Forecasting Mortgage Prepayments in Changing Interest Rate Regimes

A Hybrid Economic-Engineering Model with EMPC

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Master of Science Thesis

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Abstract

Banks such as Rabobank depend on multi-year mortgage prepayment forecasts in order to make provisions for the associated prepayment risks. The econometric models they use are fitted to historical data, and as a consequence their models are fitted to a decreasing interest rate regime. Given the current economic climate of increasing interest rates, Rabobank has ascertained that their models are underperforming. They expressed the need for an alternative modeling approach that performs better in changing interest rate regimes.

This thesis takes a systems and control approach motivated by this need. We split the development into two parts; a dynamical system for modeling mortgage payments and prepayments, and a controller for simulating mortgagor behavior.

We model the dynamical system by following the principles of economic engineering. Economic engineering is based on the method of analogs, and we develop specific analogies applicable to the mortgage market. We first derive a continuous model describing the mortgage payment and partial prepayment dynamics. This model is then extended towards a hybrid model to include the dynamics of full prepayment. The parameters of this economic engineering model can be identified with historical data and are relatively constant. The resulting model is not affected by the variation of interest rates and performs well in any interest rate regime.

We design an Economic Model Predictive Controller (EMPC) to simulate mortgagor behavior that minimizes an objective function of its costs. This controller minimizes an economic objective which is needed to simulate the behavior of mortgagors in changing interest rate regimes. For different interest rate scenarios, we forecast prepayments with the model by simulating this minimizing behavior. We perform simulations for different kinds of mortgagors by varying the model parameters and the objective function. Based on these simulations, we describe for each mortgagor both the exact cause and dynamics behind the mortgage prepayments supplied.

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Preface

This thesis marks the end of undoubtably the most challenging task of my academic career. When starting this thesis I had never envisioned the final result it turned out to be. The most difficult task was not finding the solution of a problem, but finding a problem worth solving. This required numerous mistakes to finally arrive at this result. However, this makes me only more enthusiastic to present my work in this thesis.

I received along the way numerous support and assistance to reach this result. In the first place, I want to thank my supervisor prof. em. Max Mendel for his honest opinion during the whole process. The knowledge he shares took me well beyond the standard curriculum of mechanical engineering, and lifted my knowledge of economics and finance. I would also like to thank Ir. Coen Hutters for sharing his ideas when I was stuck in my research. Most of the ideas presented in this thesis are a result of discussions with either of them.

This thesis is a result of a pleasant cooperation with Rabobank. It all started a few years ago with Jules Braat from Credit Core to make a 'touchdown' at Rabobank with the research of the TU Delft. This resulted in weekly meetings for sharing our knowledge and translating the problems of Rabobank into Systems and Control problems. I want to thank Jeroen Kleijn from Credit Risk Modeling for his constant enthusiasm and engagement in the meetings, as well ass Bert-Jan Nauta from ALM Modeling for his questions that were sometimes even too difficult for Max to answer. A special thanks to Pieter van Zwol from ALM Modeling for sharing his valuable knowledge which contributed for a large part to this thesis. I am curious how this collaboration will develop, and I wish the new graduate students good luck.

Also, I would like to thank my peers in the economic engineering group for their feedback during the weekly meetings and suggestions for improvements.

Moreover, I want to thank my parents for their support during my entire study, and for making it possible to reach my goals.

Finally, this thesis also marks the end of my time in Delft. I am very grateful that I have studied in Delft and I have built many long-lasting friendships during the past seven years in my student house, student society, Human Power Team, and hockey team.

Later!

Britt Krabbenborg

Delft, November 30, 2022

“Teach a parrot the terms ‘supply and demand’ and you’ve got an economist”
— *Thomas Carlyle*

Chapter 1

Introduction

From a bank's perspective, mortgage prepayments lead to an uncertainty in estimating their liquidity profile, and a mismatch in interest paid and received. Given the magnitude of a typical balance sheet of a bank, banks need reliable forecasts to cover their prepayment risk.

Banks forecast prepayments with econometric models based on correlations, and these correlations are estimated by fitting to historical data. In the history of interest rates there is a structurally downward trend observed in the last 20 years [4]. Econometric models fitted to this history produces a bias in the models [5]. Meaning a model fitted to this history might perform well in a decreasing rates regime but might fail if interest rates are increasing.

The increase in interest rates as announced by the European Central bank [6] is a turning point in prepayment behavior of mortgagors, and problems arise with using these econometric models to forecast prepayments.

The main idea of this thesis is to do the forecasting of mortgage prepayments with an alternative modeling approach that does not primarily rely on historical data. In order to do that, we formulate in this thesis the forecasting of mortgage prepayments as a systems and control problem. We describe the dynamical system with a model of mortgage payments and prepayments. We forecast prepayments with this model by simulation behavior of a mortgagor with a controller.

We derive the dynamics of mortgages by following the first principles of economic engineering in Chapter 2. In economic engineering, economic systems are modeled as causal dynamic systems based on analogies between economics and mechanics, similar to how mechanical and electrical systems are modeled [7]. We derive analogies between mechanics and economics applicable to the mortgage market by analyzing mortgage amortization schedules. By using the intuitive economic perspective of the Lagrangian formalism in economic engineering, we derive the economic forces responsible for prepayments. Based on these analogies, we derive the dynamics of partial and full prepayments by analyzing their impact on repayment schedules.

We formulate the forecasting of mortgage prepayments as a systems and control problem in Chapter 3. We discuss the current challenge of forecasting prepayments in increasing interest rates, and how interest rates will influence the model. We design an optimal controller known as Model Predictive Control (MPC) to simulate behavior of a mortgagor that optimizes a cost function or objective function. By implementing this controller, we simulate the dynamics of partial and full prepayments derived in chapter 2.

The simulations with MPC in Chapter 3 show that the current dynamics of the model appear to be too limited. In Chapter 4 we develop modeling techniques to include the option for mortgage porting, to model full prepayments determined by their risk drivers (mortgage rates and housing activity), and to model a new contract agreement after full prepayments. A mortgagor can now choose for mortgage porting after moving to a new house, the full loan prepayments are driving by their determinants, and it is possible to take out a new loan after a full loan prepayment. These model extensions result in the hybrid model which we use for the forecasting of mortgage prepayments.

The hybrid model is a dynamic gray-box model that relies on past data for the identification of their parameters, but not for the dynamics of the model itself. This leads to an adaptable model with elements that have an economic interpretation. These gray-box models are what sets economic engineering apart from other disciplines that seek to produce prepayment models such as survival analysis [8], and logistic regression models from econometrics [9, 10, 11, 5], and neural networks from machine learning [12]. These approaches result in a black box model for forecasting mortgage prepayments. A black box model is a useful approach if the economic system that this model describes (mortgages and its prepayments) remains similar. However, if the system structure changes substantially, as is bound to happen for mortgage prepayments due to changes in interest rate trends over time, these models lack interpretability to adapt to future scenarios [13].

We forecast prepayments for different interest rates scenarios in Chapter 5 with the hybrid model designed in Chapter 4. We design an Economic Model Predictive Controller (EMPC) to simulate behavior of a mortgagor that minimizes an objective function which is an economic objective. We forecast prepayments in a decreasing interest rate scenario, and subsequently in an increasing interest rate scenario. We perform the simulation of the increasing interest rate scenario for different types of mortgagors. The type of mortgagor is varied by simulating for different values of the parameters in the model and the objective function. In this way, we forecast prepayments for different risk-averse, creditworthy, and rational mortgagors.

These simulations show how different types of mortgagors react to increasing interest rates with prepayments. Based on these simulations, we describe the cause-and-effect relation between interest rate changes and prepayments. Ultimately, this provides the bank with better insight into the cause of varying levels of prepayments relative to different risk-averse, creditworthy, and rational mortgagors.

Chapter 2

An Economic Engineering Perspective on Mortgages

To forecast prepayments, we need causalities describing the cause of a change in mortgage repayments, and dynamics to describe the exact time change of mortgage repayments. In this thesis, we resolve both by following the first principles of economic engineering.

Economic engineering is a new field of study developed over the past few years at the Delft Center of Systems and Control, primarily by prof. em. Mendel. Together with several students that wrote theses about the subject varying from modeling the electricity market of the future [14], to using economic engineering for supply chain management [15], or developing a macroeconomic model of the US economy [16]. They have all employed tools traditionally used in engineering disciplines and physics to improve the predictive power of (macro)economic models.

To derive the dynamics of mortgages, we first need to define the kinematics. We define the kinematics of mortgages by analyzing repayment schedules of mortgage contracts till maturity. Then, we derive the dynamics of mortgages by analyzing the consequences of partial and full prepayments on repayment schedules. We use the Lagrangian formalism to derive the economic forces causing mortgage prepayments. This is because, perhaps contrary to classical mechanics, the Lagrangian formalism is the most intuitive from an economic perspective, compared to Newtonian mechanics.

The structure of this chapter is as follows. First, in Section 2-1 the components of mortgage products in the Dutch mortgage market are described. We provide the economic engineering analogies applicable to the mortgage market in Section 2-2, and subsequently simulate mortgage amortization schedules of different mortgage products in Section 2-3. In Section 2-4 the most common types of prepayments are described, the risks of prepayments for banks, and the factors that influence a mortgagor's decision to prepay. Then in Section 2-5, the dynamics of mortgage prepayments are derived based on the economic engineering analogies, where a distinction is made in the dynamics of partial prepayments and full prepayments.

2-1 The Terms of Mortgage Agreements

A mortgage loan is the largest loan contract for any individual. It is a private contract between a lender (e.g., bank, insurance company), or *mortgagee*, and the borrower, known as the *mortgagor*. The contract is an agreement between the mortgagor and mortgagee about the various characteristics of the mortgage. Examples of such characteristic are the size of the loan, amortization schedule, maturity, interest rate, collateral and so on. The role of collateral distinguishes a mortgage loan from other loans or private placements, as the collateral is often the house purchased with the funds from the mortgage. In this thesis, loan and mortgage will alternately be used to refer to a level of money which is borrowed by a mortgagor where the collateral of the agreed contract is a house.

A mortgage is a private contract as only two agents are involved in deciding upon the contract. In general, a mortgage contract consists of a principal amount of money borrowed. This principal amount is repaid within a predetermined period, or before the mortgage matures. The maturity of a mortgage is mostly 30 years, which is also the maximum possible maturity.

The payments per month of a mortgage are outlined in an *amortization schedule*. An amortization schedule shows, for each month till the mortgage contract matures, the principal balance, the interest accumulated, and the monthly payments. The monthly payments of a mortgage consist of two parts: a *principal repayment* and an *interest payment*.

The interest payment depends on the contractual interest rate (or contract rate) and the principal balance. The interest payments are the return received on the mortgage contract for the mortgagee; these are costs for a mortgagor. The interest rate is fixed for a predetermined period and could have different lengths from zero to thirty years. At the end of each fixed interest rate period, a reset happens and a new contractual interest rate is agreed. The principal repayment reduces the principal balance each month. The composition of the principal repayments is determined by the mortgage product.

The Dutch tax code is the main determinant of the available mortgage products in the Netherlands [17, 18]. In general, three types of mortgage loans can be distinguished. The full amortizing loans with maturity of mostly 30 years. These full amortizing loans are known as the annuity and linear mortgage loans. Secondly, the interest-only mortgage loans and thirdly the mortgage products including deferred principal repayment structures. In these products, capital is accumulated in a linked account to take care of a bullet principal repayment due at maturity of the loan [18]. The repayment schedule of this last mortgage product is similar to the interest-only mortgage, however an extra savings or investment account is linked in which capital is accumulated [19].

- **Annuity mortgage:** The monthly payments are fixed for the length of the fixed interest rate period, but its composition changes. At the beginning of the term of the mortgage, most of the monthly payment is made up by interest payments and just a small amount is repayment of the principal. Due to these principal repayments, the principal balance decreases as well as the interest payment. This is because the interest payment is dependent on the principal balance. At the end of the mortgage term, the interest payment gets smaller and more monthly amount goes for repaying the principal amount.

- **Linear mortgage:** Both the monthly payments and the interest payments are linear decreasing as every month the borrower repays a fixed amount of principal. This fixed amount of principal repayment is equal to the initial amount borrowed divided by the maturity of the contract in months. This means that at the beginning of the mortgage term, it has higher monthly payments and the actual debt reduces faster than with an annuity contract. The accumulated interest at the end of the term will be lower for a linear mortgage than for an annuity mortgage.
- **Interest-only mortgage:** The monthly payments are also fixed, however these monthly payments consist only of an interest payment, and so the principal repayment is zero during the mortgage term. The principal is entirely repaid at the end of the mortgage term.

2-2 Modeling Mortgages with Economic Engineering

Economic engineering is a new field of study that models economic systems similarly as mechanical systems. It has useful tools for analyzing mortgages from an engineering point of view.

This section shows how we use economic engineering for modeling mortgages. We derive analogies between mechanics and economics applicable to the mortgage market based on Lagrangian mechanics. Moreover, we point out the consequences of mortgage contracts on mortgage payments, and the succeeding section simulates different mortgage amortization schedules. Later in this chapter, the analogies derived in this section will be used to derive the dynamics of mortgage payments and prepayments.

2-2-1 A Mortgage as a Dynamical System

At a given time t, the state of a particle consists of its generalized position q and generalized velocity v [20]. This statement translates to mortgages by asserting that the state of a mortgage at a given time t is determined by its balances $\boldsymbol{q} = (q_0, q_1)$ and its payments $\boldsymbol{v} = (v_0, v_1)$. The balances consist of a principal balance q_0 and a balance of accumulated interest q_1, and the payments include a principal repayment v_0 and interest payment v_1. The monthly payment P is the sum of the principal repayment and interest payment.

Figure 2-1 shows a path of a particle moving in the two-dimensional plane spanned by q_0 and q_1 called the *principal-interest plane*. A tangent to the path of the particle is $\boldsymbol{v}$ as it is the time derivative of $\boldsymbol{q}$, and can be decomposed in v_0 and v_1. The sum of the principal repayment and interest payment is the monthly payment P, and so is the tangent to this path of the particle.

The motion of a particle in this principal-interest plane is analogous to a *mortgage amortization schedule*. A mortgage amortization schedule is a complete table of periodic mortgage payments and shows how a mortgage is repaid over a period of time. Each month (each time step t) a schedule shows the principal balance $q_0(t)$, the interest accumulated $q_1(t)$, the principal repayment $v_0(t)$, and the interest payment $v_1(t)$ until the mortgage is paid off at the end of its term [21].

The principal balance is the current balance on a loan account. It is the unpaid balance of the loan, which is the still outstanding debt for the mortgagor. The balance of accumulated interest is the already received return by the mortgagee, which are the interest costs for a mortgagor. In this thesis, a mortgage will be viewed from the perspective of a bank, and thus the principal balance is the current balance outstanding, and the balance of accumulated interest is the interest received over the loan account. Both axes are chosen positive for convenience. A principal repayment reduces the principal balance, while an interest payment increases the balance of accumulated interest.

Generally, a mortgage amortization schedule starts at the q_0-axis with a zero value for q_1, and ends at the q_1-axis with a zero value for q_0. This is because at the beginning of the mortgage term no interest is paid, and no principal is repaid, while at the end of the mortgage term the total principal balance is repaid, and all the interest on the loan is received.

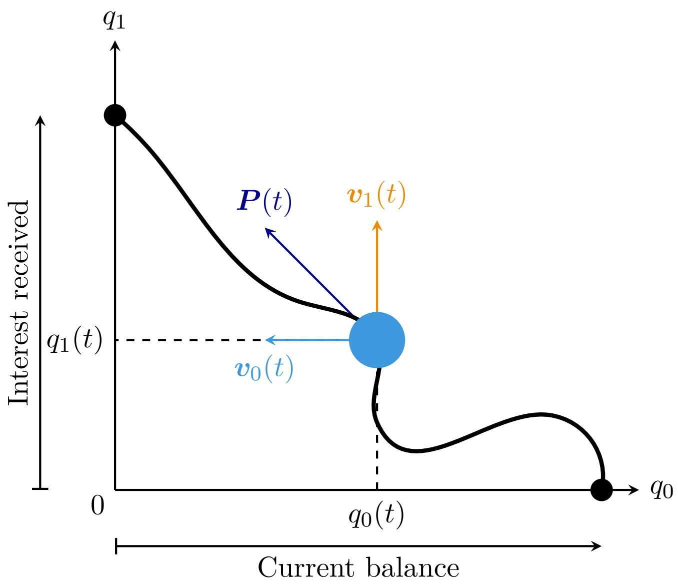

Figure 2-1: A mortgage amortization schedule as a path in the principal-interest plane spanned by $\boldsymbol{q} = (q_0, q_1)$ as analogous to the motion of a particle in the plane. The vector tangent to this path is the monthly payment P, which is decomposed in the principal repayment v_0 and the interest payment v_1, both in the units of euro per time. A principal repayment reduces the principal balance q_0, while an interest payment increases the balance of accumulated interest q_1. Both balances are in the units of euro.

Particles base their movement on the *principle of least action*. They do so by minimizing the action S:

$$S(\gamma) = \int_{t=0}^{T} L(\boldsymbol{q}, \boldsymbol{v})\, \mathrm{d}t \tag{2-1}$$

In the above equation, L will be the running cost (or the *disutility*) of the mortgage. The mortgage is as a *cost minimizing* (or *utility maximizing*) mortgage if an amortization schedule γ minimizes the accrued costs S for the period t to T. Here $t = 0$ is the starting time of the mortgage contract and T the maturity of the contract in months.

By using the tools of the calculus of variations, the solution of Equation (2-1) results in the *Euler-Lagrange equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{v}} - \frac{\partial L}{\partial \boldsymbol{q}} = 0 \tag{2-2}$$

From the Euler-Lagrangian equations in Equation (2-2), the characterizations of the marginal costs are obtained. The marginal change in running cost for a marginal change in payments $\boldsymbol{v}$

$$\boldsymbol{p} := \frac{\partial L}{\partial \boldsymbol{v}} \tag{2-3}$$

is called a *risk factor*. Risk factors have their roots in financial engineering for adjusting assets, cash flows, or payoffs, for risk [22, 23]. The risk factor p_0 is a risk factor valued to the principal repayments of the mortgage, and the factor valued to the payments of interest is p_1. These factors are personal risk factors a mortgagor values to these mortgage payments.

$$p_0 = \frac{\partial L}{\partial v_0} \qquad p_1 = \frac{\partial L}{\partial v_1} \tag{2-4}$$

The marginal change in running cost for a marginal change in balances $\boldsymbol{q}$

$$\boldsymbol{F} := \frac{\partial L}{\partial \boldsymbol{q}} \tag{2-5}$$

is referred to a *risk appetite*. Risk appetite is the amount of risk someone is willing to take or accept to achieve its objectives.

$$F_0 = \frac{\partial L}{\partial q_0} \qquad F_1 = \frac{\partial L}{\partial q_1} \tag{2-6}$$

The risk appetite for principal repayments and risk appetite for interest payments are F_0 and F_1 in Equation (2-6) respectively.

Table 2-1 summarizes the analogies between mechanics and economics, and the specific analogies applicable to the mortgage market.

Table 2-1: Analogies between mechanics and economics, and the extension of the analogies applicable to the mortgage market.

	Mechanics	Economics		Mortgage Market	Units
$\boldsymbol{q}$	Position	Balance	q_0	Principal balance	€
			q_1	Accumulated interest	€
$\boldsymbol{v}$	Velocity	Flow	v_0	Principal repayments	€/month
			v_1	Interest payments	€/month
$\boldsymbol{p}$	Momentum	Value	p_0	Risk factor principal repayments	–
			p_1	Risk factor interest payments	–
$\boldsymbol{F}$	Force	Want	F_0	Risk appetite principal repayments	1/month
			F_1	Risk appetite interest payments	1/month

A velocity or a position, especially in case of a height, will be measured as a level and said to be high or low. Thus, the level of the balances can be high or low. Normally, a flow of money is measured as an amount and said to be large or small. But for the analogy we refer to the payments as a level of supplied money that can be high or low. Momenta and forces will be measured as an amount. The risk factors are an amount of risk, where the amount of risk can be large or small. Similar a risk appetite can be large or small.

2-2-2 Mortgagor Characteristics as Model Parameters

A mortgagor's characteristics such as his risk aversion and creditworthiness are respectively analogous to masses and springs in mechanics.

The convexity of the running cost function L (or *disutility function*) measures the mortgagor's attitude towards risk. In general, the more convex the running cost function, the more risk-averse the mortgagor will be, and the less convex the running cost function, the less risk-averse the mortgagor will be.

Risk aversion is a well known term in economics for analyzing choice under uncertainty. It explains the tendency of people to prefer outcomes with a more predictable, but possible lower payoff, rather than another situation with a highly unpredictable, but possible higher payoff [23].

The convexity of the running cost function depends on the choice of the masses m_0 and m_1 in the following mass tensor.

$$\left[\frac{\partial^2 L}{\partial \boldsymbol{v}^2}\right] = \begin{bmatrix} m_0 & 0 \\ 0 & m_1 \end{bmatrix}, \tag{2-7}$$

where a heavy mass results in a more convex running cost function and a more risk-averse mortgagor. While a lighter mass results in a less convex running cost function and a less risk-averse mortgagor. The inverse of the mass tensor is known as the elasticity tensor E:

$$E = \begin{bmatrix} m_0 & 0 \\ 0 & m_1 \end{bmatrix}^{-1} \tag{2-8}$$

The supply and demand curves in economics are analogous to looking at masses in mechanics. We use the elasticity tensor to determine the level of payments supplied $\boldsymbol{v}$ at a certain risk factor $\boldsymbol{p}$:

$$\begin{gathered} \boldsymbol{v} = E\boldsymbol{p} \\ \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} m_0 & 0 \\ 0 & m_1 \end{bmatrix}^{-1} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \end{gathered} \tag{2-9}$$

We show in Figure 2-2 a typical linear supply schedule that is equally recognizable as a graph of the definition of momentum (in the usual '$p = mv$' functional form). The risk factor of principal repayments is related to the level of principal repayments supplied by a supply curve. The slope of the supply curve is the risk aversion of a mortgagor, and depends on the weight of the mass. Curve S_1 represents the supply curve of a more risk-averse mortgagor, while curve S_2 represents the supply curve of a less risk-averse mortgagor. For a similar risk factor p_0^*, a more risk-averse mortgagor will supply a lower level of principal repayments ($v_{0,1}$) than the less risk-averse mortgagor is willing to supply ($v_{0,2}$).

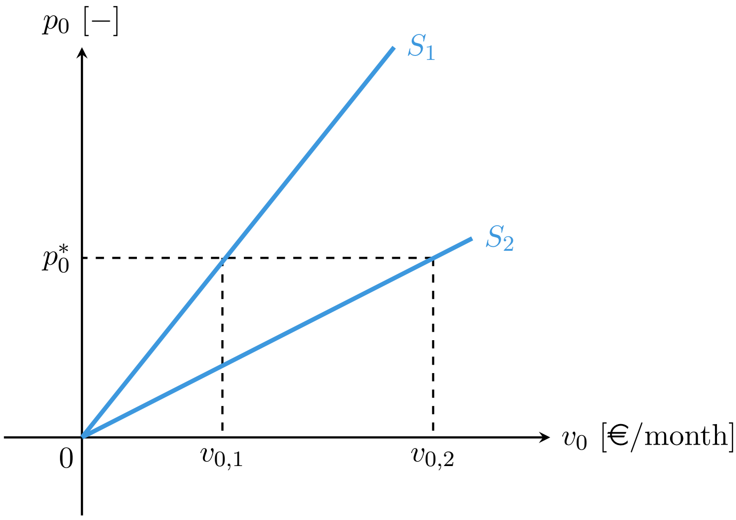

Figure 2-2: Rotation of the supply curve due to different risk aversions of mortgagors. The more risk-averse mortgagor with supply curve S_1 will supply a lower level of principal repayments $v_{0,1}$ for a similar risk factor p_0^* than the less risk-averse mortgagor is willing to supply $v_{0,2}$.

The creditworthiness of a mortgagor is analogous to the choice of the spring constants in the following spring constant tensor:

$$\left[\frac{\partial^2 L}{\partial \boldsymbol{q}^2}\right] = \begin{bmatrix} k_0 & 0 \\ 0 & k_1 \end{bmatrix}, \tag{2-10}$$

The *Creditworthiness* of a mortgagor represents how worthy a mortgagor is to receive new credit when he applies for a debt obligation. A common score that is used to determine a mortgagor's creditworthiness is the Fair Isaac Credit Organization (FICO) credit score [24], [25]. A higher FICO score is associated with a higher degree of creditworthiness [11]. The creditworthiness of a mortgagor has only significance with respect to a level of money borrowed, or the principal balance. Therefore, k_1 has no economic interpretation in this thesis.

The inverse of the spring constant tensor is known as the convenience yield tensor C.

$$C = \begin{bmatrix} k_0 & 0 \\ 0 & k_1 \end{bmatrix}^{-1} \tag{2-11}$$

In this tensor, k_0 is the creditworthiness of a mortgagor. From Equation (2-10) follows that a high spring constant translates to a mortgagor with a high creditworthiness, while a low spring constant represents a mortgagor with a low creditworthiness.

The tensor in Equation (2-11) is known as a convenience yield tensor, as it 'stores' a benefit of having a product available in storage. If the storage belongs to a mortgagor, then this mortgagor will have a benefit of borrowing money. A mortgagor with a lower creditworthiness is likely to benefit more from the borrowed money and therefore wants to repay the borrowed money slower than a mortgagor with a higher creditworthiness. A mortgagor with a higher creditworthiness will impose a larger force, or have a larger appetite for repaying the mortgage as follows from eq. (2-12).

$$\begin{aligned} \boldsymbol{F}_C &= -C^{-1}\boldsymbol{q} \\ \begin{bmatrix} F_0 \\ F_1 \end{bmatrix}_C &= -\begin{bmatrix} k_0 & 0 \\ 0 & k_1 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \end{aligned} \tag{2-12}$$

The slope of the convenience yield curves as shown in Figure 2-3 depends on the creditworthiness of a mortgagor. For a similar principal balance q_0^*, a mortgagor with a higher creditworthiness has a steeper slope of the convenience yield curve (Y_1) and a larger appetite $F_{0,C1}$ for repaying the loan, while a mortgagor with a lower creditworthiness has a less steep slope of the convenience yield curve (Y_2) and a smaller appetite $F_{0,C2}$ for repaying the loan.

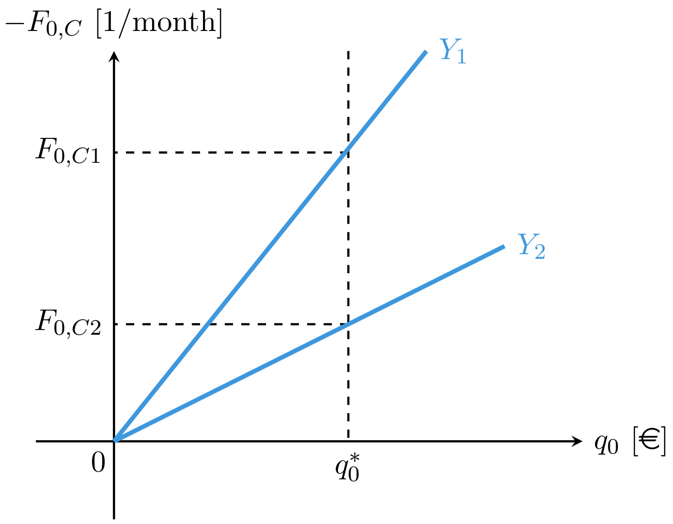

Figure 2-3: Rotation of the convenience yield curve due to different creditworthy mortgagors. The more creditworthy mortgagor with yield curve Y_1 will have a larger appetite $F_{0,C1}$ for repaying the mortgage for a similar principal balance q_0^* than the mortgagor with a lower creditworthiness which has a smaller appetite $F_{0,C2}$.

The analogies between mechanics and economics, and the specific analogies applicable to the mortgage market are shown in Table 2-2.

Table 2-2: Analogies between mechanics and economics, and the extension of the analogies applicable to the mortgage market.

	Mechanics	Economics		Mortgage Market	Units
m	mass	Price	m_0	Risk aversion principal repayments	[month/€]
		Inelasticity	m_1	Risk aversion interest payments	[month/€]
k	Spring	Inverse convenience	k_0	Creditworthiness	[1/(€-month)]
	constant	yield	k_1	-	-

2-2-3 Mortgage Contracts Impose Constraints on Payments

The type of mortgage contract determines how the mortgage is paid off during the term of the loan. A mortgage contract acts as a constraint that limits the motion of the dynamical system. The level of payments is constrained each month till maturity depending on the type of mortgage contract. In classical mechanics, constraints are classified depending on how the conditions of the constraint can be expressed. The same approach is taken as in classical mechanics to classify the constraints of mortgage contracts

A particle constrained to move along any curve or on a given surface is said to be a holonomic constraint [26]. With the equations defining the curve or surface acting as the equations of a constraint. If the conditions of constraint can be expressed as equations connecting the coordinates (and possible time) of a particle in the principal-interest plane having the form

$$f(q_0, q_1, t) = 0 \tag{2-13}$$

then the constraints are said to be holonomic.

Non-holonomic constraints are most often expressed as a differential relation of the form [27]:

$$f(v_0, v_1, q_0, q_1, t) = 0 \tag{2-14}$$

If the constraints are non-holonomic, the equations expressing the constraint can not be used to reduce the number of coordinates, or the number of degrees of freedom of the system as opposed to holonomic constaints [27]. The constraints imposed by mortgage contracts on the mortgage payments are non-holonomic constraints, because the degrees of freedom of the system can not be reduced with the constraints.

The non-holonomic constraints for the different type of contracts are summarized in Table 2-3, and we explain the constraints based on simulations in the next section.

Table 2-3: Mortgage contracts impose constraints on the payments of mortgages. The constraint on the interest payments is equal for each mortgage product. Whereas the constraint on the principal repayment differs per mortgage product.

Contract	**Constraints**
Annuity	$v_0 + v_1 = P$
	$v_1 = r_c q_0$
Linear	$v_0 = A_0/T$
	$v_1 = r_c q_0$
Interest-only	$v_0 = 0$
	$v_1 = r_c q_0$

2-3 Simulating the Kinematics of Mortgages

We model a mortgage as a point mass which is constrained to move freely due to the mortgage contract. The non-holonomic constraints constrain the payments at each time step till the contract matures, and are therefore also sufficient to constrain the possible positions (or balances) of the contract. We model the mortgage as a point mass because no external force is acting on the system and the velocities are fixed (payments are scheduled) at each time step.

To run simulations, we discretize the system with Forward Euler method, and we add the non-holonomic constraints from Table 2-3 depending on the contract simulated. One time step is equal to one month, because the payment frequency of a mortgage is monthly. The simulations are in discrete time, however, the figures are generated by using a dashed (continuous) line to make a clear distinction in the figures later on in this thesis. We use in the principal-interest plane a solid line to indicate the level of the balances each month.

We simulate a mortgage with an initial level of money borrowed A_0 equal to €$500,000$. The contract interest rate r_c is 3% per year fixed for the entire mortgage term, and the maturity T of the loan is 30 years.

Annuity Mortgage

The annuity mortgage is characterized by its fixed monthly payments P calculated with Equation (2-15). The level of the monthly payment depends on the contractual interest rate per month r_c, the maturity in months T, and the initial level of money borrowed A_0 .

$$P = A_0 \frac{r_c(1+r_c)^T}{(1+r_c)^T - 1} \tag{2-15}$$

The fixed monthly payment is the sum of a principal repayment and an interest payment:

$$v_0 + v_1 = P \tag{2-16}$$

Here, the interest payment is calculated by multiplying the contractual interest rate per month with the principal balance outstanding:

$$v_1 = r_c q_0 \tag{2-17}$$

The initial conditions of the loan balances and payments are.

$$q_0[0] = A_0 \qquad q_1[0] = 0 \qquad v_0[0] = r_c q_0[0] - P \qquad v_1[0] = r_c q_0[0] \tag{2-18}$$

The velocity v_0 will be negative, because a principal repayment decreases the principal balance q_0, while the velocity v_1 will be positive as this payment increases the balance of accumulated interest. The simulation for one iteration has the following form:

$$\begin{aligned} q_0[k+1] &= q_0[k] + v_0[k] \\ q_1[k+1] &= q_1[k] + v_1[k] \\ v_0[k+1] &= r_c q_0[k+1] - P \\ v_1[k+1] &= r_c q_0[k+1] \end{aligned} \tag{2-19}$$

We simulate the mortgage amortization scheme from the start of the contract $t = 0$ till the end of the contract $t = T = 360$ months. The mortgagor starts repaying the loan at $t = 0$.

In Figure 2-4 the balances and the payments per month are shown till the end of the contract. We plot the payments both as positive numbers for convenience. At $t = 0$, the principal balance is equal to the initial level of money borrowed, while the interest accumulated is zero. At the end of the mortgage term, the principal balance is zero, and the interest accumulated is €$258,882$ euros.

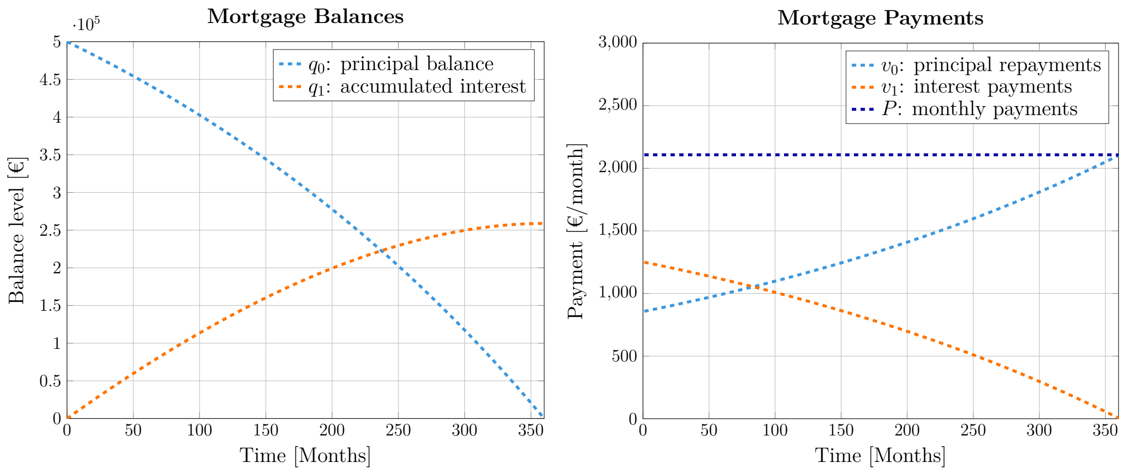

Figure 2-4: The monthly balances and payments of an annuity contract with a maturity of 30 years and an initial level of €500, 000 borrowed. An annuity contract is characterized by its fixed monthly payments till the contract matures. As a result, the interest payment is at the beginning of the contract higher than the principal repayments, and vice versa at the end of the term.

At the beginning of the contract, the principal repayments are lower than the interest payments, while at the end of the contract the principal repayments are higher than the interest payments. The total monthly payments P are constant over time till the contract matures which characterizes an annuity contract.

The motion of the annuity schedule in the principal-interest plane looks like a quarter ellipse located in the first quadrant as shown in Figure 2-5. The starting point is in the right bottom at $(q_0, q_1) = (5 \cdot 10^5, 0)$ and the mortgage matures at $(q_0, q_1) = (0, 2.59 \cdot 10^5)$. This means that at the end of the contract, a total interest of €258, 882 is received by the bank.

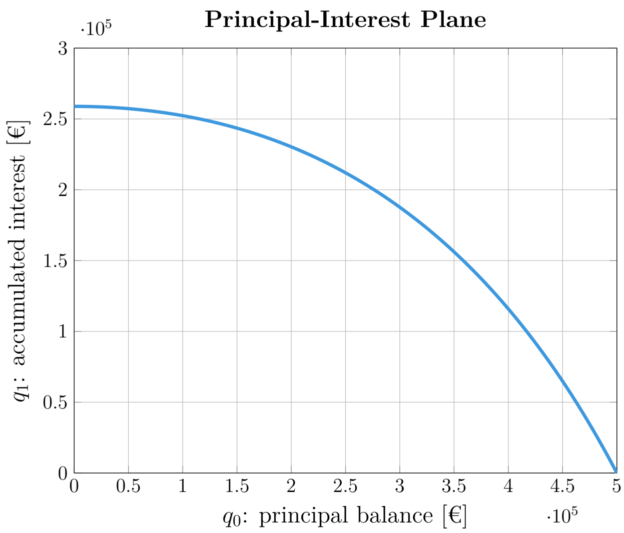

Figure 2-5: A schedule of an annuity mortgage contract in the principal-interest plane. The contract starts at $t = 0$ at $(q_0, q_1) = (5 \cdot 10^5, 0)$ and the end of the mortgage term is at $(q_0, q_1) = (0, 2.58 \cdot 10^5)$.

Linear Mortgage

A linear mortgage is characterized by its fixed monthly principal repayment v_0 calculated with the initial level of money borrowed A_0 divided by the maturity T in months:

$$v_0 = \frac{A_0}{T}$$

The interest payment is calculated by multiplying the contract interest rate per month with the principal balance: $v_1 = r_c q_0$.

The initial conditions of the loan balances and payments are:

$$q_0[0] = A_0 \qquad q_1[0] = 0 \qquad v_0[0] = -A_0/T \qquad v_1[0] = r_c q_0[0] \tag{2-20}$$

The simulation for one iteration has the following form:

$$\begin{aligned} q_0[k+1] &= q_0[k] + v_0[k] \\ q_1[k+1] &= q_1[k] + v_1[k] \\ v_0[k+1] &= -A_0/T \\ v_1[k+1] &= r_c q_0[k+1] \end{aligned} \tag{2-21}$$

In Figure 2-6 we plot the balances till the mortgage matures at $t = T = 360$ months, and the payments per month. At $t = 0$, the principal balance is equal to the initial level borrowed, while the interest accumulated is zero. At the end of the contract, the principal balance is zero, and the interest accumulated is €$225,621$. A characteristic of a linear contract is that the principal repayments are constant over time till the contract matures. Consequently, the interest payments as well as the monthly payments decrease linearly over time.

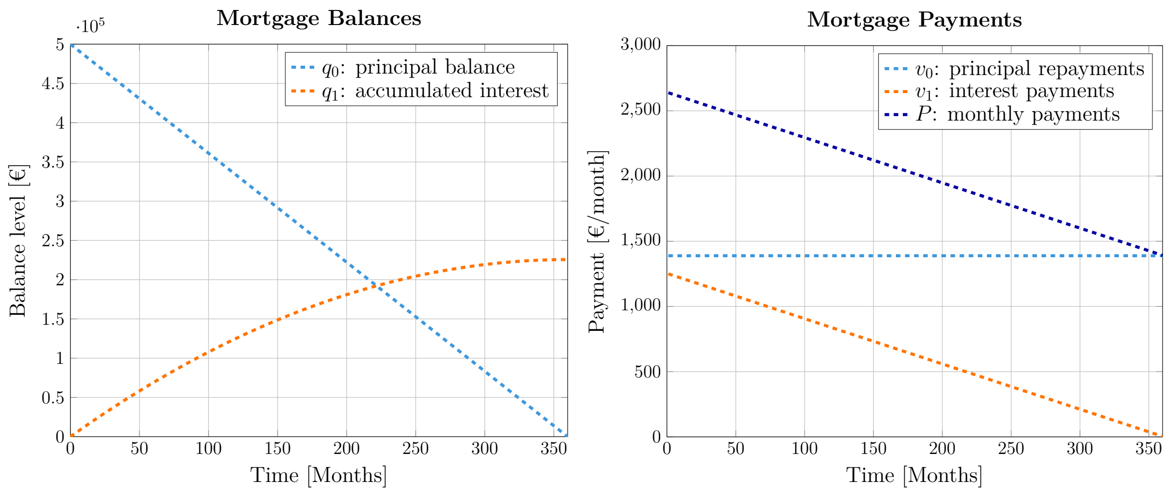

Figure 2-6: The monthly balances and payments of a linear contract with maturity of 30 years and an initial level of €$500,000$ borrowed. A linear contract is characterized by its fixed principal repayments till the contract matures. As a result, the monthly payment and interest payment is high at the beginning of the contract and decreases linearly over time.

The schedule of a linear mortgage in the principal-interest plane is quite similar to the one of an annuity contract as shown in Figure 2-7. The schedule starts is in the right bottom at $(q_0, q_1) = (5 \cdot 10^5, 0)$ and stops at $(q_0, q_1) = (0, 2.256 \cdot 10^5)$. This means that at the end of the contract, a total interest of €225,621 euros is received by the bank.

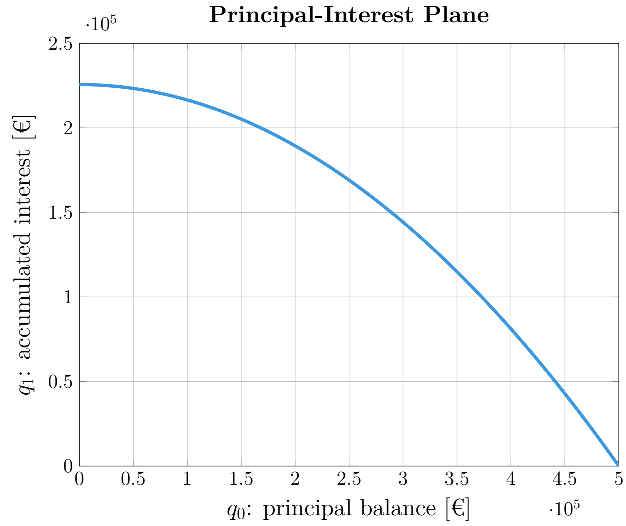

Figure 2-7: The schedule of a linear contract in the principal-interest plane. The contract starts at $t = 0$ in $(q_0, q_1) = (5 \cdot 10^5, 0)$ and the end of the contract term is in $(q_0, q_1) = (0, 2.256 \cdot 10^5)$.

Compared with the annuity mortgage, a lower total interest is paid at the end of the linear contract which means that a linear mortgage is more favorable for mortgagors. Although, often an annuity mortgage is chosen due to the fixed monthly payments and the relatively higher payments at the beginning of a linear contract.

Interest-Only Mortgage

An interest-only mortgage, or bullet loan, is characterized by a repayment of the entire principal due at the end of the loan term. Only interest is paid during the term of the mortgage and is calculated similar as with the other two mortgage types: $v_1 = r_c q_0$. The principal repayment is zero $v_0 = 0$ and the principal balance remains constant during the term of the mortgage. At the end of the term, the interest payment is zero and the principal repayment consists of the entire principal outstanding.

The initial conditions of the loan balances and payments are:

$$q_0[0] = A_0 \qquad q_1[0] = 0 \qquad v_0[0] = 0 \qquad v_1[0] = r_c q_0[0] \tag{2-22}$$

The interest-only mortgage is characterized for monthly payments consisting only of interest payments (principal repayments are zero $v_0 = 0$), and the total principal balance is paid off at the end of the contract. Therefore, we add final conditions:

$$v_0[T] = -q_0[k] \qquad v_1[T] = 0 \tag{2-23}$$

The simulation of one iteration has the following form:

$$\begin{aligned} q_0[k+1] &= q_0[k] + v_0[k] \\ q_1[k+1] &= q_1[k] + v_1[k] \\ v_0[k+1] &= 0 \\ v_1[k+1] &= r_c q_0[k+1] \\ v_0[T] &= -q_0[k] \\ v_1[T] &= 0 \end{aligned} \tag{2-24}$$

We show in Figure 2-8 the balances till the mortgage matures at $t = T = 360$ months, and the payments per month. At $t = 0$, the principal balance is equal to the initial level of money borrowed, while the interest accumulated is zero. At the end of the contract, the principal balance is zero, and the interest accumulated is €448, 750. A characteristic of an interest-only contract is that the principal repayments remain zero during the contract term and only interest is paid during the contract. The principal is entirely repaid at the end of the mortgage contract.

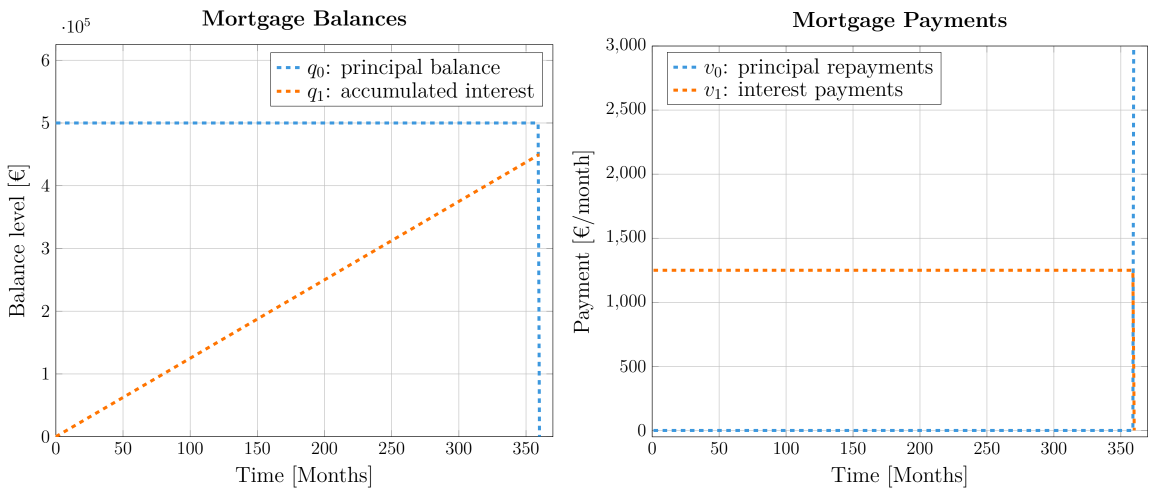

Figure 2-8: The monthly balances and payments of an interest-only contract with maturity of 30 years and an initial level of €500, 000 borrowed. An interest-only contract is characterized by its fixed interest payments and no principal repayments during the contract term. The total principal balance is repaid back at the end of the term.

The schedule of the interest-only contract in the principal-interest is shown in Figure 2-9. The mortgage contract starts in right bottom at $(q_0, q_1) = (5 \cdot 10^5, 0)$ and the end of the mortgage term is at $(q_0, q_1) = (0, 4.488 \cdot 10^5)$. This means that at the end of the contract, a total interest of €448, 750 is received by the bank.

Clearly, a bank receives the highest interest for an interest-only contract compared to an annuity contract and linear contract. This kind of mortgage was often contracted in the past due to the high tax incentives given by the state on loan interests. Nowadays, the popularity of this type of mortgage has lowered, but still a large portion of mortgage portfolios consists of interest-only mortgages [1].

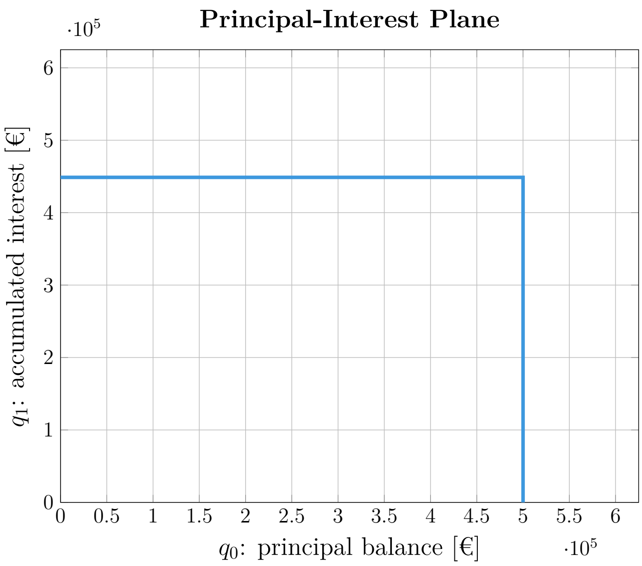

Figure 2-9: The schedule of an interest-only contract in the principal-interest plane. The contract starts at $t = 0$ in $(q_0, q_1) = (5 \cdot 10^5, 0)$ and the end of the term is in $(q_0, q_1) = (0, 4.488 \cdot 10^5)$.

2-4 Prepayments as Embedded Option in Mortgage Contracts

All mortgage contracts have a repayment schedule, but it is an incomplete contract as it does not contain clauses for all possible states of nature. A change in economic conditions may give the mortgagor incentives to deviate from that repayment schedule. An extreme deviation is a full prepayment of the mortgage loan, but partial prepayments are also possible during the term of the loan. Prepaying mortgage principal can save large interest costs for the mortgagor, and results in interest losses for a bank.

The focus of this research will be on the modeling of the types of prepayments described in Section 2-4-2, since these prepayments represent the largest amount of prepayments in the Dutch mortgage market. Although, not all determinants for prepayments were found in literature specific to the Netherlands, most, if not all the risk drivers described in Section 2-4-3 can be applied to the Dutch mortgage market. We identify the risk drivers of prepayments in this section as the most relevant ones from literature, whatever the mortgagor his demographics or mortgage type.

2-4-1 The Risk of Prepayments

Mortgage prepayments are a risk from the perspective of the issuer of the loan. In this thesis, a bank will be the issuer of loans to borrowers (individuals). A borrower has the option of prepaying a part of the principal or fully redeeming the loan. Both prepayment types result in an early mortgage termination and a deviation of the expected return on the mortgage (interest received). These two consequences of mortgage prepayments result in the following two types of prepayment risks for banks:

- **Interest rate risk** in general arises when there is a mismatch in the fixation of the interest rates paid and received by the bank [24]. A bank attracts resources from elsewhere to fund mortgages on which a certain agreed upon rate has to be paid over a fixed period. If the mortgagor decides to prepay (part of) the principal before its original maturity, the bank has to find alternative use for these funds. If market interest rates have fallen since the origination of the loan, the bank will incur a loss on these funds. Usually, fixed interest rate mortgages are hedged against market interest rate changes using Interest Rate Swaps (IRS) [24]. Two parties exchange or 'swap' one set of interest payments for another set of interest payments. The bank receives a variable interest rate and pays a fixed interest rate. However, a mortgage prepayment may expose the bank to the possibility of paying a fixed interest rate which is higher than the one obtained on the newly originated loans.
- **Liquidity risk** arises from the fact that the liquidity profile of the bank is influenced by the maturity profile of loans [24]. The uncertainty in this maturity profile subjective to prepayments has a considerable impact on a bank's liquidity profile. An incorrect estimation of this profile leads to the risk of overestimating or underestimating future liquidity requirements, as well as increased long-term liquidity costs.

In order to cover their prepayment risk, a bank must be able to forecast prepayments for the coming years in order to implement some effective hedging strategies.

2-4-2 Types of Prepayments

In Figure 2-10 the Conditional Prepayment Rate (CPR) is shown for three different types of prepayments of Rabobank's mortgage portfolio in the Dutch mortgage market from 2005 till 2016. The CPR is expressed as an annual percentage and is calculated by dividing the amount of prepayments with the amount of outstanding principal of a portfolio of loans. Therefore, the CPR represents the percentage of a mortgage portfolio's principal that is paid off prematurely. Most of the mortgage prepayments in the Netherlands are prepayments due to principal curtailments, relocation, and refinancing [1].

- **Principal curtailments**: These prepayments are defined as prepayments in excess of the principal repayments scheduled under the contractual amortization payment scheme, often also referred to as curtailments or partial prepayments [1, 13]. Some mortgagors are in the habit of repaying more than the scheduled principal repayment each month to build up equity in their homes faster and as a form of forced savings. Typically, curtailments of up to $10\% - 20\%$ of the money borrowed can take place without a prepayment penalty. Partial prepayments show up in the amortization schedule as extra repayments on top of the principal repayment [13]. Partial prepayments are an effective way for shortening the term of the mortgage. By making a partial prepayment, a loan will amortize more quickly than the nominal remaining term implies. Curtailments are relatively a small contributor to overall prepayments, simply because they are a lot smaller in amount than the full repayments related to refinancing or relocation.

- **Relocation**: These prepayments are full loan prepayments of mortgages due to mortgagors that take a *new* mortgage loan on a *new* property when relocating, and are mainly driven by home sales. Selling a home will usually trigger a full loan prepayment. However, mortgages in the Netherlands are generally portable, which means the mortgagor who sells the house and buys a new one can transfer the terms and conditions of the current mortgage to the new house (i.e., mortgage porting or mortgage transferring). There are no additional costs involved by transferring the current mortgage to a new home. However, the option of taking out a new loan on a new property is also penalty free. Taking out a new mortgage after relocating is favorable if the mortgage rate of a new contract is lower than the one of the existing mortgage contract at the moment of the selling [13]. This new loan can be contracted by a different mortgagee.

- **Refinancing**: These prepayments are full loan prepayments of mortgages due to mortgagors that take a *new* mortgage loan on the *same* property. If a loan is refinanced, the current loan is paid off with a new loan. Refinancing is attractive if the market mortgage rate is below the mortgage rate of the current mortgage contract. In the case of a refinancing, a prepayment penalty is imposed if a refinancing is not at a reset date of the contract rate, or the mortgagor is not moving to a new home. The prepayment penalty is an amount of money equal to the present value of the difference in monthly interest payments between the current loan and a newly originated loan with the same characteristics [28]. Additional costs as administration costs and tax rates are also part of the prepayment penalty. Moreover, it is possible to refinance your loan by a different lender [29].

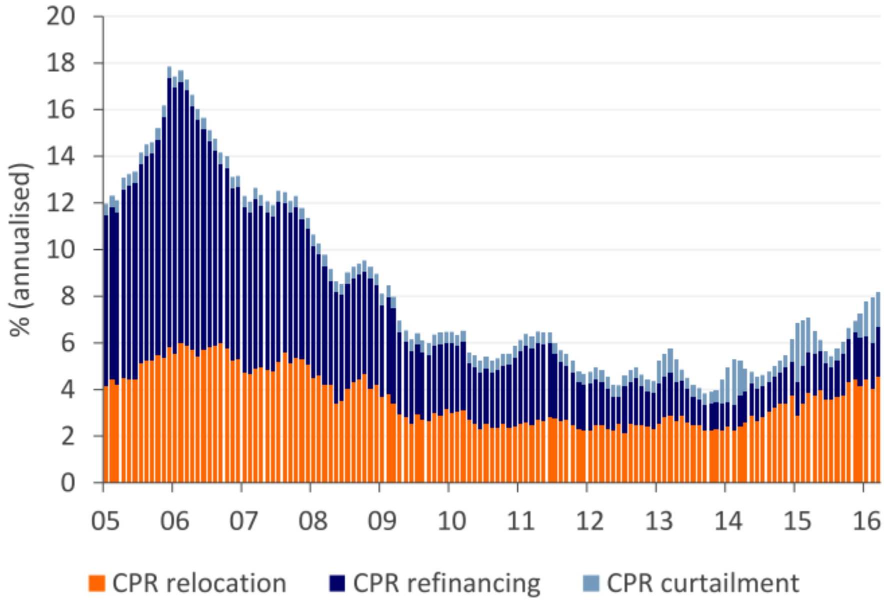

Figure 2-10: The CPR of the three different types of prepayments of Rabobank's mortgage portfolio in the Dutch mortgage market in the period 2005-2016 [1].

Prepayments due to defaults are caused by the foreclosure and subsequent liquidation of a mortgage. Defaults are relatively a minor component of aggregate prepayments in most cases [13]. The mortgage market in the Netherlands is characterized by low default rates and a low number of foreclosures [17]. Therefore, prepayments due to defaults will not be considered in the model.

2-4-3 The Risk Drivers of Mortgage Prepayments

Factors that have influence on a mortgagor's decision to prepay are known as the risk drivers or determinants of prepayments. We describe the risk drivers per type of prepayment.

- **Principal curtailments** appear not to be an optimal strategy for a borrower at first sight. If the contractual rate of a borrower is above mortgage market rates, a borrower derives maximum benefit from a full refinancing. Contrary, if the contractual rate is above the market mortgage rate, it is not beneficial to send in an amount above the monthly payment [13]. However, a refinancing can be inconvenient and may involve transaction cost. If a borrower does not want to refinance because of the transaction costs, partial prepayments represent the optimal strategy. Reasons to partially prepay are related to deleveraging. Also, principal curtailments are positively related to loan age due to a greater risk aversion of older householders with respect to holding debt, as well as an increased financial ability to make extra payments [13]. Another factor that explains the amount of partial prepayments are alternative investments. Funds used for mortgage prepayment could be used for other purposes. A borrower has options for placing this money in investments such as stocks, bonds, futures, savings accounts etc. If returns on these investments exceeds the mortgage rate, these investments represent a wiser economic choice. Even if an investment pays a lower rate than a mortgage (such as a savings deposit), it can give a borrower a benefit of having liquidity or cash available on demand. [30].

- The **relocation incentive** is mainly driven by sales of existing homes. In periods of house price appreciation, home sales and mortgage originations may increase as the expected return on investment increases and borrowers are becoming less constrained by negative equity [1, 11]. On the contrary, during house price depreciation periods, the sale of homes and mortgage originations tend to decrease as risk-averse home-buyers are reluctant to enter the market. As a result, the prepayments due to relocation will drop. The sale of existing homes has a direct relationship with prepayments due to relocation [1, 13]. However, mortgages are portable, and it is allowed to transfer the terms and conditions of the existing mortgage contract to a new house without extra costs involved. So the incentive to fully prepay due to relocation is *twofold*: first there should be an incentive for housing, and second an incentive to take out a new mortgage after housing.
- The **refinance incentive** is generally motivated by lower interest rates in the market, but can also occur because the mortgagor wants to access increased equity in the house or to take advantage of an improvement in credit [13]. The incentive is mainly driven by the difference between the contractual rate and market mortgage rates. If the market mortgage rate is lower than the contractual rate, it is favorable to refinance the loan. A refinancing can be thought of as an exercise of a call option on the existing loan by the mortgagor. However, several complications exist by stating that the refinancing decision simply involves comparing the rate on the existing loan with that available on a new loan [1, 13]. First, there is not a single mortgage rate, but many, which vary by lender, by region, by mortgage maturity, and by the creditworthiness of the borrower. Second, the means to reach a decision on whether to refinance or not differs per borrower, as borrowers are not an efficient corporation and not always behaving rationally. Third, substantial costs can be involved in taking out a new mortgage, and these costs can vary per loan depending on the borrower and loan characteristics (such as loan amount, loan age, contractual rate).

The seasonal pattern of the overall CPR observed in Figure 2-10 is mainly due to the partial prepayments and prepayments due to relocation. Prepayments peak around year-end as a large share of all housing transactions take place in December due to the gift tax framework [1]. Moreover, there might be an incentive to use wealth to pay down mortgage debt prior to the end of December, since it is the measurement date for tax authorities.

2-5 Modeling Mortgage Prepayments with Economic Engineering

A prepayment is a repayment of extra principal on top of the scheduled principal repayment. A distinction can be made between partial prepayments and full prepayments.

Borrowers are allowed to partially prepay every month (the frequency of prepayment is up to the mortgagor), and the level of the principal prepaid is up to the mortgagor, but there is a maximum allowed per year which is a certain percentage of the initial level of money borrowed. A partial prepayment reduces the principal balance faster, and this increased reduction in the principal balance results in a faster decrease in interest payments, and less interest received on the mortgage.

A full loan prepayment is one repayment of the remaining principal balance outstanding, and may involve a penalty. The main difference between full prepayments and partial prepayments is that full prepayments end the mortgage contract immediately, whereas after a partial prepayment the mortgage contract continues.

For modeling prepayments with economic engineering, we also make a distinction between the dynamics of partial prepayments and full prepayments. Partial prepayments are continuous, because the frequency can be monthly, and the level of money prepaid is up to the borrower with a maximum that is penalty-free. Full prepayments are discontinuities in the state vector, because the remaining debt is repaid in one time step, and this results in an instantaneous change in the state of the system.

From this point on, the contract constraints of an annuity mortgage will be used as an example for the simulations. However, these constraints can easily be changed to another contract type by using the contract constraints in Section 2-2-3.

2-5-1 Partial Prepayments and Newton's Second Law

Partial prepayments show up in the amortization schedule as extra repayments on top of the scheduled principal repayment [13]. These extra repayments result in a faster decrease of the principal balance. In Figure 2-11, an illustration is shown of the consequence of partial prepayments on the schedule of an annuity mortgage in the principal-interest plane. The left figure shows the principal repayments and interest payments without partial prepayments, and the right figure the consequence of partial prepayments on the schedule of the contract.

Clearly, partial prepayments increase the principal repayment per month (each time step). In mechanics, this is analogous to an acceleration that increases the velocity at one time step.

$$v_0 = v_0^* + \int a_0 \, \mathrm{d}t \tag{2-25}$$

Here, v_0 is a principal repayment including partial prepayments, v_0^* a scheduled principal repayment, and $a_0 (= \dot{v}_0)$ a partial prepayment. Due to this increase of the velocity vector v_0, the tangent vector to the curve of the contract schedule tilts more to the left. This change in tangent vector (monthly payment) results in a different mortgage repayment schedule.

Partial prepayments increase the principal repayments supplied by the mortgagor, or the velocity v_0. In Figure 2-12 we show a supply curve for principal repayments. An increase in principal repayments is caused by a change in risk factor $\dot{p}_0 = F_0$. Assuming that the risk aversion of a borrower remains a constant and differentiating with respect to time, to arrive to the relationship between a risk appetite F_0 and the consequent extension of the supply of principal repayments $\dot{v}_0$

$$\dot{v}_0 = m_0^{-1} F_0 \tag{2-26}$$

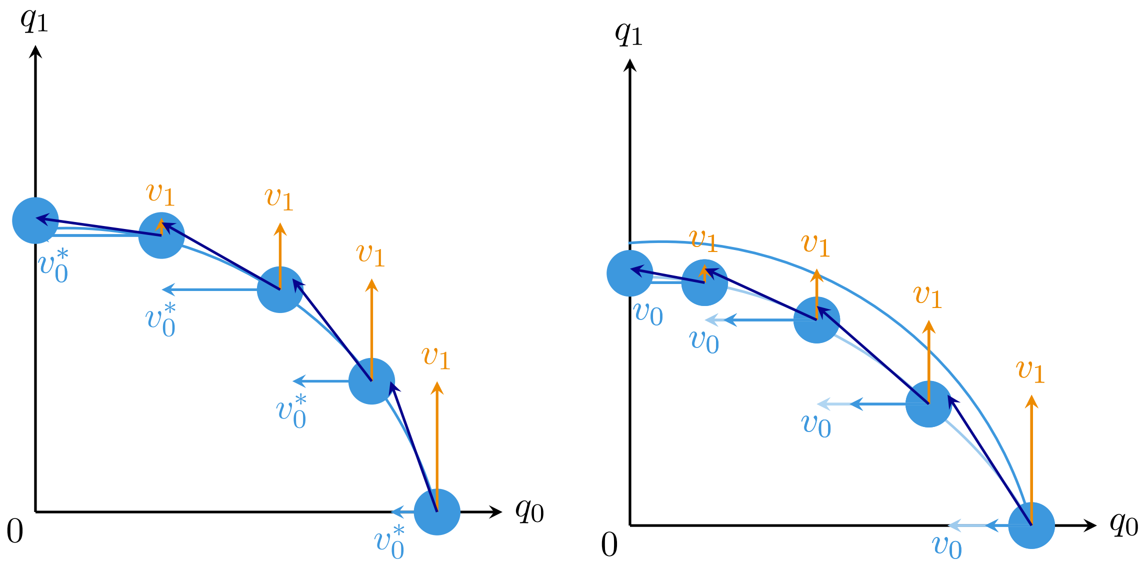

Figure 2-11: An illustration of the schedule of an annuity contract in the principal-interest plane. The left figure shows the schedule without partial prepayments and the right figure the schedule with partial prepayments. As a consequence of the partial prepayments the tangent vector to the path of the schedule tilts more to the left resulting in a different repayment schedule.

analogous to Newton's second law in engineering $F = ma$. Thus, a partial prepayment increases the principal repayment due to economic forces acting on the mortgage, analogous to forces acting on a body in mechanics. Equation (2-26) expresses how the principal repayments supplied are adjusted based on economic forces impinging on it, and this is shown graphically as a movement along the supply curve in Figure 2-12.

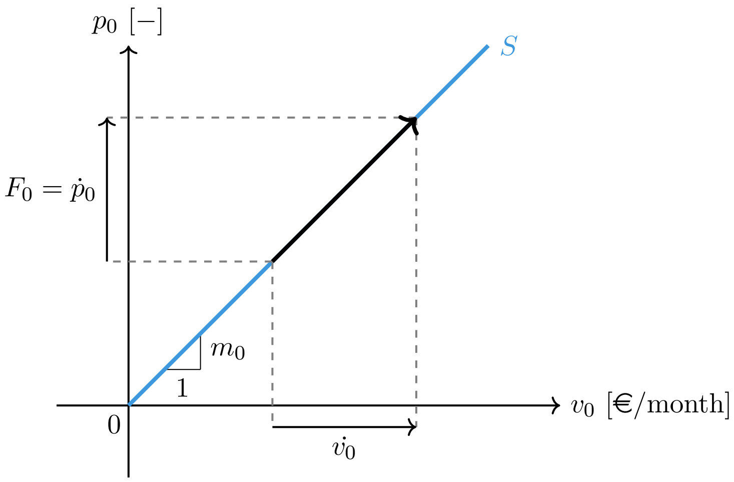

Figure 2-12: A supply curve that shows the relation between a risk factor and the level of principal repayments supplied. An increase in principal repayments is due to a change in risk factor. This is analogous to an increase in velocity $\dot{v}_0$ due to a change in momentum $\dot{p}_0$ as a consequence of forces acting on a body, or Newton's second law of motion in mechanics.

We perform simulations of an annuity contract with partial prepayments. We simulate a mortgage with a level of money borrowed A_0 equal to €500,000, a contract rate of 3%, and maturity of 10 years. A shorter maturity is chosen such that the time steps in the principal-interest plane are better visible.

The borrower partially prepays the maximum allowed level of money per month. The maximum allowed prepayments per year is a percentage ρ of the initial level of money borrowed. We assume from now on that the maximum allowed partial prepayments will be equal to $\rho/12$ per month of the initial level of money borrowed. In reality, the frequency and level of partial prepayments is up to the borrower as long as the level prepaid per year is the maximum allowed.

The simulation of one iteration has the following form:

$$\begin{aligned} q_0[k+1] &= q_0[k] + v_0[k] \\ q_1[k+1] &= q_1[k] + v_1[k] \\ v_0[k+1] &= r_c q_0[k+1] - P - \frac{\rho}{12} A_0 \\ v_1[k+1] &= r_c q_0[k+1] \end{aligned} \tag{2-27}$$

We use the following initial conditions:

$$q_0[0] = A_0 \qquad q_1[0] = 0 \qquad v_0[0] = r_c q_0[0] - P - \frac{\rho}{12} A_0 \qquad v_1[0] = r_c q_0[0] \tag{2-28}$$

Here ρ is 10%, and P is the fixed monthly payments of an annuity contract calculated with Equation (2-15).

In Figure 2-13 we show the mortgage payments per month and the balances each month. The partial prepayments result in an early mortgage termination, and less interest received by the bank due to the change in contract schedule.

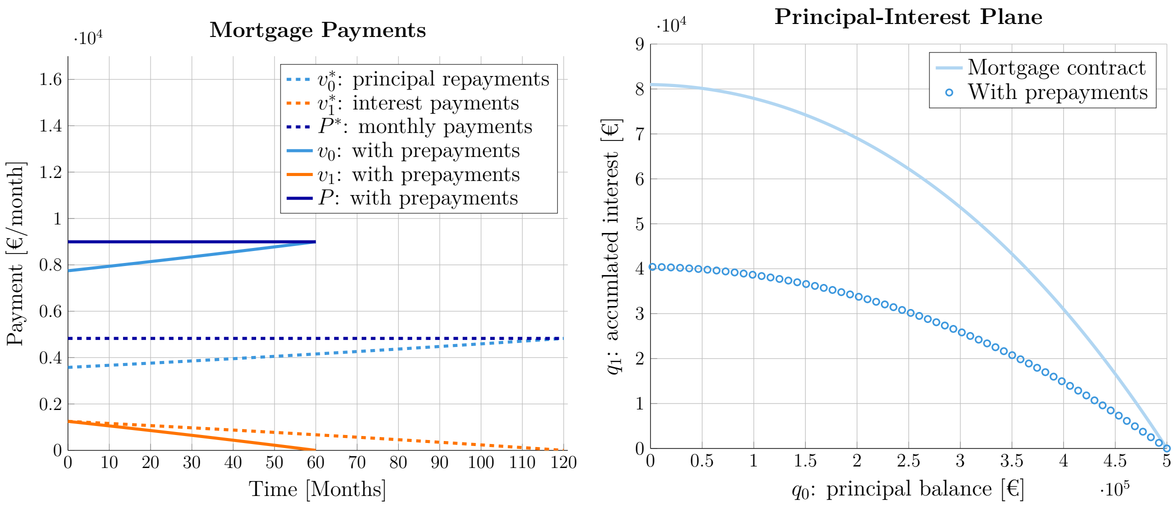

Figure 2-13: An annuity contract with and without partial prepayments. The left figure shows the scheduled payments per month $(\cdot)^*$ and the scheduled payments with prepayments $(\cdot)$. The right figure shows the schedule of the annuity mortgage without prepayments and with prepayments.

We model a mortgage as a mass m with values for m_0 and m_1, and a linear horizontal spring with a spring stiffness k_0 as shown in Figure 2-14. The mass is analogous to a borrower's risk aversion for mortgage payments, and the spring constant is analogous to a borrower's creditworthiness.

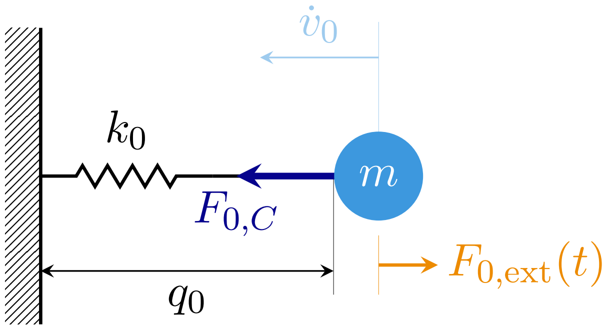

Figure 2-14: We model a mortgage as a mass-damper system. The spring constant k_0 is analogous to a mortgagor's creditworthiness, and the mass m is analogous to a mortgagor's risk aversion. The appetite $F_{0,C}$ (spring force) incentivizes the mortgagor to reduce his debt with prepayments $\dot{v}_0$. The risk-free rates act as an external market force $F_{0,\text{ext}}(t)$ on the mortgage.

The spring has a displacement equal to the principal balance q_0 of the mortgage contract. The maximum elongation of the spring is equal to $q_0 = A_0$, and the spring has a rest state of $q_0 = 0$. This means that a borrower will have a risk appetite resulting from its creditworthiness called $F_{0,C}$:

$$F_{0,C} = -k_0 q_0 \tag{2-29}$$

Partial prepayments are from the mortgagor's perspective a voluntary action that may be done at any point in time and any level less than the maximum allowed per year. Therefore, a partial prepayment can be seen as an expression of the mortgagor's desired debt level of the mortgage [31].

The risk appetite shown in Equation (2-29) is a mortgagor's appetite to reduce his debt. This appetite will be larger at the beginning of the mortgage term and smaller at the end of the mortgage term (displacement of the spring decreases over time).

The magnitude of this force (for an equal spring displacement) will depend on the creditworthiness of the mortgagor, or the spring constant k_0. A higher creditworthiness will result in a larger, and therefore in a larger appetite for principal repayments. Whereas, a lower creditworthiness will result in a smaller force, and thus in a smaller appetite for principal repayments.

The desirability or appetite of a mortgagor to reduce his debt is a determinant for partial prepayments, however, short-term risk-free rates are also a driver for partial prepayments [31, 32]. The short-term risk-free rates can be used to measure the relative attractiveness of partial prepayments as an alternative investment opportunity. A risk-free investment is assumed to be an investment that carries zero risk, where the rate of return on this investment is known as the risk-free rate. A risk-free rate is something of a theoretical concept, as every investment carries some level of risk. In practice, the interest rate paid on short-term government debt is assumed to be risk-free [33].

If the interest rate on its loan is higher than interest rates for other possible investments, prepayment can bring significant savings in interest, while if returns on these investments exceeds its contractual rate, investments represent a wiser economic choice than repaying the mortgage [30]. If a mortgagor can not refinance, the more in the money the mortgage is, the more likely the mortgagor is going to partially prepay [13]. We assume that risk-free rates act as an external market force impinging on the mortgage.

The force is supplemented with additional forces to arrive at an effective force that determines the change in its risk factor. The effective force or effective risk appetite is the sum of the appetite resulting from its creditworthiness and the appetite resulting from economic market forces or risk-free rate changes.

$$F_0 = F_{0,C} + F_{0,\text{ext}(t)} \tag{2-30}$$

To model the influence of these economic forces on the level of principal repayments supplied, the state of the model is

$$x = \begin{bmatrix} q_0 \\ q_1 \\ v_0 \\ v_1 \end{bmatrix} \tag{2-31}$$

and the state space is written in the form $\dot{x} = Ax + Bu$:

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{v}_0 \\ \dot{v}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_0}{m_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ v_0 \\ v_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_0} \\ 0 \end{bmatrix} u \tag{2-32}$$

Here, the input is equal to the external force exerted on the mass $u = F_{0,\text{ext}}(t)$. The effective force or effective risk appetite is now the sum of the appetite to reduce debt $F_{0,C}$, and the external market force $F_{0,\text{ext}}(t)$.

Additional principal repayments are allowed of maximum $\rho = 10\%$ per year of the initial level of money A_0 borrowed. Partial prepayments above this level will result in a prepayment penalty, but we assume that a borrower will only partially prepay within the allowable bounds.

This maximum allowed percentage of partial prepayments per year results in the following inequality constraint on principal repayments in Equation (2-33). The lower bound of the inequality constraint is equal to the scheduled principal repayments with the maximum allowed partial prepayment per month, and the upper bound is equal to the scheduled principal repayments.

$$-P + v_1 - \frac{\rho}{12} A_0 \leq v_0 \leq -P + v_1 \tag{2-33}$$

2-5-2 Full Prepayments and Jumps in the Continuous State

Both prepayments due to relocation and refinancing are full prepayments of a loan. A full prepayment of a loan is modeled as a discontinuity in the continuous dynamics. The continuous state jumps after a full repayment towards a different value. We model a jump in the state as a jump from the current value$\boldsymbol{x}$ to a new value $\boldsymbol{x}^+$. [34, 35]. The notation $\boldsymbol{x}^+$ represents the value of the state after an instantaneous change [36]. In this thesis, the notation $\boldsymbol{x}^+$ represents the value of the state *after* a full prepayment due to relocation or a refinancing.

The state of the model is:

$$\boldsymbol{x} = \begin{bmatrix} q_0 \\ q_1 \\ v_0 \\ v_1 \end{bmatrix} \tag{2-34}$$

And a jump in the state is

$$\boldsymbol{x} \rightarrow \boldsymbol{x}^+ \tag{2-35}$$

An illustration of the two different jump in states is shown in Figure 2-15. Here, $\boldsymbol{x} \rightarrow \boldsymbol{x}^+_{\text{RL}}$ is the jump in state for a full loan prepayment due to relocation, and $\boldsymbol{x} \rightarrow \boldsymbol{x}^+_{\text{RF}}$ is the jump in state for a full loan prepayment due to refinancing. The mortgage state can jump from a position $\boldsymbol{x}$ to one of the two different jumps in state. To which state the mortgage state jumps depends on how the loan is full prepaid.

A full loan prepayment drastically decreases the principal balance to zero, and a full prepayment due to relocation is free of a prepayment penalty. At the time the loan is fully prepaid, the principal repayment is equal to the outstanding principal balance (still outstanding debt of the mortgagor). The state $\boldsymbol{x}$ *after* a full loan prepayment due to relocation has the value $\boldsymbol{x}^+_{\text{RL}}$ as shown in Equation (2-36). The principal balance jumps to zero, the level of accumulated interest remains constant (as no interest is paid anymore), and both the principal repayment and interest payment jump to zero.

$$\boldsymbol{x}^+_{\text{RL}} = \begin{bmatrix} 0 \\ q_1 \\ 0 \\ 0 \end{bmatrix} \tag{2-36}$$

A mortgagor is allowed to refinance his loan at every moment in time, but a refinancing is only penalty-free when selling a house or at the reset date of the fixed-interest rate period. Otherwise, a penalty should be paid for the interest income the bank misses. This penalty is also known as a prepayment penalty. The prepayment penalty discourages mortgagors from paying more than their scheduled periodic payment or completely paying off their loan. This penalty is equal to the present value of the difference between the monthly interest payments

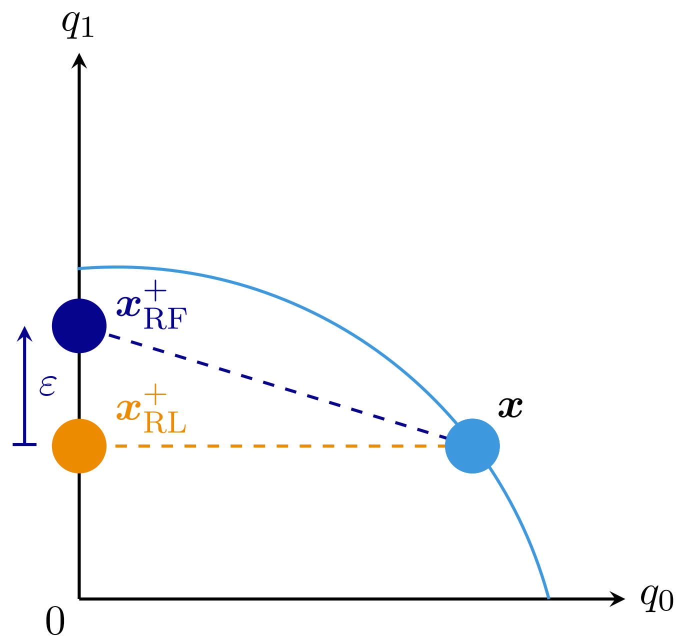

Figure 2-15: An illustration of the discontinuous dynamics of full loan prepayments in the principal-interest plane. A full loan prepayment results in a jump of the continuous state: $\boldsymbol{x} \to \boldsymbol{x}^{+}$. The state jumps from $\boldsymbol{x} \to \boldsymbol{x}_{\mathrm{RL}}^{+}$ for a full loan prepayment due to relocation, and jumps from $\boldsymbol{x} \to \boldsymbol{x}_{\mathrm{RF}}^{+}$ for a full loan prepayment due to refinancing. A prepayment penalty ε must be paid when a mortgagor refinances. The spring is left out for this figure.

of the existing mortgage contract and of a newly originated mortgage at the mortgage rate for the remaining term [28]. The penalty ε is approximated as

$$\varepsilon(t) = \underbrace{(r_c - r_m(t))}_{\text{Interest difference}} \underbrace{(q_0(t) - \rho A_0)}_{\text{Balance level}} \underbrace{(T - t)}_{\text{Remaining term}} \tag{2-37}$$

We calculate the penalty as the difference between the contractual interest rate r_c and the market mortgage interest rates $r_m(t)$ which vary over time. This is multiplied with the principal balance level on which a penalty should be paid. This balance level is the difference between the outstanding principal balance at the time a refinancing is requested, and the penalty-free part. The penalty-free part is mostly 10% (sometimes 20% depending on the contract) per year of the initial level of money A_0 borrowed. This is multiplied with the remaining term of the contract in months.

The difference with the present value calculation is that the level of money the bank misses per month for the remaining term of the contract is not discounted with a discount factor [28]. We also assume that partial prepayments do not influence the prepayment penalty. Normally, the already conducted partial prepayments in that year should be subtracted from the allowed percentage per year to calculate the part of the balance on which a penalty should be paid.

The outstanding principal balance is completely prepaid during a full loan prepayment due to refinancing. At the the moment a loan is refinanced, the principal repayment is equal to the outstanding principal balance, and the interest payment equal to the prepayment penalty

ε. The state $\boldsymbol{x}$ *after* a full loan prepayment due to refinancing has the value $\boldsymbol{x}^+_{\mathrm{RF}}$ as shown in Equation (2-38). The principal balance jumps to zero, and the level of accumulated interest jumps to the current level plus a prepayment penalty. Both the principal repayment and interest payment jump to zero.

$$\boldsymbol{x}^+_{\mathrm{RF}} := \begin{bmatrix} 0 \\ q_1 + \varepsilon \\ 0 \\ 0 \end{bmatrix} \tag{2-38}$$

2-5-3 A Combination of Partial and Full Prepayment Dynamics

The combination of modeling partial prepayments and the two types of full prepayments, results in a combination of continuous and discontinuous dynamics. In Figure 2-16, we show an illustration of the combination of these two dynamics in the principal-interest plane. We use the continuous dynamics for modeling the monthly mortgage payments and partial prepayments, and the discontinuous dynamics for modeling the full loan repayments due to a relocation and refinancing.

To derive a model that simulates these two prepayment dynamics of a mortgage, we need to combine them in one model. Here, a borrower has control over the level of prepayments he wants to supply. Therefore, the switching between the continuous dynamics and discontinuous dynamics is controlled by the borrower. In this illustration below, a borrower wants to fully prepay his mortgage when his principal balance crosses a certain lower value q_0^*. So the loan is prepaid fully if a borrower's principal balance in a certain month t is lower than this value: $q_0(t) \leq q_0^*$.

This interaction of discontinuous dynamics and continuous dynamics in systems are called hybrid systems. A simple hybrid system as for example the regulation of temperature in a house is quite similar to the hybrid system of mortgage prepayments. In a simplified description, the heating element is assumed to either to work at its maximum power or to be turned off. The switching between the heating element from "on" to "off" is controlled by the thermostat, and the evolution of the temperature in each mode ("on" and "off" mode) is described by a differential equation. The heating is turned off when the temperature in the room crosses a certain upper value (determined by the desired temperature) [35].

Similarly, the dynamics of a mortgage has a continuous mode and a discontinuous mode, where the switching between these two modes is controlled by a borrower. The evolution of the mortgage balances and payments in the continuous mode is described with differential equations, and the evolution of the mortgage balances and payments in the discontinuous mode with a jump in state. A borrower switches to a full loan prepayment when his principal balance crosses a certain lower value for example. In Chapter 3, we implement a controller that simulates these dynamics.

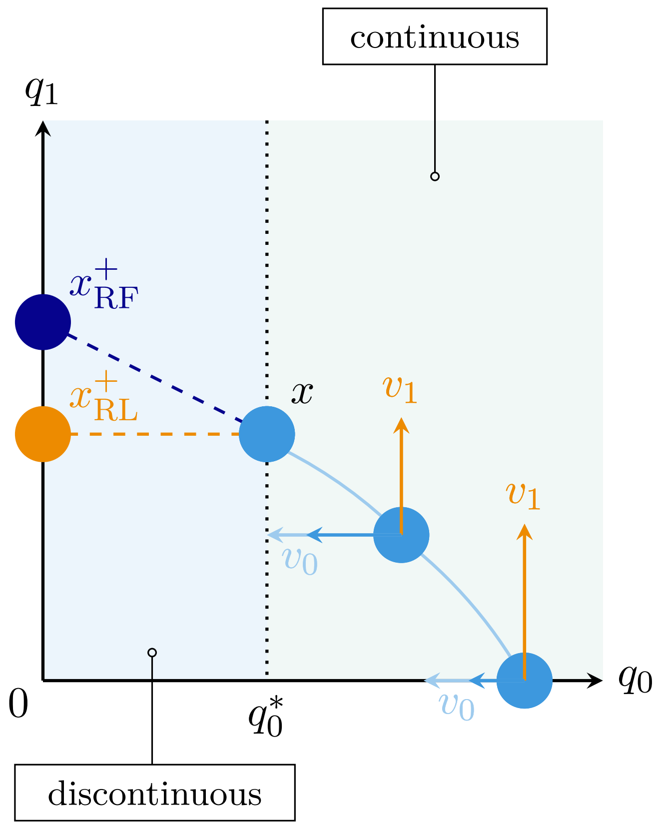

Figure 2-16: An illustration of the combination of the continuous dynamics describing mortgage payments and partial prepayments and the discontinuous dynamics describing full prepayments. The continuous state $\boldsymbol{x}$ can jump after crossing q_0^* either to a full loan prepayment due to relocation $\boldsymbol{x}^+_{\mathsf{RL}}$ or to a full loan prepayment due to refinancing $\boldsymbol{x}^+_{\mathsf{RF}}$. The spring is left out for this figure.

Chapter 3

The Systems and Control Approach for Forecasting Mortgage Prepayments

In this chapter, we formulate the forecasting of mortgage prepayments as a systems and control problem. We describe the system with a model of the dynamics of mortgage payments and prepayments derived in the previous chapter, and we simulate the behavior of a mortgagor with a controller. By simulating mortgagor behavior, we can forecast prepayments with the model for various interest rate scenarios.

First, we discuss the current challenge of forecasting prepayment in increasing interest rates in Section 3-1. Second, we formulate the forecasting of mortgage prepayments as a systems control problem in Section 3-2. Finally, we simulate the dynamics of mortgage payments and prepayments derived in Chapter 2 with optimal control in Section 3-3.

3-1 The Current Challenge of Forecasting Mortgage Prepayments

Economic conditions have a considerable influence on a mortgagor's prepayment behavior, and prepayments will fluctuate as economic conditions change. Economic conditions refer to the state of macroeconomic variables and trends in a country at a point in time. Economic conditions such as inflation, unemployment rate, interest rates, fiscal systems, etc., all find their way in prepayment behavior of mortgagors.

One of the most important risk driver of prepayments is interest rates. In 2022, the European Central Bank (ECB) decided to raise the three key interest rates of the ECB by two percentage points, marking the sharpest and most aggressive rate hike ever in the history of the ECB [6]. By raising these rates, the essence of ECB's monetary policy is to influence liquidity in the economy and subsequently reduce inflation. There exists a strong economic relationship between these interest rates decided by the ECB and other interest rates in liquidity markets (e.g, Euribor rates, risk-free rates, mortgage rates) such as the 5-years interest rates on government bonds shown in Figure 3-1

Figure 3-1: The historical 5-year interest rate on German government bonds. This is the interest paid on short-term government debt of Germany, and this interest rate can be assumed risk-free. A downward trend of interest rates is observed from 2003 till the beginning of 2022. Data is retrieved from [2].

This increase in interest rates ends an unusual and eight-year-long period of negative interest rates. When looking at the history of interest rates, is this increase in interest rate extraordinary. A raise in interest rates of 0.5 percentage point or more only occurred during the financial crisis in 2008-2009, and during the dot-com crisis[1] in 2001.

At this moment, banks use econometric models for forecasting prepayments which are fitted to historical data. By fitting to historical data, correlations are derived between interest rates and prepayments. Regression techniques such as logistic regression is a common approach for finding these correlations (a more detailed explanation of the econometric approaches is in Appendix A).

Fitting to the history of 20-years of interest rates seems large, but economic business cycles can last longer [13]. Moreover, a downward trend is observed in Figure 3-1 of the short-term 5-year interest rate from 2003 till this year. Econometric models fitted to this history of interest rates produces a bias in the models [5]. This means a model fitted to the history of interest rates might perform well in a decreasing rates regime, but might fail if interest rates are increasing.

The increase of interest rates after a long period of decreasing interest rates is a turning point in borrowers prepayment behavior. Increasing interest rates result in structurally lower prepayment rates. The first signals for higher interest rates might temporarily increase prepayment rates due to last minute switching borrowers, the long-term effect is lower prepayment rates [1].

Activity on the housing market will slow down as mortgages will be more expensive, and refinancing also becomes less attractive. Moreover, most mortgage contracts offer the borrower

[1] The dot-com crisis was a stock market bubble in the late 90's characterized by a period of massive growth

the possibility to carry the existing structure and interest rate to a new house when moving. This means that prepayments due to relocation will slow down, because more people will transfer the terms and conditions of their current mortgage to their new house, i.e, mortgage porting. Increasing interest rates will slow down prepayments significantly.

3-2 Formulating Mortgage Prepayments as a Systems and Control Problem

We formulate the forecasting of mortgage prepayments as a system and control problem as shown in Figure 3-2. The *system* is a dynamical system for modeling mortgage payments and prepayments, and the *controller* simulates mortgagor behavior.

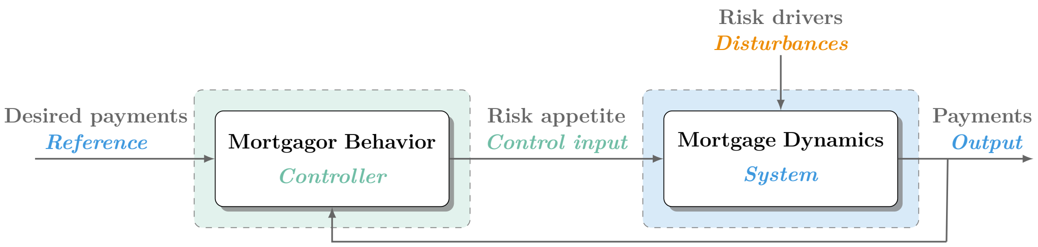

Figure 3-2: The systems and control problem for forecasting mortgage prepayments. The dynamical system is described with a model of mortgage payments and prepayments, and the controller simulates behavior of a mortgagor. This behavior is a mortgagor's risk appetite that he can adjust to reach a desired level of payments. Risk drivers such as interest rates act as disturbances to the system.

The dynamical system is described with a model of mortgage payments and prepayments, and with this model we can forecast prepayments by implementing a controller. A mortgagor determines his own risk appetite for supplying mortgage payments, which will be the control input. By determining his desire for risk a mortgagor can reach a desired level of mortgage payments while subject to disturbances from risk drivers. From now on, macroeconomic risk drivers (e.g., interest rates, mortgage rates, housing activity) that influence a borrower's decision to prepay are disturbances to the system.

By using risk drivers as disturbances to the system allows for forecasting prepayments under different interest rate scenarios. We derive the mortgage dynamics based on analogies between mechanical laws and economic laws. These economic laws apply in any market regime and to every mortgagor. Thus, we can forecast prepayments with this model without drastically changing the dynamics or structure of the model for any interest rate scenario and for any mortgagor.

Choosing a suitable controller for this control problem depends on the desired feedback application or the desired implemented behavior of a mortgagor.

Robust control is an approach for controller design that explicitly deals with uncertainty. Robust control derives control laws such that constraints are satisfied despite uncertainties in

in the use and adoption of the internet.

the system. Unfortunately, methodologies for robust stabilizing controller synthesis developed for continuous systems (such as $\mathcal{H}_\infty$) do not apply to hybrid dynamics [37]. Moreover, a robust control method does not optimize for a mortgagor's economic objective. The goal of a robust controller would be to reach a desired level of payments while having uncertainties in the model dynamics. For example, the uncertainty from the parameters representing a mortgagor's creditworthiness and risk-aversion.

Proportional–Integral–Derivative (PID) control is a classical control method that is suitable for feedback control applications such as reference tracking or disturbance rejection [38]. PID control could be used in this application to determine a borrower's risk appetite that steers the mortgage to a desired level of payments, or to reject disturbances from changing interest rates. If this desired level of payments would be the scheduled mortgage payments, a mortgagor will serve the desires from the bank with applying PID control: following the scheduled payments. However, a borrower will not necessarily benefit from this financially himself.

PID control is unable to optimize for a borrower's desires or objectives such as minimizing mortgage costs. Also, it is difficult to put constraints for example on the maximum allowable partial prepayments. Complex optimization problems can not be solved with PID control.

A mortgagor can choose between full prepayments due to refinancing, relocation or partial prepayments at each moment in time. Each with different costs associated and different consequences on the mortgage amortization schedule. A PID controller can not optimize this choice over time, because there will be a constant trade-off between the different prepayments. This requires an active optimization approach.

These limits of PID control and Robust control can be solved by using optimal control. Using optimal control, a mortgagor can trade off prepaying his mortgage with the different prepayment types at optimal moments, i.e, prepaying in low interest rate regimes and investing in high interest rate regimes. This requires a model that can be used for optimal control. We simulate the dynamics of mortgage payments and prepayments derived in the previous chapter by implementing a Model Predictive Control (MPC).

3-3 Simulating the Mortgage Dynamics with MPC

A MPC is a controller that solves optimal control problems using a receding horizon. MPC solves with a finite-horizon an open-loop optimal control problem in which the initial state is the current state of the system. The optimal control action sequence is based on the current states of the model, and only the first control action is applied at the next time step. The states update, and based on the new initial states and for the same finite horizon, the optimization is repeated. MPC is capable of solving the optimal control problem online for the current state of the system, as opposed to the classical optimization problems where a feedback policy is determined offline for all states and perfect information of the future is assumed [39].

A diagram of an MPC is shown in Figure 3-3. The dynamical system for modeling mortgage payments and prepayments is not a 'real' system, and so the model and the system will be the same. The cost function will be a general quadratic cost function. A more appropriate cost function with an economic interpretation will be designed later in Chapter 5 where we design an Economic Model Predictive Controller (EMPC). Constraints include the constraints of the mortgage contract, the state update equations of mortgage payments and partial prepayments, the state values of full prepayments, and constraints on the balance levels. The main disturbance for now will be the risk-free rates as depicted in Figure 3-1.

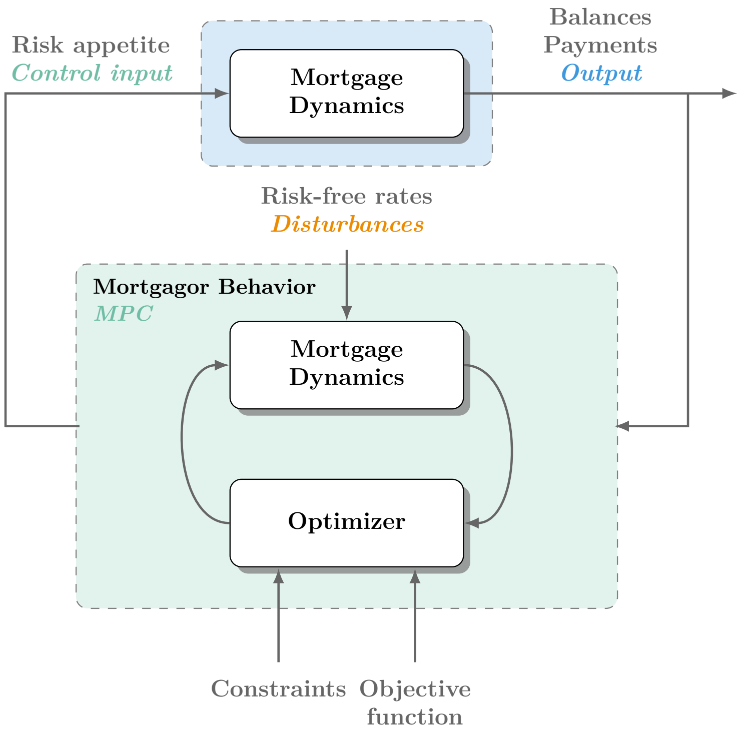

Figure 3-3: Forecasting mortgage prepayments as a systems and control problem. With MPC a mortgagor determines his risk appetite which minimizes an objective function for a prediction horizon based on the current states of the mortgage and predictions of risk-free rates. After the first risk appetite (control action) is applied, the states of the mortgage update, and based on the new initial states and for the same finite horizon, the optimization is repeated.

We assume that a mortgagor optimizes an objective function while having information about the 'dynamics' of his mortgage or in other words he is subject to the constraints of the mortgage dynamics which are mainly the constraints of his mortgage contract. A mortgagor can not arbitrarily vary his risk appetite such that it minimizes the objective function.

The mortgagor can adjust influence his repayment schedule by adding or subtracting an appetite of risk he wants to take, which will be the *control input* for the optimal controller. The *output* of the system are the states of the mortgage consisting of the mortgage balances and payments.

Alternatively, offline optimization could be used, however offline optimization assumes perfect information of the future. To simulate realistic behavior of a mortgagor, we assume that a mortgagor does not have perfect information of the future. A mortgagor has no information about future interest rates, only about current interest rates. They make decisions based on imperfect information about interest rates, and can adapt later on when they receive updates on interest rates. The receding horizon aspect is analogous to the time window a borrower uses for determining his risk strategy, and therefore we design a MPC with receding horizon instead of using offline optimization.

Additionally, MPC is suitable for developing a prepayment model that is adaptable to new mortgage market environments. The model can be adapted according to your own wishes and requirements. This is mainly due to adding and changing constraints with MPC. Constraints can be changed to account for different types of mortgage contracts and constraints can be added to include more prepayments types e.g., delinquent and default payments or partial prepayments with a higher allowed partial prepayment percentage.

We introduce the ease of adding constraints in this section by first simulating scheduled mortgage payments, and changing the model to simulate partial prepayments. Subsequently, we add constraints to simulate full prepayments, and eventually to make a distinction between full loan prepayments due to refinancing and full loan prepayments due to relocation.

3-3-1 Mortgage Payments and Partial Prepayments

The state space from Section 2-5-1 is extended with additional control inputs and written in the form $\dot{x} = Ax + Bu + Fd$:

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{v}_0 \\ \dot{v}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_0}{m_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ v_0 \\ v_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_0} & 0 \\ 0 & \frac{1}{m_1} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_0} \\ 0 \end{bmatrix} d \tag{3-1}$$

The risk-free rates are now disturbances of the model $d = F_{0,\text{ext}}(t)$

A mortgagor has a control input $u = \begin{bmatrix} u_0 & u_1 \end{bmatrix}^T$ that can be actively controlled. The mortgagor can actively determine his own appetite for risk with respect to principal repayments and interest payments, u_0 and u_1 respectively.

The economic market force from risk-free rate changes, a mortgagor's appetite to reduce his debt, and his own appetite for risk, determine the effective risk appetite of a mortgagor for principal repayments. This risk appetite leads to a change in his risk factor, which subsequently leads to a change in his level of principal repayments supplied as visualized as a movement along the supply curve in fig. 2-2.

To run simulations, we discretize the model with Forward Euler method:

$$\begin{bmatrix} q_0[k+1] \\ q_1[k+1] \\ v_0[k+1] \\ v_1[k+1] \end{bmatrix} = \begin{bmatrix} q_0[k] \\ q_1[k] \\ v_0[k] \\ v_1[k] \end{bmatrix} + T_s \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_0}{m_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_0[k] \\ q_1[k] \\ v_0[k] \\ v_1[k] \end{bmatrix} + T_s \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_0} & 0 \\ 0 & \frac{1}{m_1} \end{bmatrix} \begin{bmatrix} u_0[k] \\ u_1[k] \end{bmatrix} + T_s \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_0} \\ 0 \end{bmatrix} \hat{d}[k] \tag{3-2}$$

Here m_0, m_1, k_0 and T_s are model parameters. The control inputs are u_0 and u_1, and the predicted disturbance is $\hat{d}$.

An objective function will be optimized subject to the state update equations from the continuous dynamics, and the contract constraints. We perform simulations in MATLAB using the

YALMIP toolkit to program the MPC. The optimization problem is solved using "quadprog" solver with the interior-point-convex algorithm. The "quadprog" solver is used as the optimization problem concerns the problem of optimizing a quadratic objective function subject to linear constraints, known as Quadratic Programming (QP).

The minimization problem is given by a quadratic objective function minimized over time with constraints. The state equations, the contract constraints, and a constraint on the principal balance are constraints on the system. We add a constraint on the principal balance, because it is not possible to have a negative principal balance. A negative balance would mean that a mortgagor has no debt anymore, but is funding money to the bank. The optimization of one iteration looks as follows:

$$
\begin{aligned}
\underset{u}{\text{minimize}} \quad & \sum_{k=1}^{T_h} \frac{1}{2} m_0 v_0[k]^2 + \frac{1}{2} m_1 v_1[k]^2 - \frac{1}{2} k_0 q_0[k]^2 + \frac{1}{2} R u_0[k]^2 \\
\text{subject to} \quad & x[k+1] = Ax[k] + Bu[k] + F\hat{d}[k], \\
& x[0] = x_0 \\
& v_1[k] = r_c q_0[k], \\
-P + v_1[k] - \frac{\rho}{12} A_0 \leq \; & v_0[k] \leq -P + v_1[k], \\
& q_0[k] \geq 0,
\end{aligned}
\tag{3-3}
$$

Where the contract parameters are ρ, A_0, P and r_c. The prediction horizon is T_h, and the penalty on the control input is R.

The optimal control sequence is calculated for one iteration for $k \in [0, T_h]$. The control input for the first time step is implemented, and the new states are determined. The updated states are used as the new initial condition x_0, and the optimal control sequence for $k \in [1, T_h + 1]$ is determined, which repeats the process.

We add a penalty on the control input u_0. If the penalty on the control input is high, a mortgagor can not completely counteract the disturbances from interest rates changes with his risk appetite, resulting in a change of his risk factor which leads to partial prepayments. Meanwhile, if the penalty on the control input is close to zero, a mortgagor can counteract all the disturbances from interest rate changes with adjusting his risk appetite. The risk factor of the mortgagor will not change, and he supplies the scheduled payments.

A mortgagor does not have perfect information of future interest rates, and they make decisions based on imperfect information about interest rates. Therefore, during the prediction horizon we assume that a mortgagor only has information about the current observed risk-free interest rate. To model this imperfect information, we sample the risk-free rates with a delayed first-order hold method.

We consider the historic 5-years interest rates on government bonds of Germany as risk-free rates from Figure 3-1. We increase the risk-free rates with a scaling factor such that they act as an economic force impinging on the system. Without scaling the risk-free rates a borrower's risk factor is barely disturbed by risk-free rate changes, because of the difference in magnitudes. The risk-free rates are in decimal points and a borrower's appetite to reduce his debt is depending on the principal balance which is in hundreds of thousands. The disturbance need to be scaled to have some influence on the system. We choose a scaling factor of $\beta = 4.3 \cdot 10^7$, and will remain so for each simulation in this thesis. Alternatively, k_0 could be tuned, such that the interest rates have influence on a borrower's risk factor without scaling them.

To run simulations, we simulate a mortgage of €500,000 with a contract rate of 3% per year and a maturity of 10 years. The maximum allowed partial prepayments per year is ρ is 10%. The sampling time T_s will be one month and the prediction horizon T_h is two months. The model parameters chosen are $m_0 = \frac{1}{0.35}$, $m_1 = \frac{1}{0.1}$, and $k_0 = \frac{1}{0.1}$, and the penalty on the control input is $R = 0$. The initial conditions x_0 are:

$$q_0[0] = A_0 \qquad q_1[0] = 0 \qquad v_0[0] = r_c q_0[0] - P \qquad v_1[0] = r_c q_0[0] \tag{3-4}$$

In Figure 3-4 the mortgage payments together with the balances levels each month are shown. We use a continuously looking line to make a distinction between the scheduled contract payments and the payments with prepayments. The mortgagor is rejecting all the disturbances from risk-free rate changes, and is not adjusting his risk factor based on these interest rate changes, and supplies each month the scheduled payments. The mortgagor does not supply any prepayments and the mortgage contract is paid off as scheduled.

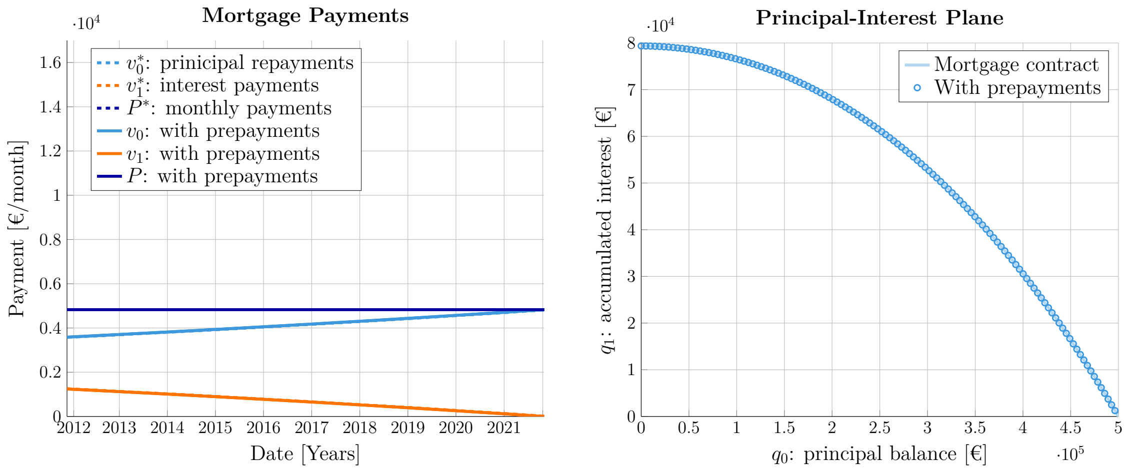

Figure 3-4: A simulation of the scheduled payments with MPC. The left figure shows the payments per month, and the right figure the schedule in the principal-interest plane. Risk-free rate changes act as a disturbance $\hat{d}$ on the system, and the penalty on the control input is $R = 0$. The mortgagor is not prone to changes in risk-free rates in the market, and rejects all the disturbances. He is not willing to adjust his risk factor and supplies the scheduled payments.

The penalty on the control input will be set to $R = 0.3$. This means that the borrower will be more prone to changes in risk-free rates and partially prepay. The controller is not able to reject all the disturbances from risk-free rates when interest rates are falling the fastest as visible Figure 3-5.

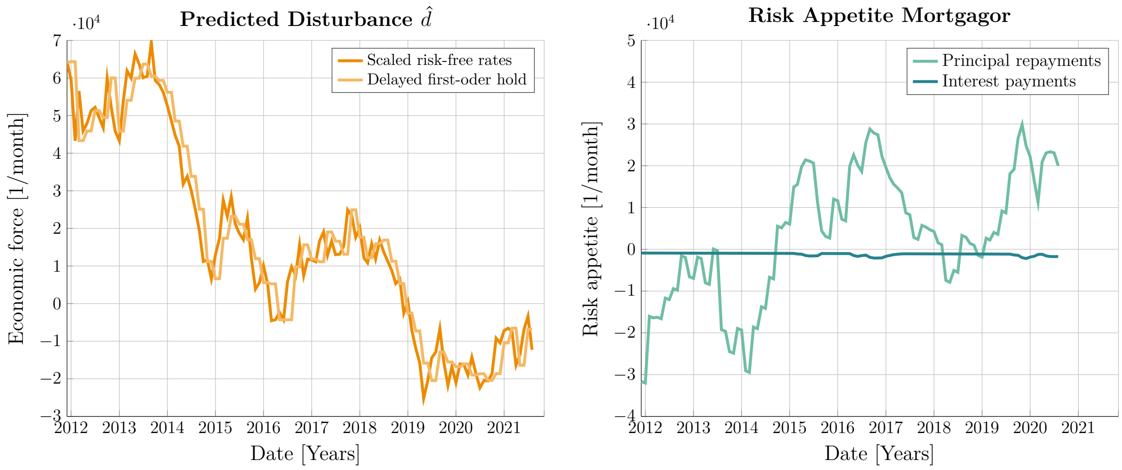

Figure 3-5: The disturbances from risk-free rate changes in the left figure, and a mortgagor his own risk appetite (control input) in the right figure. The penalty on the control input penalizes a mortgagor's risk appetite. When the interest rates decrease fast (in 2015, before 2017, and end of 2019), the mortgagor can not reject all the disturbances as visible in the right figure.

The mortgagor can not reject all the disturbances from risk-free rate changes resulting in a change in his risk factor. This change in risk factor result in an increase of the level of principal repayments supplied. The principal repayments per month are bounded by the inequality constraint in Equation (2-33), so a mortgagor must always supply the scheduled principal repayments and can deviate to the maximum allowed partial prepayments per month. A major decrease in risk-free rates results in an increase in partial prepayments, and result in a deviation from the schedule of the annuity contract as can be seen in Figure 3-6. The annuity mortgage is repaid earlier due to the partial prepayments as visible in the left figure, and the bank receives less interest on the mortgage as visible in the right figure.

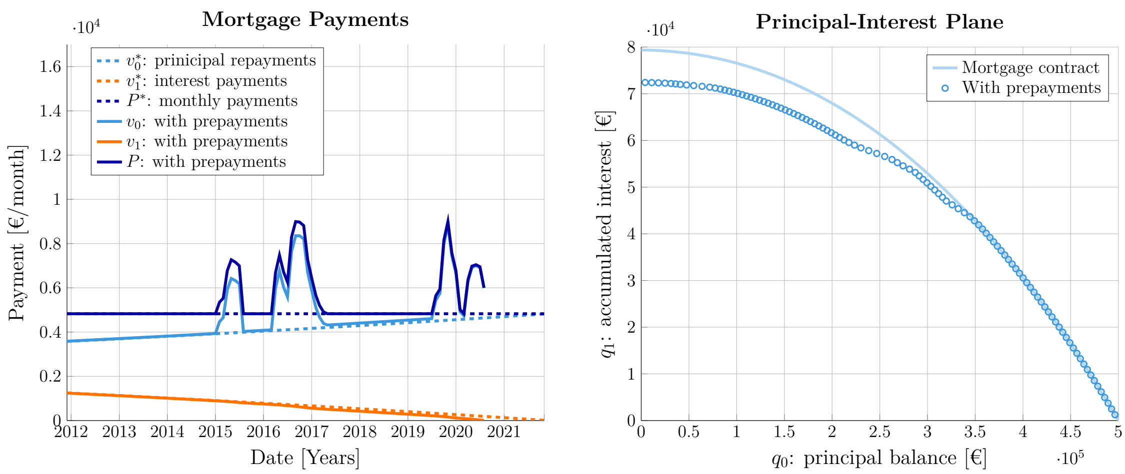

Figure 3-6: The mortgagor is prone to changes in risk-free rates in the market as he can not adjust his own risk appetite to the quantity needed to reject the disturbances from risk-free rate changes. This is due to the penalty on the control input. The risk factor of the mortgagor changes which results in an increase of the level of principal repayments supplied.

3-3-2 Combination of Partial and Full Prepayments

In this section the state update equations of the continuous model, as well as the discontinuous dynamics from the full prepayments, are constraints on the system. For simulations, we simulate the same mortgage contract and use the same model parameters as in the previous section.

An objective function will be optimized subject to the constraints from the continuous dynamics, the discontinuous dynamics, and the contract constraints. We perform the simulations in MATLAB using the YALMIP toolkit (the documentation of YALMIP is followed for the implementation in Matlab [40]), and as solver the 'Gurobi' solver is used. We use this type of solver as the combination of continuous and discontinuous dynamics results in an optimization problem containing both discrete and continuous decision variables, giving rise to a Mixed-Integer Programming (MIP) problem. MIP models with a quadratic objective but without quadratic constraints are called Mixed-Integer Quadratic Programming (MIQP) problems. The Gurobi solver uses a branch-and-bound algorithm (tree search) for MIP problems [41]. A branch-and-bound algorithm for MIQP problems consists of solving and generating new QP problems in accordance with a tree search. The nodes of the three correspond to QP subproblems. The QP subproblems are explored and solved systematically through a binary tree until the solution is an optimal solution of the original MIQP [42, 41, 35].

We add the discrete dynamics of a full loan prepayment due to relocation in Equation (2-36) to the constraints of the MPC. The MPC is now subject to two different modes, where one mode is continuous and simulating the mortgage payments per month including possible partial prepayments (**Mortgage Payments Mode**), and the other mode is a jump of the continuous state representing a full loan prepayment due to relocation (**Relocation Mode**). The discrete decision variables arise in the optimization problem as it is only possible to be in one mode at a time. In Section 5-2-3, we explain this in more detail.

The optimization of one iteration looks as follows:

$$\begin{aligned}
&\underset{u}{\text{minimize}} \quad && \sum_{k=1}^{T_h} \frac{1}{2}m_0 v_0[k]^2 + \frac{1}{2}m_1 v_1[k]^2 - \frac{1}{2}k_0 q_0[k]^2 + \frac{1}{2}R u_0[k]^2 \\
&\text{subject to} &&
\end{aligned}$$

$$\begin{aligned}
\textbf{Mortgage Payments Mode} = [x[k+1] &= Ax[k] + Bu[k] + F\hat{d}[k], \\
x[0] &= x_0 \\
v_1[k] &= r_c q_0[k], \\
-P + v_1[k] - \frac{\rho}{12}A_0 \leq v_0[k] &\leq -P + v_1[k], \\
q_0[k] &\geq A_0/2.5] \\
\textbf{Relocation Mode} = [q_0[k+1] &= 0, \\
q_1[k+1] &= q_1[k], \\
v_0[k+1] &= 0, \\
v_1[k+1] &= 0, \\
0 \leq q_0[k] &\leq A_0/2.5]
\end{aligned} \tag{3-5}$$

To allow switching between the two different modes, we add a condition on the state q_0. The mode will switch to a full prepayment if the principal balance is lower than $A_0/2.5$:

$$q_0[k] \leq A_0/2.5, \tag{3-6}$$

and we add this condition as constraint of the relocation mode. We add the following condition as constraint of the mortgage payments mode:

$$q_0[k] \geq A_0/2.5. \tag{3-7}$$

Both the mortgage payments and the contract schedules are shown in Figure 3-7. When the principal balance q_0 in the principal-interest plane is less than $A_0/2.5 = 2 \cdot 10^5$ euros the loan is fully repaid. This is visible with the jump in state $x \to x^+$ in the figures. The state jumps from the continuous state x to x^+_{RL} as in Equation (2-36).

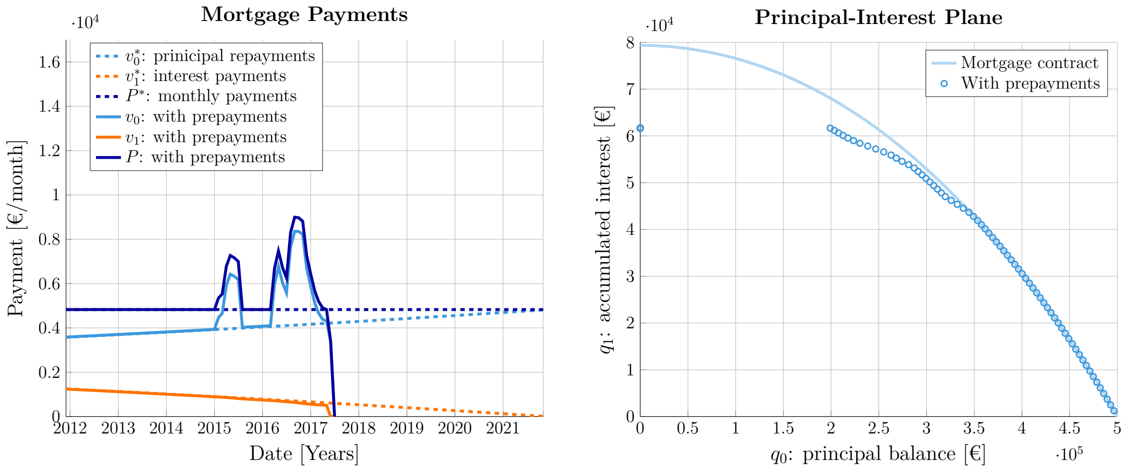

Figure 3-7: A simulation of partial prepayments and a full loan prepayment with MPC. The penalty on the control input is $R = 0.3$, and the loan is fully prepaid if the principal balance is lower than half of the borrowed money $q_0[k] \leq A_0/2.5$. In the right figure this is visible with the jump in state.

From the left figure in Figure 3-7 it is notable that the full prepayment is not visible. The principal repayment should be equal to the outstanding debt of the mortgagor. To explicitly model the full loan prepayments, extra constraints on the payments are added to the relocation mode:

$$\begin{aligned} v_0[k] &= -q_0[k] \\ v_1[k] &= 0 \end{aligned} \tag{3-8}$$

These extra constraints on the payments increase the principal repayment significantly, resulting in the peak as visible in Figure 3-8. This peak of principal repayment is equal to the principal balance before the jump in state as visible in Figure 3-7 of the mortgage contract in the principal-interest plane.

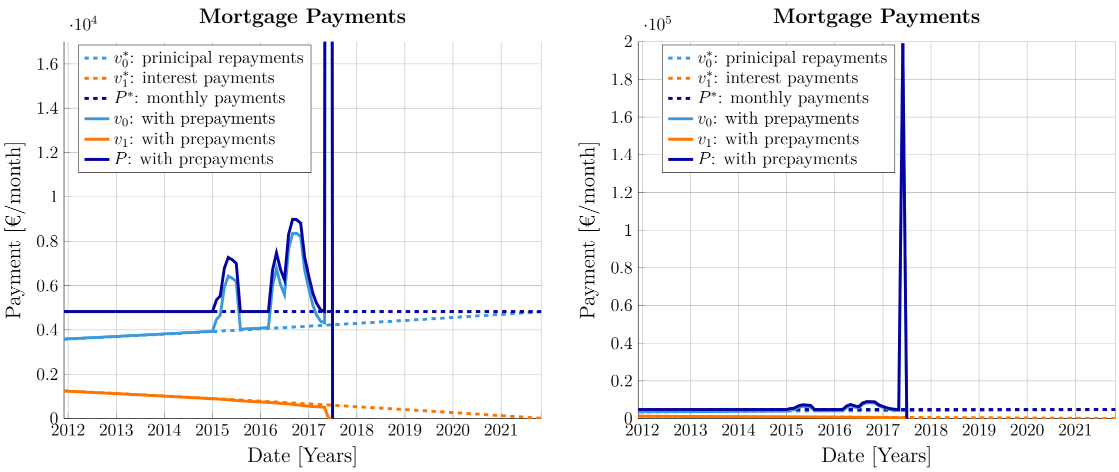

Figure 3-8: We model the full loan prepayment explicitly with extra constraints on the payments. The peak of the principal repayment is equal to the principal balance before the jump in state in Figure 3-7.

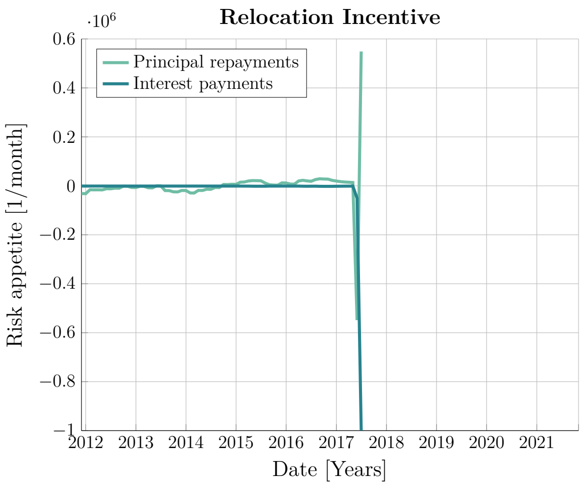

Figure 3-9: The large shocks in a mortgagor's risk appetites (relocation incentive) result in the jump in state of a full loan prepayment due to relocation. This is analogous to how forces or control inputs can be used to steer a mechanical system to a desired state.

The principal repayment equals the principal balance (still outstanding debt) due to a mortgagor's risk appetite as visible in Figure 3-8. When he fully prepays, the mortgagor decreases his risk appetite for principal repayments immediately (force in the negative q_0 direction) such that his principal repayment is equal to the still outstanding debt. Subsequently, he increases his risk appetite (force in the positive q_0 direction) to decrease the principal repayment to zero. His risk appetite for interest payments decreases (force in the negative q_1 direction) to reduce the interest payment to zero.

The large shocks in the risk appetite of a mortgagor are needed to fully prepay a loan. These shocks (or force impulses) that result in a jump of the continuous state will be called *incentives*. The relocation incentive is thus the adjustment needed from a mortgagor's risk appetite to supply a full loan prepayment due to relocation as visible in Figure 3-8. The conditions (for example the constraints in Equation (3-6), and Equation (3-7)) that 'trigger' an incentive will be called the *switching conditions.* These conditions need to be added to the constraints of the modes.

A distinction should be made between full loan prepayments due to refinancing and full loan prepayments due to relocation. We make this distinction easily by adding an extra mode, and using the value of the state in Equation (2-38) as constraints. This results in the following optimization problem for one iteration:

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & \sum_{k=1}^{T_h} \frac{1}{2} m_0 v_0[k]^2 + \frac{1}{2} m_1 v_1[k]^2 - \frac{1}{2} k_0 q_0[k]^2 + \frac{1}{2} R u_0[k]^2 \\ \text{subject to} \quad & \end{aligned}$$

$$\begin{aligned} \textbf{Mortgage Payments Mode} = [x[k+1] &= Ax[k] + Bu[k] + F\hat{d}[k], \\ x[0] &= x_0 \\ v_1[k] &= r_c q_0[k], \\ -P + v_1[k] - \frac{\rho}{12} A_0 \leq v_0[k] &\leq -P + v_1[k], \\ q_0[k] &\geq A_0/2.5] \\ \textbf{Relocation Mode} = [q_0[k+1] &= 0, \\ q_1[k+1] &= q_1[k], \\ v_0[k+1] &= 0, \\ v_1[k+1] &= 0, \\ v_0[k] &= -q_0[k], \\ v_1[k] &= 0, \\ q_0[k] &\geq A_0] \\ \textbf{Refinance Mode} = [q_0[k+1] &= 0, \\ q_1[k+1] &= q_1[k] + \varepsilon[k], \\ v_0[k+1] &= 0, \\ v_1[k+1] &= 0, \\ v_0[k] &= -q_0[k], \\ v_1[k] &= \varepsilon[k], \\ 0 \leq q_0[k] &\leq A_0/2.5] \end{aligned} \tag{3-9}$$

Here ε is the prepayment penalty. The prepayment penalty for this simulation is set equal to a constant penalty of $\varepsilon[k] = 3000$. The mortgage payments mode will switch to the refinance mode if $q_0[k] \leq A_0/2.5$. We add this switching condition as a constraint to the refinance mode.

The mortgage payments and contract schedules are shown in Figure 3-10. The interest payment is equal to the prepayment penalty, and the principal repayment is equal to the outstanding principal balance. The state q_0 jumps to zero in the principal-interest plane, and the state q_1 jumps to a higher level of accumulated interest due to the prepayment penalty.

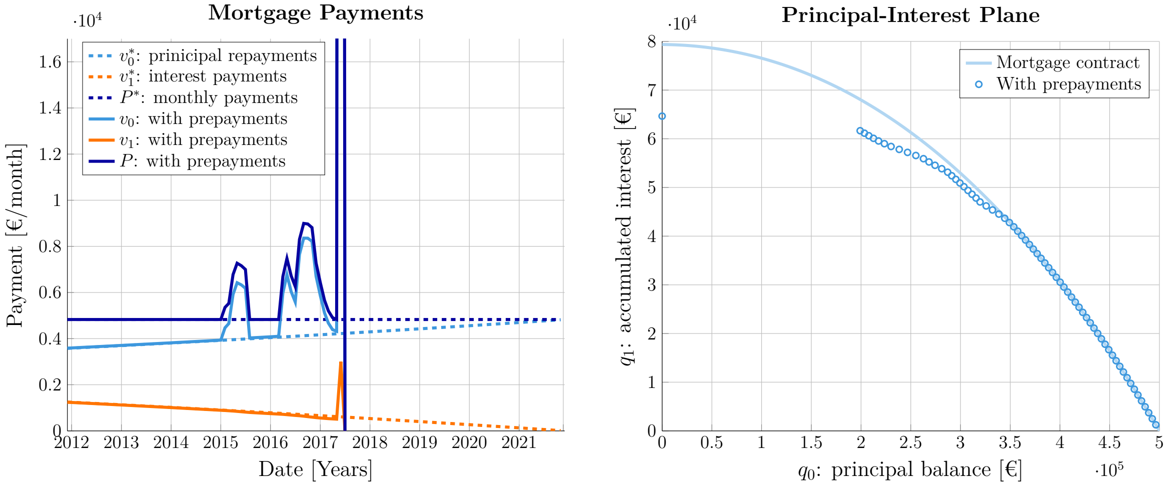

Figure 3-10: The prepayment penalty causes an increase in interest payment in the left figure and an increase in accumulated interest as visible in the right figure. The state q_1 jumps to a higher level of accumulated interest. The penalty on the control input is $R = 0.3$. The loan is fully prepaid if the condition on the principal balance holds: $q_0[k] \leq A_0/2.5$.

The incentive needed to refinance is shown in Figure 3-11. His risk appetite for interest payments first increases (force in the positive q_1 direction) such that the interest payment equals the prepayment penalty, and decreases (force in the negative q_1 direction) to decrease the interest payment to zero. His risk appetite for principal repayments first decreases (force in the negative q_0 direction), and subsequently increases (force in the positive q_0 direction) such that the principal repayment equals the still outstanding debt and reduces to zero.

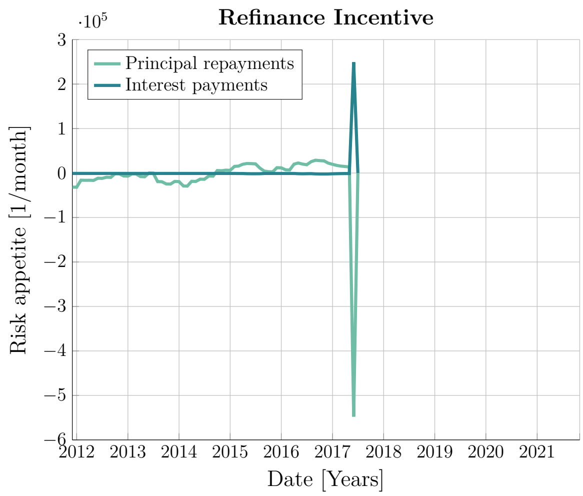

Figure 3-11: The large shocks in a mortgagor his risk appetites (refinance incentive) result in the jump in state of a full loan prepayment due to refinancing. This is analogous to how forces can be used to steer a system to a desired state.

Chapter 4

The Hybrid Economic-Engineering Model

The mortgage dynamics simulated in the previous chapter are too simplified to forecast prepayments. At this moment, a mortgagor has no option after moving to choose between fully prepaying his mortgage (and taking out a new mortgage contract), or transferring his existing contract to the new house. Moreover, the full loan prepayments are not determinant by their risk drivers: mortgage rates and housing activity.

In this chapter, we extend the dynamics of the model by modeling the possibility for mortgage porting, using the risk drivers of full loan prepayments as switching conditions, and modeling a new contract agreement after full loan prepayments. These extensions result in a hybrid model that is quite similar to what is known as jump-flow systems in hybrid systems modeling.

First in Section 4-1, we describe and visualize the hybrid model in general, and we describe the jump-flow behavior. Subsequently, we describe in Section 4-2 till Section 4-4 how we model the proposed extensions of the dynamics, and in Section 4-5 we discuss the adaptability of the model.

4-1 Modeling the Mortgage Dynamics as Hybrid System

Jump-flow systems are a possible model structure for hybrid systems and form a quite natural extension of differential equations [36, 35]. The behavior of a dynamical system that can be described by a differential equation is referred to as a *flow*. The discontinuous dynamics that can be described by a difference equation is referred to as *jumps*.

The continuous model written in differential equations derived in Section 2-5-1 and extended in Section 3-3-1 is the flow part. This flow part contains the continuous dynamics of mortgage payments and prepayments. The value of the states derived in Section 2-5-2 form the two possible jumps in the continuous state. The continuous state can either jump to the state

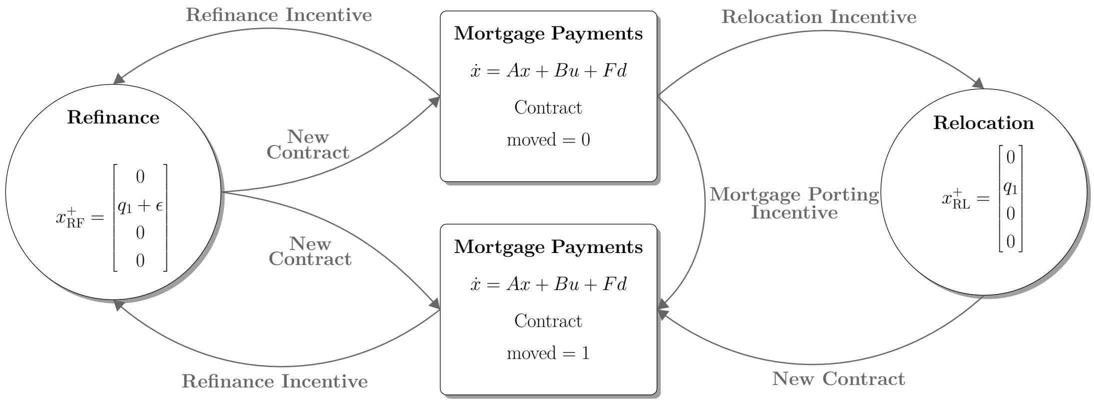

Figure 4-1: A block diagram of the hybrid economic-engineering model. The flow part consists of the state update equations in the rectangular blocks, and the two jumps are indicated with the circular blocks. The flow part contains the continuous dynamics of mortgage payments and prepayments with additional contract constraints, and the jumps the two different full loan prepayments. A full loan prepayment occurs due to either the refinance incentive or relocation incentive. After each full loan prepayment, a new contract is agreed upon. A mortgagor has the option to transfer his mortgage via the mortgage porting incentive. We add a discrete state that counts the times a mortgagor has moved.

of a full loan prepayment due to refinancing or to the state of a full loan prepayment due to relocation.

A jump in the continuous state occurs due to either a *refinance incentive* or a *relocation incentive*. These incentives are activated if the conditions of either the refinance mode or relocation mode hold. We add these conditions as constraints to these modes. These conditions will in this chapter depend on risk drivers such as mortgage rates and housing activity.

The controller, or mortgagor, ensures that the specifications of the jump in states are satisfied whenever a full loan prepayment happens. These incentives are a large force for a short period of time (shock) exerted by the controller. This behavior can be observed from the control input u during full loan prepayments as shown in the previous chapter.

We add to the model a discrete state that counts the times a mortgagor has moved. If some memory is present in the switching between jumps and flows, then one needs the explicit inclusion of a discrete state in the model [35]. We use the discrete state to indicate whether a mortgagor has already moved or not during the mortgage contract. The discrete state can jump via the *mortgage porting incentive* or the *relocation incentive*.

Moreover, we add extra states to the model such that the mortgage contract can be updated after full loan prepayments. This new mortgage contract will have different contract specifications with a new level of money borrowed, a new contractual interest rate, and a new level of monthly payments. This new contract also resets all the balances $\boldsymbol{q}$ and payments $\boldsymbol{v}$ after each jump, such that continuation is possible in the flow part of the model.

Keep in mind that still the constraints of an annuity mortgage will be used in this section as an example. These constraints can easily be changed to another contract type by using the contract constraints as derived in Section 2-2-3.

4-2 Modeling Mortgage Porting

Generally, in the Netherlands mortgages are portable, and it is allowed to transfer the terms and conditions of the existing mortgage contract to the new property without extra costs involved. Especially, in an increasing interest rate environment it is expected that more mortgages will be transferred to the new property instead of fully prepaying the current mortgage and taking out a new mortgage on the new home. Simply, this is due to the probably lower interest rate on the current mortgage than the mortgage rates in the market.

We add an incentive to the model for housing to determine when a mortgagor moves. We keep track if a mortgagor has already moved or not with a discrete state that counts the times a mortgagor moves.

4-2-1 Discrete State and Moving

We add a discrete state to the model such that the model 'remembers' if a mortgagor has already moved or not. This discrete state does not affect the continuous dynamics.

The discrete state can take the values

$$m \in \{0, 1\}, \tag{4-1}$$

and is used to indicate whether a mortgagor has *moved* ($m = 1$) or has *not moved* ($m = 0$).

We use data on housing activity to determine when the discrete state switches. The sale of existing homes is a relevant housing activity statistic for prepayment analysis as it has a direct relationship with prepayments [13, 1]. A sale of an existing home usually triggers a prepayment, unless the mortgage is transferred or the home has no mortgage.

The discrete state can only change from not moved ($m = 0$) to moved ($m = 1$). We assum that a mortgagor does not move more than one time during the contract.

4-2-2 Housing Incentive

Given data on the amount of existing houses sold per month nationally, we normalize the data by rescaling the range to the interval $[0, 1]$. We assume that this normalized data represents the probability that a mortgagor moves in a particular month.

We add a Gaussian noise to the data such that the probability a mortgagor moves varies per mortgagor. The probability a mortgagor moves in a certain month is translated to a binary signal of 0's and 1's, called h.

We calculate the average amount of existing houses sold over an interval equal to the contract maturity, called p_{average}. When the probability of moving in a certain month exceeds p_{average} a 1 is produced and a 0 otherwise. Here, a $h = 1$ means there is an inventive for housing, and a $h = 0$ means there is no incentive for housing.

The possible discrete state transitions of m are shown in Table 4-1. The discrete state transition $m = 0 \to m = 0$ indicates that a mortgagor has not moved and remains in this state, and the transition $m = 1 \rightarrow m = 1$ indicates that a mortgagor is already moved and remains in this state.

Table 4-1: The discrete state jumps due to an incentive for housing. We assume that a mortgagor moves only one time during the mortgage contract.

Discrete State Jumps	**Housing Incentive**
$m = 0 \rightarrow m = 0$	$h = 0$
$m = 0 \rightarrow m = 1$	$h = 1$
$m = 1 \rightarrow m = 1$	$h = 0$
$m = 1 \rightarrow m = 1$	$h = 1$

4-2-3 Mortgage Porting Incentive

If a mortgagor sells his existing house and moves to a new one, he has the possibility to carry the existing structure and interest rate of his current mortgage contract to the new house (i.e., porting the mortgage). The mortgage porting incentive in the model refers to a mortgagor that moves to a new home, but transfers the terms and conditions of his existing mortgage contract to this new house.

The mortgage porting incentive depends on the housing incentive, the discrete state, and the difference between the monthly payments under the current mortgage contract and a new mortgage contract at the mortgage market rates

A mortgagor has an incentive for mortgage porting if there is an incentive for housing ($h = 1$) while he has not yet moved ($m = 0$), and if

$$P_{\text{contract}} \leq P_{\text{relocation}}(t). \tag{4-2}$$

Here, P_{contract} is calculated with Equation (2-15) and $P_{\text{relocation}}$ is equal to the monthly payments of a new mortgage contract at the mortgage market rates.

$$P_{\text{relocation}}(t) = q_0(t)\frac{r_m(t)(1 + r_m(t))^{T_r}}{(1 + r_m(t))^{T_r} - 1}. \tag{4-3}$$

Here, r_m are the mortgage market rates, q_0 is the principal balance, and $T_r = T - t$ the remaining term of the mortgage.

The mortgage porting mode will have additional constraints which are $m = 0$, $h = 1$, and the condition in Equation (4-2). If these conditions hold the mortgagor moves and transfers his existing contract to the new house, because the monthly payments of his current mortgage contract are lower than the monthly payments of a new mortgage contract.

4-3 Modeling Full Prepayments with Risk Drivers

A mortgagor his decision for a full loan prepayment is influenced by many factors as described in Section 2-4-3.

Clearly, one of the most important risk driver is the interest rates on new mortgages in the market. These interest rates both influence a mortgagor's decision for taking out a new loan after selling a house, or to refinance a loan.

However, refinancing involves extra costs, known as the cost of refinancing. These cost should be taking into account as these influence a borrower's decision to refinance or not.

The conditions (refinance conditions) that lead to a switch to the refinance mode depends on the mortgage rates and the cost of refinancing. The conditions (relocation conditions) that lead to a switch to the relocation mode depends on the mortgage rates and the housing incentive. These refinance and relocation conditions need to be added as constraints to the refinance mode and relocation mode, respectively. If these conditions do not hold, a mortgagor will remain in the mortgage payment mode according to the payment conditions.

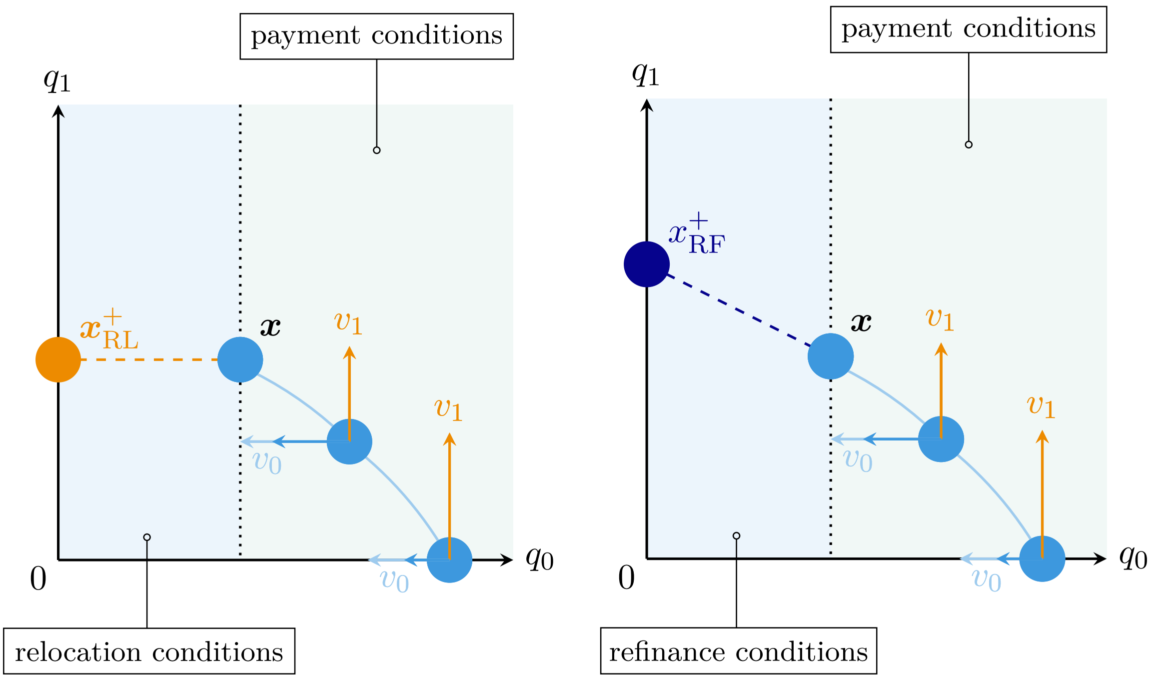

Figure 4-2: Each full prepayment type will have different conditions triggering the incentive. The conditions determine when there will be a switching between the continuous and discontinuous dynamics. The left figure shows the jump in state of a full loan prepayment due to relocation, and the right figure the jump in state of a full loan prepayment due to refinancing. The relocation conditions depend on the market mortgage rates and housing activity. The refinance conditions depend on market mortgage rates and refinancing costs. The payment conditions are all the conditions for which a mortgagor does not want to fully prepay his mortgage.

4-3-1 Refinance Incentive

Refinancing is taking a new mortgage loan on the same property. The inventive to refinance is mainly driven by the interest rate differential between a borrower's current contract and new mortgage contracts. An approach is to compare the monthly payments under the current and new mortgages, factoring in the costs of refinancing [13].

We assume that a mortgagor compares the monthly payments of his current mortgage contract with the monthly payments of a new mortgage contract for the remaining term of the mortgage and the remaining outstanding principal balance. A mortgagor refinances if the following condition holds:

$$P_{\text{contract}} \geq P_{\text{refinance}}(t) \tag{4-4}$$

The mortgagor refinances because the monthly payments of his current mortgage contract are higher than the monthly payments of a new mortgage contract. Here, P_{contract} is calculated as in Equation (2-15).

Here $P_{\text{refinance}}$ is the monthly payment of a new mortgage contract with factoring in the cost of refinancing $\varepsilon(t)$ is calculated as:

$$P_{\text{refinance}}(t) = q_0(t)\frac{r_m(t)(1+r_m(t))^{T_r}}{(1+r_m(t))^{T_r}-1} + \varepsilon(t) \tag{4-5}$$

Here, $r_m(t)$ are prevailing mortgage rates in the market, expressed as monthly decimals. The outstanding principal balance, or remaining debt of the mortgagor is $q_0(t)$, and the remaining term of the loan is T_r.

Refinancing cost are included with a prepayment penalty $\varepsilon(t)$:

$$\varepsilon(t) = \max(r_c(t) - r_m(t), 0)\max(q_0(t) - \rho A_0, 0)T_r \tag{4-6}$$

A prepayment penalty only applies if the contract rate is higher than the mortgage market rates, and if the outstanding principal balance is higher than the penalty-free part.

Extra refinancing costs could be taken into account such as administration costs. The cost of refinancing can be adjusted by simply adding the additional costs to the calculation of the prepayment penalty.

4-3-2 Relocation Incentive

If a mortgagor sells his existing house and moves to a new one, he has the possibility to fully prepay his current mortgage and take out a new mortgage on his new property. We assume that a mortgagor moves and takes out a new mortgage contract (penalty-free), if the following condition holds:

$$P_{\text{contract}} \geq P_{\text{relocation}}(t), \tag{4-7}$$

and if there is an incentive for housing ($h = 1$) while he has not yet moved ($m = 0$). Here is $P_{\text{relocation}}$ equal to the monthly payments of a new mortgage contract without factoring in extra costs:

$$P_{\text{relocation}}(t) = q_0(t)\frac{r_m(t)(1 + r_m(t))^{T_r}}{(1 + r_m(t))^{T_r} - 1}. \tag{4-8}$$

4-4 Modeling a New Mortgage Contract after Full Loan Prepayments

We assume that a new mortgage contract is agreed upon after a full loan prepayment by the same bank. A new contract agreement results in a new determined contractual interest rate, a new initial amount of money borrowed, and a new level of fixed monthly payments. We assume that a mortgagor does not want to shorten or extend the maturity of the contract, and does not want to borrow more money than he has already in debt.

In Figure 4-3, we show an illustration of how we model a new mortgage contract after a full loan prepayment. The new contract agreement after a full loan prepayment results in a reset of the continuous states. The principal balance resets to the still outstanding debt before the full loan prepayment, and the level of accumulated interest resets to zero. Also, the principal repayment and interest payment reset according to the newly determined contract rate and level of money borrowed. This means that x_0 reset with the updated A_0 and r_c.

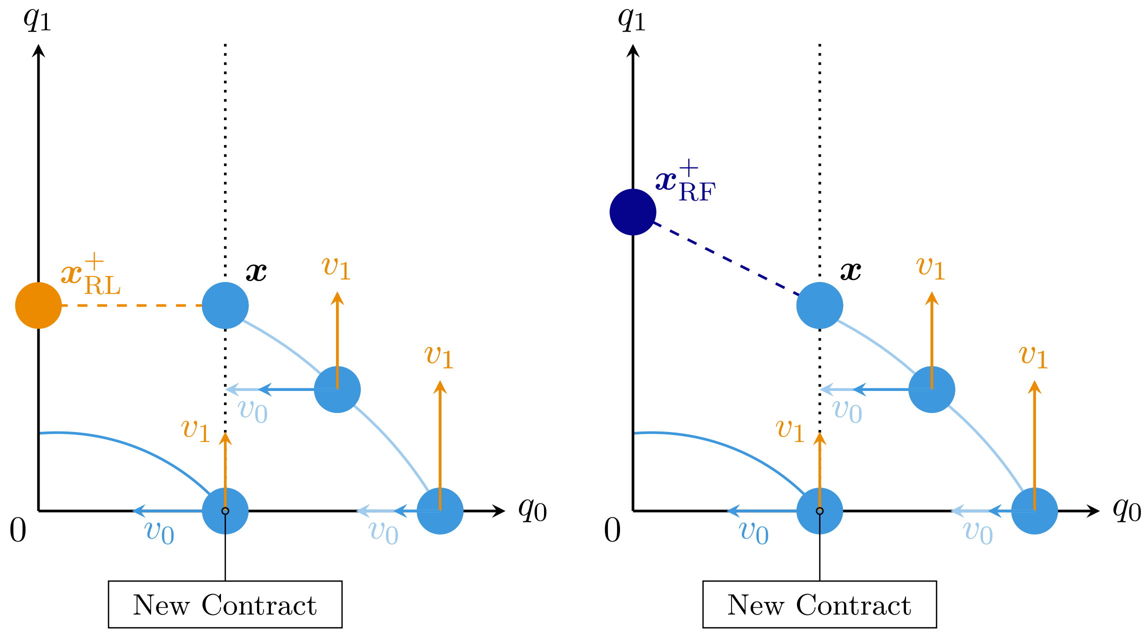

Figure 4-3: After each jump in state of either $\boldsymbol{x} \to \boldsymbol{x}^+_{\text{RL}}$ or $\boldsymbol{x} \to \boldsymbol{x}^+_{\text{RF}}$, the state resets due to the new contract agreement. The new contract has an initial level of money equal to the still outstanding debt, an updated contract rate, and thus a new schedule of principal repayments and interest payments. This new contract will also update the switching conditions.

The new contract agreement will have a contract rate equal to the prevailing mortgage market rates, a level of money borrowed equal to the still outstanding debt, and a fixed monthly payment calculated with Equation (2-15) for a new A_0 and r_c. This new fixed monthly payment also updates the conditions in Equation (4-4) and Equation (4-7).

We assume that a new mortgage is contracted at the same mortgagee. In reality, it is allowed to take out a new loan by a different mortgagee after a full loan prepayment. This model assumption could be changed by assuming that a certain amount of mortgagors contract at a different mortgagee after a full loan prepayment. These mortgagors 'leave' the system which is equal to a stop of the optimization problem for these mortgagors.

4-5 An Adaptable Economic-Engineering Model

We derived a model in this chapter that can be adjusted as desired. This is because the structure of the model is interpretable, and so assumption in the model can be changed.

To allow for moving multiple times during the contract term, the discrete state can be used as a counting variable. Each count implies that a mortgagor has moved, and the probability of moving again reduces. This would extend the model in Figure 4-1, by adding extra mortgage payments blocks (rectangular blocks). Each block contains the discrete state with a number representing the times a mortgagor has moved. If $m = 0$ the borrower has not moved, $m = 1$ has moved one time, $m = 2$ has moved two times, $m = 3$ has moved three times, etc... To reduce the probability of moving, the binary signal h should be recalculated each time a borrower has moved, and the times the binary signal produces a 1 should be reduced.

Additional mortgage payments are added with adding more circular blocks. Mortgage payments such as delinquent and default payments are straightforward to model with economic engineering. For delinquent and default payments the continuous state will not jump, but the balances will remain constant as no payments are received. The principal repayment and interest payment will be zero, meaning that the balances will remain the same. Delinquent and default payments depend on unemployment rates [11, 12]. Unemployment rates can be used as switching condition.

We assume that the conditions to switch to a full loan prepayment involves comparing the monthly payments of the current contract with the monthly payments of a new mortgage contract at the market mortgage rates. However, this full prepayment strategy is not always that favorable for a mortgagor. Sometimes the contract rate resets to a higher interest rate, meaning more money will go to interest payments and eventually a mortgagor will pay more interest to the bank. To design a more favorable full prepayment strategy, the conditions for full loan prepayments can be changed. For example, the contract rate could be compared with the mortgage market rates, or the interest payment of the current contract with the interest payment of a new contract. This will result in a full prepayment strategy of a mortgagor that only fully prepays if the new mortgage contract has a lower interest rate.

Also, we assume that a mortgagor immediately prepays if he observes that the monthly payment of his current contract exceeds the monthly payment of a new loan contract. This represents behavior of a mortgagor that fully prepays at optimal moments. To account for an inefficient exercise of this option, a probability could be added to these conditions resulting in a chance constraint. Moreover, a budget constraint can be used to constrain the level of money a borrower can spend on partial prepayments, because a borrower does not always have enough budget to prepay due to his income, or other expenses. Budget constraints are commonly used in economics to constrain the level of money a consumer can spend on goods [23].

4-6 Parameter Estimation with System Identification

System identification is a methodology for estimating the parameters in dynamic systems from experimental data. Typical gray-box system identification requires measurements of the system's input/output signals to estimate the values of adjustable parameters in the given model structure [43].

Historical data can be used for parameter identification as the parameters in the model are relatively fixed and do not specifically depend on the movement of interest rates. The parameters that are unknown do have an actual economic interpretation, and this helps with choosing appropriate values. The parameters are related to the characteristics of a mortgagor. These characteristics are a mortgagor's creditworthiness and risk aversion. A mortgagor's creditworthiness is already 'measured' by banks with a FICO score which is a type of credit score. A mortgagor's risk aversion is harder to determine, but could for example be derived from the responsiveness in prepayments of a mortgagor to interest rate changes. A more risk-averse mortgagor will respond later, and with lower levels of prepayments, than a less risk-averse mortgagor. We demonstrate this behavior of different risk-averse mortgagors with simulations in Chapter 5.

Accurate parameter estimation is not within the scope of this thesis, and so there will not be an extensive search to find proper values. We use different values for the parameters in Chapter 5 to forecast prepayments for different risk-averse and creditworthy mortgagors. The model is in this chapter used as a proof of concept.

Chapter 5

Cost-Minimizing Mortgagor Behavior with Economic MPC

In this chapter, we design an Economic Model Predictive Controller (EMPC) to simulate the behavior of a mortgagor. By simulating mortgagor behavior, we forecast prepayments with the model derived in the previous chapter for different interest rate scenarios and for different types of mortgagors. These simulations provide insights in the difference in prepayments supplied for mortgagors with various degrees of risk aversion, creditworthiness, and rationality. Based on these simulations we describe the causal and dynamic relation between interest rate changes and prepayments.

First, we argue in Section 5-1 how a mortgagor can minimize his mortgage costs by using funds for prepayments or investments, and in Section 5-2 we design an EMPC to simulate this cost minimizing behavior. Subsequently, in Section 5-3 we simulate mortgagor behavior in a decreasing interest rate regime to explain the dynamic and causal relation between interest rates changes and prepayments. In Section 5-4 we forecast prepayments for different types of mortgagors for an increasing interest rate scenario. We explain the level of prepayments supplied for each mortgagor based on these simulations.

5-1 Cost Minimization through Prepaying or Investing

Mortgage prepayments result in an early mortgage termination and a reduction in the total level of interest paid. Despite the interest costs that can be saved with prepayments, a mortgagor should consider in decisions regarding mortgage prepayments factors as alternative investments

Alternative investments is an opportunity to invest in alternative assets such as stocks, bonds, or certificates of deposit. These investments represent a wiser economic choice if returns on these investments exceed the interest rate on the mortgage. Moreover, funds used to prepay could be used for other purposes such as paying off other consumer credit loans. If a mortgagor pays a higher interest rate on these loans than the mortgage loan, prepayment of these would bring greater returns than prepayment of the mortgage.

Also, a mortgagor might choose to place money in an investment which pays a lower interest rate than the mortgage (e.g, savings account). Since savings deposits are available on demand, it guarantees a mortgagor quick access to money, and thus liquidity or cash availability. Any extra money that is spent on reducing mortgage debt faster, is money that is not used for other financial goals. Paying off a mortgage earlier can be at the expense of retirement savings, emergency funds or other higher return opportunities.

In economics, more often the term opportunity costs of funds is used, i.e., the value or benefit given up by engaging in one activity over another activity. Interest rates can be used as a measure for the opportunity cost of funds, because it measures the amount of interest (expected return) is sacrificed by not holding an alternative asset [44, 23]. Every stream of payments should be compared to the best alternative. The interest rate on the best forgone option will be compared with the interest rate on the chosen option.

A mortgagor will make a trade-off between using funds for paying off his mortgage early (and reducing his interest costs) and using funds for investments. These investments can have a higher return, but also a lower return to benefit from having liquidity available on demand.

5-2 Economic MPC Design for Minimizing Costs

In this chapter, we design an EMPC that minimizes costs of a mortgagor. By having a mortgagor minimizing his costs, we simulate mortgagor behavior. A diagram is shown in Figure 5-1. The model and the system are the same and represent the model derived in the previous chapter. The disturbance are the risk-free rates, mortgage rates, and housing activity.

5-2-1 Economic Model Predictive Control

In the traditional Model Predictive Control (MPC) used for optimization and control a two layer design is employed. A Real-Time Optimization (RTO) computes the economically optimal steady-states that are subsequently sent down to a tracking MPC layer that uses this optimal economic steady-state as a reference input. The tracking MPC layer computes control actions that are applied to the closed loop system to force the state to the optimal steady-state [45]. The objective function has usually the objective to achieve fast tracking to set point changes, and is usually unrelated to the economic costs of operating a plant or process; the steady state may be economically optimal, but the path to this steady-state is most probably not economically optimal.

An EMPC merges economic optimization and control and thus employs a one-layer approach to optimization and control [45]. Both the optimal steady-state and optimal transient (transient is the evolution of the system to the steady-state) are found in the same optimization step. The objective function is not the usual trade-off between minimizing the control input and driving the state to the reference state, but has a direct economic interpretation. Maintaining a division between economic optimization and control is still more suitable in some

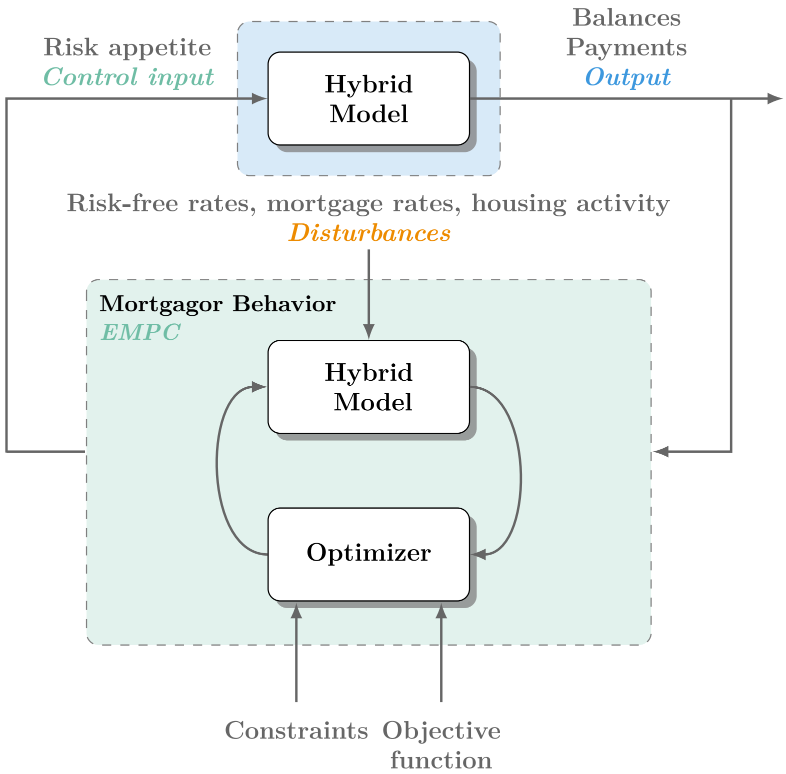

Figure 5-1: The control diagram of the EMPC. The objective function is an economic objective. The behavior of a mortgagor minimizes this economic objective over a prediction horizon. A mortgagor determines his risk appetite for the prediction horizon while constraint to the dynamics of the hybrid model and subject to disturbances from risk-free rates, mortgage rates, and activity in the housing market.

applications, especially where there is an explicit timescale separation between the system dynamics and the update frequency or timescale of evolution of economic factors. However, for problems for which dynamic economic performance is crucial, the hierarchical separation of economic analysis and control is either inefficient or inappropriate [46].

The dynamic economic performance is crucial for the economic systems modeled with economic engineering, therefore the two-layer design of RTO and MPC will not be sufficient. This is because the mortgage balances and payments are dynamic, and therefore the balances and principal payments are a state of the system as well as present in the objective function. The path of the transient has an influence on the mortgage balances and payments. We use an EMPC to compute both the economically optimal transient and steady state at the same optimization step.

5-2-2 Objective Function

An objective function that minimizes costs for a mortgagor needs to keep track of the accrued interest during the mortgage term, the opportunity of investing in alternative assets, and the liquidity benefit that arises from having money on demand. Additionally, we include a penalty on the control input as a measure of how rational the behavior of the mortgagor is. The objective function will have the following form:

$$J = \underbrace{q_{1,\text{acc}}[T_h]}_{\text{Accumulated interest}} + \sum_{k=1}^{T_h} [\underbrace{(r[k] - r_c[k])v_0[k]}_{\text{Opportunity cost}} - \underbrace{\frac{1}{2}k_0 q_0[k]^2}_{\text{Liquidity benefit}} + \underbrace{\frac{1}{2}R u_0[k]^2]}_{\text{Rationality}} \tag{5-1}$$

The principal repayments are continuously changing based on a mortgagor's risk appetite and disturbances from interest rates. The dynamics of mortgage payments and prepayments included in the economic engineering model therefore immediately influence the objective function. This means that an economically optimal steady-state within the horizon of the MPC can not be computed if these payments are continuously changing as a result of control inputs and disturbances. This shows the importance of an EMPC compared to the two-layer design. The system is controlled to move along the economically optimal path over the time horizon of the EMPC.

Accumulated interest

The costs of a mortgage are the interest payments each month. In the current model, the state q_1 representing the balance of accumulated interest. This state resets to zero each new contract agreement. However, the costs incurred on interest payments and prepayment penalties should accumulate over the whole simulation time. We add an extra state $q_{1,\text{acc}}$ to the model that accumulates the interest and penalty costs over the simulation time. This state will not reset to zero after a full loan prepayment.

Opportunity costs

We calculate the opportunity cost by taking the difference of the return on risk-free investments r and the interest rate of the mortgage contract r_c multiplied with the stream of principal repayments v_0. If the return on risk-free investments is higher than the contract rate, the difference will be positive, and therefore investing in risk-free assets is a wiser economic choice. Contrary, if the return on risk-free investments is lower than the contract rate, the difference will be negative, and prepayments are a wiser economic choice. This relationship is well-known for mortgage prepayments, where prepayments increase in low interest rate environments and decrease in high interest rate environments [13]. This part of the objective function is linear.

Liquidity benefit

Liquidity benefit refers to the benefit of having cash available. In economic engineering, we say that potential energy represents the benefit of holding a good. Here, the good is money, and therefore the potential energy represents the benefit of having liquidity available. A mortgagor wants to maximize this benefit, therefore the minimization of this term has a minus sign in front of it. This part of the objective function is quadratic and shaped concave, because it is the negative of a convex function.

Rationality

The option to prepay can be viewed as an exercise of a call option in which the value of the outstanding loan repayments at a market interest rate is compared with the value of the outstanding loan repayments at the original contract interest rate [47]. If interest rates go down then the option is in the money (positive payoff), and if interest rates go up than the option is out of the money. Rationally, the call should only be exercise if the above-mentioned difference is positive, otherwise the option is worth more if you do not exercise it. Meaning that it is more convenient to put money for example in a savings account rather than prepaying the mortgage. However, people do not always act economically rationally though, and the term irrational will be used to identify mortgagor behavior that differs from the economically rationally behavior described above.

The penalty R on the control input, we call it the rationality parameter of a mortgagor, determines how easily a mortgagor can add a risk appetite on top of his appetite to reduce debt. This additional risk appetite is a control input in the form of an economic want that will change a mortgagor's risk factor.

If the penalty on the control input is close to zero, the mortgagor is able to add a risk appetite to any quantity he sees fit to reject the disturbances from risk-free rate changes to supply the scheduled payments. This means he will pay the scheduled principal and interest payments of the mortgage contract without deviating from it if interest rates are going up or down. This is in some way irrational behavior, because the mortgagor is not exercising the partial prepayment option. However, the mortgagor is still prone to the full prepayment option and can not reject this prepayment exercise even if the control penalty is zero.

The risk aversion parameter and rationality parameter both determine to which extent a mortgagor is subject to external market forces that result in prepayments. The difference is that the penalty on the control input determines to which extent a mortgagor can counteract the economic forces from interest rate changes, while the mass determines to which extent a mortgagor supplies addition repayments due to economic forces. Despite the chosen mass, the penalty on the control input can be used to simulate mortgagors that are in some sense irrational, e.g, mortgagors that are not partially prepaying and only fully prepaying.

This part of the objective function that penalizes the control input is quadratic and shaped convex.

5-2-3 Constraints

The minimization problem is given by the objective function J minimized over time subject to constraints. The states can not just arbitrarily be changed to determine an optimal control sequence, since the system is constrained to its own dynamics (the state equations of the continuous and discontinuous dynamics), and additional constraints. The control sequence that minimizes the objective function changes the states of the system after the control input for the first timestep is implemented, and therefore changes the objective function. The optimization of one iteration with the economic objective function and including all the constraints is shown in Appendix B. The hybrid economic-engineering model has eight possible modes in which it can operate.

The control variable is not explicitly in the entire objective function, and so the state equations need to be included in the objective function to be able to solve the optimization problem. A costate variable $\mu[k]-$ similar to a Lagrange multiplier$-$ is used, according to Pontryagin's maximum principle [48]. The state equations become soft constraints on the objective function, and the minimization problem is solved both for $u[k]$ and $\mu[k]$. This method is similar to how Lagrangian multiplies $\lambda[k]$ are used for optimization problems with non-dynamical constraints [49]. The Hamiltonian or resulting objective function H that is formed by using a time-varying auxiliary multiplier $\mu[k]$ is decomposed in the objective function J and the state equations f in the following form:

$$H[x, u, k, \mu] = J[x, u, k] + \mu[k] f[x, u, k] \tag{5-2}$$

Here, J is the original objective function and f are the state update equations. The costate variable can then be interpreted as the marginal costs for disobeying the state update equations. Hence, these equations are soft constraints in this optimization problem.

Binary variables

The hybrid model has eight modes, and it is only possible to be in one mode at a time. We introduce binary variables in the constraints of the model for representing the 'or' operator between the different modes. This method indicates more clearly where binary variables are introduced. Here d is a binary variable that can only take two values, for example 0 or 1, true or false, yes or no. The binary variable can only select one mode that is 'true', due to the extra constraint that the sum of the binary variables should be equal to 1. In other words, if a mode is selected the binary variable will take a value of 1 for that specific mode and the binary variables for the other modes will take a value of 0. The 'or' operator is essentially equivalent by adding the following constraints to the model [40].

```
implies(d[k](1), Mortgage Payments Mode
implies(d[k](2), Mortgage Porting Mode
implies(d[k](3), Mortgage Payments Mode (and moved=1)
implies(d[k](4), Mortgage Payments Mode (and moved=1)
implies(d[k](5), Refinance Mode
implies(d[k](6), Refinance Mode
implies(d[k](7), Refinance Mode
implies(d[k](8), Relocation Mode
sum(d[k]) == 1
```

Model mode

We add a variable mdl to the constraints to keep track of the mode of the model. Each mode has its own number assigned, i.e, for the mortgage payments mode $mdl[k] = 1$, mortgage porting mode $mdl[k] = 2$, mortgage payments mode while moved $mdl[k] = 3$, etc. . .

Accumulate interest without resets

We include additional constraints to accumulate interest without resetting the state to zero after a new contract agreement. During mortgage payments mode, the state accumulates interest with the general state update equation:

$$q_{1,\text{acc}}[k+1] = q_{1,\text{acc}}[k] + T_s v_1[k] \tag{5-3}$$

During a refinancing, the prepayment penalty will add to the accumulated interest:

$$q_{1,\text{acc}}[k+1] = q_{1,\text{acc}}[k] + \epsilon[k] \tag{5-4}$$

And during a relocation

$$q_{1,\text{acc}}[k+1] = q_{1,\text{acc}}[k] \tag{5-5}$$

Terminal constraints

A terminal constraint can be used to improve performance, and also acts in a system stabilizing manner as a terminal constraint forces the state to the best equilibrium point at the end of the horizon [50]. For an economic system, this could be considered as the economically optimal steady-state at the end of the horizon.

In case of a mortgagor it forces a mortgagor to push their balance levels to a certain steady state deemed economically optimal at the end of their control horizon. This optimal state indicates a preference for a specific long-term behavior. It ensures that mortgagors do not engage in unexpected behavior to generate short-term results. However, a mortgagor is already quite restricted due to the constraints imposed by mortgage contracts, and we are particular interested in unexpected behavior of mortgagors. There is not one optimal balance level (principal balance and balance of accumulated interest), and therefore we can relax this constraint by imposing a region constraint on the terminal state [50].

We add a terminal region to the principal balance that should always be positive and less than the initial amount of money borrowed. The balance of accumulated interest should always be positive, but the upper limit will not be restricted, because there is no restriction in the costs accumulated on a mortgage contract.

5-2-4 Stability

An MPC is used to find the control sequence that minimizes the objective function J subject to system equations and additional constraints as described in the previous section. This results in a MIQP problem, and these are classified as NP-hard [49, 51, 52]. This means that there is no algorithm that answers the question of stability in polynomial time. Moreover, the stability of the submodels do not say something about the stability of the whole switched system [35]. It is not possible to study stability properties of the subsystems only, the switching structure has to be taken into account as well.

For an EMPC, the system is stable if the system converges to the optimal trajectory. However, the stabilization conditions of the tracking objective function used in a standard MPC design do not apply [46], and the stabilization conditions do not apply when hybrid models are used for prediction [37]. Determining if the system always converges to its optimal trajectory, by using Lyapunov-based stability proofs and LaSalle's invariance principles [34, 35, 37, 53], is well beyond the scope of this research. A sufficient condition for optimal operation is that a system is dissipative [46], however applying this dissipative condition to systems for which the operating regime fails to be an equilibrium or steady-state [46], is more challenging. Economic engineering systems do not have an optimal steady-state within the prediction horizon. The principal repayments are essential for determining optimal operation, and these repayments are continuously changing due to the model dynamics.

In this thesis we assume that the EMPC with the hybrid model is stable with the chosen parameters. The terminal region does guarantee that at least the balance levels return to their steady-state region after the end of the prediction horizon. Convergence behavior can be verified through trial-and-error with simulations. During the process of deriving and tuning this model with simulations, the terminal constraints, and the constraints on the level of payments and balances are important to have a stable model. The extra constraints on the payments as proposed at the end in Section 3-3 appear to be too restrictive to obtain solutions, and we do not use them for the eventual model simulation in this chapter.

Assuming that the system is stable and converges to some optimal path is admissible since the current application is restricted purely to simulations. EMPC is also used to include economic goals in real-life systems such as chemical processes and chemical plants [46, 54]. For operation of these systems it is of greater importance to proof stability for safety and economic reasons.

5-2-5 Tuning

The sampling time is a trade-off between simulation accuracy and computation time. A smaller value of the sampling time T_s does come at the expense of computational power. Economically, the sampling time represents the smallest reaction time of a mortgagor to adjust his risk strategy for prepayments. Currently, partial prepayments can be done online at any time of the month. But, full prepayments due to relocation or refinancing is a longer process, and often a meeting with an adviser from the bank is needed. Moreover, macroeconomic variables such as interest rate update on a monthly frequency, and thus it is not needed that a mortgagor has a smaller reaction time than one month. A mortgagor can adjust his risk strategy monthly, and so the sampling T_s is chosen to be 1 month.

The length of the prediction horizon T_h is economically a trade-off between uncertainty on predicting interest rates and a mortgagor's cost minimization performance. A longer prediction horizon result in a higher uncertainty on interest rates, but most likely a better performance in minimizing a mortgagor's costs. Interest rates update monthly, and looking too far in the future will suffer from a higher uncertainty on these interest rates. It is difficult to predict what interest rates will do in the upcoming half year, one year, or even five years. Especially, short-term interest rates fluctuate more than long-term interest rates.

Additionally, how far in the future a mortgagor speculates about interest rates is mortgagor specific. Some mortgagors will only look at the interest rates of the current month, others speculate for three months, or others even half a year. The further ahead a mortgagor speculates about interest rates, the uncertainty on the interest rate predictions will increase. If a longer prediction horizon is chosen, noise should be added to the interest rates with an increasing deviation for values far in the future. This is a method to represent a higher uncertainty on interest rates for predictions further away. For simulations in this thesis, a mortgagor will look at the level of current interest rates and the slope (upward or downward) to determine his payment strategy for the upcoming two months, resulting in a prediction horizon of 2 months.

The control horizon is set equal to the prediction horizon. For a typical MPC design used for tasks such as reference tracking the control horizon is chosen less than the prediction horizon [55, 56]. This can reduce the computational time. The system reaches a steady-state within the time frame of the control horizon, and the control input remains constant for the remainder of the prediction horizon. If we choose the control horizon shorter than the prediction horizon the performance of the optimization reduces, because this steady-state is never reached within the prediction horizon.

5-3 The Dynamic Effect of Interest Rate Changes on Prepayments

We forecast prepayments with the hybrid model and the EMPC. By simulating mortgagor behavior with an EMPC, we forecast prepayments with the hybrid model. We describe the effect of economic forces on the level of mortgage repayments supplied. These economic forces are forces from interest rate changes (*disturbances*), a mortgagor his cost minimizing behavior (*control inputs*), and a mortgagor his appetite to reduce debt (*spring force*). In Figure 5-2, we show the order of the causal and dynamic relation between interest rates changes and prepayments. This relation will be clarified in this section.

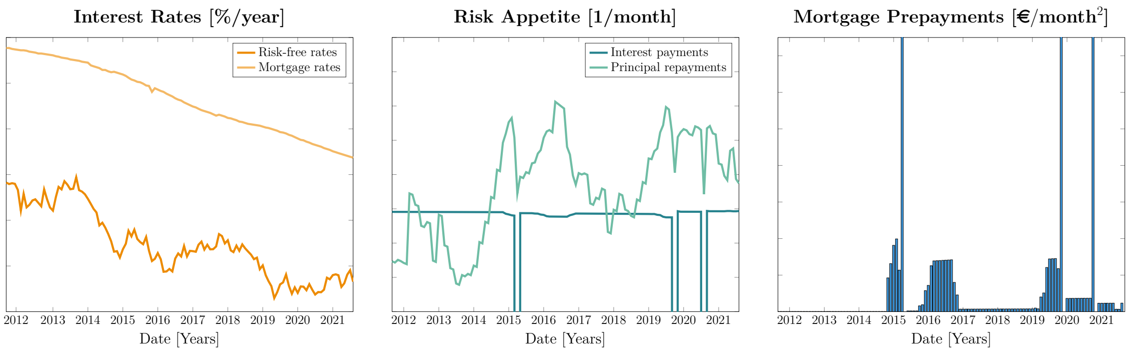

Figure 5-2: The dynamic effect of interest rate changes on prepayments. Interest rates are the *disturbances* of the model. We simulate the *control inputs* or behavior of a mortgagor which is his risk appetite with an EMPC. When interest rates substantially decrease, a mortgagor will not reject all the disturbances from interest rate changes with his risk appetite resulting in a change in his risk factor. A small change in his risk factor causes partial prepayments (*low accelerations*), while a substantial change in his risk factor causes full prepayments (*high accelerations*).

We simulate an annuity mortgage contract with a maximum partial prepayment of 10% per year of $A_0 = €500,000$. The maturity of the contract is 10 years, and the contract rate is 3%

per year. The sampling time is one month, and the prediction horizon is two months. We use the same model parameters as in the Chapter 3, and the penalty on the control input is $R = 0.3$. The initial conditions are:

$$q_0[0] = A_0 \qquad q_1[0] = 0 \qquad v_0[0] = r_c q_0[0] - P \qquad v_1[0] = r_c q_0[0] \tag{5-6}$$

Here, P is calculated as in Equation (2-15).

5-3-1 Generating Interest Rate Scenario

First, we generate interest rate scenarios of risk-free rates and mortgage rates. We use the historical interest rates of German government bonds as risk-free rates [2], and the interest rate on outstanding mortgages by Dutch banks to households in the same time frame [4]. Both interest rates are reconstructed with a delayed first-order hold method as shown in Figure 5-3. We scale the risk-free rates with the same factor as in Section 3-3-1 and these interest rates act as an external force, while we use the mortgage rates as switching conditions for full loan prepayments.

We use data on housing sales per month in the Netherlands as shown in the right figure of Figure 5-3. We transform this housing activity to a binary signal as explained in Section 4-2-2. We use this binary signal to determine when a mortgagor has an incentive for housing.

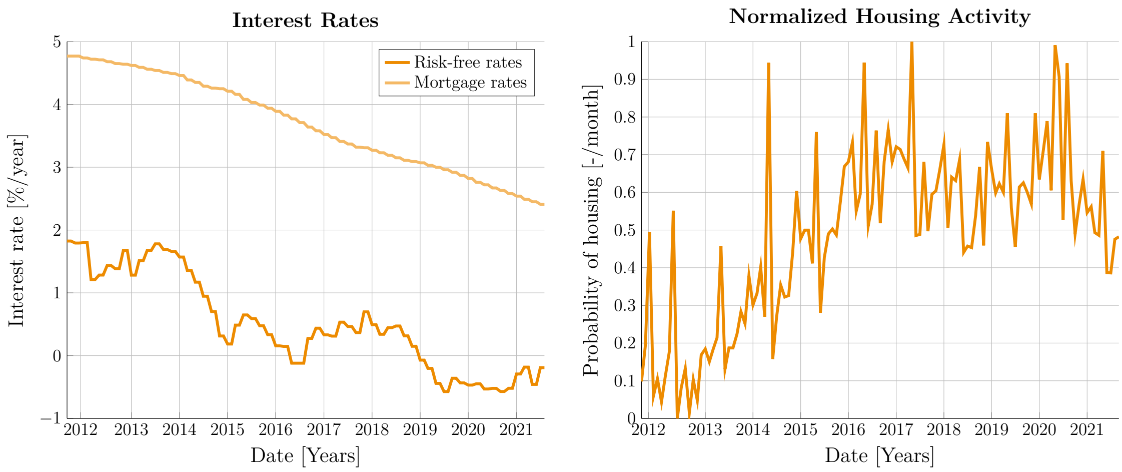

Figure 5-3: The risk-free rates are external force of the model, and the mortgage rates determine the switching conditions for full loan prepayments. We use the activity in the housing market to determine a mortgagor his incentive for housing. This activity in the housing market are the housing sales per month in the Netherlands. We retrieve data on housing sales in the Netherlands from [3].

We use the mortgage market rates to calculate the switching conditions for full loan prepayments. These conditions determine the switching behavior of the model, and are made visible in Figure 5-4. In this figure, P_{contract} is the fixed monthly payment of the current mortgage contract calculated with Equation (2-15), and updates after each full loan prepayment. The monthly payment of a newly originated contract without prepayment penalty is $P_{\text{relocation}}$, and these are calculated with the mortgage market rates as in Equation (4-8). The monthly payments of a new contract including a prepayment penalty is $P_{\text{refinance}}$ calculated with Equation (4-4).

These conditions determine when a mortgagor switches to a full loan prepayments. A mortgagor switches to a full loan prepayment if he can reduce his monthly payment by taking out a new mortgage contract. The mortgage will be fully prepaid at the beginning of 2015, at the end of 2019, and in the middle of 2020. These are all full loan prepayments due to refinancing. The mortgagor fully prepays because he observes that:

$$P_{\text{contract}} \geq P_{\text{refinance}}(t), \tag{5-7}$$

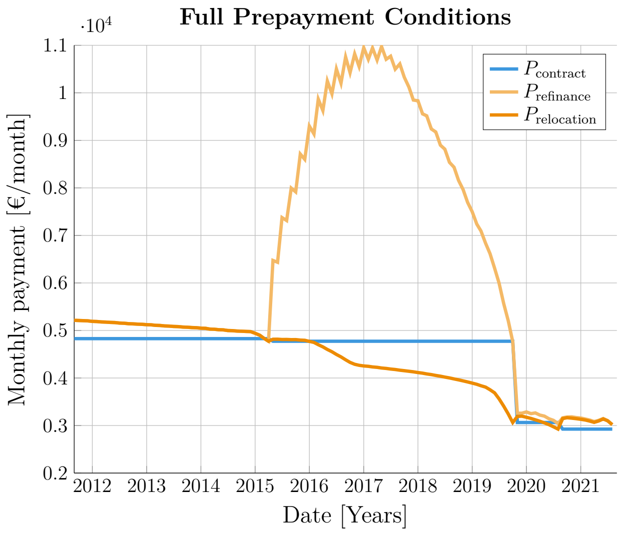

Figure 5-4: The switching conditions determine when a mortgagor fully prepays. A mortgagor compares his monthly payment of his current contract with the monthly payment of a new contract with and without a prepayment penalty. If a mortgagor can reduce his monthly payment by taking out a new mortgage contract he fully prepays, and the model switches to the full prepayment dynamics.

5-3-2 Simulating Mortgagor Behavior

We simulate the behavior of a mortgagor with the EMPC designed in this chapter. The controller determines a mortgagor his cost minimizing risk appetite. In mechanics, these are the control inputs or reaction forces that steer a system's state to a desired state.

A mortgagor his risk appetite for principal repayments (or control input u_0) clearly follows a similar pattern as the disturbances from the risk-free rates but mirrored as visible in the left figure of Figure 5-5. The mortgagor tries to reject the disturbances from the risk-free rate changes. However, due to the penalty on his risk appetite for principal repayments the disturbances are not fully rejected. When risk-free rates are falling fast (end of 2015, in 2016, and in 2019), the mortgagor can not fully reject these market forces which results in a change in his risk factor, and will supply partial prepayments as visible in Figure 5-6.

When a mortgagor supplies a full loan prepayment (begin 2015, end of 2019, and mid 2020), his appetite for principal repayments decreases significantly such that the principal repayment is equal to the principal balance outstanding. This behavior is observed in his appetite for principal repayments in Figure 5-5. When the mortgage is fully prepaid, this risk appetite decreases significantly which results in a substantial change in his risk factor to supply full prepayments. This full prepayment is exactly equal to the still outstanding mortgage debt as visible with the jumps in state in Figure 5-7. These shocks in risk appetite are the forces needed to steer the mortgage to a full loan prepayment.

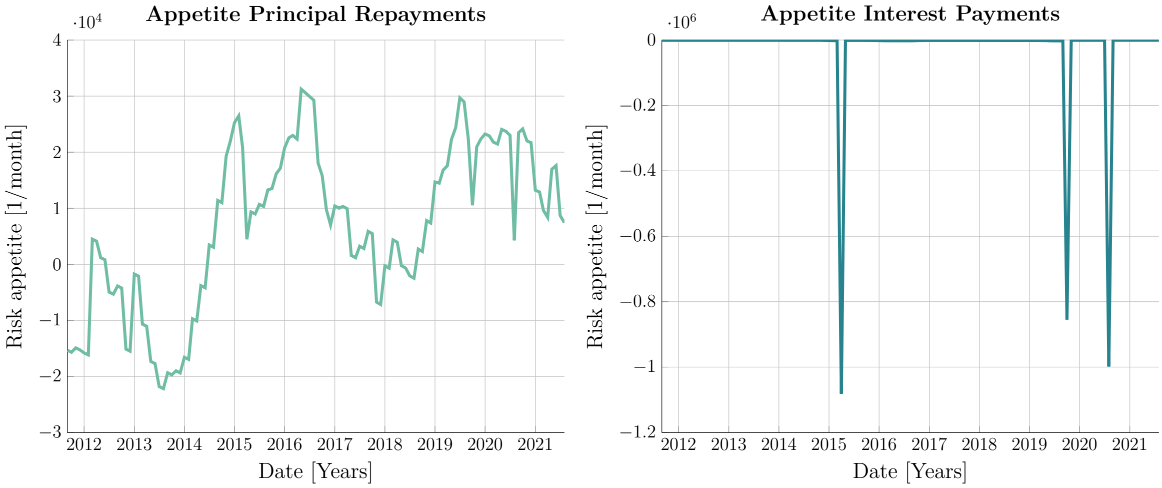

Figure 5-5: The control inputs of the model, analogous to a mortgagor's risk appetite. A mortgagor his risk appetite is influenced by disturbances from interest rate changes.

A mortgagor his risk appetite for interest payments (or control input u_1) steers the states of the mortgage such that the constraints of an annuity contract are always satisfied. When the mortgage is fully prepaid, this risk appetite decreases significantly to steer the interest payment to zero while the principal repayment is equal to the principal balance outstanding. After a full loan prepayment, this risk appetite steers the mortgage to the new interest payment determined with the new contract rate. These shocks in risk appetite are the forces needed to steer the mortgage to a full prepayment and to subsequently update the interest payment according to the new contract.

5-3-3 Deriving Mortgage Prepayments

We use the states of the model as model outputs. In Figure 5-6 the balance and payment levels of the mortgage are shown till all mortgage debt is paid off. We simulate the scheduled payments with the method described in Section 2-3, and we also update these after each new contract agreement.

With the simulation shown in Figure 5-6, we derive prepayments with Equation (2-25) by calculating for each month (for each time step t) the difference between the scheduled principal repayments v_0^* and the principal repayments supplied by the mortgagor v_0. This is the difference between the solid lines and dashed lines of the principal repayments in Figure 5-6:

$$\text{Prepayments}(t) = v_0(t) - v_0^*(t) \tag{5-8}$$

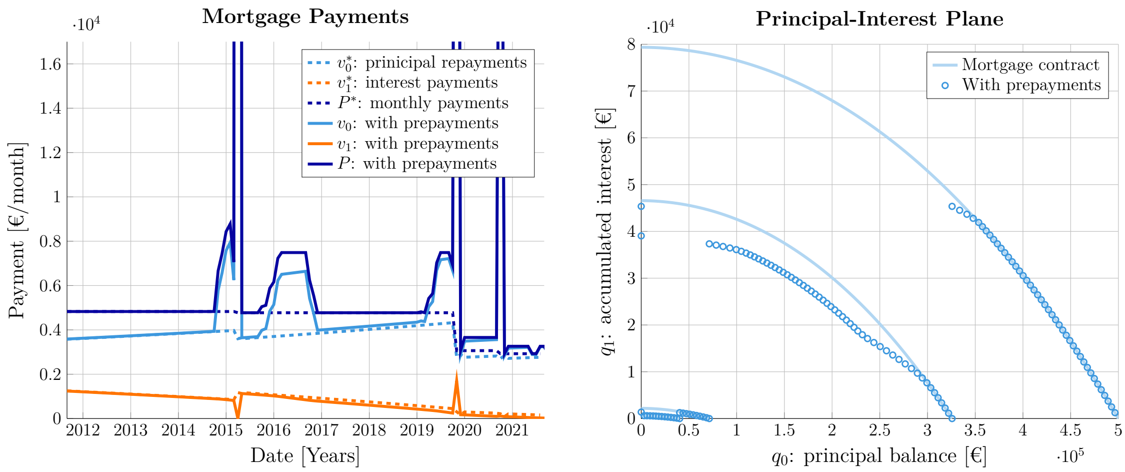

Figure 5-6: The mortgage payments and balances per month of the mortgage contracts. In the left figure, the dashed lines are the scheduled payments, and the solid lines the scheduled payments with prepayments. The peaks (outside y-axis limit) are the full prepayments equal to the principal balance outstanding before the jump of the state in the right figure. After each full prepayment a new contract is agreed upon with a new schedule. The repayment schedules with and without prepayments are shown in the right figure.

We visualize these prepayments in Figure 5-7.

The Conditional Prepayment Rate (CPR) is a mortgage prepayment rate that estimates the proportion of a loan's notional that be prepaid in each period. The CPR is an annual percentage, and can be derived from the Single Monthly Mortality (SMM). The SMM can be derived from the prepayments per month, and is defined as the percentage by which prepayments reduce the month-end principal balance compared to what it would have been with no prepayments:

$$\text{SMM}(t) = \frac{\text{Prepayments}(t)}{\text{Scheduled principal balance}(t)} \tag{5-9}$$

In this equation, the prepayments are the partial prepayments and full prepayments per month as shown in Figure 5-7. The scheduled principal balance is the level of principal balance outstanding at the end of each month, and thus with the scheduled principal repayment already subtracted. We derive this scheduled principal balance from the state q_0 without prepayments in Figure 5-6.

From the SMM, the CPR can be derived:

$$\text{CPR}(t) = 1 - (1 - \text{SMM}(t))^{12} \tag{5-10}$$

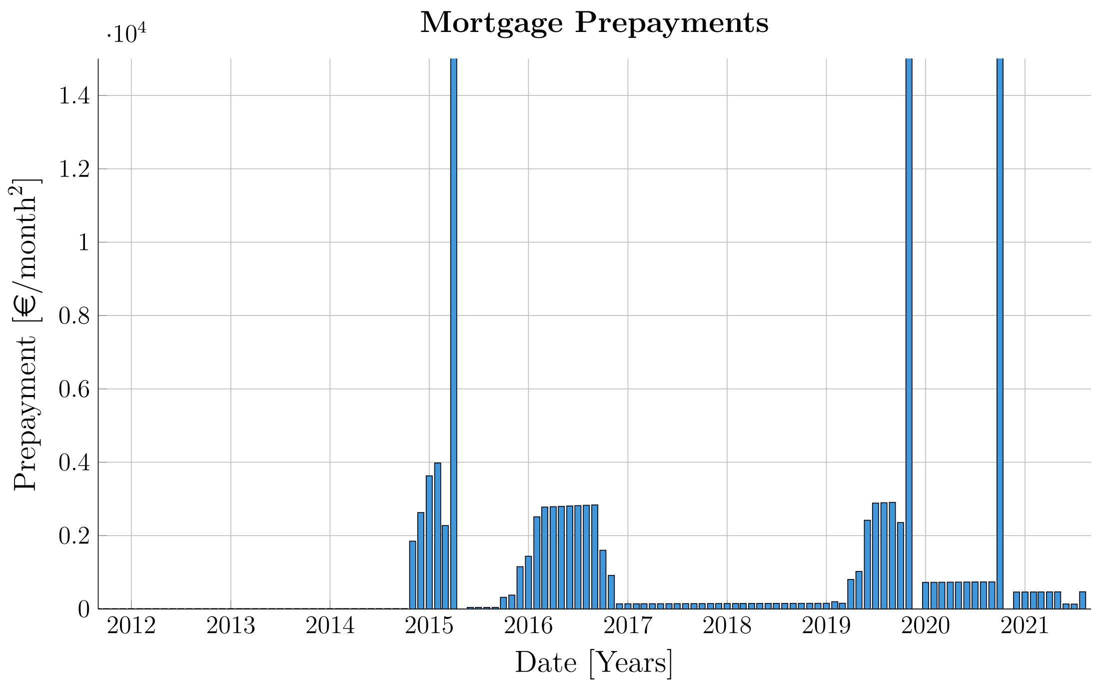

Figure 5-7: The partial and full prepayments per month supplied by the mortgagor. The full loan prepayments (the peaks outside y-axis limit) are equal to the principal balance before each jump in state in Figure 5-6.

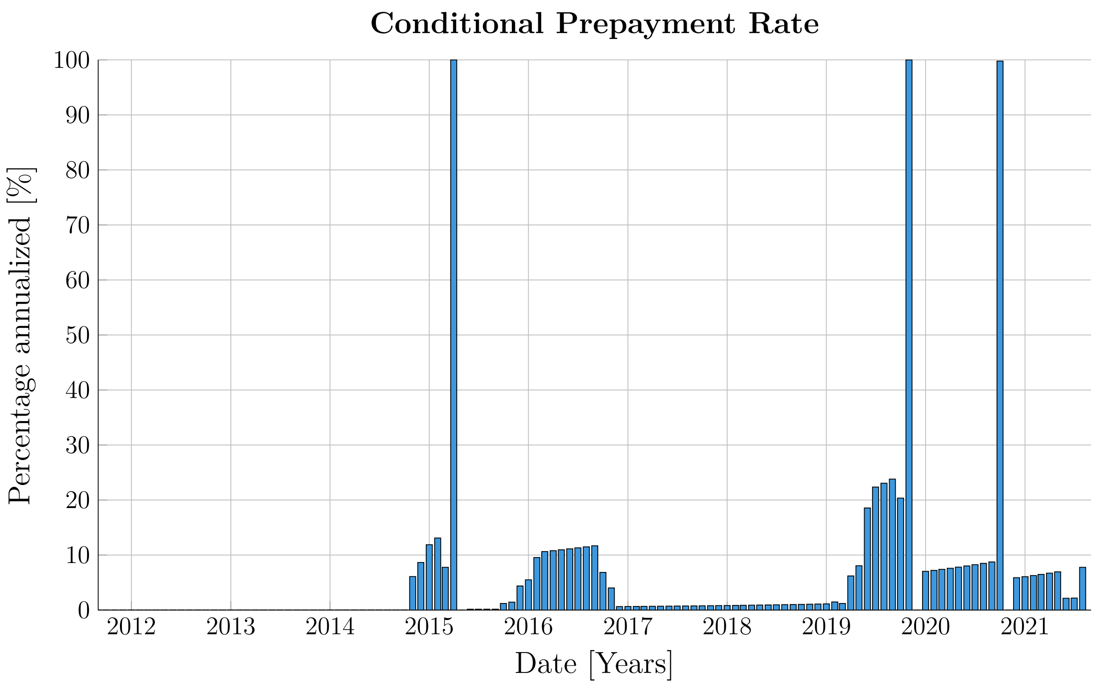

Figure 5-8: We derive the CPR from the states of the model. The CPR is an annual percentage that expresses the loan's principal that is paid off prematurely due to prepayments.

The CPR expresses the annual percentage of a loan's principal that is likely to be paid off prematurely. In Figure 5-8, the CPR is shown for the simulated mortgage contract.

We calculate the CPR of one mortgage, and therefore the peaks are 100%. This is because the loan's principal is paid off completely during a full loan prepayment, and therefore 100% of the loan's principal is paid off prematurely.

5-4 Forecasting Mortgage Prepayments in Increasing Interest Rates

In this section, we simulate and explain the response of different kinds of mortgagors to increasing interest rates with prepayments. We generate an increasing interest rate scenario by using the historical data of the risk-free rates and mortgage rates from the previous section. In this section, we use the interest rate scenario shown in Figure 5-9.

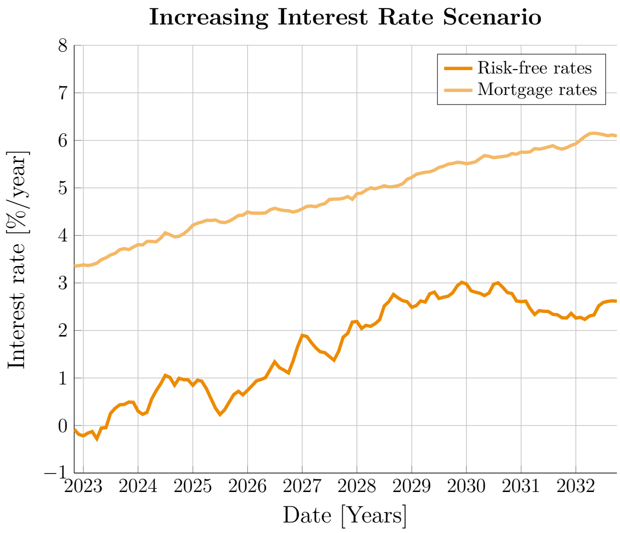

Figure 5-9: An interest rate scenario generated to forecast prepayments in increasing interest rates. These interest rates are disturbances of the model.

To generate this scenario, we flip the historical trends from both interest rates from Figure 5-3. By flipping the trends we create an increasing interest rate trend. We shift both trends upwards 1% representing a shift in interest rates, and add a cosine to these trends. The cosine is added to represent the cyclical behavior of short-term interest rates. Interest rates are closely related to business cycles, and therefore have a cyclical behavior [44, 33, 57]. Subsequently, a Brownian noise is added representing the uncertainty in predicting interest rates. Brownian noise is chosen over white noise, because the deviation in the interest rates should not be completely random. Brownian motion is commonly used for modeling short-term (zero-coupon) interest rates such as in the Hull-White model and Heath-Jarrow-Morton model [58].

We perform the simulations in Matlab using the YALMIP toolkit to program the MPC and we use the 'gurobi' solver that uses a branch-and-bound algorithm to solve Mixed-Integer Programming (MIP) problems.

We use for the simulations a prediction horizon of two months. A mortgagor determines his optimal risk appetite for two months ahead based on predictions of interest rates. After the optimal control sequence is calculated for the prediction horizon and implemented for the first time step, an update is received on interest rates for $k \in [1, T_h + 1]$. The deviation of the noise will not increase for values further away. If a longer prediction horizon is preferred, the deviation of the noise should be increased for interest rates far in the future, because the predictions of these values will be more uncertain.

We run simulations for different values of the parameters m_0, m_1, k_0, and R. This changes the dynamics of the model and the objective function. The optimizer object will be called with different values of these parameters following the approach described in the YALMIP documentation [59, 60]. We define the parameters as decision variables such that the optimization problem can be solved for different values of the parameters simultaneously.

The optimizer object will be called three times simultaneously representing three different types of mortgagors. As a result, the state update equations, objective function, switching conditions, and the new contract agreements will differ per mortgagor. Based on these simulations, we explain the causes of the differences in mortgage prepayments supplied with the derived causalities, and compare the results with literature.

5-4-1 Different Risk-Averse Mortgagors

We vary the value of the parameter representing a mortgagor's risk aversion (*the mass*) to model different risk-averse mortgagors. We use the following values listed in Table 5-1. Each mortgagor will be equally creditworthy with $k_0 = 0.05$. We simulate an annuity mortgage of €500,000 with maturity of 10 years, and a contract interest rate of 3% per year. The penalty on the control input is $R = 0.3$.

Table 5-1: A different value for the parameter representing a mortgagor's risk aversion is analogous to a different value for a mass in mechanics. A heavier mass corresponds to a more risk-averse mortgagor and a lighter mass to a less risk-averse mortgagor.

Risk-Averse Mortgagor	Risk Aversion [month/€]
Least	$m_0 = 10, m_1 = 10$
Medium	$m_0 = 100, m_1 = 100$
Most	$m_0 = 1000, m_1 = 1000$

We derive the CPR from the model states following the method described in Section 5-3-3. The CPR of each mortgage is shown in Figure 5-10.

Risk aversion is related to the tendency of people to prefer outcomes with low uncertainty to those outcomes with high uncertainty, even if the average payoff of the latter is equal or higher in value than the more certain outcome. In short, risk aversion is the tendency of people to avoid risk. The value of this model parameter can be changed to forecast prepayments in situations when a change in the risk aversion of mortgagors is expected.

Crises such as the financial crisis or COVID-19 crisis have influence on the risk aversion of mortgagors. During the COVID-19 crisis the savings of households increased sharply, but,

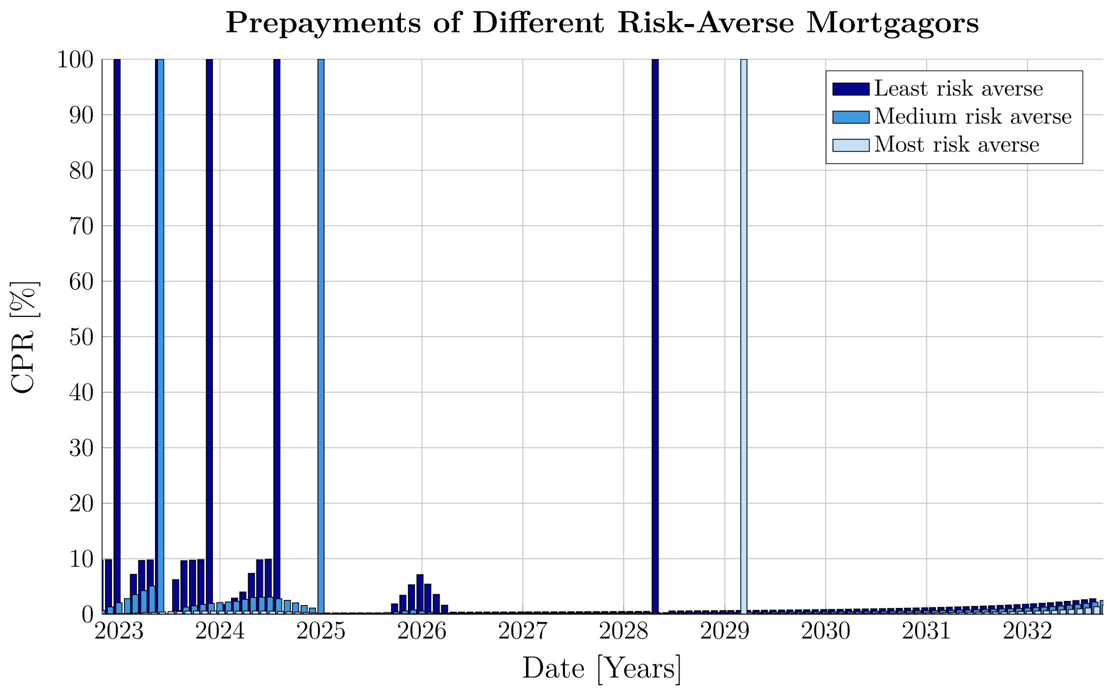

Figure 5-10: The CPR of different risk-averse mortgagors. The most risk-averse mortgagor supplies less partial prepayments and full prepayments than the least risk-averse mortgagor supplies. The most risk-averse mortgagor corresponds to the largest value for the mass, and is therefore less influenced by the economic forces impinging on the mass.

mortgage prepayments did not increase during the same period [61]. This is remarkable, as prepayments usually account for a substantial share of free savings. The decrease in prepayments and the increase in total savings suggest an increased preference for more liquid savings in households as a precautionary measure against the high uncertainty surrounding the development of the pandemic and the economy [61]. Unlike deposits, assets accumulated in a home are not immediately withdrawable, and so households cannot use them to absorb income drops or unexpected expenses. This means that households during the COVID-19 crisis were more risk averse.

In Figure 5-10, the most risk-averse mortgagor barely supplies partial prepayments, because the economic forces from risk-free rate changes have no influence on him. He supplies one time a full prepayment, because he can reduce his monthly payments by refinancing his mortgage according to the condition in Equation (5-7). In general, low-income persons are found to be more risk-averse because they have little margin for error or loss [62].

In mechanics, when a similar force is exerted on a mass, a lighter mass will have a higher acceleration than a heavier mass. This explains the very low levels of partial prepayments supplied by the most risk-averse mortgagor. The economic forces from risk-free rate changes and a mortgagor's appetite to reduce debt cause lower levels of partial prepayments supplied from the most risk-averse mortgagor than from the least risk-averse mortgagor.

The least risk-averse mortgagor clearly supplies the highest levels of partial prepayments and the most full prepayments. This is because the value of the risk aversion parameter is the lowest for this mortgagor and therefore the mass is the lightest. The least risk-averse mortgagor reduces his principal balance faster with partial prepayments, meaning he has earlier opportunities to reduce his monthly payments. The condition shown in Equation (5-7) is more frequently satisfied for this mortgagor, and therefore there are more frequently switches to the dynamics of full prepayments.

The increasing economic force from increasing interest rates together with the decreasing appetite of a mortgagor to reduce his debt (spring force), explains the very low levels of partial prepayments after mid 2026. The least risk averse supplies as only one partial prepayments at the end of 2025, this is because of the fast decrease of risk-free rates in the middle of 2025. These partial prepayments are delayed because of the delayed signal of the risk-free rates which is used as input.

5-4-2 Irrational Mortgagors

In this section, we only change the penalty on the control input to $R = 0$ for all three mortgagors to show the difference between the risk aversion parameter and the rationality parameter. We keep all other parameters equal and simulate the same mortgage contract as in the previous section.

The prepayments supplied of all the three mortgagors is shown in Figure 5-11. All three mortgagors do not supply any partial prepayments anymore. This is because they can adjust their risk appetite to any quantity needed such that their risk factor does not change. They reject all the disturbances from economic forces. In mechanics, this means that the reaction forces from the controller reject all the disturbances from the forces acting on the mass.

All three mortgagors follow the scheduled contract, and only supply a full prepayment at the same time when they can reduce their monthly payments. They still have an incentive to refinance, because the mortgage rates determine the switching behavior between the model dynamics. This switching behavior can not be rejected, and the mortgagors adjust their risk factor to supply a full loan prepayment.

People do not act rationally, meaning they do not prepay whenever there is an incentive to do so, and they sometimes prepay when it is not optimal. In literature, an explanatory variable named the burnout effect tries to explain and take into account that not all mortgagor behave rationally when presented an incentive to refinance [63, 64, 11].

This term is used to describe for a pool of mortgages that has experienced an earlier exposure to refinancing opportunities will, other things being equal, have lower refinancing rates than a pool with no such prior exposure. Burnout can be explained as the effect of changes in the composition of the pool of mortgages caused by refinancings which remove the most capable or most eager mortgagors from the pool, and the remaining mortgagors have less of a tendency to refinance [13]. The most aware mortgagors refinance immediately, while other mortgagor may not be aware that a refinancing opportunity has been presented. They may be reacting slow, and thus acting irrationally [63, 64].

We expect that the penalty on the control input together with the switching conditions can be used to model the rationality of mortgagors, or the burnout effect in a pool of mortgages.

A high penalty on the control input will result in less irrational behavior, i.e, make use of the partial prepayment option and full loan prepayment option when interest rates are decreasing.

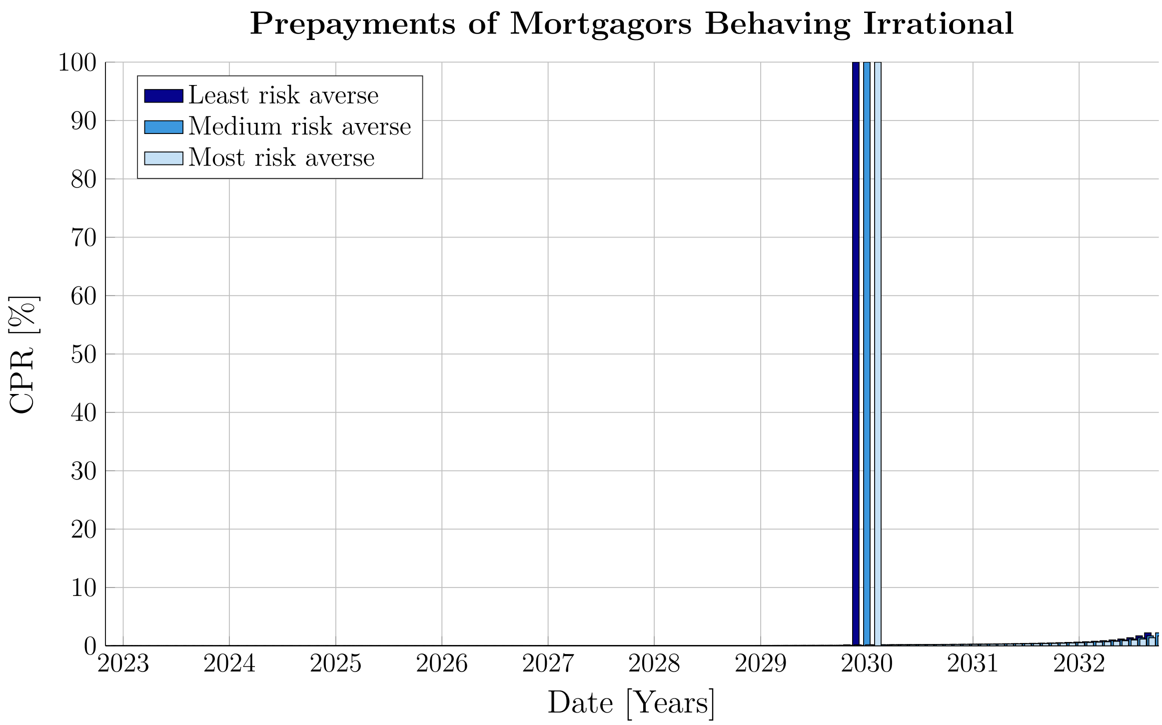

Figure 5-11: The CPR of different risk-averse mortgagors that behave irrational. We set the penalty on the control input to $R = 0$ to simulate irrational behavior. The mortgagors can reject all the disturbances from risk-free rate changes, meaning that they do not supply any partial prepayments when interest rates are decreasing. They still supply a full loan prepayment when optimal, because they can not reject the mortgage rates changes that determine the switching to a full prepayment.

A low control penalty will result in more irrational behavior, because the mortgagor will not make use of the partial prepayment option when interest rates are decreasing, and will only exercise the full prepayment option to reduce his monthly payments.

The switching conditions can be used to model the tendency of people to refinance. For the most eager and aware mortgagors an optimal switching condition should be used, while a margin on the switching condition should be added to model mortgagors with a less tendency to refinance.

5-4-3 Different Creditworthy Mortgagors

We vary the value of the parameter representing a mortgagor's creditworthiness (*the spring constant*) to model different creditworthy mortgagors. We use the following values listed in Table 5-2. Each mortgagor will be equally risk averse with $m_0 = 100$ and $m_1 = 100$. We simulate the same mortgage contract as in the previous sections, namely an annuity mortgage of €$500,000$. The maturity of the mortgage is 10 years and the contract interest rate is 3% per year. The penalty on the control input is $R = 0.3$.

From the states of the model we derive the CPR. In Figure 5-12, the CPR of each mortgage is shown.

Table 5-2: A different value for the parameter representing a mortgagor's creditworthiness is analogous to a different value for a spring constant in mechanics. A higher spring constant corresponds to a more creditworthy mortgagor and a lower spring constant to a less creditworthy mortgagor.

Creditworthy Mortgagor	**Creditworthiness** [1/(**€ – month**)]
Most	$k_0 = 0.1$
Medium	$k_0 = 0.05$
Least	$k_0 = 0.01$

In literature, a mortgagor's creditworthiness is an explanatory variable that tries to explain partial prepayments based on data on FICO scores [11, 5, 13, 5]. A higher FICO score is associated with a higher degree of creditworthiness [11]. Previous studies have shown that FICO score at mortgage origination is positively associated with partial prepayments [32]. Meaning that these prepayments are more likely for mortgagors with higher scores [32]. These mortgagors have an increased financial ability to make extra mortgage payments when it is beneficial to do so [13, 32].

The most creditworthy clearly supplies the highest levels of partial prepayments and the most full prepayments in Figure 5-12. This is because the value of this parameter is the highest for this mortgagor and therefore his appetite to reduce is debt is the largest compared to the other mortgagors for a similar principal balance.

In mechanics, for a similar displacement, a higher spring constant results in a larger restoring force than for a spring with a lower spring constant. The most creditworthy mortgagor has therefore the largest appetite to reduce his debt which causes a higher level of partial prepayment supplied. For the least creditworthy mortgagor his appetite to reduce debt is the smallest and therefore a lower level of partial prepayments is supplied.

This economic force from the spring gradually decreases over time, because the principal balance decreases over time. Together with the increasing economic force from increasing interest rates explains the very low levels of partial prepayments after 2026.

The most creditworthy mortgagor also supplies the most full prepayments. This is because he reduces his principal balance faster with partial prepayments, meaning he has earlier opportunities to reduce his monthly payments with fully prepaying and taking out a new mortgage contract. The condition shown in Equation (5-7) is more frequently satisfied for this mortgagor, and therefore there are more frequently switches to the dynamics of full prepayments compared to mortgagors with a lower creditworthiness.

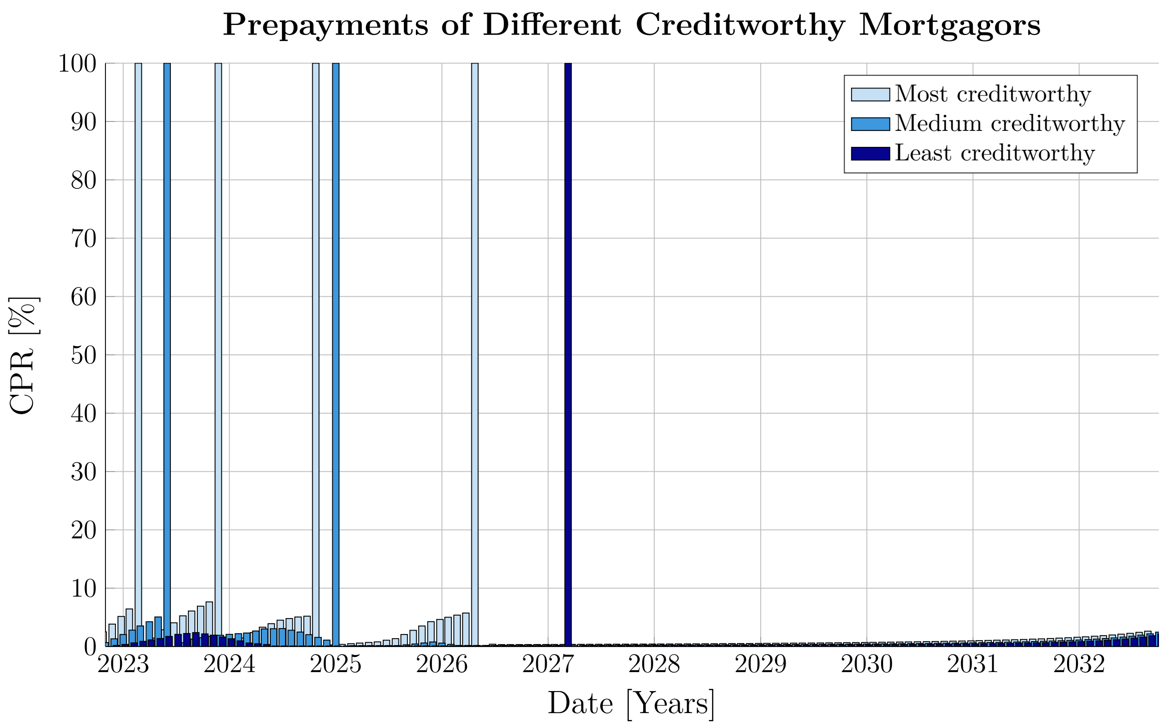

Figure 5-12: The CPR of different creditworthy mortgagors. The least creditworthy mortgagor supplies less partial prepayments and full prepayments than the most creditworthy mortgagor supplies. The least creditworthy mortgagor corresponds to the lowest value for the spring constant, and therefore has the smallest appetite to reduce his debt.

Chapter 6

Conclusions and Recommendations

Conclusions

Human behavior is inherently hard to forecast and so are mortgage prepayments. Instead of traditional forecasting methods using an all-inclusive econometric model, this thesis contributes to developing a systems and control approach for the forecasting of mortgage prepayments. By following the principles of economic engineering, a method is developed to build a dynamical model of mortgage prepayments which captures both the cause and dynamics behind changing mortgage repayments, and therefore performs in changing interest rate regimes.

In this thesis, we derive that the cause of changing mortgage repayments is a change in a mortgagor's risk factor. This causality remains reliable even when interest rates drastically change. This is because the law of demand and supply remains valid in any interest rate regime. This thesis contributes to developing a causal model for forecasting mortgage prepayments. In equilibrium situations are econometric models based on correlations efficient to forecast prepayments, but economic situations such as the financial crisis have shown that econometric models underperform due to changing parameters. The current economic situation is not an equilibrium situation as interest rates are drastically increasing, and the causal model derived in this thesis is able to dynamically forecast prepayments for disequilibrium.

We derive the dynamics of mortgage prepayments by specifying a risk appetite—an economic force—as the time derivative of a risk factor. This is based on Newton's laws of motion. As a consequence, we derive differential equations that describe the dynamics of mortgage payments and partial prepayments. We demonstrate the functioning of these dynamics in Chapter 3 by simulating repayment schedules including partial prepayments. These differentials describe in a fully deterministic manner the exact time change of mortgage repayments caused by economic forces. In economics, such economic forces are not defined explicitly and, hence, econometric models do not describe the dynamics for this changing.

The dynamics of mortgage payments and prepayments depend on economic interpretable parameters. These parameters are personal characteristics of mortgagors, namely their risk aversion and creditworthiness, and these remain relatively constant. Personal characteristics do not specifically depend on interest rates, and therefore these parameters can be estimated with historical data. The values of these parameters are economically interpretable as opposed to the parameters of econometric models which only reflect the degree of association between variables. For different creditworthy and risk-averse mortgagors, we forecast prepayments in an increasing interest rate scenario in Section 5-4. We explicitly explain prepayment behavior for each mortgagor based on the values of these parameters. As a result, this model can be used as a forward-looking model. This means that we can forecast prepayments for economic situations leading to more risk-averse mortgagors such as in a financial crisis.

The structure of the hybrid model allows implementing the kinematics of any mortgage contract, and the dynamics of any prepayment type. We specify the kinematics with constraints which limit the mortgage payments, and use an inequality constraint to define the maximum allowable percentage of partial prepayments. A jump of the continuous state describes the dynamics of full prepayments. This results in a hybrid model that switches between continuous and discontinuous dynamics. This hybrid model is a preliminary design. We have not estimated the model parameters, and the conditions that determine the switching between the different dynamics are simplified. However, this model shows the ease with which different prepayment dynamics can be added, and how switching conditions can be used to determine if and when mortgagors fully prepay. As a result, the model can be adapted to any mortgage contract and prepayment type, even nonexistent ones, to forecast prepayments. In these situations, only what-if scenarios are possible with econometric models because of data unavailability.

In addition, we have designed an EMPC to simulate the economic forces from the behavior of a mortgagor. The choice of an EMPC is essential because of the receding horizon aspect, the minimization of an objective function directly representing a mortgagor's costs, and adding constraints easily to account for limitations to either states or inputs. The objective function is an economic objective of the mortgagor. This can be changed to simulate mortgagor behavior with other optimizing needs such as maximizing tax deductions or minimizing environmental footprint. Repeatedly solving a minimization problem over a receding horizon is not possible with other constrained optimization or control methods.

For optimization problems in finance such as multi-period portfolio optimization, EMPC is a useful tool. Maximizing profits or minimizing costs is a common objective in finance, and with a EMPC it is possible to optimize an objective over multiple periods while assuming imperfect information of the future. This is contrary to (offline) constrained optimization methods that assume perfect information of the future. However, when perfect information of the future can be assumed, constrained optimization methods are easier to implement.

Overall, the developed economic engineering model in this thesis positively contributes to forecasting prepayments using a completely different modeling approach. Instead of relying on correlations as a basic premise, we rely on causalities. This economic engineering approach is useful for producing a challenger model for banks to uncover potential other biases in their models from relying on historical data (e.g., tax regimes, client behavior). As a challenger model, both the forecasting power of economic engineering models and the performance of the models of banks can be tested by comparing prepayment forecasts.

Recommendations

Economic engineering and asset-liability management

In this thesis, we have shown how to develop specific analogies for the mortgage market by following the principles of economic engineering. We derived the dynamics of a single mortgage with these analogies. For deriving the dynamics, we used the linear economic engineering analog, and we viewed the motion of a mortgage amortization schedule as a translation horizontally and vertically each month. However, similar as with a pendulum in mechanics, it could be more convenient to derive the dynamics of a mortgage with rotational mechanics, because the motion of mortgage contracts in the principal-interest plane is rotational.

When modeling a balance sheet consisting of multiple assets and liabilities and for the management of the balance sheet, the rotational analog yields especially interesting insights. Specifying the economic interpretation of the rotational analog applicable to banks, translating a pool of assets and liabilities to a mechanical system of rigid bodies, and formulating a bank's practice of managing assets and liabilities as a systems and control problem, is a way to extend the economic engineering research for banks. The economic interpretation of rotational mechanics and rigid body dynamics will also contribute to the field of economic engineering in general.

The hybrid economic-engineering model

The focus of this research was on formulating the forecasting of mortgage prepayments as a systems and control problem, and describing the system with a model of mortgage payments and prepayments derived with economic engineering. To facilitate this process, some assumptions and or simplifications were made.

The simulated mortgage contract is simplified by assuming that the interest rate of the contract is fixed till the mortgage matures. Normally, the interest rate is fixed for a predetermined period which is not necessarily equal to the term of the contract, but could have different lengths from zero to thirty years.

Also, we assumed that a new contract agreement is at the same bank after each full loan prepayment. However, a mortgagor is allowed to contract at another bank during a refinancing or relocation, and so an outflow of mortgages is expected. A probability should be taken into account that a mortgagor takes out a new mortgage contract at another bank.

The conditions that determine the switching between different dynamics in the model are an important feature. We assumed that a mortgagor compares his monthly payment (without partial prepayments) with the monthly payment of a new mortgage contract at the prevailing mortgage rates. In this way, we could easily add a prepayment penalty (which discourages mortgagors to refinance) on top of the monthly payment.

In reality, a mortgagor compares the interest rate of his current mortgage with that available on a new mortgage, or compares the value of his mortgage with the value of other mortgages in the market. A mortgagor's incentive to refinance changes over time, depending on macroeconomic conditions. It is not a simple comparison of interest rates. It is not always exercised. When designing these switching conditions, these uncertainties should be taken into account.

Moreover, we assumed that the incentive for housing remains constant and does not change for different interest rate scenarios. Housing activity is negatively correlated with mortgage rates, and so it is more realistic to also generate future expected housing activity scenarios. It is expected that activity in the housing market reduces with increasing mortgage rates. Also, the assumption that a mortgagor moves only one time during the mortgage contract of 10 years should be revised. For example, by assuming that a mortgagor can move unlimited times, but the probability a mortgagor moves reduces each time after moving.

Defining uncertainty with control methods

The current EMPC design uses a deterministic objective function, deterministic constraints and deterministic disturbances. The disturbances from risk-free rates, mortgage rates and housing activity do vary over time using noise, but at every optimization step, the MPC assumes everything completely deterministic.

However, uncertainty is present in the objective function of a mortgagor, in predictions of macroeconomic variables such as interest rates, the parameters of the model, and the switching conditions. To account for uncertainties when using MPC, there are two common possibilities.

Robust Model Predictive Control (RMPC) is a methodology to design controllers for uncertain systems, where the uncertainty is assumed to be bounded, and a control law is computed that satisfies the constraints for every possible uncertainty realization [65]. This leads to a large set of deterministic constraints and while robustness against the bounded uncertainty can be ensured, however, this requirement may significantly degrade the overall controller performance by the need to protect against low probability outliers.

Often uncertainties are of probabilistic nature, like the uncertainties in the refinance, relocation and mortgage porting incentives, and the uncertainty that a mortgagor takes out a new contract by another bank. Stochastic Model Predictive Control (SMPC) is a relaxation of RMPC, in which the objective function is formulated as an expectation and the constraints are interpreted probabilistically via chance constraints, allowing for a small constraint violation probability. The optimization will be a trade-off between fulfilling the control objectives and satisfying the constraints, a mix of performance and robustness. Unfortunately, chance constrained control problems are difficult to model, and must often be approximated [65].

Defining the uncertainty in the systems and control problem formulated in this thesis is an important step. This will not only contribute to obtain more realistic forecasts of mortgage prepayments with the method proposed in this thesis, but also contribute to dealing with uncertainty in economic engineering models in general.

Appendix A

Econometric Models

The two main approaches for modeling mortgage termination and subsequently forecasting prepayment rates are option theoretic models and exogenous models.

A-1 Optional Theoretical Models

Optional theoretical models only look at the financial considerations of a mortgagor, which means that a mortgagor would prepay only if the actual value of the asset is greater than the remaining mortgage principal balance with possible transaction costs. The prepayment option is viewed as a call option which is assumed to be exercised only under optimal conditions. Option theoretic models assume that borrowers behave purely rational and model prepayments endogenously, without incorporating driving factors that trigger mortgage prepayments. Using optional theoretical model as modeling approach removes all the irrational behavior of mortgagors. Exogenous models on the other hand model mortgage termination exogenous and can incorporate borrower heterogeneity.

A-2 Exogenous Models

The exogenous models make use of two different statistical frameworks. These two frameworks are logistic regression and survival analysis. The logistic regression models simply add exogenous calls to the optimal prepayment models in order to take into account the non-optimal behavior of mortgagors. The latter framework looks into the observed prepayments to determine which explanatory variables (i.e. driving factors) can explain such prepayment behavior. The main difference between the two frameworks is that the dependent variable in survival analysis is the survival time to prepayment, while the dependent variable in a regression model is a binary variable, indicating the occurrence of a prepayment. An extension of the logistic regression models are Markov models or conditional logistic regression models that can account for dependence between observations.

A-2-1 Logistic Regression

Logistic regression indicates the occurrence of a certain event, in this case a prepayment, as a binary variable. In the case of prepayment modeling, we are interested in whether a mortgage will prepay ($Y_{it} = 1$) or not ($Y_{it} = 0$). Here Y_{it} is the dependent variable (also known as a response variable or output variable) which can only take values of 0 or 1 [12].

Logistic regression applied to multi class problems (i.e. with more than two possible discrete outcomes) is known as Multinomial Logit (MNL) models. A MNL model measures the relative probability of being in one state compared to another [5].

The dependent variable is a binary variable indicating the occurrence of a prepayment $Y_{it} = j$ for possible payment states $j = 1, ..., n$ for each mortgagor $i = 1, ..., N$. These models estimate the probability that either one of the categories in the dependent variable occurs $P(Y_{it} = j)$ using a set of explanatory variables. The logistic model assumes a logistic distribution, where the logistic is defined as

$$\ln\left(\frac{F(.)}{1-F(.)}\right) = \boldsymbol{X}'_{it}\boldsymbol{\beta}_j, \tag{A-1}$$

where $F(.)$ is the logistic cumulative distribution function. The vector $\boldsymbol{X}_{it}$ (also called independent variables or predictor variables) contains the explanatory variables and the vector $\boldsymbol{\beta}_j$ the regression coefficient estimates. Consequently, the probability that mortgage i at time t classifies as category j is defined as

$$P\left(Y_{it} = j\right) = \frac{e^{\boldsymbol{X}_{it}\boldsymbol{\beta}_j}}{\sum_j e^{\boldsymbol{X}'_{it}\boldsymbol{\beta}_j}}. \tag{A-2}$$

The log-likelihood of the MNL model is defined as

$$\ln L(\boldsymbol{\beta}) = \sum_t \sum_i \sum_j d_{ijt} \ln P\left(Y_{it} = j\right), \tag{A-3}$$

for all mortgagors i over all time periods t and possible states j. Here, d_{ijt} is the dummy variable for the category of Y_{it}. The dummy variable takes values of 1 if j is equal to the payment state of mortgagor i on any time t, and 0 otherwise.

Typically, the regression coefficients are estimated by some sort of optimization procedure, e.g. Maximum Likelihood Estimation (MLE). The estimates are obtained by maximizing eq. (A-3) with respect to $\boldsymbol{\beta}_j$. This method finds regression coefficients that best fit the observed data, and give the most accurate predictions for the data already observed [5], [11].

For prepayment modeling, a three state classification and a five state classification could be used [11]. The three state classification leads to a dependent variable with three states: contractual payment, default, and prepayment. The reference state or the default category is contractual payment. The probability of the ith mortgage at time t being classified as being in the reference state reads as:

$$P\left(Y_{it} = 0\right) = \frac{1}{1 + \sum_{i=1} e^{\boldsymbol{X}'_{it}\boldsymbol{\beta}_j}} \tag{A-4}$$

The five state classification includes five states to the dependent variable: contractual payment, principal curtailment, prepayment, delinquent, and default.

The MNL models are the most common approach as starting point for modeling mortgage termination due to the relative ease with which the models can be estimated [11, 5, 12, 66]. Moreover, contrary to the survival analysis, logistic models are able to include time varying covariates. However, the MNL models have the drawback that it is assumed that all draws to be independent observations. So the models are unable to take into account dependence in observations. However, mortgage termination cannot be assumed to be independent over time, as the same mortgage contract is observed over multiple time periods.

A-2-2 Survival Analysis

Survival models focus on modeling the time until the occurrence of a certain event. In prepayment modeling, we are interested in the time until the occurrence of a prepayment event. Within the survival analysis, the link between the survival function and the covariates is expressed on the basis of the proportional hazard model as introduced by Cox [9]. The quantity of interest is the hazard (risk) of a prepayment occurring in the next month, given the mortgage has not been prepaid yet. The hazard rate is modeled as the product of a baseline hazard rate and a multiplier. The baseline hazard rate describes the time development of a 'typical' prepayment profile (depends on mortgage age and has a so-called S-shape[1]). The multiplier of the hazard rate contains borrower-specific factors and loan-specific factors, which are outlined in section 2-4-3.

The hazard rate is defined as the probability of a mortgage termination for cause j at time t given the nonoccurence of prepayment event until time t.

$$h_j(t,x) = \lim_{\Delta t \to 0} \frac{P(t \leq T_j < t + \Delta t \mid T_j \geq t)}{\Delta t} = h_{j0}(t) e^{-\boldsymbol{X}'_{it}\boldsymbol{\beta}_j}. \tag{A-5}$$

The first part $h_{j0}(t)$ captures the distribution of the failure time when the driving factors explaining prepayments are zero (e.g. the baseline hazard rate). It should reflect the 'typical' prepayment rate and varies only with the age of the loan. The multiplier in the hazard rate contains the explanatory variables $\boldsymbol{X}'_{it}$ describing observation i, and $\boldsymbol{\beta}_j$ the weights (or regression coefficients) corresponding to mortgage termination j. The multiplier is a proportionality factor that includes the effects of the driving factors on the hazard rate, and it incorporates both borrower-specific and loan-specific effects per mortgage i. The survival time is a discrete random variable T, analogous to the classification of the dependent variable in the MNL model.

The survival function is the probability of surviving from failure type j up to time t and is given by

$$S_j(t,x) = e^{-H_j(t,x)} \tag{A-6}$$

[1]Prepayment rates often exhibit an S-shaped relation with loan age. Prepayment rates are generally low shortly after origination of the loan and increase as the mortgage matures to finally arrive at a steady state level at the end of the mortgage term.

Here, $H_j(t, x)$ is the cumulative hazard for cause j:

$$H_j(t, x) = \sum_t \sum_i h_j(t, x) \tag{A-7}$$

Combining these results, the cause-specific density function $f_j(t, x)$ describes the unconditional probability that a mortgage i is terminated at time t due to cause j.

$$f_j(t, x) = h_j(t, x) S_j(t, x) \tag{A-8}$$

Mortgages face multiple reasons for contract termination. When there is more than one hazard affecting the survival of a mortgage, they compete. Simultaneous estimation of these hazards produces a competing risk model [67]. The log likelihood of the competing risk model is given by

$$\ln L(\boldsymbol{\beta}_j) = \sum_t \sum_i \sum_j d_{ijt} \ln f_j(t, x) \tag{A-9}$$

Here, d_{ijt} is a dummy variable indicating the reason for contract termination. This likelihood is very similar to the likelihood of the MNL model in eq. (A-3). When the effect of the baseline hazard rate is small relative to the effect of the covariates, the competing risk model and the MNL model are comparable [11].

One of the problems in using these type of models is the difficulty of including time varying covariates in the analysis [11, 12, 13]. As the dependent variable in survival analysis is the survival time of mortgage i from cause j: T_{ij}. This variable does not depend on time t over which mortgages are observed. So for each mortgage, only one observation on each covariate can be used. Meis [11] transforms a historic data set of mortgages (containing data of loan and borrower characteristics, and prepayment rates) into a cross-sectional dataset by combining information per time-varying covariate per mortgage. The time varying covariates per mortgage are averaged or randomly selected for a time period t for each mortgage i from which covariate information is included.

Moreover, these competing risk models implicit assume an independence between consecutive observations [11, 12]. Which means that the next state of the model is independent of the current state. Dependence is accounted for by conditioning on the current state in mortgage termination. Markov models can be used to account for this dependence between observations [11].

A-2-3 Markov Models

Both logistic models and survival models lack the ability to account for dependence between observations. The simplest Markov model is the Markov chain. It is a stochastic model describing a sequence of possible events (i.e. states). The probability of each event is modeled with a random variable that changes through time. The future states depend only on the current state and not on the events that occurred before it.

A stochastic process Y_t with $t \geq 0$ is a discrete time Markov chain if for all states i, j

$$P(Y_{t+1} = j | Y_t = i). \tag{A-10}$$

Hence, the future state Y_{t+1} only depends on the present state Y_t and not on the past states Y_{t-1}. The (one-step) transition probabilities from state i to state j are defined as

$$p_{ij} = P(Y_{t+1} = j | Y_t = i), \tag{A-11}$$

and are conditional probabilities. For a Markov chain with m states, the one-step transition matrix is given by

$$\mathbf{P} = \begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0m} \\ p_{10} & p_{11} & \cdots & p_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ p_{i0} & p_{i1} & \cdots & p_{im} \\ p_{m0} & p_{m1} & \cdots & p_{mm} \end{pmatrix}, \tag{A-12}$$

where the transition probabilities p_{ij} can be estimated from data by maximizing a likelihood function (MLE) or by using covariate information [11, 5]. Using Markov chains, a dependency is built in between the current state and the next state. The model does not exhibit the memoryless property anymore as with logistic regression models and survival analysis models. The model is estimated by conditioning on a transient state and performing a logistic regression. These type of models are often referred to as a conditional logistic regression model.

For modeling mortgage termination, Meis [11] uses a Markov chain with three states where the borrower can decide to continue with contractual payments, fully prepay the mortgage or default on the mortgage. The states of full prepayment and default are end states. The Markov model can be extended to five states as suggested by Meis [11] and Wesseling [5] by including partial prepayments and delinquency state of the borrower. In this Markov chain model the transient states are partial prepayments, delinquent and contractual payment. Therefore, a complete estimation of this model thus entails estimating a conditional logistic regression model for partial prepayment, delinquent and contractual payment.

A-2-4 Time-Varying Markov Models

The problem with Markov models is that the transition probabilities are time independent. A time-varying Markov model can be used to correct for the time-dependency of driving factors. Since, macroeconomic variables differ during different economic regimes, and the state of the economy, hence time, has an effect on mortgage prepayments.

Wesseling [5] develops a time-varying Markov model for mortgage prepayments. In a regular Markov model the transition probabilities are assumed to be the same at all time. In a time-varying Markov model, multiple Markov models are set up for which there exist transition probabilities between different payment states and different economic states. So for each economic state, a regular Markov model (as described in the previous section) is set up, where transition probability matrices represent the switching between the different economic regimes. A downside of this method is that more parameters need to be estimated compared to the regular Markov model, and therefore a less accuracy in terms of variance. However, the advantage of this method is that it is able to give a more precise estimate for different time intervals.

Appendix B

The Objective Function with Constraints

The minimization problem is given by the objective function J minimized over time with constraints. The state update equations of the continuous dynamics, state values of discontinuous dynamics, contract constraints, and terminal constraints are constraints on the system. Moreover, constraints are added for the switching conditions of the modes, to keep track of the model mode, and to accumulate interest without resetting the state. The optimization of one iteration looks as follows:

$$\underset{u}{\text{minimize}} \quad q_{1,\text{acc}}[T_h] + \sum_{k=1}^{T_h} [(r[k] - r_c[k])v_0[k] - \frac{1}{2}k_0 q_0[k]^2 + \frac{1}{2}Ru_0[k]^2]$$

$$\text{subject to}$$

$$\begin{aligned}
\textbf{Mortgage Payments Mode} = [x[k+1] &= Ax[k] + Bu[k] + F\hat{d}[k], \\
q_{1,\text{acc}}[k+1] &= q_{1,\text{acc}}[k] + T_s v_1[k], \\
x[0] &= x_0 \\
0 \leq q_0[T_h] &\leq A_0 \\
q_1[T_h] &\geq 0 \\
v_1[k] &= r_c[k] q_0[k], \\
-P_{\text{contract}}[k] + v_1[k] - \frac{\rho}{12}A_0[k] &\leq v_0[k] \leq -P_{\text{contract}}[k] + v_1[k], \\
P_{\text{contract}}[k] &\leq P_{\text{refinance}}[k], \\
m[k] &= 0, \\
h[k] &= 0, \\
mdl[k] &= 1] \\
&\ldots
\end{aligned} \tag{B-1}$$

$$
\begin{aligned}
\textbf{Mortgage Porting Mode} = [x[k+1] &= Ax[k] + Bu[k] + F\hat{d}[k], \\
q_{1,\text{acc}}[k+1] &= q_{1,\text{acc}}[k] + T_s v_1[k], \\
x[0] &= x_0 \\
0 \leq q_0[T_h] &\leq A_0 \\
q_1[T_h] &\geq 0 \\
v_1[k] &= r_c[k] q_0[k], \\
-P_{\text{contract}}[k] + v_1[k] - \frac{\rho}{12} A_0[k] &\leq v_0[k] \leq -P_{\text{contract}}[k] + v_1[k], \\
q_0[k] &\geq 0, \\
P_{\text{contract}}[k] &\leq P_{\text{relocation}}[k], \\
m[k] &= 0, \\
h[k] &= 1 \\
mdl[k] &= 2] \\
\textbf{Mortgage Payments Mode} = [x[k+1] &= Ax[k] + Bu[k] + F\hat{d}[k], \\
q_{1,\text{acc}}[k+1] &= q_{1,\text{acc}}[k] + T_s v_1[k], \\
x[0] &= x_0 \\
0 \leq q_0[T_h] &\leq A_0 \\
q_1[T_h] &\geq 0 \\
v_1[k] &= r_c[k] q_0[k], \\
-P_{\text{contract}}[k] + v_1[k] - \frac{\rho}{12} A_0[k] &\leq v_0[k] \leq -P_{\text{contract}}[k] + v_1[k], \\
q_0[k] &\geq 0, \\
P_{\text{contract}}[k] &\leq P_{\text{refinance}}[k], \\
m[k] &= 1, \\
h[k] &= 0 \\
mdl[k] &= 3] \\
\textbf{Mortgage Payments Mode} = [x[k+1] &= Ax[k] + Bu[k] + F\hat{d}[k], \\
q_{1,\text{acc}}[k+1] &= q_{1,\text{acc}}[k] + T_s v_1[k], \\
x[0] &= x_0 \\
0 \leq q_0[T_h] &\leq A_0 \\
q_1[T_h] &\geq 0 \\
v_1[k] &= r_c[k] q_0[k], \\
-P_{\text{contract}}[k] + v_1[k] - \frac{\rho}{12} A_0[k] &\leq v_0[k] \leq -P_{\text{contract}}[k] + v_1[k], \\
q_0[k] &\geq 0, \\
P_{\text{contract}}[k] &\leq P_{\text{refinance}}[k], \\
m[k] &= 1, \\
h[k] &= 1 \\
mdl[k] &= 4] \\
&\ldots
\end{aligned}
$$

$$
\begin{aligned}
\textbf{Refinance Mode} = [q_0[k+1] &= 0, \\
q_1[k+1] &= q_1[k] + \epsilon[k], \\
v_0[k+1] &= 0, \\
v_1[k+1] &= 0, \\
q_{1,\text{acc}}[k+1] &= q_{1,\text{acc}}[k] + \epsilon[k], \\
P_{\text{contract}}[k] &\geq P_{\text{refinance}}[k], \\
m[k] &= 0, \\
h[k] &= 0 \\
mdl[k] &= 5] \\
\textbf{Refinance Mode} = [q_0[k+1] &= 0, \\
q_1[k+1] &= q_1[k] + \epsilon[k], \\
v_0[k+1] &= 0, \\
v_1[k+1] &= 0, \\
q_{1,\text{acc}}[k+1] &= q_{1,\text{acc}}[k] + \epsilon[k], \\
P_{\text{contract}}[k] &\geq P_{\text{refinance}}[k], \\
m[k] &= 1, \\
h[k] &= 0 \\
mdl[k] &= 6] \\
\textbf{Refinance Mode} = [q_0[k+1] &= 0, \\
q_1[k+1] &= q_1[k] + \epsilon[k], \\
v_0[k+1] &= 0, \\
v_1[k+1] &= 0, \\
q_{1,\text{acc}}[k+1] &= q_{1,\text{acc}}[k] + \epsilon[k], \\
P_{\text{contract}}[k] &\geq P_{\text{refinance}}[k], \\
m[k] &= 1, \\
h[k] &= 1 \\
mdl[k] &= 7] \\
\textbf{Relocation Mode} = [q_0[k+1] &= 0, \\
q_1[k+1] &= q_1[k], \\
v_0[k+1] &= 0, \\
v_1[k+1] &= 0, \\
q_{1,\text{acc}}[k+1] &= q_{1,\text{acc}}[k], \\
P_{\text{contract}}[k] &\geq P_{\text{relocation}}[k], \\
m[k] &= 0, \\
h[k] &= 1 \\
mdl[k] &= 8]
\end{aligned}
$$

Appendix C

Matlab Files

```
%% Pre Processing

yalmip('clear') ;clear all;
clc;close all

% Data inputs
load rfr.mat          % risk-free rates
load housing.mat      % housing sales
load mortgage.mat     % mortgage rates

%% Mortgage Contract

maturity = 10;                  % maturity in years
T        = 120;                 % maturity in months
interest = 3;                   % interest rate in percentage per year
rc       = interest/100/12;   % contract rate per month
A0       = 500000;              % initial amount of money borrowed
rho      = 10/100;              % max prepayment per year in percentage
P        = A0*(rc*(1+rc)^T)/(((1+rc)^T)-1);    % monthly payment

%% Initialization

Ts = 1;                       % sampling time of 1 month
Horizon = 2;                  % prediction horizon of 2 months
Timesteps = T/Ts;             % timesteps
terminal_constraint = 0;% terminal constraint

```

```
%% Cost of refinancing

% To include additional cost of refinancing
penalty_extra = 0;

%% Forward Euler for simulating initial scheduled contract

q0_0(1) = A0;             % starting principal balance
q1_0(1) = 0;              % starting accumulated interest
v1_0(1) = rc*q0_0(1);     % starting interest payment
v0_0(1) = -P+v1_0(1);     % starting principal payment

for k = 1:T
    q0_0(k+1) = q0_0(k) + Ts*(v0_0(k));
    q1_0(k+1) = q1_0(k) + Ts*(v1_0(k));
    v1_0(k+1) = rc*q0_0(k+1);
    v0_0(k+1) =  v1_0(k+1)-P;
end

P        = P*ones(1,length(T));  % intiial fixed montlhy costs
beta     = -v0_0(1)*100*120;     % sensitivity partial
   prepayments and risk-free rates

x0 = [q0_0(1);q1_0(1);v0_0(1);v1_0(1)];  % intial state

%% Dimensions

dim.N        = Horizon;                 % prediction horizon
dim.M        = Timesteps;               % number of timesteps

%% Decision variables dimensions

% State space states, inputs and disturbance
q_0         = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1));
q_1         = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1));
v_0         = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1));
v_1         = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1));
dhat        = sdpvar(repmat(1,1,dim.N),repmat(1,1,dim.N));
u_0         = sdpvar(repmat(1,1,dim.N),repmat(1,1,dim.N));
u_1         = sdpvar(repmat(1,1,dim.N),repmat(1,1,dim.N));

% Penalty
e = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1));

% Contract cost
P_refinance    = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1));
```

```
P_relocation     = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1)
   );
P_contract       = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1)
   );

% Accumulated interest costs
q_1_interest = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1));

% Risk-free rate real and observed
r        = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1));
r_obs    = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1));

% Housing
h = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1));

% Binary
d        = binvar(repmat(8,1,dim.N),repmat(1,1,dim.N));

% Mode
mdl      = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1));

% Contract reset
r_c  = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1));
P    = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1));
A    = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1));

% Moved
m    = sdpvar(repmat(1,1,dim.N+1),repmat(1,1,dim.N+1));

% Model parameters
m0    = sdpvar(repmat(1,1,dim.N),repmat(1,1,dim.N));
m1    = sdpvar(repmat(1,1,dim.N),repmat(1,1,dim.N));
k0    = sdpvar(repmat(1,1,dim.N),repmat(1,1,dim.N));
R_rational = sdpvar(repmat(1,1,dim.N),repmat(1,1,dim.N));

constraints.mortgagor = [];
J.mortgagor = 0;

%% Define objective function, hybrid dynamics, and controller

for k = 1:dim.N


   J.mortgagor = J.mortgagor + q_1_interest{dim.N} + (r{k}- r_c{
      k})*v_0{k} - (1/2)*k0{k}*q_0{k}^2  + (1/2)*R_rational{k}*
      u_0{k}^2;

```

```
% Mortgage payments: scheduled payments and partial prepayments
  Model1 = [

        % Dynamics
        q_0{k+1} == q_0{k} + Ts*(v_0{k})
        q_1{k+1} == q_1{k} + Ts*(v_1{k})
        v_0{k+1} == v_0{k} + Ts*(-1/m0{k}*k0{k}*q_0{k}+ 1/m0{k}*
           u_0{k} + 1/m0{k}*dhat{k})
        v_1{k+1} == v_1{k} + Ts*(1/m1{k}*u_1{k})
        q_0{k}(1) >= 0

        % Accumulated interest
        q_1_interest{k+1} == q_1_interest{k} + Ts*(v_1{k})

        % Contract constraints
        -P{k} + v_1{k} - rho/12*A{k} <= v_0{k} <= -P{k} + v_1{k}
      v_1{k} == r_c{k}*q_0{k}

        % Conditions
        P_contract{k} <= P_refinance{k}
        m{k} == 0,
        h{k} == 0,

        % Mode
        mdl{k} == 1];

% Mortgage porting
  Model2 = [

        % Dynamics
        q_0{k+1} == q_0{k} + Ts*(v_0{k})
        q_1{k+1} == q_1{k} + Ts*(v_1{k})
        v_0{k+1} == v_0{k} + Ts*(-1/m0{k}*k0{k}*q_0{k}+ 1/m0{k}*
           u_0{k} + 1/m0{k}*dhat{k})
        v_1{k+1} == v_1{k} + Ts*(1/m1{k}*u_1{k})
        q_0{k}(1) >= 0

        % Accumulated interest
        q_1_interest{k+1} == q_1_interest{k} + Ts*(v_1{k})

        % Contract constraints
        -P{k} + v_1{k} - rho/12*A{k} <= v_0{k} <= -P{k} + v_1{k}
      v_1{k} == r_c{k}*q_0{k}

        % Conditions
        P_contract{k} <= P_relocation{k}
        m{k} == 0,
```

```
        h{k} == 1,

        % Mode
        mdl{k} == 2];

 % Mortgage payments: scheduled payments and partial prepayments
 % Moved
  Model3 = [

        % Dynamics
        q_0{k+1} == q_0{k} + Ts*(v_0{k})
        q_1{k+1} == q_1{k} + Ts*(v_1{k})
        v_0{k+1} == v_0{k} + Ts*(-1/m0{k}*k0{k}*q_0{k}+ 1/m0{k}*
           u_0{k} + 1/m0{k}*dhat{k})
        v_1{k+1} == v_1{k} + Ts*(1/m1{k}*u_1{k})
        q_0{k}(1) >= 0

        % Accumulated interest
        q_1_interest{k+1} == q_1_interest{k} + Ts*(v_1{k})

        % Contract constraints
        -P{k} + v_1{k} - rho/12*A{k} <= v_0{k} <= -P{k} + v_1{k}
      v_1{k} == r_c{k}*q_0{k}

        % Conditions
        P_contract{k} <= P_refinance{k}
        m{k} == 1
        h{k} == 0

        % Mode
        mdl{k} == 3];

% Mortgage payments: scheduled payments and partial prepayments
% Moved
  Model4 = [

        % Dynamics
        q_0{k+1} == q_0{k} + Ts*(v_0{k})
        q_1{k+1} == q_1{k} + Ts*(v_1{k})
        v_0{k+1} == v_0{k} + Ts*(-1/m0{k}*k0{k}*q_0{k}+ 1/m0{k}*
           u_0{k} + 1/m0{k}*dhat{k})
        v_1{k+1} == v_1{k} + Ts*(1/m1{k}*u_1{k})
        q_0{k}(1) >= 0

        % Accumulated interest
        q_1_interest{k+1} == q_1_interest{k} + Ts*(v_1{k})
```

```
        % Contract constraints
        -P{k} + v_1{k} - rho/12*A{k} <= v_0{k} <= -P{k} + v_1{k}
      v_1{k} == r_c{k}*q_0{k}

        % Conditions
        P_contract{k} <= P_refinance{k}
        m{k} == 1
        h{k} == 1

        % Mode
        mdl{k} == 4];

% Refinance
  Model5 = [

        % Discrete dynamics
        q_0{k+1} == 0
        q_1{k+1} == q_1{k} + e{k}
        v_0{k+1} == 0
        v_1{k+1} == 0
        q_0{k} >= 0

        % Accumulated interest
        q_1_interest{k+1} == q_1_interest{k} + Ts*(e{k})

        % Conditions
        P_contract{k} >= P_refinance{k}
        m{k} == 0
        h{k} == 0

        % Mode
        mdl{k} == 5];

% Refinance
  Model6 = [

        % Discrete dynamics
        q_0{k+1} == 0
        q_1{k+1} == q_1{k} + e{k}
        v_0{k+1} == 0
        v_1{k+1} == 0
        q_0{k} >= 0

        % Accumulated interest
        q_1_interest{k+1} == q_1_interest{k} + Ts*(e{k})

        % Conditions
```

```
        P_contract{k} >= P_refinance{k}
        m{k} == 1
        h{k} == 0

        % Mode
        mdl{k} == 6];

% Refinance
  Model7 = [

        % Discrete dynamics
        q_0{k+1} == 0
        q_1{k+1} == q_1{k} + e{k}
        v_0{k+1} == 0
        v_1{k+1} == 0
        q_0{k} >= 0

        % Accumulated interest
        q_1_interest{k+1} == q_1_interest{k} + Ts*(e{k})

        % Conditions
        P_contract{k} >= P_refinance{k}
        m{k} == 1
        h{k} == 1

        % Mode
        mdl{k} == 7];

% Relocation
  Model8 = [

        % Dynamics
        q_0{k+1} == 0
        q_1{k+1} == q_1{k}
        v_0{k+1} ==0
        v_1{k+1} == 0
        q_0{k}(1) >= 0

        % Accumulated interest
        q_1_interest{k+1} == q_1_interest{k}

        % Conditions
        P_contract{k} >= P_relocation{k}
        m{k} == 0
        h{k} == 1

        % Mode
```

```
        mdl{k} == 8];



    constraints.mortgagor = [constraints.mortgagor,
           implies(d{k}(1), Model1) % Mortgage payments
           implies(d{k}(2), Model2) % Mortgage porting
           implies(d{k}(3), Model3) % Mortgage payments and moved
           implies(d{k}(4), Model4) % Mortgage payments and moved
           implies(d{k}(5), Model5) % Refinance
           implies(d{k}(6), Model6) % Refinance
           implies(d{k}(7), Model7) % Refinance
           implies(d{k}(8), Model8) % Relocation
                 0 <= q_0{k} <= A0
                 0 <= q_1{k}
               -A0 <= v_0{k} <= 0
                 0 <= v_1{k} <= A0
                 0 <= q_1_interest{k}
                 1 <= mdl{k} <= 8
         sum(d{k}) == 1];

  % Terminal constraint
  if k == dim.N && terminal_constraint == 1
     constraints.mortgagor = [constraints.mortgagor,
       0 <= q_0{k+1} <= A0
       0 <= q_1{k+1}
       0 <= q_1_interest{k+1}];
  end


end

     constraints.mortgagor = [constraints.mortgagor,
                             0 <= q_0{k+1} <= A0
                             0 <= q_1{k+1}
                             0 <= q_1_interest{k+1}];

% Parameters in
par.mortgagor = {[q_0{1}],[q_1{1}],[dhat{1:dim.N}],[v_1{1}],[v_0{1}],[r{1:dim.N}],[e{1:dim.N}],[P_contract{1:dim.N}],[P_refinance{1:dim.N}],[P_relocation{1:dim.N}],[r_c{1:dim.N}],[P{1:dim.N}],[A{1:dim.N}],[m{1:dim.N}],[h{1:dim.N}],[m0{1:dim.N}],[m1{1:dim.N}],[k0{1:dim.N}],[q_1_interest{1}],[R_rational{1:dim.N}]};

```

```
% Parameters out
sol.mortgagor = {[v_0{:}],J.mortgagor,[q_0{:}],[r{:}],[v_1{:}],[
    q_1{:}],[mdl{:}],[u_0{:}],[u_1{:}],[dhat{:}],[e{:}],[
    P_contract{:}],[P_refinance{:}],[P_relocation{:}],[r_c{:}],[
    q_1_interest{:}]};

% Optimizer Object
mortgagor = optimizer(constraints.mortgagor, J.mortgagor,
    sdpsettings('solver','gurobi'),par.mortgagor,sol.mortgagor);



%% Data inputs

% define data inputs
ir = ... % riskfree rates
p_m = ... % market mortgage rates
q_housing = ... % housing sales per month

% create riskfree rate and mortgage rate disturbance
riskfreerate = ir(end-T-3:end-4)'/12/100;
mortgagerate = p_m(end-T:end-1)'/12/100;

% delayed first-order hold
r_rfr = [];
rfr_delay = [];
r_m = [];
r_rfr(1:dim.N) = beta*riskfreerate(1);
rfr_delay(1:dim.N) = riskfreerate(1);
r_m(1:dim.N) = mortgagerate(1);

for i = 1:dim.M/Horizon+1
    if i > 1
        r_rfr(i*Horizon-1:i*Horizon) = beta*riskfreerate(Horizon
            *i-Horizon);
        r_m(i*Horizon-1:i*Horizon) = mortgagerate(Horizon*i-
            Horizon);
        rfr_delay(i*Horizon-1:i*Horizon) = riskfreerate(Horizon*
            i-Horizon);
    end
end

% Housing activity (houses sold per month)
housingturnover     = q_housing(end-T:end)';
norm_housing        = normalize(housingturnover,'range');
P_average           = sum(norm_housing)/length(norm_housing);

```

```

%% Initial conditions

P = A0*(rc*(1+rc)^T)/(((1+rc)^T)-1);
m = 0;

%% Vary borrower characteristics

% m1: Mortgagor 1
% Risk aversion
m0_m1 = 10*ones(1,dim.N);
m1_m1 = 10*ones(1,dim.N);

% Creditworthiness
k0_m1 = 0.05*ones(1,dim.N);

% Rationality
R_m1 = 0.3*ones(1,dim.N);

% Housing sales
housing_noise_m1 = housingturnover + 2000*randn(size(
    housingturnover));
norm_housing_noise_m1  = normalize(housing_noise_m1,'range');

% Initial states
x_m1 = x0;
x0_m1 = x0;

% Contract
rc_m1 = rc;
A0_m1 = A0;
P_m1 = P;

% Accumulated interest
x_interest1 = x0;

% Moved
moved_m1 = m;

% m2: Mortgagor 2
% Risk averison
m0_m2 = 100*ones(1,dim.N);
m1_m2 = 100*ones(1,dim.N);

% Creditworthiness
k0_m2 = 0.05*ones(1,dim.N);

```

```
% Rationality
R_m2 = 0.03*ones(1,dim.N);

housing_noise_m2 = housingturnover + 2000*randn(size(
    housingturnover));
norm_housing_noise_m2  = normalize(housing_noise_m2,'range');

% Initial states
x_m2 = x0;
x0_m2 = x0;

% Contract
rc_m2 = rc;
A0_m2 = A0;
P_m2 = P;

% Accumulated interest
x_interest2 = x0;

% Moved
moved_m2 = m;

% m3: Mortgagor 3
% Risk aversion
m0_m3 = 1000*ones(1,dim.N);
m1_m3 = 1000*ones(1,dim.N);

% Creditworthiness
k0_m3 = 0.05*ones(1,dim.N);

% Rationality
R_m3 = 0.3*ones(1,dim.N);

housing_noise_m3 = housingturnover + 2000*randn(size(
    housingturnover));
norm_housing_noise_m3  = normalize(housing_noise_m3,'range');

% Initial states
x_m3 = x0;
x0_m3 = x0;

% Contract
rc_m3 = rc;
A0_m3 = A0;
P_m3 = P;

% Accumulated interest
```

```
x_interest3 = x0;

% Moved
moved_m3 = m;

%% Run MPC

for i = 1:dim.M
      if i <= T-(dim.N-1)
    rfr_obs = rfr_delay(i:i+dim.N-1);   % observed risk free rates
    rfr = r_rfr(i:i+dim.N-1);   % disturbance risk free rates
    r_m_obs = r_m(i:i+dim.N-1);     % observed mortgage rates
    rm = mortgageraterate(i:i+dim.N-1); % real mortgage rates
      else
       rfr_obs=rfr_obs;
       rfr = rfr;
       rm = rm;
       r_m_obs = r_m_obs;
      end

    % Remaining term
    if i < T
    Tr = T - i;
    else
        Tr = 1;
    end

    % Mortgagor 1:
    % housing data
    if norm_housing_noise_m1(i) < P_average
       p_housing_m1 = 0;
    elseif norm_housing_noise_m1(i) > P_average
       p_housing_m1 = 1;
    end
    h0_m1 = p_housing_m1*ones(1,dim.N);

    % Initial reset variables
    P0_m1 = P_m1*ones(1,dim.N);
    A00_m1 = A0_m1*ones(1,dim.N);
    rc0_m1 = rc_m1*ones(1,dim.N);
    moved0_m1 = moved_m1*ones(1,dim.N);

    % Current states for scheduled payments
    q0m1 = x0_m1(1);
    q1m1 = x0_m1(2);
    v0m1 = x0_m1(3);
```

```
    v1m1 = x0_m1(4);

    % Current states
    q0_m1 = x_m1(1);
    q1_m1 = x_m1(2);
    v0_m1 = x_m1(3);
    v1_m1 = x_m1(4);

    % Accumulated interest
    q1_interest1 = x_interest1(2);

    % Prepayment penalty
    epsilon_m1 = penalty_extra + (max(rc0_m1(1)-r_m_obs(1),0)*Tr
       *max((x_m1(1)-rho*A00_m1(1)),0))*ones(1,dim.N);

    % Current contract costs
    Pcontract_m1 = P_m1*ones(1,dim.N);

    % Contract costs relocation
    Prelocation_m1 = x_m1(1).*(r_m_obs(1).*(1+r_m_obs(1)).^Tr)
       ./(((1+r_m_obs(1)).^Tr)-1)*ones(1,dim.N);

    % Contract costs refinancing
    Prefinancing_m1 = Prelocation_m1 + epsilon_m1;

    % Solve optimization problem mortgagor 1
    [solutions.mortgagor1,diagnose.mortgagor1] = mortgagor{q0_m1
       ,q1_m1,rfr,v1_m1,v0_m1,rfr_obs,epsilon_m1,Pcontract_m1,
       Prefinancing_m1,Prelocation_m1,rc0_m1,P0_m1,A00_m1,
       moved0_m1,h0_m1,m0_m1,m1_m1,k0_m1,q1_interest1,R_m1};

    % Retrieve solutions mortgagor 1
    v0hist_m1 = solutions.mortgagor1{1}(1);
    v1hist_m1 = solutions.mortgagor1{5}(1);
    q0hist_m1 = solutions.mortgagor1{3}(1);
    q1hist_m1 = solutions.mortgagor1{6}(1);
    u0hist_m1 = solutions.mortgagor1{8}(1);
    u1hist_m1 = solutions.mortgagor1{9}(1);
    mode_m1  = solutions.mortgagor1{7}(1);
    ehist_m1 = solutions.mortgagor1{11}(1);
    qaccinterest1 = solutions.mortgagor1{16}(1);
    costs  = solutions.mortgagor1{2}(1);

     % Mortgage payments: scheduled payments and partial
       prepayments
    if mode_m1 == 1

```

```
        % Update state equations
        x_m1 = [q0_m1 + Ts*(v0hist_m1(1));
                q1_m1 + Ts*(v1hist_m1(1));
                v0_m1  + Ts*(-1/m0_m1(1)*k0_m1(1)*q0hist_m1(1) +
                    1/m0_m1(1)*u0hist_m1(1) + 1/m0_m1(1)*rfr_obs
                   (1))
                v1_m1 + Ts*(1/m1_m1(1)*u1hist_m1(1))];

        states_m1 = [states_m1 x_m1];

        % Accumulated interest
        x_interest1(2) = q1_interest1 + Ts*v1hist_m1(1);
        q1accinterest1 = [q1accinterest1 x_interest1(2)];

        % Update state equations scheduled payments
        x0_m1 = [q0m1 + Ts*(v0m1);
                q1m1 + Ts*(v1m1);
                rc_m1*q0m1 - P_m1;
                rc_m1*q0m1];

        states0_m1 = [states0_m1 x0_m1];

    % Mortgage porting
    elseif mode_m1 == 2

        % Update state equations
        x_m1 = [q0_m1 + Ts*(v0hist_m1(1));
                q1_m1 + Ts*(v1hist_m1(1));
                v0_m1  + Ts*(-1/m0_m1(1)*k0_m1(1)*q0hist_m1(1) +
                    1/m0_m1(1)*u0hist_m1(1) + 1/m0_m1(1)*rfr_obs
                   (1))
                v1_m1 + Ts*(1/m1_m1(1)*u1hist_m1(1))];

        states_m1 = [states_m1 x_m1];

        % Accumulated interest
        x_interest1(2) = q1_interest1 + Ts*v1hist_m1(1);
        q1accinterest1 = [q1accinterest1 x_interest1(2)];

        % Reset to moved
        moved_m1 = 1;

        % Update state equations scheduled payments
        x0_m1 = [q0m1 + Ts*(v0m1);
                q1m1 + Ts*(v1m1);
                rc_m1*q0m1 - P_m1;
```

```
                rc_m1*q0m1];

        states0_m1 = [states0_m1 x0_m1];

    % Mortgage payments: scheduled payments and partial prepayments
    % and moved
    elseif mode_m1 == 3 || mode_m1 == 4

        % Update state equations
        x_m1 = [q0_m1 + Ts*(v0hist_m1(1));
                q1_m1 + Ts*(v1hist_m1(1));
                v0_m1  + Ts*(-1/m0_m1(1)*k0_m1(1)*q0hist_m1(1) + 1/m0_m1(1)*u0hist_m1(1) + 1/m0_m1(1)*rfr_obs(1))
                v1_m1 + Ts*(1/m1_m1(1)*u1hist_m1(1))];

        states_m1 = [states_m1 x_m1];

         % Accumulated interest
        x_interest1(2) = q1_interest1 + Ts*v1hist_m1(1);
        q1accinterest1 = [q1accinterest1 x_interest1(2)];

        % Update state equations scheduled payments
        x0_m1 = [q0m1 + Ts*(v0m1);
                q1m1 + Ts*(v1m1);
                rc_m1*q0m1 - P_m1;
                rc_m1*q0m1];

        states0_m1 = [states0_m1 x0_m1];

    % Refinance
    elseif mode_m1 == 5 || mode_m1 == 6 || mode_m1 == 7

       % Reset contract rate
       rc_m1 = rm(1);

       % Update money borrowed
       A_ref_m1 = x_m1(1);
       A0_m1 = A_ref_m1;

       % Update monthly cost
       P_m1 = A_ref_m1*(rc_m1*(1+rc_m1)^Tr)/(((1+rc_m1)^Tr)-1);

       % Jump in state
        x_m1=  [0;
            q1hist_m1(1) + ehist_m1(1);
```

```
            -q0hist_m1(1) ;
            ehist_m1(1)];

      states_m1 = [states_m1 x_m1];

        % Accumulated interest
        x_interest1(2) = q1_interest1 + Ts*ehist_m1(1);
        q1accinterest1 = [q1accinterest1 x_interest1(2)];


       % Update state equations scheduled payments
       x0_m1 = [q0m1 + Ts*(v0m1);
                q1m1 + Ts*(v1m1);
                rc_m1*q0m1 - P_m1;
                rc_m1*q0m1];

        states0_m1 = [states0_m1 x0_m1];

      % Contract reset
      x_m1(1) = A_ref_m1;
      x_m1(2) = 0;
      x_m1(3) = -P_m1 + rc_m1*A_ref_m1;
      x_m1(4) = rc_m1*A_ref_m1;

      states_m1 = [states_m1 x_m1];

      % Accumulated interest
      x_interest1(2) = q1_interest1 + Ts*x_m1(4);
      q1accinterest1 = [q1accinterest1 x_interest1(2)];

      % Update state equations scheduled payments
      x0_m1 = [q0m1 + Ts*(x_m1(3));
               q1m1 + Ts*(x_m1(4));
               rc_m1*A_ref_m1 - P_m1;
               rc_m1*A_ref_m1];

      states0_m1 = [states0_m1 x0_m1];

    % Relocation
    elseif mode_m1 == 8


       % Reset contract rate
       rc_m1 = rm(1);

       % Update money borrowed
       A_rel_m1 = x_m1(1);
```

```
        A0_m1 = A_rel_m1;

        % Update monthly cost
        P_m1 = A_rel_m1*(rc_m1*(1+rc_m1)^Tr)/(((1+rc_m1)^Tr)-1);

        % Jump in state
        x_m1=  [0;
             q1hist_m1(1);
             -q0hist_m1(1) ;
             0];

        states_m1 = [states_m1 x_m1];

        % Accumulated interest
        x_interest1(2) = q1_interest1;
        q1accinterest1 = [q1accinterest1 x_interest1(2)];

        % Update state equations scheduled payments
        x0_m1 = [q0m1 + Ts*(v0m1);
                 q1m1 + Ts*(v1m1);
                 rc_m1*q0m1 - P_m1;
                 rc_m1*q0m1];

         states0_m1 = [states0_m1 x0_m1];

       % Contract reset
       x_m1(1) = A_rel_m1;
       x_m1(2) = 0;
       x_m1(3) = -P_m1 + rc_m1*A_rel_m1;
       x_m1(4) = rc_m1*A_rel_m1;

       states_m1 = [states_m1 x_m1];

       % Accumulated interest
       x_interest1(2) = q1_interest1 + Ts*(x_m1(4));
       q1accinterest1 = [q1accinterest1 x_interest1(2)];

       % Update state equations scheduled payments
       x0_m1 = [q0m1 + Ts*(x_m1(3));
                q1m1 + Ts*(x_m1(4));
                rc_m1*A_rel_m1 - P_m1;
                rc_m1*A_rel_m1];

       states0_m1 = [states0_m1 x0_m1];

       % Reset to moved
       moved_m1 = 1;
```

```
    end


    % Mortgagor 2:

    % housing data
    if norm_housing_noise_m2(i) < P_average
        p_housing_m2 = 0;
    elseif norm_housing_noise_m2(i) > P_average
        p_housing_m2 = 1;
    end
    h0_m2 = p_housing_m2*ones(1,dim.N);

    % Initial reset variables
    P0_m2 = P_m2*ones(1,dim.N);
    A00_m2 = A0_m2*ones(1,dim.N);
    rc0_m2 = rc_m2*ones(1,dim.N);
    moved0_m2 = moved_m2*ones(1,dim.N);

    % Current states for scheduled payments
    q0m2 = x0_m2(1);
    q1m2 = x0_m2(2);
    v0m2 = x0_m2(3);
    v1m2 = x0_m2(4);

    % Current states
    q0_m2 = x_m2(1);
    q1_m2 = x_m2(2);
    v0_m2 = x_m2(3);
    v1_m2 = x_m2(4);

    % Accumulated interest
    q1_interest2 = x_interest2(2);

    % Prepayment penalty
    epsilon_m2 = penalty_extra + (max(rc0_m2(1)-r_m_obs(1),0)*Tr
        *max((x_m2(1)-rho*A00_m2(1)),0))*ones(1,dim.N);

    % Current contract costs
    Pcontract_m2 = P_m2*ones(1,dim.N);

    % Contract costs relocation
    Prelocation_m2 = x_m2(1).*(r_m_obs(1).*(1+r_m_obs(1)).^Tr)
        ./(((1+r_m_obs(1)).^Tr)-1)*ones(1,dim.N);

    % Contract costs refinancing
```

```
    Prefinancing_m2 = Prelocation_m2 + epsilon_m2;

    % Solve optimization problem mortgagor 2
    [solutions.mortgagor2,diagnose.mortgagor] = mortgagor{q0_m2,
       q1_m2,rfr,v1_m2,v0_m2,rfr_obs,epsilon_m2,Pcontract_m2,
       Prefinancing_m2,Prelocation_m2,rc0_m2,P0_m2,A00_m2,
       moved0_m2,h0_m2,m0_m2,m1_m2,k0_m2,q1_interest2,R_m2};

    % Retrieve solutions mortgagor 2
    v0hist_m2 = solutions.mortgagor2{1}(1);
    v1hist_m2 = solutions.mortgagor2{5}(1);
    q0hist_m2 = solutions.mortgagor2{3}(1);
    q1hist_m2 = solutions.mortgagor2{6}(1);
    u0hist_m2 = solutions.mortgagor2{8}(1);
    u1hist_m2 = solutions.mortgagor2{9}(1);
    mode_m2  = solutions.mortgagor2{7}(1);
    ehist_m2 = solutions.mortgagor2{11}(1);
    qinterest2 = solutions.mortgagor2{16}(1);

    % Mortgage payments: scheduled payments and partial
       prepayments
    if  mode_m2 == 1

        % Update state equations
        x_m2 = [q0_m2 + Ts*(v0hist_m2(1));
                q1_m2 + Ts*(v1hist_m2(1));
                v0_m2  + Ts*(-1/m0_m2(1)*k0_m2(1)*q0hist_m2(1) +
                    1/m0_m2(1)*u0hist_m2(1) + 1/m0_m2(1)*rfr_obs
                   (1))
                v1_m2 + Ts*(1/m1_m2(1)*u1hist_m2(1))];

        states_m2 = [states_m2 x_m2];

       % Accumulated interest
       x_interest2(2) = q1_interest2 + Ts*v1hist_m2(1);
       q1accinterest2 = [q1accinterest2 x_interest2(2)];


       % Update state equations scheduled payments
       x0_m2 = [q0m2 + Ts*(v0m2);
                q1m2 + Ts*(v1m2);
                rc_m2*q0m2 - P_m2;
                rc_m2*q0m2];

       states0_m2 = [states0_m2 x0_m2];

    % Mortgage porting
```

```
    elseif mode_m2 == 2

        % Update state equations
        x_m2 = [q0_m2 + Ts*(v0hist_m2(1));
                q1_m2 + Ts*(v1hist_m2(1));
                v0_m2  + Ts*(-1/m0_m2(1)*k0_m2(1)*q0hist_m2(1) + 1/m0_m2(1)*u0hist_m2(1) + 1/m0_m2(1)*rfr_obs(1))
                v1_m2 + Ts*(1/m1_m2(1)*u1hist_m2(1))];

        states_m2 = [states_m2 x_m2];

       % Accumulated interest
       x_interest2(2) = q1_interest2 + Ts*v1hist_m2(1);
       q1accinterest2 = [q1accinterest2 x_interest2(2)];

       % Update state equations scheduled payments
       x0_m2 = [q0m2 + Ts*(v0m2);
                q1m2 + Ts*(v1m2);
                rc_m2*q0m2 - P_m2;
                rc_m2*q0m2];

        states0_m2 = [states0_m2 x0_m2];

        % Reset to moved
        moved_m2 = 1;

    % Mortgage payments: scheduled payments and partial prepayments
    % and moved
    elseif mode_m2 == 3 || mode_m2 == 4

        % Update state equations
        x_m2 = [q0_m2 + Ts*(v0hist_m2(1));
                q1_m2 + Ts*(v1hist_m2(1));
                v0_m2  + Ts*(-1/m0_m2(1)*k0_m2(1)*q0hist_m2(1) + 1/m0_m2(1)*u0hist_m2(1) + 1/m0_m2(1)*rfr_obs(1))
                v1_m2 + Ts*(1/m1_m2(1)*u1hist_m2(1))];

        states_m2 = [states_m2 x_m2];

       % Accumulated interest
       x_interest2(2) = q1_interest2 + Ts*v1hist_m2(1);
       q1accinterest2 = [q1accinterest2 x_interest2(2)];

       % Update state equations scheduled payments
```

```
        x0_m2 = [q0m2 + Ts*(v0m2);
                 q1m2 + Ts*(v1m2);
                 rc_m2*q0m2 - P_m2;
                 rc_m2*q0m2];

         states0_m2 = [states0_m2 x0_m2];

    % Refinance
    elseif mode_m2 == 5 || mode_m2 == 6 || mode_m2 == 7

       % Reset contract rate
       rc_m2 = rm(1);

       % Update money borrowed
       A_ref_m2 = x_m2(1);
       A0_m2 = A_ref_m2;

       % Update monthly cost
       P_m2 = A_ref_m2*(rc_m2*(1+rc_m2)^Tr)/(((1+rc_m2)^Tr)-1);

       % Jump in state
        x_m2=  [0;
            q1hist_m2(1) + ehist_m2(1);
            -q0hist_m2(1) ;
            ehist_m2(1)];

      states_m2 = [states_m2 x_m2];

       % Accumulated interest
       x_interest2(2) = q1_interest2 + Ts*ehist_m2(1);
       q1accinterest2 = [q1accinterest2 x_interest2(2)];

       % Update state equations scheduled payments
       x0_m2 = [q0m2 + Ts*(v0m2);
                q1m2 + Ts*(v1m2);
                rc_m2*q0m2 - P_m2;
                rc_m2*q0m2];

        states0_m2 = [states0_m2 x0_m2];

      % Contract reset
      x_m2(1) = A_ref_m2;
      x_m2(2) = 0;
      x_m2(3) = -P_m2 + rc_m2*A_ref_m2;
      x_m2(4) = rc_m2*A_ref_m2;

      states_m2 = [states_m2 x_m2];
```

```

      % Accumulated interest
      x_interest2(2) = q1_interest2 + Ts*x_m2(4);
      q1accinterest2 = [q1accinterest2 x_interest2(2)];

      % Update state equations scheduled payments
      x0_m2 = [q0m2 + Ts*(x_m2(3));
               q1m2 + Ts*(x_m2(4));
               rc_m2*A_ref_m2 - P_m2;
               rc_m2*A_ref_m2];

       states0_m2 = [states0_m2 x0_m2];

    % Relocation
    elseif mode_m2 == 8


       % Reset contract rate
       rc_m2 = rm(1);

       % Update money borrowed
       A_rel_m2 = x_m2(1);
       A0_m2 = A_rel_m2;

       % Update monthly cost
       P_m2 = A_rel_m2*(rc_m2*(1+rc_m2)^Tr)/(((1+rc_m2)^Tr)-1);

       % Jump in state
       x_m2=  [0;
            q1hist_m2(1);
            -q0hist_m2(1);
            0];

       states_m2 = [states_m2 x_m2];

       % Accumulated interest
       x_interest2(2) = q1_interest2;
       q1accinterest2 = [q1accinterest2 x_interest2(2)];

       % Update state equations scheduled payments
       x0_m2 = [q0m2 + Ts*(v0m2);
                q1m2 + Ts*(v1m2);
                rc_m2*q0m2 - P_m2;
                rc_m2*q0m2];

        states0_m2 = [states0_m2 x0_m2];

```

```
        % Contract reset
        x_m2(1) = A_rel_m2;
        x_m2(2) = 0;
        x_m2(3) = -P_m2 + rc_m2*A_rel_m2;
        x_m2(4) = rc_m2*A_rel_m2;

        states_m2 = [states_m2 x_m2];

        % Accumulated interest
        x_interest2(2) = q1_interest2 + Ts*x_m2(4);
        q1accinterest2 = [q1accinterest2 x_interest2(2)];

         % Update state equations scheduled payments
         x0_m2 = [q0m2 + Ts*(x_m2(3));
                  q1m2 + Ts*(x_m2(4));
                  rc_m2*A_rel_m2 - P_m2;
                  rc_m2*A_rel_m2];

          states0_m2 = [states0_m2 x0_m2];

        % Reset to moved
        m_m2 = 1;

      end


      % Mortgagor 3:

      % housing data
      if norm_housing_noise_m3(i) < P_average
         p_housing_m3 = 0;
      elseif norm_housing_noise_m3(i) > P_average
         p_housing_m3 = 1;
      end
      h0_m3 = p_housing_m3*ones(1,dim.N);

      % Initial reset variables
      P0_m3 = P_m3*ones(1,dim.N);
      A00_m3 = A0_m3*ones(1,dim.N);
      rc0_m3 = rc_m3*ones(1,dim.N);
      moved0_m3 = moved_m3*ones(1,dim.N);

      % Current states for scheduled payments
      q0m3 = x0_m3(1);
      q1m3 = x0_m3(2);
      v0m3 = x0_m3(3);
      v1m3 = x0_m3(4);
```

```

    % Current states
    q0_m3 = x_m3(1);
    q1_m3 = x_m3(2);
    v0_m3 = x_m3(3);
    v1_m3 = x_m3(4);

    % Accumulated interest
    q1_interest3 = x_interest3(2);

    % Prepayment penalty
    epsilon_m3 = penalty_extra + (max(rc0_m3(1)-r_m_obs(1),0)*Tr
        *max((x_m3(1)-rho*A00_m3(1)),0))*ones(1,dim.N);

    % Current contract costs
    Pcontract_m3 = P_m3*ones(1,dim.N);

    % Contract costs relocation
    Prelocation_m3 = x_m3(1).*(r_m_obs(1).*(1+r_m_obs(1)).^Tr)
        ./(((1+r_m_obs(1)).^Tr)-1)*ones(1,dim.N);

    % Contract costs refinancing
    Prefinancing_m3 = Prelocation_m3 + epsilon_m3;

    % Solve optimization problem mortgagor 2
    [solutions.mortgagor3,diagnose.mortgagor] = mortgagor{q0_m3,
        q1_m3,rfr,v1_m3,v0_m3,rfr_obs,epsilon_m3,Pcontract_m3,
        Prefinancing_m3,Prelocation_m3,rc0_m3,P0_m3,A00_m3,
        moved0_m3,h0_m3,m0_m3,m1_m3,k0_m3,q1_interest3,R_m3};

    % Retrieve solutions mortgagor 2
    v0hist_m3 = solutions.mortgagor3{1}(1);
    v1hist_m3 = solutions.mortgagor3{5}(1);
    q0hist_m3 = solutions.mortgagor3{3}(1);
    q1hist_m3 = solutions.mortgagor3{6}(1);
    u0hist_m3 = solutions.mortgagor3{8}(1);
    u1hist_m3 = solutions.mortgagor3{9}(1);
    mode_m3  = solutions.mortgagor3{7}(1);
    ehist_m3 = solutions.mortgagor3{11}(1);
    qinterest3 = solutions.mortgagor3{16}(1);

    % Mortgage payments: scheduled payments and partial
        prepayments
    if  mode_m3 == 1

        % Update state equations
        x_m3 = [q0_m3 + Ts*(v0hist_m3(1));
```

```
                q1_m3 + Ts*(v1hist_m3(1));
                v0_m3  + Ts*(-1/m0_m3(1)*k0_m3(1)*q0hist_m3(1) +
                    1/m0_m3(1)*u0hist_m3(1) + 1/m0_m3(1)*rfr_obs
                   (1))
                v1_m3 + Ts*(1/m1_m3(1)*u1hist_m3(1))];

        states_m3 = [states_m3 x_m3];

        % Accumulated interest
       x_interest3(2) = q1_interest3 + Ts*v1hist_m3(1);
       q1accinterest3 = [q1accinterest3 x_interest3(2)];

       % Update state equations scheduled payments
       x0_m3 = [q0m3 + Ts*(v0m3);
                q1m3 + Ts*(v1m3);
                rc_m3*q0m3 - P_m3;
                rc_m3*q0m3];

        states0_m3 = [states0_m3 x0_m3];

    % Mortgage porting
    elseif mode_m3 == 2

        % Update state equations
        x_m3 = [q0_m3 + Ts*(v0hist_m3(1));
                q1_m3 + Ts*(v1hist_m3(1));
                v0_m3  + Ts*(-1/m0_m3(1)*k0_m3(1)*q0hist_m3(1) +
                    1/m0_m3(1)*u0hist_m3(1) + 1/m0_m3(1)*rfr_obs
                   (1))
                v1_m3 + Ts*(1/m1_m3(1)*u1hist_m3(1))];

        states_m3 = [states_m3 x_m3];

       % Accumulated interest
       x_interest3(2) = q1_interest3 + Ts*v1hist_m3(1);
       q1accinterest3 = [q1accinterest3 x_interest3(2)];

       % Update state equations scheduled payments
       x0_m3 = [q0m3 + Ts*(v0m3);
                q1m3 + Ts*(v1m3);
                rc_m3*q0m3 - P_m3;
                rc_m3*q0m3];

       states0_m3 = [states0_m3 x0_m3];

        % Reset to moved
        moved_m3 = 1;
```

```matlab

    % Mortgage payments: scheduled payments and partial
       prepayments
    % and moved
    elseif mode_m3 == 3 || mode_m3 == 4

        % Update state equations
        x_m3 = [q0_m3 + Ts*(v0hist_m3(1));
                q1_m3 + Ts*(v1hist_m3(1));
                v0_m3  + Ts*(-1/m0_m3(1)*k0_m3(1)*q0hist_m3(1) +
                    1/m0_m3(1)*u0hist_m3(1) + 1/m0_m3(1)*rfr_obs
                   (1))
                v1_m3 + Ts*(1/m1_m3(1)*u1hist_m3(1))];

        states_m3 = [states_m3 x_m3];

       % Accumulated interest
       x_interest3(2) = q1_interest3 + Ts*v1hist_m3(1);
       q1accinterest3 = [q1accinterest3 x_interest3(2)];

       % Update state equations scheduled payments
       x0_m3 = [q0m3 + Ts*(v0m3);
                q1m3 + Ts*(v1m3);
                rc_m3*q0m3 - P_m3;
                rc_m3*q0m3];

       states0_m3 = [states0_m3 x0_m3];

    % Refinance
    elseif mode_m3 == 5 || mode_m3 == 6 || mode_m3 == 7

       % Reset contract rate
       rc_m3 = rm(1);

       % Update money borrowed
       A_ref_m3 = x_m3(1);
       A0_m3 = A_ref_m3;

       % Update monthly cost
       P_m3 = A_ref_m3*(rc_m3*(1+rc_m3)^Tr)/(((1+rc_m3)^Tr)-1);

       % Jump in state
       x_m3=  [0;
                q1hist_m3(1) + ehist_m3(1);
                -q0hist_m3(1) ;
                ehist_m3(1)];

```

```
       states_m3 = [states_m3 x_m3];

       % Accumulated interest
       x_interest3(2) = q1_interest3 + Ts*ehist_m3(1);
       q1accinterest3 = [q1accinterest3 x_interest3(2)];

       % Update state equations scheduled payments
       x0_m3 = [q0m3 + Ts*(v0m3);
                q1m3 + Ts*(v1m3);
                rc_m3*q0m3 - P_m3;
                rc_m3*q0m3];

       states0_m3 = [states0_m3 x0_m3];

      % Contract reset
      x_m3(1) = A_ref_m3;
      x_m3(2) = 0;
      x_m3(3) = -P_m3 + rc_m3*A_ref_m3;
      x_m3(4) = rc_m3*A_ref_m3;

      states_m3 = [states_m3 x_m3];

      % Accumulated interest
      x_interest3(2) = q1_interest3 + Ts*x_m3(4);
      q1accinterest3 = [q1accinterest3 x_interest3(2)];

      % Update state equations scheduled payments
      x0_m3 = [q0m3 + Ts*(x_m3(3));
               q1m3 + Ts*(x_m3(4));
               rc_m3*A_ref_m3 - P_m3;
               rc_m3*A_ref_m3];

      states0_m3 = [states0_m3 x0_m3];

    % Relocation
    elseif mode_m3 == 8


       % Reset contract rate
       rc_m3 = rm(1);

       % Update money borrowed
       A_rel_m3 = x_m3(1);
       A0_m3 = A_rel_m3;

       % Update monthly cost
       P_m3 = A_rel_m3*(rc_m3*(1+rc_m3)^Tr)/(((1+rc_m3)^Tr)-1);
```

```
        % Jump in state
        x_m3 =  [0;
                 q1hist_m3(1);
                 -q0hist_m3(1);
                 0];

        states_m3 = [states_m3 x_m3];

        % Accumulated interest
        x_interest3(2) = q1_interest3;
        q1accinterest3 = [q1accinterest3 x_interest3(2)];

        % Update state equations scheduled payments
        x0_m3 = [q0m3 + Ts*(v0m3);
                 q1m3 + Ts*(v1m3);
                 rc_m3*q0m3 - P_m3;
                 rc_m3*q0m3];

        states0_m3 = [states0_m3 x0_m3];

       % Contract reset
       x_m3(1) = A_rel_m3;
       x_m3(2) = 0;
       x_m3(3) = -P_m3 + rc_m3*A_rel_m3;
       x_m3(4) = rc_m3*A_rel_m3;

       states_m3 = [states_m3 x_m3];

       % Accumulated interest
       x_interest3(2) = q1_interest3 + Ts*x_m3(4);
       q1accinterest3 = [q1accinterest3 x_interest3(2)];

       % Update state equations scheduled payments
       x0_m3 = [q0m3 + Ts*(x_m3(3));
                 q1m3 + Ts*(x_m3(4));
                 rc_m3*A_rel_m3 - P_m3;
                 rc_m3*A_rel_m3];

       states0_m3 = [states0_m3 x0_m3];

       % Reset to moved
       moved_m3 = 1;

        end
```

```
end
```

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Glossary

List of Acronyms

MPC	Model Predictive Control
CPR	Conditional Prepayment Rate
ECB	European Central Bank
IRS	Interest Rate Swaps
MNL	Multinomial Logit
MLE	Maximum Likelihood Estimation
Euribor	Euro InterBank Offer Rate
MPC	Model Predictive Control
MIP	Mixed-Integer Programming
MIQP	Mixed-Integer Quadratic Programming
EMPC	Economic Model Predictive Controller
QP	Quadratic Programming
SMM	Single Monthly Mortality
CPR	Conditional Prepayment Rate
RTO	Real-Time Optimization
PID	Proportional–Integral–Derivative
RMPC	Robust Model Predictive Control
SMPC	Stochastic Model Predictive Control
FICO	Fair Isaac Credit Organization

List of Symbols

ρ	Partial prepayment percentage

ε	Prepayment penalty
β	Scaling factor risk-free rates
$\boldsymbol{x}_{\text{RL}}^{+}$	Value state after a full prepayment due to relocation
$\hat{d}$	Predicted risk-free rates
$\boldsymbol{x}$	State variable
A_0	Initial level of money borrowed
F_0	Risk appetite principal repayments
F_1	Risk appetite interest payments
$F_{0,C}$	Appetite to reduce debt
h	Housing incentive
k_0	Creditworthiness
L	Lagrangian
m	Discrete state for moving
m_0	Risk aversion principal repayments
m_1	Risk aversion interest payments
P	Monthly payment
p_0	Risk factor principal repayments
p_1	Risk factor interest payments
P_{contract}	Monthly payment current contract
$P_{\text{refinance}}$	Monthly payment new contract with refinancing costs
$P_{\text{relocation}}$	Monthly payment new contract
q_0	Principal balance
q_1	Accumulated interest
R	Rationality mortgagor
r	Risk-free rate
r_c	Contractual interest rate
$r_m(t)$	Market mortgage rates
S	Action
T	Maturity
t	Time
T_h	Prediction horizon
T_r	Remaining mortgage term
T_s	Sampling time
u_0	Risk appetite mortgagor principal repayments
u_1	Risk appetite mortgagor interest payments
v	Velocity
v_0	Principal repayment
v_1	Interest payment
x_0	Initial state
q	Position
$\boldsymbol{x}_{\text{RF}}^{+}$	Value state after a full prepayment due to refinancing
k	Discrete time