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Asymptotically efficient estimation under local constraint in Wicksell's problem

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ABSTRACT

We consider nonparametric estimation of the distribution function F of squared sphere radii in the classical Wicksell problem. Under smoothness conditions on F in a neighborhood of x, in Gili et al. (2024) it is shown that the Isotonic Inverse Estimator (IIE) is asymptotically efficient and attains rate of convergence $\sqrt{n/\log n}$. If F is constant on an interval containing x, the optimal rate of convergence increases to $\sqrt{n}$ and the IIE attains this rate adaptively, i.e. without explicitly using the knowledge of local constancy. However, in this case, the asymptotic distribution is not normal. In this paper, we introduce three *informed* projection-type estimators of F, which use knowledge on the interval of constancy and show these are all asymptotically equivalent and normal. Furthermore, we establish a local asymptotic minimax lower bound in this setting, proving that the three *informed* estimators are asymptotically efficient and a convolution result showing that the IIE is not efficient. We also derive the asymptotic distribution of the difference of the IIE with the efficient estimators, demonstrating that the IIE is *not* asymptotically equivalent to the *informed* estimators. Through a simulation study, we provide evidence that the performance of the IIE closely resembles that of its competitors, supporting the use of the IIE as the standard choice when no information about F is available.

1. Introduction

In the field of stereology, scientists study the three-dimensional properties of materials and objects by interpreting their two-dimensional cross-sections. This may allow to estimate three-dimensional quantities without the use of expensive 3D reconstructions. In the Wicksell problem, a number of spheres are embedded in an opaque three-dimensional medium. Because the medium is opaque, we are not able to observe the spheres directly. However, we can observe a cross-section of the medium, which shows the circular sections of the spheres that happen to be cut by the plane. It is assumed that the spheres' squared radii are realizations from a cumulative distribution function (cdf) F, the object to be estimated. Following the same notation as in Groeneboom and Jongbloed (1995) and Gili et al. (2024), a version of the density g of the observable squared circle radii Z is given by (cf. Watson (1971)):

$$g(z) = \frac{1}{2m_0}\int_z^\infty \frac{dF(s)}{\sqrt{s-z}}, \tag{1.1}$$

where $0 < m_0 = \int_0^\infty \sqrt{s}\,dF(s) < \infty$ is the expected sphere radius under F. Wicksell (Wicksell, 1925) inverted this equation, by recognizing an Abel-type integral, and found an expression for F in terms of g, given by:

$$F(x) = 1 - \frac{\int_x^\infty (z-x)^{-1/2} g(z)\,dz}{\int_0^\infty z^{-1/2} g(z)\,dz} = 1 - \frac{V(x)}{V(0)}, \quad x > 0, \tag{1.2}$$

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where

$$V(x) := \int_x^\infty \frac{g(z)}{\sqrt{z-x}}\, dz. \tag{1.3}$$

(Furthermore $F(x) = 0$ for all $x \le 0$). Therefore, in order to estimate F at a point $x > 0$, the essential object to be estimated is the function V at x and at 0. For a more detailed introduction to Wicksell's problem, we refer the reader to Groeneboom and Jongbloed (1995), Stoyan and Kendall (1987) and Gili et al. (2024), where the asymptotic analysis of the reconstruction of F from V is also illustrated.

In this paper, we focus on the case in which the underlying cdf F (and therefore V) is known to be constant on an interval of positive length. We conduct the analysis on a single interval $[\underline{x}, \bar{x}]$, for $\underline{x} < \bar{x}$, of constancy. The same procedures can be applied if the underlying cdf F is constant on multiple intervals. Note that even if F is constant on $[\underline{x}, \bar{x}]$, we do not observe any *gap* in the data. Indeed, the density g in (1.1) is positive throughout the set $\{x : F(x) < 1\}$. In this setting, we compare an estimator (the "Isotonic Inverse Estimator" IIE, cf. Groeneboom and Jongbloed (1995) and Gili et al. (2024) for a detailed introduction) that attains the $\sqrt{n}$-rate of convergence adaptively, *without* using the information that V is constant on $[\underline{x}, \bar{x}]$, to three other *informed* estimators that *do use* this information and are therefore constrained to be constant on this interval.

In Section 3, we then show that all the informed estimators are asymptotically equivalent and normal, but not asymptotically equivalent to the IIE. Next in Section 4, we derive a lower bound for the local asymptotic minimax risk of any estimator sequence in this setting, and show that this is attained by the informed estimators. In contrast, the IIE is inefficient. In fact, we show that its (non Gaussian) limit distribution is the convolution of the limit distribution of the informed estimators and a nondegenerate factor. However, the difference between the estimators is small, as we illustrate in a simulation study, in Section 5.

A summary of the findings is that in those rare cases, in which it is known that the distribution function F is constant on one or more intervals, any of the informed estimators should be used for estimation on such intervals, resulting in efficient normal asymptotic behavior at $\sqrt{n}$-rate. Whenever this information is not available, it is preferable to use the isotonic estimator, which behaves closely enough to the informed estimators.

Motivation, connections with the literature and conclusions.

There is a vast literature on Wicksell's problem, see for instance (Groeneboom and Jongbloed, 1995, 2014; Jongbloed, 2001; Stoyan and Kendall, 1987; Golubev and Levit, 1998; Watson, 1971; Hall and Smith, 1988; Golubev and Enikeeva, 2001; Chan and Qin, 2016; Sen and Woodroofe, 2012; Deng et al., 2021; Gili et al., 2024). Wicksell's problem has present-day applications as the estimation of the distribution of stars in a galactic cluster (cf. Sen and Woodroofe (2012)) or the estimation of the 3D microstructure of materials (cf. Lopez-Sanchez and LLana-Fúnez (2016) and Cuzzi and Olson (2017)). In applications, it is common to see the use of the so-called Saltykov methods, based on numerical discretization. These methods are far from being efficient and in some cases not even consistent. The estimation of F given data from g is of interest not only for stereological procedures, but also from the mathematical point of view, both because of the unusual rates of convergence and because of the non-standard efficiency theory.

For nonparametric estimation the best attainable rate of convergence for estimation of the cdf F at a point is $\sqrt{n/\log n}$. Many authors contributed to this problem (Groeneboom and Jongbloed, 1995; Jongbloed, 2001; Golubev and Levit, 1998; Watson, 1971; Hall and Smith, 1988; Golubev and Enikeeva, 2001; Chan and Qin, 2016; Sen and Woodroofe, 2012; Deng et al., 2021; Gili et al., 2024), introducing various estimators. We believe the state-of-the-art is the Isotonic Inverse Estimator, introduced in Groeneboom and Jongbloed (1995). (This regularizes the naive plug-in estimator obtained by replacing $g(z)\,dz$ in (1.3) by the empirical distribution of the inverse radii.) As proved in Gili et al. (2024), this estimator maintains the rate of convergence $\sqrt{n/\log n}$ across a range of smoothness conditions on F, with varying asymptotic variance, and in this sense automatically adapts to smoothness, without needing the selection of a bandwidth parameter.

The scope of this paper is to understand the theoretical limits of estimation when information of the constancy of F on $[\underline{x}, \bar{x}]$ is available, with the final objective of gaining a better understanding of the IIE, which does not use this information. As shown in the paper, the situation with known $[\underline{x}, \bar{x}]$ is particularly favorable, because it is characterized by normal limits, which makes the classical efficiency theory applicable (in contrast with the limit attained by the IIE which is non-normal, making even the definition of efficiency unclear). In this sense, the informed estimators serve as theoretical benchmarks. This is in the spirit of the *oracle* estimation theory. We show that they are asymptotically efficient and that the IIE is not. However, our simulation study presents a typical example in which the IIE performs close to the efficient estimators, suggesting that in practice, the loss incurred by using the IIE should be small. Therefore, except in the rare cases in which $\underline{x}$ and $\bar{x}$ are known (e.g. it is known F is the c.d.f. of a discrete random variable with known support points), we do not envision a direct practical utility for these estimators. In all other cases, precisely in view of this paper, we would recommend practitioners to use the IIE, as it behaves close enough to the *oracle* informed estimators.

2. Construction of the estimators

In this section, we introduce three projection-type *informed* estimators (2.7)–(2.9). We provide explicit constructions and prove that they solve determined minimization problems.

In all that follows, we assume that V is supported on $[0, M]$, so $V(y) = 0$ for all $y \in [M, \infty)$. Relation (1.3) suggests a natural naive (empirical plug-in) estimator for V, which For $\mathbb{G}_n$ the cdf of a sample $Z_1, \dots, Z_n$ from the density g, let:

$$V_n(x) := \int_x^\infty \frac{d\mathbb{G}_n(z)}{\sqrt{z-x}}, \tag{2.1}$$

Using the "naive estimator", we construct all the other estimators as repeated projections of V_n into designated spaces. Let $\underline{x} < \bar{x}$, $V_n(x) = 0$ for $x > M$, and a fixed partition of $[0, M]$ into intervals. Let $\mathscr{P} = \{[x_i, x_{i+1}) \;:\; i = 1, \ldots, I\}$ be such that $0 = x_0 < x_1 < x_2 < \cdots < x_I = M$, and $[\underline{x}, \bar{x}] = [x_i, x_{i+1}]$ for some i. Define the spaces:

$$\mathcal{V} := \left\{ V : [0, \infty) \mapsto [0, \infty) \;:\; V \;\text{ decreasing, right-cont.} \right\}, \tag{2.2}$$

$$\overline{\mathcal{V}}_{\underline{x},\bar{x}} := \left\{ V : [0, \infty) \mapsto [0, \infty) \;:\; V \text{ right-cont., piecewise-const. on } \mathscr{P} \right\}, \tag{2.3}$$

$$\mathcal{V}_{\underline{x},\bar{x}} := \left\{ V \in \mathcal{V} \;:\; V \text{ constant on } [\underline{x}, \bar{x}] \right\}. \tag{2.4}$$

All the above-defined spaces are convex cones. We define projections relative to the discrepancy measure $Q^f : \mathbb{L}_2[0, \infty) \mapsto \mathbb{R}$, for a fixed function $f \in \mathbb{L}_1[0, \infty)$:

$$h \mapsto Q^f(h) := \int_0^\infty h(s)\,(h(s) - 2f(s))\; ds \tag{2.5}$$

Since $\|f - h\|_2^2 = Q^f(h) + \|f\|_2^2$, minimizing $h \to Q^f(h)$ is equivalent to minimizing the $\mathbb{L}_2$-distance to f in the case that f is square-integrable. We use the discrepancy Q^f, because in our context the naive estimator V_n is *not* square-integrable. We consider as estimators the projections of the naive estimator V_n onto the three spaces (2.2)–(2.4) and its repeated projection onto $\mathcal{V}$ and next on $\mathcal{V}_{\underline{x},\bar{x}}$:

$$\hat{V}_n = \underset{V\in\mathcal{V}}{\operatorname{argmin}}\, Q^{V_n}(V), \tag{2.6}$$

$$\hat{V}_n^{\Pi} = \underset{V\in\mathcal{V}_{\underline{x},\bar{x}}}{\operatorname{argmin}}\, Q^{\hat{V}_n}(V). \tag{2.7}$$

$$V_n^{\mathscr{P}} = \underset{V\in\overline{\mathcal{V}}_{\underline{x},\bar{x}}}{\operatorname{argmin}}\, Q^{V_n}(V), \tag{2.8}$$

$$V_n^{\Pi} = \underset{V\in\mathcal{V}_{\underline{x},\bar{x}}}{\operatorname{argmin}}\, Q^{V_n}(V), \tag{2.9}$$

(We note that projecting on a cone and then projecting again onto a subcone does not need to give the same result as projecting directly onto the subcone). Existence and uniqueness of the solutions of these minimization problems can be shown along the lines of Theorem 1.2.1 in Robertson et al. (1988). The first estimator $\hat{V}_n$ is the Isotonic Inverse Estimator (IIE) introduced in Groeneboom and Jongbloed (1995). The other three estimators are informed estimators that take the local constancy of F into account.

The following proposition gives explicit constructions of the estimators (2.6)–(2.9) (the first statement of the following proposition has been proved in Groeneboom and Jongbloed (1995)). For a continuous function k on $[0, M]$, the Least Concave Majorant (LCM) and its right-hand side derivative, are given by:

$$k^*(x) := \min\{f(x) \;:\; f(y) \ge k(y) \text{ for } y \in [0, M],\; f \text{ concave}\}, \quad x \in [0, M]$$

$$(k^*)'(x) := \inf_{u<x} \sup_{v\ge x} \frac{k(v) - k(u)}{v - u}.$$

By definition, $k^*(x) \ge k(x)$ for all $x \in [0, M]$, and $k^*(0) = k(0)$, $k^*(M) = k(M)$. Define functions U_n and U as

$$U_n(x) = \int_0^x V_n(y)\, dy = 2\int_0^\infty \sqrt{z}\, d\mathbb{G}_n(z) - 2\int_x^\infty \sqrt{z - x}\, d\mathbb{G}_n(z), \tag{2.10}$$

$$U(x) = \int_0^x V(y)\, dy = \frac{\pi}{2m_0} \int_0^x (1 - F(y))\, dy. \tag{2.11}$$

The last equality follows from (1.2).

Proposition 1 (*Construction of estimators*). *Let U_n be as in (2.10), $\underline{x} < \bar{x}$ with $[\underline{x}, \bar{x}) \in \mathscr{P}$ and $x \in [0, \infty)$. The solutions of the minimizations (2.6)–(2.9) are:*

$$\hat{V}_n(x) = (U_n^*)'(x). \tag{2.12}$$

$$\hat{V}_n^{\Pi}(x) = \begin{cases} \frac{U_n^*(\bar{x}) - U_n^*(\underline{x})}{\bar{x} - \underline{x}}, & x \in [\underline{x}, \bar{x}], \\ \hat{V}_n(x), & x \notin [\underline{x}, \bar{x}]. \end{cases} \tag{2.13}$$

$$V_n^{\mathscr{P}}(x) = V_n^{(x_i, x_{i+1})}(x), \quad x \in [x_i, x_{i+1}), \quad i = 1, 2, \ldots, I, \tag{2.14}$$

where for $x \in [x_i, x_{i+1}) \in \mathscr{P}$:

$$V_n^{(x_i, x_{i+1})}(x) = \frac{U_n(x_{i+1}) - U_n(x_i)}{x_{i+1} - x_i}, \tag{2.15}$$

$$V_n^{\Pi}(x) = (\widetilde{U}_n^*)'(x), \tag{2.16}$$

where:

$$\widetilde{U}_n(t) := \begin{cases} \frac{U_n(\bar{x})-U_n(\underline{x})}{\bar{x}-\underline{x}}(t-\underline{x}) + U_n(\underline{x}), & t \in [\underline{x}, \bar{x}], \\ U_n(t), & t \notin [\underline{x}, \bar{x}]. \end{cases} \tag{2.17}$$

The proof of Proposition 1 is given in Appendix A and the asymptotic properties of the constructed estimators are given in Theorem 1 below.

Remark 1. For any $v \in [\underline{x}, \bar{x}]$ and $x \in [\underline{x}, \bar{x}]$ with $v \neq x$, we have:

$$U(v) - U(x) - V(x)(v - x) = 0,$$

and thus by evaluating the above relation at $v = \bar{x}$ and $x = \underline{x}$ we get:

$$U(\bar{x}) - U(\underline{x}) - V(x)(\bar{x} - \underline{x}) = 0. \tag{2.18}$$

This gives an intuition for (2.15) and explains why $V_n^{(\underline{x},\bar{x})}(x)$, i.e. $V_n^{\mathscr{P}}(x)$ for $x \in [\underline{x}, \bar{x})$, can be called "empirical slope" on $[\underline{x}, \bar{x})$. It what follows we use the notation $V_n^{(\underline{x},\bar{x})}(x)$ to indicate $V_n^{\mathscr{P}}(x)$ for $x \in [\underline{x}, \bar{x})$.

The isotonic estimator in (2.6) can be implemented using (2.12), where the least concave majorant of a function can be computed by classical algorithms like the PAVA algorithm (cf. Groeneboom and Jongbloed (2014)). Similarly, the solutions to (2.7)–(2.9) can be easily implemented using (2.13), (2.15) and (2.16). Alternatively, the solution to the minimization problem (2.9) can be computed algorithmically using a *profile procedure* as follows. First, fix $a > 0$ and compute the minimizer of $Q^{V_n}(V)$ over the space of decreasing, right-continuous functions V that have value a on $[\underline{x}, \bar{x}]$, and call such minimizer V_n^a. Next we optimize the mapping $a \mapsto Q^{V_n}(V_n^a)$ over $a > 0$ and obtain the desired projection. We give a detailed version of this algorithm at the end of the Appendix.

3. Asymptotic distributions of the estimators

In Gili et al. (2024) (cf. Theorem 2.2), it was shown that if $V \in \mathcal{V}_{\underline{x},\bar{x}}$ where $x \in (\underline{x}, \bar{x})$, the biggest interval that contains x on which V is constant (and assuming (3.3) below), then as $n \to \infty$:

$$\sqrt{n}\left(\hat{V}_n(x) - V(x)\right) \rightsquigarrow L_x, \tag{3.1}$$

where, for any $a \in \mathbb{R}$: $\mathbb{P}(L_x \leq a) = \mathbb{P}(\text{argmax}_{s\in[\underline{x},\bar{x}]}\left\{\mathbb{Z}_x(s) - as\right\} \leq x)$, for $(x,t) \mapsto \mathbb{Z}_x(t)$ a centered continuous Gaussian Process with covariance structure given by:

$$\text{Cov}\left(\mathbb{Z}_y(t), \mathbb{Z}_x(s)\right) = 4\,\text{Cov}\left(\sqrt{(Z_y)_+} - \sqrt{(Z_t)_+}, \sqrt{(Z_x)_+} - \sqrt{(Z_s)_+}\right). \tag{3.2}$$

where $(\cdot)_+ = \max\{0, \cdot\}$ and $Z_u := Z - u$, $Z \sim g$, $u \in \mathbb{R}$. In the present section, we use the characterizations of the *informed* estimators to obtain the first main result of this paper that puts the different estimators in perspective. Assume that:

$$\int_0^\infty s^{\frac{3}{2}}\, dF(s) < \infty. \tag{3.3}$$

Theorem 1 (*Asymptotics*)**.** *Let F be the distribution function of the squared sphere radii and g the corresponding density of the squared circle radii Z according to (1.1). Let $x \geq 0$, and $K := [\underline{x}, \bar{x}]$, for $\underline{x} < \bar{x}$, the biggest interval that contains x on which F is constant and let (3.3) hold true. Then, as $n \to \infty$:*

a. *The estimators $V_n^{(\underline{x},\bar{x})}$, V_n^{Π} and $\hat{V}_n^{\Pi}$ are all asymptotically equivalent:*

$$\sqrt{n}\left(V_n^{\Pi}(x) - V_n^{(\underline{x},\bar{x})}(x)\right) = o_p(1), \qquad \sqrt{n}\left(\hat{V}_n^{\Pi}(x) - V_n^{(\underline{x},\bar{x})}(x)\right) = o_p(1).$$

Furthermore, they all attain the same limiting distribution, given by:

$$\sqrt{n}\left(V_n^{(\underline{x},\bar{x})}(x) - V(x)\right) \rightsquigarrow N(0, \sigma^2_{\underline{x},\bar{x}})$$

where:

$$\sigma^2_{\underline{x},\bar{x}} = \text{Var}\left(2\,(\bar{x} - \underline{x})^{-1}\left\{\sqrt{(Z_{\underline{x}})_+} - \sqrt{(Z_{\bar{x}})_+}\right\}\right). \tag{3.4}$$

b. *The isotonic estimator $\hat{V}_n$ is **not** asymptotically equivalent to $V_n^{(\underline{x},\bar{x})}$, as:*

$$\sqrt{n}\left(\hat{V}_n(x) - V_n^{(\underline{x},\bar{x})}(x)\right) \rightsquigarrow W, \tag{3.5}$$

where, for any $a \in \mathbb{R}$, $\mathbb{P}(W \leq a) = \mathbb{P}\left(\text{argmax}_{s\in K}\{\mathbb{Z}(s) - as\} \leq x\right)$, for

$$\mathbb{Z}(t) = \frac{\bar{x} - t}{\bar{x} - \underline{x}}\mathbb{Z}_{\underline{x}}(t) + \frac{t - \underline{x}}{\bar{x} - \underline{x}}\mathbb{Z}_{\bar{x}}(t), \tag{3.6}$$

and $(t,x) \mapsto \mathbb{Z}_x(t)$ the centered Gaussian Process with covariance structure (3.2).

c. *Under the assumptions of Proposition 2, the sequences*

$$\sqrt{n}\left(V_n^{(\underline{x},\bar{x})}(x) - V(x)\right) \text{ and } \sqrt{n}\left(\hat{V}_n(x) - V_n^{(\underline{x},\bar{x})}(x)\right),$$

are asymptotically independent.

The proofs of Theorems 1.a, 1.b and 1.c are given in Appendix A. Combining 1.a, 1.b and 1.c of the Theorem, we see that the limit L_x of the isotonic estimator in (3.1) is the convolution of the normal limit $N(0, \sigma^2_{\underline{x},\bar{x}})$ of the informed estimators and the distribution of W in (3.5). Thus the limit distribution of the isotonic estimator is less concentrated than the limit distribution of the informed estimators.

4. Lower bound for the local asymptotic minimax risk

In this section, we show that the estimator (2.8) is asymptotically efficient, in the sense that its asymptotic variance is the smallest attainable. This proves that also (2.7) and (2.9), which are asymptotically equivalent to (2.8), are efficient. The claim is valid in the sense of both the local asymptotic minimax theorem and the convolution theorem (c.f. Chapter 25 of van der Vaart (1998)). The key is a LAN expansion for a submodel constructed by perturbing the true function V in the least favorable direction.

We consider the estimation of V at a point x in $(\underline{x}, \bar{x})$, knowing that F, and thus V, are constant on this interval (cf. (B.1)). The associated version of the density of the observations in Wicksell's problem is expressed in V as:

$$g_V(z) = -\int_z^M \frac{dV(s)}{\pi\sqrt{s-z}} \quad \left(= \frac{1}{2m_0}\int_z^M \frac{dF(s)}{\sqrt{s-z}}\right). \tag{4.1}$$

We denote the corresponding probability measure by G_V. We describe a so called "least favorable" submodel in terms of V. We assume that the true parameter V satisfies the following assumption.

Assumption 1. Let $V \in \mathcal{V}$ be supported on $[0, M]$, constant on $[\underline{x}, \bar{x}]$, for $\underline{x} < \bar{x}$ and possess Lipschitz continuous density v on $[\underline{x}, \bar{x}]^c$. Moreover $\exists\, \eta > 0$ such that:

$$\int_{[\underline{x},\bar{x}]^c} \frac{1}{|v(s)|}\,ds + \int_{\underline{x}-\eta}^{\underline{x}} \frac{\log^2(\underline{x}-s)}{|v(s)|}\,ds + \int_{\bar{x}}^{\bar{x}+\eta} \frac{\log^2(s-\bar{x})}{|v(s)|}\,ds < \infty. \tag{4.2}$$

Define functions $k^{(\underline{x},\bar{x})}$, h and h_0 by, for $\underline{x} < \bar{x}$:

$$k^{(\underline{x},\bar{x})}(z) := 2\frac{\sqrt{(z-\underline{x})_+} - \sqrt{(z-\bar{x})_+}}{\bar{x}-\underline{x}},$$

$$h(z) := \begin{cases} -\frac{1}{\sqrt{\pi}v(z)}\int_z^M \frac{(k^{(\underline{x},\bar{x})}g_V)'(s)}{\sqrt{s-z}}\,ds, & z \notin [\underline{x},\bar{x}],\\ 0, & z \in [\underline{x},\bar{x}], \end{cases} \tag{4.3}$$

$$h_0(z) := h(z) + \frac{2}{\pi}\int_0^M \sqrt{s}h(s)\,dV(s). \tag{4.4}$$

The *least favorable submodel* is defined by:

$$dV_t(s) = \frac{1}{c_t}(1 + th_0(s))_+\,dV(s), \quad c_t := -\frac{2}{\pi}\int_0^M \sqrt{s}(1+th_0(s))_+\,dV(s). \tag{4.5}$$

Note that $\int_0^\infty \sqrt{s}\,dV_t(s) = -\frac{\pi}{2}$, that $c_t \geq 0$ and that the obtained V_t is decreasing. In Appendix B we show that the score function of the model $t \mapsto G_{V_t}$ at $t = 0$ is the centered function $k^{(\underline{x},\bar{x})}$, the influence function of the informed estimators. Assumption 1 ensures that indeed h_0 is mapped to $k^{(\underline{x},\bar{x})}$ via the score operator and that $-\frac{2}{\pi}\int h_0^2(s)\sqrt{s}\,dV(s) < \infty$. A detailed explanation is contained in the study of the tangent spaces in the context of our problem in Appendix B.

The next result shows the model $t \mapsto G_{V_t}$ is locally asymptotically normal (LAN, see van der Vaart (1998), Chapter 7). This result is fundamental because both Theorems 1.c and 2 rely on it. The proof of Proposition 2 is given in Appendix A.

Proposition 2 (*LAN expansion*). *Let V_t be as in (4.5) and $t_n = t/\sqrt{n}$ with $t \in \mathbb{R}$. If (3.3) and Assumption 1 hold true, then:*

$$\sum_{i=1}^n \log\frac{g_{V_{t_n}}(Z_i)}{g_V(Z_i)} = t\,\Delta_n - \frac{t^2}{2}\sigma^2_{\underline{x},\bar{x}} + o_p(1) \tag{4.6}$$

where: $\Delta_n := \frac{1}{\sqrt{n}}\sum_{i=1}^n\left(k^{(\underline{x},\bar{x})}(Z_i) - \mathbb{E}_{G_V}k^{(\underline{x},\bar{x})}\right) \rightsquigarrow N(0,\sigma^2_{\underline{x},\bar{x}})$ and $\sigma^2_{\underline{x},\bar{x}}$ is as in (3.4).

The second fundamental result of this paper gives a lower bound for the local asymptotic minimimax (LAM) risk of an arbitrary estimator, and the related convolution theorem.

Theorem 2 (*LAM & Convolution*). *Let* (3.3) *and* Assumption 1 *hold true and let* $\sigma^2_{\underline{x},\bar{x}}$ *be as in* (3.4). *Let then* $\ell : \mathbb{R} \mapsto [0,\infty)$ *be symmetric and subconvex. Then for any* $x \in (\underline{x},\bar{x})$, *and every estimator sequence* $(V_n(x))_{n\in\mathbb{N}}$:

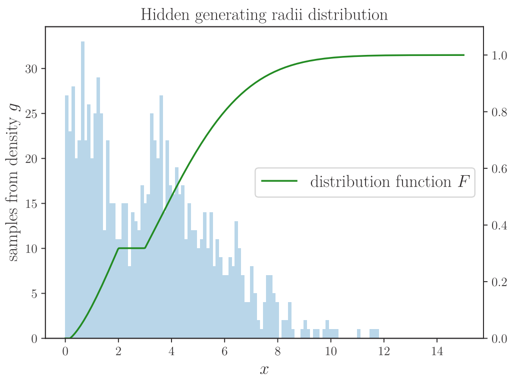
$$\sup_{\substack{I\subset\mathbb{R}\\ I \text{ finite}}} \liminf_{n\to\infty} \sup_{t\in I} \mathbb{E}_{G_{t/\sqrt{n}}} \ell\left(\sqrt{n}\left(V_n(x) - V_{t/\sqrt{n}}(x)\right)\right) \geq \int \ell \, dN(0,\sigma^2_{\underline{x},\bar{x}}), \tag{4.7}$$

Moreover, for any estimator sequence such that $\sqrt{n}\left(V_n(x) - V_{t/\sqrt{n}}(x)\right) \rightsquigarrow K$ *under* $G_{t/\sqrt{n}}$, *for every* $t \in \mathbb{R}$, *and some given distribution* K, *this distribution is the convolution of* $N(0,\sigma^2_{\underline{x},\bar{x}})$ *and some other distribution.*

The proof of Theorem 2 is given in Appendix A. By Theorem 1 the informed estimators attain equality in the LAM risk (for bounded continuous loss functions), and their limit distribution is $N(0,\sigma^2_{\underline{x},\bar{x}})$ without extra convolution factor. On the other hand, the limit distribution of the isotonic estimator contains the distribution of $\tilde{W}$ as in (3.5) as extra factor. Hence the isotonic estimator is not LAM.

5. Simulation study

In this section, we present a simulation study that shows that the isotonic inverse estimator, even if not efficient, behaves very closely to the efficient estimator. In practice, the information whether or not the cdf F is constant on a specific interval is generally unavailable. In that situation the best choice would be to use the isotonic estimator, given the efficiency theory and the adaptivity results developed in Gili et al. (2024). However, if the cdf is known to be constant on some known interval, then the isotonic estimator will incur a loss in terms of efficiency. In this section, we perform a simulation study to illustrate the difference in terms of performance between the isotonic estimator $\hat{V}_n$ and the efficient informed estimator $V^{(\underline{x},\bar{x})}$. We consider an underlying distribution F of the squared sphere radii, which is constant on $[2,3]$, namely (see Fig. 1). :



$$F(x) = \begin{cases} 1 - e^{-x^2/20}, & 0 \leq x < 2 \\ 1 - e^{-1/5}, & x \in [2,3] \\ 1 - e^{-(x-1)^2/20}, & x > 3 \end{cases}$$

$$g_V(x) = \frac{1}{2m_0}\int_x^M \frac{se^{-\frac{s^2}{10}}\mathbf{1}_{[0,2]}(s)}{10\sqrt{s-x}}\,ds + \frac{1}{2m_0}\int_x^M \frac{(s-1)e^{-\frac{(s-1)^2}{20}}\mathbf{1}_{[3,M]}(s)}{10\sqrt{s-x}}\,ds$$

Fig. 1. Underlying cdf F and in light blue a histogram of a sample of size 1000 from g.

Fig. 2 shows the true underlying V with all estimators defined in this paper within the specified interval. As $\hat{V}_n^{\Pi}$, V_n^{Π}, and $V_n^{(\underline{x},\bar{x})}$ are all asymptotically equivalent by Theorem 1, we focus on the comparison between $V_n^{(\underline{x},\bar{x})}$ and $\hat{V}_n$.

In Fig. 3, the scatter plot reveals a strong positive correlation between the two estimators, while the Tukey mean-difference plot indicates a lack of inherent bias. Taken together, the two plots in Fig. 3 demonstrate a substantial agreement between the two estimation methods. In Fig. 4, we observe that the distributions of $\sqrt{n}(V_n^{(\underline{x},\bar{x})}(x) - V(x))$ and $\sqrt{n}(\hat{V}_n(x) - V(x))$ closely resemble each other. Moreover, we observe a strong resemblance not only between the estimators but also between their respective limiting distributions, L_x and $N(0,\sigma^2_{\underline{x},\bar{x}})$. However, the violin plot with the standard percentiles, the second histogram with $\sqrt{n}(\hat{V}_n(x) - V_n^{(\underline{x},\bar{x})}(x))$ and W and Table 1 indicate that the variance of $\hat{V}_n(x)$ is slightly bigger than the one of $V_n^{(\underline{x},\bar{x})}(x)$.

The values in Table 1 were computed using varying sample size n from g as indicated, with 2000 samples from $\hat{V}_n(x)$ and $V_n^{(\underline{x},\bar{x})}(x)$. All the Kernel Density Estimator plots were computed using Scott's rule (where the bin size $h \sim 3.5\hat{\sigma}n^{-1/3}$) for the chosen bandwidth. For W and L_x, 20000 samples were used.

Taken together, the simulation study confirms the theoretical findings of Theorems 1 and 2: the informed estimators are efficient and not asymptotically equivalent to the IIE. However, the empirical performances of $V_n^{(\underline{x},\bar{x})}$ and $\hat{V}_n$ show a high degree of proximity in the estimation scenario where the underlying function V is constant on an interval. Even if $\hat{V}_n$ is used in cases where it is indeed true that V is constant within an interval, the incurred loss appears to be small.

Table 1
Standard deviations.

n	100	200	400	1000	2000		Limit
$\sqrt{n}(V_n^{(\underline{x},\bar{x})} - V)(x)$	0.5441	0.5439	0.5465	0.5467	0.5464	$N(0,\sigma^2_{\underline{x},\bar{x}})$	0.5466
$\sqrt{n}(\hat{V}_n - V)(x)$	0.5537	0.5592	0.5627	0.5664	0.5697	L_x	0.5703

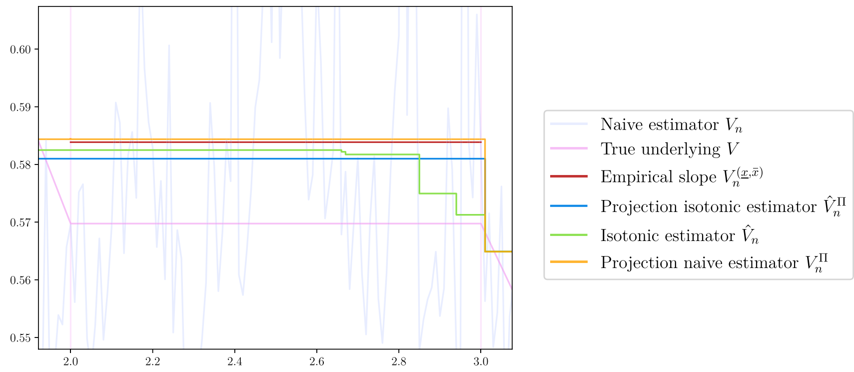


Fig. 2. All estimators and the true function V, which is constant on $[2,3]$, for a sample of size $n = 300$.

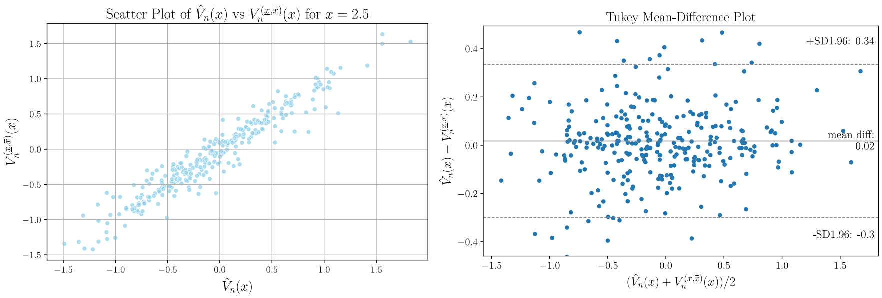


Fig. 3. Scatter and Tukey mean-difference plots for $n = 1000$ and 300 repetitions of $\hat{V}_n$ and $V_n^{(\underline{x},\bar{x})}$.

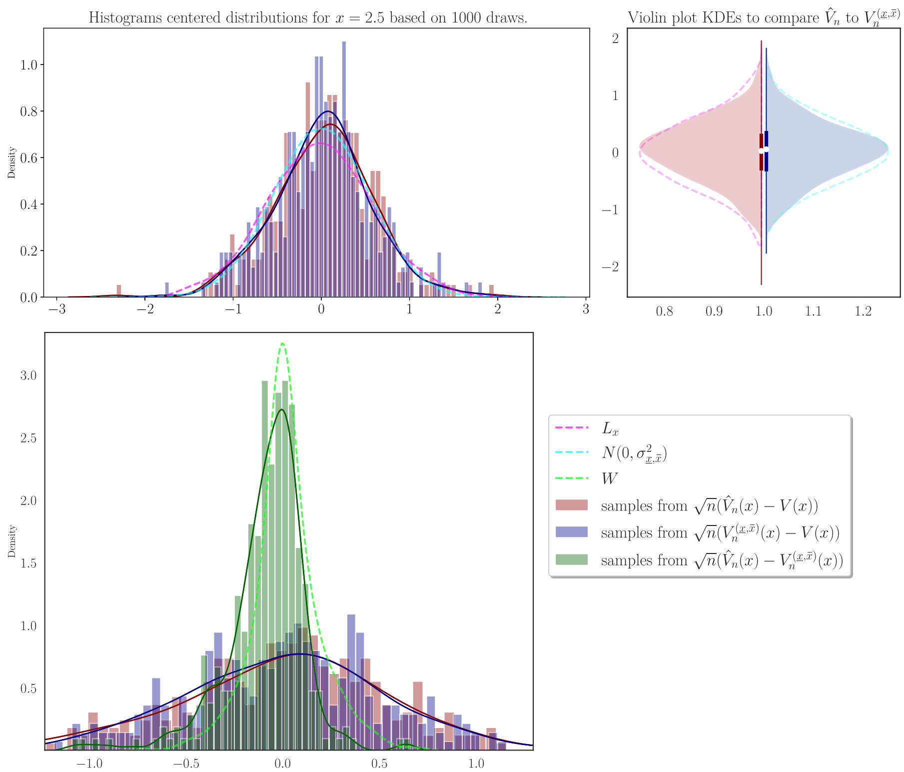


Fig. 4. Based on 1000 samples from g, centered and rescaled histograms based on 300 repetitions, $V_n^{(\underline{x},\bar{x})}$ and their difference compared with respective limiting distributions L_x, $N(0, \sigma^2_{\underline{x},\bar{x}})$ and W.

6. Conclusion

We conclude by stressing the fact that the local asymptotically minimax variance in Theorem 2 coincides with the one obtained in Theorem 1.a. This shows that in the setting in which V is constant on $[\underline{x}, \bar{x}]$, for $\underline{x} < \bar{x}$, the informed estimators (2.8)–(2.7) are asymptotically efficient, whereas the isotonic inverse estimator is not, as it is clearly shown by Theorem 2. However, we conjecture that the isotonic estimator remains asymptotically efficient in the situations in which the boundary points $\underline{x}, \bar{x}$ are allowed to converge to x at an arbitrarily polynomial rate (or slower). In this case, the rate of convergence drops to $\sqrt{n/\log(\bar{x} - \underline{x})^{-1}}$.

From a practical point of view, both the results of the current paper and Theorems 2.1–2.2 in Groeneboom and Jongbloed (1995) are difficult to use to obtain confidence bands, as they require estimating the density g, either at x or globally to compute $\sigma^2_{\underline{x},\bar{x}}$ or to compute the covariance structure of the limiting Gaussian process. We envision Theorems 2.1–2.2 in Groeneboom and Jongbloed (1995) and the asymptotic results of the current paper as the theoretical foundation, while the practical computation of confidence bands may be better addressed through bootstrap methods. A potential direction for further research would be proving formally why the bootstrap works. We believe that a formal justification would combine our theoretical results with the techniques used in Groeneboom and Jongbloed (2014) (which already shows consistency of the bootstrap for $\gamma_x = 1$ in the limiting variance of Theorem 2.1 in Groeneboom and Jongbloed (1995)).

Another potential direction for further investigation is whether the IIE is efficient in the setting in which there exists an interval of constancy for V but the boundary points $\underline{x}, \bar{x}$ are unknown. However, the definition and assessment of *efficiency* in this context are unclear. For practical purposes, if the boundary points $\underline{x}, \bar{x}$ are unknown, the IIE remains the state-of-art as it behaves very closely to the efficient estimator (see Section 5) and in the $\sqrt{n/\log n}$ rate setting is efficient and adaptive to the level of smoothness of V.

Acknowledgments

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Appendix A. Proofs.

Proofs of Section 2.

To prove Proposition 1, we need an additional lemma that will be essential to show that the estimators (2.13)–(2.16) are indeed the projections. The proof of this lemma uses similar reasoning as contained in Anevski and Soulier (2011) and Groeneboom and Jongbloed (2010). Below for a function f, $f(t^-)$ indicates the left limit of f at t.

Lemma 1. *Let $\overline{V} \in \mathbb{L}_1$ be given, define $V \mapsto Q^{\overline{V}}(V)$ by (2.5) and $\hat{V} \in \mathcal{V}_{\underline{x},\bar{x}}$. Suppose the following conditions hold true, for $\underline{x} < \bar{x}$:*

1. $\int_0^t (\hat{V}(x) - \overline{V}(x))\,dx \geq 0, \quad \forall\, t \in [0,\infty) \cap [\underline{x}, \bar{x}]^c$,
2. $\int_0^t (\hat{V}(x) - \overline{V}(x))\,dx = 0, \quad \forall\, t \in [0,\infty) \cap [\underline{x}, \bar{x}]^c$ *with* $\hat{V}(t) < \hat{V}(t^-)$.
3. *The measure defined via $\hat{V}$ and the one defined via $\overline{V}$ are compactly supported on the positive real line.*

Then $\hat{V} = \arg\min_{V \in \mathcal{V}_{\underline{x},\bar{x}}} Q^{\overline{V}}(V)$.

Proof. Let us recall the concept of Gateaux differentiability at a point $V \in \mathcal{V}$, for some function space $\mathcal{V}$. Let $Q : \mathcal{V} \mapsto \mathbb{R}$ be an arbitrary functional. It is called Gateaux differentiable at the point $V \in \mathcal{V}$ if the limit:

$$\partial Q(V,h) = \lim_{\varepsilon \to 0} \frac{Q(V + \varepsilon h) - Q(V)}{\varepsilon}$$

exists for every h such that $V + \varepsilon h \in \mathcal{V}$ for small enough ε. By straightforward computations, one can verify that:

$$Q^{\overline{V}}(\hat{V} + \varepsilon \mathbf{1}_{[0,t]}) - Q^{\overline{V}}(\hat{V}) = 2\varepsilon \int_0^t \hat{V}(x)\,dx + \varepsilon^2 t - 2\varepsilon \int_0^t \overline{V}(x)\,dx$$

Therefore the Gateaux derivative of $Q^{\overline{V}}$ is given by:

$$\partial Q^{\overline{V}}(\hat{V}, h) = 2 \int_0^\infty h(x) \left(\hat{V}(x) - \overline{V}(x) \right) dx \tag{A.1}$$

Note that, since $\partial Q^{\overline{V}}(\hat{V}, \mathbf{1}_{[0,t]}) = \int_0^t \hat{V}(x) - \overline{V}(x)\,dx$, the first two conditions of Lemma 1 entail the Gateaux derivative of the functional $Q^{\overline{V}}$ along the test functions $\mathbf{1}_{[0,t]}$, $\forall\, t \geq 0$ but $t \notin [\underline{x}, \bar{x}]$.

Now we show that for any $V \in \mathcal{V}_{\underline{x},\bar{x}}$, $\partial Q^{\overline{V}}(\hat{V}, V - \hat{V}) \geq 0$. First notice:

$$\hat{V}(x) - \overline{V}(x) = \frac{d(\hat{U} - \overline{U})}{dx}(x)$$

Without loss of generality, we assume that the two supports of the measures associated with the functions $\hat{V}$ and $\overline{V}$ are contained in $[0, M_1]$ and $[0, M_2]$. Defining $M := \max\{M_1, M_2\}$. Using integration by parts and assumption 3, $\hat{U}(0) - \overline{U}(0) = \hat{U}(M) - \overline{U}(M) = 0$:

$$\partial Q^{\overline{V}}(\hat{V}, V - \hat{V}) = 2\int_0^\infty \left(V(x) - \hat{V}(x)\right)\left(\hat{V}(x) - \overline{V}(x)\right) dx$$

$$= 2\int_0^M \left(V(x) - \hat{V}(x)\right) d(\hat{U} - \overline{U})(x) = \underbrace{2\left(V(x) - \hat{V}(x)\right)\left(\hat{U}(x) - \overline{U}(x)\right)\Big|_0^M}_{=0} - 2\int_0^M \left(\hat{U}(x) - \overline{U}(x)\right) d\left(V(x) - \hat{V}(x)\right)$$

By assumption 2, the support of the measure $-d\hat{V}$ is included in the set $\{x : \overline{U}(x) = \hat{U}(x)\}$ (cf. Lemma 1 in Anevski and Soulier (2011)), therefore: $\int_0^\infty \left(\hat{U}(x) - \overline{U}(x)\right) d\hat{V}(x) = 0$. On the other hand, because $V \in \mathcal{V}_{\underline{x},\bar{x}}$ the negative measure defined based on V does not have mass in $[\underline{x}, \bar{x}]$ and then, by using Assumption 1, we obtain:

$$\int_0^\infty \left(\overline{U}(x) - \hat{U}(x)\right) dV(x) \geq 0$$

which proves indeed $\partial Q^{\overline{V}}(\hat{V}, V - \hat{V}) \geq 0$. Now define the convex function $t \mapsto u(t) = Q^{\overline{V}}(\hat{V} + t(V - \hat{V}))$ where $t \in [0,1]$. Using (A.1) and what we just proved, we have that $u'(0) = \partial Q^{\overline{V}}(\hat{V}, V - \hat{V}) \geq 0$. Since u is convex, the fact $u'(0) \geq 0$ implies $u(1) \geq u(0)$, which indeed is $Q^{\overline{V}}(V) \geq Q^{\overline{V}}(\hat{V})$. □

Proof of Proposition 1. The proof of (2.6) is provided in Groeneboom and Jongbloed (1995) (c.f. Lemma 2 in Groeneboom and Jongbloed (1995)).

Proof of (2.7). First, we show that $\hat{V}_n^{\Pi} \in \mathcal{V}_{\underline{x},\bar{x}}$. By definition, for $x \in [\underline{x}, \bar{x}]$, for $\underline{x} < \bar{x}$:

$$\hat{V}_n^{\Pi}(x) = \frac{U_n^*(\bar{x}) - U_n^*(\underline{x})}{\bar{x} - \underline{x}}$$

Let Z_k^* for $k \in \{1, \dots, m-1\}$ be the points where U_n^* changes slope in $[\underline{x}, \bar{x}]$. Denote by Z_0^* the closest point where U_n^* changes slope to the left of $\underline{x}$. Similarly, denote by Z_m^* the closest point where U_n^* changes slope to the right of $\bar{x}$. Now consider the fact that:

$$U_n^*(\underline{x}) = \frac{U_n(Z_1^*) - U_n(Z_0^*)}{Z_1^* - Z_0^*}(\underline{x} - Z_1^*) + U_n(Z_1^*)$$

$$U_n^*(\bar{x}) = \frac{U_n(Z_m^*) - U_n(Z_{m-1}^*)}{Z_m^* - Z_{m-1}^*}(\bar{x} - Z_{m-1}^*) + U_n(Z_{m-1}^*)$$

By telescoping we have:

$$U_n(Z_1^*) - U_n(Z_{m-1}^*) = -\sum_{k=2}^{m-1} U_n(Z_k^*) - U_n(Z_{k-1}^*)$$

And therefore all together we have on $[\underline{x}, \bar{x}]$, for $\underline{x} < \bar{x}$:

$$\hat{V}_n^{\Pi}(x) := \sum_{k=2}^{m-1} \frac{Z_k^* - Z_{k-1}^*}{\bar{x} - \underline{x}} \frac{U_n(Z_k^*) - U_n(Z_{k-1}^*)}{Z_k^* - Z_{k-1}^*} + \frac{Z_1^* - \underline{x}}{\bar{x} - \underline{x}} \frac{U_n(Z_1^*) - U_n(Z_0^*)}{Z_1^* - Z_0^*} + \frac{\bar{x} - Z_{m-1}^*}{\bar{x} - \underline{x}} \frac{U_n(Z_m^*) - U_n(Z_{m-1}^*)}{Z_m^* - Z_{m-1}^*} \tag{A.2}$$

The fact that $\hat{V}_n^{\Pi} \in \mathcal{V}_{\underline{x},\bar{x}}$ is now clear, because the computations given above prove that $\hat{V}_n^{\Pi}$ on $[\underline{x}, \bar{x}]$ is just a weighted average of the values chosen by the isotonic estimator over $[\underline{x}, \bar{x}]$. This ensures both the monotonicity constraint and right-continuity, other than the fact that $\hat{V}_n^{\Pi}$ is constant on $[\underline{x}, \bar{x}]$. Let us now verify condition 1 of Lemma 1. Clearly, for $t \leq \underline{x}$, the condition is satisfied as it is zero. Take $t \geq \bar{x}$. The primitive of $\hat{V}$ is U_n^*. We verify the main condition. We split the integral into three bits. Clearly $\int_0^{\underline{x}} \hat{V}_n^{\Pi}(x) - \hat{V}_n(x)\,dx = \int_{\bar{x}}^t \hat{V}_n^{\Pi}(x) - \hat{V}_n(x)\,dx = 0$. Now for $[\underline{x}, \bar{x}]$: $\int_{\underline{x}}^{\bar{x}} \hat{V}_n^{\Pi}(x) - \hat{V}_n(x)\,dx = 0$ by (2.13). Condition 2 can be verified immediately using the same strategy and condition 3 is verified as well by the definition of the isotonic estimator.

Proof of (2.8). If we minimize Q^{V_n} over a space where we do not require the solution to be decreasing (but still constant on each $[x_i, x_{i+1})$), then we can split the minimization problem into different integrals over each $[x_i, x_{i+1})$. When we minimize over $[x_i, x_{i+1})$ we obtain, setting $V(x) = c \in \mathbb{R}$ for $x \in [x_i, x_{i+1})$:

$$\int_{x_i}^{x_{i+1}} V(x)\left(V(x) - 2V_n(x)\right) dx = \int_{x_i}^{x_{i+1}} c^2 - 2cV_n(x)dx = (x_{i+1} - x_i)c^2 - 2c\frac{1}{n}\sum_{i=1}^n \int_{x_i}^{x_{i+1}} \left(Z_i - x\right)^{-1/2} dx$$

$$= (x_{i+1} - x_i)c^2 - 2c\left(U_n(x_{i+1}) - U_n(x_i)\right).$$

By optimizing, we obtain $c = \frac{U_n(x_{i+1}) - U_n(x_i)}{x_{i+1} - x_i}$.

Proof of (2.9). We use Lemma 1. By (2.16), $V_n^{\Pi} \in \mathcal{V}_{\underline{x},\bar{x}}$. Moreover, we have that condition 1 is obtained because it requires that $\forall\, t \in [\underline{x}, \bar{x}]^c \cap [0, \infty)$:

$$\int_0^t V_n^{\Pi}(x) - V_n(x)\,dx = \widetilde{U}_n^*(t) - U_n(t) \geq 0$$

but by construction $\widetilde{U}_n^*(t) \geq \widetilde{U}_n(t) = U_n(t)$ for $t \notin [\underline{x}, \bar{x}]$. For condition 2, if $V_n^{\Pi}(t) < V_n^{\Pi}(t^-)$ then t is a support point of the $\widetilde{U}_n^*$ and we also know by construction that all the support points of $\widetilde{U}_n^*$ satisfy $\widetilde{U}_n^*(t) = U_n(t)$. Condition 3 is trivially satisfied. □

Proofs of Section 3.

All the proofs that follow make use of the explicit constructions of the estimators (2.6)–(2.7) given in Proposition 1. We present the proof of Theorem 1.b first, as part of the proof of Theorem 1. List a can be viewed as a special case of it.

To prove Theorems 1.a and 1.b we need the following additional lemma.

Lemma 2. *Let $\ell^\infty(K)$ be endowed with the uniform norm and $\forall\, s \in K$:*

$$Z_n(s) := \frac{\sqrt{n}}{\bar{x}-\underline{x}}\left((U_n(s)-U_n(\underline{x}))(\bar{x}-s)-(U_n(\bar{x})-U_n(s))(s-\underline{x})-(U(s)-U(\underline{x}))(\bar{x}-s)-(U(\bar{x})-U(s))(s-\underline{x})\right). \tag{A.3}$$

Then

$$Z_n \rightsquigarrow \mathbb{Z} \quad \text{in } \ell^\infty(K), \tag{A.4}$$

where $\mathbb{Z}$ is the zero mean Gaussian Process given in (3.6) and has a version with continuous sample paths, with unique point of maximum.

Proof. Because

$$\begin{aligned}
&(U_n(s)-U_n(\underline{x}))(\bar{x}-s)-(U_n(\bar{x})-U_n(s))(s-\underline{x})\\
&=\frac{2}{n}\sum_{i=1}^n\left\{\left(\sqrt{(Z_i-\underline{x})_+}-\sqrt{(Z_i-s)_+}\right)(\bar{x}-s)-\left(\sqrt{(Z_i-s)_+}-\sqrt{(Z_i-\bar{x})_+}\right)(s-\underline{x})\right\},
\end{aligned}$$

the stated convergence is equivalent to saying that the class of functions $z \to (\sqrt{(z-\underline{x})_+}-\sqrt{(z-s)_+})(\bar{x}-s)-(\sqrt{(z-s)_+}-\sqrt{(z-\bar{x})_+})(s-\underline{x})$, $s \in K$ is Donsker. From Lemma 2.6.16 in van der Vaart and Wellner (1996), it follows that the class of functions $\{z-s \;:\; s \geq 0\}$ is VC, next by Lemma 2.6.18 (ii) in van der Vaart and Wellner (1996) we see that the class $\{(z-s)_+ \;:\; s \geq 0\}$ is VC. Again by Lemma 2.6.18 (vii), (iv) and (v) in van der Vaart and Wellner (1996) we conclude that the class of functions $\{\sqrt{(z-x)_+}-\sqrt{(z-s)_+} \;:\; s \geq 0\}$ is VC. Similarly, we conclude that the class of functions $z \mapsto \sqrt{(z-s)_+}-\sqrt{(z-\bar{x})_+}$ is VC. The two above-mentioned classes of functions have respective envelopes: $0 \leq \sqrt{(z-x)_+}-\sqrt{(z-s)_+} \leq \sqrt{(z-x)_+}$ and $0 \leq \sqrt{(z-s)_+}-\sqrt{(z-\bar{x})_+} \leq \sqrt{z}$. Therefore these classes are P-Donsker for any P with $\int_0^\infty z\,dP(z) < \infty$. This is the case by assumption (3.3). Again by Lemma 2.6.16 in van der Vaart and Wellner (1996) we have that the classes of functions: $\{(\bar{x}-s) \;:\; s \in K\}$ and $\{(\underline{x}-s) \;:\; s \in K\}$ are VC. They have clearly square integrable envelope as $K = [\underline{x}, \bar{x}]$ is compact. We therefore conclude that the uniform entropy integral for the class of functions of our interest is finite, using Lemma 7.21 (i) and (ii) from Sen (2022).

To prove that $\mathbb{Z}$ possesses a version with continuous sample paths, it suffices to show:

$$\mathbb{E}\,(\mathbb{Z}(s)-\mathbb{Z}(t))^2 \lesssim |s-t|+|s-t|^2. \tag{A.5}$$

This follows from the fact that the square root is Hölder continuous of degree $1/2$ and the fact that:

$$\begin{aligned}
&\frac{1}{\bar{x}-\underline{x}}\left(\sqrt{(z-\underline{x})_+}-\sqrt{(z-s)_+}(\bar{x}-s)-(\sqrt{(z-s)_+}-\sqrt{(z-\bar{x})_+})(s-\underline{x})\right)\\
&=\sqrt{(z-\underline{x})_+}-\sqrt{(z-s)_+}+\frac{1}{\bar{x}-\underline{x}}(\sqrt{(z-\underline{x})_+}-\sqrt{(z-\bar{x})_+})(\underline{x}-s)
\end{aligned}$$

The covariance function of $\mathbb{Z}$ is given by, for $Z_x := Z - x$:

$$\begin{aligned}
&\mathrm{Cov}\,(\mathbb{Z}(t),\mathbb{Z}(s))\\
&=4\mathrm{Cov}\left(\frac{(\sqrt{(Z_{\underline{x}})_+}-\sqrt{(Z_t)_+})(\bar{x}-t)-(\sqrt{(Z_t)_+}-\sqrt{(Z_{\bar{x}})_+})(t-\underline{x})}{\bar{x}-\underline{x}},\frac{(\sqrt{(Z_{\underline{x}})_+}-\sqrt{(Z_s)_+})(\bar{x}-s)-(\sqrt{(Z_s)_+}-\sqrt{(Z_{\bar{x}})_+})(s-\underline{x})}{\bar{x}-\underline{x}}\right).
\end{aligned}$$

This coincides with the covariance function of the process in (3.6).

By (A.5), we deduce that $\mathrm{Var}(\mathbb{Z}(t)-\mathbb{Z}(s)) \neq 0$ for $s \neq t$. Since $\mathbb{Z}$ is indexed by a σ-compact metric space, we obtain from Lemma 2.6 in Kim and Pollard (1990) that the location of the maximum of the sample paths of $\mathbb{Z}$ is a.s. unique. □

Proof of Theorem 1.b. For $a > 0$, the "switch relation" (see Groeneboom and Jongbloed (2014)) is gives:

$$\hat{V}_n(x) \leq a \quad \Leftrightarrow \quad \underset{s\geq 0}{\operatorname{argmax}^-}\left\{U_n(s)-as\right\} \leq x. \tag{A.6}$$

Here the argmax is defined as:

$$\underset{s\geq 0}{\operatorname{argmax}^-}\left\{U_n(s)-as\right\} := \inf\left\{s \geq 0 : U_n(s)-as \text{ is maximal}\right\}.$$

Therefore for each fixed $a \in \mathbb{R}$ we can, for all n sufficiently large, write:

$$\begin{aligned}
&\sqrt{n}\left(\hat{V}_n(x)-V_n^{(\underline{x},\bar{x})}(x)\right) \leq a\\
\Leftrightarrow\;& \underset{s\geq 0}{\operatorname{argmax}^-}\left\{U_n(s)-\left(V_n^{(\underline{x},\bar{x})}(x)+a/\sqrt{n}\right)s\right\} \leq x
\end{aligned}$$

$$\Leftrightarrow \quad \underset{s\geq 0}{\operatorname{argmax}^-} \Big\{ \sqrt{n}(U_n(s) - U_n(\underline{x}) - U(s) + U(\underline{x})) - \sqrt{n}((U_n(\bar{x}) - U_n(\underline{x}))/(\bar{x} - \underline{x}) - V(x))(s - \underline{x}) + \sqrt{n}(U(s) - U(\underline{x}) - V(x)(s - \underline{x})) - as \Big\} \leq x.$$

Here we used that the location of a maximum of a function is equivariant under translations and invariant under multiplication by a positive number and addition of a constant. We obtain that the above is equivalent to:

$$\underset{s\geq 0}{\operatorname{argmax}^-} \left\{ Z_n(s) - as - \sqrt{n}h_x(s) \right\} \leq x,$$

where:

$$\begin{aligned} Z_n(s) &:= \sqrt{n}(U_n(s) - U_n(\underline{x}) - U(s) + U(\underline{x})) - \sqrt{n}((U_n(\bar{x}) - U_n(\underline{x}))/(\bar{x} - \underline{x}) - V(x))(s - \underline{x}), \\ h_x(s) &:= U(\underline{x}) - U(s) - V(x)(\underline{x} - s). \end{aligned}$$

One can easily show that Z_n can be written as in (A.3). Therefore by Lemma 2 and the above computations, we conclude that the limiting behavior of:

$$\widetilde{Z}_n(s) := Z_n(s) - as - \sqrt{n}h_x(s),$$

is determined by the process $\mathbb{W}$ defined by, for $\underline{x} < \bar{x}$:

$$\mathbb{W}(s) = \begin{cases} \mathbb{Z}(s) - as, & \text{for } s \in [\underline{x}, \bar{x}] \\ -\infty, & \text{for } s \notin [\underline{x}, \bar{x}]. \end{cases}$$

This is because $h_x(s) > 0$ for all $s \in (0, \underline{x}) \cup (\bar{x}, \infty)$, and therefore multiplied by $-\sqrt{n}$ will go to $-\infty$, while $h_x(s) = 0,\ \forall s \in [\underline{x}, \bar{x}]$, and thus on that interval the limiting behavior is completely determined by $\mathbb{Z}(s) - as$.

Let:

$$\hat{s}_n := \underset{s\geq 0}{\operatorname{argmax}^-} \left\{ Z_n(s) - as - \sqrt{n}h_x(s) \right\}, \tag{A.7}$$

$$\hat{s} := \underset{s\geq 0}{\operatorname{argmax}} \{\mathbb{W}(s)\} = \underset{s\in[\underline{x},\bar{x}]}{\operatorname{argmax}} \{\mathbb{Z}(s) - as\}. \tag{A.8}$$

To prove convergence in distribution of $\hat{s}_n$ to $\hat{s}$, we use the Portmanteau Lemma as done in Gili et al. (2024) (cf. proof of Theorem 3) and show for every closed subset F:

$\limsup_{n\to\infty} \mathbb{P}\left(\hat{s}_n \in F\right) \leq \mathbb{P}(\hat{s} \in F)$. As in Gili et al. (2024) we choose a sequence $\epsilon_n \downarrow 0$ such that $\sqrt{n}h_x(\underline{x} - \epsilon_n) \to \infty$ and $\sqrt{n}h_x(\bar{x} + \epsilon_n) \to \infty$. For $K_n := [\underline{x} - \epsilon_n, \bar{x} + \epsilon_n]$ and for $K := [\underline{x}, \bar{x}]$:

$$\begin{aligned} &\limsup_{n\to\infty} \mathbb{P}\left(\hat{s}_n \in F\right) \leq \limsup_{n\to\infty} \mathbb{P}\left(\hat{s}_n \in F \cap K_n\right) + \limsup_{n\to\infty} \mathbb{P}\left(\hat{s}_n \in K_n^c\right) \\ &\leq \underbrace{\limsup_{n\to\infty} \mathbb{P}\left(\sup_{s\in F\cap K_n} \widetilde{Z}_n(s) \geq \sup_{s\in K_n} \widetilde{Z}_n(s) \right)}_{(1)} + \underbrace{\limsup_{n\to\infty} \mathbb{P}\left(\hat{s}_n \in K_n^c\right)}_{(2)}. \end{aligned} \tag{A.9}$$

We show that (2) in (A.9) goes to zero. If $\forall s \in K^c$ we have Z_n as in (A.3), then:

$$\sup_{s\in K_n^c} \widetilde{Z}_n(s) = \sup_{s\in K_n^c} \left\{ Z_n(s) - as - \sqrt{n}h_x(s) \right\} \xrightarrow{\mathbb{P}} -\infty \tag{A.10}$$

Note that:

$$\begin{aligned} &\sup_{s\in K_n^c} \left\{ Z_n(s) - \sqrt{n}h_x(s) - as \right\} \leq \sup_{s\in K^c} Z_n(s) - \inf_{\substack{s<\underline{x}-\epsilon_n \\ s>\bar{x}+\epsilon_n}} \left\{ \sqrt{n}h_x(s) + as \right\} \\ &\leq \sup_{s\in K^c} \{ \sqrt{n}(U_n(s) - U_n(\underline{x}) - U(s) + U(\underline{x})) \} - \inf_{\substack{s<\underline{x}-\epsilon_n \\ s>\bar{x}+\epsilon_n}} \left\{ \sqrt{n}h_x(s) + as \right\} + \sup_{s\geq 0} \left\{ -\sqrt{n} \left(U_n(\bar{x}) - U_n(\underline{x}) - U(\bar{x}) + U(\underline{x}) \right) \frac{s - \underline{x}}{\bar{x} - \underline{x}} \right\} \\ &= \sup_{s\in K^c} \{ \sqrt{n}(U_n(s) - U_n(\underline{x}) - U(s) + U(\underline{x})) \} - \inf_{\substack{s<\underline{x}-\epsilon_n \\ s>\bar{x}+\epsilon_n}} \left\{ \sqrt{n}h_x(s) + as \right\} + \sqrt{n} \frac{U_n(\bar{x}) - U_n(\underline{x}) - U(\bar{x}) + U(\underline{x})}{\bar{x} - \underline{x}} \underline{x} \end{aligned}$$

where the first and the last term on the right-hand side are $O_p(1)$ by the convergence given in Lemma 1 in Gili et al. (2024). The rest of the proof is the same as in Gili et al. (2024), and it shows that: $-\inf_{\substack{s<\underline{x}-\epsilon_n \\ s>\bar{x}+\epsilon_n}} \left\{ \sqrt{n}h_x(s) + as \right\} \to -\infty$. Using (A.10):

$$\mathbb{P}\left(\hat{s}_n \in K_n^c\right) \leq \mathbb{P}\left(\sup_{s\in K_n^c} \widetilde{Z}_n(s) \geq \sup_{s\in K_n} \widetilde{Z}_n(s) \right) \leq \mathbb{P}\left(\sup_{s\in K_n^c} \widetilde{Z}_n(s) \geq \sup_{s\in K} \left\{ Z_n(s) - as \right\} \right) \to 0. \tag{A.11}$$

The last convergence is a consequence of (A.10) and the convergence $Z_n \rightsquigarrow \mathbb{Z}$ on $\ell^\infty(K)$, combined with the continuous mapping theorem (which implies that the term on the right-hand side of the last inequality is $O_p(1)$).

Now we argue the behavior of (1) in (A.9). For that, we need an additional convergence in probability given in (A.12):

$$\sup_{s\in F\cap K_n}\left\{Z_n(s)-\sqrt{n}h_x(s)-as\right\}-\sup_{s\in F\cap K}\left\{Z_n(s)-as\right\}\xrightarrow{\mathbb{P}}0. \tag{A.12}$$

First note:

$$0\le\sup_{s\in F\cap K_n}\left\{Z_n(s)-\sqrt{n}h_x(s)-as\right\}-\sup_{s\in F\cap K}\left\{Z_n(s)-as\right\}\le\sup_{s\in F\cap K_n}\left\{Z_n(s)-as\right\}-\sup_{s\in F\cap K}\left\{Z_n(s)-as\right\}.$$

Define the process, for $\underline{x}<\bar{x}$:

$$\widetilde{Z}_n(s):=\begin{cases}Z_n(s)-as, & \text{if } s\in[\underline{x},\bar{x}],\\ Z_n(\underline{x})-a\underline{x}, & \text{if } s\in[0,\underline{x}],\\ Z_n(\bar{x})-a\bar{x}, & \text{if } s\in[\bar{x},\infty).\end{cases}$$

Because: $\sup_{s\in F\cap K_n}\widetilde{Z}_n(s)=\sup_{s\in F\cap K}\widetilde{Z}_n(s)=\sup_{s\in F\cap K}\left\{Z_n(s)-as\right\}$, it follows that

$$\left|\sup_{s\in F\cap K_n}\left\{Z_n(s)-as\right\}-\sup_{s\in F\cap K}\left\{Z_n(s)-as\right\}\right|=\left|\sup_{s\in F\cap K_n}\left\{Z_n(s)-as\right\}-\sup_{s\in F\cap K_n}\widetilde{Z}_n(s)\right|\le\sup_{s\in F\cap K_n}\left|Z_n(s)-as-\widetilde{Z}_n(s)\right|$$
$$\le\sup_{s\in F\cap K_n\cap[0,\underline{x}]}\left|Z_n(s)-as-\widetilde{Z}_n(\underline{x})\right|\vee\sup_{s\in F\cap K_n\cap[\bar{x},\infty)}\left|Z_n(s)-as-\widetilde{Z}_n(\bar{x})\right|\to 0,$$

where we used the asymptotic equicontinuity of the Z_n. Using that:

$$\sup_{s\in F\cap K}\left\{Z_n(s)-as\right\}\rightsquigarrow\sup_{s\in F\cap K}\left\{\mathbb{Z}_x(s)-as\right\},$$

we conclude using Theorem 2.7 (iv) from van der Vaart (1998) and (A.12):

$$\sup_{s\in F\cap K_n}\left\{Z_n(s)-\sqrt{n}h_x(s)-as\right\}\rightsquigarrow\sup_{s\in F\cap K}\left\{\mathbb{Z}_x(s)-as\right\}.$$

Term (1) in (A.9) can be upper bounded by:

$$\limsup\mathbb{P}\left(\sup_{s\in F\cap K_n}\left\{Z_n(s)-\sqrt{n}h_x(s)-as\right\}\ge\sup_{s\in K}\left\{Z_n(s)-as\right\}\right)$$
$$\le\mathbb{P}\left(\sup_{F\cap K}\left\{\mathbb{Z}_x(s)-as\right\}\ge\sup_{s\in K}\left\{\mathbb{Z}_x(s)-as\right\}\right)\le\underbrace{\mathbb{P}(\hat{s}\in K^c)}_{=0}+\mathbb{P}\left(\hat{s}\in F\right), \tag{A.13}$$

where we used the above derivations, together with Lemma 2 and Theorem 2.7 (v) from van der Vaart (1998). Therefore by (A.9) combined with (A.11), (A.13) and the Portmanteau Lemma we obtain the desired convergence in distribution. □

To prove Theorem 1.a we need the following additional lemma, which uses similar arguments as in Carolan (2002).

Lemma 3. *If U is linear with positive slope on $[\underline{x},\bar{x}]$, for $\underline{x}<\bar{x}$, and there does not exist a larger interval containing $[\underline{x},\bar{x}]$ on which U is linear, then:*

$$U_n^*(\bar{x})-U_n(\bar{x})=o_p(1/\sqrt{n})\quad\textit{and}\quad U_n^*(\underline{x})-U_n(\underline{x})=o_p(1/\sqrt{n})$$

Proof. Because $U_n(x)=\frac{2}{n}\sum_{i=1}^n\left\{\sqrt{(Z_i)_+}-\sqrt{(Z_i-x)_+}\right\}$, the same arguments as in the proof of Lemma 2 give in $\ell^\infty(0,\infty)$:

$$\sqrt{n}\left(U_n-U\right)\rightsquigarrow\widetilde{\mathbb{Z}} \tag{A.14}$$

for a Gaussian Process $\widetilde{\mathbb{Z}}$. By assumption U is linear on $[\underline{x},\bar{x}]$. Let $l(t)=\frac{U(\bar{x})-U(\underline{x})}{\bar{x}-\underline{x}}(t-\underline{x})+U(\underline{x})$ be the line which is equal to $U(t)$ on $[\underline{x},\bar{x}]$. Recall that the upper script $*$ denotes the operation of taking the least concave majorant, for instance U^* is the least concave majorant of U. Then consider:

$$\sqrt{n}\left(U_n^*(\underline{x})-U_n(\underline{x})\right)=\sqrt{n}\left[\left\{U_n-U+U-l+l\right\}^*(\underline{x})-U_n(\underline{x})\right]=\left[\sqrt{n}\left\{U_n-U\right\}+\sqrt{n}\left\{U-l\right\}\right]^*(\underline{x})-\sqrt{n}\left(U_n(\underline{x})-U(\underline{x})\right), \tag{A.15}$$

where we use that the least concave majorant of the sum of any function h and a linear function l is equal to the sum of the linear function l and the least concave majorant of h.

By Skorokhod's representation theorem there exists a sequence of random variables $\widetilde{Z}_n$, with same distribution as $\sqrt{n}\left\{U_n-U\right\}$, which converges almost surely to $\widetilde{\mathbb{Z}}$ (or see Rio (1993) for strong approximation type of result). Furthermore taking the Least Concave Majorant is a continuous function (see Beare and Fang (2017)). Thus the expression in (A.15) has the same limiting distribution as:

$$\left[\widetilde{\mathbb{Z}}+\sqrt{n}\left\{U-l\right\}\right]^*(\underline{x})-\widetilde{\mathbb{Z}}(\underline{x}).$$

We proceed by an analytic argument to show that the above tends to zero in probability. Fix a sample path k of $\widetilde{Z}$, which is a bounded and continuous function on $[0, M]$. Because $0 \leq \sqrt{n}(U_n^*(\underline{x}) - U_n(\underline{x}))$, for $\varepsilon > 0$ we need to show $\forall\, n$ large enough:

$$\{k + \sqrt{n}[U - l]\}^*(\underline{x}) < k(\underline{x}) + \varepsilon \tag{A.16}$$

To do so, we construct a concave majorant of $k + \sqrt{n}[U - l]$ on $(\underline{x}, M]$ passing through the point $(\underline{x}, k(\underline{x}) + \varepsilon)$ and we show that for all n big enough it constitutes a concave majorant of $k + \sqrt{n}[U - l]$ also on $[0, \underline{x}]$, thus eventually on the whole $[0, M]$.

Let S_n denote the largest slope of lines connecting the point $(\underline{x}, k(\underline{x}) + \varepsilon)$ with $(t, k(t) + \sqrt{n}[U - l](t))$ for $t > \underline{x}$. Note that $S_n < \infty$ for all n. Since U is concave, increasing and linear on $[\underline{x}, \bar{x}]$, $U - l \leq 0$ and thus $\sqrt{n}[U - l]$ becomes progressively smaller on $(\bar{x}, M]$, implying $S_1 \geq S_2 \geq \cdots$.

Let l_n be the line with slope S_n that passes through the point $(\underline{x}, k(\underline{x}) + \varepsilon)$. (A.16) holds true if and only if $k(t) + \sqrt{n}[U(t) - l(t)] < l_n(t)$ for all $t \in [0, \underline{x}]$. Since $S_1 \geq S_n$, it suffices to show that for all $t \in [0, \underline{x}]$ for all n large enough

$$k(t) + \sqrt{n}[U(t) - l(t)] < l_1(t), \tag{A.17}$$

as for $t > \underline{x}$, $k + \sqrt{n}[U - l] < l_1$ by construction and $S_1 \geq S_n$. These imply:

$$\{k + \sqrt{n}[U - l]\}^* \leq l_1.$$

We prove (A.17). Because $[U - l] \leq 0$, $\forall\, \delta > 0$, as $n \to \infty$

$$\sup_{0 \leq t < \underline{x} - \delta} (k + \sqrt{n}[U - l])(t) \to -\infty$$

this implies that for any $\delta > 0$, on $[0, \underline{x} - \delta)$ for sufficiently large n: $k + \sqrt{n}[U - l] < l_1$. For $\underline{x} - \delta \leq t < \underline{x}$ and $\delta < \underline{x} - \varepsilon/(2S_1)$,

$$l_1(t) = k(\underline{x}) + \varepsilon + (t - \underline{x})S_1 \geq k(\underline{x}) + \varepsilon/2.$$

For δ small enough on $[\underline{x} - \delta, \underline{x})$ and sufficiently large n, by the continuity of k:

$$\sup_{\underline{x} - \delta \leq t < \underline{x}} (k + \sqrt{n}[U - l])(t) \leq \sup_{\underline{x} - \delta \leq t < \underline{x}} k(t) \leq k(\underline{x}) + \varepsilon/2.$$

This implies that for δ small enough, on $[\underline{x} - \delta, \underline{x})$ for sufficiently large n: $k + \sqrt{n}[U - l] < l_1$.

The result for $\bar{x}$ follows analogously. □

Proof of Theorem 1.a. First, we show:

$$\sqrt{n}\left(V_n^{\Pi}(x) - V_n^{(\underline{x}, \bar{x})}(x)\right) = o_p(1), \tag{A.18}$$

Because V_n^{Π} is the left derivative of the least concave majorant of $\widetilde{U}_n$, the switch relation gives, for $a \in \mathbb{R}$ and sufficiently large n,

$$\sqrt{n}\left(V_n^{\Pi}(x) - V_n^{(\underline{x}, \bar{x})}(x)\right) \leq a$$

$$\Leftrightarrow \operatorname*{argmax}^-_{s \geq 0} \left\{ \sqrt{n}(\widetilde{U}_n(s) - \widetilde{U}_n(\underline{x}) - U(s) + U(\underline{x})) - \sqrt{n}(V_n^{(\underline{x}, \bar{x})}(x) - V(x))(s - \underline{x}) + \sqrt{n}(U(s) - U(\underline{x}) - V(x)(s - \underline{x})) - as \right\} \leq x,$$

where we used the usual properties of the argmax^- as in the proof of Theorem 1.b. By the definition of $\widetilde{U}_n$ in (2.17) and $V_n^{(\underline{x}, \bar{x})}(x)$ in (2.15), the above can be rewritten as:

$$\operatorname*{argmax}^-_{s \geq 0} \left\{ Z_n(s) - as - \sqrt{n} h_x(s) \right\} \leq x,$$

where:

$$Z_n(s) := \begin{cases} \frac{\sqrt{n}}{\bar{x} - \underline{x}} \Big((U_n(s) - U_n(\underline{x}))(\bar{x} - s) - (U_n(\bar{x}) - U_n(s))(s - \underline{x}) \\ \quad -(U(s) - U(\underline{x}))(\bar{x} - s) - (U(\bar{x}) - U(s))(s - \underline{x}) \Big), & s \notin [\underline{x}, \bar{x}], \\ 0, & s \in [\underline{x}, \bar{x}], \end{cases}$$

$$h_x(s) := U(\underline{x}) - U(s) - V(x)(\underline{x} - s).$$

From here on, we apply the argument of Theorem 1.b, using also for the current case the convergence in Eq. (A.10). Presently

$$\operatorname*{argmax}^-_{s \geq 0} \left\{ Z_n(s) - as - \sqrt{n} h_x(s) \right\} \rightsquigarrow \operatorname*{argmax}_{s \in K} \{-as\},$$

as in this case the limiting process $\mathbb{Z}$ is constantly 0 on K. The argmax on the right is $\bar{x}$ if $a < 0$ and $\underline{x}$ if $a > 0$. Since $x \in (\underline{x}, \bar{x})$ it follows that

$$\mathbb{P}\left(\operatorname*{argmax}^-_{s \geq 0} \left\{ Z_n(s) - as - \sqrt{n} h_x(s) \right\} \leq x \right) \xrightarrow{n \to \infty} \mathbb{P}\left(\operatorname*{argmax}_{s \in K} \{-as\} \leq x \right) = \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a < 0. \end{cases}$$

This concludes the proof of (A.18).

Second we prove:

$$\sqrt{n}\left(\hat{V}_n^{\Pi}(x) - V_n^{(\underline{x},\bar{x})}(x)\right) = o_p(1). \tag{A.19}$$

By the definitions of $\hat{V}_n^{\Pi}$ and $V_n^{(\underline{x},\bar{x})}$ in (2.13) and (2.15):

$$\sqrt{n}\left(\hat{V}_n^{\Pi}(x) - V_n^{(\underline{x},\bar{x})}(x)\right) = \frac{1}{\bar{x}-\underline{x}}\left\{\sqrt{n}\left(U_n^*(\bar{x}) - U_n(\bar{x})\right) + \sqrt{n}\left(U_n^*(\underline{x}) - U_n(\underline{x})\right)\right\}.$$

Both terms on the right tend to zero by Lemma 3.

We conclude by giving the asymptotic distributions. By the Central Limit Theorem:

$$\sqrt{n}\left(V_n^{(\underline{x},\bar{x})}(x) - V(x)\right) \rightsquigarrow N(0, \sigma^2_{\underline{x},\bar{x}}). \tag{A.20}$$

The estimators $V_n^{\Pi}(x)$ and $\hat{V}_n^{\Pi}(x)$ have the same limiting behavior by Slutsky's lemma and (A.18)–(A.19). □

The proof Theorem 1.c is deferred to the next section because it relies on Lemma 4, Lemma 5 and Proposition 2, which can be understood only in the context of Section 4.

Proofs of Section 4

Proof of Proposition 2. Define:

$$J_{n,i} := \left(1 - \frac{1}{c_{t_n}}\right) + \frac{1}{g_V(Z_i)}\int_{\{s\in[Z_i,M],\,(1+t_nh_0(s))\le 0\}} c_{t_n}^{-1}(1+t_nh_0(s))\frac{dV(s)}{\pi\sqrt{s-Z_i}}, \tag{A.21}$$

$$\Delta_{n,i} := \frac{1}{c_{t_n}\sqrt{n}}\left(k^{(\underline{x},\bar{x})}(Z_i) - \mathbb{E}_{G_V} k^{(\underline{x},\bar{x})}(Z)\right).$$

Using the definition of h_0:

$$\frac{t_n}{c_{t_n}}\int_{Z_i}^{M} -\frac{h_0(s)v(s)}{\pi\sqrt{s-Z_i}}\,ds = t\Delta_{n,i}\,g_V(Z_i). \tag{A.22}$$

Now note that for any i, using $1-\frac{(1+x)_+}{c} = -\frac{x}{c} + 1 - \frac{1}{c} + \frac{(1+x)_-}{c}$ and (A.21)–(A.22):

$$\begin{aligned}\frac{g_{V_{t_n}}(Z_i)}{g_V(Z_i)} - 1 &= \frac{1}{g_V(Z_i)}\left(\int_{Z_i}^{M}\left(1 - \frac{(1+t_nh_0(s))_+}{c_{t_n}}\right)\frac{dV(s)}{\pi\sqrt{s-Z_i}}\right)\\ &= -\frac{t_n}{c_{t_n}g_V(Z_i)}\int_{Z_i}^{M}\frac{h_0(s)v(s)}{\pi\sqrt{s-Z_i}}\,ds + J_{n,i} = t\Delta_{n,i} + J_{n,i}.\end{aligned} \tag{A.23}$$

Through the Taylor expansion $\log\{1+x\} = x - \frac{x^2}{2} + x^2R(2x)$ where $R(x)\to 0$ as $x\to 0$, we can express the log-likelihood ratio:

$$\begin{aligned}&\sum_{i=1}^{n}\log\frac{g_{V_{t_n}}(Z_i)}{g_V(Z_i)}\\ &= \sum_{i=1}^{n}\left\{t\Delta_{n,i} + J_{n,i} - \frac{1}{2}(t\Delta_{n,i}+J_{n,i})^2 + (t\Delta_{n,i}+J_{n,i})^2R(2(t\Delta_{n,i}+J_{n,i}))\right\}.\end{aligned} \tag{A.24}$$

We start by showing that:

$$c_{t_n} = -\frac{2}{\pi}\int_0^M \sqrt{s}\left(1+t_nh_0(s)\right)_+ dV(s) = 1 + o\left(\frac{1}{n}\right). \tag{A.25}$$

Let $S\sim\mu_V$ for μ_V defined in (B.2). Since $\int h_0\,d\mu_V = 0$:

$$c_{t_n} \ge -\frac{2}{\pi}\int_0^M \sqrt{s}\left(1+t_nh_0(s)\right)dV(s) = 1.$$

Now note:

$$c_{t_n} = -\frac{2}{\pi}\int_{\{s\in[0,M],\,(1+t_nh_0(s))>0\}} \sqrt{s}\left(1+t_nh_0(s)\right)dV(s) = 1 + \frac{2}{\pi}\int_{\{s\in[0,M],\,(1+t_nh_0(s))\le 0\}} \sqrt{s}\left(1+t_nh_0(s)\right)dV(s) = 1 + \mathbb{E}_{\mu_V}\left[\left|1+t_nh_0(S)\right|\mathbf{1}_{\{t_nh_0(S)\le -1\}}\right].$$

If $t_nh_0(s)\le -1$ then: $|1+t_nh_0(s)| \le |t_nh_0(s)| \le t_n^2(h_0(s))^2$, so that:

$$\frac{1}{t_n^2}\mathbf{1}_{\{t_nh_0(s)\le -1\}}\left|1+t_nh_0(s)\right| \le (h_0(s))^2\mathbf{1}_{\{t_nh_0(s)\le -1\}}. \tag{A.26}$$

Now $\mathbb{E}_{\mu_V}\big[h_0^2(S)\big] \le \mathbb{E}_{\mu_V}\big[h^2(S)\big]$, which by (B.10) is bounded by (4.2), which is finite by assumption. Therefore:

$$\limsup_{n\to\infty} \frac{1}{t_n^2}\mathbb{E}_{\mu_V}\Big[\big|1+t_n h_0(S)\big|\,\mathbf{1}_{\{t_n h_0(S)\le -1\}}\Big] = 0, \tag{A.27}$$

by the Dominated Convergence theorem. This concludes the proof of (A.25) as $t_n^2 = t^2/n$.

By (A.25) and the Central Limit theorem: $\sum_{i=1}^n \Delta_{n,i} \rightsquigarrow N(0,\sigma^2_{\underline{x},\bar{x}})$. Now note that because the measure induced by dV is a nonpositive measure (since V is decreasing), $\forall\, i$:

$$\frac{1}{g_V(Z_i)}\int_{\{s\in[Z_i,M],\,(1+t_n h_0(s))\le 0\}} c_{t_n}^{-1}(1+t_n h_0(s))\,\frac{dV(s)}{\pi\sqrt{s-Z_i}} \ge 0, \qquad \text{a.s.} \tag{A.28}$$

Again by (A.25), by Markov's inequality and Fubini's theorem, $\sum_{i=1}^n J_{n,i} = o_p(1)$ because $\forall\,\varepsilon>0$:

$$\mathbb{P}_{G_V}\left(\sum_{i=1}^n \frac{1}{g_V(Z_i)}\int_{\{s\in[Z_i,M],\,(1+t_n h_0(s))\le 0\}} c_{t_n}^{-1}(1+t_n h_0(s))\,\frac{dV(s)}{\pi\sqrt{s-Z_i}} > \varepsilon\right) \le \frac{n}{\varepsilon}\int_0^M\int_z^M \mathbf{1}_{\{(1+t_n h_0(s))\le 0\}}\, c_{t_n}^{-1}\left(1+t_n h_0(s)\right)\frac{v(s)}{\pi\sqrt{s-z}}\,ds\,dz$$
$$= \frac{n}{\varepsilon\pi}\int \mathbf{1}_{\{(1+t_n h_0(s))\le 0\}}\, c_{t_n}^{-1}\left(1+t_n h_0(s)\right)v(s)\int_0^s \frac{1}{\sqrt{s-z}}\,dz\,ds = \frac{n}{\varepsilon c_{t_n}}\mathbb{E}_{\mu_V}\Big[\big|1+t_n h_0(S)\big|\,\mathbf{1}_{\{t_n h_0(S)\le -1\}}\Big] \overset{(A.27)}{=} o(1).$$

All together this proves, for Δ_n as in (4.6): $\sum_{i=1}^n (t\Delta_{n,i}+J_{n,i}) = t\Delta_n + o_p(1)$, proving the behavior of the linear term.

For the quadratic term, if $\max_{1\le i\le n}|t\Delta_{n,i}+J_{n,i}| \xrightarrow{\mathbb{P}} 0$, by property of the function R, the sequence $\max_{1\le i\le n}|R(t\Delta_{n,i}+J_{n,i})| \xrightarrow{\mathbb{P}} 0$ as well. By the triangle inequality:

$$\max_{1\le i\le n}|t\Delta_{n,i}+J_{n,i}| \le |t|\max_{1\le i\le n}|\Delta_{n,i}| + \sum_{i=1}^n |J_{n,i}|.$$

The first term tends to zero in probability because for $\{X_i,\ i\ge 1\}$ i.i.d. random variables with finite variance: $\max_{1\le i\le n}\left\{\frac{|X_i|}{\sqrt{n}}\right\} \xrightarrow{\mathbb{P}} 0$. The second term tends to 0 by (A.25) and (A.29).

By the Law of Large numbers, $\sum_{i=1}^n \Delta_{n,i}^2 \to \sigma^2_{\underline{x},\bar{x}}$ in probability. Therefore we are left with showing:

$$\sum_{i=1}^n \Delta_{n,i}J_{n,i} \xrightarrow{\mathbb{P}} 0 \qquad \text{and} \qquad \sum_{i=1}^n (J_{n,i})^2 \xrightarrow{\mathbb{P}} 0. \tag{A.29}$$

The second follows because $\sum_{i=1}^n (J_{n,i})^2 \le \max_{1\le i\le n}|J_{n,i}|\sum_{i=1}^n J_{n,i}$ and the first next follows with the help of the Cauchy–Schwarz inequality. □

For the proof of the LAM (locally asymptotically minimax) theorem, the first statement of Theorem 2, and for the proof of Theorem 1.c, we need the derivative of the map we are interested in estimating, in our case $V(x)$ (cf. definition 1.10 in Bolthausen et al. (1999)). This is given in the following lemma.

Lemma 4. *Along the path given in (4.5), as $n\to\infty$,*

$$\sqrt{n}\left(V_{t_n}(x) - V(x)\right) \to t\,\sigma^2_{\underline{x},\bar{x}},$$

where $\sigma^2_{\underline{x},\bar{x}}$ is as in (3.4).

Proof. Because $\int_x^s \frac{\sqrt{w-x}}{\sqrt{s-w}}\,dw = \frac{\pi}{2}(s-x)$ for every x, we have, $\forall\, x\in[\underline{x},\bar{x}]$ by Fubini's theorem:

$$\int_0^M 2\frac{\sqrt{(w-\underline{x})_+}-\sqrt{(w-\bar{x})_+}}{\bar{x}-\underline{x}}\,g_V(w)\,dw = -\frac{2}{\pi}\frac{1}{\bar{x}-\underline{x}}\left(\int_{\underline{x}}^M\int_{\underline{x}}^s \frac{\sqrt{w-\underline{x}}}{\sqrt{s-w}}\,dw\,dV(s) - \int_{\underline{x}}^M\int_{\bar{x}}^s \frac{\sqrt{w-\bar{x}}}{\sqrt{s-w}}\,dw\,dV(s)\right)$$
$$= -\frac{2}{\pi}\frac{1}{\bar{x}-\underline{x}}\int_{\underline{x}}^M \frac{\pi}{2}(\bar{x}-\underline{x})\,dV(s) = V(\underline{x}) = V(x). \tag{A.30}$$

Using the same computations as done for (A.23) we have:

$$\left(g_{V_t}(z)-g_V(z)\right) = -\frac{t_n}{c_{t_n}}\int_z^M \frac{h_0(s)v(s)}{\pi\sqrt{s-z}}\,ds + \left(1-\frac{1}{c_{t_n}}\right)\int_z^M \frac{dV(s)}{\pi\sqrt{s-z}} + \int_{\{s\in[z,M],\,(1+t_n h_0(s))\le 0\}} c_{t_n}^{-1}(1+t_n h_0(s))\,\frac{dV(s)}{\pi\sqrt{s-z}}.$$

Because $(\mathbb{E}_{G_V}k^{(\underline{x},\bar{x})})\int_0^M (g_{V_t}(w)-g_V(w))\,dw = 0$, using (A.30), we obtain:

$$\sqrt{n}\left(V_{t_n}(\underline{x})-V(\underline{x})\right) = \sqrt{n}\int_0^M \left(k^{(\underline{x},\bar{x})}(z)-\mathbb{E}_{G_V}k^{(\underline{x},\bar{x})}\right)\left(g_{V_{t_n}}(z)-g_V(z)\right)dz$$
$$= \frac{t}{c_{t_n}}\int_0^M \left(k^{(\underline{x},\bar{x})}(z)-\mathbb{E}_{G_V}k^{(\underline{x},\bar{x})}\right)\left(-\int_z^M \frac{h_0(s)v(s)}{\pi\sqrt{s-z}}\,ds\right)dz + \sqrt{n}\left(\frac{1}{c_{t_n}}-1\right)\underbrace{\int_0^M \left(k^{(\underline{x},\bar{x})}(z)-\mathbb{E}_{G_V}k^{(\underline{x},\bar{x})}\right)g_V(z)\,dz}_{=0}$$

$$+\sqrt{n}\int_0^M \left(k^{(\underline{x},\bar{x})}(z)-\mathbb{E}_{G_V} k^{(\underline{x},\bar{x})}\right)\int_{\substack{c_{t_n}^{-1}(1+t_n h_0(s))\\ \{s\in[z,M],\,(1+t_n h_0(s))\le 0\}}} \frac{dV(s)}{\pi\sqrt{s-z}}\,dz \xrightarrow{n\to\infty} t\int_0^M \left(k^{(\underline{x},\bar{x})}(w)-\mathbb{E}_{G_V} k^{(\underline{x},\bar{x})}\right)^2 g_V(w)\,dw = t\,\sigma^2_{\underline{x},\bar{x}}.$$

by (A.22) and (A.27), where we use that $k^{(\underline{x},\bar{x})}(z)-\mathbb{E}_{G_V} k^{(\underline{x},\bar{x})}$ is bounded on $[0,M]$. □

The proof of Theorem 1.c, is based on the fact that the isotonic estimator is regular on the submodel $t\mapsto G_{V_t}$, i.e. the limit distribution of the sequence $\sqrt{n}(\hat{V}_n(x)-V_{G_{V_{t_n}}}(x))$ under $G_{V_{t_n}}$ is the same, for all $t\in\mathbb{R}$.

Lemma 5. *The IIE is regular in the submodel defined in (4.5).*

Proof. For clarity of notation, in this proof we denote $U(x)$ as: $U_V(x)=\int_0^x V(y)\,dy$ and thus $U_{V_{t_n}}(x)=\int_0^x V_{t_n}(y)\,dy$. We show, under $G_{V_{t_n}}$: $\sqrt{n}(\hat{V}_n(x)-V_{t_n}(x))\rightsquigarrow L_x$, which is independent of t. We use the same steps as in the proof of Theorem 4 in Gili et al. (2024). Using the switch relation, we study:

$$\inf\left\{\, s\ge 0 : (U_n(s)-U_n(x)-U_{V_{t_n}}(s)+U_{V_{t_n}}(x))+(U_{V_{t_n}}(s)-U_{V_{t_n}}(x)-V_{t_n}(x)(s-x))-(as)/\sqrt{n} \text{ is maximal } \right\}\le x.$$

Using the same proof of Lemma 1 in Gili et al. (2024) it follows that, under $G_{V_{t_n}}$:

$$\sqrt{n}(U_n(s)-U_n(x)-U_{V_{t_n}}(s)+U_{V_{t_n}}(x)\ :\ s\ge 0)\rightsquigarrow \mathbb{Z}_x \quad \text{in } \ell^\infty[0,\infty).$$

Furthermore:

$$U_{V_{t_n}}(s)-U_{V_{t_n}}(x)-V_{t_n}(x)(s-x)=\int_x^s \left(V_{t_n}(y)-V_{t_n}(x)\right)dy=\begin{cases}0, & s,x\in[\underline{x},\bar{x}],\\ <0, & x\leqslant\bar{x}<s,\quad s<\underline{x}\leqslant x.\end{cases}$$

Now for $s=\bar{x}+\varepsilon_n$ and $\varepsilon_n\downarrow 0$ as in the proof of Theorem 4 in Gili et al. (2024) and using the same reasoning as in the proof of (A.25), we obtain:

$$\begin{aligned}&\sqrt{n}\int_{\bar{x}}^{\bar{x}+\varepsilon_n}\left(V_{t_n}(y)-V_{t_n}(\bar{x})\right)dy\\ &=-\sqrt{n}\int_{\bar{x}}^{\bar{x}+\varepsilon_n}\int_{\bar{x}}^{y} c_{t_n}^{-1}\left(1+\frac{t}{\sqrt{n}}h_0(s)\right)_+ dV(s)dy=-\sqrt{n}\int_{\bar{x}}^{\bar{x}+\varepsilon_n}\int_{\bar{x}}^{y} c_{t_n}^{-1}\left(1+\frac{t}{\sqrt{n}}h_0(s)\right)dV(s)dy+o\left(\frac{1}{\sqrt{n}}\right)\to-\infty,\end{aligned}$$

since by the proof of Theorem 4 in Gili et al. (2024) we have: $\sqrt{n}\int_{\bar{x}}^{\bar{x}+\varepsilon_n} V(y)-V(\bar{x})\,dy\to-\infty$. The same reasoning holds also for: $\sqrt{n}\int_{\underline{x}-\varepsilon_n}^{\underline{x}}\left(V_{t_n}(y)-V_{t_n}(\underline{x})\right)dy\to-\infty$. Using further the same computations as in the proof of Theorem 4 in Ghosal and van der Vaart (2017) we conclude:

$$\mathbb{P}\left(\sqrt{n}(\hat{V}_n(x)-V_{t_n}(x))\le a\right)\xrightarrow{n\to\infty}\mathbb{P}\left(\underset{s\in[\underline{x},\bar{x}]}{\operatorname{argmax}}\{\mathbb{Z}_x(s)-as\}\le x\right).\quad \square$$

Proof of Theorem 1.c (*c.f. Theorem 1 in Section 2.3 of Bickel et al. (1998)*)**.** The efficient estimator satisfies:

$$\sqrt{n}\left(V_n^{(\underline{x},\bar{x})}(x)-V_0(x)\right)=\Delta_n+o_p(1)$$

where $V_0\equiv V$ but this notation clarifies that it coincides with the perturbed V_t in (4.5) evaluated at $t=0$. $\Delta_n\rightsquigarrow N(0,\sigma^2_{\underline{x},\bar{x}})$ is the linear term in (4.6) in the expansion of $\Lambda_n(t):=\sum_{i=1}^n\log\frac{g_{V_{t_n}}(Z_i)}{g_V(Z_i)}$. The joint sequence

$$\left(\sqrt{n}(\hat{V}_n(x)-V_n^{(x,x)}(x)),\ \sqrt{n}(V_n^{(x,x)}(x)-V_0(x))\right)$$

is uniformly tight (because marginally so). It suffices to show that every joint limit point under V_0 has independent coordinates. A joint limit point takes the form $(T-\Delta,\Delta)$ for (T,Δ) a joint limit of $\left(\sqrt{n}\left(\hat{V}_n(x)-V_0(x)\right),\Delta_n\right)$ under V_0. We have:

$$\left(\sqrt{n}\left(\hat{V}_n(x)-V_0(x)\right),\Lambda_n(t)\right)\overset{V_0}{\rightsquigarrow}\left(T,t\Delta-\frac{1}{2}t^2\sigma^2_{\underline{x},\bar{x}},\right).$$

Therefore by Le Cam's 3rd lemma (e.g. Theorem 6.6 in van der Vaart (1998)):

$$\sqrt{n}\left(\hat{V}_n(x)-V_0(x)\right)\overset{V_{t/\sqrt{n}}}{\rightsquigarrow}L_t,\qquad L_t(B)=\mathbb{E}\mathbf{1}_B(T)e^{t\Delta-\frac{1}{2}t^2\sigma^2_{\underline{x},\bar{x}}}$$

because $\sqrt{n}\left(V_{t/\sqrt{n}}(x)-V_0(x)\right)\xrightarrow{n\to\infty}t\sigma^2_{\underline{x},\bar{x}}$ (see Lemma 4) we conclude:

$$\sqrt{n}\left(\hat{V}_n(x)-V_{t/\sqrt{n}}(x)\right)\overset{V_{t/\sqrt{n}}}{\rightsquigarrow}\bar{L}_t,\qquad \bar{L}_t(B)=L_t\left(B-t\sigma^2_{\underline{x},\bar{x}}\right).$$

By regularity of the IIE (see Lemma 5): $\bar{L}_t=\bar{L}_0$, independent of t, so $\forall\,t,u$

$$\int e^{iuy}d\bar{L}_0(y)=\mathbb{E}e^{iu\left(T-t\sigma^2_{\underline{x},\bar{x}}\right)}e^{t\Delta-\frac{1}{2}t^2\sigma^2_{\underline{x},\bar{x}}}.$$

Choose $t = -iu$ to find: $\int e^{iuy} d\bar{L}_0(y) = \mathbb{E}e^{iu(T-\Delta)}e^{-\frac{1}{2}u^2\sigma^2_{\underline{x},\bar{x}}}$. Choose $t = -iu + iv$ to find: $\int e^{iuy} d\bar{L}_0(y) = \mathbb{E}e^{iu(T-\Delta)+iv\Delta}e^{\frac{1}{2}(v^2-u^2)\sigma^2_{\underline{x},\bar{x}}}$. Then:

$$\mathbb{E}e^{iu(T-\Delta)+iv\Delta} = \mathbb{E}e^{iu(T-\Delta)}e^{-\frac{1}{2}v^2\sigma^2_{\underline{x},\bar{x}}} = \mathbb{E}e^{iu(T-\Delta)}\mathbb{E}e^{iv\Delta}. \quad \square$$

Proof of Theorem 2. The proof is a direct application of Theorems 25.20 and 25.21 from van der Vaart (1998), using Proposition 2 and Lemma 4. The variance of the limiting random variable in (4.7) is given by $\dot{v}(\sigma^2_{\underline{x},\bar{x}})^{-1}\dot{v} = \sigma^2_{\underline{x},\bar{x}}$, where $\dot{v} = \sigma^2_{\underline{x},\bar{x}}$ is the derivative of the functional of interest and $(\sigma^2_{\underline{x},\bar{x}})^{-1}$ is the Fisher information for t, given in Proposition 2. $\square$

Appendix B. Efficiency theory

In this section we derive the least favorable submodel (4.5) from the semiparametric score calculus, as in Van Der Vaart (1991) (or Chapter 25 of van der Vaart (1998)).

The parameter V belongs to the class (c.f. Groeneboom and Jongbloed (1995)), for $\underline{x} < \bar{x}$:

$$\mathscr{V}_{\underline{x},\bar{x}} = \left\{ V \in \mathcal{V}_{\underline{x},\bar{x}}, \ : \ \int_0^\infty \sqrt{s}\, dV(s) = -\frac{\pi}{2}, \text{ zero on } [M,\infty) \right\}. \tag{B.1}$$

Because the measure induced by dV is not a probability measure, it is convenient to work with its associated probability measure μ_V defined for every measurable $A \subseteq [0,\infty)$ by:

$$\mu_V(A) := \int_A -\frac{2}{\pi}\sqrt{s}\, dV(s). \tag{B.2}$$

We consider the associated space of "underlying" probability measures $\mathscr{M}_{\underline{x},\bar{x}} := \{\mu_V \ : \ V \in \mathscr{V}_{\underline{x},\bar{x}}\}$ and the space of the "observed" probability measures $\mathscr{G}_{\underline{x},\bar{x}} := \{g_V \ : \ \mu_V \in \mathscr{M}_{\underline{x},\bar{x}}\}$. The tangent set $\dot{\mathscr{M}}_{\underline{x},\bar{x}}$ consists of all functions $h \in \mathbb{L}_2(\mu_V)$ such that $\int h\, d\mu_V = 0$. The derivative of the map $\mu_V \mapsto g_V$ (the score operator) maps score functions in the tangent set $\dot{\mathscr{M}}_{\underline{x},\bar{x}}$ into scores in the tangent set $\dot{\mathscr{G}}_{\underline{x},\bar{x}}$. For any $h \in \mathbb{L}_1(\mu_V)$ such that $-\int_z^M \frac{|h(s)|}{\sqrt{s-z}}\, dV(s) < \infty$, define:

$$(Th)(z) := \int_z^M \frac{h(s)}{\sqrt{s-z}}\, dV(s). \tag{B.3}$$

Then the score operator $A_V \ : \ \dot{\mathscr{M}}_{\underline{x},\bar{x}} \to \dot{\mathscr{G}}_{\underline{x},\bar{x}}$, for any $h \in \dot{\mathscr{M}}_{\underline{x},\bar{x}}$, is given by:

$$A_V h(z) = \frac{\int_z^M \frac{h(s)}{\sqrt{s-z}}\, dV(s)}{\int_z^M \frac{dV(s)}{\sqrt{s-z}}} = \frac{(Th)(z)}{(T1)(z)}. \tag{B.4}$$

For instance, for bounded $h \in \dot{\mathscr{M}}_{\underline{x},\bar{x}}$ define the following perturbation:

$$dV_t(s) = (1 + th(s))\, dV(s). \tag{B.5}$$

Then:

$$\frac{d}{dt}\log\left\{ -\int_z^M \frac{(1+th(s))}{\pi\sqrt{s-z}} dV(s) \right\}\Bigg|_{t=0} = \frac{\int_z^M \frac{h(s)}{\sqrt{s-z}} dV(s)}{\int_z^M \frac{dV(s)}{\sqrt{s-z}}} =: A_V h(z).$$

We check that A_V is a linear operator between the $\mathbb{L}_2$ spaces claimed above. Linearity is clear. We can check by using the Cauchy–Schwarz inequality and that $h \in \mathbb{L}^2(\mu_V)$:

$$\begin{aligned}
\int (A_V h)^2 dG_V &= \int_0^\infty \left(\frac{\int_z^\infty \frac{h(s)dV(s)}{\sqrt{s-z}}}{\int_z^\infty \frac{dV(s)}{\sqrt{s-z}}} \right)^2 \left(-\frac{1}{\pi}\int_z^\infty \frac{dV(s)}{\sqrt{s-z}} \right) dz \\
&\leqslant \int_0^\infty \left(\frac{\sqrt{\int_z^\infty -\frac{h^2(s)}{\sqrt{s-z}} dV(s)}\sqrt{\int_z^\infty -\frac{1}{\sqrt{s-z}} dV(s)}}{-\int_z^\infty \frac{dV(s)}{\sqrt{s-z}}} \right)^2 \left(-\frac{1}{\pi}\int_z^\infty \frac{dV(s)}{\sqrt{s-z}} \right) dz \\
&= -\frac{1}{\pi}\int_0^\infty \int_z^\infty \frac{h^2(s)}{\sqrt{s-z}} dV(s)dz = -\frac{2}{\pi}\int_0^\infty h^2(s)\sqrt{s}dV(s) = \int h^2\, d\mu_V < \infty.
\end{aligned} \tag{B.6}$$

The function $k^{(\underline{x},\bar{x})}$ is the influence function of the (regular) estimator (2.15). To show that this estimator is asymptotically efficient, it suffices to show that $k^{(\underline{x},\bar{x})}$ is contained in the closed linear span of the set of all score functions $A_V h$. We shall show that in fact $k^{(\underline{x},\bar{x})} - \mathbb{E}k^{(\underline{x},\bar{x})} = A_V h_0$ for h_0 given in (4.4). Equivalently, we show that h_0 solves

$$\begin{cases} (Th_0) = \left(k^{(\underline{x},\bar{x})} - \mathbb{E}_{G_V} k^{(\underline{x},\bar{x})} \right) \cdot (T1) \\ \int_0^M -|h_0(s)|\, dV(s) < \infty, \qquad \int_0^M \sqrt{s}h_0(s)\, dV(s) = 0. \end{cases} \tag{B.7}$$

Proposition 3. *Under Assumption 1, the solution to (B.7) is given by (4.4).*

Proof of Proposition 3. We show that the function h in (4.3) solves: $Th = q$, for $q = k^{(\underline{x},\bar{x})} \cdot T1$. Then the solution h_0 is found by centering h, i.e. $h_0(z) = h(z) - m$ (since $\int_0^M \sqrt{s}\, dV(s) = -\frac{\pi}{2}$) where m is such that:

$$\int_0^M \sqrt{s}\,(h(s) - m)\, dV(s) = 0 \quad \implies \quad m = -\frac{2}{\pi}\int_0^M \sqrt{s}h(s)\, dV(s)$$

The equation $Th = q$, for $q = k^{(\underline{x},\bar{x})} \cdot T1$ is the following Abel's integral equation:

$$\int_z^M \frac{h(s)v(s)}{\sqrt{s-z}}\, ds = 2\frac{\sqrt{(z-\underline{x})_+} - \sqrt{(z-\bar{x})_+}}{\bar{x}-\underline{x}} \int_z^M \frac{v(s)}{\sqrt{s-z}}\, ds$$

By Theorem 2.1 in Samko et al. (1993) this equation is solvable for $hv \in \mathbb{L}_1[0, M]$ iff:

$$z \mapsto \int_z^M \frac{q(s)}{\sqrt{s-z}}\, ds$$

is absolutely continuous. Furthermore the unique in $\mathbb{L}_1$ solution is given by $h(z)v(z) = -\frac{d}{dz}\int_z^M \frac{q(s)}{\sqrt{s-z}}\, ds$, where the derivative is understood in the sense of absolute continuity (AC). Moreover, if $q \in \mathrm{AC}[0, M]$, then this condition is satisfied and for $z \notin [\underline{x}, \bar{x}]$:

$$h(z)v(z) = \frac{1}{\sqrt{\pi}}\left(\frac{q(M)}{\sqrt{M-z}} - \int_z^M \frac{q'(s)}{\sqrt{s-z}}\, ds\right). \tag{B.8}$$

Now we prove that under the given conditions $k^{(\underline{x},\bar{x})} \cdot T1 = q \in \mathrm{AC}[0, M]$. Because $\sqrt{(z-x)_+} = \int_0^z \frac{1}{2\sqrt{s-x}}\mathbf{1}_{s>x}\, ds$ we see that $k^{(\underline{x},\bar{x})} \in \mathrm{AC}[0, M]$. $T1$ is the $1/2$-fractional integral of v. Because $v \in \mathrm{AC}[\underline{x}, \bar{x}]^c$ and $v = 0$ on $[\underline{x}, \bar{x}]$, we know that $T1 \in \mathrm{AC}[0, M]$ (c.f. property 3.2.(6).(b) pp. 209 in Bonilla et al. (1999)). The product of absolutely continuous functions is absolutely continuous.

Finally we show:

$$\mathbb{E}_{G_V} k^{(\underline{x},\bar{x})} = -\frac{2}{\pi}\int_0^M \sqrt{s}h(s)v(s)\, ds.$$

For the given h in (4.3) we just proved:

$$\int_z^M -\frac{h(s)v(s)}{\pi\sqrt{s-z}}\, ds = 2\frac{\sqrt{(z-\underline{x})_+} - \sqrt{(z-\bar{x})_+}}{\bar{x}-\underline{x}} g_V(z). \tag{B.9}$$

By integrating both sides between 0 and M we get that the r.h.s. is $\mathbb{E}_{G_V} k^{(\underline{x},\bar{x})}$, whereas the l.h.s.

$$\int_0^M \int_z^M -\frac{h(s)v(s)}{\pi\sqrt{s-z}}\, ds\, dz = \int_0^M \int_0^s -\frac{h(s)v(s)}{\pi\sqrt{s-z}}\, dz\, ds = -\frac{2}{\pi}\int_0^M \sqrt{s}h(s)v(s)\, ds.$$

All together this proves that h_0 solves (B.7). □

Lemma 6. *For a fixed $\eta > 0$, there exist constants $c_1, c_2, c_3 > 0$ such that, for $z \in [\underline{x}, \bar{x}]^c$:*

$$|h(z)v(z)| \le c_1 + c_2 \log\left(\frac{1}{\underline{x}-z}\right)\mathbf{1}_{\{0<\underline{x}-z\le\eta\}} + c_3 \log\left(\frac{1}{z-\bar{x}}\right)\mathbf{1}_{\{0<z-\bar{x}\le\eta\}}. \tag{B.10}$$

Proof. Because $\lim_{z\to M}(T1)(z) = 0$, the function q in (B.8) satisfies $q(M) = 0$. Hence, for $z \in [\underline{x}, \bar{x}]^c$:

$$h(z) = -\frac{1}{\sqrt{\pi}v(z)}\Bigg(\underbrace{\int_z^M \frac{(k^{(\underline{x},\bar{x})})'(T1)(s)}{\sqrt{s-z}}\, ds}_{(1)} + \underbrace{\int_z^M \frac{(k^{(\underline{x},\bar{x})})(T1)'(s)}{\sqrt{s-z}}\, ds}_{(2)}\Bigg)$$

where:

$$(k^{(\underline{x},\bar{x})})'(s) = \frac{1}{(\bar{x}-\underline{x})}\left(\frac{1}{\sqrt{(s-\underline{x})_+}} - \frac{1}{\sqrt{(s-\bar{x})_+}}\right).$$

We prove (2) is bounded. By the same proof of Corollary 2.1 in Samko et al. (1993) (pp. 32), since $(T1) \in \mathrm{AC}[0, M]$ (see proof of Proposition 3), $\lim_{z\to M} v(z) = 0$, and denoting by $\mathrm{I}^{\frac{1}{2}}(f)$ the half–integral of f (see Samko et al. (1993) for an introduction to fractional calculus):

$$\frac{d}{dz}\int_z^M \frac{v(s)\, ds}{\sqrt{s-z}} = \int_z^M \frac{v'(s)\, ds}{\sqrt{s-z}} - \lim_{s\to M}\frac{v(s)}{\sqrt{s-z}} = \mathrm{I}^{\frac{1}{2}}(v'),$$

where the derivative is understood in the sense of absolute continuity (AC). Thus (2) coincides with $\mathrm{I}^{\frac{1}{2}}(k^{(\underline{x},\bar{x})}\,\mathrm{I}^{\frac{1}{2}}(v'))$. By Theorem 3.6 in Samko et al. (1993), $\mathrm{I}^{\frac{1}{2}} : \mathbb{L}_p \mapsto H^{1/2-1/p}$, where H^{α} denotes the space of Hölder continuous functions of degree α. Thus if $k^{(\underline{x},\bar{x})}\,\mathrm{I}^{\frac{1}{2}}(v') \in \mathbb{L}_p$ for $p > 2$ we obtain the claim. Because $k^{(\underline{x},\bar{x})}$ is bounded we show $\mathrm{I}^{\frac{1}{2}}(v') \in \mathbb{L}_p$ for $p > 2$. By theorem 3.5 in Samko et al. (1993), $\mathrm{I}^{\frac{1}{2}}(v') \in \mathbb{L}_{q/(1-q/2)}$ if $v' \in \mathbb{L}_q$. Because v is Lipschitz on $[\underline{x},\bar{x}]^c$ and 0 on $[\underline{x},\bar{x}]$, $\exists\, q > 1$ such that $v' \in \mathbb{L}_q$. As $q > 1 \iff q/(1-q/2) > 2$, $\mathrm{I}^{\frac{1}{2}}(v') \in \mathbb{L}_p$ for some $p > 2$ and we obtain the claim.

Regarding the absolute value of (1), using the fact that $(T1)(s) = -\frac{\pi}{2} g_V(s)$, which is bounded as well as v:

$$\left| -\frac{1}{\sqrt{\pi}} \int_{z\vee\underline{x}}^{M} \frac{(T1)(s)}{\sqrt{(s-\underline{x})(s-z)}}\, ds \right| \lesssim \left| \int_{z\vee\underline{x}}^{M} \frac{ds}{\sqrt{(s-\underline{x})(s-z)}} \right|.$$

For $z > \bar{x}$ the previous display is bounded. For $z < \underline{x}$ it is upper bounded (up to constants) by:

$$\left| \sin^{-1}\left(\sqrt{\frac{M-z}{\underline{x}-z}} \right) \right| = 2\log\sqrt{\frac{M-z}{\underline{x}-z}} + O(1) + O\left(\frac{\underline{x}-z}{M-z}\right) \quad \text{as} \quad z \uparrow \underline{x}.$$

The integral $\int_{z\vee\bar{x}}^{M} \frac{(T1)(s)}{\sqrt{(s-\bar{x})(s-z)}}\, ds$ can be handled analogously. □

Minimization algorithm to find Q^{V_n} projection of the naive estimator

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Algorithm 1: Compute $V_n^{a^*} \equiv V_n^{\Pi}$

Input: vector observations $\mathbf{Z}$ and **Algorithm 2:** Compute V_n^a.
Output: $V_n^{a^*} \equiv V_n^{\Pi}$
Using the preferred minimization procedure of the reader:
$a^* = \arg\min_{a\in\mathbb{R}^+} Q^{V_n}(V_n^a) = \arg\min_{a\in\mathbb{R}^+} \int_0^\infty V_n^a(x)(V_n^a(x) - 2V_n(x))\, dx$,
where V_n^a is computed using **Algorithm 2**.

Algorithm 2: Compute V_n^a.

Input: $a > 0$, vector observations $\mathbf{Z}$.
Output: V_n^a
Compute U_n and U_n^*, the least concave majorant of U_n.
Initialize an evaluation grid $\mathbf{x} = [x_1, \dots, x_N]$.

L-LCM Compute the restricted Least Concave Majorant of U_n from the Left, ${}^lU_n^*$.
Initialize appropriately ${}^lU_n^*$, and $i = 1$. Let $(U_n^*)'$ be the r.h.s. derivative of U_n^*.

while $(U_n^*)'(x_i) > a$ **do**
  ${}^lU_n^*(x_i) \leftarrow U_n^*(x_i)$
  $i \leftarrow i + 1$

$i^* = \max\{i \in \{1,\dots,N\} \ :\ ({}^lU_n^*)'(x_i) > a\}$
for $j \in \{i^*+1,\dots,N\}$ **do**
  ${}^lU_n^*(x_i) \leftarrow a \cdot x_i$

R-LCM Compute the restricted Least Concave Majorant of U_n from the Right, ${}^rU_n^*$.
Initialize appropriately ${}^rU_n^*$, and $i = N$.

while $(U_n^*)'(x_i) < a$ **do**
  ${}^rU_n^*(x_i) \leftarrow U_n^*(x_i)$
  $i \leftarrow i - 1$

$i_* = \min\{i \in \{1,\dots,N\} \ :\ ({}^rU_n^*)'(x_i) < a\}$
for $j \in \{1,\dots,i_*-1\}$ **do**
  ${}^rU_n^*(x_i) \leftarrow a \cdot x_i$

$i_{\bar{x}} = \arg\min_{i\,:\,x_i\geq\bar{x}}\{|x_i - \bar{x}|\}$
$i_{\underline{x}} = \arg\min_{i\,:\,x_i\leq\underline{x}}\{|x_i - \underline{x}|\}$
if $x_{i^*} < \underline{x}$ *or* $x_{i^*} > \bar{x}$ **then**

$$V_n^a = \begin{cases} ({}^lU_n^*)'(x_i) & \text{for } i \in \{1,\dots,i_{\underline{x}}-1\} \\ a & \text{for } x_i \ :\ i \in \{i_{\underline{x}},\dots,i_{\bar{x}}\} \\ ({}^rU_n^*)'(x_i) & \text{for } i \in \{i_{\bar{x}}+1,\dots,N\} \end{cases}$$

else if $x_{i^*} \in [\underline{x},\bar{x}]$ **then**

$$V_n^a = \begin{cases} (U_n^*)'(x_i) & \text{for } i \in \{1,\dots,i_{\underline{x}}-1\} \cup \{i_{\bar{x}}+1,\dots,N\} \\ a & \text{for } x_i \ :\ i \in \{i_{\underline{x}},\dots,i_{\bar{x}}\} \end{cases}$$

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