

ON THE SYNTHESIS OF THREE-TERMINAL NETWORKS COMPOSED OF TWO KINDS OF ELEMENTS

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1. Introduction

From the classical work of Foster ¹⁾ and Cauer ²⁾, network synthesis has developed to a remarkable state of complexity and generality. Yet, in spite of the achievements, there are some notable gaps in the theory — problems skipped over, for which we have no general solution, and in connection with which our knowledge of even quite simple cases is unfortunately meagre. The theory of the synthesis of linear, constant, passive, reciprocal networks has developed along two main streams of generalization. One is the extension of results valid for two-terminal networks, or two-poles as they are frequently called, to three-poles, four-poles and, in general, n -poles; the other is the extension of results pertaining to networks composed of two kinds of elements to those consisting of three kinds. These extensions all follow the pattern exemplified by Foster's reactance theorem ¹⁾.

Foster's theorem, in fact, gives a solution to the problem of finding a set of necessary and sufficient conditions for a given function of frequency to be the impedance function of a two-pole composed of two kinds of elements, and of constructing such a two-pole corresponding to any function satisfying these conditions. From this result, the work of Brune ³⁾ in connection with two-poles composed of three kinds of elements and the work of Cauer ²⁾ and Gewertz ⁴⁾ relating to four-poles appear as natural extensions. Yet, from a practical viewpoint, their work is open to a serious objection, namely, the inclusion of transformers, especially ideal transformers, in their networks. True, in the case of the Brune problem, Bott and Duffin ⁵⁾, many years later, were able to show that transformers are not necessary for realization; but this still leaves unanswered the question to what extent one can dispense with them in the case of four-poles. Another noteworthy fact is that although much has been done in connection with four-poles, and more generally n -terminal-pair networks, very little investigation of the properties of three-poles has taken place.

There are various reasons for this. One is that the early filter theory was evolved by analogy with transmission-line theory, and the transmission line is definitely a two-terminal-pair device. Moreover, the ideal transformer, so essential in the early synthesis theory, is also a two-terminal-pair device, and this fact has further enhanced the influence of the transmission-line analogy. But perhaps most important, the three-pole can be regarded as a special kind of four-pole; it can be analysed by means of four-pole theory, with the result that a separate three-pole theory has seldom been considered necessary. This point of view is not always to be recommended for, as we

shall show in sec. 2, three-poles containing no transformers have an important special property which is inevitably lost sight of if general four-pole analysis is applied.

Now the three-pole is a very important system in practice. A large number of amplifier circuits and their associated coupling and corrective networks are three-poles. Moreover, many of such three-poles are resistance-capacitance networks, so that it becomes important to know precisely what one can do with three-poles constructed of two kinds of elements. In fact, it would be very useful to know the answer to the problem of finding a set of necessary and sufficient conditions for a set of three functions to characterize a three-pole composed of two kinds of elements, and of constructing such a three-pole corresponding to any set of functions satisfying the conditions. This problem has not yet been solved. The difficulty is that there are, as yet, no general methods of synthesis from the functions that are able to guarantee the exclusion of transformers in the network. During recent years some progress has been made with the study of networks by means of algebraic topology ⁶⁾, and since this is the most fundamental treatment of electrical networks yet given, there is some hope that a general method will be forthcoming; but at the present time this viewpoint is insufficiently developed to be applied to synthesis problems.

With no general method available, the only alternative is to consider a simpler special case. One such special case is the synthesis for a prescribed transfer ratio of a three-pole composed of two kinds of elements. This simpler problem was solved by Fialkow and Gerst ⁷⁾, who gave necessary and sufficient conditions for the exclusion of transformers. Ozaki ⁸⁾ specialized the problem in a different way. He considered the case when two of the transfer admittances bear a constant (frequency-independent) ratio to one another while the real parts of the zeros of the third transfer admittance are non-positive, and was able to obtain a set of necessary and sufficient conditions as well as a method of constructing the network. Lucal ⁹⁾, following the same path, attempted to remove Ozaki's restrictions on the functions. He was able to show that, for some simple functions outside the class considered by Ozaki, the technique could still be applied, but he did not give a set of necessary and sufficient conditions. Darlington ¹⁰⁾, in a review of the whole field of realization techniques, considered the following essential features of such techniques:

- (i) a class of networks,
- (ii) a class of functions (of the networks in the network class),
- (iii) necessary and sufficient conditions which define the function class in mathematical terms,

- (iv) a "canonical configuration" or sub-class of the network class which is sufficient for the realization of the entire function class,
- (v) a straightforward technique for finding the element values of the canonical configuration, given specific functions within the function class.

In terms of these five items, Ozaki's function class is a sub-class of the class of RC three-pole functions; his canonical configuration is a sub-class of the class of series-parallel *) three-poles. Lucal's network class is the class of RC three-poles of the series-parallel type, but his function class is not completely defined. In connection with Lucal's work Darlington was led to conjecture that the series-parallel type of RC three-pole constitutes a canonical sub-class of the class of RC three-poles. This conjecture has neither been proved nor disproved.

In this paper we consider a special case of the three-pole two-element problem, but of a type different from that considered by Ozaki. Our network class is the class of three-poles composed of capacitances and inductances without mutual coupling, and of the series-parallel type. Our function class is the class of sets of transfer admittance functions, of the sixth and lower degrees **), that define a three-pole composed of capacitances and inductances without mutual coupling. We give necessary and sufficient conditions for the realization of the function class by networks of the network class. In proving the sufficiency of the conditions we give a procedure for computing the element values of a network corresponding to any set of functions of the function class that satisfy these necessary and sufficient conditions. We have chosen the LC type of network in preference to the RC type in order to make use of certain symmetry properties that are not so obvious in the case of RC-networks.

The classification of three-poles according to the degree of the defining functions may seem rather artificial. A classification according to the number of elements would be more useful for practical applications and would be a first step towards the solution of Darlington's "price list" problem ¹⁰⁾. However, this approach proves to be far less successful with the present algebraic methods available. For more than five elements it becomes very difficult to formulate a set of necessary and sufficient conditions or to obtain any general relationships.

In secs 2, 3 and 4 the fundamental concepts are introduced and the problem is set up. In sec. 5 some general theorems applicable to functions of arbitrary

*) This term will be defined in sec. 4.

**) This term will be defined in sec. 3.

trary degree are given. In secs 6 to 9 the third, fourth, fifth and sixth degrees are investigated with the help of the theorems of sec. 5. The second and lower degrees are treated as special cases of the third. For the fifth and lower degrees it is shown that the conditions given by Cauer²⁾ and Fialkow and Gerst⁷⁾ are both necessary and sufficient. In sec. 9 some extra necessary conditions are derived for the sixth degree. These conditions together with the earlier known ones are shown to be sufficient.

2. The polynomial notation and the three-pole equations

Let us examine a general three-pole network from first principles with the aim of expressing its essential properties in algebraic form. For this purpose it will be convenient to make use of a polynomial notation introduced by Tellegen¹¹⁾, and used by him in the study of four-poles.

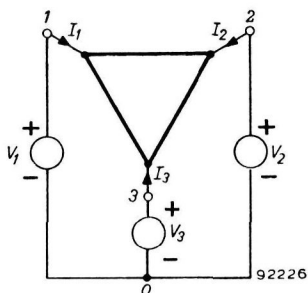


Fig. 2.1. Linear, reciprocal three-pole connected to three external voltage sources.

Consider a general linear, reciprocal three-pole connected to three external voltage sources as shown in fig. 2.1. Since the network is linear, the relations between the voltages and the currents can be expressed in the form

$$\left. \begin{aligned} I_1 &= Y_{11}V_1 - Y_{12}V_2 - Y_{13}V_3, \\ I_2 &= -Y_{21}V_1 + Y_{22}V_2 - Y_{23}V_3, \\ I_3 &= -Y_{31}V_1 - Y_{32}V_2 + Y_{33}V_3, \end{aligned} \right\} \quad (2.1)$$

where the Y 's depend only on the network, and not on the I 's or V 's. Also, since the three-pole is reciprocal, we have

$$Y_{23} = Y_{32}, \quad Y_{31} = Y_{13}, \quad Y_{12} = Y_{21}.$$

Kirchhoff's current law applied at the point O leads us to conclude that

$$I_1 + I_2 + I_3 = 0. \quad (2.2)$$

Since V_1, V_2, V_3 are arbitrary, it follows from (2.2) that

$$\left. \begin{aligned} Y_{11} - Y_{21} - Y_{31} &= 0, \\ -Y_{12} + Y_{22} - Y_{32} &= 0, \\ -Y_{13} - Y_{23} + Y_{33} &= 0. \end{aligned} \right\} \quad (2.3)$$

Thus only three of the Y quantities are independent, with the result that Y_{23}, Y_{31}, Y_{12} can be taken as defining the three-pole. Consequently, eqs (2.1) can be represented by the equivalent circuit shown in fig. 2.2, in which it is to be noted that Y_{23}, Y_{31}, Y_{12} are not necessarily realizable driving-point admittances. For a network composed of more than one kind of element these functions will be rational functions of the complex frequency parameter λ . Let D be a polynomial divisible by the denominators of

$$\begin{aligned} &Y_{23}, Y_{31}, Y_{12}, \\ &Y_{31}Y_{12} + Y_{12}Y_{23} + Y_{23}Y_{31}. \end{aligned}$$

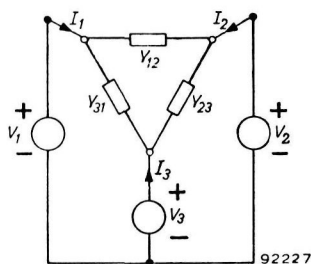


Fig. 2.2. Network equivalent to the network of fig. 2.1.

We now define the polynomials F, G, H, C by the relations

$$\left. \begin{aligned} \frac{F}{D} &= Y_{23}, & \frac{G}{D} &= Y_{31}, & \frac{H}{D} &= Y_{12}, \\ \frac{C}{D} &= Y_{31}Y_{12} + Y_{12}Y_{23} + Y_{23}Y_{31}, \end{aligned} \right\} \quad (2.4)$$

from which it follows immediately that

$$GH + HF + FG = CD. \quad (2.5)$$

Dually, one can consider the three-pole connected to external current sources as shown in fig. 2.3. With reference to this diagram, we can write the equations

$$\left. \begin{aligned} V_{23} &= Z_{11}I_{23} - Z_{12}I_{31} - Z_{13}I_{12}, \\ V_{31} &= -Z_{21}I_{23} + Z_{22}I_{31} - Z_{23}I_{12}, \\ V_{12} &= -Z_{31}I_{23} - Z_{32}I_{31} + Z_{33}I_{12}, \end{aligned} \right\} \quad (2.6)$$

where the Z 's depend only on the network, and where

$$Z_{23} = Z_{32}, \quad Z_{31} = Z_{13}, \quad Z_{12} = Z_{21}.$$

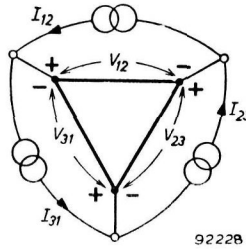


Fig. 2.3. Linear, reciprocal three-pole connected to three external current sources.

By Kirchhoff's voltage law, we have

$$V_{23} + V_{31} + V_{12} = 0, \quad (2.7)$$

so that, since I_{23} , I_{31} , I_{12} are independent,

$$\left. \begin{aligned} Z_{11} - Z_{21} - Z_{31} &= 0, \\ -Z_{12} + Z_{22} - Z_{32} &= 0, \\ -Z_{13} - Z_{23} + Z_{33} &= 0. \end{aligned} \right\} \quad (2.8)$$

Thus only three of the Z quantities are independent, with the result that Z_{23} , Z_{31} , Z_{12} can be taken as defining the three-pole. Equations (2.6) can now be represented by the equivalent circuit shown in fig. 2.4, where Z_{23} ,

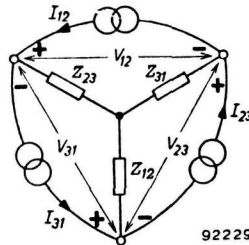


Fig. 2.4. Network equivalent to the network of fig. 2.3.

Z_{31} , Z_{12} are not necessarily realizable driving-point impedances. If the delta representation of the three-pole depicted in fig. 2.2 is transformed into the star representation shown in fig. 2.4, the values of the impedances will be given by

$$\left. \begin{aligned} Z_{23} &= \frac{Y_{23}}{Y_{31}Y_{12} + Y_{12}Y_{23} + Y_{23}Y_{31}}, \\ Z_{31} &= \frac{Y_{31}}{Y_{31}Y_{12} + Y_{12}Y_{23} + Y_{23}Y_{31}}, \\ Z_{12} &= \frac{Y_{12}}{Y_{31}Y_{12} + Y_{12}Y_{23} + Y_{23}Y_{31}}, \end{aligned} \right\} \quad (2.9)$$

from which it is clear that

$$Z_{23} = \frac{F}{C}, \quad Z_{31} = \frac{G}{C}, \quad Z_{12} = \frac{H}{C}. \quad (2.10)$$

Thus we see that, if F/D , G/D , H/D are the three short-circuit transfer admittances of the three-pole, where

$$GH + HF + FG = CD,$$

then the three open-circuit transfer impedances are F/C , G/C , H/C , and vice versa. That is to say, the dual connections of fig. 2.5 are equally valid representations of the same three-pole, where it is to be understood that F/C , G/C , H/C , F/D , G/D , H/D are not necessarily realizable driving-point immittances. From eq. (2.5) and fig. 2.5 it is clear that an interchange of any two of F , G , H has the effect of merely permuting the terminal numbering of the three-pole. In sec. 5.2 we shall prove that, for planar networks,

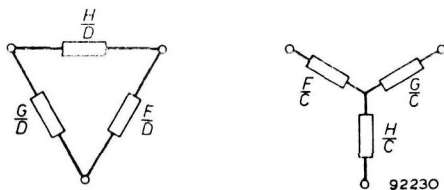


Fig. 2.5. Equivalent representations of the general three-pole network.

the interchange of C and D corresponds to taking the dual of the three-pole network. These simple results will prove to be extremely useful when we come to study special cases in secs 6 to 9. For planar networks, it means that twelve different configurations can be synthesized from five given polynomials by the same algebraic operations, a fact which is not immediately obvious if the usual four-pole parameters Z_{11} , Z_{12} , Z_{22} are employed.

3. Necessary conditions in terms of the polynomials

From eqs (2.1), (2.3), (2.4), and (2.6), (2.8), (2.10) it follows that

$$\frac{G+H}{D}, \quad \frac{H+F}{D}, \quad \frac{F+G}{D}, \quad \frac{G+H}{C}, \quad \frac{H+F}{C}, \quad \frac{F+G}{C} \quad (3.1)$$

are driving-point immittance functions. For a network composed exclusively of capacitances and inductances, classical theory predicts:

- I. F, G, H are of even and C, D of odd degree, or vice versa.
- II. Each of the functions (3.1) has the properties: its poles and zeros lie exclusively on the imaginary axis, the poles separating the zeros (and vice versa); it is non-negative for positive real λ .
- III. If f_i, g_i, h_i are the residues of $F/D, G/D, H/D$, respectively, at a pole, $\lambda = \lambda_i$, then

$$(g_i + h_i)(h_i + f_i) - h_i^2 \geq 0,$$

that is,

$$g_i h_i + h_i f_i + f_i g_i \geq 0. \quad (3.2)$$

Also, if f'_i, g'_i, h'_i are the residues of $F/C, G/C, H/C$, respectively, at a pole, $\lambda = \lambda'_i$, then

$$g'_i h'_i + h'_i f'_i + f'_i g'_i \geq 0. \quad (3.3)$$

Conditions (3.2) and (3.3) will be called the *D-Cauer condition* and the *C-Cauer condition*, respectively ²⁾.

Furthermore, if the network contains no mutual inductances, then

- IV. the functions $F/D, G/D, H/D$ are positive for positive real values of λ .

This result, first proved by Fialkow and Gerst ⁷⁾ for RC networks, also applies to LC networks. It is conceivable that some of the coefficients of F, G, H will be negative. However, Fialkow and Gerst proved in the case of RC networks that it is always possible to find a polynomial with negative real zeros whose product with F is a polynomial with non-negative coefficients. We therefore conclude that in the LC case it is always possible to find a polynomial with zeros exclusively on the imaginary axis such that its product with F is a polynomial with entirely non-negative coefficients. Since the multiplier itself contains only positive coefficients, it is clear that such a common multiplier can be found for all five polynomials F, G, H, C, D . Since the network is defined by the ratio of the polynomials, the introduction of common factors does not affect the three-pole.

At this point it will be convenient to introduce two definitions which will help to simplify our subsequent work.

Definitions

- (a) The highest degree of C, D, F, G, H after division by all factors common to the five polynomials is called the *order of the three-pole* ¹¹⁾.
- (b) If, after cancellation of the common factors, the polynomials are multiplied by a polynomial having zeros exclusively on the imaginary axis, and of the lowest degree necessary to produce polynomials with exclusively non-negative coefficients, then the highest degree of the resulting polynomials will be called the *degree of the three-pole (or polynomials)*, or simply, the *degree*. In the sequel we shall use the word *degree* in this special sense, except where otherwise stated.

4. Statement of the problem

Before we can give a complete statement of the problem it will be necessary to introduce the concept of series-parallel three-pole.

4.1. Series-parallel LC three-poles

In what follows, we shall denote two-poles and three-poles by the numbers assigned to their terminals. Thus a three-pole with terminals 1, 2, 3 will be denoted by *three-pole (1,2,3)*. By *elementary three-pole* we shall understand one of the configurations shown in fig. 4.1, viz:

elementary three-pole (a): terminals 1,2 short-circuited, terminal 3 isolated;
 elementary three-pole (b): terminals 1,2,3 isolated;
 elementary three-pole (c): terminals 1,2,3 short-circuited.

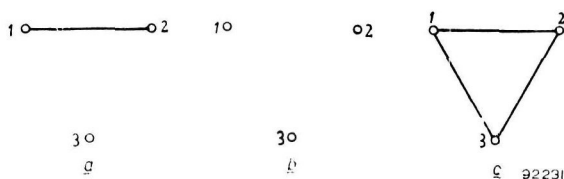


Fig. 4.1. Elementary three-poles.

By *elementary two-pole* we shall understand two terminals connected by either a single element, or two elements in series, or two elements in parallel. The elements are restricted to inductances and capacitances.

We next consider three elementary connections:

- (i) The series connection of a three-pole and an elementary two-pole is a three-pole (1,2,3) formed from a three-pole (1',2,3) and an elementary two-pole (1,1') by connecting terminals (1', 1'') together (fig. 4.2).

- (ii) The parallel connection of a three-pole and an elementary two-pole is a three-pole $(1,2,3)$ formed from a second three-pole $(1,2,3)$ and an elementary two-pole $(2',3')$ by connecting the terminals $(2,2')$ and $(3,3')$ pairwise together (fig. 4.3). This connection is the dual of (i).
- (iii) The parallel connection of two three-poles is a three-pole $(1,2,3)$ formed from two three-poles $(1,2,3)$ and $(1',2',3')$ by connecting the terminals $(1,1')$, $(2,2')$, $(3,3')$ pairwise together (fig. 4.4). This connection has no dual, as a series connection of three-poles is not a three-pole.

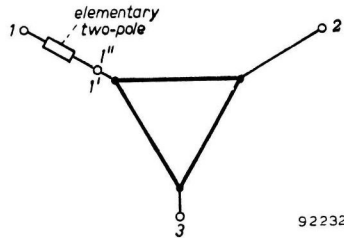


Fig. 4.2. Series connection of a three-pole and an elementary two-pole.

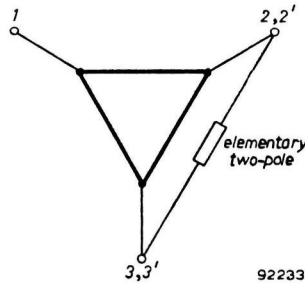


Fig. 4.3. Parallel connection of a three-pole and an elementary two-pole.

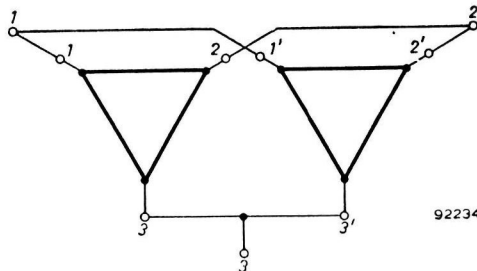


Fig. 4.4. Parallel connection of two three-poles.

Definition (a)

A series-parallel LC three-pole is a three-pole network that can be built up by the parallel and series connections of elementary three-poles and elementary two-poles. Examples of series-parallel and non-series-parallel three-poles are shown in figs 4.5, 4.6, respectively.

Definition (b)

If a three-pole defined by a given set of polynomials can be constructed from only capacitances and (self) inductances, then the set of polynomials is said to be realizable, and the network is called the realization of the set of polynomials.

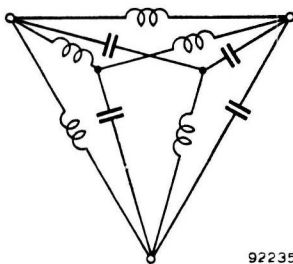


Fig. 4.5. A series-parallel three-pole.

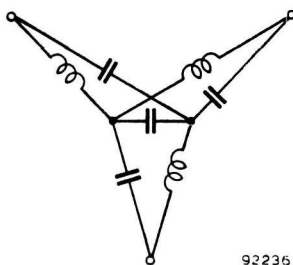


Fig. 4.6. A non-series-parallel three-pole.

4.2. The problem

The problem can now be stated as follows:

- I. To find a set of necessary and sufficient conditions for a given set of polynomials to be realizable by a series-parallel LC three-pole.
- II. To construct a series-parallel LC three-pole corresponding to any set of polynomials satisfying these necessary and sufficient conditions.

Our answer to this problem is incomplete. A set of necessary and sufficient conditions for realizability by series-parallel three-poles and a method

of construction have been obtained only for polynomials of the sixth and lower degrees. For higher degrees the number of special cases is very large and not easily handled without further development of the general methods of sec. 5.

5. General properties of the polynomials

Before proceeding to a study of the polynomials of the 0th, 1st, ..., 6th degrees we shall consider some general results which hold for polynomials of any degree. Throughout it will be supposed that all the necessary conditions of sec. 3 are satisfied; however, some of the results will be valid under less stringent conditions. In particular, some of the results will be valid for the polynomials of networks containing mutual inductance, in which case it will be mentioned explicitly.

5.1. Interchange of λ and μ

Suppose in a network composed exclusively of capacitances and self inductances we replace each capacitance C_i by an inductance $L'_i = 1/C_i$ and each inductance L_i by a capacitance $C'_i = 1/L_i$; then each branch admittance $\lambda C_i + 1/\lambda L_i$ will be replaced by $1/\lambda L'_i + \lambda C'_i = C_i/\lambda + \lambda/L_i$. The effect of the operation is thus to interchange λ and $1/\lambda$; consequently the polynomials of the new network will be the same as those of the old with $1/\lambda$ in place of λ . In what follows, we shall frequently write μ in place of $1/\lambda$ and the polynomials, as far as possible, as functions with like powers of λ and μ . Thus if a network realization can be found for a given set of polynomials, then we know immediately that a corresponding realization can be found for the set of polynomials obtained by interchanging λ and μ .

5.2. Interchange of C and D

If a given set of polynomials is realizable by a planar network, then the set obtained by interchanging C and D is realizable by the dual network. To see this we consider the mesh equations of the complete network, i.e., the three-pole plus the external voltage sources. In solving for the currents I_1, I_2, I_3 of fig. 2.1 we obtain the admittance equations (2.1), in which the common denominator, D , of the admittances is the determinant of the mesh equations. For the dual network the node equations will have precisely the same coefficients as the previously considered mesh equations. When we solve for the voltages V_{23}, V_{31}, V_{12} of fig. 2.3, we obtain the impedance equations (2.6), in which the common denominator, C' , of the impedances is the determinant of the node equations, and is therefore equal to D . On the other hand, F, G, H are the same functions of the coefficients in

the two cases and have therefore the same values. Thus the operation of taking the dual has solely the effect of interchanging C and D .

If, however, the network is non-planar, no dual network exists. We shall see later that for certain fourth-degree non-planar networks the interchange of C and D still has network significance. In this case the "pseudo-dual", that is to say, a network with the same F, G, H but with C and D interchanged, is of precisely the same configuration as the original. However, the values of corresponding elements do not bear any particularly simple relationship to one another.

5.3. Residue conditions

We now prove an important theorem concerning the Caue conditions (3.2) and (3.3).

Theorem 1

If C, D, F, G, H are polynomials such that

- (a) $GH + HF + FG = CD$,
 - (b) $\lambda + a_k\mu$ ($a_k > 0$) is a simple factor of D ,
 - (c) $(G+H)/D, (H+F)/D, (F+G)/D$ are driving-point reactance functions,
- then $g_k h_k + h_k f_k + f_k g_k = 0$, where f_k, g_k, h_k are the coefficients of $(\lambda + a_k\mu)^{-1}$ in the partial-fraction expansions of $F/D, G/D, H/D$, respectively.

Proof:

From (c), we can expand $(G+H)/D, (H+F)/D, (F+G)/D$ in the form

$$\left. \begin{aligned} y_\infty \lambda + y_0 \mu + \sum_{i=1}^r \frac{y_i}{\lambda + a_i \mu}, \\ \text{where } y_\infty, y_0, y_i \text{ are non-negative,} \\ a_i > 0; \quad a_i \neq a_j \quad (i \neq j), \quad (i = 1, \dots, r). \end{aligned} \right\} \quad (5.1)$$

But $F = \frac{1}{2}[-(G+H) + (H+F) + (F+G)]$, etc., so that $F/D, G/D, H/D$ can be expanded in the forms

$$\left. \begin{aligned} \frac{F}{D} &= f_\infty \lambda + f_0 \mu + \sum_{i=1}^r \frac{f_i}{\lambda + a_i \mu}, \\ \frac{G}{D} &= g_\infty \lambda + g_0 \mu + \sum_{i=1}^r \frac{g_i}{\lambda + a_i \mu}, \\ \frac{H}{D} &= h_\infty \lambda + h_0 \mu + \sum_{i=1}^r \frac{h_i}{\lambda + a_i \mu}. \end{aligned} \right\} \quad (5.2)$$

From (a) we have

$$GH + HF + FG \equiv 0 \pmod{D}, \quad (5.3)$$

so that, by (b),

$$GH + HF + FG \equiv 0 \pmod{[\lambda + a_k \mu]}. \quad (5.4)$$

From (b) and (5.2) we have *)

$$\left. \begin{aligned} F &\equiv f_k \prod_{\substack{i=1 \\ i \neq k}}^r (\lambda + a_i \mu) \\ G &\equiv g_k \prod_{\substack{i=1 \\ i \neq k}}^r (\lambda + a_i \mu) \\ H &\equiv h_k \prod_{\substack{i=1 \\ i \neq k}}^r (\lambda + a_i \mu) \end{aligned} \right\} \pmod{[\lambda + a_k \mu]}. \quad (5.5)$$

From (5.4) and (5.5) we now obtain

$$(g_k h_k + h_k f_k + f_k g_k) \left[\prod_{\substack{i=1 \\ i \neq k}}^r (\lambda + a_i \mu) \right]^2 \equiv 0 \pmod{[\lambda + a_k \mu]}.$$

Since by (5.1) the a_i ($i = 1, \dots, r$) are distinct, we must have

$$g_k h_k + h_k f_k + f_k g_k = 0. \quad (5.6)$$

Remark

$$\text{Since } \frac{f_k}{\lambda + a_k \mu} = \frac{1}{2} \left[\frac{f_k}{\lambda + \sqrt{-a_k}} + \frac{f_k}{\lambda - \sqrt{-a_k}} \right],$$

f_k is equal to twice the value of the residues of F/D at the poles $\lambda = \pm (-a_k)^{\frac{1}{2}}$. Thus (5.6) will still be valid if f_k, g_k, h_k are replaced by the corresponding residues of $F/D, G/D, H/D$. Accordingly, we shall call (5.6) the *D-residue condition*. A dual result holds for *C* residues and will be called the *C-residue condition*. These results are special cases of the Cauchy conditions (sec. 3). If, however, the *D*-Cauchy condition takes the form

$$g_k h_k + h_k f_k + f_k g_k > 0,$$

then $\lambda + a_k \mu$ must be a multiple factor of D . For the purposes of realization, this possibility does not give rise to any special difficulties.

*) If $\lambda + a_k \mu$ is a multiple factor of D of multiplicity s , then from (5.2) it must be a factor of F of multiplicity $s-1$; whence it then follows that $F \equiv 0 \pmod{[\lambda + a_k \mu]}$.

5.4. Separation conditions

In the last section we saw that the coefficients y_i are non-negative, i.e.,

$$\left. \begin{aligned} g_i + h_i &\geq 0, \\ h_i + f_i &\geq 0, \\ f_i + g_i &\geq 0. \end{aligned} \right\} \quad (5.7)$$

We shall refer to (5.7) as the *D-separation conditions*, since they follow from the separation of the poles and zeros of driving-point reactance functions. Similar results hold for the functions $(G+H)/C$, $(H+F)/C$, $(F+G)/C$, and will be called the *C-separation conditions*.

It follows from Theorem 1 that, if $\lambda + a_j\mu$ ($a_j > 0$) is a simple factor of D , either

$$\left. \begin{aligned} &\text{(i) } f_j = g_j = h_j = 0, \\ &\text{or} \\ &\text{(ii) two of the coefficients (say } g_j \text{ and } h_j) \text{ are zero,} \\ &\quad \text{and the remaining, } f_j, \text{ is positive,} \\ &\text{or} \\ &\text{(iii) two of the coefficients (say } g_j \text{ and } h_j) \text{ are positive,} \\ &\quad \text{and the remaining, } f_j, \text{ is negative.} \end{aligned} \right\} \quad (5.8)$$

In case (iii), we have from (5.6) that

$$\frac{1}{|f_j|} = \frac{1}{g_j} + \frac{1}{h_j},$$

from which it follows that

$$g_j + f_j > 0, \quad h_j + f_j > 0.$$

5.5. Interdependence of the necessary conditions

The conditions formulated in the preceding two sections are rather numerous, and it may be asked whether they are in fact all independent. That they are not so is proved in the following two algebraic theorems.

Theorem 2

If C, D, F, G, H are polynomials such that

- (a) $GH + HF + FG = CD$,
 - (b) F and D have no common factor of the form $\lambda + a_i\mu$ ($a_i > 0$),
 - (c) $(G+H)/D$, $(H+F)/D$, $(F+G)/D$ are driving-point reactance functions,
 - (d) the coefficients of the highest and lowest powers of λ in CD are positive,
- then
- $(G+H)/C$, $(H+F)/C$, $(F+G)/C$ are driving-point reactance functions.

Proof:

All the factors of D of the form $\lambda + \alpha_i \mu$ ($\alpha_i > 0$) are simple; for, by (c), a multiple factor would divide $H + F$, $F + G$, and thus F . Thus it follows from Theorem 1 that

$$g_i h_i + h_i f_i + f_i g_i = 0.$$

(i) We consider first a special case in which D has no factor of the form $\lambda + \alpha_i \mu$ ($\alpha_i > 0$) in common with either F , G , or H .

Since $(F + G)/D$ is a driving-point reactance function, $D/(F + G)$ is also, and may be expanded in partial fractions in the form

$$\sum_i \frac{d_i}{m_i \lambda + n_i \mu},$$

where $d_i > 0$, $m_1 \geq 0$, $n_2 \geq 0$; $m_i > 0$ ($i \neq 1$); $n_i > 0$ ($i \neq 2$).

Here, $F + G$ has no factor of the form $\lambda + \alpha_i \mu$ ($\alpha_i > 0$) in common with D ; for, if $m_j \lambda + n_j \mu$ were a factor of D , the coefficient $f_j + g_j$ in the partial-fraction expansion of $(F + G)/D$ would be zero, and thus by (5.8) we would have

$$f_j = g_j = 0,$$

in contradiction to (b).

It follows from (c) that if D is odd, F , G , H are all even, and therefore from (a), C is odd. Similarly, if D is even, C is even. Moreover, $F + G$ cannot have multiple factors, for otherwise $F + G$ and D would have a factor in common. Hence $C/(F + G)$ can be expanded in partial fractions in the form

$$\sum_i \frac{c_i}{m_i \lambda + n_i \mu}.$$

Also, from (a) and (c), any other binomial or trinomial factor of $F + G$ must also be a factor of C .

Now $F/(F + G)$ can be written as a function of λ^2 , since F and G are either both odd or both even. The partial-fraction expansion of $F/(F + G)$ is thus

$$q_\infty \lambda^2 + \sum_i \frac{q_i}{m_i \lambda^2 + n_i} = q_\infty \lambda^2 + \sum_i \frac{q_i \mu}{m_i \lambda + n_i \mu}.$$

Hence we have the congruences:

$$\left. \begin{aligned} D &\equiv d_k \prod_{\substack{i=1 \\ i \neq k}}^r (m_i \lambda + n_i \mu) \\ C &\equiv c_k \prod_{\substack{i=1 \\ i \neq k}}^r (m_i \lambda + n_i \mu) \\ F &\equiv q_k \mu \prod_{\substack{i=1 \\ i \neq k}}^r (m_i \lambda + n_i \mu) \end{aligned} \right\} \pmod{[m_k \lambda + n_k \mu]}.$$

From (a) we have

$$CD \equiv FG \equiv -F^2 \pmod{[F+G]}.$$

Since the factors $m_i \lambda + n_i \mu$ are all distinct, we have

$$c_k d_k \equiv -q_k^2 \mu^2 \pmod{[m_k \lambda + n_k \mu]},$$

for all k except the special cases when $m_k = 0$, or $n_k = 0$.

Thus

$$c_k d_k = q_k^2 \left(\frac{m_k}{n_k} \right) \geq 0 \quad (m_k n_k \neq 0).$$

Since

$$d_k > 0, \quad c_k \geq 0 \quad (m_k n_k \neq 0).$$

By (c) and (d), the coefficients of the highest and lowest powers of λ in C and $F+G$ are of the same sign. Hence the quantities c_∞ , c_0 , corresponding to $m_i = 0$, $n_i = 0$, respectively, are non-negative. Hence $C/(F+G)$ is a driving-point reactance function. Similarly, $C/(G+H)$, $C/(H+F)$ are driving-point reactance functions, and thus also $(G+H)/C$, $(H+F)/C$, $(F+G)/C$.

(ii) We now suppose that D has factors of the form $\lambda + a_i \mu$ ($a_i > 0$) in common with G , H but not with F . That is to say, we now permit case (ii) of (5.8). (Case (i) of (5.8) is contrary to condition (a)).

Then

$$f_i g_i h_i \neq 0 \quad (i = 1, \dots, p),$$

$$f_i > 0, \quad g_i = h_i = 0 \quad (i = p+1, \dots, r).$$

We can apply the argument of case (i) to the functions $C/(F+G)$, $C/(H+F)$, but not to $C/(G+H)$. To deal with the last-mentioned function we consider

$$\frac{F}{D} = \frac{F_1}{D_1} + X,$$

$$\frac{G}{D} = \frac{G_1}{D_1},$$

$$\frac{H}{D} = \frac{H_1}{D_1},$$

where

$$\frac{F_1}{D_1} = \sum_{i=1}^p \frac{f_i}{\lambda + a_i \mu}, \quad X = \sum_{i=p+1}^r \frac{f_i}{\lambda + a_i \mu}.$$

Thus X is a driving-point reactance function.

Let

$$\prod_{i=p+1}^r (\lambda + a_i \mu) = D_2;$$

D_1 is then defined by $D = D_1 D_2$.

Then

$$F = F_1 D_2 + X D_1 D_2,$$

$$G = G_1 D_2,$$

$$H = H_1 D_2,$$

and thus

$$C = (G_1 H_1 + H_1 F_1 + F_1 G_1) \frac{D_2}{D_1} + X(G_1 + H_1) D_2.$$

Let

$$C_1 = \frac{G_1 H_1 + H_1 F_1 + F_1 G_1}{D_1};$$

then

$$\frac{G}{C} = \frac{G_1}{C_1 + X(G_1 + H_1)},$$

and

$$\frac{H}{C} = \frac{H_1}{C_1 + X(G_1 + H_1)}.$$

Hence

$$\frac{C}{G + H} = \frac{C_1}{G_1 + H_1} + X.$$

But $C_1/(G_1 + H_1)$ is a driving-point reactance function by case (i); since X is a driving-point reactance function, it follows that $C/(G + H)$ and therefore $(G + H)/C$ is a driving-point reactance function.

Theorem 3

If C, D, F, G, H are polynomials such that

(a) $GH + HF + FG = CD,$

(b) $(G + H)/D, (H + F)/D, (F + G)/D, (G + H)/C, (H + F)/C, (F + G)/C$
are driving-point reactance functions,

then $g_i h_i + h_i f_i + f_i g_i \geq 0$, where f_i, g_i, h_i are the coefficients of $(\lambda + \alpha_i \mu)^{-1}$ in the partial-fraction expansions of $F/D, G/D, H/D$, respectively.

Proof:

We suppose that for a particular value k of i , the coefficients satisfy

$$g_k h_k + h_k f_k + f_k g_k < 0.$$

From (b) we have

$$g_k + h_k \geq 0, \quad h_k + f_k \geq 0, \quad f_k + g_k \geq 0.$$

Thus two of f_k, g_k, h_k must be positive and one negative. Let

$$f_k < 0, \quad \text{and} \quad g_k, h_k > 0.$$

We rewrite the polynomials in the form

$$\frac{F}{D} = \frac{F'}{D'} + \frac{f_k + g_k h_k / (g_k + h_k)}{\lambda + \alpha_k \mu} = \frac{F'}{D} + Y, \quad \text{say,}$$

$$\frac{G}{D} = \frac{G'}{D'},$$

$$\frac{H}{D} = \frac{H'}{D'}.$$

Now let

$$C' = \frac{G'H' + H'F' + F'G'}{D};$$

then in the same way as in the proof of case (ii), Theorem 2, we find

$$\frac{C}{G + H} = \frac{C'}{G' + H'} + Y.$$

Suppose that $\lambda + \alpha_k \mu$ is a factor of $G' + H'$ of multiplicity $s-1$. Then, since $g_k + h_k \neq 0$, $\lambda + \alpha_k \mu$ must be a factor of D of multiplicity s .

We can therefore write

$$\left. \begin{aligned} F' &\equiv - \frac{g_k h_k}{g_k + h_k} (\lambda + \alpha_k \mu)^{s-1} \prod_{i \neq k} (\lambda + \alpha_i \mu) \\ G' &\equiv g_k (\lambda + \alpha_k \mu)^{s-1} \prod_{i \neq k} (\lambda + \alpha_i \mu) \\ H' &\equiv h_k (\lambda + \alpha_k \mu)^{s-1} \prod_{i \neq k} (\lambda + \alpha_i \mu) \end{aligned} \right\} \pmod{[\lambda + \alpha_k \mu]^s}.$$

Thus

$$G'H' + H'F' + F'G' \equiv 0 \pmod{[\lambda + \alpha_k \mu]^{2s-1}},$$

whence

$$C' \equiv 0 \pmod{[\lambda + \alpha_k \mu]^{s-1}}.$$

Hence the factor $(\lambda + a_k\mu)^{s-1}$ can be cancelled out of the numerator and denominator of $C'/(G' + H')$. But the coefficient of $(\lambda + a_k\mu)^{-1}$ in the partial-fraction expansion of Y is

$$\frac{g_k h_k + h_k f_k + f_k g_k}{g_k + h_k} < 0, \text{ by hypothesis.}$$

Hence $C'/(H' + G') + Y$ cannot be a driving-point reactance function. But this violates condition (b) and thus $g_k h_k + h_k f_k + f_k g_k$ cannot be negative.

Note that C and D can be interchanged throughout, and that the conditions of validity of Theorems 2 and 3 hold also for the polynomials of three-pole networks containing capacitances, self and mutual inductances. The significance of the theorems is as follows:

Suppose we start with a set of polynomials satisfying the conditions of sec. 3 and Theorem 2. We then split this set into two simpler sets of lower degree than the original (in a way to be explained in the next section) such that the conditions of Theorem 2 and condition IV of sec. 3 hold for the two simpler sets. Then we know from Theorems 2 and 3 that *all* the conditions of sec. 3 will be satisfied for these simpler sets. That is to say, under the conditions of Theorem 2, it is sufficient to ensure that condition IV of sec. 3 is satisfied. Theorem 2 then guarantees the C -separation conditions, while Theorem 3 and its dual guarantee the C - and D -Cauer conditions.

5.6. Elementary realization operations

We are now ready to introduce three algebraic operations*) which correspond to the network operations of dividing a given three-pole into two component three-poles in parallel or into a three-pole and a two-pole in series. These operations will be called D pole-removal, C pole-removal and partitioning. Throughout we shall suppose that the conditions of sec. 3 are satisfied for all sets of polynomials.

D pole-removal

If the functions can be expressed in the form

$$\frac{F}{D} = \frac{F_1}{D_1} + \frac{F_2}{D_2},$$

$$\frac{G}{D} = \frac{G_2}{D_2},$$

*) These operations are equivalent to the six operations given by Ozaki⁸⁾.

$$\frac{H}{D} = \frac{H_2}{D_2},$$

$$D = D_1 D_2,$$

such that F_1/D_1 is a driving-point reactance function, and such that $(F_2, G_2, H_2, C_2, D_2)$ is a set of polynomials of lower degree than (F, G, H, C, D) , and known to be realizable, then the given set can be realized by the configuration shown in fig. 5.1. This readily follows from (2.1) and (2.4). The operation consists in removing the term F_1/D_1 from the functions and in realizing it by a two-pole admittance between terminals 2 and 3. The operation is called D pole-removal since in many cases the expression F_1/D_1 is a single term $f_\infty \lambda$, $f_0 \mu$ or $f_1(\lambda + a_1 \mu)^{-1}$ and the operation has the effect of removing a pole from the expression F/D . When, however, F_2/D_2 has a pole in common with F_1/D_1 , we shall say that the pole has been “incompletely removed”, and the operation will be referred to as *partial D pole-removal*.

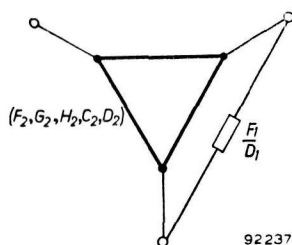


Fig. 5.1. Realization by D pole-removal.

C pole-removal

C pole-removal is the dual operation of D pole-removal. If D is replaced throughout by C in the above argument, then the realization takes the form shown in fig. 5.2.

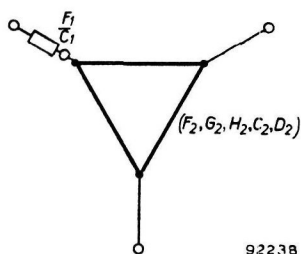


Fig. 5.2. Realization by C pole-removal.

Partitioning

Partitioning is a more general operation than D pole-removal. It is more difficult to carry out and leads to more complicated networks than does pole-removal, so that we shall use it only when the other operations fail. There is no dual operation, so that it is not surprising to learn that it leads to non-planar networks. It consists in writing the functions in the form

$$\frac{F}{D} = \frac{F_1}{D} + \frac{F_2}{D},$$

$$\frac{G}{D} = \frac{G_1}{D} + \frac{G_2}{D},$$

$$\frac{H}{D} = \frac{H_1}{D} + \frac{H_2}{D},$$

where (F_1, G_1, H_1, C_1, D) and (F_2, G_2, H_2, C_2, D) are polynomial sets of lower degree than (F, G, H, C, D) , and are known to be realizable by three-poles Γ_1, Γ_2 . From (2.1) and (2.4) it follows that the given polynomial set (F, G, H, C, D) can then be realized by the parallel connection of Γ_1 and Γ_2 , as shown in fig. 5.3. We shall refer to (F_1, G_1, H_1, C_1, D) and (F_2, G_2, H_2, C_2, D) as partitions of (F, G, H, C, D) .

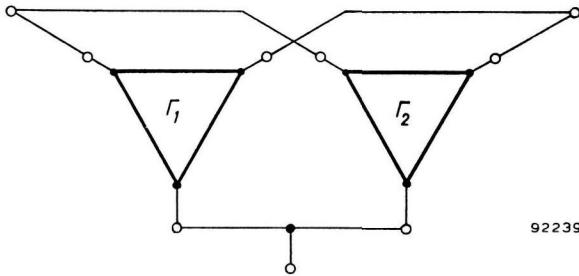


Fig. 5.3. Realization by partitioning.

These three operations will constitute the basis of our synthesis technique. It will be noted that the phrase *known to be realizable* occurs in all the statements of the operations. The technique, however, will be to make the realizability of a given degree depend upon that of a lower degree by means of one of the operations. The realization of this lower degree can then be made to depend on that of a still lower degree by a subsequent operation, and proceeding in this way one eventually arrives at a degree low enough for the network to be realized by inspection. These three operations are the inverses of the operations of building up a three-pole

by the elementary connections of sec. 4.1. We conclude that, if a given three-pole network can be decomposed into its elements and elementary three-poles by the use of the three operations, then it must be a series-parallel three-pole. Conversely, all series-parallel three-poles can be decomposed into their elements and elementary three-poles by application of the three operations.

At this point it will be fitting to introduce a theorem that limits the range of applicability of the partitioning operation.

Theorem 4 (the partitioning theorem)

Let f, g, h be the coefficients of $(\lambda + a\mu)^{-1}$ in the partial-fraction expansions of $F/D, G/D, H/D$; let the residue condition, $gh + hf + fg = 0$, and the separation conditions be satisfied, and let $f \neq 0$.

Then a partitioning of (f, g, h) into (f_1, g_1, h_1) and $(f - f_1, g - g_1, h - h_1)$, such that the separation and Cauchy conditions hold for the corresponding partitions of (F, G, H, C, D) , is possible if and only if

$$1 \geq \frac{f_1}{f} = \frac{g_1}{g} = \frac{h_1}{h} \geq 0.$$

Proof:

We have

$$gh + hf + fg = 0. \quad (5.9)$$

If the Cauchy conditions hold for the partitions of (F, G, H, C, D) , then

$$g_1 h_1 + h_1 f_1 + f_1 g_1 = \varepsilon_1 \geq 0, \quad (5.10)$$

and

$$(g - g_1)(h - h_1) + (h - h_1)(f - f_1) + (f - f_1)(g - g_1) = \varepsilon_2 \geq 0. \quad (5.11)$$

Expanding (5.11) with the aid of (5.9) and (5.10), we obtain

$$\varepsilon_1 - \varepsilon_2 = gh_1 + g_1 h + hf_1 + h_1 f + fg_1 + f_1 g. \quad (5.12)$$

Since $f \neq 0$, we have either $f_1 \neq 0$ or $f - f_1 \neq 0$. Let $f_1 \neq 0$; then, in virtue of (5.9) and (5.10),

$$\begin{aligned} \varepsilon_1 - \varepsilon_2 &= f_1 g + f g_1 - \frac{fg}{f + g} (f_1 + g_1) + (f + g) \left(\frac{\varepsilon_1}{f_1 + g_1} - \frac{f_1 g_1}{f_1 + g_1} \right), \\ \varepsilon_1 - \varepsilon_2 &= \frac{(f_1 g - f g_1)^2}{(f + g)(f_1 + g_1)} + \frac{\varepsilon_1(f + g)}{f_1 + g_1}. \end{aligned}$$

Thus, since $\varepsilon_2 \geq 0$,

$$\frac{-\varepsilon_1}{f_1 + g_1} (f - f_1 + g - g_1) \geq \frac{(f_1 g - f g_1)^2}{(f + g)(f_1 + g_1)}.$$

If the separation conditions hold, we have, from (5.9), (5.10), and the fact that $f_1 f \neq 0$,

$$f + g > 0, \quad f_1 + g_1 > 0, \quad f - f_1 + g - g_1 \geq 0.$$

Since $\varepsilon_1 \geq 0$, the only possibility is

$$\varepsilon_1 = 0, \quad f_1 g - f g_1 = 0.$$

By similar reasoning we obtain, with h in place of g ,

$$f_1 h - f h_1 = 0.$$

Thus

$$\frac{f_1}{f} = \frac{g_1}{g} = \frac{h_1}{h} = m. \quad (5.13)$$

But $(f_1 + g_1) = m(f + g) > 0$. Hence $m > 0$, since $f + g > 0$. If $f - f_1 = 0$, then, by (5.13), $m = 1$.

Similarly, replacing f_1 by $f - f_1$ in the above argument, we obtain $1 - m \geq 0$.

Conversely, if (5.13) holds, then $\varepsilon_1 = \varepsilon_2 = 0$, i.e., the Cauchy conditions hold. Also if $1 \geq m \geq 0$, then $f_1 + g_1 \geq 0$ and $f - f_1 + g - g_1 \geq 0$, etc. Thus the separation conditions hold.

Corollary (i)

If $f = g = h = 0$, then $f_1 = g_1 = h_1 = 0$.

For, by the separation conditions, $f_1 + g_1 \geq 0$, $f - f_1 + g - g_1 = -(f_1 + g_1) \geq 0$, and similar expressions for $g_1 + h_1$, $h_1 + f_1$. Hence $g_1 + h_1 = h_1 + f_1 = f_1 + g_1 = 0$, i.e., $f_1 = g_1 = h_1 = 0$.

Corollary (ii)

If the order of a set of polynomials is lower than the degree, then the set is unrealizable by a series-parallel network. For, this implies that if the common factors are cancelled out, a set of polynomials will be obtained at least one of which has negative coefficients. If this set is partitioned then at least one of the partitions will contain some negative coefficients. Otherwise, if the common factors are re-introduced and the system again partitioned, then by corollary (i) the partitions will also contain the same common factors, and after these have been cancelled out the negative coefficients will be recovered. Pole-removal and further partitioning can lead eventually only to a polynomial of the first degree *) consisting of a single negative

*) Here, we use the word *degree* in the ordinary sense and not with the special meaning of definition (b), sec. 3.

term. But the product of this term with any common factor containing only positive coefficients is a polynomial with negative coefficients and thus the set to which it belongs cannot be realizable. Since any sequence of the three operations leads to the same result, it follows that the given set is unrealizable by a series-parallel network.

Thus the condition that F, G, H, C, D are *polynomials with non-negative coefficients*, is necessary for realization by series-parallel networks. Whether or not it is also necessary for the realization by networks of arbitrary structure, as has been conjectured¹⁰⁾, is a question which this analysis does not answer.

5.7. Realization when one of the polynomials is zero

Before considering the realization of polynomials of the third, fourth and higher degrees, we will consider a trivial case of arbitrary degree.

Let

$$C \equiv 0;$$

then

$$GH + HF + FG \equiv 0.$$

Thus at least two of the polynomials, say G and H , must be zero, since none can be negative for positive real values of λ . From sec. 3 it then follows that F/D is a driving-point reactance function. Thus the system is realizable as shown in fig. 5.4.

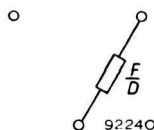


Fig. 5.4. Realization when $C \equiv 0$.

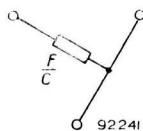


Fig. 5.5. Realization when $D \equiv 0$.

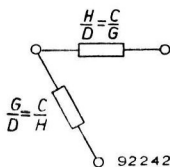


Fig. 5.6. Realization when $F \equiv 0$.

Similarly if $D \equiv 0$, the realization is the dual of fig. 5.4, viz, fig. 5.5. On the other hand, if $F \equiv 0$, then G/D and H/D are driving-point reactance functions. The system is then realizable as shown in fig. 5.6.

6. The third degree

The third-degree polynomials are of a sufficiently simple form to be realized directly without first investigating the lower degrees. Consequently, we shall begin our study of the various degrees with an investigation of the third degree. The zeroth, first, and second degrees will then be treated as degenerate cases.

The general third-degree polynomials can be written as either (i)

$$\begin{aligned} F &= p_3\lambda^3 + p_1\lambda, & C &= x_2\lambda^2 + x_0, \\ G &= q_3\lambda^3 + q_1\lambda, & D &= y_2\lambda^2 + y_0, \\ H &= r_3\lambda^3 + r_1\lambda, \end{aligned}$$

or (ii)

$$\begin{aligned} F &= p_2\lambda^2 + p_0, & C &= x_3\lambda^3 + x_1\lambda, \\ G &= q_2\lambda^2 + q_0, & D &= y_3\lambda^3 + y_1\lambda. \\ H &= r_2\lambda^2 + r_0, \end{aligned}$$

From eq. (2.5) it follows that in case (i)

$$\left. \begin{aligned} q_3r_3 + r_3p_3 + p_3q_3 &= 0, \\ x_0y_0 &= 0, \end{aligned} \right\} \quad (6.1)$$

while in case (ii)

$$\left. \begin{aligned} q_0r_0 + r_0p_0 + p_0q_0 &= 0, \\ x_3y_3 &= 0. \end{aligned} \right\} \quad (6.2)$$

Since the coefficients are all non-negative, it follows that at least two of p_3, q_3, r_3 and two of p_0, q_0, r_0 must be zero. Let $q_3 = r_3 = 0$ and $q_0 = r_0 = 0$. We also take $y_0 = 0$ and $y_3 = 0$. Then the polynomials of case (i) may be rewritten as

$$\left. \begin{aligned} F &= f_1\lambda + f_{-1}\mu, \\ G &= g_{-1}\mu, \\ H &= h_{-1}\mu, \end{aligned} \right\} \begin{aligned} C &= c_0 + c_{-2}\mu^2, \\ D &= d_0 \neq 0. \end{aligned} \quad (6.3)$$

Case (ii) is then obtained from this set by interchanging λ and μ . From the non-negativeness of the coefficients it follows that (6.3) can be realized by the network shown in fig. 6.1.

The other possibilities resulting from (6.1) and (6.2) lead to polynomials that may be obtained from the standard form (6.3) by the interchange of F , G , H , or of C and D . Since the dual of the network of fig. 6.1 exists, it follows that a realization is possible in all these cases.

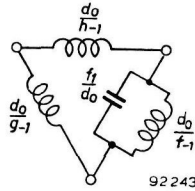


Fig. 6.1. Realization of the standard form of the third-degree polynomials.

The second-degree polynomials are obtained by putting either (iii) $f_1 = 0$, in which case $c_0 = 0$;
or (iv) $c_{-2} = 0$, in which case $f_{-1} = 0$ and either g_{-1} or $h_{-1} = 0$.

The first degree results from the additional conditions

$$c_{-2} = 0 \quad \text{in} \quad (iii),$$

or $f_1 = 0$, or $g_{-1} = h_{-1} = 0$ in (iv).

Thus for the first degree it is necessary that either $C \equiv 0$ or $D \equiv 0$, a case which was considered in sec. 5.7. This condition is also necessary for the zeroth degree.

We conclude that all systems of the third and lower degrees and satisfying the conditions of sec. 3 are realizable. The network of fig. 6.1 serves as a basic type for all these realizations; they can be derived from it by the simple transformations of secs 5.1 and 5.2 or by letting some of the element values become zero or infinite.

7. The fourth degree

In this and the following sections we use a decimal system of classification of the various cases which arise. The initial figure denotes the degree. Moreover, we shall consider only certain basic cases from which the others can be derived by C , D , or λ , μ , or F , G , H interchanges. We have two cases to consider according as F , G , H are even and C , D odd (case 4.1), or F , G , H are odd and C , D even (case 4.2).

Case 4.1

The polynomials are

$$F = p_4\lambda^4 + p_2\lambda^2 + p_0,$$

$$C = x_3\lambda^3 + x_1\lambda,$$

$$G = q_4\lambda^4 + q_2\lambda^2 + q_0,$$

$$D = y_3\lambda^3 + y_1\lambda,$$

$$H = r_4\lambda^4 + r_2\lambda^2 + r_0,$$

where

$$q_0r_0 + r_0p_0 + p_0q_0 = 0,$$

$$q_4r_4 + r_4p_4 + p_4q_4 = 0.$$

Let $q_0 = r_0 = 0$.

There are two sub-cases to consider, viz., $q_4 = r_4 = 0$ (case 4.11) and $p_4 = q_4 = 0$ (case 4.12).

Case 4.11

$$q_4 = r_4 = 0.$$

The polynomials may be rewritten in the form

$$\left. \begin{aligned} \frac{F}{D} &= f_\infty \lambda + f_0 \mu + \frac{f_1}{\lambda + \alpha_1 \mu}, \\ \frac{G}{D} &= \frac{g_1}{\lambda + \alpha_1 \mu}, \\ \frac{H}{D} &= \frac{h_1}{\lambda + \alpha_1 \mu}, \\ D &= \lambda + \alpha_1 \mu. \end{aligned} \right\} \quad (7.1)$$

(Here, we suppose that $y_1 y_3 \neq 0$; otherwise the system is of the third degree or falls under sec. 5.7.) By Theorem 1,

$$g_1 h_1 + h_1 f_1 + f_1 g_1 = 0,$$

so that

$$C = (g_1 + h_1) (f_\infty \lambda + f_0 \mu).$$

Here, the only possibility is $f_1 < 0$, $g_1 > 0$, $h_1 > 0$; otherwise either negative coefficients would be present or this case would fall under sec. 5.7.

This case is further divided into two sub-cases, case 4.111 and case 4.112.

Case 4.111

$$\left. \begin{aligned} &\text{If either} \\ &\text{or} \end{aligned} \right\} \begin{aligned} &\alpha_1 f_\infty + f_1 \geq 0, \\ &f_0 + f_1 \geq 0, \end{aligned} \quad (7.2)$$

then, from (7.1), $f_0 \mu$ or $f_\infty \lambda$ can be removed to leave a third-degree residual with non-negative coefficients. Theorems 2 and 3 then guarantee that this residual satisfies all the conditions of sec. 3.

Case 4.112

If (7.2) does not hold we partition into two third-degree systems according to the following scheme:

$$\left. \begin{aligned} \frac{F_1}{D} &= f_\infty \lambda + \frac{f_{11}}{\lambda + \alpha_1 \mu}, & \frac{F_2}{D} &= f_0 \mu + \frac{f_{12}}{\lambda + \alpha_1 \mu}, \\ \frac{G_1}{D} &= \frac{g_{11}}{\lambda + \alpha_1 \mu}, & \frac{G_2}{D} &= \frac{g_{12}}{\lambda + \alpha_1 \mu}, \\ \frac{H_1}{D} &= \frac{h_{11}}{\lambda + \alpha_1 \mu}, & \frac{H_2}{D} &= \frac{h_{12}}{\lambda + \alpha_1 \mu}, \end{aligned} \right\} \quad (7.3)$$

where the introduced constants are given, in accordance with Theorem 4, by

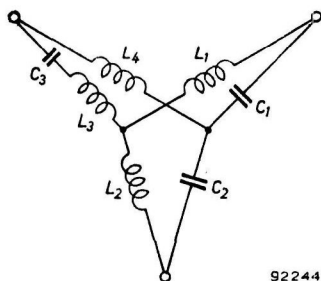
$$\left. \begin{aligned} \frac{f_{11}}{f_1} &= \frac{g_{11}}{g_1} = \frac{h_{11}}{h_1} = \frac{\alpha_1 f_\infty}{|f_1|}, \\ \frac{f_{12}}{f_1} &= \frac{g_{12}}{g_1} = \frac{h_{12}}{h_1} = 1 - \frac{\alpha_1 f_\infty}{|f_1|}. \end{aligned} \right\} \quad (7.4)$$

Theorems 2,3,4 guarantee the separation and Causer conditions for both partitions. We must, however, check that condition IV of sec. 3 is not violated.

Since (7.2) does not hold, we have

$$1 - \frac{\alpha_1 f_\infty}{|f_1|} > 0,$$

so that the G and H partitions have positive coefficients. Also, $F_1 = f_\infty \lambda^2$ and $F_2 = \alpha_1 f_0 \mu^2 + (f_0 + f_1 + \alpha_1 f_\infty)$. But $f_0 + f_1 + \alpha_1 f_\infty \geq 0$, since this is the middle term of F . Thus the F partitions also satisfy IV of sec. 3. Both partitions are of the third degree. Hence this case is realizable; the network is shown in fig. 7.1. The C polynomials of the two partitions are $f_\infty \lambda(g_{11} + h_{11})$ and $f_0 \mu(g_{12} + h_{12})$, respectively. The values of the elements are then readily calculated:



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Fig. 7.1. Realization of case 4.112.

$$\left. \begin{aligned} L_1 &= \frac{g_{12}}{f_0(g_{12} + h_{12})} = \frac{g_1}{f_0(g_1 + h_1)}, \\ L_2 &= \frac{h_{12}}{f_0(g_{12} + h_{12})} = \frac{h_1}{f_0(g_1 + h_1)}, \\ L_3 &= \frac{f_0 + f_{12}}{f_0(g_{12} + h_{12})}, \\ L_4 &= \frac{1}{g_{11} + h_{11}}, \\ C_1 &= \frac{f_\infty (g_{11} + h_{11})}{g_{11}} = \frac{f_\infty (g_1 + h_1)}{g_1}, \\ C_2 &= \frac{f_\infty (g_{11} + h_{11})}{h_{11}} = \frac{f_\infty (g_1 + h_1)}{h_1}, \\ C_3 &= \frac{g_{12} + h_{12}}{a_1}. \end{aligned} \right\} \quad (7.5)$$

This is not the only possible way of partitioning unless $a_1 f_\infty + f_0 + f_1 = 0$; for, the range of f_{11} is limited by the condition that F_1 and F_2 should have non-negative coefficients, i.e.,

$$\begin{aligned} a_1 f_\infty + f_{11} &\geq 0, \\ f_0 + f_{12} &\geq 0, \end{aligned}$$

from which it follows that

$$f_0 + f_1 \geq f_{11} \geq -a_1 f_\infty.$$

When $a_1 f_\infty + f_0 + f_1 = 0$, only one value is possible for f_{11} . In this case, both F_1 and F_2 consist of single terms with the result that only six elements are used for the realization. Otherwise, if $a_1 f_\infty + f_0 + f_1 > 0$ and

$$\begin{aligned} f_{11} &\neq -a_1 f_\infty, \\ f_{11} &\neq f_0 + f_1, \quad *) \end{aligned}$$

F_1 and F_2 consist of two terms each, so that eight elements are then used for realization.

The network of fig. 7.1, consisting of seven elements, contains two so-called redundant elements. There are precisely five independent parameters, viz, f_∞ , f_0 , f_1 , g_1 , a_1 . Thus we would expect five independent elements. This is indeed the case, for two relations between the element values can be found, viz,

*) The case $f_{11} = f_0 + f_1$ is derived from (7.3) and (7.4) by interchanging λ and μ .

$$\left. \begin{aligned} L_1 C_1 &= L_2 C_2, \\ L_4 (C_1 + C_2)^2 &= C_3 (L_1 C_1 + L_3 C_1 + L_3 C_2). \end{aligned} \right\} \quad (7.6)$$

Since C is a polynomial of the same form as D , it follows that the pseudo-dual is realizable by a network of the same structure, and we need not discuss it further.

It was suggested by Ozaki⁸⁾ that in some cases partial C pole-removal might enable D pole-removal to be performed in place of partitioning. Since partitioning introduces more elements than D pole-removal, the net effect would be a saving in the number of elements. In this case, however, it is not possible, as the following reasoning shows. Since C and D are of the same form it is sufficient to consider the effect of partial D pole-removal on the possibility of C pole-removal. From (7.1) it follows that the effect of this is to reduce the values of either f_∞ or f_0 , or both. We find that

$$\left. \begin{aligned} \frac{F}{C} &= \frac{1}{g_1 + h_1} \left[\lambda + a_1 \mu + \frac{f_1}{f_\infty \lambda + f_0 \mu} \right], \\ \frac{G}{C} &= \frac{g_1}{(g_1 + h_1) (f_\infty \lambda + f_0 \mu)}, \\ \frac{H}{C} &= \frac{h_1}{(g_1 + h_1) (f_\infty \lambda + f_0 \mu)}. \end{aligned} \right\} \quad (7.7)$$

From (7.7), C pole-removal is possible if and only if either

$$a_1 f_\infty + f_1 \geq 0,$$

or

$$f_0 + f_1 \geq 0.$$

But if this condition is not initially satisfied, it will certainly not be after f_∞ or f_0 have been reduced, and thus the effect of the operation is adverse. We are forced to the conclusion that, in the general case, the circuit of fig. 7.1 uses the least number of elements required for a series-parallel realization.

Case 4.12

$$p_4 = q_4 = 0.$$

The polynomials can be rewritten as

$$\begin{aligned} \frac{F}{D} &= f_0 \mu + \frac{f_1}{\lambda + a_1 \mu}, \\ \frac{G}{D} &= \frac{g_1}{\lambda + a_1 \mu}, \\ \frac{H}{D} &= h_\infty \lambda + \frac{h_1}{\lambda + a_1 \mu}. \end{aligned}$$

By sec. 3 and (5.8), $g_1 \geq 0$ and one of f_1, h_1 is non-negative. Let $f_1 > 0$; then we can remove $f_0\mu$ to leave a third-degree residual. This is most easily seen by calculating the resulting polynomials F', G, H, C' , taking $D = \lambda + a_1\mu$. Theorems 2, 3 and sec. 6 guarantee the realization (see fig. 7.2).

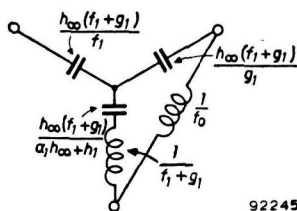


Fig. 7.2. Realization of case 4.12.

Case 4.2

The polynomials are

$$F = p_3\lambda^3 + p_1\lambda,$$

$$C = x_4\lambda^4 + x_2\lambda^2 + x_0,$$

$$G = q_3\lambda^3 + q_1\lambda,$$

$$D = y_4\lambda^4 + y_2\lambda^2 + y_0.$$

$$H = r_3\lambda^3 + r_1\lambda,$$

Thus

$$x_4y_4 = 0, \quad x_0y_0 = 0.$$

We take $y_4 = 0$. There are two sub-cases, viz, $y_0 = 0$ (case 4.21) and $x_0 = 0$ (case 4.22).

Case 4.21

$$y_0 = 0.$$

Rewrite as

$$\frac{F}{D} = \frac{f_\infty\lambda + f_0\mu}{d_0},$$

$$\frac{G}{D} = \frac{g_\infty\lambda + g_0\mu}{d_0},$$

$$\frac{H}{D} = \frac{h_\infty\lambda + h_0\mu}{d_0}.$$

This is clearly realizable as a delta connection (fig. 7.3).

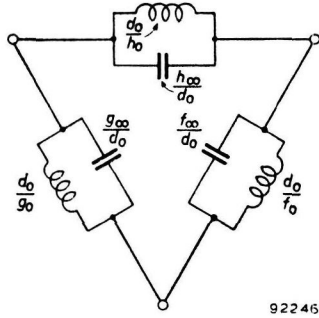


Fig. 7.3. Realization of case 4.21.

Case 4.22

$$x_0 = 0.$$

Rewrite as

$$\frac{F}{D} = f_\infty \lambda + \frac{f_1}{\lambda + \alpha_1 \mu},$$

$$\frac{G}{D} = g_\infty \lambda + \frac{g_1}{\lambda + \alpha_1 \mu},$$

$$\frac{H}{D} = h_\infty \lambda + \frac{h_1}{\lambda + \alpha_1 \mu}.$$

From (5.8), we can take $f_1 \geq 0$, $g_1 \geq 0$. Then $f_\infty \lambda$ and $g_\infty \lambda$ can be removed to leave a third-degree residual (fig. 7.4).

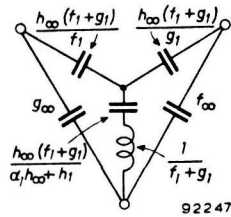


Fig. 7.4. Realization of case 4.22.

8. The fifth degree

Two forms of the polynomials are possible:

$$(i) \quad F = p_5 \lambda^5 + p_3 \lambda^3 + p_1 \lambda,$$

$$C = x_4 \lambda^4 + x_2 \lambda^2 + x_0,$$

$$G = q_5 \lambda^5 + q_3 \lambda^3 + q_1 \lambda,$$

$$D = y_4 \lambda^4 + y_2 \lambda^2 + y_0.$$

$$H = r_5 \lambda^5 + r_3 \lambda^3 + r_1 \lambda,$$

$$x_0 y_0 = 0,$$

$$q_5 r_5 + r_5 p_5 + p_5 q_5 = 0.$$

We take $y_0 = 0$ and $q_5 = r_5 = 0$.

$$(ii) \quad \begin{aligned} F &= p_4 \lambda^4 + p_2 \lambda^2 + p_0, & C &= x_5 \lambda^5 + x_3 \lambda^3 + x_1 \lambda, \\ G &= q_4 \lambda^4 + q_2 \lambda^2 + q_0, & D &= y_5 \lambda^5 + y_3 \lambda^3 + y_1 \lambda. \\ H &= r_4 \lambda^4 + r_2 \lambda^2 + r_0, \\ x_5 y_5 &= 0, \\ q_0 r_0 + r_0 p_0 + p_0 q_0 &= 0. \end{aligned}$$

We take $y_5 = 0$ and $q_0 = r_0 = 0$.

Both cases can now be standardized in the form

$$\begin{aligned} \frac{F}{D} &= f_\infty \lambda + f_0 \mu + \frac{f_1}{\lambda + a_1 \mu}, \\ \frac{G}{D} &= g_0 \mu + \frac{g_1}{\lambda + a_1 \mu}, \\ \frac{H}{D} &= h_0 \mu + \frac{h_1}{\lambda + a_1 \mu}. \end{aligned}$$

Case 5.1

If $f_1 \geq 0$, we can remove $f_\infty \lambda + f_0 \mu$ to leave a fourth-degree residual (case 4.22).

Case 5.2

If $f_1 < 0$, then $g_1, h_1 > 0$; we can remove $g_0 \mu$ and $h_0 \mu$ to leave a fourth-degree residual (case 4.11).

9. The sixth degree

We have two principal cases to consider, namely, case 6.1 in which F, G, H are odd and C, D even, and case 6.2 in which F, G, H are even and C, D odd. The further sub-division of these cases is shown in fig. 9.1.

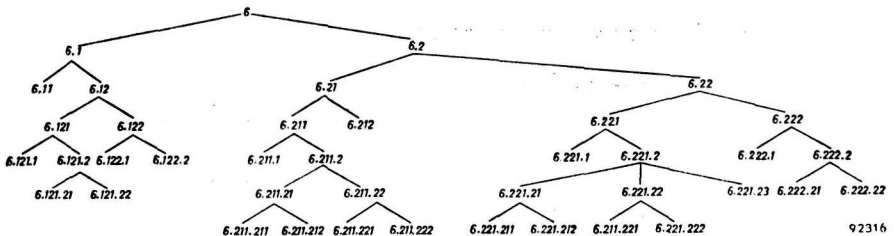


Fig. 9.1. The sub-cases of the sixth degree.

Case 6.1

$$F = p_5\lambda^5 + p_3\lambda^3 + p_1\lambda,$$

$$C = x_6\lambda^6 + x_4\lambda^4 + x_2\lambda^2 + x_0,$$

$$G = q_5\lambda^5 + q_3\lambda^3 + q_1\lambda,$$

$$D = y_6\lambda^6 + y_4\lambda^4 + y_2\lambda^2 + y_0.$$

$$H = r_5\lambda^5 + r_3\lambda^3 + r_1\lambda,$$

$$x_0y_0 = 0, \text{ and } x_6y_6 = 0. \text{ Take } y_0 = 0.$$

We consider two sub-cases, viz, $y_6 = 0$ (case 6.11)

$$\text{and } x_6 = 0 \text{ (case 6.12).}$$

Case 6.11

$$y_6 = 0.$$

We can write

$$\frac{F}{D} = f_\infty \lambda + f_0\mu + \frac{f_1}{\lambda + \alpha_1\mu},$$

$$\frac{G}{D} = g_\infty \lambda + g_0\mu + \frac{g_1}{\lambda + \alpha_1\mu},$$

$$\frac{H}{D} = h_\infty \lambda + h_0\mu + \frac{h_1}{\lambda + \alpha_1\mu}.$$

By (5.8), we may take $f_1 \geq 0$, $g_1 \geq 0$, so that $f_\infty \lambda + f_0\mu$ and $g_\infty \lambda + g_0\mu$ can be removed to leave a fourth-degree residual (case 4.11).

Case 6.12

$$x_6 = 0.$$

We consider two sub-cases, viz, D does not (case 6.121) or does (case 6.122) contain multiple factors.

Case 6.121

If D has no multiple factors of the form $\lambda + \alpha_i\mu$ ($\alpha_i > 0$), then Theorem 1 and (5.8) are applicable. We have

$$\frac{F}{D} = f_0\mu + \frac{f_1}{\lambda + \alpha_1\mu} + \frac{f_2}{\lambda + \alpha_2\mu},$$

$$\frac{G}{D} = g_0\mu + \frac{g_1}{\lambda + \alpha_1\mu} + \frac{g_2}{\lambda + \alpha_2\mu},$$

$$\frac{H}{D} = h_0\mu + \frac{h_1}{\lambda + a_1\mu} + \frac{h_2}{\lambda + a_2\mu}.$$

We have two sub-cases to consider.

Case 6.121.1

$$\begin{aligned} \text{If} \quad & f_1 \geq 0, \quad f_2 \geq 0, \\ & g_1 \geq 0, \quad g_2 \geq 0, \end{aligned}$$

then $f_0\mu$ and $g_0\mu$ can be removed to leave a fifth-degree *) residual.

Case 6.121.2

Otherwise, we consider

$$\left. \begin{aligned} f_1 > 0, \quad g_1 > 0, \quad h_1 < 0, \\ f_2 < 0, \quad g_2 > 0, \quad h_2 > 0. \end{aligned} \right\} \quad (9.1)$$

We take $a_2 > a_1$. (If $a_1 > a_2$, interchange F and H .) First remove $g_0\mu$. (This does not lower the degree.)

Since F and H have non-negative coefficients we have

$$f_0 + f_1 + f_2 \geq 0, \quad (9.2)$$

$$(a_1 + a_2)f_0 + a_2f_1 + a_1f_2 \geq 0, \quad (9.3)$$

$$h_0 + h_1 + h_2 \geq 0, \quad (9.4)$$

$$(a_1 + a_2)h_0 + a_2h_1 + a_1h_2 \geq 0. \quad (9.5)$$

From Theorem 1 and (9.1) we have

$$\frac{1}{|f_2|} = \frac{1}{g_2} + \frac{1}{h_2}, \quad (9.6)$$

$$\frac{1}{|h_1|} = \frac{1}{f_1} + \frac{1}{g_1}. \quad (9.7)$$

Since $a_2 > a_1$, (9.2) implies (9.3). Consequently, (9.2) to (9.5) are equivalent to

$$f_0 + f_1 \geq \frac{g_2h_2}{g_2 + h_2}, \quad (9.8)$$

$$h_0 + h_2 \geq \frac{f_1g_1}{f_1 + g_1}, \quad (9.9)$$

*) This is most easily seen by calculating the highest powers of λ, μ in F, G, H, D , from which the corresponding powers in C may be deduced.

$$(a_1 + a_2)h_0 + a_1h_2 \geq \frac{a_2f_1g_1}{f_1 + g_1}. \quad (9.10)$$

We have two sub-cases to consider.

Case 6.121.21

If either

$$f_1 \geq |f_2|,$$

or

$$h_2 \geq \frac{a_2}{a_1} |h_1|,$$

we can remove $f_0\mu$ or $h_0\mu$, respectively, since (9.8) is, or (9.9), (9.10) are, then satisfied for $f_0 = 0$ or $h_0 = 0$, respectively. This then leaves a fifth-degree residual.

Case 6.121.22

Otherwise,

$$f_1 < \frac{g_2h_2}{g_2 + h_2} = |f_2|, \quad (9.11)$$

and

$$h_2 < \frac{a_2}{a_1} \frac{f_1g_1}{f_1 + g_1} = \frac{a_2}{a_1} |h_1|. \quad (9.12)$$

From (9.11),

$$\frac{1}{f_1} > \frac{1}{g_2} + \frac{1}{h_2};$$

hence

$$\frac{1}{f_1} > \frac{1}{h_2},$$

$$\frac{1}{f_1} + \frac{1}{g_1} > \frac{1}{h_2},$$

and

$$h_2 > \frac{f_1g_1}{f_1 + g_1} = |h_1|. \quad (9.13)$$

Also, from (9.12),

$$\frac{1}{h_2} > \frac{a_1}{a_2} \left(\frac{1}{f_1} + \frac{1}{g_1} \right);$$

hence

$$\frac{1}{g_2} + \frac{1}{h_2} > \frac{a_1}{a_2} \frac{1}{f_1},$$

i.e.,

$$|f_2| < \frac{\alpha_2}{\alpha_1} f_1. \quad (9.14)$$

Rewriting (9.11) to (9.14) we have

$$\left. \begin{aligned} f_1 &< |f_2| < \frac{\alpha_2}{\alpha_1} f_1, \\ |h_1| &< h_2 < \frac{\alpha_2}{\alpha_1} |h_1|. \end{aligned} \right\} \quad (9.15)$$

We now partition according to the scheme

$$\left. \begin{aligned} \frac{F}{D} &= f_0\mu + \frac{f_{11}}{\lambda + \alpha_1\mu} + \frac{f_{21}}{\lambda + \alpha_2\mu} + \frac{f_{12}}{\lambda + \alpha_1\mu} + \frac{f_{22}}{\lambda + \alpha_2\mu}, \\ \frac{G}{D} &= \frac{g_{11}}{\lambda + \alpha_1\mu} + \frac{g_{21}}{\lambda + \alpha_2\mu} + \frac{g_{12}}{\lambda + \alpha_1\mu} + \frac{g_{22}}{\lambda + \alpha_2\mu}, \\ \frac{H}{D} &= \frac{h_{11}}{\lambda + \alpha_1\mu} + \frac{h_{21}}{\lambda + \alpha_2\mu} + h_0\mu + \frac{h_{12}}{\lambda + \alpha_1\mu} + \frac{h_{22}}{\lambda + \alpha_2\mu}, \end{aligned} \right\} \quad (9.16)$$

where

$$\left. \begin{aligned} \frac{f_{11}}{f_1} &= \frac{g_{11}}{g_1} = \frac{h_{11}}{h_1} = \xi_1, \\ \frac{f_{12}}{f_1} &= \frac{g_{12}}{g_1} = \frac{h_{12}}{h_1} = \eta_1, \end{aligned} \right\} \quad (9.17)$$

$$\xi_1 = \frac{-\alpha_1 h_2 (f_1 + f_2)}{\alpha_2 f_2 h_1 - \alpha_1 f_1 h_2}, \quad (9.18)$$

$$\eta_1 = 1 - \xi_1 = \frac{f_2 (\alpha_2 h_1 + \alpha_1 h_2)}{\alpha_2 f_2 h_1 - \alpha_1 f_1 h_2}, \quad (9.19)$$

$$\left. \begin{aligned} \frac{f_{21}}{f_2} &= \frac{g_{21}}{g_2} = \frac{h_{21}}{h_2} = \xi_2, \\ \frac{f_{22}}{f_2} &= \frac{g_{22}}{g_2} = \frac{h_{22}}{h_2} = \eta_2, \end{aligned} \right\} \quad (9.20)$$

$$\xi_2 = \frac{\alpha_2 |h_1|}{\alpha_1 h_2} \xi_1, \quad \eta_2 = 1 - \xi_2 = \frac{f_1}{|f_2|} \eta_1. \quad (9.21)$$

From (9.15) it follows that $\xi_1, \xi_2, \eta_1, \eta_2$ are positive. The partitioning is in conformity with the conditions of Theorems 2, 3 and 4; consequently it is only necessary to check that the polynomials all have non-negative

coefficients. The details are given in Appendix 1. Since the partitions are of the fifth degree, we conclude that this case is realizable.

It will be noted that when D pole-removal failed we tried partitioning next, thereby introducing more elements than would be necessary if C pole-removal were possible. Unfortunately, it is not possible to obtain simple relations between the D residues and the C residues, so that it is not known under what conditions C and D pole-removal are simultaneously impossible.

Case 6.122

If D has multiple factors of the form $\lambda + \alpha_i \mu$ ($\alpha_i > 0$), then

$$\frac{F}{D} = f_0 \mu + \frac{f_1}{\lambda + \alpha_1 \mu},$$

$$\frac{G}{D} = g_0 \mu + \frac{g_1}{\lambda + \alpha_1 \mu},$$

$$\frac{H}{D} = h_0 \mu + \frac{h_1}{\lambda + \alpha_1 \mu},$$

where, from Theorem 3 and since the system is of the sixth degree, we have

$$g_1 h_1 + h_1 f_1 + f_1 g_1 > 0. \quad (9.22)$$

There are two sub-cases to consider.

Case 6.122.1

If all three residues are non-negative, then the system can be realized as a delta connection of two-pole admittances.

Case 6.122.2

Otherwise, we consider $f_1 < 0$. Then by (5.7), $g_1 > 0$, $h_1 > 0$. By (9.22),

$$\frac{1}{|f_1|} > \frac{1}{g_1} + \frac{1}{h_1}.$$

Let

$$\frac{1}{g'_1} = \frac{1}{|f_1|} - \frac{1}{h_1} > \frac{1}{g_1} > 0.$$

Then

$$g''_1 = g_1 - g'_1 > 0.$$

Thus we can remove $g_1''/(\lambda + \alpha_1\mu)$ from G/D leaving a residual such that

$$g_1'h_1 + h_1f_1 + f_1g_1' = 0.$$

Now $F = (\alpha_1f_0\mu^2 + f_0 + f_1)(\lambda + \alpha_1\mu)$ is a polynomial with non-negative coefficients. It follows that the coefficients of $\alpha_1f_0\mu^2 + f_0 + f_1$ are non-negative. Hence the residual set of polynomials is of the fourth degree and is therefore realizable.

Case 6.2

$$\begin{aligned} F &= p_6\lambda^6 + p_4\lambda^4 + p_2\lambda^2 + p_0, & C &= x_5\lambda^5 + x_3\lambda^3 + x_1\lambda, \\ G &= q_6\lambda^6 + q_4\lambda^4 + q_2\lambda^2 + q_0, & D &= y_5\lambda^5 + y_3\lambda^3 + y_1\lambda. \\ H &= r_6\lambda^6 + r_4\lambda^4 + r_2\lambda^2 + r_0, \\ & q_6r_6 + r_6p_6 + p_6q_6 = 0, \\ & q_0r_0 + r_0p_0 + p_0q_0 = 0. \end{aligned}$$

We take $q_0 = r_0 = 0$. There are two sub-cases to consider, namely, $p_6 = q_6 = 0$ (case 6.21) and $q_6 = r_6 = 0$ (case 6.22).

Case 6.21

$$p_6 = q_6 = 0.$$

We have two sub-cases to consider, viz, D does not (case 6.211) or does (case 6.212) contain multiple factors.

Case 6.211

If D has no multiple factors of the form $\lambda + \alpha_i\mu$ ($\alpha_i > 0$), then Theorem 1 and (5.8) are applicable. We have

$$\begin{aligned} \frac{F}{D} &= f_0\mu + \frac{f_1}{\lambda + \alpha_1\mu} + \frac{f_2}{\lambda + \alpha_2\mu}, \\ \frac{G}{D} &= \frac{g_1}{\lambda + \alpha_1\mu} + \frac{g_2}{\lambda + \alpha_2\mu}, \\ \frac{H}{D} &= h_\infty\lambda + \frac{h_1}{\lambda + \alpha_1\mu} + \frac{h_2}{\lambda + \alpha_2\mu}. \end{aligned}$$

There are two sub-cases to consider.

Case 6.211.1

If either

$$f_1 \geq 0, \quad f_2 \geq 0,$$

or

$$h_1 \geq 0, \quad h_2 \geq 0,$$

then $f_0\mu$ or $h_\infty\lambda$, respectively, can be removed to leave a fifth-degree residual.

Case 6.211.2

Otherwise, we consider

$$\begin{aligned} f_1 &> 0, & g_1 &> 0, & h_1 &< 0, \\ f_2 &< 0, & g_2 &> 0, & h_2 &> 0. \end{aligned}$$

The conditions that F and H should have non-negative coefficients become

$$(a_1 + a_2)f_0 + a_2f_1 + a_1f_2 \geq 0, \quad (9.23)$$

$$f_0 + f_1 + f_2 \geq 0, \quad (9.24)$$

$$(a_1 + a_2)h_\infty + h_1 + h_2 \geq 0, \quad (9.25)$$

$$a_1a_2h_\infty + a_2h_1 + a_1h_2 \geq 0. \quad (9.26)$$

There are two sub-cases to consider, viz, $a_2 > a_1$ (case 6.211.21) and $a_1 > a_2$ (case 6.211.22).

Case 6.211.21

$$a_2 > a_1.$$

Then (9.24) implies (9.23) and (9.26) implies (9.25). Thus it is sufficient to consider only the inequalities

$$\left. \begin{aligned} f_0 + f_1 + f_2 &\geq 0, \\ a_1a_2h_\infty + a_2h_1 + a_1h_2 &\geq 0. \end{aligned} \right\} \quad (9.27)$$

This case is further sub-divided into two.

Case 6.211.211

If either

$$\left. \begin{aligned} f_1 &\geq |f_2|, \\ a_1h_2 &\geq a_2|h_1|, \end{aligned} \right\} \quad (9.28)$$

$f_0\mu$ or $h_\infty\lambda$, respectively, can be removed to leave a fifth-degree residual.

Case 6.211.212

Otherwise,

$$\left. \begin{aligned} |f_2| &> f_1, \\ |h_1| &> \frac{a_1}{a_2} h_2. \end{aligned} \right\} \quad (9.29)$$

We partition according to the scheme

$$\left. \begin{aligned} \frac{F}{D} &= f_0\mu + \frac{f_{11}}{\lambda + a_1\mu} + \frac{f_{21}}{\lambda + a_2\mu} + \frac{f_{12}}{\lambda + a_1\mu} + \frac{f_{22}}{\lambda + a_2\mu}, \\ \frac{G}{D} &= \frac{g_{11}}{\lambda + a_1\mu} + \frac{g_{21}}{\lambda + a_2\mu} + \frac{g_{12}}{\lambda + a_1\mu} + \frac{g_{22}}{\lambda + a_2\mu}, \\ \frac{H}{D} &= \frac{h_{11}}{\lambda + a_1\mu} + \frac{h_{21}}{\lambda + a_2\mu} + h_\infty\lambda + \frac{h_{12}}{\lambda + a_1\mu} + \frac{h_{22}}{\lambda + a_2\mu}, \end{aligned} \right\} \quad (9.30)$$

where f_{ij} , g_{ij} , h_{ij} ($i, j = 1, 2$) are given by (9.17) to (9.21). By (9.29), ξ_1 , ξ_2 , η_1 , η_2 are positive. The partitions are of the fifth degree and the partitioning is in accordance with Theorems 2,3,4. Moreover, the coefficients are all non-negative (for details see Appendix 1); hence this case is realizable.

Case 6.211.22

$$a_1 > a_2.$$

This case is further sub-divided into two.

Case 6.211.221

If either

$$\left. \begin{aligned} a_2f_1 + a_1f_2 &\geq 0, \\ f_1 + f_2 &\geq 0, \end{aligned} \right\}$$

or

$$\left. \begin{aligned} h_1 + h_2 &\geq 0, \\ a_2h_1 + a_1h_2 &\geq 0, \end{aligned} \right\}$$

we can remove $f_0\mu$ or $h_\infty\lambda$, respectively, since (9.23), (9.24) or (9.25), (9.26) are then satisfied for $f_0 = 0$ or $h_\infty = 0$, respectively. This then leaves a fifth-degree residual.

Case 6.211.222

Otherwise,

$$\left. \begin{aligned} f_1 &< \frac{a_1}{a_2}|f_2|, \\ h_2 &< |h_1|. \end{aligned} \right\} \quad (9.31)$$

and

Thus

$$\frac{1}{f_1} > \frac{a_2}{a_1} \left(\frac{1}{g_2} + \frac{1}{h_2} \right).$$

Hence

$$\frac{1}{f_1} + \frac{1}{g_1} > \frac{\alpha_2}{\alpha_1} \frac{1}{h_2},$$

i.e.,

$$h_2 > \frac{\alpha_2}{\alpha_1} |h_1|.$$

Similarly, from (9.31),

$$\frac{1}{h_2} > \frac{1}{|h_1|} = \frac{1}{f_1} + \frac{1}{g_1}.$$

Hence

$$\frac{1}{h_2} + \frac{1}{g_2} > \frac{1}{f_1},$$

i.e.,

$$f_1 > |f_2|.$$

Thus, with (9.31), we have

$$\left. \begin{aligned} |f_2| &< f_1 < \frac{\alpha_1}{\alpha_2} |f_2|, \\ \frac{\alpha_2}{\alpha_1} |h_1| &< h_2 < |h_1|. \end{aligned} \right\} \quad (9.32)$$

We employ the partitioning of (9.30) but this time with different values for the partitioned residues, viz,

$$\left. \begin{aligned} \frac{f_{11}}{f_1} &= \frac{g_{11}}{g_1} = \frac{h_{11}}{h_1} = \varphi_1, \\ \frac{f_{12}}{f_1} &= \frac{g_{12}}{g_1} = \frac{h_{12}}{h_1} = \psi_1, \end{aligned} \right\} \quad (9.33)$$

$$\varphi_1 = \frac{-h_2(a_1 f_2 + a_2 f_1)}{a_1 f_2 h_1 - a_2 f_1 h_2}, \quad (9.34)$$

$$\psi_1 = 1 - \varphi_1 = \frac{a_1 f_2 (h_1 + h_2)}{a_1 f_2 h_1 - a_2 f_1 h_2}, \quad (9.35)$$

$$\left. \begin{aligned} \frac{f_{21}}{f_2} &= \frac{g_{21}}{g_2} = \frac{h_{21}}{h_2} = \varphi_2, \\ \frac{f_{22}}{f_2} &= \frac{g_{22}}{g_2} = \frac{h_{22}}{h_2} = \psi_2, \end{aligned} \right\} \quad (9.36)$$

$$\varphi_2 = \frac{|h_1|}{h_2} \varphi_1, \quad \psi_2 = 1 - \varphi_2 = \frac{a_2 f_1}{a_1 |f_2|} \psi_1, \quad (9.37)$$

By (9.32), $\varphi_1, \varphi_2, \psi_1, \psi_2$ are positive. In Appendix 1 it is proved that the coefficients of the partitions are non-negative. The partitions are of the fifth degree and in accordance with Theorems 2,3 and 4. Hence this case is realizable.

Case 6.212

If D has multiple factors of the form $\lambda + a_i\mu$ ($a_i > 0$) then

$$\frac{F}{D} = f_0\mu + \frac{f_1}{\lambda + a_1\mu},$$

$$\frac{G}{D} = \frac{g_1}{\lambda + a_1\mu},$$

$$\frac{H}{D} = h_\infty\lambda + \frac{h_1}{\lambda + a_1\mu},$$

where

$$g_1h_1 + h_1f_1 + f_1g_1 > 0.$$

This case is then treated in the same way as case 6.122.

Case 6.22

$$q_6 = r_6 = 0.$$

We consider two sub-cases, viz, D does not (case 6.221) or does (case 6.222) contain multiple factors.

Case 6.221

If D has no multiple factors of the form $\lambda + a_i\mu$ ($a_i > 0$), then Theorem 1 and (5.8) are applicable. We have

$$\frac{F}{D} = f_\infty\lambda + f_0\mu + \frac{f_1}{\lambda + a_1\mu} + \frac{f_2}{\lambda + a_2\mu},$$

$$\frac{G}{D} = \frac{g_1}{\lambda + a_1\mu} + \frac{g_2}{\lambda + a_2\mu},$$

$$\frac{H}{D} = \frac{h_1}{\lambda + a_1\mu} + \frac{h_2}{\lambda + a_2\mu}.$$

The conditions that F should have non-negative coefficients are

$$(a_1 + a_2)f_\infty + f_0 + f_1 + f_2 \geq 0, \quad (9.38)$$

$$a_1a_2f_\infty + (a_1 + a_2)f_0 + a_2f_1 + a_1f_2 \geq 0. \quad (9.39)$$

Case 6.221.1

If either

$$(a_1 + a_2)f_\infty + f_1 + f_2 \geq 0, \quad \} \quad (9.40)$$

$$a_1 a_2 f_\infty + a_2 f_1 + a_1 f_2 \geq 0, \quad \} \quad (9.41)$$

or

$$f_0 + f_1 + f_2 \geq 0, \quad \} \quad (9.42)$$

$$(a_1 + a_2)f_0 + a_2 f_1 + a_1 f_2 \geq 0, \quad \} \quad (9.43)$$

then $f_0\mu$ or $f_\infty\lambda$, respectively, can be removed to leave a fifth-degree residual. (If in addition $f_1 \geq 0$, $f_2 \geq 0$ then $f_0\mu + f_\infty\lambda$ can be removed to leave a fourth-degree residual.)

Case 6.221.2

Otherwise, at least one of (9.40), (9.41) and at least one of (9.42), (9.43) are violated. Thus at least one of f_1, f_2 is negative. We have three sub-cases to consider, viz,

$$f_1 < 0, f_2 < 0 \text{ (case 6.221.21),}$$

$$f_1 < 0, f_2 \geq 0, a_2 > a_1 \text{ (case 6.221.22),}$$

$$f_1 < 0, f_2 \geq 0, a_2 < a_1 \text{ (case 6.221.23).}$$

Case 6.221.21

$$f_1 < 0, f_2 < 0.$$

Without loss of generality we can take $a_2 > a_1$.

From (9.40) to (9.43), the conditions that neither $f_\infty\lambda$ nor $f_0\mu$ can be removed are

$$\begin{aligned} a_1 a_2 f_\infty + a_2 f_1 + a_1 f_2 &< 0, \\ f_0 + f_1 + f_2 &< 0, \end{aligned} \quad \} \quad (9.44)$$

since these conditions are implied by (but do not imply)

$$(a_1 + a_2)f_\infty + f_1 + f_2 < 0,$$

$$(a_1 + a_2)f_0 + a_2 f_1 + a_1 f_2 < 0,$$

respectively.

We shall now show that reduction of the degree by partitioning is not always possible. If the system is to be partitioned into two systems of lower degree, the only possibility is

$$\left. \begin{aligned} \frac{F}{D} &= f_{\infty} \lambda + \frac{f_{11}}{\lambda + \alpha_1 \mu} + \frac{f_{21}}{\lambda + \alpha_2 \mu} + f_0 \mu + \frac{f_{12}}{\lambda + \alpha_1 \mu} + \frac{f_{22}}{\lambda + \alpha_2 \mu}, \\ \frac{G}{D} &= \frac{g_{11}}{\lambda + \alpha_1 \mu} + \frac{g_{21}}{\lambda + \alpha_2 \mu} + \frac{g_{12}}{\lambda + \alpha_1 \mu} + \frac{g_{22}}{\lambda + \alpha_2 \mu}, \\ \frac{H}{D} &= \frac{h_{11}}{\lambda + \alpha_1 \mu} + \frac{h_{21}}{\lambda + \alpha_2 \mu} + \frac{h_{12}}{\lambda + \alpha_1 \mu} + \frac{h_{22}}{\lambda + \alpha_2 \mu}. \end{aligned} \right\} \quad (9.45)$$

The conditions that the partitions should have non-negative coefficients are

$$(\alpha_1 + \alpha_2)f_{\infty} + f_{11} + f_{21} \geq 0, \quad (9.46)$$

$$\alpha_1 \alpha_2 f_{\infty} + \alpha_2 f_{11} + \alpha_1 f_{21} \geq 0, \quad (9.47)$$

$$f_0 + f_{12} + f_{22} \geq 0, \quad (9.48)$$

$$(\alpha_1 + \alpha_2)f_0 + \alpha_2 f_{12} + \alpha_1 f_{22} \geq 0. \quad (9.49)$$

Furthermore, Theorem 4 requires that $f_{ij} \leq 0$ ($i, j = 1, 2$). (9.50)

We note that (9.47) implies (9.46) and (9.48) implies (9.49). From (9.47), (9.48) we have

$$\alpha_1 \alpha_2 f_{\infty} + \alpha_2 f_0 + \alpha_2 f_1 + \alpha_1 f_2 + (\alpha_2 - \alpha_1)f_{22} \geq 0.$$

By (9.50), $f_{22} \leq 0$. Hence, since $\alpha_2 > \alpha_1$,

$$\boxed{\alpha_1 \alpha_2 f_{\infty} + \alpha_2 f_0 + \alpha_2 f_1 + \alpha_1 f_2 \geq 0.} \quad (9.51)$$

Also from (9.47), (9.48) we have

$$\alpha_1 \alpha_2 f_{\infty} + \alpha_1 f_0 + \alpha_1 f_2 + \alpha_1 f_1 + (\alpha_2 - \alpha_1)f_{11} \geq 0.$$

By (9.50), $f_{11} \leq 0$. Hence $\alpha_1 \alpha_2 f_{\infty} + \alpha_1 f_0 + \alpha_1 f_2 + \alpha_1 f_1 \geq 0$, i.e.,

$$\boxed{\alpha_2 f_{\infty} + f_0 + f_1 + f_2 \geq 0.} \quad (9.52)$$

Thus (9.51) and (9.52) are necessary for partitioning. We note that they are essentially new conditions, being stronger than (9.39) and (9.38), respectively. To prove that they are sufficient we must show that it is possible to choose f_{11} and f_{21} consistent with (9.47), (9.48) and subject to (9.50) when (9.51) and (9.52) hold. We consider two sub-cases, viz, (9.51) and (9.52) hold (case 6.221.211) and one of (9.51), (9.52) is violated (case 6.211.212).

Case 6.221.211

Conditions (9.51) and (9.52) hold.

By (9.44), *either* it is possible to choose f_{21}, f_{11} such that

$$\left. \begin{aligned} \alpha_2 f_\infty + f_{21} &= 0, \\ f_{11} &= 0, \end{aligned} \right\}$$

so that (9.47) is satisfied, while at the same time it follows from (9.52) that

$$f_0 + f_1 + f_{22} \geq 0,$$

i.e., (9.48) is satisfied;

or it is possible to choose f_{11}, f_{22} such that

$$\left. \begin{aligned} \alpha_1 \alpha_2 f_\infty + \alpha_1 f_2 + \alpha_2 f_{11} &= 0, \\ f_{22} &= 0, \end{aligned} \right\}$$

so that (9.47) is satisfied, while at the same time it follows from (9.51) that

$$\alpha_2 f_0 + \alpha_2 f_{12} \geq 0,$$

i.e., (9.48) is satisfied.

This proves that (9.51) and (9.52) are sufficient for the partitioning of F without inducing negative coefficients. Since G and H have non-negative residues the conditions are sufficient for partitioning the complete set of polynomials.

Case 6.221.212

At least one of conditions (9.51) and (9.52) is violated. Partitioning is now impossible, but what can one say of C pole-removal? In Appendix 2.1 it is proved that C pole-removal at either infinity or zero is impossible. (Partial or complete C or D pole-removal at any other point can lead only to a violation of the Cauchy conditions.)

We now proceed to examine the effects of partial pole-removal. The effect of partial D pole-removal at infinity or zero is simply to reduce f_∞ or f_0 or both, that is, to reduce the left-hand sides of (9.51) and (9.52). It follows that, if partitioning is initially impossible, it will remain so after partial D pole-removal has been carried out, and therefore, in virtue of Appendix 2.1, C pole-removal will be impossible.

We next have to consider the effect of *partial* C pole-removal. Before doing this, however, it will be necessary to consider the other sub-cases of case 6.221.2.

Case 6.221.22

$$f_1 < 0, \quad f_2 \geq 0, \quad \alpha_2 > \alpha_1.$$

We take $g_1 > 0, g_2 \leq 0$, so that $h_1 > 0, h_2 \geq 0$.

For this case also we shall show that partitioning is not always possible.

We note that if (9.40) is violated, then (9.41) is violated. Thus, from (9.41), the condition that $f_0\mu$ cannot be removed is

$$a_1a_2f_\infty + a_2f_1 + a_1f_2 < 0. \quad (9.53)$$

From (9.42), (9.43), the condition that $f_\infty\lambda$ cannot be removed is either (case 6.221.221)

$$\left. \begin{aligned} (a_1 + a_2)f_0 + a_2f_1 + a_1f_2 < 0, \\ f_0 + f_1 + f_2 \geq 0, \end{aligned} \right\}$$

or (case 6.221.222)

$$f_0 + f_1 + f_2 < 0.$$

Case 6.221.221

$$\left. \begin{aligned} (a_1 + a_2)f_0 + a_2f_1 + a_1f_2 < 0, \\ f_0 + f_1 + f_2 \geq 0. \end{aligned} \right\} \quad (9.54)$$

We partition according to the scheme of (9.45) such that

$$\left. \begin{aligned} \frac{f_{11}}{f_1} = \frac{g_{11}}{g_1} = \frac{h_{11}}{h_1} = \frac{f_{21}}{f_2} = \frac{g_{21}}{g_2} = \frac{h_{21}}{h_2} = \zeta, \\ \zeta = \frac{-a_1a_2f_\infty}{a_1f_2 + a_2f_1}. \end{aligned} \right\} \quad (9.55)$$

By (9.53), $\zeta > 0$.

The partitioning is in accordance with Theorem 4, since $1 - \zeta > 0$, by (9.53). The non-negativeness of the coefficients is proved in Appendix 3. Hence this case is realizable.

Case 6.221.222

$$f_0 + f_1 + f_2 < 0. \quad (9.56)$$

From (9.47) and (9.48) we have

$$a_1a_2f_\infty + a_2f_0 + a_2f_1 + a_2f_2 + (a_1 - a_2)f_{21} \geq 0.$$

By Theorem 4,

$$f_{21} \geq 0.$$

Hence

$$a_1a_2f_\infty + a_2f_0 + a_2f_1 + a_2f_2 \geq 0,$$

i.e.,

$$\boxed{a_1f_\infty + f_0 + f_1 + f_2 \geq 0.} \quad (9.57)$$

Thus (9.57) is a necessary condition for partitioning. To prove its sufficiency we employ the partitioning of (9.45), but this time with the values

$$f_{21} = g_{21} = h_{21} = 0; \quad \frac{f_{11}}{f_1} = \frac{g_{11}}{g_1} = \frac{h_{11}}{h_1} = \frac{\alpha_1 f_\infty}{|f_1|}. \quad (9.58)$$

In Appendix 3 it is proved that the partitioning is in accordance with Theorem 4 and that the coefficients are non-negative.

If, however, (9.57) is violated, then C pole-removal at either zero or infinity is impossible (proof in Appendix 2.2). Partial or complete C or D pole-removal at any other point can lead only to a violation of the Cauer conditions. Moreover, the effect of partial D pole-removal at infinity or zero is simply to reduce f_∞ or f_0 or both, so that this operation cannot induce the possibilities of partitioning or C pole-removal if (9.57) is initially violated. We note in passing that (9.57) holds for case 6.221.221 also; it is implied by (9.54).

Case 6.221.23

$$f_1 < 0, \quad f_2 \geq 0, \quad \alpha_2 < \alpha_1.$$

This is not an essentially new case, for, by the transformations

$$\begin{aligned} \lambda &= \frac{1}{\lambda''}, & \alpha_1 &= \frac{1}{\alpha_1''}, & f_1 &= \frac{f_1''}{\alpha_1''}, \\ f_0 &= f_\infty'', & \alpha_2 &= \frac{1}{\alpha_2''}, & f_2 &= \frac{f_2''}{\alpha_2''}, \\ f_\infty &= f_0'', \end{aligned}$$

F/D transforms to

$$f_0'' \mu'' + f_\infty'' \lambda'' + \frac{f_1''}{\lambda'' + \alpha_1'' \mu''} + \frac{f_2''}{\lambda'' + \alpha_2'' \mu''},$$

where $\alpha_2'' > \alpha_1''$, $f_1'' < 0$, $f_2'' \geq 0$.

This is then in the form of case 6.221.22.

Cases 6.221.212, 6.221.22 (continued)

We are now ready to take up the question of partial C pole-removal. We find that if partitioning is initially impossible, then it is also impossible after carrying out partial C pole-removal (the details of the proof may be found in Appendix 4).

Case 6.221.2 (continued)

Thus, to sum up this discussion, we can say

- (i) if $f_1 < 0$, $f_2 < 0$, $\alpha_2 > \alpha_1$, then a series-parallel realization is possible if and only if

$$\left. \begin{aligned} \alpha_1 \alpha_2 f_\infty + \alpha_2 f_0 + \alpha_2 f_1 + \alpha_1 f_2 &\geq 0, \\ \alpha_2 f_\infty + f_0 + f_1 + f_2 &\geq 0; \end{aligned} \right\}$$

(ii) if $f_1 < 0$, $f_2 \geq 0$, $a_2 > a_1$, then a series-parallel realization is possible if and only if

$$a_1 f_\infty + f_0 + f_1 + f_2 \geq 0.$$

These are the extra conditions required for case 6.221.2.

Case 6.222

If D has multiple factors of the form $\lambda + a_i \mu$ ($a_i > 0$), then

$$\frac{F}{D} = f_\infty \lambda + f_0 \mu + \frac{f_1}{\lambda + a_1 \mu},$$

$$\frac{G}{D} = \frac{g_1}{\lambda + a_1 \mu},$$

$$\frac{H}{D} = \frac{h_1}{\lambda + a_1 \mu},$$

where $g_1 h_1 + h_1 f_1 + f_1 g_1 > 0$.

There are two sub-cases to consider.

Case 6.222.1

If $f_1 \geq 0$, then a realization by a delta connection is possible (cf. cases 6.212 and 6.122).

Case 6.222.2

Otherwise,

$$f_1 < 0.$$

We consider two sub-cases.

Case 6.222.21

If the middle coefficient of $F/(\lambda + a_1 \mu)$ is non-negative, i.e., if

$$a_1 f_\infty + f_0 + f_1 \geq 0,$$

then the method of case 6.122.2 can be applied.

Case 6.222.22

Otherwise,

$$a_1 f_\infty + f_0 + f_1 < 0.$$

This case is unrealizable by a series-parallel network. This is most easily seen by considering it as a limiting case of 6.221.21, with $a_2 = a_1$ and $f_1 + f_2$ replaced by f_1 . Then both (9.51) and (9.52) reduce to

$$a_1 f_\infty + f_0 + f_1 \geq 0.$$

Consequently, if this condition is violated neither partitioning nor C pole-removal is possible. The same result is obtained if this case is regarded as a limiting case of 6.221.22.

10. Conclusion

It has been shown that, for polynomials of the sixth or lower degrees, the following conditions are necessary and sufficient for realization by a series-parallel three-pole network:

- (i) Conditions I, II, III, IV of sec. 3;
- (ii) When F/D , G/D , H/D are written in the form

$$f_{\infty} \lambda + f_0 \mu + \frac{f_1}{\lambda + \alpha_1 \mu} + \frac{f_2}{\lambda + \alpha_2 \mu}^*),$$

the conditions

- (a) $\alpha_1 \alpha_2 f_{\infty} + \alpha_2 f_0 + \alpha_2 f_1 + \alpha_1 f_2 \geq 0, \left. \begin{array}{l} \alpha_2 f_{\infty} + f_0 + f_1 + f_2 \geq 0, \end{array} \right\} \text{ if } \alpha_2 \geq \alpha_1, f_1 < 0, f_2 < 0,$
- (b) $\alpha_1 f_{\infty} + f_0 + f_1 + f_2 \geq 0, \text{ if } \alpha_2 \geq \alpha_1, f_1 < 0, f_2 \geq 0,$
- (c) $\alpha_1 \alpha_2 f_{\infty} + \alpha_2 f_0 + \alpha_2 f_1 + \alpha_1 f_2 \geq 0, \text{ if } \alpha_2 \leq \alpha_1, f_1 < 0, f_2 \geq 0.$

The last condition, (c), follows from (b) by applying the transformations given under the heading case 6.221.23. If any of f_{∞} , f_0 , f_1 , f_2 is zero, then conditions (a), (b), (c) are automatically implied by the condition that the coefficients of F , G , H be non-negative.

Two important questions remain unanswered, namely:

- (i) Can the procedure developed here be applied to higher degrees than the sixth?
- (ii) Are conditions (a), (b), (c) necessary for realization by a three-pole network of arbitrary structure?

Concerning the first question, the series-parallel realization of the seventh degree can comparatively easily be made to depend upon that of the sixth degree. However, the eighth degree is much more formidable. The number of possible combinations is very much greater, and the discussion of partitioning for all of them would be very laborious in the absence of a general theorem giving necessary and sufficient conditions for partitioning without introducing negative coefficients. Such a theorem would replace the process adopted here of first partitioning and then checking whether the coefficients are non-negative.

Very little can be said concerning question (ii). To answer this question for even so simple a network as that of fig. 4.6, consisting of only seven elements, would require, with the methods at present available, a long and detailed investigation.

*) Here we permit any of f_{∞} , f_0 , f_1 , f_2 to be zero as well as α_2 and α_1 to be equal.

Appendix 1

The partitions of cases 6.121.22, 6.211.212 and 6.211.222 have non-negative coefficients

For all three cases we note that

$$\left. \begin{aligned} f_{11} + f_{12} &= f_1, \\ f_{21} + f_{22} &= f_2, \end{aligned} \right\} \quad (11.1)$$

and similar expressions hold for g_1, g_2, h_1, h_2 .

Further, from (9.15, (9.29), (9.32) we have that

$$\xi_1, \xi_2, \eta_1, \eta_2, \varphi_1, \varphi_2, \psi_1, \psi_2$$

are all positive.

Since g_1 and g_2 are positive, it follows that the partitions of G have non-negative coefficients in all cases.

Appendix 1.1. Case 6.121.22

Considering F , we need to show that

$$f_0 + f_{11} + f_{21} \geq 0, \quad (11.2)$$

$$(\alpha_1 + \alpha_2)f_0 + \alpha_2 f_{11} + \alpha_1 f_{21} \geq 0, \quad (11.3)$$

$$f_{12} + f_{22} \geq 0, \quad (11.4)$$

$$\alpha_2 f_{12} + \alpha_1 f_{22} \geq 0. \quad (11.5)$$

We have

$$\begin{aligned} f_0 + f_{11} + f_{21} &\geq -f_{12} - f_{22} \quad (\text{by (9.2) and (11.1)}) \\ &= -\eta_1 f_1 - \eta_2 f_2 \quad (\text{by (9.17) and (9.20)}) \\ &= 0 \quad (\text{by (9.21)}). \end{aligned}$$

This proves (11.2) and (11.4); (11.3) follows from (11.2), and (11.5) follows from (11.4), since $\alpha_2 > \alpha_1$.

Next, considering H , we need to show that

$$h_{11} + h_{21} \geq 0, \quad (11.6)$$

$$\alpha_2 h_{11} + \alpha_1 h_{21} \geq 0, \quad (11.7)$$

$$h_0 + h_{12} + h_{22} \geq 0, \quad (11.8)$$

$$(\alpha_1 + \alpha_2)h_0 + \alpha_2 h_{12} + \alpha_1 h_{22} \geq 0. \quad (11.9)$$

We have

$$\begin{aligned} \alpha_2 h_{11} + \alpha_1 h_{21} &= \alpha_2 \xi_1 h_1 + \alpha_1 \xi_2 h_2, \\ &= 0, \end{aligned} \quad (11.10)$$

by (9.21).

This proves (11.7); (11.6) then follows since $\alpha_2 > \alpha_1$.

Also,

$$\begin{aligned}
 h_0 + h_{12} + h_{22} &\geq -\frac{\alpha_2}{\alpha_1 + \alpha_2} h_1 - \frac{\alpha_1}{\alpha_1 + \alpha_2} h_2 + h_{12} + h_{22} && \text{(by (9.5))} \\
 &= \frac{\alpha_1 h_{12} - \alpha_2 h_{11} + \alpha_2 h_{22} - \alpha_1 h_{21}}{\alpha_1 + \alpha_2} && \text{(by (11.1))} \\
 &= \frac{\alpha_1 h_{12} + \alpha_2 h_{22}}{\alpha_1 + \alpha_2} && \text{(by (11.10))} \\
 &= \frac{\eta_1}{\alpha_1 + \alpha_2} \left(\alpha_1 h_1 - \frac{\alpha_2 f_1}{f_2} h_2 \right) && \text{(by (9.20) and (9.21))} \\
 &> 0 && \text{(by (9.15)).}
 \end{aligned}$$

Finally,

$$\begin{aligned}
 (\alpha_1 + \alpha_2) h_0 + \alpha_2 h_{12} + \alpha_1 h_{22} &\geq -\alpha_2 h_{11} - \alpha_1 h_{21} \quad \text{(by (9.5) and (11.1))} \\
 &= 0 \quad \text{(by (11.10)).}
 \end{aligned}$$

Appendix 1.2. Case 6.211.212

By (9.16), (9.30) and Appendix 1.1 it follows that the partitions of F have non-negative coefficients.

Considering H , we need to show that

$$\alpha_2 h_{11} + \alpha_1 h_{21} \geq 0, \quad (11.11)$$

$$\alpha_1 \alpha_2 h_\infty + \alpha_2 h_{12} + \alpha_1 h_{22} \geq 0, \quad (11.12)$$

$$h_{11} + h_{21} \geq 0, \quad (11.13)$$

$$(\alpha_1 + \alpha_2) h_\infty + h_{12} + h_{22} \geq 0. \quad (11.14)$$

Since $\alpha_2 > \alpha_1$, (11.11) implies (11.13) and (11.12) implies (11.14). Now

$$\begin{aligned}
 \alpha_2 h_{11} + \alpha_1 h_{21} &= \alpha_2 \xi_1 h_1 + \alpha_1 \xi_2 h_2 \\
 &= 0 \quad \text{(by (9.21));}
 \end{aligned} \quad (11.15)$$

$$\begin{aligned}
 \alpha_1 \alpha_2 h_\infty + \alpha_2 h_{12} + \alpha_1 h_{22} &\geq -\alpha_2 h_{11} - \alpha_1 h_{21} \quad \text{(by (9.26) and (11.1))} \\
 &= 0 \quad \text{(by (11.15)).}
 \end{aligned}$$

This proves (11.11) and (11.12).

Appendix 1.3. Case 6.211.222

For F we need to show that

$$(\alpha_1 + \alpha_2) f_0 + \alpha_2 f_{11} + \alpha_1 f_{21} \geq 0, \quad (11.16)$$

$$f_0 + f_{11} + f_{21} \geq 0, \quad (11.17)$$

$$\alpha_2 f_{12} + \alpha_1 f_{22} \geq 0, \quad (11.18)$$

$$f_{12} + f_{22} \geq 0. \quad (11.19)$$

We have

$$\begin{aligned} (\alpha_1 + \alpha_2)f_0 + \alpha_2 f_{11} + \alpha_1 f_{21} &\geq -\alpha_2 f_{12} - \alpha_1 f_{22} \text{ (by (9.23), (11.1))} \\ &= -\alpha_2 \psi_1 f_1 - \alpha_1 \psi_2 f_2 \text{ (by (9.33) and (9.36))} \\ &= 0 \text{ (by (9.37)).} \end{aligned} \quad (11.20)$$

This establishes (11.16) and (11.18); (11.19) then follows since $\alpha_1 > \alpha_2$. Furthermore, we have

$$\begin{aligned} f_0 + f_{11} + f_{21} &\geq -\frac{\alpha_1}{\alpha_1 + \alpha_2} f_2 - \frac{\alpha_2}{\alpha_1 + \alpha_2} f_1 + f_{11} + f_{21} \text{ (by (9.23))} \\ &= \frac{\alpha_1 f_{11} - \alpha_2 f_{12} + \alpha_2 f_{21} - \alpha_1 f_{22}}{\alpha_1 + \alpha_2} \text{ (by (11.1))} \\ &= \frac{\alpha_1 f_{11} + \alpha_2 f_{21}}{\alpha_1 + \alpha_2} \text{ (by (11.10))} \\ &= \frac{\varphi_1}{\alpha_1 + \alpha_2} \left(\alpha_1 f_1 - \frac{\alpha_2 h_1}{h_2} f_2 \right) \text{ (by (9.33), (9.36) and (9.37))} \\ &> 0 \text{ (by (9.32)).} \end{aligned}$$

This establishes (11.17).

For H we need to show that

$$h_{11} + h_{21} \geq 0, \quad (11.21)$$

$$\alpha_2 h_{11} + \alpha_1 h_{21} \geq 0, \quad (11.22)$$

$$(\alpha_1 + \alpha_2)h_{\infty} + h_{12} + h_{22} \geq 0, \quad (11.23)$$

$$\alpha_1 \alpha_2 h_{\infty} + \alpha_2 h_{12} + \alpha_1 h_{22} \geq 0. \quad (11.24)$$

We have

$$\begin{aligned} h_{11} + h_{21} &= \varphi_1 h_1 + \varphi_2 h_2, \\ &= 0 \text{ (by (9.37)).} \end{aligned} \quad (11.25)$$

This proves (11.21); (11.22) then follows since $\alpha_1 > \alpha_2$. Furthermore, we have

$$\begin{aligned} (\alpha_1 + \alpha_2)h_{\infty} + h_{12} + h_{22} &\geq -h_{11} - h_{21} \text{ (by (9.25), (11.1))} \\ &= 0 \text{ (by (11.25));} \end{aligned}$$

$$\alpha_1 \alpha_2 h_{\infty} + \alpha_2 h_{12} + \alpha_1 h_{22} \geq \frac{-\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} (h_1 + h_2) + \alpha_2 h_{12} + \alpha_1 h_{22} \quad \text{(by (9.25))}$$

$$\begin{aligned}
 &= \frac{1}{a_1 + a_2} (a_2^2 h_{12} - a_1 a_2 h_{11} + a_1^2 h_{22} - a_1 a_2 h_{21}) \quad (\text{by (11.1)}) \\
 &= \frac{1}{a_1 + a_2} (a_2^2 h_{12} + a_1^2 h_{22}) \quad (\text{by (11.25)}) \\
 &= \frac{a_2 \psi_1}{a_1 + a_2} \left(a_2 h_1 - \frac{a_1 f_1 h_2}{f_2} \right) \quad (\text{by (9.33) and (9.37)}) \\
 &> 0 \quad (\text{by (9.32)}).
 \end{aligned}$$

This proves (11.24); (11.23) then follows since $a_1 > a_2$.

Appendix 2

Conditions (9.51), (9.52) and (9.57) are necessary conditions for C pole-removal at infinity or zero in case 6.221.2.

Appendix 2.1. C pole-removal at either zero or infinity is impossible if either (9.51) or (9.52) is violated (case 6.221.212)

We have

$$\begin{aligned}
 C &= (f_\infty \lambda + f_0 \mu) [(g_1 + h_1) (\lambda + a_2 \mu) + (g_2 + h_2) (\lambda + a_1 \mu)] + \\
 &\quad + f_1 (g_2 + h_2) + f_2 (g_1 + h_1) + g_1 h_2 + g_2 h_1 \\
 &= f_\infty (g_1 + g_2 + h_1 + h_2) \lambda^2 + [f_\infty \{a_2 (g_1 + h_1) + a_1 (g_2 + h_2)\} + \\
 &\quad + f_0 (g_1 + g_2 + h_1 + h_2) + g_1 h_2 + g_2 h_1 + h_1 f_2 + h_2 f_1 + f_1 g_2 + f_2 g_1] + \\
 &\quad + f_0 [a_2 (g_1 + h_1) + a_1 (g_2 + h_2)] \mu^2, \quad (12.1)
 \end{aligned}$$

$$\begin{aligned}
 F &= f_\infty \lambda^3 + [(a_1 + a_2) f_\infty + f_0 + f_1 + f_2] \lambda + [a_1 a_2 f_\infty + \\
 &\quad + (a_1 + a_2) f_0 + a_2 f_1 + a_1 f_2] \mu + a_1 a_2 f_0 \mu^3. \quad (12.2)
 \end{aligned}$$

Thus

$$\begin{aligned}
 \frac{F}{C} &= \frac{\lambda}{g_1 + g_2 + h_1 + h_2} + \\
 &\quad + \frac{1}{C} \left\{ \left[\frac{f_\infty \{a_2 (g_2 + h_2) + a_1 (g_1 + h_1)\} - (h_1 + h_2)(g_1 + g_2)}{g_1 + g_2 + h_1 + h_2} \right] \lambda + \right. \\
 &\quad \left. + \left[a_1 a_2 f_\infty + a_2 f_1 + a_1 f_2 + \frac{f_0 \{a_1 (g_1 + h_1) + a_2 (g_2 + h_2)\}}{g_1 + g_2 + h_1 + h_2} \right] \mu + a_1 a_2 f_0 \mu^3 \right\}. \quad (12.3)
 \end{aligned}$$

Also, we have

$$\begin{aligned}
 \frac{F}{C} &= \frac{a_1 a_2 \mu}{a_2 (g_1 + h_1) + a_1 (g_2 + h_2)} + \\
 &\quad + \frac{1}{C} \left\{ \left[\frac{f_0 \{a_2^2 (g_1 + h_1) + a_1^2 (g_2 + h_2)\} - (a_1 g_2 + a_2 g_1) (a_1 h_2 + a_2 h_1)}{a_2 (g_1 + h_1) + a_1 (g_2 + h_2)} \right] \mu + \right. \\
 &\quad \left. + \left[f_0 + f_1 + f_2 + \frac{f_\infty \{a_1^2 (g_2 + h_2) + a_2^2 (g_1 + h_1)\}}{a_2 (g_1 + h_1) + a_1 (g_2 + h_2)} \right] \lambda + f_\infty \lambda^3 \right\}. \quad (12.4)
 \end{aligned}$$

We now suppose (9.51) is violated, i.e.,

$$\alpha_1 \alpha_2 f_\infty + \alpha_2 f_0 + \alpha_2 f_1 + \alpha_1 f_2 < 0. \quad (12.5)$$

Then, from (12.3), the coefficient of μ in the numerator of

$$F/C - \lambda/(g_1 + g_2 + h_1 + h_2)$$

is

$$[(g_1 + h_1) (\alpha_1 \alpha_2 f_\infty + \alpha_1 f_0 + \alpha_2 f_1 + \alpha_1 f_2) + (g_2 + h_2) (\alpha_1 \alpha_2 f_\infty + \alpha_2 f_0 + \alpha_2 f_1 + \alpha_1 f_2)] < 0, \quad \text{since } \alpha_2 > \alpha_1.$$

Thus (12.5) implies that it is impossible to remove the pole at $\lambda = \infty$ from the function F/C . In abbreviated form we shall say that λ -C pole-removal is impossible. (Similarly, if it is impossible to remove the pole at $\mu = \infty$ from the function F/C we shall say that μ -C pole-removal is impossible.)

Now, if (12.5) and (9.52) both hold, then the elimination of f_∞ from these two inequalities leads to

$$(\alpha_1 - \alpha_2) (f_0 + f_1) > 0,$$

that is,

$$f_0 < |f_1|, \quad \text{since } \alpha_2 > \alpha_1.$$

Hence the coefficient of μ in the numerator of

$$\begin{aligned} & F/C - \alpha_1 \alpha_2 \mu / [\alpha_2 (g_1 + h_1) + \alpha_1 (g_2 + h_2)] \quad \text{is less than} \\ & |f_1| [\alpha_2^2 (g_1 + h_1) + \alpha_1^2 (g_2 + h_2)] - (\alpha_1 g_2 + \alpha_2 g_1) (\alpha_1 h_2 + \alpha_2 h_1) \\ & = \alpha_1^2 (g_2 + h_2) \frac{g_1 h_1}{g_1 + h_1} - \alpha_1^2 g_2 h_2 - \alpha_1 \alpha_2 (g_2 h_1 + g_1 h_2) \\ & = \frac{\alpha_1 (\alpha_1 - \alpha_2) g_1 h_1 (g_2 + h_2)}{g_1 + h_1} - \alpha_1^2 g_2 h_2 - \frac{\alpha_1 \alpha_2 (g_2 h_1^2 + g_1^2 h_2)}{g_1 + h_1} \\ & < 0, \quad \text{since } \alpha_2 > \alpha_1. \end{aligned}$$

On the other hand, if (12.5) holds but (9.52) is violated, i.e., if

$$\alpha_2 f_\infty + f_0 + f_1 + f_2 < 0, \quad (12.6)$$

then the coefficient of λ in the numerator of

$$F/C - \alpha_1 \alpha_2 \mu / [\alpha_2 (g_1 + h_1) + \alpha_1 (g_2 + h_2)] \quad \text{is}$$

$$[\alpha_1 (g_2 + h_2) (\alpha_1 f_\infty + f_0 + f_1 + f_2) + \alpha_2 (g_1 + h_1) (\alpha_2 f_\infty + f_0 + f_1 + f_2)] < 0.$$

Thus, in either case (12.5) implies the impossibility of μ -C pole-removal. At the same time, we note that (12.6) also implies the impossibility of μ -C pole-removal.

Next, if (9.51) and (12.6) both hold, the elimination of f_0 from these inequalities leads to

$$(a_1 - a_2) (a_2 f_\infty + f_2) > 0,$$

that is,

$$a_2 f_\infty < |f_2|.$$

The coefficient of λ in the numerator of $F/C - \lambda/(g_1 + g_2 + h_1 + h_2)$ is then less than

$$\begin{aligned} & |f_2| [g_2 + h_2 + \frac{a_1}{a_2} (g_1 + h_1)] - (h_1 + h_2) (g_1 + g_2) \\ &= \frac{a_1}{a_2} (g_1 + h_1) \frac{g_2 h_2}{g_2 + h_2} - (h_1 g_1 + h_1 g_2 + h_2 g_1) \\ &= \frac{(a_1/a_2 - 1) g_2 h_2 (g_1 + h_1)}{g_2 + h_2} - h_1 g_1 - \frac{h_1 g_2^2 + h_2^2 g_1}{g_2 + h_2} < 0. \end{aligned}$$

Thus (9.51) and (12.6) together imply the impossibility of λ -C pole-removal. Moreover, if (12.6) and (12.5) both hold, λ -C pole-removal is impossible, since this is implied by (12.5).

We conclude that C pole-removal is impossible if at least one of (9.51), (9.52) is violated. That is, (9.51) and (9.52) are necessary conditions for C pole-removal.

Appendix 2.2. C pole-removal at either zero or infinity is impossible if (9.57) is violated (case 6.221.222)

If (9.57) is violated, then

$$a_1 f_\infty + f_0 + f_1 + f_2 < 0. \quad (12.7)$$

We note that (12.7) implies (12.5); hence, λ -C pole-removal is impossible. The coefficient of μ in the numerator of $F/C - a_1 a_2 \mu / [a_2 (g_1 + h_1) + a_1 (g_2 + h_2)]$ is, by (12.4) and (12.7), less than

$$\begin{aligned} & - (f_1 + f_2) [a_2^2 (g_1 + h_1) + a_1^2 (g_2 + h_2)] - (a_2 g_1 + a_1 g_2) (a_2 h_1 + a_1 h_2) \\ &= -a_2^2 f_2 (g_1 + h_1) - a_1^2 f_1 (g_2 + h_2) - a_1 a_2 (g_1 h_2 + g_2 h_1) \\ &= -a_1^2 h_2 \left(f_1 + \frac{a_2}{a_1} g_1 \right) - a_1 a_2 h_1 \left(g_2 + \frac{a_2}{a_1} f_2 \right) - a_2^2 f_2 g_1 - a_1^2 f_1 g_2 < 0, \end{aligned}$$

since

$$\begin{aligned} f_1 &< 0, & g_1 &> 0, & h_1 &> 0, \\ f_2 &\geq 0, & g_2 &\leq 0, & h_2 &\geq 0, \end{aligned} \quad a_2 > a_1.$$

We conclude that (9.57) is a necessary condition for C pole-removal.

Appendix 3

The coefficients of the partitions of F , G in case 6.221.22 are non-negative

Considering F , we observe that (9.47) implies (9.46), so that it is sufficient to prove (9.47), (9.48) and (9.49).

Appendix 3.1. Case 6.221.221

From (9.55) we have

$$a_1 a_2 f_\infty + a_2 f_{11} + a_1 f_{21} = a_1 a_2 f_\infty - \frac{a_1 a_2 f_\infty}{a_1 f_2 + a_2 f_1} (a_2 f_1 + a_1 f_2) = 0.$$

$$f_0 + f_{12} + f_{22} = f_0 + (1-\zeta)(f_1 + f_2) > 0, \quad \text{if } f_1 + f_2 > 0.$$

Otherwise, if $f_1 + f_2 < 0$, then

$$f_0 + f_{12} + f_{22} = f_0 + f_1 + f_2 + \frac{a_1 a_2 f_\infty (f_1 + f_2)}{a_2 f_1 + a_1 f_2} > 0 \quad (\text{by (9.54)}),$$

$$\begin{aligned} (a_1 + a_2)f_0 + a_2 f_{12} + a_1 f_{22} &= (a_1 + a_2)f_0 + \frac{a_1 a_2 f_\infty + a_2 f_1 + a_1 f_2}{a_2 f_1 + a_1 f_2} (a_2 f_1 + a_1 f_2) \\ &= (a_1 + a_2)f_0 + a_1 a_2 f_\infty + a_2 f_1 + a_1 f_2 \geq 0 \quad (\text{by (9.39)}). \end{aligned}$$

This establishes the non-negativeness of the coefficients of the F partitions.

For the G partitions, it is sufficient to note that

$$G_1 = \zeta G, \quad G_2 = (1-\zeta)G,$$

from which it follows that the coefficients of G_1 and G_2 are non-negative.

Appendix 3.2. Case 6.221.222

From (9.53) we have

$$0 < \frac{a_1 f_\infty}{|f_1|} < 1.$$

From (9.58),

$$a_1 a_2 f_\infty + a_2 f_{11} + a_1 f_{21} = 0,$$

$$f_0 + f_{12} + f_{22} = f_0 + f_1 + f_2 + a_1 f_\infty > 0 \quad (\text{by (9.57)}),$$

$$(a_1 + a_2)f_0 + a_2 f_{12} + a_1 f_{22} = (a_1 + a_2)f_0 + a_2 f_1 + a_1 f_2 + a_1 a_2 f_\infty > 0 \quad (\text{by (12.2)}).$$

This establishes the non-negativeness of the coefficients of the F partitions.

For G , it is sufficient to show that $g_{12} + g_{22} \geq 0$, since $a_2 > a_1$.

By the separation conditions and Theorem 4,

$$g_{12} + g_{22} > |f_{12}| - f_{22} = -a_1 f_\infty - f_1 - f_2 \geq 0.$$

Appendix 4

Partial pole-removal in case 6.221.2 is impossible

Under the operation of partial C pole-removal C, G, H remain invariant while F and D take on new values. Denoting these by $F + \delta F$ and $D + \delta D$, we have, from (2.5),

$$\left. \begin{aligned} \delta F(G + H) &= C\delta D. \\ \text{By (12.3), (12.4),} \quad \delta F &= -(\lambda + m\mu)C, \end{aligned} \right\} \quad (13.1)$$

where

$$\left. \begin{aligned} 0 \leq l &< \frac{1}{g_1 + g_2 + h_1 + h_2}, \\ 0 \leq m &< \frac{\alpha_1 \alpha_2}{\alpha_2(g_1 + h_1) + \alpha_1(g_2 + h_2)}. \end{aligned} \right\} \quad (13.2)$$

$$\text{Thus} \quad \delta D = -(\lambda + m\mu)(G + H). \quad (13.3)$$

$$\text{Since} \quad F/D = f_\infty \lambda + f_0 \mu + f_1/(\lambda + \alpha_1 \mu) + f_2/(\lambda + \alpha_2 \mu),$$

$$\left. \begin{aligned} \text{we can put} \quad F &= (f_\infty \lambda + f_0 \mu)D + x\lambda + y\mu, \\ (F + \delta F) &= (f'_\infty \lambda + f'_0 \mu)(D + \delta D) + x'\lambda + y'\mu. \end{aligned} \right\} \quad (13.4)$$

$$\left. \begin{aligned} \text{Also,} \quad G + H &= p\lambda + q\mu, \\ \text{where} \quad p &= g_1 + g_2 + h_1 + h_2, \quad q = \alpha_2(g_1 + h_1) + \alpha_1(g_2 + h_2). \end{aligned} \right\} \quad (13.5)$$

From (13.1), (13.3), (13.4), (13.5),

$$F - (\lambda + m\mu)C = (f'_\infty \lambda + f'_0 \mu)[D - (\lambda + m\mu)(p\lambda + q\mu)] + x'\lambda + y'\mu. \quad (13.6)$$

Comparing the coefficients of λ^3, μ^3 on the two sides of (13.6), by means of (12.1), we find

$$\begin{aligned} f_\infty(1 - lp) &= f'_\infty(1 - lp), \\ f_0(\alpha_1 \alpha_2 - mq) &= f'_0(\alpha_1 \alpha_2 - mq), \end{aligned}$$

$$\text{whence} \quad f'_\infty = f_\infty, \quad f'_0 = f_0, \quad (13.7)$$

$$\text{since } l < \frac{1}{p}, \quad m < \frac{\alpha_1 \alpha_2}{q}.$$

From (13.4), (13.6) and (13.7),

$$x\lambda + y\mu - (\lambda + m\mu)C = x'\lambda + y'\mu - (f_\infty \lambda + f_0 \mu)(p\lambda + q\mu)(\lambda + m\mu).$$

But, from (12.1),

$$C = (f_{\infty} \lambda + f_0 \mu) (p \lambda + q \mu) + \Delta,$$

where

$$\begin{aligned} \Delta &= f_1 g_2 + f_2 g_1 + g_1 h_2 + g_2 h_1 + h_1 f_2 + h_2 f_1 \\ &= \frac{-g_1 h_1}{g_1 + h_1} (g_2 + h_2) - \frac{g_2 h_2}{g_2 + h_2} (g_1 + h_1) + g_1 h_2 + g_2 h_1 \\ &= \frac{(g_1 h_2 - g_2 h_1)^2}{(g_1 + h_1)(g_2 + h_2)} > 0. \end{aligned}$$

Thus

$$\left. \begin{aligned} x' &= x - l \Delta \leq x, \\ y' &= y - m \Delta \leq y. \end{aligned} \right\} \quad (13.8)$$

Next, from (13.3), (13.5), we have

$$D' = (1 - lp) \lambda^2 + (\alpha_1 + \alpha_2 - lq - mp) + (\alpha_1 \alpha_2 - mq) \mu^2. \quad (13.9)$$

Let the factors of D' be $\lambda + \alpha'_1 \mu$, $\lambda + \alpha'_2 \mu$; then, substituting $-\alpha_2/(1 - lp)$ for λ in D' , we obtain

$$\begin{aligned} &\alpha_1 - lq - mp - \frac{1}{\alpha_2} (\alpha_1 \alpha_2 - mq) (1 - lp) \\ &= l(\alpha_1 p - q) - \frac{lmpq}{\alpha_2} + m \left(\frac{q}{\alpha_2} - p \right) < 0, \text{ in virtue of (13.5).} \end{aligned}$$

Hence either $\frac{\alpha_2}{1 - lp} < \alpha'_1$, or $\frac{\alpha_2}{1 - lp} > \alpha'_2$, where $\alpha'_2 > \alpha'_1$.

But α'_2 is a continuous function of l, m in the ranges

$$0 \leq l < \frac{1}{p}, \quad 0 \leq m < \frac{\alpha_1 \alpha_2}{q}.$$

Since for $l = m = 0$, $\alpha'_2 = \alpha_2$, the only possibility is

$$\left. \begin{aligned} &\frac{\alpha_2}{1 - lp} > \alpha'_2. \\ \text{Also,} \quad &\alpha'_1 \alpha'_2 = \frac{\alpha_1 \alpha_2 - mq}{1 - lp} \leq \frac{\alpha_1 \alpha_2}{1 - lp}. \end{aligned} \right\} \quad (13.10)$$

We now consider conditions (9.51), (9.52), (9.57) after partial C pole-removal has been carried out. Denoting the new values of f_1, f_2 by f'_1, f'_2 ,

we have, from (13.4), (12.2) and the fact that the coefficient of λ^2 in $D + \delta D$ is $1-lp$,

$$\left. \begin{aligned} x' &= (f'_1 + f'_2) (1 - lp), \\ y' &= (a'_2 f'_1 + a'_1 f'_2) (1 - lp). \end{aligned} \right\} \quad (13.11)$$

Hence, from (13.4), (13.8), (13.10),

$$\left. \begin{aligned} a'_1 a'_2 f_\infty + a'_2 f_0 + a'_2 f'_1 + a'_1 f'_2 &< \frac{a_1 a_2 f_\infty + a_2 f_0 + a_2 f_1 + a_1 f_2}{1 - lp}, \\ a'_2 f_\infty + f_0 + f'_1 + f'_2 &< \frac{a_2 f_\infty + f_0 + f_1 + f_2}{1 - lp}. \end{aligned} \right\} \quad (13.12)$$

Thus in case 6.221.21, if partitioning is initially impossible, it will remain so after performing partial C pole-removal, provided

$$f'_1 < 0, \quad f'_2 < 0.$$

Since $x' < 0$, at least one of f'_1, f'_2 is negative. We thus have to consider the possibility of 6.221.21 going over to 6.221.22 or to 6.221.23.

In Appendix 5 we prove that

$$[a'_1 f_\infty + f_0 + f'_1 + f'_2 - (a_1 f_\infty + f_0 + f_1 + f_2)] (1 - lp)$$

decreases as l, m increase, so that if (12.7) holds for $l = m = 0$, it will hold for all other l, m . Now (12.6) implies (12.7). Also, if $f'_2 > 0$ for particular values of l, m , then $f'_2 = 0$ for some intermediate, i.e., smaller values of l, m , since f'_2 is a continuous function of l, m . But (12.5) with f_2 zero is equivalent to (12.7) with f_2 zero. Therefore, if either (12.5) or (12.6) holds, and if, as l, m increase, case 6.221.21 is carried over into case 6.221.22, then (12.7) will hold under the new conditions.

Next, we observe that 6.221.22 cannot go over to 6.221.23 without first going over to 6.221.21, since, otherwise, f_1 and f_2 would have to be simultaneously zero for a particular l, m , in contradiction to (12.7). Finally, we observe that (12.7) implies (12.5).

We therefore conclude that if partitioning is initially impossible then no amount of partial C pole-removal can induce conditions under which partitioning becomes possible.

Appendix 5

The function $[(a'_1 - a_1) f_\infty + f'_1 + f'_2 - (f_1 + f_2)] (1 - lp)$ decreases as l, m increase in the intervals

$$0 \leq l < \frac{1}{p}, \quad 0 \leq m < \frac{a_1 a_2}{q}.$$

From (13.9), we have

$$a'_1 = \frac{1}{2(1-lp)} \{a_1 + a_2 - lq - mp - [(a_1 + a_2 - mp - lq)^2 - 4(a_1 a_2 - mq)(1-lp)]^{1/2}\}.$$

$$\text{Let } W = (1-lp) [(a'_1 - a_1)f_\infty + f'_1 + f'_2 - f_1 - f_2].$$

From (13.8), (13.11),

$$\frac{\partial W}{\partial l} = \frac{1}{2} f_\infty \left\{ -q + 2pa_1 + \frac{2q(a_1 + a_2 - mp - lq) - 4p(a_1 a_2 - mq)}{2[(a_1 + a_2 - mp - lq)^2 - 4(a_1 a_2 - mq)(1-lp)]^{1/2}} \right\} + p(f_1 + f_2) - \Delta,$$

$$\frac{\partial W}{\partial m} = \frac{1}{2} f_\infty \left\{ -p + \frac{2p(a_1 + a_2 - mp - lq) - 4q(1-lp)}{2[(a_1 + a_2 - mp - lq)^2 - 4(a_1 a_2 - mq)(1-lp)]^{1/2}} \right\},$$

$$\frac{\partial^2 W}{\partial l \partial m} = \frac{f_\infty}{2X} \{pq[(a_1 + a_2 - mp - lq)^2 - 4(a_1 a_2 - mq)(1-lp)] + [p(a_1 + a_2 - mp - lq) - 2q(1-lp)][q(a_1 + a_2 - mp - lq) - 2p(a_1 a_2 - mq)]\},$$

$$\text{where } X = [(a_1 + a_2 - mp - lq)^2 - 4(a_1 a_2 - mq)(1-lp)]^{3/2} > 0.$$

(By (13.2), (13.5), (13.9) a'_1, a'_2 are positive, so that X is real.)

Thus

$$\frac{\partial^2 W}{\partial l \partial m} = \frac{f_\infty}{X} (a_1 + a_2 - mp - lq) (a_2 p - q) (q - a_1 p) > 0,$$

$$\text{for all } 0 \leq l < 1/p, \quad 0 \leq m < a_1 a_2 / q.$$

Thus $\partial W / \partial l$ increases with m , attaining its maximum at $m = a_1 a_2 / q$, and $\partial W / \partial m$ increases with l , attaining its maximum at $l = 1/p$. Hence $\partial W / \partial l \leq a_1 p f_\infty + p(f_1 + f_2) - \Delta < 0$ (by (12.7)), and $\partial W / \partial m \leq 0$. Hence W decreases as l, m increase.

REFERENCES

- 1) R. M. Foster, Bell Syst. tech. J. **3**, 259-267, 1924.
- 2) W. Cauer, S.B. preuss. Akad. Wiss. (phys.-math. Klasse) **30**, 673-681, 1931.
- 3) O. Brune, J. Math. Phys. **10**, 191-236, 1931.
- 4) C. M. Gewertz, J. Math. Phys. **12**, 1-257, 1933.
- 5) R. Bott and R. J. Duffin, J. appl. Phys. **20**, 816, 1949.
- 6) F. M. Reza, Proc. Symp. Circuit Analysis, Univ. of Illinois, Urbana, Illinois, 7-1 to 7-33, 1955.
- 7) A. Fialkow and I. Gerst, Quart. appl. Math. **10**, 113-127, 1952.
- 8) H. Ozaki, Technol. Rep. Osaka Univ. **3**, 57-77, 1953.
- 9) H. M. Lucal, IRE Trans. Circuit Theory, **CT-2**, 308-316, 1955.
- 10) S. Darlington, IRE Trans. Circuit Theory, **CT-2**, 291-297, 1955.
- 11) B. D. H. Tellegen, Philips Res. Rep. **1**, 169-184, 1946.

Summary

The synthesis of series-parallel LC three-terminal networks is investigated. A set of necessary and sufficient conditions and a method of realization of all sets of series-parallel LC three-terminal-network functions from the zeroth to the sixth degree are given. Some of these conditions are essentially new, that is, independent of any previously derived conditions. They are necessary for the synthesis of series-parallel LC three-terminal networks, but it is not known whether they are also necessary for the synthesis of three-terminal networks of arbitrary structure. In principle, it is possible to apply the method to functions of higher degree than the sixth, but the amount of computation required increases in general very rapidly with the degree.

Résumé

Etude de la synthèse des tripôles série-parallèle composés d'inductances et de capacités. On donne des conditions nécessaires et suffisantes et une méthode de réalisation de tous les systèmes de fonctions des degrés zéro à six qui définissent des tripôles LC série-parallèle. Quelques-unes de ces conditions sont essentiellement nouvelles, c.-à-d. indépendantes de conditions déjà connues. Elles sont nécessaires à la synthèse des tripôles LC série-parallèles, mais on ignore si elles sont nécessaires également à la synthèse des tripôles de structure arbitraire. En principe, il est possible d'appliquer la méthode aux fonctions d'un degré supérieur au sixième, mais en général le calcul devient rapidement plus compliqué à degré croissant.

Zusammenfassung

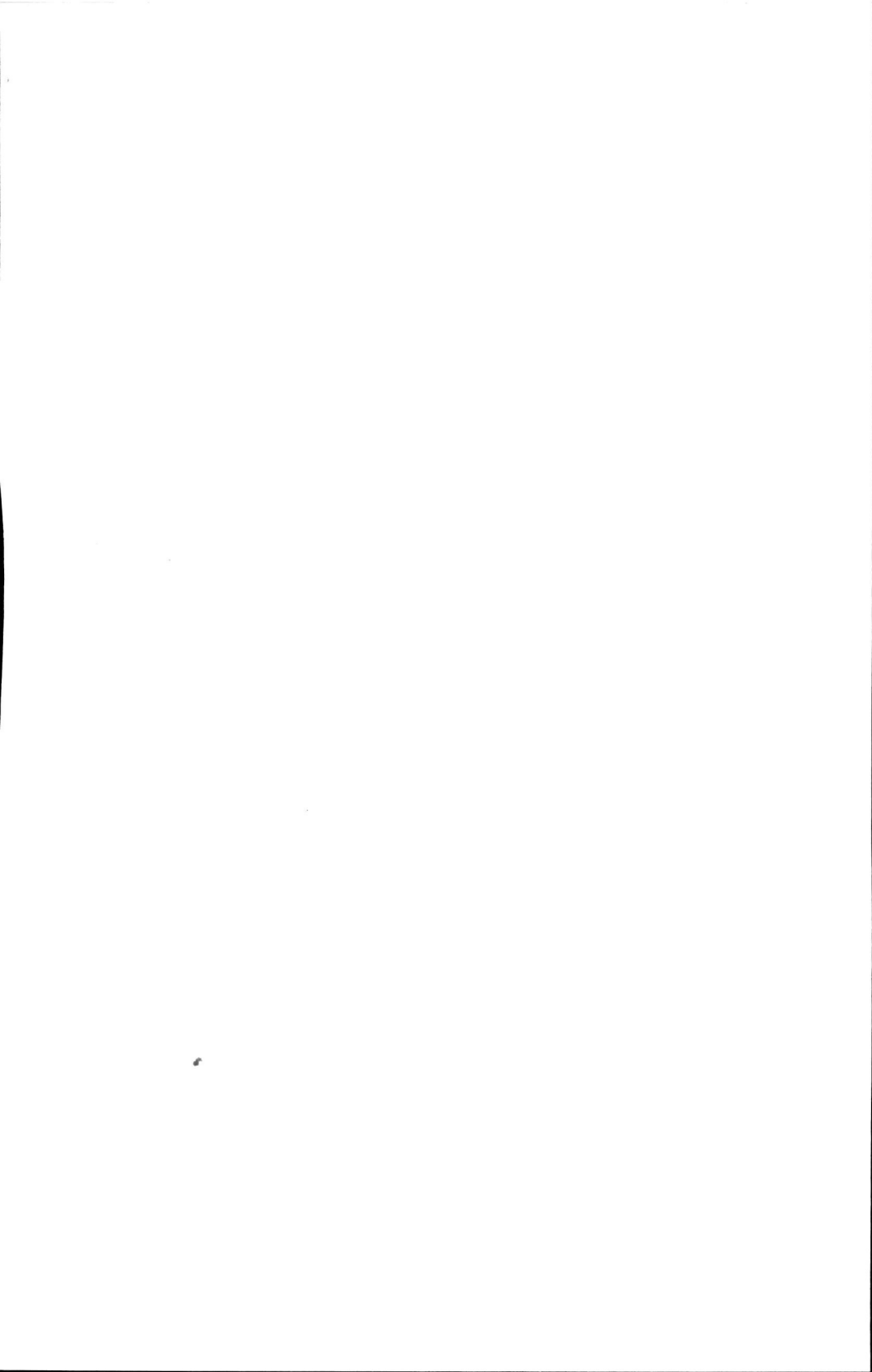
Die Synthese der ausschließlich aus Kapazitäten und Induktivitäten bestehenden Serie-Parallel-Netzwerke mit drei Klemmen wird untersucht. Notwendige und hinreichende Bedingungen und eine Methode zur Verwirklichung aller Funktionensysteme vom nullten bis zum sechsten Grad für diese Dreipole werden angegeben. Einige dieser Bedingungen sind neu und nicht aus bereits bekannten Bedingungen ableitbar. Für die Synthese von Serie-Parallel-LC-Dreipolen sind diese Bedingungen notwendig; es ist jedoch unbekannt, ob sie für die Synthese von Dreipolen willkürlicher Struktur notwendig sind. Die angegebene Methode kann an sich auch bei Funktionen von höherem als dem sechsten Grad angewendet werden, jedoch nimmt das Ausmaß der erforderlichen Berechnungen im allgemeinen mit Erhöhung des Grades ausserordentlich schnell zu.

Samenvatting

De synthese van serie-parallel-dreipolen, uitsluitend opgebouwd uit zelfinducties en capaciteiten, wordt onderzocht. Noodzakelijke en voldoende voorwaarden alsmede een methode voor de realisatie van alle stelsels van functies van de nulde tot de zesde graad, die deze driepolen beschrijven, worden aangegeven. Enige van deze voorwaarden zijn wezenlijk nieuw, d.w.z. onafhankelijk van de tot nu toe bekende voorwaarden. Voor de synthese van serie-parallel-LC-dreipolen zijn deze voorwaarden nodig, doch het is onbekend of zij voor de synthese van driepolen van willekeurige structuur nodig zijn. In principe is het mogelijk de methode op functies van hogere dan de zesde graad toe te passen, echter wordt in het algemeen de vereiste berekening snel ingewikkelder bij verhoging van de graad.

Curriculum Vitae

Keith Meredith Adams was born in 1930 at Auckland, New Zealand. He attended Victoria University College, Wellington, and studied chemistry, physics and mathematics for the degree of Bachelor of Science which he obtained in 1952. In 1953 he obtained the degree of Master of Science in Mathematics and in the same year was awarded an Aeronautical Research Scholarship by the New Zealand Government. From 1953 to 1955 he studied at the College of Aeronautics, Cranfield, England where he specialized in aerodynamics and aircraft electrical engineering. Since 1955 he has been working under Prof. ir B. D. H. Tellegen at the Philips' Research Laboratories, Eindhoven, Netherlands on problems in the field of electrical network synthesis.



STELLINGEN

I

The properties of parallel-T networks given by Smith follow directly from eqs (7.1) et seq. of this thesis.

D. H. Smith, Electron. Eng. 71-77, Feb., 1957.

II

The theorem concerning current distribution in a network, proved by Vratsanos, can be very simply established by means of a general network theorem of Tellegen.

J. Vratsanos, A.E.Ü. 11, 76-80, 1957.

B. D. H. Tellegen, Philips Res. Rep. 7, 259-269, 1952.

III

A low frequency *ideal* gyrator can be realized by a point-contact transistor and a resistance network.

IV

The calculation by Rotow of the signal-to-noise ratio of an image Orthicon is based on erroneous physical assumptions.

A. A. Rotow, IRE Conv. Record, 4, III, Electron devices and receivers, 41-49, 1956.

V

The Binet-Cauchy Theorem of matrix algebra is not employed in engineering research as much as its simplicity and usefulness would justify.

VI

The development of a topological theory of multi-coloured linear graphs would be of great help in the search for general methods of network synthesis.

VII

The excursion into homology theory in Roth's proof of the validity of Kron's method of tearing is not necessary for establishing the main result.

J. P. Roth, Proc. Nat. Acad. Sc. **41**, 518-521, 1955.
41, 599-600, 1955.

VIII

The advent of the Kron process of tearing opens up the possibility of replacing analogue computers by digital computers in guided missile simulation.

IX

The Fialkow and Gerst method of synthesis of transfer functions often introduces extra common factors when partitioning is carried out, thereby leading to unnecessarily many elements in the realization.

A. Fialkow and I. Gerst, Quart. appl. Math. **10**, 113-127, 1952.

X

The various equivalent circuits for anti-symmetrical electro-mechanical transducers, given by Hunt, cannot be regarded as satisfactory representations of the given physical system.

F.V. Hunt, "Electroacoustics", Wiley N.Y., 1954, chap. 3.

XI

The setting up of the Schmidt cameras for tracking the proposed artificial satellite provides an opportunity, which should not be missed, for the accurate observation of unidentified flying objects.

XII

A closer co-operation between the various firms of the aircraft industry of Western Europe would lead to favourable consequences for the industry as a whole.

XIII

The existing custom in New Zealand whereby it is usual for students during their long vacation to engage in some kind of manual work has an important influence upon their general development.

XIV

The system of technical education at present existing in the Soviet Union would lead to undesirable long-term consequences if adopted in Western Europe.