n} < \alpha_i, \psi(\alpha_i) >$.

Proof. Consider the element ψ^*, derived from ψ in the following manner

$$\psi^* (\alpha_i) = \psi (\alpha_i) \qquad i = 1 \rightarrow n - 1$$
$$\psi^* (0) = \psi (\alpha_n).$$

Due to the fact that $\alpha_i \neq 0$ for $i = 1 \rightarrow n - 1$, we have $\psi^* \in KB$ and also $\psi = \psi^* + < \alpha_n, \psi (\alpha_n) >$ which may be proved in a similar way as above by considering $\bigcup_{\mu+\nu=\alpha} \psi^* (\mu) < \alpha_n, \psi (\alpha_n) > (\nu)$.

$\alpha = \mu + \nu$	μ	ν	$\psi^* (\mu) < \alpha_n, \psi (\alpha_n) > (\nu)$		
$\alpha_i + \alpha_n$	α_i	α_n	$\psi (\alpha_i)$	$\psi (\alpha_n)$	$= 0$
α_i	α_i	0	$\psi (\alpha_i)$	$\psi (\alpha_n)'$	$= \psi (\alpha_i)$
α_n	0	α_n	$\psi (\alpha_n)$	$\psi (\alpha_n)$	$= \psi (\alpha_n)$
0	0	0	$\psi (\alpha_n)$	$\psi (\alpha_n)'$	$= 0$

Since ψ^* fulfils the requirements of THEOREM 3.1 we now have $\psi = \sum_{i=1}^{n-1} < \alpha_i, \psi (\alpha_i) > + < \alpha_n, \psi (\alpha_n) >$, or $\psi = \sum_{i=1}^{n} < \alpha_i, \psi (\alpha_i) >$.
Combining THEOREMS 3.1 and 3.2 we have

3.3 THEOREM. *An element ψ of KB assuming n values α_i ($i = 1 \rightarrow n$) can be written as $\psi = \sum_{i=1}^{n} < \alpha_i, \psi (\alpha_i) >$.*
If ψ does not assume the value zero we have $\alpha_i \neq 0$ for $i = 1 \rightarrow n$.
If ψ assumes the value zero on $\psi (\alpha_n)$ we have $\alpha_i \neq 0$ for $i = 1 \rightarrow n - 1$ and $\alpha_n = 0$, so that $< \alpha_n, \psi (\alpha_n) > = 0$.

3.4 DEFINITION. *A sum $\sum_{i=1}^{n} < \xi_i, b_i >$ is called an* ORTHOGONAL SUM *provided*

(i) $b_i \, b_j = 0$ for $i \neq j$.

If also (ii) $\xi_i \neq \xi_j$ for $i \neq j$

the sum is called a SIMPLIFIED ORTHOGONAL SUM.

3.5 DEFINITION. *A* REPRESENTATION *of an element $\psi \in KB$ is a sum $\sum_{i=1}^{n} < \xi_i, b_i >$ such that $\psi = \sum_{i=1}^{n} < \xi_i, b_i >$.*
According to the preceding theorems we now have

3.6 THEOREM. *Every element* ψ *of* KB *has a unique simplified orthogonal representation* $\psi = \sum_{i=1}^{n} < \xi_i, b_i >$, $\xi_i \neq 0$, $b_i \neq 0$, *while* ψ *assumes the value* ξ_i *on* b_i.

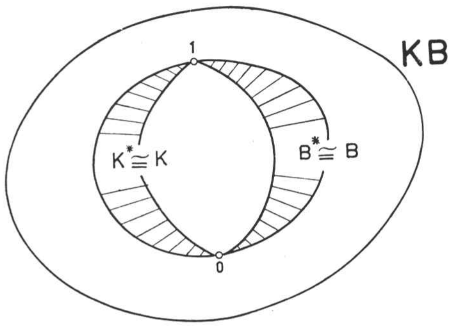
If $\bigcup_{i=1}^{n} b_i = 1$, ψ *does not assume the value zero.*

If $\bigcup_{i=1}^{n} b_i \neq 1$, ψ *assumes the value zero on* $[\bigcup_{i=1}^{n} b_i]'$.

Moreover: $\psi = \sum_{i=1}^{n} < \xi_i, b_i > = \sum_{i=1}^{n} < \xi_i, 1 > < 1, b_i > =$

$$\sum_{i=1}^{n} \xi_i < 1, b_i >.$$

We thus see that every element ψ of KB can be written as a linear combination of elements from B* with coefficients in K*, or the other way around: as a linear combination of elements from K* with coefficients in B*. Anyway, KB is generated by K* $\cong$ K and by B* $\cong$ B.



CHAPTER III

ASSOCIATE RINGS THAT ARE RINGS OF PARTITIONAL MAPPINGS

1. Unitary subrings of DB*.

Let D be a commutative integral domain with identity and B a complete Boolean algebra. Let R be a unitary subring of DB*; i.e. let R be a ring of partitional mappings of D into B containing the identity of DB*.

1.1 THEOREM. *A unitary subring* R *of the ring of all partitional mappings of a commutative integral domain* D *with identity into a complete Boolean algebra* B *is a Boolean valued ring.*

Proof. Let $\psi \in$ R. Define $\varphi(\psi) = \psi(0)'$.

(i) $\varphi(0) = 0(0)' = 0$; $\varphi(\psi) = 0$ implies $\psi(0)' = 0$ or $\psi(0) = 1$; i.e. $\psi = 0$.

(ii) $\varphi(\psi_1 \psi_2) = \psi_1 \psi_2 (0)' = [\bigcup_{\mu\nu=0} \psi_1(\mu)\, \psi_2(\nu)]'$;

$\varphi(\psi_1)\, \varphi(\psi_2) = \psi_1(0)'\, \psi_2(0)' = [\psi_1(0) \cup \psi_2(0)]'$.

We have to show $\bigcup_{\mu\nu=0} \psi_1(\mu)\, \psi_2(\nu) = \psi_1(0) \cup \psi_2(0)$.

$\bigcup_{\mu} \psi_1(\mu) = 1$; $\psi_2(0) = \psi_2(0) \bigcup_{\mu} \psi_1(\mu) = \bigcup_{\mu} \psi_1(\mu)\, \psi_2(0)$.

$\bigcup_{\nu} \psi_2(\nu) = 1$; $\psi_1(0) = \psi_1(0) \bigcup_{\nu} \psi_2(\nu) = \bigcup_{\nu} \psi_1(0)\, \psi_2(\nu)$.

Therefore we have $\psi_1(0) \cup \psi_2(0) = [\bigcup_{\nu} \psi_1(0)\, \psi_2(\nu)] \cup [\bigcup_{\mu} \psi_1(\mu)\, \psi_2(0)$.

But since D is an integral domain it holds that

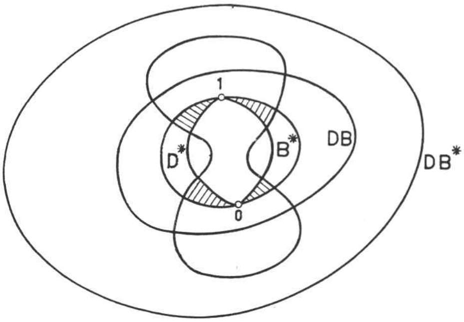
$[\bigcup_{\nu} \psi_1(0)\, \psi_2(\nu)] \cup [\bigcup_{\mu} \psi_1(\mu)\, \psi_2(0)] = \bigcup_{\mu\nu=0} \psi_1(\mu)\, \psi_2(\nu)$.

(iii) $\varphi(\psi_1 + \psi_2) = (\psi_1 + \psi_2)(0)' = [\bigcup_{\mu+\nu=0} \psi_1(\mu)\, \psi_2(\nu)]'$.

$\varphi(\psi_1) \cup \varphi(\psi_2) = \psi_1(0)' \cup \psi_2(0)' = [\psi_1(0)\, \psi_2(0)]'$.

But $\psi_1(0)\, \psi_2(0) \leq \bigcup_{\mu+\nu=0} \psi_1(\mu)\, \psi_2(\nu)$ so that $\varphi(\psi_1 + \psi_2) \leq \varphi(\psi_1) \cup \varphi(\psi_2)$.

The situation is the following:



The Boolean algebra used for the valuation of R is B*. Since we assumed that R is a unitary subring of DB* R contains the subring D_0* of D*, generated by 1.
R ∩ D* is a subring of D* containing D_0*.
R ∩ B* is a Boolean subalgebra of B*.
Some properties of R are:

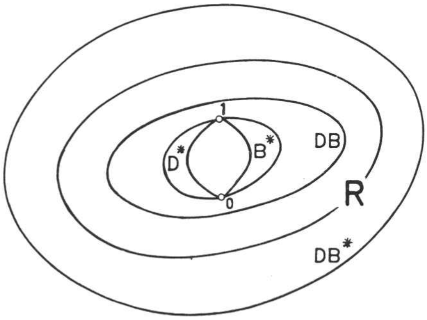
1.1.1 $\varphi(u) = u$ for all elements $u \in R \cap B^*$.

1.1.2 $\varphi(\alpha) = 1$ for all elements $\alpha \in R \cap D^*$, $\alpha \neq 0$.

1.1.3 $\bigcup\limits_{\alpha \in D^*} \varphi(\psi - \alpha)' = 1$.

1.1.1. and 1.1.2 follow immediately from the definition of φ, u and α. To show 1.1.3 we have $\varphi(\psi) = \psi(0)'$; $\varphi(\psi - \alpha) = (\psi - \alpha)(0)'$; $\varphi(\psi - \alpha)' = (\psi - \alpha)(0) = \bigcup\limits_{\mu+\nu=0} \psi(\mu)\,\alpha(-\nu) = \bigcup\limits_{\mu=\alpha} \psi(\mu) = \psi(\alpha)$, from which $\bigcup\limits_{\alpha \in D^*} \varphi(\psi - \alpha)' = \bigcup\limits_{\alpha \in D^*} \psi(\alpha) = 1$.

We already know that as soon as R ≥ D* and R ≥ B* we must have R ≥ DB since DB is generated by D* and B*. In that case the situation would be:



Since DB and DB* have the same set of idempotents B*, B* is also the Boolean algebra of idempotents of R. Thus R is an associate ring and we have

1.2 THEOREM. *If* R *is a unitary subring of the ring* DB* *of all partitional mappings of a commutative integral domain* D *with identity into a complete Boolean algebra* B *and if in addition* R *contains the sets* D* *and* B*, R *is an associate ring whose set of idempotents is* B*, *while* $DB \leq R \leq DB^*$.

1.3 COROLLARY. DB *and* DB* *are associate rings.

2. Homogeneous Boolean valued rings.

2.1 DEFINITION. *A Boolean valued ring* R *containing an integral domain* D *with identity such that*

$$\text{(i)}\quad \varphi(\alpha) = 1 \qquad \text{for all } \alpha \in D,\ \alpha \neq 0$$

$$\text{(ii)}\quad \bigcap_{\alpha \in D} \varphi(x - \alpha) = 0 \qquad \text{for all } x \in R$$

will be called a HOMOGENEOUS BOOLEAN VALUED RING. *The elements* α *of the integral domain will be referred to as* CONSTANTS. *It is understood that if* D *is an infinite set, the Boolean algebra* B *should be complete.* We now proceed to prove several lemmas, needed for our next theorem, all referring to a homogeneous Boolean valued ring R.

2.2 LEMMA. $\varphi(x - \alpha) = \varphi(y - \alpha)$ *for all* $\alpha \in D$ *implies* $x = y$.

Proof. $\varphi(x - y) \leq \varphi(x - \alpha) \cup \varphi(y - \alpha) = \varphi(x - \alpha)$ for all $\alpha \in D$. Thus $\varphi(x - y) \leq \bigcap_{\alpha \in D} \varphi(x - \alpha) = 0$. Therefore $\varphi(x - y) = 0$, or $x = y$.

2.3 LEMMA. $\varphi(x - \mu) \cup \varphi(x - \nu) = 1;\ x \in R;\ \mu,\nu \in D;\ \mu \neq \nu$.
Proof. $\varphi(x - \mu) \cup \varphi(x - \nu) \geq \varphi(\mu - \nu) = 1$ since $\mu,\nu \in D$ and $\mu \neq \nu$.

2.4 LEMMA. $\varphi(x_1 x_2 - \mu\nu) \leq \varphi(x_1 - \mu) \cup \varphi(x_2 - \nu);\ x_1,x_2 \in R;\ \mu,\nu \in D$.

Proof.
$$\begin{aligned} x_1 x_2 - \mu\nu &= (x_1 - \mu)(x_2 - \nu) + \nu x_1 + \mu x_2 - 2\mu\nu \\ &= (x_1 - \mu)(x_2 - \nu) + \nu(x_1 - \mu) + \mu(x_2 - \nu). \end{aligned}$$

Therefore $\varphi(x_1 x_2 - \mu\nu) \leq \varphi(x_1 - \mu)\,\varphi(x_2 - \nu) \cup \varphi(\nu)\,\varphi(x_1 - \mu) \cup \varphi(\mu)\,\varphi(x_2 - \nu) \leq \varphi(x_1 - \mu) \cup \varphi(x_2 - \nu)$.

2.5 LEMMA. $\varphi(x_1 - \mu) \cup \varphi(x_2 - \nu) \cup \varphi(x_1 x_2 - \alpha) = 1;\ \alpha,\mu,\nu \in D;\ \alpha \neq \mu\nu$.

Proof. $\varphi(x_1 - \mu) \cup \varphi(x_2 - \nu) \cup \varphi(x_1 x_2 - \alpha) \geq \varphi(x_1 x_2 - \mu\nu) \cup \varphi(x_1 x_2 - \alpha) = 1$ according to LEMMA 2.4 and LEMMA 2.3 resp. and since $\alpha \neq \mu\nu$.

2.6 LEMMA. $\varphi(x_1 x_2 - \alpha) = \bigcap_{\mu\nu=\alpha} (\varphi(x_1 - \mu) \cup \varphi(x_2 - \nu))$.

Proof. $\bigcap_{\mu} \bigcap_{\nu} (\varphi(x_1 x_2 - \alpha) \cup \varphi(x_1 - \mu) \cup \varphi(x_2 - \nu)) = \varphi(x_1 x_2 - \alpha)$ using $\bigcap_{\mu} \varphi(x_1 - \mu) = 0$ and $\bigcap_{\nu} \varphi(x_2 - \nu) = 0$. But also:

$$\bigcap_{\mu} \bigcap_{\nu} (\varphi(x_1 x_2 - \alpha) \cup \varphi(x_1 - \mu) \cup \varphi(x_2 - \nu)) = \bigcap_{\mu\nu=\alpha} (\varphi(x_1 x_2 - \alpha) \cup \varphi(x_1 - \mu) \cup \varphi(x_2 - \nu)) =$$
$$\bigcap_{\mu\nu=\alpha} (\varphi(x_1 x_2 - \mu\nu) \cup \varphi(x_1 - \mu) \cup \varphi(x_2 - \nu)) = \bigcap_{\mu\nu=\alpha} (\varphi(x_1 - \mu) \cup \varphi(x_2 - \nu)),$$

from which the result.
Similarly one proves

2.7 LEMMA. $\varphi(x_1 + x_2 - \alpha) = \bigcap_{\mu+\nu=\alpha} (\varphi(x_1 - \mu) \cup \varphi(x_2 - \nu))$.

2.8 THEOREM. *If* R *is a homogeneous complete associate ring then* R *is isomorphic to a ring of partitional mappings* R^* *such that* $DB \leq R^* \leq DB^*$, *where* D *is the integral domain of constants contained in* R *and* B *the complete Boolean algebra of idempotents of* R.
Proof. Let $x \in R$. Then define the mapping ψ

$$\psi: \alpha \to \varphi(x - \alpha)', \quad \alpha \in D, \ \varphi(x - \alpha) \in B,$$

of D into B.
This is a partitional mapping according to LEMMA 2.3 and prop. (ii) of DEFINITION 2.1. Let the set of all partitional mappings ψ so obtained be denoted by R^*. Then there is a one to one correspondence between R and R^* due to LEMMA 2.2; furthermore $R^* \leq DB^*$. To show that $R \cong R^*$ we have

(i) $\varphi(x_1 x_2 - \alpha) = \bigcup_{\mu\nu=\alpha} \varphi(x_1 - \mu)' \varphi(x_2 - \nu)'$ (LEMMA 2.6),

(ii) $\varphi(x_1 + x_2 - \alpha) = \bigcup_{\mu+\nu=\alpha} \varphi(x_1 - \mu)' \varphi(x_2 - \nu)'$ (LEMMA 2.7).

To prove $DB \leq R^*$ it is sufficient to show $D^* \leq R^*$ and $B^* \leq R^*$. Let $\mu \in D \leq R$; then we have $\psi \in R^*$, $\psi: \alpha \to \varphi(\mu - \alpha)'$, and also $< \mu,1 > \in D^*$, $< \mu,1 >: \alpha \to < \mu,1 > (\alpha)$ with $\varphi(\mu - \alpha)' = < \mu,1 > (\alpha)$ for all $\alpha \in D$, from which $D^* \leq R^*$.

Now let $b \in B \leq R$ and consider $\varphi(b - \alpha)$;
$\varphi(b - 1) = \varphi(1 - b) = \varphi(b') = b'$,
$\varphi(b - 0) = \varphi(b) = b$.

We will prove that $\varphi(b - \alpha) = 1$ for $0 \neq \alpha \neq 1$.
Suppose $\varphi(b - \alpha) \neq 1$ for $0 \neq \alpha \neq 1$.
$\varphi(b - \alpha) \cup \varphi(b - 0) = 1$ since $\alpha \neq 0$ (LEMMA 2.3);
$\varphi(b - \alpha) = \varphi(b - \alpha) \cup \varphi(b - 1)\, \varphi(b - 0) =$
$(\varphi(b-\alpha) \cup \varphi(b-1)) \cap (\varphi(b-\alpha) \cup \varphi(b-0)) = \varphi(b-\alpha) \cup \varphi(b-1)$.

But $\varphi(b - \alpha) \cup \varphi(b - 1) \neq 1$ would contradict LEMMA 2.3 since $\alpha \neq 1$. Thus $\varphi(b - \alpha) = 1$ for $0 \neq \alpha \neq 1$.

Now it follows readily that for $b \in B$ $< 1,b > \in R^*$, so that $B^* \leq R^*$. For we have $\psi \in R^*$, $\psi\colon \alpha \to \varphi(b - \alpha)'$, and also $< 1,b > \in B^*$, $< 1,b >\colon \alpha \to < 1,b > (\alpha)$. We just proved $\varphi(b - \alpha)' = < 1,b > (\alpha)$ for all $\alpha \in D$, so that $B^* \leq R^*$.

3. p-Rings.

3.1 THEOREM. *If* R *is a homogeneous associate ring whose integral domain* D *of constants is finite, we have* $R \cong DB$, *where* B *is the Boolean algebra of idempotents of* R. *If the set of constants of* R *is a finite field,* R *is regular.*

Proof. According to THEOREM 2.8 we have $R \cong R^*$, where $DB \leq R^* \leq DB^*$. But since D is finite, we have $DB = DB^*$ from which $DB = R^* = DB^*$, so that $R \cong DB$.
Note that since D is finite the Boolean algebra B of idempotents of R need not be complete.
If the constants of R form a finite field F, we know that FB is regular, so that the same holds for R.

The following results concerning p-rings were obtained by Foster [9] and Zemmer [20]. Since they follow from our preceding discussion (and thus independently of the fact that p-rings may be considered as subdirect sums of fields I_p [1])) they will be mentioned here.
Therefore let I_p be the residue class of integers mod p for any prime p. Then we have

3.2 THEOREM. *A ring* R *is a p-ring with identity if and only if* $R \cong I_pB$ *for some Boolean algebra* B.

[1]) [15], Theorem 45, p. 146.

Proof. Suppose $x \in I_pB$ with the following simplified orthogonal representation

$$x = \sum_{i=0}^{p-1} i < 1,b_i >, i \in I_p, b_i \in B.$$

Then $$x^p = \sum_{i=0}^{p-1} i^p < 1,b_i > = \sum_{i=0}^{p-1} i < 1,b_i > = x \quad \ldots\ldots \quad \text{(i)}$$

since $i^p = i$ for all $i \in I_p$.

Also $$px = \sum_{i=0}^{p-1} pi < 1,b_i > = 0 \quad \ldots\ldots\ldots\ldots \quad \text{(ii)}$$

since $pi = 0$ for all i I_p.

(i) and (ii) together with the fact that $1 \in I_pB$ yield the result that I_pB (and thus any $R \cong I_pB$) is a p-ring with identity.

Now, conversely, let R be a p-ring with identity, whose set of idempotents is B. In the first place we established previously that p-rings are associate rings.

Let $0,1,2,\ldots\ldots, p-1$ be the zero of R and the identity of R with its successive summands. Denote this set by F. Then we will show

(i) F is a field I_p,

(ii) $\varphi(i) = 1, i = 1 \to p-1$,

(iii) $\bigcap_{i=0}^{p-1} \varphi(x-i) = 0$ for all $x \in R$,

thus establishing the fact that p-rings are homogeneous Boolean valued rings. F being a finite field we may apply Theorem 3.1, from which we have $R \cong I_pB$.

(i) follows from the fact that R has prime characteristic p.

(ii) follows from the fact that R is a p-ring, so that we have $\varphi(i) = i^{p-1} = 1$ since $i \in I_p$ and $i \neq 0$.

(iii) $\bigcap_{i=0}^{p-1} \varphi(x-i) = \bigcap_{i=1}^{p} \varphi(x-i) = \varphi[(x-1)(x-2)\ldots\ldots(x-p)]$.

But $(x-1)(x-2)\ldots\ldots(x-p) = x^p - x = 0$ for all x, from which the result.

3.3 Theorem. *Let ψ_1 and ψ_2 be two elements of a p-ring* $R = I_pB$ *with identity whose simplified orthogonal representations are*:

$$\psi_1 = \sum_{i=1}^{p} i < 1,\psi_1(i) >, i \in I_p, \psi_1(i) \in B, \textit{ and}$$

$$\psi_2 = \sum_{i=1}^{p} i < 1,\psi_2(i) >, i \in I_p, \psi_2(i) \in B.$$

Then we have for the simplified orthogonal representations of $\psi_1 \psi_2$ *and* $\psi_1 + \psi_2$:

$$\psi_1 \psi_2 = \sum_{i=1}^{p} i < 1, \pi(i) > \text{ with } \pi(i) = \sum_{mn=i} \psi_1(m) \psi_2(n), \text{ and}$$

$$\psi_1 + \psi_2 = \sum_{i=1}^{p} i < 1, \sigma(i) > \text{ with } \sigma(i) = \sum_{m+n=i} \psi_1(m) \psi_2(n).$$

All integers are residue classes mod p.

Proof. $\psi_1 \psi_2 (i) = \bigcup_{mn=i} \psi_1(m) \psi_2(n)$,

but since $b_1 \cup b_2 = b_1 + b_2 - b_1 b_2 = b_1 + b_2$ if $b_1 b_2 = 0$
we have $\psi_1 \psi_2 (i) = \sum_{mn=i} \psi_1(m) \psi_2(n)$, and similarly

$$(\psi_1 + \psi_2)(i) = \sum_{m+n=i} \psi_1(m) \psi_2(n).$$

Now apply THEOREM 3.6. chap. II.

CHAPTER IV

CONVEX BOOLEAN METRIC SPACES

Throughout this chapter M stands for an associate Boolean metric space; i.e. a Boolean metric space obtained from an associate ring R by defining d (a,b) = φ (a — b) for a,b $\in$ R. The Boolean algebra of idempotents of R will be denoted by B. For a,b $\in$ B it holds: d (a,b) = a′b $\cup$ b′a.

1. The relations weakly-between and between.

1.1 DEFINITION. *We say that an element* x $\in$ M *is weakly-between two distinct elements* a and b *of* M: B* (a,x,b), *provided*

(i) d (a,b) = d (a,x) $\cup$ d (x,b),
(ii) a $\neq$ x $\neq$ b.

It follows from the definition that d(a,b) $\geq$ d(a,x) and d (a,b) $\geq$ d(b,x).
It also follows from the definition that B* (a,x,b) and B* (b,x,a) are equivalent; i.e. the relation weakly-between is symmetric in the outer-points.
By considering the set of all functions of a set Ω with values in a commutative ring with identity one obtains examples of associate Boolean metric spaces that contain isoceles and equilateral triples. Blumenthal showed that a Boolean metric 2-space M_2 does not contain isoceles triples [1]). This result also holds for the Boolean metric 2-space B associated with any associate Boolean metric space M.
A few elementary properties of the relation weakly-between may be mentioned here. Their proves are all straight forward.

1.1.1 B* (a,b,x) and B* (b,a,x) imply d (a,x) = d (b,x).
1.1.2 If B* (a,b,c), B* (b,c,a) and B* (c,a,b) hold, then d (a,b) = d (b,c) = d (c,a) and conversely.

[1]) [4], § 131, p. 331.

1.1.3 $B^*(a,x,b)$ with $d(a,x) = d(b,x)$ imply $d(a,x) = d(b,x) = d(a,b)$.

1.1.4 If $d(a,x) = d(b,x)$, $B^*(a,b,x)$ is equivalent to $B^*(b,a,x)$.

We thus see that to obtain also uniqueness of the inner-point for the relation between one has to impose stronger requirements. Therefore

1.2 DEFINITION. *We say that the element* $x \in M$ *is between two distinct elements* a *and* b *of* M: $B(a,x,b)$, *provided* $d(a,b) > d(a,x)$ *and* $d(a,b) > d(b,x)$.

It follows from the definition that $B(a,x,b)$ implies $B^*(a,x,b)$ and that $B(a,x,b)$ is unique for the inner-point. In a Boolean metric 2-space M_2, $B^*(a,b,c)$ implies $B(a,b,c)$, since a space M_2 does not contain isoceles triples, so that $d(a,c) = d(a,b)$ would imply $b = c$ and $d(a,c) = d(b,c)$ would imply $a = b$.

From THEOREM 3.4 chap. I and from a result obtained by Blumenthal [1]) it follows

1.3 THEOREM. *For* $a,b,c \in B \leq M$ *the relation* $B(a,x,b)$ *is equivalent to* $ab \leq x \leq a \cup b$, $a \neq x \neq b$.

This, fortunately, implies that for $a,b \in B$ such that $a < b$, metric-betweeness coincides with order-betweeness, as in this case $ab = a$ and $a \cup b = b$, so that $B(a,x,b)$ and $a < x < b$ are equivalent.

For the relation weakly-between in an associate Boolean metric space M the following property holds

1.4 PROPERTY. $B^*(a,x,b)$ *and* $B^*(a,y,b)$ *and* $B^*(x,z,y)$, $a \neq z \neq b$, *imply* $B^*(a,z,b)$.

Proof. $B^*(a,x,b)$ is equivalent to $\varphi(a-b) \geq \varphi(a-x)$ and $\varphi(a-b) \geq \varphi(b-x)$.

$B^*(a,y,b)$ is equivalent to $\varphi(a-b) \geq \varphi(a-y)$ and $\varphi(a-b) \geq \varphi(b-y)$.

$B^*(x,z,y)$ is equivalent to $\varphi(x-y) \geq \varphi(x-z)$ and $\varphi(x-y) \geq \varphi(y-z)$.

It follows $\varphi(a-b) \geq \varphi(a-x) \cup \varphi(a-y) \geq \varphi(x-y) \geq \varphi(x-z)$. Also $\varphi(a-b) \geq \varphi(a-x)$ so that $\varphi(a-b) \geq \varphi(x-z) \cup \varphi(a-x) \geq \varphi(a-z)$.

Similarly $\varphi(a-b) \geq \varphi(b-z)$. Thus $\varphi(a-b) \geq \varphi(a-z) \cup \varphi(b-z)$. Together with $\varphi(a-b) \leq \varphi(a-z) \cup \varphi(b-z)$ this yields $\varphi(a-b) = \varphi(a-z) \cup \varphi(b-z)$: $B^*(a,z,b)$.

[1]) [4], § 132, p. 333.

In a similar manner the following two properties are proved

1.5 PROPERTY. B^* (a,b,c) *and* B^* (a,c,d), $b \neq d$, *imply* B^* (a,b,d).

1.6 PROPERTY. *If* B^* (a,b,c) *and* B^* (b,c,d) *then* B^* (a,b,d) *and* B^* (a,c,d) *are equivalent.*

In case of a Boolean metric 2-space M_2 we have the additional properties

1.7 PROPERTY. B (a,b,c) *and* B (a,c,d) *imply* B (b,c,d).

Proof. B (a,b,c) is equivalent with $ac \leq b \leq a \cup c$, $a \neq b \neq c$.
B (a,c,d) is equivalent with $ad \leq c \leq a \cup d$, $a \neq c \neq d$.

It follows $bd \leq ad \cup cd \leq c \cup cd = c$, and also $c \leq ac \cup cd \leq b \cup cd \leq b \cup d$. Thus $bd \leq c \leq b \cup d$ or B (b,c,d) since $b \neq c \neq d$.

1.8 PROPERTY. B (a,x,b) *and* B (a,p,x) *and* (B (x,q,b), $b \neq c \neq d$, *imply* B (p,x,q).

Proof. From the premisses it follows that $pq \leq (a \cup x)(b \cup x) = ab \cup x = x$ and also $p \cup q \geq ax \cup bx = (a \cup b)\, x = x$, from which the result.

2. Convexity.

2.1 DEFINITION. *An associate Boolean metric space* M *is said to be* CONVEX *provided for every two distinct elements* a *and* b *of* M *there exists an element* $x \in M$ *such that* B (a,x,b).
Similarly the concept WEAKLY-CONVEX is defined.
Blumenthal proved [1]) for a Boolean metric 2-space that such a space is convex if and only if the underlying Boolean algebra is atom-free.
For associative Boolean metric spaces we have

2.2 THEOREM. *The fact that the Boolean metric space* M *is atom-free implies that* M *is weakly-convex and is implied by the convexity of* M.

Proof. First suppose M is convex. Let u be an atom of B. There must, however, be an element $a \in M$ such that B (0,a,u), since M is convex. That is we have: $u > \varphi(a)$ and $u > \varphi(a - u)$. Both $\varphi(a) = 0$ and $\varphi(a - u) = 0$ are excluded since $a \neq 0$ and $a \neq u$, so that we have $0 < \varphi(a) < u$ and also $0 < \varphi(a - u) < u$, each contradicting the fact that u was an atom of B.

[1]) BG II.

Now suppose B is atom-free and let $a,b \in M$. Consider $\varphi(a-b) \neq 0$, since we suppose $a \neq b$. There must be an element $u \in B$ such that $0 < u < \varphi(a-b)$. We assert $B^*(a, a+u, b)$.

To prove this we have:

$\varphi(a-b+u) \leq \varphi(a-b) \cup \varphi(u) = \varphi(a-b)$ since $\varphi(u) = u < \varphi(a-b)$; thus $\varphi(a-b+u) \cup \varphi(u) \leq \varphi(a-b)$ from which $\varphi(a-b) = \varphi(a-b+u) \cup \varphi(u)$.

Furthermore: $a \neq a+u$ since $u \neq 0$; and also $a-u \neq b$ since $u \neq a-b$. This completes the proof.

3. Convectification.

Blumenthal showed [1]) that the requirement for a Boolean metric 2-space to be convex is less restrictive than it might appear since he indicated a way to embed every Boolean metric 2-space isomorphically and isometrically in a convex Boolean metric 2-space. His argument is easily extended to the more general case.

3.1 THEOREM. *Every associate Boolean metric space* M *is isomorphically and isometrically embeddable in a convex associate Boolean metric space* M.

Proof. Consider the set M_1 of all ordered pairs (a,b) of elements a,b of $M = M_0$.

Define $(a_1,b_1)(a_2,b_2) = (a_1 a_2, b_1 b_2)$,

$$(a_1,b_1) + (a_2,b_2) = (a_1 + a_2, b_1 + b_2),$$

$$\varphi(a,b) = (\varphi(a), \varphi(b)), \text{ so that}$$

$$d[(a_1,b_1),(a_2,b_2)] = \varphi[(a_1,b_1) - (a_2,b_1)] =$$

$$\varphi[(a_1 - a_2),(b_1 - b_2)] = (\varphi(a_1 - a_2), \varphi(b_1 - b_2)) =$$

$$(d(a_1,a_2), d(b_1,b_2)).$$

Then it is easily seen that M_1 is an associate Boolean metric space whose underlying set R_1 is an associate ring whose set B_1 of idempotents consists of all ordered pairs (u,v) with $u,v \in B = B_0$. The identification of $(a,a) \in R_1$ with $a \in R = R_0$ embeds M_0 isomorphically and isometrically in M_1.

By induction, repetition of this procedure yields a sequence of associate Boolean metric spaces $\{M_i\}$, each of which is embedded isomorphically and isometrically in the following. Let R^* be the union of all sets R_i: $R^* = \bigcup_{i=1}^{\infty} R_i$. Define multiplication, addition

[1]) BG II.

and valuation for elements of R^* in the same way in which these operations were defined in the ring R_k of smallest index k that contains all the elements involved, then R^* is an associate ring. If M^* is the associate Boolean metric space obtained from R^* and if B^* is the Boolean algebra of idempotents of R^*, one may easily establish that $B^* = \bigcup_{i=1}^{\infty} B_i$ and that B^* is atom free. For suppose $u \in B^*$ is an atom and B_k is the Boolean algebra of idempotents of R_k with smallest index k containing u; then it holds $0 < (u,0) < (u,u)$ (and also $0 < (0,u) < (u,u)$) for $(u,u) \in B_{k+1} < B^*$. Thus u cannot be an atom. Since B^* is atom-free M^* is weakly-convex. But M is even convex in this case. To prove this, let $x,y \in M^*$, $x \neq y$. Consider (x,x), (x,y) and (y,y). We assert B ((x,x),(x,y),(y,y)), which is equivalent to $\varphi((x,x) - (y,y)) > \varphi((x,x) - (x,y))$ and $\varphi((x,x) - (y,y)) > \varphi((y,y) - (x,y))$. For these expressions we have:

$$\varphi((x,x) - (y,y)) = \varphi(x - y, x - y) = (\varphi(x - y), \varphi(x - y)),$$
$$\varphi((x,x) - (x,y)) = \varphi(0, x - y) \quad = (0, \varphi(x - y)),$$
$$\varphi((y,y) - (x,y)) = \varphi(y - x, 0) \quad = (\varphi(y - x), 0),$$

from which the result.

CHAPTER V

INTRODUCTION OF A TOPOLOGY

1. Order-convergence. [1])

Let $\{ x_i \}$, $i = 1,2\ldots$, be a sequence of elements of a partially ordered set P. An element $u \in P$ is called a LOWER-BOUND for the sequence $\{ x_i \}$, provided $u \leq x_i$ for all i. Dually an UPPER-BOUND v is defined. A sequence $\{ x_i \}$ is called BOUNDED provided the set U of all lower-bounds and the set V of all upper-bounds are not empty.

An element $\pi \in P$ is called a SUB-ELEMENT for a sequence $\{ x_i \}$ in case $\pi \leq x_j$ for all $j \geqq N$. Dually a SUPER-ELEMENT σ is defined. Note that the set U of all lower-bounds is contained in the set Π of all sub-elements and that the set V of all upper-bounds is contained in the set Σ of all super-elements: $U \leq \Pi$ and $V \leq \Sigma$. Also: each element of Π precedes each element of Σ. An INTER-ELEMENT ρ for a sequence $\{ x_i \}$ is an element of P such that $\pi \leq \rho \leq \sigma$ for all sub- and super-elements π and σ of the sequence $\{ x_i \}$. In case the set R of all inter-elements has a smallest resp. a largest element, we denote these by $\underline{\lim}\ x_i$ (limes inferior) resp. $\overline{\lim}\ x_i$ (limes superior). A sequence is called CONVERGENT provided $\underline{\lim}\ x_i = \overline{\lim}\ x_i$; in other words: provided the sequence has one and only one inter-element. This inter-element is called the LIMIT of the sequence. In case both sets Π and Σ are void the sequence $\{ x_i \}$ can not be convergent since every element of P is inter-element. In case of bounded sequences Π and Σ are not empty, since U and V are not empty.

Suppose we have a sequence $\{ x_i \}$ such that $x_i = x$ for all i. The set Π of all sub-elements consists of all elements of P preceding the element x. The set of all super-elements consists of all elements of P following x. The only inter-element, therefore, is the element x

[1]) [14].

itself, from which it follows that the sequence $\{ x_i \}$ converges to the limit x.
From the definition of convergence it follows immediately that if a sequence converges to a limit x and also to a limit y, x and y must be equal, so that for convergent sequences the limit is uniquely determined, this being the only inter-element of the sequence.
If $\{ x_{n_i} \}$ is a subsequence of $\{ x_i \}$ and if Π,Σ and R denote the set of sub-elements, super-elements and inter-elements resp. for the sequence $\{ x_i \}$ and Π^*,Σ^* and R^* similarly for the subsequence $\{ x_{n_i} \}$, we have $\Pi \leq \Pi^*, \Sigma \leq \Sigma^*$ and $R^* \leq R$. From this it follows that if a subsequence of a convergent sequence converges, it must have the same limit as the sequence itself, since R only contains one element and hence R^* contains at most one element. That there may be situations in which subsequences of convergent sequences do not converge (R^* void) will be shown at the end of this section.

RESULT. *If one introduces in a partially ordered set the above described* ORDER-CONVERGENCE *for sequences, one obtains a space in which the following three conditions hold*:

(i) $\{ x_i \}, x_i = x$ for all i, converges to x;
(ii) $\{ x_i \}$ converges to a and $\{ x_i \}$ converges to b implies a = b;
(iii) every convergent subsequence $\{ x_{n_i} \}$ of a convergent sequence $\{ x_i \}$ converges to the same limit as the sequence does.

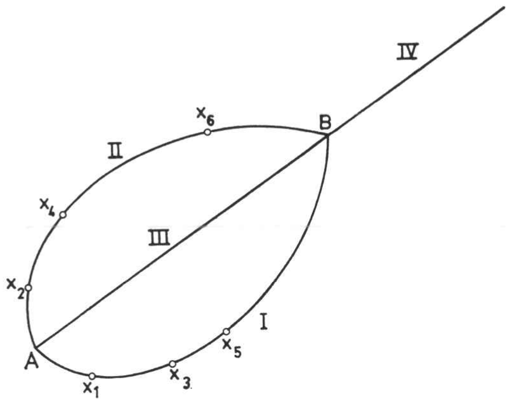
Because of the resemblance these three conditions bear with the three conditions imposed on a space to be a Fréchet L-space [1]), we might say that a partially ordered set P with the order-convergence constitutes a WEAK FRÉCHET L-SPACE.

The term "order-convergence" for sequences in complete lattices was first introduced by G. Birkhoff [2]) and independently by L. Kantorovich [3]). H. Löwig, however, introduced this concept for partially ordered sets [4]). It will be shown in section 3 that for complete lattices these notions coincide.
See for these references [16], footnote 15 on page 59, and also footnote 18 on page 62.

[1]) [4], chap. I, § 4.
[2]) [2], esp. THEOREM 29; also [3].
[3]) [13]; also [12].
[4]) [14].

The Birkhoff-Kantorovich procedure was previously used in set theory by Borel.

EXAMPLE.



The partially ordered set P consists of all points on the curves I, II, III and IV with the following partial ordering:

all	points	on	I	precede	all	points	on	IV;
,,	,,	,,	II	,,	,,	,,	,,	IV;
,,	,,	,,	III	,,	,,	,,	,,	IV;
,,	,,	,,	I	,,	,,	,,	,,	II;
,,	,,	,,	I	,,	,,	,,	,,	III;

and further:
all points precede points on the same curve more to the right.
Now let $\{ x_i \}$ be a sequence with the odd-numbered terms on I and the even-numbered terms on II, in such a fashion that the terms move to the right and approach the point B, both on I as on II. We then have

$$\Sigma: \ \mathrm{IV} \cup \mathrm{B},$$
$$\Pi: \ \mathrm{I} \cup \mathrm{B}.$$

The only inter-element therefore is B, so that $\{ x_i \}$ converges to B. Taking the subsequence of odd-terms we have

$$\Sigma^*: \ \mathrm{III} \cup \mathrm{B} \cup \mathrm{IV},$$
$$\Pi^*: \ \mathrm{I} \cup \mathrm{B},$$

so that there is no inter-element.

2. Order-convergence in lattices.

In this section we will investigate some properties of sequences in case the underlying set is a lattice L. Then we can say that for any sequence $\{ x_i \}$ the set Π of all sub-elements and the set Σ of all super-elements constitute an additive, resp. a multiplicative ideal [1]). Also we can assert that the set R of all inter-elements is a sublattice of L. This follows directly from the definitions of all concepts involved.

2.1 Lemma. *If* $\{ x_i \}$ *is a sequence for which* $\underline{\lim}\, x_i$ *exists,* $\pi \leq a$ *for all elements* $\pi \in \Pi$ *implies* $\underline{\lim}\, x_i \leq a$.

Proof. $\pi \leq a$ for all $\pi \in \Pi$;

$\pi \leq \underline{\lim}\, x_i$ (def. of $\underline{\lim}\, x_i$).

So $\pi \leq a\,\underline{\lim}\, x_i \leq \underline{\lim}\, x_i$ for all $\pi \in \Pi$ . . . . (i)

Therefore $a\,\underline{\lim}\, x_i$ is an inter-element so that $\underline{\lim}\, x_i \leq a\,\underline{\lim}\, x_i$.

(i) and (ii) together yield $a\,\underline{\lim}\, x_i = \underline{\lim}\, x_i$ or equivalently $\underline{\lim}\, x_i \leq a$.

A consequence of this lemma is established in the following

2.2 Theorem. *Let* $\{ x_i \}$ *and* $\{ y_i \}$ *be two sequences such that* $x_i \leq y_i$ *for all* i. *Then* $\underline{\lim}\, x_i \leq \underline{\lim}\, y_i$ *and* $\overline{\lim}\, x_i \geq \overline{\lim}\, y_i$ *provided the involved expressions exist.*

Proof. It is clear that $\Pi_x \leq \Pi_y$.

Thus from $\pi_y \leq \underline{\lim}\, y_i$ for all $\pi_y \in \Pi_y$

it follows $\pi_x \leq \underline{\lim}\, y_i$ for all $\pi_x \in \Pi_x$.

According to the previous lemma this implies $\underline{\lim}\, x_i \leq \underline{\lim}\, y_i$.

Dually one establishes $\overline{\lim}\, x_i \geq \overline{\lim}\, y_i$.

2.3 Corollary. *If* $\{ x_i \}$ *and* $\{ y_i \}$ *are two convergent sequences with* $x_i \leq y_i$ *for all* i, *then* $\lim x_i \leq \lim y_i$.

2.4 Theorem. $\underline{\lim}\, x_i$ *exists if and only if* $\cup \{ \pi : \pi \in \Pi \}$ *exists. Dually*: $\overline{\lim}\, x_i$ *exists if and only if* $\cap \{ \sigma : \sigma \in \Sigma \}$ *exists.*

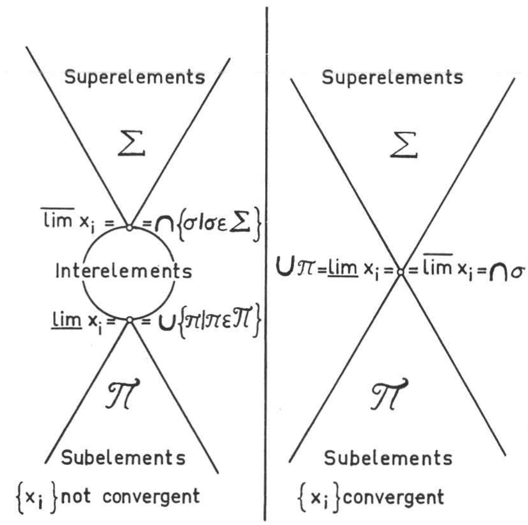
Proof. Suppose $\underline{\lim}\, x_i$ exists. Then we have $\pi \leq \underline{\lim}\, x_i$ for all $\pi \in \Pi$. Suppose further $\pi \leq a$ for all $\pi \in \Pi$. It then follows that $\underline{\lim}\, x_i \leq a$. This shows that $\underline{\lim}\, x_i$ is indeed the l.u.b. of all elements of Π.

1) [10], § 12.

On the other hand suppose $\cup \{ \pi : \pi \in \Pi \}$ exists. It is then obvious that $\pi \leq \cup \pi \leq \sigma$ for all $\pi \in \Pi$ and for all $\sigma \in \Sigma$. This means that $\cup \pi$ is an inter-element. Let a be an arbitrary inter-element: $\pi \leq a$ for all $\pi \in \Pi$. Then $\cup \pi \leq a$, so that $\cup \pi$ is indeed the smallest inter-element: $\cup \{ \pi : \pi \in \Pi \} = \underline{\lim}\, x_i$.

2.5 Corollary. *A sequence is convergent if and only if both* $\cup \{ \pi : \pi \in \Pi \}$ *and* $\cap \{ \sigma : \sigma \in \Sigma \}$ *exist and are equal.*

The two situations may be depicted as follows:



3. Order-convergence in complete lattices.

In case the lattice L is complete, all sequences $\{ x_i \}$ are bounded since

$$\cap x_i \leq x_j \leq \cup x_i \text{ for all } j.$$

This implies that for any sequence $\{ x_i \}$ the set Π and Σ are non void. Furthermore, the completeness of L and Theorem 2.4 imply that $\underline{\lim}\, x_i = \cup \{ \pi : \pi \in \Pi \}$ and $\overline{\lim}\, x_i = \cap \{ \sigma : \sigma \in \Sigma \}$ so that, in case of a complete lattice L, any sequence always has a non void set of inter-elements. This implies that every subsequence $\{ x_{n_i} \}$ of a convergent sequence $\{ x_i \}$ converges to the same limit as $\{ x_i \}$ does. Thus we have

3.1 Theorem. *The order-convergence, defined in a complete lattice* L, *makes* L *into a Fréchet* L*-space.*

Consider
$$\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} x_i \text{ and } \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} x_i.$$

Let $\sigma_k = \bigcup_{i=k}^{\infty} x_i$ and $\pi_k = \bigcap_{i=k}^{\infty} x_i$.

The following three properties hold

(i) $\pi_k \leq \sigma_k$ for all k;

(ii) $x_j \leq \sigma_k$ for all $j \geqq k$;

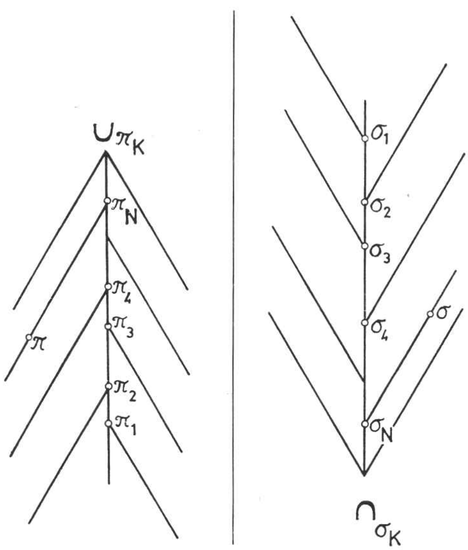
(iii) $\pi_k \leq x_j$ for all $j \geqq k$.

It follows from (ii) and (iii) that π_k and σ_k are sub- resp. super-elements for the sequence $\{ x_i \}$. The sequences $\{ \pi_i \}$ and $\{ \sigma_i \}$ are resp. monotone increasing and monotone decreasing sequences. Suppose π is a sub-element for the sequence $\{ x_i \} : \pi \leq x_j$ for $j \geqq N$. Then it follows $\pi \leq \pi_N \leq \cup \pi_k$. So any sub-element precedes some sub-element π_N and hence $\cup \pi_k$. Dually any super-element follows some super-element σ_N and hence $\cap \sigma_k$.

If $(\cup \pi_k)$ denotes the additive ideal generated by the element $\cup \pi_k$ (the so called principle ideal consisting of all elements preceding the element $\cup \pi_k$) and dually, if $(\cap \sigma_k)$ denotes the multiplicative ideal generated by the element $\cap \sigma_k$ (the so called principle ideal consisting of all elements following the element $\cap \sigma_k$) we may state
$$\Pi \leqslant \left(\bigcup_{k=1}^{\infty} \pi_k \right) \text{ and } \Sigma \leqslant \left(\bigcap_{k=1}^{\infty} \sigma_k \right).$$

This fishbone situation can be depicted as follows:

All sub-elements π_N are on the backbone, whereas the other sub-elements may also be on the sidebones.



All super-elements σ_N are on the backbone, whereas the other super-elements may also be on the sidebones.

3.2 Theorem. $\bigcup \{ \pi : \pi \in \Pi \} = \bigcup_{k=1}^{\infty} \pi_k$ *and* $\bigcap \{ \sigma : \sigma \in \Sigma \} = \bigcap_{k=1}^{\infty} \pi_k$.

Proof. $\pi \leq \bigcup \pi_k$ for all $\pi \in \Pi$;
therefore $\bigcup \{ \pi : \pi \in \Pi \} \leq \bigcup \pi_k$;
but since $\pi_k \in \Pi$ for all k we also have $\bigcup \pi_k \leq \bigcup \{ \pi : \pi \in \Pi \}$, so that $\bigcup \{ \pi : \pi \in \Pi \} = \bigcup_{k=1}^{\infty} \pi_k$ and dually.

3.3 Corollary. *A sequence* $\{ x_i \}$ *of elements of a complete lattice* L *is convergent if and only if* $\bigcup_{k=1}^{\infty} \pi_k = \bigcap_{k=1}^{\infty} \sigma_k$ *or if and only if* $\bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} x_i = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} x_i$.

3.4 Lemma. *If* $\{ x_i \}$ *is a monotone increasing sequence, it has the limit* $\bigcup_{i=1}^{\infty} x_i$. *Dually: if* $\{ x_i \}$ *is a monotone decreasing sequence it has the limit* $\bigcup_{i=1}^{\infty} x_i$.

Proof. $\pi_k = \bigcap_{i=k}^{\infty} x_i = x_k$; so $\bigcup_{k=1}^{\infty} \pi_k = \bigcup_{k=1}^{\infty} x_k$;

$\sigma_k = \bigcup_{i=k}^{\infty} x_i = \bigcup_{i=1}^{\infty} x_i$; so $\bigcap_{k=1}^{\infty} \sigma_k = \bigcup_{i=1}^{\infty} x_i$.

3.5 Theorem. *A sequence* $\{ x_i \}$ *converges to the limit* x *if and only if for* each element x_k there exist elements u_k and v_k such that $u_k \leq x_k \leq v_k$, where $\{ u_i \} \uparrow x$ and $\{ v_i \} \downarrow x$; i.e. where $\{ u_i \}$ is a monotone increasing sequence with the limit x and where $\{ v_i \}$ is a monotone decreasing sequence with the limit x also.

Proof. Suppose $\{ x_i \} \rightarrow x$. Take $u_k = \pi_k = \bigcap_{i=k}^{\infty} x_i$ and $v_k = \sigma_k = \bigcup_{i=k}^{\infty} x_i$. Then $\{ \pi_i \}$ and $\{ \sigma_i \}$ are monotone increasing resp. decreasing sequences with $\bigcup \pi_i = \underline{\lim}\, x_i = x = \overline{\lim}\, x_i = \bigcap \sigma_i$, so that $\lim \pi_i = \lim \sigma_i = x$.
Now suppose $u_k \leq x_k \leq v_k$, where $\{ u_k \} \uparrow x$ and $\{ v_k \} \downarrow x$. Apparently all elements u_k and v_k are sub- resp. super-elements for the sequence $\{ x_i \}$. According to the definition of $\underline{\lim}\, x_i$ and $\overline{\lim}\, x_i$ we have for all k

$u_k \leq \underline{\lim}\, x_i$ from which $\bigcup u_k \leq \underline{\lim}\, x_i$ and

$v_k \geq \overline{\lim}\, x_i$ from which $\bigcap v_k \geq \overline{\lim}\, x_i$.

But since $\bigcup u_i = x = \bigcap v_i$ we have $\underline{\lim}\, x_i = \overline{\lim}\, x_i$.
Thus $\{ x_i \}$ is convergent to the limit x.

RESULT. *In case of a complete lattice* L, *we have the following equivalent criteria for a sequence* $\{ x_i \}$ *to be convergent with limit* x:

I $\quad \underline{\lim}\, x_i = \overline{\lim}\, x_i = x;$

II $\quad \bigcup \{ \pi : \pi \in \Pi \} = \bigcap \{ \sigma : \sigma \in \Sigma \} = x;$

III $\quad \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} x_i = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} x_i = x;$

IV $\quad u_k \leq x_k \leq v_k$, where $\{u_k\}\uparrow x$ and $\{ v_k \}\downarrow x$.

4. The continuity of the lattice operations.

Let L be a complete lattice with the following properties

(i) $\quad \{ x_i \}\uparrow x$ implies $a \cap \lim x_i \leq \lim (a \cap x_i)$;

(ii) $\quad \{ x_i \}\downarrow x$ implies $a \cup \lim x_i \geq \lim (a \cup x_i)$.

These lattices were called TOPOLOGICAL LATTICES by Birkhoff[1]). He established the fact that in topological lattices the lattice operations are continuous. The proof for the $\cup$-operation will be given here; the proof for the $\cap$-operation is dual.

4.1 LEMMA. $\bigcup (x_i \cup y_i) = \bigcup x_i \cup \bigcup y_i$.

Proof. $x_k \leq \bigcup x_i$ and $y_k \leq \bigcup y_i$ so that $x_k \cup y_k \leq \bigcup x_i \cup \bigcup y_i$ and hence $\bigcup (x_i \cup y_i) \leq \bigcup x_i \cup \bigcup y_i$;

$x_k \leq \bigcup (x_i \cup y_i)$, therefore $\bigcup x_i \leq \bigcup (x_i \cup y_i)$ and similarly $\bigcup y_i \leq \bigcup (x_i \cup y_i)$ so that $\bigcup x_i \cup \bigcup y_i \leq \bigcup (x_i \cup y_i)$.

4.2 COROLLARY. $\{ x_i \}\uparrow x$ *and* $\{y_i \}\uparrow y$ *imply* $\lim (x_i \cup y_i) = \lim x_i \cup \lim y_i$.

4.3 LEMMA. $\bigcap x_i \cup \bigcap y_i \leq \bigcap (x_i \cup y_i)$.

Proof. $\bigcap x_i \leq x_k$ and $\bigcap y_i \leq y_k$ so that $\bigcap x_i \cup \bigcap y_i \leq x_k \cup y_k$ for all k, from which the result.

4.4 COROLLARY. $\{ x_i \}\downarrow x$ *and* $\{ y_i \}\downarrow y$ *imply* $\lim x_i \cup \lim y_i \leq \lim (x_i \cup y_i)$.

4.5 LEMMA. $\{ x_i \}\downarrow x$ *implies* $a \cup \lim x_i = \lim (a \cup x_i)$.

Proof. $a \cup x_k \geq a \cup \bigcap x_i$ for all k, hence $\bigcap (a \cup x_i) \geq a \cup \bigcap x_i$.

[1]) [1], p. 63.

Since L is a topological lattice: $a \cup \lim x_i \geq \lim (a \cup x_i)$ or equivalently $a \cup \bigcap x_i \geq \bigcap (a \cup x_i)$ from which the result.

4.6 LEMMA. $\{x_i\} \downarrow x$ *and* $\{y_i\} \downarrow y$ *imply* $\lim x_i \cup \lim y_i \geq \lim (x_i \cup y_i)$.

Proof. $x_i \cup y_i = x_i \cup y_i$ so that $x_j \cup y_i \geq x_i \cup y_i$ for $j \leqq i$;
therefore $\lim (x_j \cup x_i) \geq \lim (x_i \cup y_i)$ (COROLLARY 2.3);
$x_j \cup \lim y_i = \lim (x_j \cup x_i)$ (LEMMA 4.5);
$x_j \cup y \geq \lim (x_i \cup y_i)$ for all j;
$\lim (x_j \cup y) \geq \lim (x_i \cup y_i)$ (COROLLOARY 2.3);
$y \cup \lim x_i \geq \lim (x_i \cup y_i)$ (LEMMA 4.5);
$y \cup x \geq \lim (x_i \cup y_i)$ or $\lim x_i \cup \lim y_i \geq \lim (x_i \cup y_i)$.

From COROLLARY 4.4 and from LEMMA 4.6 we now have

4.7 LEMMA. $\{x_i\} \downarrow x$ *and* $\{y_i\} \downarrow y$ *imply* $\lim x_i \cup \lim y_i = \lim (x_i \cup y_i)$.
This enables us to state the main

4.8 THEOREM. $\{x_i\} \to x$ *and* $\{y_i\} \to y$ *imply* $\lim (x_i \cup y_i) = \lim x_i \cup \lim y_i$.

Proof. $\{x_i\} \to x$ implies $\pi_k^x \leq x_k \leq \sigma_k^x$ such that $\{\pi_k^x\} \uparrow x$ and $\{\sigma_k^x\} \downarrow x$. Similarly $\{y_i\} \to y$ implies $\pi_k^y \leq y_k \leq \sigma_k^y$ such that $\{\pi_k^y\} \uparrow y$ and $\{\sigma_k^y\} \downarrow y$.
It follows $\pi_k^x \cup \pi_k^y \leq x_k \cup y_k \leq \sigma_k^x \cup \sigma_k^y$
with $\{\pi_k^x \cup \pi_k^y\} \uparrow x \cup y$ from COROLLARY 4.2
and $\{\sigma_k^x \cup \sigma_k^y\} \downarrow x \cup y$ from COROLLARY 4.7.

But this means $\lim (x_i \cup y_i) = \lim x_i \cup \lim y_i$.

Now that we have proved that the operations $\cap$ and $\cup$ are continuous with respect to the order-convergence we also can say that the relation $\leq$ is continuous, as this relation may be expressed in terms of the operation $\cap$ (or $\cup$).

Concluding this section we want to show that complete Boolean algebras are topological lattices. This follows immediately from the fact that in complete Boolean algebras the following two distributive laws hold

$$a \cap \bigcup x_i = \bigcup ax_i \text{ and } a \cup \bigcap x_i = \bigcap (a \cup x_i),$$

as we already pointed out in section 1 of chap. II.
Thus we have

4.9 THEOREM. *A complete Boolean algebra is a topological lattice in which the Boolean operations are continuous with respect to the order-convergence.*

Proof. It has already been shown that the lattice operations $\cap$ and $\cup$ and the lattice relation $\leq$ are continuous. Since, however, the Boolean operation of complementation can also be expressed in terms of the lattice operations $\cap$ and $\cup$ the result follows.

5. The metric topology of $\mathcal{M}$.

Let M be a complete, associate Boolean metric space whose underlying complete associate ring is R. Let B be the complete Boolean algebra of idempotents of R. For $a,b \in R$ we have $d(a,b) = \varphi(a - b) \in B$ and more specifically $d(a,b) = a'b \cup b'a$ for $a,b \in B$. The order-convergence makes B into a Fréchet L-space: $\mathcal{B}$. We will refer to this topology of $\mathcal{B}$ as the ORDER-TOPOLOGY.
In a similar fashion we want to introduce a topology in M.

5.1 DEFINITION. *We say that a sequence* $\{x_i\}$ *of elements* $x_i \in M$ *converges to the element* $x \in M$: $\lim x_i = x$, *provided* $\lim d(x_i,x) = 0$ *in the order-topology of* $\mathcal{B}$. *The topology of* M, *induced by this notion of convergence, will be referred to as the* METRIC-TOPOLOGY *of* M. It will be proved below that the metric-topology makes M into a Fréchet L-space: $\mathcal{M}$.
It should be noted that due to definition 5.1 there are two notions of convergence now for elements of $\mathcal{B}$. Fortunately we have

5.2 THEOREM. *For the elements of* $\mathcal{B}$ *the metric-topology coincides with the order-topology*; *i.e.* $\lim x_i = x$ *if and only if* $\lim d(x_i,x) = 0$.

Proof. From the fact that the distance function for elements of $\mathcal{B}$ can be expressed in terms of the Boolean operations: $d(a,b) = a'b \cup b'a$ and since these Boolean operations are continuous in the order-topology it follows that the distance function is also continuous in the order-topology. Thus if $\lim x_i = x$ then also $\lim d(x,x_i) = d(x,\lim x_i) = d(x,x) = 0$. Conversely:
$\lim x_i = \lim d\{d(x,x_i),x\}$ [1] $= d\{\lim d(x,x_i),x\} = d(0,x) = x$.

5.3 THEOREM. $\mathcal{M}$ *is a Fréchet* L-*space.*

Proof. We have to show

(i) $\{x_i\}$, $x_i = x$, converges to x;
(ii) $\lim x_i = a$ and $\lim x_i = b$ imply $a = b$;
(iii) $\lim x_i = x$ implies $\lim x_{n_i} = x$ for any subsequence $\{x_{n_i}\}$ of the sequence $\{x_i\}$.

[1]) [4], Theorem 131.2, p. 332.

(i) is obvious.

(ii) follows from $d(a,b) \leq d(x_i,a) \cup d(x_i,b)$, for all i, which implies $d(a,b) \leq \lim d(x_i,a) \cup \lim d(x_i,b) = 0$. Thus $d(a,b) = 0$ or $a = b$.

(iii) follows from the fact that $\mathcal{B}$ is a Fréchet L-space.

5.4 THEOREM. *The ring operations of* R *are continuous in* $\mathcal{M}$.

Proof. Let $\lim x_i = x$; i.e. $\lim d(x,x_i) = 0$; and let $\lim y_i = y$; i.e. $\lim d(y,y_i) = 0$.

Then we have $d(xy,x_i y_i) = \varphi(xy - x_i y + x_i y - x_i y_i) \leq$
$\varphi[(x - x_i) y] \cup \varphi[x_i (y - y_i] =$
$\varphi(y) \varphi(x - x_i) \cup \varphi(x_i) \varphi(y - y_1) \leq \varphi(x - x_i) \cup \varphi(y - y_i) =$
$d(x,x_i) \cup d(y,y_i)$.

From this it follows that $\lim d(xy,x_i y_i) = 0$ so that $\lim x_i y_i = xy$.

Similarly we have $d(x + y,x_i + y_i) =$
$\varphi(x + y - x_i - y_i) \leq \varphi(x - x_i) \cup \varphi(y - y_i) =$
$d(x,x_i) \cup d(y,y_i)$, from which $\lim (x_i + y_i) = x + y$.

5.5 THEOREM. $\mathcal{M}$ *has the property that if a sequence* $\{x_i\}$ *does not converge to* x, *there is a subsequence* $\{x_{n_i}\}$ *not containing a sub-subsequence converging to* x.

Proof. We will prove the equivalent statement: if every subsequence $\{x_{n_i}\}$ has a subsubsequence $\{y_i\}$ with limit x, then the original sequence $\{x_i\}$ has the limit x. Let, therefore, $d(x_i x) = d_i$ and $d(x_{n_i},x) = d_{n_i}$ and also $d(y_j,x) = \delta_j$.
Then we can say $\lim \delta_j = 0$, which means $\delta_j \leq u_j$ with $\{u_j\} \downarrow 0$. For every subsequence $\{x_{n_i}\}$ there is such a sequence $\{u_i\}$. This implies

$$\bigcap_{i=k}^{\infty} d_i \leq \bigcap_{j=1}^{\infty} \delta_j \leq \bigcap_{j=1}^{\infty} u_j \text{ from which } \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} d_i \leq \bigcup_{l=1}^{\infty} \bigcap_{j=l}^{\infty} u_j = 0;$$

$$\bigcup_{i=k}^{\infty} d_i \leq \bigcup_{j=1}^{\infty} \delta_j \leq \bigcup_{j=1}^{\infty} u_j \text{ from which } \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} d_i \leq \bigcap_{l=1}^{\infty} \bigcup_{j=l}^{\infty} u_j = \bigcap_{l=1}^{\infty} u_l = 0.$$

Since $\bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} d_i = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} d_i = 0$, we have $\lim d_i = 0$ and hence $\lim x_i = x$.

5.6 DEFINITION. *An element* $x \in \mathcal{M}$ *is called an* ACCUMULATION-*element of a subset* X *of* $\mathcal{M}$, *provided there exists a sequence* $\{x_i\}$ *of elements of* X, *all different from* x, *such that* $\lim x_i = x$.

The CLOSURE $\overline{X}$ *of a subset* X *of* $\mathcal{M}$ *is the set* X *together with all its accumulation elements.*
A set X *will be called* CLOSED *provided* $X = \overline{X}$.
A set X *will be called* OPEN *provided its compliment is closed.*
One may easily establish that the so defined family of open sets induces an open-set-topology in $\mathcal{M}$; i.e. a topology defined in terms of open sets satisfying the usual requirements for open sets. Even more, as one may also easily see,

5.7 THEOREM. *The open-set-topology of* $\mathcal{M}$, *induced by the metric topology of* $\mathcal{M}$ *by* DEFINITION 5.6 *makes* $\mathcal{M}$ *a* FRÉCHET SPACE (T_1-SPACE).

5.8 DEFINITION. *A mapping* f (x) *of* $\mathcal{M}$ *into* $\mathcal{M}$ *is called* CONTINUOUS AT x_0, *provided* $\lim x_i = x_0$ *implies* $\lim f(x_i) = f(x_0)$.

5.9 THEOREM. *The counterimage of a closed (open) set under a continuous mapping is closed (open). More precise: if* A *and* B *are two subsets of* $\mathcal{M}$ *and if* f *is a continuous mapping of* A *onto* B, $f(A) = B$, *then a set closed (open) in* B *will have a counterimage that is closed (open) in* A.

Proof. Let Y be a closed subset of B and let $f^{-1}(Y) = X \leq A$. Consider the sequence $\{x_i\}$ with limit x, $x_i \in X, x \in A$. The continuity of f implies $\lim f(x_i) = f(x), f(x_i) \in Y,\ f(x) \in B$. But since Y is closed in B, we must have $f(x) \in Y$, and hence $x \in X = f^{-1}(Y)$. Therefore X is closed in A.
By taking relative complements one sees that the same holds for open sets.

Note: the converse of this theorem need not be true, i.e. if the counterimages of closed (open) sets are closed, the mapping need not be continuous. We have to be careful therefore, not to apply certain topological theorems that are based upon the equivalence of the statements:

(i) f is a continuous mapping,
(ii) counterimages of closed (open) sets are closed (open).

5.10 COROLLARY. *A homeomorphism between two subsets* A *and* B *of* $\mathcal{M}$ *carries sets closed (open) in* A *over into sets, closed (open) in* B.

5.11 THEOREM. *The Boolean algebra* $\mathcal{B}$ *is closed.*

Proof. Let x be an accumulation element of $\mathcal{B}$: $\lim x_i = x, x_i \in \mathcal{B}$.

Since $x_i x_i = x_i$ we also have
$x = \lim x_i = \lim x_i x_i = \lim x_i \lim x_i = xx$. Thus $x \in \mathcal{B}$.

5.12 DEFINITION. *A subset* A *of* $\mathcal{M}$ *is compact, provided every infinite subset of* A *contains at least one accumulation-element in the set* A.

It follows that all accumulation-elements of A necessarily belong to A, so that a compact set A is automatically closed.

5.13 THEOREM. *A homeomorphism* f *carries a compact set* A *over into a compact set* f (A) = B.

Proof. Let X be an infinite subset of B. Then f^{-1} (X) is an infinite subset of A that must have at least one accumulation-element x in A since A is compact. It follows that the element f (x) is an accumulation element of X in B.

5.14 THEOREM. *If* x *is an accumulation-element of the set* X, *every open set containing* x *contains at least one element of* X *different from* x.

Proof. Suppose this is not the case. Let U be an open set containing x but not containing any elements of X different from x. The complement U′ of U is a closed set containing all elements of X but not the accumulation-element x, which is a contradiction.

CHAPTER VI

MAXIMAL CHAINS

In this chapter $\mathcal{B}$ stands for a complete, atom-free Boolean algebra with the order-topology induced by the order-convergence. As was shown in THEOREM 5.2 chap. V this order-topology coincides with the metric-topology induced by the distance function $d(a,b) = a'b \cup b'a$.

1. Characterization.

1.1 DEFINITION. *An* ORDERED *subset of $\mathcal{B}$ is a subset of $\mathcal{B}$ in which every two elements are comparable. A* CHAIN $C(\alpha,\beta)$ *in $\mathcal{B}$ is an ordered subset of $\mathcal{B}$ with a first element α and a last element β. A chain is also called an ordered subset of $\mathcal{B}$ with end-points α and β. A* MAXIMAL CHAIN $\Gamma(\alpha,\beta)$ *in $\mathcal{B}$ is a chain that is not a proper subset of a chain with the same end-points. We also say that a maximal chain is irreducible between its end-points.*

1.2 THEOREM. *A maximal chain is convex.*

Proof. Let γ and $\delta, \gamma < \delta$, be two distinct elements of a maximal chain $\Gamma(\alpha,\beta)$ and suppose there is no element ξ on the maximal chain such that $B(\gamma,\xi,\delta)$, which, here, is equivalent to $\gamma < \xi < \delta$. Consider the motion m of $\mathcal{B}$ defined by $m(x) = d(\gamma,x)$, $x \in \mathcal{B}$. Then we have $m(\gamma) = 0$ and $m(\delta) = d(\gamma,\delta)$. Since $\mathcal{B}$ is atom-free there must be an element $\varepsilon \in \mathcal{B}$ such that $0 < \varepsilon < m(\delta)$, which implies $B(0,\varepsilon,m(\delta))$. The motion m is involutory, so that we also have $B(\gamma,m^{-1}(\varepsilon),\delta)$ which is equivalent to $\gamma < m^{-1}(\varepsilon) < \delta$. This would imply that $m^{-1}(\varepsilon)$ is an element of $\mathcal{B}$ between α and β comparable to all elements of the maximal chain $\Gamma(\alpha,\beta)$ that is not on $\Gamma(\alpha,\beta)$, which is a contradiction.

1.3 THEOREM. *A maximal chain is algebraically complete.*

Proof. If $A \leq \Gamma(\alpha,\beta)$, then $\cap A = \bigcap_{a \in A} a$ exists in $\mathcal{B}$ since $\mathcal{B}$ is complete.

Let $x \in \Gamma(\alpha,\beta)$ such that $x \geq a$ for some element $a \in A$. Then $x \geq \cap A$. If $x \in \Gamma(\alpha,\beta)$ such that for no element $a \in A$, $x \geq a$, then for all $a \in A$, $x < a$ and thus $x \leq \cap A$. It follows that $\cap A$ is comparable to all elements of the maximal chain. It must therefore be an element of it. In a similar way we prove that $\cup A = \bigcup_{a \in A} a \in \Gamma(\alpha,\beta)$.

1.4 THEOREM. *A chain that is convex and algebraically complete is a maximal chain.*

Proof. Let $x \in \mathcal{B}$, $\alpha \leq x \leq \beta$, comparable to all elements of the convex and algebraically complete chain $C(\alpha,\beta)$. We will show that then x must be an element of $C(\alpha,\beta)$. This would establish that $C(\alpha,\beta)$ is a maximal chain. Let $C(x,\beta)$ be the set of all elements of $C(\alpha,\beta)$ following x and let $C(\alpha,x)$ be the set of all elements of $C(\alpha,\beta)$ preceding x. Then we have $C(\alpha,x) \cup C(x,\beta) = C(\alpha,\beta)$. Consider both $\cap C(x,\beta)$ and $\cup C(\alpha,x)$. These elements of $\mathcal{B}$ must belong to $C(\alpha,\beta)$, due to its algebraic completeness. We have $\cup C(\alpha,x) \leq x \leq \cap C(x,\beta)$. Suppose $\cup C(\alpha,x) \neq \cap C(x,\beta)$. Then there must be an element $y \in \mathcal{B}$ such that $\cup C(\alpha,x) < y < \cap C(x,\beta)$ because $C(\alpha,\beta)$ is convex. It follows that $y \notin C(\alpha,x)$ and $y \notin C(x,\beta)$, which contradicts the fact that $C(\alpha,x) \cup C(x,\beta) = C(\alpha,\beta)$.
Therefore $\cup C(\alpha,x) = x = \cap C(x,\beta)$, from which $x \in C(\alpha,\beta)$.
It follows from THEOREMS 1.2, 1.3 and 1.4 that maximal chains in complete, atom-free Boolean algebras are characterized as convex and algebraically closed chains. Some additional properties of maximal chains are given in the following theorems.

1.5 THEOREM. *A maximal chain is closed.*

Proof. Suppose $\lim x_i = x$, $x_i \in \Gamma(\alpha,\beta)$, $x \in \mathcal{B}$. The elements x_i are comparable to all elements of the maximal chain. Because of the continuity of the order relation $\leq$ x is also comparable to all elements of the maximal chain and hence x must be an element of it.

1.6 THEOREM. *A maximal chain is compact.*

Proof. Let $\Gamma(\alpha,\beta)$ be a maximal chain and let $X \leq \Gamma(\alpha,\beta)$ be an infinite subset of it. We will construct a monotone sequence $\{ x_i \}$ of elements of X. Since monotone sequences are convergent and since maximal chains are closed the limit must be an element of $\Gamma(\alpha,\beta)$, which would prove the compactness of the maximal chain. The monotone sequence is constructed by successive bisection of

the maximal chain. Of the successively generated two parts of the maximal chain, at least one must contain infinitely many elements of X. If both of them do, select "the left interval" to proceed with. The first step is performed by taking $x_1 \in X$, $x_1 \neq \alpha$, $x_1 \neq \beta$, so that $\alpha_1 = \alpha < x_1 < \beta = \beta_1$. At least one of these intervals $\alpha_1 < \gamma < x_1$ and $x_1 < \gamma < \beta_1$ contains infinitely many elements of X. We denote the selection by $\alpha_2 < \gamma < \beta_2$. Now we take $x_2 \in X$ such that $\alpha_2 < x_2 < \beta_2$ etc. The n^{th} step is $\alpha_n < x_n < \beta_n$, which means either $\alpha_{n-1} < x_n < x_{n-1}$ or $x_{n-1} < x_n < \beta_{n-1}$.
We then have the following situation

$$\begin{array}{ll} \alpha_1 = \alpha < x_1 < \beta = \beta_1, & \\ \alpha_2 < x_2 < \beta_2, & \\ \alpha_3 < x_3 < \beta_3, & \\ \ldots\ldots\ldots\ldots & \\ \alpha_n < x_n < \beta_n & \text{etc.,} \\ \ldots\ldots\ldots\ldots, & \end{array}$$

where $\alpha_i = \alpha_{i-1}$ and $\beta_i = x_{i-1}$;
or $\alpha_i = x_{i-1}$ and $\beta_i = \beta_{i-1}$.

We may distinguish two cases:

(i) The selection of intervals is eventually a selection of the type

$$\alpha_{k-1} < x_k < x_{k-1}$$

(after a certain index only this kind of intervals appears).
(ii) The sequence of intervals contains infinitely often the type

$$x_{k-1} < x_k < \beta_{k-1}.$$

In the first case it is easily seen that the sequence $\{ x_i \}$ is eventually monotone decreasing.
In the second case: delete all intervals of type (i). The subsequence of intervals so obtained yields a subsequence $\{ x_{n_i} \}$ that is monotone increasing.

1.7 THEOREM. *A maximal chain cannot have a connected proper subset with the same end-points.*

Proof. Let $\Gamma(\alpha,\beta)$ be a maximal chain and let $\Delta(\alpha,\beta)$ be a connected proper subset containing α and β. Suppose $x \in \Gamma(\alpha,\beta)$, $x \notin \Delta(\alpha,\beta)$. Denoting by P (x) the set of all maximal chain elements y such that $y \leq x$ and similarly denoting by F (X) the set of all maximal chain elements y such that $y \geq x$, we will consider the sets

$P(x) \cap \Delta(\alpha,\beta)$ and $F(x) \cap \Delta(\alpha,\beta)$ both of which are closed in $\Delta(\alpha,\beta)$. Furthermore these sets are disjoint, non void while their union is $\Delta(\alpha,\beta)$. They form a closed partition of $\Delta(\alpha,\beta)$ which contradicts the assumption that $\Delta(\alpha,\beta)$ was connected.

1.8 THEOREM. *A chain that is connected is a maximal chain.*

Proof. Let $x \in \mathcal{B}$, $\alpha < x < \beta$, comparable to all elements of a connected chain $C(\alpha,\beta)$. Consider the sets $P(x) \cap C(\alpha,\beta)$ and $F(x) \cap C(\alpha,\beta)$, where $P(x)$ and $F(x)$ have the same meaning as in THEOREM 1.7. If x is not an element of $C(\alpha,\beta)$, the two sets form a closed partition of $C(\alpha,\beta)$ which cannot be. Hence $x \in C(\alpha,\beta)$.

1.9 THEOREM. *Two maximal chains* $\Gamma_1(\alpha_1,\beta_1)$ *and* $\Gamma_2(\alpha_2,\beta_2)$ *for which there exists a one to one mapping* f *of* Γ_1 *onto* Γ_2 *that is order preserving and continuous, are homeomorphic.*

Proof. Suppose $\{\xi_i^2\} \uparrow \xi^2, \xi_i^2 \in \Gamma_2$. Then $\{f^{-1}(\xi_i^2)\}$ is a monotone increasing sequence of elements of Γ_1 that must have a limit ξ^1 on Γ_1. Then, because of the continuity of f, it follows $f(\xi^1) = \xi^2$ or equivalently $f^{-1}(\xi^2) = \xi^1$. This means that the mapping f^{-1} is continuous for monotone increasing sequences. In the same way the continuity of f^{-1} for monotone decreasing sequences is proved. But then f^{-1} is continuous for arbitrary sequences.

2. Separable Boolean algebras.

2.1 DEFINITION. *An ordered subset* A *of* $\mathcal{B}$ *is called* SEPARABLE, *provided the* g.l.b. *and the* l.u.b. *of* A *can be written as a* g.l.b. *and a* l.u.b. *of at most countably many elements of* A. *Thus, provided*

(i) $\bigcap A = \bigcap_{a \in A} a = \bigcap_{i=1}^{\infty} x_i, x_i \in A$; *and*

(ii) $\bigcup A = \bigcup_{a \in A} a = \bigcup_{i=1}^{\infty} y_i, y_i \in A$.

2.2 DEFINITION. $\mathcal{B}$ *is called separable, provided every ordered subset of* $\mathcal{B}$ *is separable.*

2.3 LEMMA. *An ordered, closed and separable subset* A *of* $\mathcal{B}$ *is algebraically complete.*

Proof. We have $\bigcap A = \bigcap_{i=1}^{\infty} x_i, x_i \in A$ and $\bigcup A = \bigcup_{i=1}^{\infty} y_i, y_i \in A$.
It is then always possible to select a monotone decreasing sequence

$\{x_{n_i}\} \downarrow \cap A$ and similarly a monotone increasing sequence $\{y_{n_i}\} \uparrow \cup A$. Since A is closed we have $\cap A \in A$ and $\cup A \in A$.

2.4 THEOREM. *If $\mathcal{B}$ is separable every maximal chain is connected.*

Proof. Suppose $\Gamma(\alpha,\beta) = A \cup B$ where $A \cap B = 0$, A and B both closed in $\Gamma(\alpha,\beta)$; $A \neq 0$, $B \neq 0$.
As $\Gamma(\alpha,\beta)$ is closed in $\mathcal{B}$, A and B are both closed in $\mathcal{B}$. They are also both ordered. Since $\mathcal{B}$ is separable A and B are algebraically complete (LEMMA 2.3). Consider $\cap B = b \in B$. Assume $b \neq \alpha$. Denote by A* the set of all maximal chain elements preceding and not equal b. Then A* is not empty since $b \neq \alpha$ and $\Gamma(\alpha,\beta)$ is convex. A* is a subset of A, obviously closed in A, hence closed in $\Gamma(\alpha,\beta)$ and thus closed in $\mathcal{B}$. A* is also ordered. Since $\mathcal{B}$ is separable A* is algebraically complete. Therefore $\cup A^* \in A^* \leq A$; but $\cup A^* = b \in B$, yielding a contradiction. We have to assume therefore that $b = \alpha$, which implies $\alpha \in B$. In exactly the same way we may prove that $\alpha \in A$. This contradicts the assumption that $\Gamma(\alpha,\beta)$ was not connected.

2.5 THEOREM. *If $\mathcal{B}$ is separable, a closed and convex chain is a maximal chain.*

Proof. Follows from LEMMA 2.3 and THEOREM 1.4.

2.6 COROLLARY. *If $\mathcal{B}$ is separable, a maximal chain cannot have a closed and convex proper subset with the same end-points.*
Thus we have obtained for complete, atom-free and separable Boolean algebras the following characterization of maximal chains

2.7 THEOREM. *If $\mathcal{B}$ is separable, a chain is a maximal chain if and only if it is closed and convex.*

Proof. Follows from THEOREMS 1.2, 1.5 and 2.5.

CHAPTER VII

ARCS

Throughout this chapter $\mathcal{M}$ stands for a complete, separable, associate, convex Boolean metric space whose complete, separable Boolean algebra of idempotents is $\mathcal{B}$.

1. Arc-length.

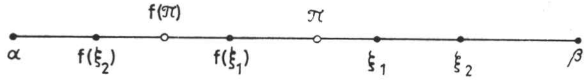
1.1 DEFINITION. *An* ARC A (a,b) *is a subset of* $\mathcal{M}$ *that is homeomorphic with a maximal chain* $\Gamma(\alpha,\beta)$ *in* $\mathcal{B}$: $A(a,b) = f(\Gamma(\alpha,\beta))$ *where* f *is a homeomorphism such that* $f(\alpha) = a$ *and* $f(\beta) = b$ [1]).

Since a maximal chain is irreducible between its end-points the same is true for an arc. Furthermore, since the property of being connected is solely expressed in terms of closed (open) sets and since a homeomorphism carries sets closed (open) in the maximal chain over into sets closed (open) in the arc and conversely, we may say that an arc is connected since a maximal chain is connected. For the same reasons it is impossible for an arc to have a connected proper subset with the same end-points.

1.2 THEOREM. *An arc is compact* (*and hence closed*).
Proof. Follows from the fact that a maximal chain is compact and that compactness is invariant under a homeomorphism.

1.3 LEMMA. *A homeomorphism* $f(\Gamma(\alpha,\beta)) = \Gamma(\alpha,\beta)$ *of a maximal chain onto itself such that* $f(\alpha) = \alpha$ *and* $f(\beta) = \beta$ *preserves the order.*

Proof.



Assume the contrary: $\xi_1 < \xi_2$ and $f(\xi_1) > f(\xi_2)$, $\xi_1 \neq \alpha$, $\xi_2 \neq \beta$; then we will produce a contradiction.

1) BG II.

Let X be the set of all elements ξ such that $f(\xi) \geq f(\xi_2)$. This set is not void since $\xi_1 \in X$. Consider $\cap X = \pi$. Then $\pi \neq \alpha$; for if $\pi = \alpha$, we would have $\alpha = f(\alpha) = f(\pi) \geq f(\xi_2)$, which is impossible. Since X is ordered and (obviously) closed and since $\mathcal{B}$ is separable, X is algebraically complete. Therefore $\pi \in X$, which means $f(\pi) \geq f(\xi_2)$; $f(\pi) = f(\xi_2)$ would imply $\pi = \xi_2$, thus yielding $\xi_1 < \xi_2 = \pi$ which contradicts the definition of π. Therefore: $f(\pi) > f(\xi_2)$.

Let $P(\pi)$ denote the set of all maximal chain elements preceding π. Since $\pi \neq \alpha$ this set is not empty. It holds that $\cup P(\pi) = \pi$. Since $\mathcal{B}$ is separable it is always possible to write $\pi = \bigcup_{i=1}^{\infty} \pi_i$, $\pi_i \in P(\pi)$. Furthermore it is always possible to select a monotone increasing subsequence $\{\pi_{n_i}\} \uparrow \pi$. Then we have $\lim f(\pi_{n_i}) = f(\pi)$. Together with $f(\pi) > f(\xi_2)$ this implies the existence of an element π_{n_k} such that $f(\pi_{n_k}) > f(\xi_2)$. But $\pi_{n_k} \notin X$ since $\pi_{n_k} \in P(\pi)$. This is a contradiction.

1.4 DEFINITION. *A* NORMALLY ORDERED INSCRIBED POLYGON $P(a_0,a_1,\ldots\ldots,a_n)$ *of an arc* $A(a,b) = f(\Gamma(\alpha,\beta))$ *is a subset* $P(a_0,a_1,\ldots\ldots,a_n)$ *of the arc* $A(a,b)$ *such that* $a_0 = a$ *and* $a_n = b$ *and such that* $f^{-1}(a_i) = \alpha_i \in \Gamma(\alpha,\beta)$, *whereas* $i < j$ *implies* $f^{-1}(a_i) < f^{-1}(a_j)$ *or* $\alpha_i < \alpha_j$.

It follows from LEMMA 1.3 that this order on the arc $A(a,b)$ is independent of the homeomorphism f.

Let $\lambda(P) = \bigcup_{i=0}^{n-1} d(a_i,a_{i+1})$. This quantity is then independent of the homeomorphism f, soleley an intrinsic value of the point set $P(a_0,a_1,\ldots\ldots,a_n)$ and the intrinsic normal ordering induced by the order of $\Gamma(\alpha,\beta)$.

Finally let $l(A(a,b)) = \bigcup_{P} \lambda(P)$, where the union is to be extended over all normally ordered inscribed polygons of the arc $A(a,b)$. This will be our definition of the length of an arc $A(a,b)$, independent of the homeomorphism f [1]).

1.5 DEFINITION. *A* COMPLEX C *of a subset* $X \leq \mathcal{M}$ *is a set of unordered pairs of elements of* X.

A sequence $(x_1,x_2,\ldots\ldots,x_n)$ *of elements of a subset* $X \leq \mathcal{M}$ *is said*

[1]) BG II.

TO BELONG TO A COMPLEX C *of* X, *provided* $(x_i, x_{i+1}) \in C$, $i = 1 \rightarrow n-1$.
A sequence $(x_1, x_2, \ldots, x_n)$ *of elements of a subset* $X \leq \mathcal{M}$ *is said* TO CONNECT p WITH q, *where* p *and* q *are elements of* X, *provided* $x_1 = p$ *and* $x_n = q$.
A subset $X \leq \mathcal{M}$ *is said* TO BE CONNECTED WITH RESPECT TO A COMPLEX C *of* X *provided every two elements* p *and* q *of* X *are connected by a sequence belonging to* C.

Note that by taking C as a complex of the set C* consisting of all the elements forming the unordered pairs of the set C, the complex C may be connected with respect to itself. In that case we say that C is a CONNECTED COMPLEX.

1.6 DEFINITION. *The* LINEAR CONTENT *of a subset* $X \leq \mathcal{M}$ *is the element of* $\mathcal{B}$: $l(X) = \bigcup_{a,b \in X} d(a,b)$.
The COMPLEX CONTENT *of a complex* C *is the element of* $\mathcal{B}$:

$$\lambda(C) = \bigcup_{(a,b) \in C} d(a,b).$$

1.7 THEOREM. *The linear content of a set* $X \leq \mathcal{M}$ *is equal to the complex content of any complex* C *of* X *with respect to which* X *is connected.*

Proof. Obviously $\lambda(C) \leq l(X)$. For $p,q \in X$ let $(x_1, x_2, \ldots, x_n)$ be the sequence belonging to C that connects $p = x_1$ with $q = x_n$. Then we have

$$d(p,q) \leq d(p,x_2) \cup d(x_2,x_3) \ldots \cup d(x_{n-1},q) \leq \lambda(C).$$

This holds for any $p,q \in X$. Therefore $l(X) \leq \lambda(C)$.

In case the set X only contains finitely many elements we will refer to a permutation of X as a polygon $P(x_1, x_2, \ldots, x_n)$, $x_i \in X$, of X. Each permutation will yield a different polygon P of X, the underlying finite set X remaining the same. A polygon, therefore, is a finite set with an ordering. As a polygon P of X clearly is a connected complex we may apply THEOREM 1.7: $\lambda(P) = l(X)$. We will now apply the foregoing to the arc-length by taking for X a finite subset of the arc:

1.8 THEOREM. $l(A(a,b)) = \bigcup_{x,y \in A} d(x,y)$.

Proof. $l(A(a,b)) \bigcup_{P<A} \lambda(P)$, according to the definition. The union is extended over all normally ordered polygons P inscribed in A (a,b).

But now we have λ (P) = l (X), where X is the underlying point set of P. So we have: $l(A(a,b)) = \bigcup_{X<A} l(X)$, the union now being extended over all finite subsets X of A; $l(X) \leq \bigcup_{x,y \in A} d(x,y)$ for any subset X of A; therefore $\bigcup_{X<A} l(X) \leq \bigcup_{x,y \in A} d(x,y)$. We also have $d(x,y) \leq l(X) \leq \bigcup l(X)$ for any $x,y \in A(a,b)$. Therefore $\bigcup_{x,y \in A} d(x,y) \leq \bigcup_{X<A} l(X)$.

Result: $\bigcup_{x,y \in A} d(x,y) = \bigcup_{X<A} l(X) = \bigcup_{P<A} l(P) = l(A(a,b))$.

In some instances it will be convenient to have still another expression for arc-length. Therefore let p be a fixed element of the arc A (a,b):

1.9 Theorem. $l(A(a,b)) = \bigcup_{y \in A} d(p,y)$, $p \in A(a,b)$.

Proof. The arc A (a,b) is connected with respect to the complex C consisting of all pairs (p,x) where p is a fixed element of A (a,b), and x is any element of the arc. The theorem now follows from Theorem 1.7: $l(A(a,b)) = \lambda(C) = \bigcup d(p,y)$. Sometimes it will be desirable to take for the fixed element p the element a of A (a,b): $l(A\ a,b) = \bigcup_{x \in A} d(a,x)$.

2. Continuity of arc-length.

It is immediately obvious from the above established theorems that arc-length is a congruence invariant, monotone function of arcs. In addition we have

2.1 Theorem. $l(A(a,x)) \cup l(A(x,b)) = l(A(a,b))$, $x \in A(a,b)$.

Proof. $l(A(a,b)) = \bigcup_{y} d(x,y)$ with $y \in A(a,b)$;

$l(A(a,x)) = \bigcup_{y} d(x,y)$ with $y \in A(a,x)$;

$l(A(x,b)) = \bigcup_{y} d(x,y)$ with $y \in A(x,b)$,

from which the statement follows immediately.

It must be mentioned that arc-length need not be a strictly monotone function of arcs. Similar examples as the one constructed in chap. IX will show this. In order to prove that arc-length is a lower semi-continuous function of arcs, we need the notion of limit of a sequence of sets. However, all subsets of $\mathcal{M}$ form a complete

Boolean algebra in which we can introduce the order-convergence described in chap. V.

2.2 THEOREM. *The linear content of a subset* $X \leq \mathcal{M}$ *is a lower semi-continuous function of* X; *i.e. if* $\{X_i\}$ *is a sequence of subsets of* $\mathcal{M}$ *such that* $\lim X_i = X$ *it holds that* $l(X) = \underline{\lim}\, l(X_i)$.

Proof. For every sequence $\{X_i\}$ of subsets of $\mathcal{M}$ we can construct monotone sequences $\{U_i\}$ and $\{V_i\}$ such that $U_i \leq X_i \leq V_i$, while

$$\{U_i\} \uparrow \underline{\lim}\, X_i = \bigcup_{i=1}^{\infty} U_i \text{ and } \{V_i\} \downarrow \overline{\lim}\, X_i = \bigcap_{i=1}^{\infty} V_i.$$

Let us consider the linear contents of all these sets.

$$l(U_i) \leq l(X_i) \leq l(V_i).$$

$\{l(U_i)\}$ is a monotone increasing sequence of elements of $\mathcal{B}$ with limit $\bigcup_{i=1}^{\infty} l(U_i) = \underline{\lim}\, l(X_i)$. Since $U_n \leq \bigcup_{i=1}^{\infty} U_i = \underline{\lim}\, X_i$, for all n, we have $l(U_n) \leq l(\underline{\lim}\, X_i)$, for all n, which implies
$\underline{\lim}\, l(X_i) \leq l(\underline{\lim}\, X_i)$.
We proceed to prove $l(\underline{\lim}\, X_i) \leq \underline{\lim}\, l(X_i)$.

Let x,y be elements of $\underline{\lim}\, X_i = \bigcup_{i=1}^{\infty} U_i$. Since $\{U_i\}$ is an monotone increasing sequence there must be a set U_n containing both x and y. Therefore we have $d(x,y) \leq l(U_n) \leq \underline{\lim}\, l(X_i)$.
As this holds for any two elements of $\underline{\lim}\, X_i$ we have
$l(\underline{\lim}\, X_i) \leq \underline{\lim}\, l(X_i)$. Thus we proved $l(\underline{\lim}\, X_i) = \underline{\lim}\, l(X_i)$. But as $\lim X_i = X$ implies $\underline{\lim}\, X_i = \overline{\lim}\, X_i = X$, we have $l(X) = \underline{\lim}\, l(X_i)$.

2.3 COROLLARY. *Arc-length is a lower semi-continuous function of arcs.*

Proof. Since the length of an arc is equal to its linear content we may apply the previous theorem.

2.4 DEFINITION. *A* CONTINUOUS ARC $A(a,b)$ *is an arc with the property that* $\lim x_i = x$, $x_i \in A(a,b)$, *implies* $\lim l(A(a,x_i)) = l(A(a,x))$.

Let $x \in A(a,b)$. Consider the mapping l

$$l\colon x \to l(x) = l(A(a,x)),\ x \in A(a,b),\ l(x) \in \mathcal{B},$$

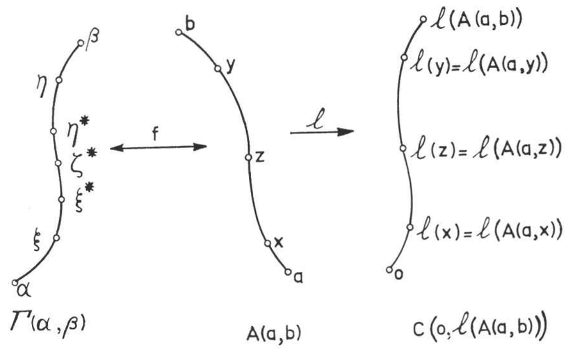
of an arc A (a,b) into $\mathcal{B}$. Obviously the image of the arc A (a,b) under this mapping l is a chain C (0,l (A (a,b)).

2.5 THEOREM. C (0,l (A (a,b))) *is a maximal chain if* A (a,b) *is continuous.*

Proof. We will show the chain C to be convex and closed, from which the theorem would follow.

(i) C (0,l (A (a,b))) is closed. Suppose $\lim l(x_i) = l$, $x_i \in A(a,b)$. As A (a,b) is compact there must be a subsequence $\{ x_{ni} \}$ on A (a,b) with limit $x \in A(a,b)$. But $\lim x_{n_i} = x$ implies $\lim l(A(a,x_{n_i})) = l(A(a,x))$ if A (a,b) is continuous. As $l(A(a,x_{n_i})) = l(x_{n_i})$ and $\lim l(x_{n_i}) = l$, we have $l = l(A(a,x)) = l(x)$. It thus follows that l is the image of $x \in A(a,b)$. Hence $l \in C$.

(ii) C (0,l (A (a,b))) is convex. Suppose therefore $l(A(a,x)) < l(A(a,y))$, which implies $x \neq y$.



Let f be the homeomorphism such that $A(a,b) = f(\Gamma(\alpha,\beta))$, with $f(\alpha) = a$ and $f(\beta) = b$. Let $\xi = f^{-1}(x)$ and $\eta = f^{-1}(y)$. Then $\xi \neq \eta$. Let X be the set of all elements $\zeta \in \Gamma(\alpha,\beta)$ such that $l(f(\zeta)) \leq l(x)$. Consider the element $\xi^* = \cup X$. Then $\xi^* \neq \eta$; for if $\xi^* = \eta$, we could select a sequence $\{ \eta_i \} \uparrow \eta$ which would imply $f(\eta_i) = f(\eta)$ or $\lim y_i = y$, which, in turn, would imply $\lim l(y_i) = l(y)$. But $l(y_i) = l(x)$ and $l(x) \neq l(y)$. Similarly let Y be the set of all elements $\zeta \in \Gamma(\alpha,\beta)$ such that $l(f(\zeta)) \geq l(y)$ and let $\eta^* = \cap Y$. Then also $\eta^* \neq \xi$. Since for any $\zeta_1 \in X$ and for any $\zeta_2 \in Y$ we have $l(f(\zeta_1) \leq l(x) < l(y) \leq l(f(\zeta_2))$, we have $\zeta_1 \leq \zeta_2$ from which $\xi^* \leq \eta^*$.

It is easily proved, however, by the same kind of continuity

argument as used above that $\xi^* \neq \eta^*$. Thus we have $\xi \leq \xi^* = \cup X < \cap Y = \eta^* \leq \eta$.
Since $\Gamma(\alpha,\beta)$ is convex there must exist an element $\zeta^* \in \Gamma(\alpha,\beta)$ such that $\xi^* < \zeta^* < \eta^*$. Let $z = f(\zeta^*)$, then $l(x) < l(z) < l(y)$, due to the definitions of ξ^* and η^*. This proves the convexity.

2.6 DEFINITION. *A* STRICTLY MONOTONE ARC $A(a,b)$ *is an arc with the property that for* $x,y \in A(a,b)$, $A(a,x) < A(a,y)$ *implies* $l(A(a,x)) < l(A(a,y))$.

2.7 THEOREM. *A continuous, strictly monotone arc may be parametrized with respect to arc-length.*

Proof. Let $x \in A(a,b)$. Consider the mapping l

$$l\colon x \rightarrow l(x) = l(A(a,x))$$

of the arc $A(a,b)$ onto the maximal chain $\Gamma(0,l(A(a,b)))$. Due to the fact that we assumed the arc $A(a,b)$ to be continuous and strictly monotone, this mapping l is a continuous one to one mapping of $A(a,b)$ onto $\Gamma(0,l(A(a,b)))$. If now $\Gamma(\alpha,\beta)$ is the maximal chain of which $A(a,b)$ is the homeomorphic image we have two maximal chains $\Gamma(\alpha,\beta)$ and $\Gamma(0,l(A(a,b)))$ with a continuous one to one mapping of $\Gamma(\alpha,\beta)$ onto $\Gamma(0,l(A(a,b)))$ that is order-preserving.
THEOREM 1.9 chap. VI then implies that $\Gamma(\alpha,\beta)$ and $\Gamma(0,l(A(a,b)))$ and consequently $A(a,b)$ and $\Gamma(0,l(A(a,b)))$ are homeomorphic. $A(a,b)$ is now the homeomorphic image of a maximal chain $\Gamma(0,l(A(a,b)))$ such that $x \in A(a,b)$ corresponds with its arc-length $l(A(a,x)) \in \Gamma(0,l(A(a,b)))$. We thus have obtained a parametrization with respect to arc-length for continuous, strictly monotone arcs.

CHAPTER VIII

SEGMENTS

Throughout this chapter $\mathcal{M}_2$ stands for a complete, separable, convex Boolean metric 2-space, whose complete, separable Boolean algebra of idempotents is $\mathcal{B}$. If the situation is also valid for complete, separable, associate, convex Boolean metric spaces in general we will write $\mathcal{M}$. However, since not many results are available yet concerning the motions of such a space $\mathcal{M}$, most of our following results only apply to a space $\mathcal{M}_2$.

1. Characterization.

1.1 Definition. *A* SEGMENT S (a,b) *is a subset of* $\mathcal{M}$ *that is congruent to a maximal chain* $\Gamma(\alpha,\beta)$ in $\mathcal{B}$: $S(a,b) = g(\Gamma(\alpha,\beta))$, *where* g *is a congruence such that* $g(\alpha) = a$ *and* $g(\beta) = b$ [1]).

The following properties of segments are immediate consequences of the definition:

(i) segments are convex, closed and connected;
(ii) $l(S(a,b)) = d(a,b)$;
(iii) segments are continuous arcs.

Since all segments are arcs our first attempt will be to establish a characterization of segments among arcs. We will show that segments are characterized as convex arcs.

1.2 Lemma. *Let* A *be a closed, non void subset of* $\mathcal{B}$ *with the property that for every non zero element* $a \in A$ *there exists another element* $a^* \in A$ *such that* $0 < a^* < a$. *Then* $0 \in A$.

Proof. Let $a \in A$ and let $\Gamma(0,a)$ be a maximal chain in $A^* = A \cup \{0\}$, i.e. an ordered set in A^* that does not contain an ordered proper subset in A^*. Let Γ^* denote the set obtained from $\Gamma(0,a)$ after deleting the element zero. Then $\Gamma^* \leq A$. Γ^* is closed in A. But since A is closed Γ^* is also closed in $\mathcal{B}$. Since Γ^* is also ordered

[1]) BG I.

Γ^* is algebraically closed. Let $\cap \Gamma^* = b \in \Gamma^*$. Then b must be zero. For if $b \neq 0$ there must be an element $b^* \in A$ such that $0 < b^* < b$, contradicting the fact that Γ (0,a) is maximal in A^*.

1.3 Corollary. *Let* A *be a closed, non void subset of a Boolean metric 2-space* $\mathcal{M}_2$. *Let* $p \in \mathcal{M}_2$ *and suppose* A *has the property that for every element* $a \in A$ *there exists an element* $a^* \in A$ *such that* B (p,a*,a). *Then* $p \in A$.

Proof. The motion m (x) = d (p,x) transforms p into 0. The set m (A) has the property: for every element m (a) $\in$ m (A) there is an element m (a)* $\in$ m (A) such that B (0,m (a)*, m (a)) or 0 < m (a)* < m (a). Our previous lemma then yields m (p) = 0 $\in$ m (A) or $p \in A$.

1.4 Lemma. *A convex and closed subset* A *of a Boolean metric 2-space* $\mathcal{M}_2$ *is connected.*

Proof. Suppose A is not connected: $A = X \cup Y$, $X \cap Y = 0$, X and Y being non empty, closed subsets of A.
Let $x \in X$. It cannot be that between x and every element of Y there is an element of Y, as this would imply $x \in Y$. So there must be an element $y \in Y$ such that there is no element of Y between x and y. Nevertheless, because of the convexity of A, there must be an element of A between x and y. This has to be an element of X therefore. Between this element and y there has to be another element of X etc. This implies $x \in Y$, which contradicts $X \cap Y = 0$. Therefore A is connected.

1.5 Corollary. *An arc* A (a,b) *in* $\mathcal{M}_2$ *cannot have a proper subset with the same end-points that is convex and closed.*

Proof. If this were possible we would have a subset of A (a,b) with the same end-points which is connected. This is impossible.

1.6 Lemma. *Every inner-element of a convex arc* A (a,b) *in* $\mathcal{M}_2$ *is between* a *and* b.

Proof. Consider the subset of A (a,b) of all elements that are between a and b, a and b included: A* (a,b). If we can show that A* (a,b) is convex and closed, A* (a,b) must coincide with A (a,b). Obviously A* (a,b) is closed. To prove the convexity: suppose $p,q \in$ A* (a,b); i.e. B (a,p,b) and B (a,q,b). Since A (a,b) is convex there must be an element $r \in$ A (a,b) such that B (p,r,q). But then we have B (a,r,b), as follows from Property 1.8 chap. IV. Thus $r \in$ A* (a,b), which proves the convexity of A* (a,b).

1.7 LEMMA. *If* A (a,b) *is a convex arc in* $\mathcal{M}_2$, *then for all distinct inner-elements* x,y *of the arc either* B (a,x,y) *or* B (a,y,x) *holds.*

Proof. Consider the inner-elements x and y of the arc. Since A (a,b) is convex B (a,x,b) holds. Consider the subset A* (a,b) of A (a,b) consisting of the elements z of A (a,b) such that B (a,z,x) or B (x,z,b) holds, including the elements a,b and x. This subset A* (a,b) of A (a,b) has to coincide with the arc A (a,b) as soon as we have shown that A* (a,b) is convex and closed. A* (a,b) is obviously closed. To prove that A* (a,b) is also convex, let p,q ∈ A* (a,b).
We will distinguish three cases:

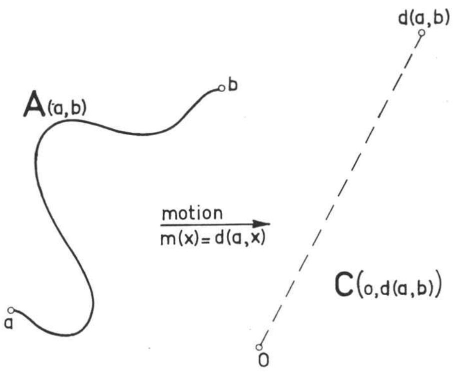
(i) B (a,p,x) and B (a,q,x) hold. Let r ∈ A (a,b) such that B (p,r,q). According to PROPERTY 1.4 chap. IV B (a,r,x) subsists, so that r ∈ A* (a,b).
(ii) B (x,p.b) and B (x,q,b) hold. This case is treated similarly as case (i).
(iii) B (a,p,x) and B (x,q,b) hold. Since also B (a,x,b) holds we may apply PROPERTY 1.12 chap. IV so that B (p,x,q) holds.

This establishes the convexity of A* (a,b), so that A* (a,b) = A (a,b) and hence: for every two distinct inner-elements x,y ∈ A (a,b), B (a,x,y) or B (x,y,b) holds. From PROPERTY 1.11 chap. IV it follows that B (x,y,b) and B (a,x,b) imply B (a,x,y), which completes the proof.

We are now able to proof our main theorem of this section

1.8 THEOREM. *A convex arc* A (a,b) *in* $\mathcal{M}_2$ *is a segment.*

Proof. Consider the motion m (x) = d (a,x) such that m (a) = 0 and m (b) = d (a,b).



From our preceding lemmas it follows that for two arbitrary, distinct inner-elements x,y of the convex arc A (a,b) we must have either B (a,x,y) or B (a,y,x), so that we must have either B (0,m (x),m (y)) or B (0,m (y),m (x)),which reduces to $0 < m(x) < m(y)$ or $0 < m(y) < m(x)$. This implies that the image of the arc A (a,b) is a chain C (0,d (a,b)). Since A (a,b) is convex and compact, the same holds for C (0,d (a,b)). Therefore C (0,d (a,b)) is a maximal chain and A (a,b) a segment.

2. Segment-like arcs.

One might expect that an arc A (a,b) with the property l (A (a,b) = d (a,b) is a segment, in which case we would have another characterization of segments. However, this is not true as will be shown by an example in chap. IX.

2.1 Definition. *An arc* A (a,b) *in* $\mathcal{M}_2$ *with the property* l (A (a,b,)) = d (a,b) *is called* SEGMENT-LIKE.

2.2 Theorem. *A continuous arc* A (a,b) *in* $\mathcal{M}_2$ *such that every subarc* A (a,x), $x \in$ A (a,b), *is segment-like is a segment.*

Proof. The arc is clearly strictly monotone. Since the arc is also continuous the motion m (x) = d (x,a) transforms A (a,b) into a maximal chain as was shown in chap. VII. In fact, we obtain the parametrization of the arc with respect to arc-length: $x \in$ A (a,b) is mapped into l (A (a,x)) = d (a,x). Since A (a,b) is congruent with a maximal chain, the arc is a segment.

2.3 Definition. *The* EXCES E_p *of an* ELEMENT p *of an arc* A (a,b) *in* $\mathcal{M}$ *is defined by*: $E_p = \{d(a,p) \cup d(p,b)\}\, d'(a,b)$.
The EXCES E (a,b) *of the* ARC A (a,b) *is defined by*: $E(a,b) = \bigcup_{p \in A} E_p$.
Direct computation shows that for $\mathcal{M}_2$ $E_p = abp' \cup a'b'p$. From the definition it follows readily that for $\mathcal{M}_2$ $E_p = 0$ if and only if B (a,p,b) holds.

2.4 Lemma. E (a,b) = l (A (a,b)) d'(a,b).

Proof. $E(a,b) = \bigcup_p E_p = \bigcup_p \{[\, d(a,p) \cup d(p,b)]\, d'(a,b)\} =$

$\{\bigcup_p d(a,p)\, d'(a,b)\} \cup \{\bigcup_p d(p,b)\, d'(a,b)\} =$

$l(A(a,b))\, d'(a,b) \cup l(A(a,b))\, d'(a,b) = l(A(a,b))\, d'(a,b)$.

It is easily seen that E (a,b) = 0 if and only if l (A (a,b) = d (a,b);

i.e. if and only if the arc is segment-like. Furthermore we may say that E (a,b) $=0$ if and only if $E_p = 0$ for all $p \in A$ (a,b), so that l (A (a,b)) $=$ d (a,b) holds if and only $E_p = 0$ for all $p \in A$ (a,b), or, in case of a space $\mathcal{M}_2$, if and only if B (a,p,b) holds for all $p \in A$ (a,b). Thus we have

2.5 THEOREM. *An arc* A (a,b) *in* $\mathcal{M}_2$ *is segment-like if and only if every inner-element* p *of the arc is between the end-points of the arc.*

CHAPTER IX

EXAMPLES

Let Ω be the left open interval (0,1] on the real line. Let B denote the class of all subsets of Ω that are unions of finitely many left open intervals (a,b], $0 \leqq a \leqq 1$ and $0 \leqq b \leqq 1$. Then B is an atom-free Boolean algebra whose Boolean operations are the usual set operations.

Another example [1] may be obtained by taking for Ω the closed interval [0,1] on the real line. Let B denote the set of all regular open subsets of Ω. An (open) set is called regular provided it is equal to the interior of its closure. Using the following notation

cX: complement of X
hX: closure of X
iX: interior of X

we have: X is regular provided $X = ihX$, which is equivalent to $X = chchX$. These regular open subsets of [0,1] form a complete and atom-free Boolean algebra with the following operations

$$X \cap Y = X \dot{\cap} Y$$
$$X \cup Y = ih\,(X \dot{\cup} Y)$$
$$\bigcap X_\alpha = i\,(\dot{\bigcap} X_\alpha)$$
$$\bigcup X_\alpha = ih\,(\dot{\bigcup} X_\alpha).$$

To establish that this Boolean algebra B is also separable we have to show that for any ordered subset $\{X_\alpha\}$ of B it holds that

$$\bigcap_\alpha X_\alpha = \bigcap_{i=1}^{\infty} X_i \text{ and } \bigcup_\alpha X_\alpha = \bigcup_{j=1}^{\infty} X_j.$$

We will only give the proof for the g.l.b., the proof for the l.u.b. being similar.

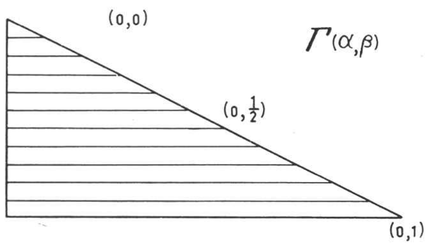
LEMMA. Let A_1 and A_2 be two regular (open) subsets of [0,1] such that $A_1 < A_2$. Then $\mu\,(A_1) < \mu\,(A_2)$, where μ denotes the Lebesgue measure on the real line.

[1] [10], Beispiel 24.2, p. 133.

Proof. Since $A_1 < A_2$ there must be a point $x_0 \in A_2$ such that $x_0 \notin A_1$. Let (x_1,x_2) be an open interval in A_2 containing x_0.
Now we have $A_1 = chchA_1$ since A_1 is regular. Thus $cA_1 = hchA_1 = hicA_1$. Therefore $x_0 \in hicA_1$. Let ε be such that the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ is contained in the interval (x_1,x_2). Then $(x_0 - \varepsilon, x_0 + \varepsilon)$ contains elements of icA_1 since $x_0 \in hicA_1$. Since, however, icA_1 is an open set, $(x_0 - \varepsilon, x_0 + \varepsilon)$ contains an interval of cA_1, i.e. a set with positive measure. Now $(x_0 - \varepsilon, x_0 + \varepsilon)$ is contained in A_2; therefore A_2 contains a set of positive measure that is not in A_1. Thus $\mu(A_2) > \mu(A_1)$.

Due to the above established lemma we may say that there exists a one to one, order-preserving correspondence between the sets $\{X_\alpha\}$ and $\{\mu(X_\alpha)\}$; i.e. $X_\alpha < X_\beta$ if and only if $\mu(X_\alpha) < \mu(X_\beta)$.
Let $\mu(X_\alpha) = \mu_\alpha$ and let m be the infimum of the set $\{\mu_\alpha\}$. We then may select a monotone decreasing sequence $\{\mu_i\} \downarrow m, \mu_i = \mu(X_i)$.
We now assert $\bigcap_{i=1}^{\infty} X_i = \bigcap X_\alpha$. In order to prove this we only have to show $\overset{\frown}{\cdot}\, X_i \leq \overset{\frown}{\cdot}\, X_\alpha$. Suppose therefore $x \in \overset{\frown}{\cdot}\, X_i$. Let X_α be an arbitrary element of $\{X_\alpha\}$. There must exist an element $\mu_n \in \{\mu_i\}$ such that $\mu_n \leqq \mu_\alpha$, which implies $X_n \leq X_\alpha$. Since $x \in X_n$, we also have $x \in X_\alpha$. Hence $\overset{\frown}{\cdot}\, X_i \leq \overset{\frown}{\cdot}\, X_\alpha$, from which $\overset{\frown}{\cdot}\, X_i = \overset{\frown}{\cdot}\, X_\alpha$ and thus $i\overset{\frown}{\cdot}\, X_i = i\overset{\frown}{\cdot}\, X_\alpha$ which means $\bigcap_{i=1}^{\infty} X_i = \bigcap X_\alpha$.

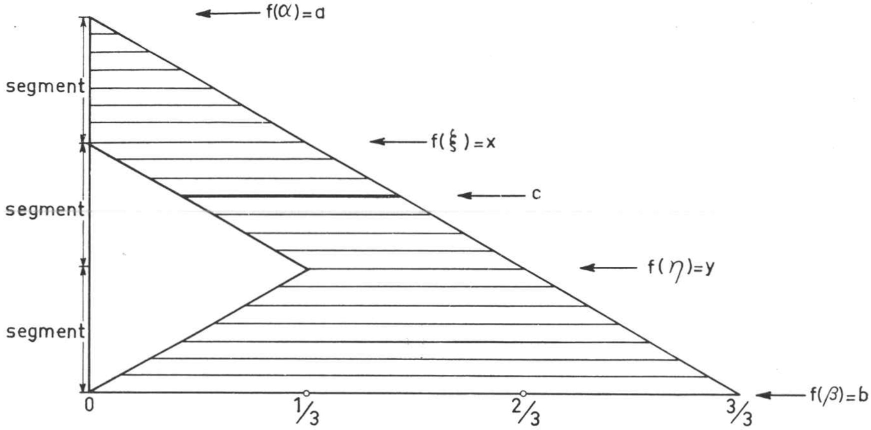
We will now construct an example showing that segment-like arcs need not be segments.
Let B denote the complete, separable and atom-free Boolean algebra of regular open subsets of $\Omega = [0,1]$. Let $\Gamma(\alpha,\beta)$ be the maximal chain whose elements are the open intervals $(0,t)$, $t = 0 \rightarrow 1$.



Let $A(a,b)$ be the arc obtained from $\Gamma(\alpha,\beta)$ by the following mapping f:

$$
\begin{array}{llll}
(0,t) \in \Gamma(\alpha,\beta) & \rightarrow (0,t) \in A(a,b) & \text{for } t = 0 \rightarrow \frac{1}{3} \\
(0,t) & \rightarrow (t - \frac{1}{3},t) & \text{for } t = \frac{1}{3} \rightarrow \frac{2}{3} \\
(0,t) & \rightarrow (1 - t,t) & \text{for } t = \frac{2}{3} \rightarrow 1.
\end{array}
$$

This yields the following picture of the arc A (a,b,):



$$
\begin{array}{ll}
\alpha = (0,0), \text{ or the empty set;} & f(\alpha) = a = (0,0) \\
\xi = (0,\tfrac{1}{3}) & f(\xi) = x = (0,\tfrac{1}{3}) \\
\eta = (0,\tfrac{2}{3}) & f(\eta) = y = (\tfrac{1}{3},\tfrac{2}{3}) \\
\beta = (0,\tfrac{3}{3}) & f(\beta) = b = (0,\tfrac{3}{3})
\end{array}
$$

The arc A (a,b) is obviously strictly monotone and has the property that its arc-length equals the distance between its end-points: both are equal to (0,1). The subarcs A (a,x), A (x,y) and A (y,b) are segments since they are convex arcs.
The arc A (a,b) is not a segment since it is not convex as there is no element between the arc-elements a and c.

We conclude with the remark that if B denotes the complete, separable and atom-free Boolean algebra of regular open subsets of $\Omega = [0,1]$ and if D stands for a commutative integral domain with identity, the Boolean metric spaces $\mathcal{M}$ and $\mathcal{M}^*$, obtained from the rings DB and DB* respectively, are complete, separable, associate, weakly-convex Boolean metric spaces that are even regular if D is a field.

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BIOGRAPHY

The author of this thesis was born in 1931. After graduating from the Lyceum in 1950 he attended the University of Leiden where he followed the lectures of Prof. Dr. J. Droste, Prof. Dr. J. Haantjes, Prof. Dr. H. D. Kloosterman and others. In 1957 he passed his preliminaries for his Ph. D. and was appointed part-time instructor at the University of Missouri, U.S.A., where he prepared part of this thesis under the guidance of Prof. Dr. L. M. Blumenthal, during the terms 1957-1958 and 1958-1959.

In September 1959 he was appointed instructor at the Institute of Technology in Delft, the Netherlands, where he completed this thesis.

STELLINGEN

behorende bij C. J. Penning, Boolean metric spaces,
Delft, 21 december 1960.

I

Pauc definieert de „aplatissement" (p,q,r) van een drietal punten in een metrische ruimte als de som van de twee kleinste hoeken van de door p, q en r bepaalde driehoek.
Zijn 1, 2, 3 en 4 de hoekpunten van een (eventueel ontaard) viervlak en ε_i de „aplatissement" van de drie hoekpunten tegenover het hoekpunt i, dan beweert Pauc ten onrechte dat het bewijs van de ongelijkheid

$$\varepsilon_i + \varepsilon_j + \varepsilon_k \geqq \varepsilon_l$$

triviaal is.
De ongelijkheid blijkt ook te gelden als ε_i voorstelt:

(i) de som van de twee grootste hoeken in de betreffende driehoek;
(ii) de som van de grootste en de kleinste hoek in de betreffende driehoek.

Ch. Pauc, Les méthodes directes en géométrie différentielle, p. 128 en p. 133.
L. M. Blumenthal, A budget of curiosa metrica, Am. Math. Monthly 66 (1959), p. 453.

II

Onder $2m^2 - 6m + 6$ mensen zijn er altijd m die elkaar kennen of m die elkaar niet kennen. Voor $m = 3$ is dit tevens het minimale aantal met deze eigenschap.

III

Birkhoff beweert ten onrechte dat elke partieel geordende verzameling door invoering van de „order-topology" een Hausdorff-ruimte wordt.

G. Birkhoff, Lattice theory, revised edition, Theorem 13, p. 60.
E. E. Floyd, Boolean algebras with pathological order-topologies, Pac. Journ. of Math. 5 (1955), p. 687-689.

IV

Zij H een Hilbert-ruimte met een volledig orthonormaal systeem $\underset{\sim}{Q}$ in H. Dan is er een Boolese valuatiering R aan te geven die isomorf is met H. Voor de Boolese metrische ruimte M, verkregen uit R, geldt dat de afstand tussen de elementen f en g van M gelijk is aan de verzameling van die elementen van $\underset{\sim}{Q}$, waarvoor de Fouriercoëfficiënten van f en g verschillen.
De deelverzameling $\underset{\sim}{Q}$ van H is isomorf met de atomen van R.
Bevat $\underset{\sim}{Q}$ oneindig veel elementen, dan is R niet regulier.

V

Dat elke eindige Abelse groep van de orde 2 (2n + 1) cyclisch moet zijn, is geheel op elementaire wijze aan te tonen.

VI

Birkhoff beweert ten onrechte dat de deellichamen van een eindig lichaam een lineair geordend systeem vormen met de inclusie-relatie als ordenings-relatie.

Birkhoff & McLane, A survey of modern algebra, exercise 5, p. 431.

VII

Zijn I(S) en I(R) de roosters van idealen in een commutative ring S met eenheid, resp. een unitaire onderring R van S, dan zijn de afbeelding ρ van I(S) in I(R) en de afbeelding σ van I(R) in I(S), gedefinieerd door resp.

$$\rho\,(A) = A \cap R \quad \text{en} \quad \sigma\,(A) = AS$$

een $\cap$-homomorfie, resp. een $\cup$-homomorfie.
Is S bovendien een geassocieerde ring, dan is de afbeelding σ tevens een $\cap$-homomorfie.
Is S een complete directe som van lichamen: $S = \sum_{\omega \in \Omega} F\,(\omega)$, dan is een ideaal A in een unitaire onderring R van S dan en slechts dan het ideaal van alle functies in R die nul zijn op een zekere deelverzameling $Z \leq \Omega$, als geldt $\rho\sigma\,(A) = A$.

VIII

De bewering dat de meetkundige plaats der buigpunten van de algemene integraalkrommen van een differentiaal vergelijking

$F(x,y,p) = 0$ van de eerste orde en van hogere graad zou moeten voldoen aan

$$F(x,y,p) = 0$$
$$\text{en} \quad F_p(x,y,p) = 0$$

is onjuist.

B. Meulenbeld en W. K. Baart, Analyse voor propaedeutische examens deel 2, § 100, blz. 256.

IX

Om te voorkomen dat belangrijke Nederlandse kunstcollecties uiteenvallen of naar het buitenland verdwijnen is het wenselijk fiscale faciliteiten te scheppen voor erfgenamen van zulke kunstcollecties die deze zouden willen afstaan aan een museum.

X

Het is niet gerechtvaardigd de Nederlandse Technische Hogescholen de naam „Technische Universiteit" te geven.

XI

Het verdient aanbeveling aan het Engelse werkwoord toe te kennen:

a) een tijd, welke absoluut dan wel relatief kan zijn en waarbij zowel de absolute als de relatieve tijd zijn onder te verdelen in verleden, heden en toekomst;

b) een vorm, welke gesloten of open kan zijn.

Hierbij dient opgemerkt te worden dat de bijzondere combinatie „to be going" alleen een (geïnverteerde) relatieve tijd kent in open vorm.

XII

De kiem van Heidegger's opvatting aangaande het „Dasein" in de tijd vindt men reeds in Kierkegaard's dagboekaantekeningen

Sören Kierkegaard, Tagebücher, p. 129 en p. 174.
M. Heidegger, Sein und Zeit.

XIII

De wijze waarop Madelung over de delta-functie schrijft schept verwarring.

E. Madelung, Die mathematischen Hilfsmittel des Physikers, 6. Aufl., 1957, S. 18.