# NON-LINEAR ANALYSIS OF FRICTIONAL MATERIALS

# PROEFSCHRIFT

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DR. IR. P.A. VERMEER heeft als begeleider in hoge mate bijgedragen aan het totstandkomen van het proefschrift. Het College van Dekanen heeft hem als zodanig aangewezen.

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### 1. INTRODUCTION

The past decade has witnessed the development of sophisticated constitutive models for engineering materials. An overwhelming number of models which aim at describing the mechanical behaviour of steel, soils, concrete, rock etc. has been put forward. At present, the evolution has come to a stage at which the development of constitutive laws itself is no longer the limiting factor in an engineering analysis. Rather, a paucity of experimental data to support the models and an inability to obtain a converged solution in a numerical analysis pose the major limitations. Indeed, most civil engineering materials (soils, rock, concrete) are difficult materials in the sense that complicated constitutive models are needed to describe their basic characteristics. Moreover, when we scrutinise such models, we observe that neither stability nor uniqueness is guaranteed for all load levels, but that these favourable properties can only be established below a threshold load level. These instabilities are also encountered in a numerical analysis, and even when we are able to obtain a converged solution we often achieve this goal at the sacrifice of extremely high computational costs.

## 1.1. Aims and scope of this study

In this study we shall develop constitutive models for continua and numerical techniques which can be used in the analysis of granular materials. Here, the conception of a continuum is taken rather wide as we will also consider cracked concrete as a continuum. As in the sequel of this study, we shall understand by 'granular materials', materials with a granular structure, either loose like sand or cemented like concrete, sandstone or rock. A common characteristic of these materials is that their strength significantly depends upon the stress level, or in other words, the behaviour of these materials is different in compression than in tension.

Ideally, the development of constitutive models and their application in numerical programs should go hand in hand with each other. Yet, we observe an increasing discrepancy between the relatively simple models which are employed by numerical analysts and the often very complicated material models developed by materials scientists. This study aims to bridge this gap and so the development of the constitutive models will be such that on the one hand the basic characteristics which we observe in testing devices can be represented, but that on the other hand the degree of sophistication does not preclude successful use of the models - 4 -

When we consider constitutive models for granular materials, we observe that uniqueness and stability are guaranteed only below a threshold level, unless the models are so rough that they cannot reasonably represent the material behaviour. For instance, when we try to construct a plasticity model of the response of a granular material in triaxial compression, we are forced to abandon the associated flow rule of classical plasticity as it is not able to describe the inelastic volume changes which are measured in experiments. Accordingly, Drucker's Postulate is no longer valid and non-unique solutions are possible already in the hardening regime. A similar situation occurs with respect to the cohesive strength of cemented granular materials. Here, continued loading results in micro-cracking and ultimately in a degradation of the strength with accumulation of inelastic deformation (softening).

The lack of uniqueness and stability above some threshold level of loading is also reflected in a numerical analysis and we encounter bifurcations and softening branches also in discretized systems. Indeed, the spatial discretization, the numerical integration of the stress-strain law, the iterative solution procedure and so on tend to destabilize the numerical solution already before bifurcation or limit points of the underlying continuum are encountered. The iterative solution procedure then breaks down and the structure is said to have "failed". In this study we adopt the philosophy that such a judgement is not adequate and that an analysis of the post-bifurcation or the post-limit path is required for a proper assessment of the structural behaviour. Tracing of these paths is notoriously difficult and is only feasible if a constitutive model is employed which strikes a balance between simplicity and an accurate description of the material behaviour, if the numerical integration of the differential stress-strain law does not entail significant inaccuracies, if the mechanical system which arises upon discretization of the underlying continuum resembles the original system closely enough and if the iterative solution techniques permit tracing such paths.

# 1.2. Contents of this study

This study starts with a brief description of the basic kinematic and static relationships of continuum mechanics whereby restriction is made to small displacement gradients. Equilibrium is formulated by means of the virtual work principle in a form which is attributable to Piola. The class of constitutive laws to which we will confine our attention is discussed. In particular, we will restrict attention to rate type laws and we will not employ functional type constitutive laws. Furthermore, conditions for uniqueness and stability under dead loading are formulated.

Chapter 3 addresses the numerical representation of these laws of continuum mechanics. The stability condition is elaborated for discrete systems, and an interpretation of unstable behaviour is given. It appears that the theory for non-symmetric systems is much less satisfactory than that for symmetric systems as the latter not only allows for the establishment of a sufficient condition, but also of a necessary condition for instability under dead loading. It is furthermore shown that the response of discretized systems may differ fundamentally from the response of the underlying continuum.

In Chapter 4, we will outline the constitutive models employed in this study. The most important feature is perhaps the addition of strain rates due to the different non-linear phenomena. Fracture in cohesive granular materials is treated using a smeared concept and a new model which permits non-orthogonal cracks is outlined. For compressive loadings, a hardening-softening plasticity model with a non-associated flow rule is used. The fracture model and the plasticity model are then combined in a plastic-fracture model and some consequences of the use of a nonassociated flow rule and of strain-softening, both of which are employed in the model, are reviewed.

Chapter 5 is concerned with the derivation of sound numerical algorithms for the constitutive models which were discussed in the preceding chapter. On a local level, particular attention is devoted to singularities in a yield surface and to the combination of fracture and plasticity. On a structural level, techniques are discussed which permit overcoming limit points in an economic and elegant manner and which permit branching off on alternative equilibrium branches at a bifurcation point. These techniques are combined with fast iterative procedures to achieve convergence within a loading step. The last two chapters present applications to some typical bifurcation and limit problems in soil and concrete mechanics. We will concentrate on the ability to trace post-bifurcation and post-failure branches and we will give some solutions not presented before.

#### 1.3. Notation

In this study we will use tensor as well as matrix-vector notation, where the former notation will primarily be used for continuous systems while the latter notation is usually adopted for the description of discretized systems. Restriction will be made to Cartesian tensors in order not to obscure the physics behind mathematical expressions. Hence, all indices will be lower indices, and the summation convention is adopted for repeated latin subscripts. Matrices and vector are distinguished by boldfaced symbols. It is furthermore noted that a global list of symbols is not included because several symbols have more than one meaning. Instead, symbols are defined when they first appear in the text.

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# 2. FORMULATION OF THE BOUNDARY VALUE PROBLEM

In this chapter we shall derive the differential equations for the nonlinear behaviour of a continuum. We shall restrict ourselves to a class of materials for which the material response can be formulated using a local rate law. Frictional materials exhibit all sorts of 'undesirable' phenomena like softening, cracking, dilatancy and so on. This implies that stability and uniqueness are guaranteed only below a certain load level. Therefore, conditions for stability and bifurcation are discussed in detail.

# 2.1. Kinematic and static preliminaries

In the light of the derivation of the finite element equations in the next chapter, equilibrium is most conveniently expressed via the principle of virtual work. In particular, we shall employ a version of this principle which is attributable to Piola<sup>13</sup> and which reads;

"The virtual work of the external forces is in case of equilibrium equal to zero for all virtual displacements which yield no deformations".

Mathematically, this is expressed by the condition that

$$\int_{S} t_i \delta u_i dS + \int_{V} \rho g_i \delta u_i dV = 0 \tag{2.1}$$

for all virtual displacements  $\delta u_i$  subject to the subsidiary condition that the associated virtual strain field  $\delta \varepsilon_{ii}$ 

$$\delta \varepsilon_{ij} = \frac{1}{2} \left[ \frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i} \right]$$
(2.2)

vanishes  $(\delta \varepsilon_{ij} = 0)$  for all points of V. Here,  $t_i$  are the boundary tractions,  $\rho$  is the specific mass of the material,  $g_i$  is the gravity acceleration and  $x_i$  are spatial coordinates. In this study, attention is restricted to static problems, and consequently, a term due to inertia forces has been omitted in equation (2.1).

Introducing a tensor field of Lagrangean multipliers  $\sigma_{ij}$ , we can comprise equations (2.1) and (2.2) to a single equation:

$$\int_{S} t_{i} \delta u_{i} dS + \int_{V} \rho g_{i} \delta u_{i} dV - \int_{V} \sigma_{ij} \delta \varepsilon_{ij} dV = 0$$
(2.3)

It follows that equation (2.3) indeed expresses equilibrium as inserting equation (2.2) and application of the divergence theorem to equation (2.3) gives:

$$\int_{S} (t_i - \sigma_{ij} n_j) \delta u_i dS + \int_{V} \left[ \rho g_i + \frac{\partial \sigma_{ij}}{\partial x_j} \right] \delta u_i dV = 0$$
(2.4)

with  $n_j$  the outward normal to the surface of the body. A sufficient condition for this equation to hold for any virtual displacement  $\delta u_i$  is that

$$\rho g_i + \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \tag{2.5}$$

in the interior of the body and

 $t_i - \sigma_{ij} n_j = 0 \tag{2.6}$ 

on the boundary S. We can thus identify the Lagrangean multipliers  $\sigma_{ij}$  with the so-called Cauchy stress tensor  $^{10,11,13,111}$ 

For deformation problems in solid mechanics, equation (2.3) is not very suitable as it takes the current, yet unknown configuration as the reference configuration. It is more convenient to take a previous configuration as reference configuration, e.g. the original configuration of the undeformed body or some intermediate configuration. Such a choice is reasonable as long as the inelastic strains remain small. Indeed, we will assume throughout this study that all strains, inelastic and elastic, remain small. Transforming equation (2.3) to the reference configuration  $V^0$  yields

$$\int_{S^{0}} t_{i}^{0} \delta u_{i} dS^{0} + \int_{V^{0}} \rho^{0} g_{i} \delta u_{i} dV^{0} - \int_{V^{0}} S_{ij} \delta \gamma_{ij} dV^{0} = 0$$
(2.7)

with  $\gamma_{ij}$  the Green-Lagrange strain tensor

$$\gamma_{ij} = \frac{1}{2} \left( \frac{\partial x_k}{\partial \xi_i} \frac{\partial x_k}{\partial \xi_j} - \delta_{ij} \right)$$
(2.8)

so that

$$\delta \gamma_{ij} = \frac{1}{2} \left[ \frac{\partial x_k}{\partial \xi_i} \frac{\partial \delta x_k}{\partial \xi_j} + \frac{\partial \delta x_k}{\partial \xi_i} \frac{\partial x_k}{\partial \xi_j} \right]$$
(2.9)

and  $S_{ij}$  the second Piola-Kirchhoff stress tensor which is related to the Cauchy stress tensor by

$$S_{ij} = J \frac{\partial \xi_i}{\partial x_k} \sigma_{kl} \frac{\partial \xi_j}{\partial x_l}$$
(2.10)

 $\xi_i$  are material coordinates which are related to the spatial coordinates  $x_i$  by

$$x_i = \xi_i + u_i \tag{2.11}$$

and J represents the functional determinant of the mapping  $x_i = x_i(\xi_j)$ .  $t_i^0$  is the so-called nominal traction and represents the current traction, but referred to the reference configuration and  $\rho^0$  is the specific mass of the material in the reference configuration  $V^0$ .

In order to clarify the physical meaning of equation (2.7) and the employed stress measure, we rewrite equation (2.7) as follows:

$$\int_{S^0} t_i^0 \delta u_i dS^0 + \int_{V^0} \rho^0 g_i \delta u_i dV^0 - \int_{V^0} \Sigma_{ij} \frac{\partial \delta u_i}{\partial \xi_j} dV^0 = 0$$
(2.12)

where the first Piola-Kirchhoff stress tensor  $\Sigma_{ij}$  ,

$$\Sigma_{ij} = J \sigma_{ik} \frac{\partial \xi_j}{\partial x_k} \tag{2.13}$$

has been introduced. Proceeding in the same way as when formulating equilibrium with respect to the current configuration, we can derive that

$$\rho^0 g_i + \frac{\partial \Sigma_{ij}}{\partial \xi_j} = 0 \tag{2.14}$$

in the interior of the body and

$$t_i^0 - \Sigma_{ij} n_j^0 = 0$$
 (2.15)

on the boundary  $S^0$ .  $n_j^0$  is the outward normal to the surface of the body in the reference configuration.

Inelastic processes are often path-dependent and the stress tensor must then be integrated along the loading path. To this end we rewrite equation (2.7) as follows

$$\int_{V^0} \left[ S_{ij}^{t-\Delta t} + \int_{t-\Delta t}^t \dot{S}_{ij}(\tau) d\tau \right] \delta\gamma_{ij} dV^0 - \int_{S^0} t_i^0 \delta u_i dS^0 - \int_{V^0} \rho^0 g_i \delta u_i dV^0 = 0 \quad (2.16)$$

with  $\dot{S}_{ij}$  the material derivative of the stress tensor and  $S_{ij}^{t-\Delta t}$  the stress tensor at  $\tau = t - \Delta t$ . It is now important to recall that although all quantities are referred to some past reference configuration, the virtual work equation has been set up at  $\tau = t$ . This implies that the variation  $\delta \gamma_{ij}$  has to be evaluated for  $\tau = t$  and consequently, the spatial coordinates  $x_k$ which enter equation (2.9) are coordinates at  $\tau = t$ . Using identity (2.11) and operating on the displacements in a similar manner as on the stress tensor, namely by putting

$$u_i^t = u_i^{t-\Delta t} + \Delta u_i \tag{2.17}$$

we can rewrite  $\delta \gamma_{ij}$  as follows:

$$\delta \gamma_{ij} = \delta \gamma'_{ij} + \frac{1}{2} \left[ \frac{\partial \delta \Delta u_k}{\delta \xi_i} \frac{\partial \Delta u_k}{\delta \xi_j} + \frac{\partial \Delta u_k}{\delta \xi_i} \frac{\partial \delta \Delta u_k}{\delta \xi_j} \right]$$
(2.18)

with

$$\delta\gamma'_{ij} = \frac{1}{2} \left[ \frac{\partial\delta\Delta u_i}{\delta\xi_j} + \frac{\partial\delta\Delta u_j}{\delta\xi_i} + \frac{\partial\delta\Delta u_k}{\delta\xi_i} \frac{\partial u_k^{t-\Delta t}}{\delta\xi_j} + \frac{\partial u_k^{t-\Delta t}}{\delta\xi_i} \frac{\partial\delta\Delta u_k}{\delta\xi_j} \right] (2.19)$$

Inserting these identities in the virtual work expression (2.16), we obtain:

$$\int_{V^0 t - \Delta t} \int_{S^0}^{t} \dot{S}_{ij}(\tau) d\tau \delta\gamma_{ij} dV^0 + \int_{V^0} \frac{\partial \delta \Delta u_k}{\delta \xi_i} S_{ij}^{t - \Delta t} \frac{\partial \Delta u_k}{\delta \xi_j} dV^0$$
$$= \int_{S^0} t_i^0 \delta u_i dS^0 + \int_{V^0} \rho^0 g_i \delta u_i dV^0 - \int_{V_0} S_{ij}^{t - \Delta t} \delta\gamma'_{ij} dV^0$$
(2.20)

This equation can only be developed further if we make assumptions on how  $\dot{S}_{ij}$  depends on the strain rate and the strain history. This matter will be considered in the next section and the evaluation of the integrand will be treated in Chapter 5.

The derivative  $\dot{S}_{ij}$  of the second Piola-Kirchhoff stress tensor merits some further discussion as the issue of the choice of a proper stress rate is somewhat controversial in continuum mechanics. With definition (2.10) we obtain

$$\dot{S}_{ij} = J \frac{\partial \xi_i}{\partial x_k} \left[ \dot{\sigma}_{kl} + \sigma_{kl} \frac{\partial \dot{u}_m}{\partial x_m} - \sigma_{ml} \frac{\partial \dot{u}_k}{\partial x_m} - \sigma_{km} \frac{\partial \dot{u}_l}{\partial x_m} \right] \frac{\partial \xi_j}{\partial x_l}$$
(2.21)

with  $\dot{u}_i$  the velocity of a material point,

$$\dot{u}_i = \dot{x}_i - \dot{\xi}_i \tag{2.22}$$

The stress rate  $\dot{S}_{ij}$  is an example of an objective stress rate. It contains the so-called Truesdell stress rate<sup>11</sup>

$$S_{ij}^{\dagger} = \dot{\sigma}_{ij} + \sigma_{ij} \frac{\partial \dot{u}_m}{\partial x_m} - \sigma_{mj} \frac{\partial \dot{u}_i}{\partial x_m} - \sigma_{im} \frac{\partial \dot{u}_j}{\partial x_m}$$
(2.23)

as a special case which is obtained when the reference configuration is instantaneously updated to coincide with the actual configuration because then  $\frac{\partial \xi_i}{\partial x_j} = \delta_{ij}$  and J=1. The above stress rates are merely examples of objective stress rates as there is an abundance of other possible choices. Prominent amongst these is the so-called Jaumann derivative of the Cauchy stress tensor

$$\sigma_{ij}^{v} = \dot{\sigma}_{ij} - \frac{1}{2} \sigma_{kj} \left( \frac{\partial \dot{u}_{i}}{\partial x_{k}} - \frac{\partial \dot{u}_{k}}{\partial x_{i}} \right) + \frac{1}{2} \sigma_{ik} \left( \frac{\partial \dot{u}_{k}}{\partial x_{j}} - \frac{\partial \dot{u}_{j}}{\partial x_{k}} \right)$$
(2.24)

the difference with the Truesdell stress rate being given by

$$S_{ij}^{\mathbf{v}} - \sigma_{ij}^{\mathbf{v}} = \sigma_{ij} \frac{\partial \dot{u}_k}{\partial x_k} - \frac{1}{2} \sigma_{kj} \left[ \frac{\partial \dot{u}_i}{\partial x_k} + \frac{\partial \dot{u}_k}{\partial x_i} \right] - \frac{1}{2} \sigma_{ik} \left[ \frac{\partial \dot{u}_k}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_k} \right]$$
(2.25)

We observe that objective stress rates differ merely by terms 'stress component times velocity gradient component'  $^{56}$ . In fact, it does not matter very much which objective stress rate is employed provided that the constitutive law is properly adapted to the choice of the stress rate tensor.

## 2.2. Constitutive equations

Broadly speaking, there are two ways to formulate constitutive laws<sup>78,111</sup>. The most general approach is probably to consider the stress in a material point with coordinates  $\xi_k$  to be a functional F of the deformation history of all points (with material coordinates  $\eta_k$ ) of the body B:

$$\sigma_{ij}(\xi_k, t) = \mathop{\mathbb{F}}_{\tau=0}^{\infty} \left[ x_m(\eta_m, t-\tau), \xi_k \right]$$
(2.26)

where  $\tau$  is the time parameter. Some caution should be exercised when referring to  $\tau$  as a time because for short-term loadings, time-dependent effects can be disregarded and  $\tau$  attains the role of a parameter which merely orders the loading process. In this study for instance, timedependent effects are not considered, but when we divide the (continuous) loading process in a number of finite load increments, we will speak of this discretization as a 'temporal discretization'.

The above concept can be simplified if we assume that only the neighbourhood of a material point influences the stress. Then,  $x_k$  can be developed in a Taylor series. The most simple approach is of course to retain only the linear term. In doing so, we can rewrite equation (2.26) as:

$$\sigma_{ij}(t) = \prod_{\tau=0}^{\infty} (\gamma_{ij}(\xi_k, t - \tau), \xi_k)$$
(2.27)

In Coleman's terminology<sup>27</sup>, such a material in which the stress in a material point is determined only by the strain history of the same material point, is called a 'simple solid'.

Especially the advent of digital computers and numerical techniques have rendered the functional approach less suitable for practical applications, as in a numerical analysis such a model requires the storage of the entire strain history, which clearly exceeds storage capacity even of modern computers. As this study primarily aims at developing tools for practical analyses of granular materials, we will follow the differential approach, in which constitutive laws are formulated via rate laws. In this approach, the stress rate tensor is a function of the stress tensor, the strain tensor, the strain rate tensor and a finite number of internal variables  $H_{\alpha}$ , which intrinsically reflect the strain history:

$$\dot{S}_{ij} = \dot{S}_{ij} \left( S_{kl}, \gamma_{kl}, \dot{\gamma}_{kl}, H_{\alpha} \right) \tag{2.28}$$

Furthermore, we will only consider the case that this expression is linear in the strain-rate tensor  $\dot{\gamma}_{kl}$ , so that

$$\dot{S}_{ij} = D_{ijkl} \left( S_{kl}, \gamma_{kl}, H_{\alpha} \right) \dot{\gamma}_{kl} \tag{2.29}$$

with  $D_{ijkl}$  a fourth-order tensor which contains the stiffness moduli and which is a function of  $S_{kl}$ , $\gamma_{kl}$  and  $H_{\alpha}$ . Equation (2.29) is a linear relationship between the stress-rate tensor and the strain-rate tensor of a particular material. In the sequel, we will only consider constitutive laws which are expressible in such a form. This means that for instance incrementally-nonlinear constitutive laws such as Valanis' endochronic model<sup>112</sup> are not considered. Also, non-local rate equations are excluded. Recently, there have been some attempts to model softening in concrete by means of non-local rate equations<sup>8,102</sup>, so exclusion of these laws is to a certain extent questionable.

Restricting the treatment to incrementally-linear, local rate laws, we can derive the governing field equation. To this end, we substitute equation (2.29) in equation (2.20). This gives:

$$\int_{V^0 t} \int_{-\Delta t}^{t} D_{ijkl} \dot{\gamma}_{kl} d\tau \delta \gamma_{ij} dV^0 + \int_{V^0} \frac{\partial \delta \Delta u_k}{\delta \xi_i} S_{ij}^{t-\Delta t} \frac{\partial \Delta u_k}{\delta \xi_j} dV^0$$

$$= \int_{S^0} t_i^0 \delta u_i dS^0 + \int_{V^0} \rho^0 g_i \delta u_i dV^0 - \int_{V_0} S_{ij}^{t-\Delta t} \delta \gamma'_{ij} dV^0 \qquad (2.30)$$

## 2.3. Stability and uniqueness

An equilibrium state is called stable if the response on a vanishingly small disturbance also remains vanishingly small<sup>25,56,63</sup>. Suppose now that we have an equilibrium state at  $\tau = t$  with a stress field  $\Sigma_{ij}$ . We consider an infinitesimal displacement field  $\delta u_i = \dot{u}_i \delta t$ . A stress rate  $\dot{\Sigma}_{ij}$  can be calculated by multiplying the velocity gradient  $\frac{\partial \dot{u}_i}{\partial x_j}$  with a stiffness tensor. All rates are referred to time  $\tau = t$  and we assume that the external forces do not depend on the position (dead loading). In an infinitesimal time  $\delta t$  the increase in internal energy minus the work of the external forces equals

(to second order)55,56

$$U = \int_{V} (\Sigma_{ij} + \frac{1}{2} \dot{\Sigma}_{ij} \delta t) \frac{\partial \dot{u}_i}{\partial x_j} \delta t dV - \int_{S} t_i \dot{u}_i \delta t dS - \int_{V} \rho g_i \dot{u}_i \delta t dV \quad (2.31)$$

along any kinematically admissible path which starts in the direction  $\dot{u}_i$ . Subtracting the equilibrium equation

$$\int_{V} \Sigma_{ij} \frac{\partial \dot{u}_{i}}{\partial x_{j}} \delta t dV - \int_{S} t_{i} \dot{u}_{i} \delta t dS - \int_{V} \rho g_{i} \dot{u}_{i} \delta t dV = 0$$
(2.32)

results in

$$U = \frac{1}{2} (\delta t)^2 \int_V \dot{\Sigma}_{ij} \frac{\partial \dot{u}_i}{\partial x_j} dV$$
(2.33)

We will henceforth assume that stability under dead loading is ensured if U is positive for all kinematically admissible velocity fields, while the equilibrium is unstable under dead loading if U becomes negative for at least one kinematically admissible velocity distribution. Put differently, the condition

$$\int_{V} \dot{\Sigma}_{ij} \frac{\partial \dot{u}_{i}}{\partial x_{j}} dV > 0$$
(2.34)

for all kinematically admissible velocity gradient distributions  $\frac{\partial \dot{u}_i}{\partial x_j}$  is sufficient for stability under dead loading, and the beginning of an unstable branch is marked by the vanishing of this expression for at least one kinematically admissible velocity gradient distribution.

So far, the stability condition has been expressed in terms of the rate of the first Piola-Kirchhoff stress tensor  $\dot{\Sigma}_{ij}$  and the velocity gradient  $\frac{\partial \dot{u}_i}{\partial x_j}$ , both referred to the configuration at  $\tau = t$ . A number of alternative, but essentially equivalent formulations exist. For instance, a rather long but straightforward derivation shows that U is also given by

$$U = \frac{1}{2} (\delta t)^2 \int_{V_0} \left[ \dot{S}_{ij} \dot{\gamma}_{ij} + \frac{\partial \dot{u}_k}{\partial \xi_i} S_{ij} \frac{\partial \dot{u}_k}{\partial \xi_j} \right] dV^0$$
(2.35)

Another elegant and useful formulation can be derived when the constitutive equation is phrased in terms of the Truesdell stress rate  $S_{ij}$  and the rate of deformation tensor  $\dot{\varepsilon}_{ij}$ .

$$\dot{\varepsilon}_{ij} = \frac{1}{2} \left( \frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i} \right)$$
(2.36)

such that

$$S_{ij} = D_{ijkl} \dot{\varepsilon}_{kl}$$
(2.37)

Then, we can deduce that

$$U = \frac{1}{2} (\delta t)^2 \int_{V} \frac{\partial \dot{u}_i}{\partial x_j} \overline{D}_{ijkl} \frac{\partial \dot{u}_k}{\partial x_l} dV$$
(2.38)

where

$$\overline{D}_{ijkl} = D_{ijkl} + \sigma_{jl} \delta_{ik} \tag{2.39}$$

Next consider uniqueness. We again suppose that we have an equilibrium state and that there exist two kinematically admissible strain rate distributions which both satisfy compatibility and which do not differ merely by a rigid body motion. Let  $\Delta \left( \frac{\partial \dot{u}_i}{\partial x_j} \right)$  be the difference between both velocity gradient distributions and let  $\Delta \dot{\Sigma}_{ij}$  be the difference between between both stress rate distributions. Now consider the integral

$$W = \int_{V} \Delta \dot{\Sigma}_{ij} \Delta \left[ \frac{\partial \dot{u}_i}{\partial x_j} \right] dV$$
(2.40)

By virtue of the divergence theorem we obtain

$$W = \int_{S} \Delta \dot{\Sigma}_{ij} n_j \Delta \dot{u}_i dS - \int_{V} \Delta \dot{u}_i \frac{\partial (\Delta \dot{\Sigma}_{ij})}{\partial x_j} dV$$
(2.41)

In order that both solutions follow an equilibrium path, they must both satisfy equation (2.14) within the body and equation (2.15) on the part of the boundary where the tractions  $t_i$  are prescribed. Subtracting the equilibrium equation of one stress rate field from the equilibrium equation of the other stress distribution yields

$$\frac{\partial(\Delta \hat{\Sigma}_{ij})}{\partial x_i} = 0 \tag{2.42}$$

for points within the body and

$$\Delta \dot{\Sigma}_{ij} n_j = 0 \tag{2.43}$$

for points on the part of the boundary where tractions are prescribed.

Together with the observation that on the remainder of the boundary, we have  $\Delta \dot{u}_i = 0$ , this leads to the conclusion that

$$\int_{V} \Delta \dot{\Sigma}_{ij} \Delta \left[ \frac{\partial \dot{u}_{i}}{\partial x_{j}} \right] dV = 0$$
(2.44)

for two different solutions. Uniqueness is therefore guaranteed if<sup>55,56</sup>

$$\int_{V} \Delta \dot{\Sigma}_{ij} \Delta \left( \frac{\partial \dot{u}_i}{\partial x_j} \right) dV > 0 \tag{2.45}$$

and we have a bifurcation point if the integral vanishes for two different admissible solutions. It is noted that the condition that W < 0 is also a sufficient condition for uniqueness, but this possibility seems not important as according to condition (2.34) such a situation will probably be unstable. It is noted that alternative formulations such as derived for the stability condition, can also be deduced for the uniqueness condition.

We observe that stability and uniqueness may give rise to different requirements. Essentially, the stability requirement is single-valued, that is the stress rate  $\dot{\Sigma}_{ij}$  can be associated with a unique velocity gradient  $\frac{\partial \dot{u}_i}{\partial x_j}$ . However, the uniqueness requirement is multi-valued when both

possible velocity gradient distributions are related to stress rates by different values for  $D_{ijkl}$ , which happens when we have different behaviour in loading and unloading, e.g. elastic-plastic solids. Strictly speaking, we have to investigate all possible combinations of loading and unloading for such a multi-valued constitutive law in order to determine whether the uniqueness integral vanishes for some combination.

# 2.4. Geometrical linearization

Unstable structural behaviour and non-unique solutions may either be caused by geometrical nonlinearities or by physical nonlinearities or by a combination of both. It depends on the material and the type of structure which cause (physical or geometrical) will prevail. In steel structures, the material behaviour is usually stable and geometrical nonlinearities as represented by the second term in equations (2.20) or (2.35) are the major factor which cause unstable structural behaviour. For materials like sand, rock and concrete, we only have material stability below some threshold load level and geometrical nonlinearities play a minor role in causing structural instability. An exception is formed by slender concrete structures such as tall reinforced or prestressed columns, but for mass

concrete and earth and rock masses, geometrical nonlinearities are unlikely to contribute significantly to unstable behaviour.

As for the class of materials and kind of structures which we will consider, material instability will be the governing factor which causes instability, and as the constitutive models unfortunately still entail significant errors in describing the materials considered here, we will omit geometrically nonlinear terms in the sequel of this study. A further advantage of neglecting possible geometrical nonlinear effects is the fact that we are then able to concentrate fully on the consequences of the material models for structural stability. By neglecting geometrical nonlinearities, we in fact isolate the consequences of material behaviour for structural stability.

When introducing the assumption that the geometrical nonlinear terms can be disregarded, the equilibrium equation (2.30) and the stability and uniqueness conditions (2.34) and (2.45) simplify considerably. A first consequence of the assumption that the displacement gradients remain small throughout the loading process is that the difference between the Cauchy and Piola-Kirchhoff stress tensors vanishes and the additional terms in the expressions for the stress rates can also be disregarded. Furthermore, no distinction need be made between the current and the reference configuration when evaluating the integrals, and the linearized 'engineering' strain  $\varepsilon_{ij}$ ,

$$\varepsilon_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right]$$
(2.46)

replaces the Green-Lagrange strain  $\gamma_{ij}.$  With these simplifications, the virtual work expression reduces to

$$\int_{V} \int_{-\Delta t} \int_{U} D_{ijkl} \dot{\varepsilon}_{kl} d\tau \delta \varepsilon_{ij} dV = \int_{S} t_i \delta u_i dS + \int_{V} \rho g_i \delta u_i dV - \int_{V} \sigma_{ij}^{t-\Delta t} \delta \varepsilon_{ij} dV \quad (2.47)$$

while the stability and the uniqueness condition respectively reduce to

$$\int_{V} \dot{\sigma}_{ij} \dot{\varepsilon}_{ij} dV > 0 \tag{2.48}$$

and

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$$\int_{V} \Delta \dot{\sigma}_{ij} \Delta \dot{\varepsilon}_{ij} dV > 0 \tag{2.49}$$

#### 3. DISCRETIZATION OF THE BOUNDARY VALUE PROBLEM

In this chapter we shall consider a finite element discretization of the differential equations for a non-linear continuum which were derived in the preceding chapter. As material rather than geometrical instabilities are the central theme of this study and as the examples which are considered in the final chapters have been performed under the assumption of small displacement gradients, we will take the linearized differential equations (in the sense that geometry changes are neglected) as starting point of the derivation. Furthermore, we will investigate the implications of the conditions for stability and uniqueness discussed in the preceding chapter for discretized systems.

## 3.1. Finite element representation

Let the continuum be divided in an arbitrary number of finite elements, and let the continuous displacement field  $\mathbf{u}$  be interpolated as follows

$$\boldsymbol{u} = \boldsymbol{H}\boldsymbol{a} \tag{3.1}$$

in which the matrix H contains the interpolation polynomials and a is a vector which contains the nodal displacements (see for instance Bathe<sup>4</sup>). The relation between the displacement field a and the linearized strain  $\varepsilon$  can formally be written as

$$\boldsymbol{\varepsilon} = \boldsymbol{L} \boldsymbol{u} \tag{3.2}$$

with L a matrix which contains differential operators. The relation between the nodal displacements and the linearized strain then becomes

$$\boldsymbol{\varepsilon} = \boldsymbol{B} \boldsymbol{a}$$
 (3.3)

or upon differentiation

$$\dot{\boldsymbol{\varepsilon}} = \boldsymbol{B} \, \dot{\boldsymbol{a}}$$
 (3.4)

while the notation B = L H has been introduced for the strain-nodal displacement matrix.

Substitution of equation (3.4) in the linearized (in the sense that geometry changes are neglected) virtual work expression (2.47)

$$\int_{V} \boldsymbol{\delta}\boldsymbol{\varepsilon}^{T} \int_{t-\Delta t}^{t} \boldsymbol{D} \, \boldsymbol{\dot{\varepsilon}} d\tau dV = \int_{S} \boldsymbol{\delta}\boldsymbol{u}^{T} \boldsymbol{t} dS + \int_{V} \rho \, \boldsymbol{\delta}\boldsymbol{u}^{T} \boldsymbol{g} dV - \int_{V} \boldsymbol{\delta}\boldsymbol{\varepsilon}^{T} \boldsymbol{\sigma}_{0} dV \quad (3.5)$$

gives after rearranging

$$\delta \boldsymbol{a}^{T} \left\{ \int_{V \boldsymbol{t} - \Delta \boldsymbol{t}}^{\boldsymbol{t}} \boldsymbol{B}^{T} \boldsymbol{D} \boldsymbol{B} \dot{\boldsymbol{a}} d\tau dV - \int_{S} \boldsymbol{H}^{T} \boldsymbol{t} dS - \int_{V} \rho \boldsymbol{H}^{T} \boldsymbol{g} dV + \int_{V} \boldsymbol{B}^{T} \boldsymbol{\sigma}_{0} dV \right\} = 0 \quad (3.6)$$

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We note that  $\sigma^{t-\Delta t}$  has been replaced by the shorter notation  $\sigma_0$ . As equation (3.6) must hold for any virtual displacement  $\delta \alpha$ , we obtain the following set of algebraic equations

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$$\int_{V} \int_{t-\Delta t} \boldsymbol{B}^{T} \boldsymbol{D} \boldsymbol{B} \dot{\boldsymbol{a}} d\tau dV = \int_{S} \boldsymbol{H}^{T} \boldsymbol{t} dS + \int_{V} \rho \boldsymbol{H}^{T} \boldsymbol{g} dV - \int_{V} \boldsymbol{B}^{T} \boldsymbol{\sigma}_{0} dV \quad (3.7)$$

On integration point level, accurate evaluation of the integral  $\int \mathbf{B}^T \mathbf{D} \mathbf{B} \dot{\mathbf{a}} d\tau$  is very important and various possibilities exist, ranging from a simple Euler forward method to sophisticated Runge-Kutta methods. A simple, but accurate method is derived in Section 5.1 where we will devote special attention to singularities which may complicate the evaluation of the integral. On structural level, however, other methods than a forward Euler method are seldom employed. Hence, for the solution of the non-linear equation on a structural level which is described in Section 5.2, we can integrate equation (3.7) in a straightforward manner, yielding:

$$\int_{V} \boldsymbol{B}^{T} \boldsymbol{D}_{0} \boldsymbol{B} dV \boldsymbol{\Delta a} = \int_{S} \boldsymbol{H}^{T} \boldsymbol{t} dS + \int_{V} \rho \boldsymbol{H}^{T} \boldsymbol{g} dV - \int_{V} \boldsymbol{B}^{T} \boldsymbol{\sigma}_{0} dV \qquad (3.8)$$

where the notation  $D_0$  means that the stress-strain matrix D is evaluated at the beginning of the loading step. We next introduce the notations

$$\mathbf{K}_{0} = \int_{V} \mathbf{B}^{T} \mathbf{D}_{0} \mathbf{B} dV \tag{3.9}$$

for the stiffness matrix evaluated at the beginning of the loading step and

$$\boldsymbol{q} = \int_{S} \boldsymbol{H}^{T} \boldsymbol{t} dS + \int_{V} \rho \boldsymbol{H}^{T} \boldsymbol{g} dV$$
(3.10)

for the external load vector. For proportional loading we may replace  $\boldsymbol{q}$  by  $\mu \boldsymbol{q}^*$  with  $\boldsymbol{q}^*$  a normalized load vector and  $\mu$  a load parameter. With the additional definition

$$\boldsymbol{p}_0 = \mu_0 \boldsymbol{q}^* - \int_V \boldsymbol{B}^T \boldsymbol{\sigma}_0 dV \tag{3.11}$$

we have for the initial estimate of the increment in nodal displacements within a loading step  $\pmb{\Delta a}_1$ 

$$\boldsymbol{\Delta \boldsymbol{a}}_{1} = \boldsymbol{K}_{0}^{-1} \left\{ \Delta \boldsymbol{\mu}_{1} \boldsymbol{q}^{*} + \boldsymbol{p}_{0} \right\}$$
(3.12)

Anticipating the treatment in Chapter 5, the load parameter  $\mu$  has also been labelled, as it may change from iteration to iteration. In particular,  $\mu_0$  is the value of the load parameter at the end of the previous loading step and  $\Delta\mu_1$  is the value of the load increment in the first iteration within the current step. As we will see in Chapter 5,  $\Delta\mu_1$  need not be equal to the value of the load increment in subsequent iterations  $\Delta\mu_i$ .

Because of the forward integration of equation (3.7),  $\Delta a_1$  mostly does not lead to a stress field  $\sigma_1 = \sigma_0 + \Delta \sigma_1$  which satisfies equilibrium. Therefore, a correction  $\delta a_2$ ,

$$\boldsymbol{\delta a}_{2} = \boldsymbol{K}_{1}^{-1} \left[ \Delta \mu_{2} \boldsymbol{q}^{*} + \boldsymbol{p}_{1} \right]$$
(3.13)

is calculated with

$$\boldsymbol{p}_1 = \mu_0 \boldsymbol{q}^* - \int_V \boldsymbol{B}^T \boldsymbol{\sigma}_1 dV \tag{3.14}$$

and  $K_1$  a possibly updated stiffness matrix. It is further noted that the notation  $\delta$  no longer denotes a virtual quantity, but a small increment. The correction  $\delta a_2$  is added to the first estimate of nodal displacements  $\Delta a_1$ , so that we obtain as improved estimate of the incremental displacements:

$$\Delta \boldsymbol{a}_2 = \Delta \boldsymbol{a}_1 + \delta \boldsymbol{a}_2 \tag{3.15}$$

This process can be repeated until convergence has been achieved. For iteration number i, the process can be summarised by the equations

$$\boldsymbol{p}_{i-1} = \mu_0 \boldsymbol{q}^* - \int_{\boldsymbol{V}} \boldsymbol{B}^T \boldsymbol{\sigma}_{i-1} d\boldsymbol{V}$$
(3.16)

$$\boldsymbol{\delta a_i} = \boldsymbol{K_{i-1}^{-1}} \left( \Delta \mu_i \boldsymbol{q}^* + \boldsymbol{p_{i-1}} \right)$$
(3.17)

$$\Delta \boldsymbol{a}_i = \Delta \boldsymbol{a}_{i-1} + \delta \boldsymbol{a}_i \tag{3.18}$$

## 3.2. Bifurcation and limit points

In Section 2.3 a structure was defined to be in a state of stable equilibrium if

$$\int_{V} \dot{\boldsymbol{\varepsilon}}^{T} \dot{\boldsymbol{\sigma}} dV > 0 \tag{3.19}$$

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for all kinematically admissible strain-rate vectors  $\dot{\boldsymbol{z}}$ , while it was said to be in a critical state of neutral equilibrium if  $\int \dot{\boldsymbol{z}}^T \dot{\boldsymbol{\sigma}} dV$  vanishes for at least one kinematically admissible strain-rate vector. With the notations and the definitions of the preceding section, we can rewrite the integral of (3.19) as

$$\int_{V} \dot{\boldsymbol{\varepsilon}}^{T} \dot{\boldsymbol{\sigma}} dV = \int_{V} \dot{\boldsymbol{a}}^{T} \boldsymbol{B}^{T} \boldsymbol{D} \boldsymbol{B} \dot{\boldsymbol{a}} dV = \dot{\boldsymbol{a}}^{T} \boldsymbol{K} \dot{\boldsymbol{a}}$$
(3.20)

so that a discrete mechanical system is in a state of stable equilibrium under dead loading if

$$\dot{\boldsymbol{a}}^T \boldsymbol{K} \dot{\boldsymbol{a}} > 0 \tag{3.21}$$

for all admissible velocity vectors  $\dot{\pmb{\alpha}}$  while it is said to be in a critical state of neutral equilibrium if

$$\dot{\boldsymbol{a}}^{T} \boldsymbol{K} \dot{\boldsymbol{a}} = 0 \tag{3.22}$$

for at least one admissible vector  $\dot{\boldsymbol{a}}$ . A sufficient condition for equation (3.22) to be satisfied is that

$$\det(\mathbf{K}) = 0 \tag{3.23}$$

which according to Vieta's rule,

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$$\det(\mathbf{K}) = \prod_{i=1}^{n} \lambda_i \tag{3.24}$$

with  $\lambda_i$  the eigenvalues of K, implies that at least one eigenvalue vanishes. It is noted that the vanishing of det(K) is a sufficient condition for equation (3.22) to hold, but is only a necessary condition in case of symmetric matrices. For non-symmetric matrices, which for instance arise in non-associated plasticity,  $\dot{a}^T K \dot{a}$  also vanishes when  $K \dot{a}$  is orthogonal to  $\dot{a}$ . We will come back to this issue in a subsequent paragraph.

Let now  $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$  be the right eigenvectors and  $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$  be the left eigenvectors of  $\boldsymbol{K}$  corresponding to the eigenvalues  $\lambda_1, \lambda_2, \ldots, \lambda_n$  (in ascending order):

$$\boldsymbol{K}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i \tag{3.25}$$

$$\boldsymbol{w}_i^T \boldsymbol{K} = \lambda_i \, \boldsymbol{w}_i^T \tag{3.26}$$

the summation convention not being implied in this case. Between the left and right eigenvectors  $\boldsymbol{w}_{j}$  and  $\boldsymbol{v}_{i}$ , there exists the relationship (see for instance Ralston and Rabinowitz<sup>87</sup>)

$$\boldsymbol{w}_{j}^{T}\boldsymbol{v}_{i}=0 \qquad i\neq j \tag{3.27}$$

Further,  $\boldsymbol{w}_i$  and  $\boldsymbol{v}_i$  can be normalized such that

$$\boldsymbol{w}_i^T \boldsymbol{v}_i = 1 \tag{3.28}$$

It is noted that the left and right eigenvectors coincide for the special case of a symmetric matrix, i.e.  $\boldsymbol{v}_i = \boldsymbol{w}_i$ . If  $\boldsymbol{K}$  is not defect<sup>87</sup>, i.e. if the n (either right or left) eigenvectors span a n-dimensional vector space, any vector can be written as a linear combination of the right eigenvectors

$$\dot{\boldsymbol{a}} = \sum_{i=1}^{n} (\boldsymbol{w}_{i}^{T} \dot{\boldsymbol{a}}) \boldsymbol{v}_{i}$$
(3.29)

or alternatively of the left eigenvectors

$$\dot{\boldsymbol{a}} = \sum_{j=1}^{n} (\boldsymbol{v}_{j}^{T} \dot{\boldsymbol{a}}) \boldsymbol{w}_{j}$$
(3.30)

The vanishing of  $det(\mathbf{K})$  implies that the corresponding set of algebraic equations

$$\mathbf{K}\dot{\mathbf{a}} = \dot{\mu}\mathbf{q}^* \tag{3.31}$$

becomes singular. It is noted that in equation (3.31) we consider a velocity field  $\dot{\boldsymbol{a}}$  and a loading rate  $\dot{\boldsymbol{\mu}}$  rather than a finite displacement increment  $\boldsymbol{\Delta a}$  and finite load increment  $\Delta \boldsymbol{\mu}$ . It is furthermore assumed that we have a converged equilibrium state in our numerical process, so that at the beginning of the new loading step, de unbalanced force vector  $\boldsymbol{p}_0$ is vanishingly small. The velocity vector  $\dot{\boldsymbol{a}}$  and the normalized load vector  $\boldsymbol{q}^*$  are now decomposed in the sense of equation (3.29). Substituting the result in (3.31) gives

$$\boldsymbol{K}\left[\sum_{i=1}^{n} (\boldsymbol{w}_{i}^{T} \dot{\boldsymbol{a}}) \boldsymbol{v}_{i}\right] = \mu \sum_{i=1}^{n} (\boldsymbol{w}_{i}^{T} \boldsymbol{q}^{*}) \boldsymbol{v}_{i}$$
(3.32)

With aid of equation (3.25), we can modify (3.32) as follows

$$\sum_{i=1}^{n} \left[ \lambda_{i} (\boldsymbol{w}_{i}^{T} \dot{\boldsymbol{a}}) - \dot{\boldsymbol{\mu}} (\boldsymbol{w}_{i}^{T} \boldsymbol{q}^{*}) \right] \boldsymbol{v}_{i} = \boldsymbol{0}$$
(3.33)

As **K** is not defect, the eigenvectors  $\boldsymbol{v}_i$   $(i=1,\ldots,n)$  constitute a set of n linearly independent vectors, which implies that equation (3.33) can only be satisfied if

$$\lambda_i(\boldsymbol{w}_i^T \dot{\boldsymbol{a}}) - \dot{\boldsymbol{\mu}}(\boldsymbol{w}_i^T \boldsymbol{q}^*) = 0 \tag{3.34}$$

for each eigenvector  $\boldsymbol{v}_i$ . In particular, we have for i=1

$$\lambda_1(\boldsymbol{w}_1^T \boldsymbol{\dot{a}}) - \dot{\boldsymbol{\mu}}(\boldsymbol{w}_1^T \boldsymbol{q}^*) = 0 \tag{3.35}$$





When the eigenvalues are in ascending order, the vanishing of det( $\mathbf{K}$ ) implies that  $\lambda_1=0$ .  $\lambda_2,\lambda_3$  etc. may also vanish, but we shall restrict attention to the case that only  $\lambda_1$  vanishes in the understanding that generalisation to more vanishing eigenvalues poses no serious problem. With  $\lambda_1=0$  we obtain instead of equation (3.35)

$$\dot{\mu}(\boldsymbol{w}_{1}^{T}\boldsymbol{q}^{*})=0 \tag{3.36}$$

This equation is satisfied when either

or

$$\boldsymbol{w}_{1}^{T}\boldsymbol{q}^{*}=0 \tag{3.38}$$

or when both conditions are met. The first possibility is usually called a limit point and is by far the most common. A typical example of such a point, in which the load becomes stationary ( $\mu$ =0), is plotted in Figure 3.1. Condition (3.38) determines a bifurcation point of equilibrium states from which various equilibrium branches emanate. Equation (3.38) implies that the load vector is orthogonal to the left eigenvector  $\mathbf{w}_1$  so that equation (3.36) is satisfied while the load does not necessarily become stationary for all solutions at the bifurcation point. Figure 3.2 shows some possibilities of bifurcation points and post-bifurcation behaviour. The classification of Figure 3.2 stems from the field of elastic stability<sup>110</sup>, but does not have much meaning for bifurcations of elastic-plastic or elastic-fracturing solids, as for such types of material behaviour we mostly can only establish a load level beyond which bifurcation points. In





reality, bifurcation points are rather rare, as they mainly occur in perfect structures, and they are transferred into limit points upon introduction of an imperfection. The third case, i.e.  $\mu = 0$  and  $\boldsymbol{w}_{1}^{T}\boldsymbol{q}^{*} = 0$  for all possible solutions, constitutes a point which is a limit point as well as a bifurcation point. This somewhat rare case will not be considered here in detail.

The vanishing of  $\dot{\mu}$  for a limit point implies that

$$K\dot{a} = 0$$
 (3.39)

or substituting equations (3.25) and (3.29)

$$\sum_{i=1}^{n} \lambda_i (\boldsymbol{w}_i^T \dot{\boldsymbol{a}}) \boldsymbol{v}_i = \boldsymbol{0}$$
(3.40)

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With the same arguments as used in the preceding, we can deduce that

$$\boldsymbol{w}_{i}^{T} \boldsymbol{\dot{a}} = 0 \tag{3.41}$$

for each  $i \neq 1$  since then  $\lambda_i > 0$ . Equation (3.41) can only be satisfied if

 $\dot{\boldsymbol{a}} = \alpha \boldsymbol{v}_1$ 

(3.42)

with  $\alpha$  some indeterminate scalar. Hence, a multiple of the right eigenvector belonging to the lowest eigenvalue is the only possible solution at a limit point.

With regard to bifurcation points, we assume that  $\pmb{a}^*$  is a solution to equation (3.31) so that

$$\mathbf{Ka}^* = \mu \mathbf{q}^* \tag{3.43}$$

and we will henceforth call this solution the fundamental or basic solution. It is noted that  $\dot{\mu}\neq 0$  for this solution. We will now investigate whether the equations

$$\lambda_i(\boldsymbol{w}_i^T \dot{\boldsymbol{a}}) - \dot{\mu}(\boldsymbol{w}_i^T \boldsymbol{q}^*) = 0 \tag{3.44}$$

for  $i \neq 1$  admit other solutions than the fundamental solution  $a^{\circ}$ . Substitution of equation (3.43) yields:

$$\lambda_i(\boldsymbol{w}_i^T \dot{\boldsymbol{a}}) - (\boldsymbol{w}_i^T \boldsymbol{K} \boldsymbol{a}^*) = 0 \tag{3.45}$$

and decomposing  $\boldsymbol{a}^*$  in the sense of equation (3.29) and using (3.25) results in

$$\lambda_{i}(\boldsymbol{w}_{i}^{T}\boldsymbol{\dot{a}}) - \lambda_{i} \boldsymbol{w}_{i}^{T} \left( \sum_{j=1}^{n} (\boldsymbol{w}_{j}^{T}\boldsymbol{a}^{*}) \boldsymbol{v}_{j} \right) = 0$$
(3.46)

Because  $\lambda_i > 0$  for  $i \neq 1$  and because of the orthogonality of  $\boldsymbol{w}_i$  and  $\boldsymbol{v}_j$ , equation (3.46) reduces to

 $\boldsymbol{w}_{i}^{T}(\boldsymbol{\dot{a}}-\boldsymbol{a}^{*})=0 \tag{3.47}$ 

This equation will be satisfied for each  $\boldsymbol{w}_i$  if and only if

$$\dot{\boldsymbol{a}} - \boldsymbol{a}^* = \beta \boldsymbol{v}_1 \tag{3.48}$$

with  $\beta$  some scalar. Hence, all solutions

$$\dot{\boldsymbol{a}} = \boldsymbol{a}^* + \beta \boldsymbol{v}_1 \tag{3.49}$$

are possible at a bifurcation point. It is noted that equation (3.49) contains the fundamental solution as a special case when  $\beta=0$ , but that

 $\dot{\boldsymbol{a}} = \boldsymbol{\beta} \boldsymbol{v}_1 \tag{3.50}$ 

is not contained in the set of solutions (3.49). Nevertheless, (3.50) always constitutes a solution to (3.31). To show this, we first substitute (3.50) in equations (3.44), yielding

$$\beta \lambda_i (\boldsymbol{w}_i^T \boldsymbol{v}_1) - \dot{\mu} (\boldsymbol{w}_i^T \boldsymbol{q}^*) = 0$$
(3.51)

Because of orthogonality of  $\boldsymbol{w}_i$  and  $\boldsymbol{v}_1$  this reduces to

 $\dot{\mu}(\boldsymbol{w}_{i}^{T}\boldsymbol{q}^{*})=0 \qquad i>1 \tag{3.52}$ 

and as  $\boldsymbol{q}^*$  is not a null vector by definition, this implies that  $\dot{\mu}$  must vanish. Hence, a solution of the kind (3.50) must be associated with a bifurcation branch with a zero slope at the bifurcation point. The observation that the set of solutions (3.49) as well as solution (3.50) are permitted, can be comprised in the single equation

$$\dot{\boldsymbol{a}} = \alpha \boldsymbol{a}^* + \beta \boldsymbol{v}_1 \tag{3.53}$$

where the scalars  $\alpha$  and  $\beta$  may both vanish (but of course not simultaneously). A procedure for determining  $\alpha$  and  $\beta$  for a finite displacement increment **Aa** will be outlined in Chapter 5.

In numerical processes, limit or bifurcation points are extremely difficult to isolate. Rather, distinction is made between stable equilibrium states for which equation (3.21) holds, and equilibrium states which are unstable under dead loading, i.e.

$$\dot{\boldsymbol{a}}^T \boldsymbol{K} \dot{\boldsymbol{a}} < 0$$
 (3.54)

for at least one kinematically admissible field  $\dot{a}$ . Substituting expressions (3.29) and (3.30) in inequality (3.54) gives

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (\boldsymbol{v}_{i}^{T} \dot{\boldsymbol{a}}) (\boldsymbol{w}_{j}^{T} \dot{\boldsymbol{a}}) \boldsymbol{w}_{j}^{T} \boldsymbol{K} \boldsymbol{v}_{i} < 0$$
(3.55)

or using equations (3.25) to (3.28)

$$\sum_{i=1}^{n} (\boldsymbol{w}_{i}^{T} \dot{\boldsymbol{a}}) (\boldsymbol{w}_{i}^{T} \dot{\boldsymbol{a}}) \lambda_{i} < 0$$
(3.56)

This inequality will certainly be satisfied if one or more eigenvalues become negative ‡. Suppose for instance that we have encountered two

 $<sup>\</sup>ddagger$  It is noted that the fact that the stiffness matrix K may be non-symmetric implies that the eigenvalues of the stiffness matrix may be complex even if the stiffness matrix only has real coefficients. The bulk of the literature on finite element methods only deals with the sub-class of symmetric, real matrices and then the eigenvalues can be proved to be real. It seems however, that the possibility of complex eigenvalues of a tangent stiffness matrix which arises in structural mechanics is rather academic since this would according to equation (3.57) imply

negative eigenvalues  $\lambda_1$  and  $\lambda_2$  with corresponding (right) eigenvectors  $\boldsymbol{v}_1$  and  $\boldsymbol{v}_2$ . Choosing  $\dot{\boldsymbol{a}} = \alpha \boldsymbol{v}_1$  with  $\alpha$  some real scalar, we obtain with (3.27) and (3.28)

(3.57)



Figure 3.3. Unstable branches after a bifurcation.(a) The load rises on the basic path and there are *m* alternative equilibrium states.(b) The load decreases on the basic path after bifurcation.

Similarly, if we choose  $\dot{\boldsymbol{a}} = \beta \boldsymbol{v}_2$  with  $\beta$  some other scalar, then

$$\sum_{i=1}^{n} (\boldsymbol{v}_{i}^{T} \dot{\boldsymbol{a}}) (\boldsymbol{w}_{i}^{T} \dot{\boldsymbol{a}}) \lambda_{i} = \beta^{2} \boldsymbol{v}_{2}^{T} \boldsymbol{v}_{2} \lambda_{2} < 0$$
(3.58)

that the corresponding eigenvector would also be complex, which seems physically inconceivable.



Figure 3.4. Load versus transverse displacement for perfect Euler strut.

so that there exist at least two independent kinematically admissible velocity vectors for which inequality (3.54) holds, and consequently, the equilibrium is unstable under dead loading.

There are two possible interpretations of negative eigenvalues. The first is that we have passed a limit point, which implies that the load is descending. In this case, we find a negative eigenvalue which is associated with the descending branch (see Figure 3.1). The other possibility is that the negative eigenvalues belong to alternative equilibrium states and that we have passed a bifurcation point. Again, two possibilities arise, as the basic path after bifurcation may either be ascending or descending. If it is still ascending (Figure 3.3a), all the, say m negative eigenvalues can be associated with m alternative equilibrium states which can in principle be reached through a suitable combination of the incremental displacement vector of the basic path and the corresponding eigenvector  $\ddagger$ . If the

<sup>&</sup>lt;sup>‡</sup> The question now arises whether the alternative equilibrium states are indeed accessible. If a mechanical system is undergoing a continuous process, such an alternative equilibrium state can only be reached via an equilibrium path. If a bifurcation point has been passed and the system is in a state of unstable equilibrium thereafter, the point will continue on this unstable path because other equilibrium states cannot be reached under static dead loading conditions. This implies that for a continuous loading process branching off on other equilibrium paths can only take place at a bifurcation point. If a *temporal* discretization of the loading process is employed, this is no longer true as alternative equilibrium states can in that case also be reached via non-equilibrium paths, because we then essentially deal with equilibrium states and not with equilibrium paths. In fact, we actually follow a non-equilibrium path when iterating to a converged solution. We will come back to this issue in Chapter 5 and we will give an example in Chapter 7.

basic path is descending after passing a bifurcation point (Figure 3.3b), one negative eigenvalue is associated with the descending basic path, and the remaining m-1 negative eigenvalues correspond to m-1 alternative equilibrium paths.

An example of a bifurcation after which all negative eigenvalues belong to alternative equilibrium states which can in principle be reached via some kinematically admissible displacement vector, is the perfect Euler strut. When we load a perfect strut centrically, we can continue the solution indefinitely. After the first bifurcation point (labelled "A" in Figure 3.4) we encounter a negative eigenvalue, which indicates the existence of a (in this case stable) alternative equilibrium branch (dashed in Figure 3.4). Similarly, we get two negative eigenvalues when the second bifurcation point ("B" in Figure 3.4) is passed as beyond point B there exist two alternative equilibrium branches, namely the branch which emanates from point A (dashed) and the branch which emanates from point B (dash-dotted). It is noted that in the terminology of elastic stability as quoted above, the bifurcation points of this example are stable symmetric bifurcation points.



Figure 3.5. Load-displacement curve for axially loaded bar.

An example in which one negative eigenvalue belongs to the descending branch and the remaining negative eigenvalues correspond to alternative equilibrium branches is the axially loaded bar of softening material as shown in Figure 3.5. In this figure, the most shallow post-peak curve corresponds to the basic path at which the bar deforms homogeneously. As we will derive in greater detail in Sections 4.4 and 6.1, the employed softening model permits localisation in one or more elements. These alternative equilibrium paths after peak load are the steeper curves of Figure 3.5 and the remaining negative eigenvalues correspond to these paths. For convenience of readers who are more familiar with bifurcations in elastic-plastic solids, it is noted that the above example is not a necking type bifurcation.

We have seen that the existence of negative eigenvalues is a sufficient condition for unstable structural behaviour under dead loading, i.e. the structure is in a state of unstable equilibrium if we extract negative eigenvalues from the tangent stiffness matrix. It is not a necessary condition, as inequality (3.56) may be satisfied for particular choices of  $\dot{\boldsymbol{a}}$  with all eigenvalues  $\lambda_i$  still being positive. For the particular case of symmetric systems however, the existence of negative eigenvalues is not only a sufficient, but also a necessary condition for unstable behaviour under dead loading. In this case, the difference between the left and the right eigenvectors vanishes  $(\boldsymbol{v}_i = \boldsymbol{w}_i)$  and (3.56) reduces to

$$\sum_{i=1}^{n} (\boldsymbol{v}_{i}^{T} \dot{\boldsymbol{a}})^{2} \lambda_{i} < 0 \tag{3.59}$$

This inequality can only be satisfied if one or more eigenvalues become negative. Hence, the case of symmetric matrices is more satisfactory than the more general case of unsymmetric matrices, as for nonsymmetric matrices instability may occur prior to the occurrence of negative eigenvalues, whereas this cannot happen for symmetric matrices. The type of instability for which  $\dot{a}^T K \dot{a}$  vanishes for at least one admissible vector  $\dot{a}$ , but in which all eigenvalues are still positive, may be called a 'flutter' type instability<sup>77</sup>, as it is not necessarily a divergence type instability (i.e. a bifurcation or an unstable post-limit response). The possibility that for a non-symmetric system the stability expression vanishes for some admissible velocity field while all eigenvalues are still positive is not merely an academic case. We will derive in Chapter 4 that this actually happens for some plasticity models with a non-associated flow rule.

If the constitutive law is multi-valued, there exist other stiffness matrices which may produce more negative eigenvalues than the number found for the current tangent stiffness matrix. Strictly speaking, we will only detect bifurcations for which the tangent moduli show, at least initially, loading on the localisation branch. Yet, we will probably locate all bifurcation points by only considering such a solid, as the case that all points show loading gives the weakest response to additional loading. Indeed, for some models it can be proved that the case that all material points show loading is always critical<sup>117</sup>. Such a solid in which loading is assumed for all material points, has been named a 'linear comparison solid' <sup>56</sup>. In conclusion, we can state that there may be more alternative equilibrium states than calculated on basis of the stability expression,

but this is not likely to occur. Nevertheless, we should investigate all possible combinations of loading and unloading for a rigorous establishment of uniqueness.

Finally, it is emphasised that the above statements only hold for load-controlled problems, but not necessarily when the load is applied by prescribing displacements as we then deal with a different stiffness matrix. Negative eigenvalues which correspond to descending branches disappear and we only retain negative eigenvalues which correspond to alternative equilibrium states.

## 3.3. Consequences of spatial discretization

In spite of the fact that a continuum can be approximated by a discrete system to an arbitrary degree of accuracy, it should be realised that when we analyse a discrete system, the response will never be exactly that of the underlying continuum. Strictly speaking, we can only calculate limit and bifurcation points of the discrete system, but we can not rigorously identify them with limit or bifurcation points of the underlying continuum, although it may be expected that upon mesh refinement, i.e. when we improve the spatial approximation, 'spurious' limit or bifurcation points gradually vanish. Such observations have for instance been made in fluid mechanics<sup>3</sup>, where non-physical bifurcations appeared to vanish upon mesh refinement. An example within the realm of solid mechanics was encountered by the Author<sup>22,23</sup> when analyzing a reinforced concrete beam. In this case, a 'snap-back' phenomenon appeared to vanish on mesh refinement.

As the results, at least below some threshold level, are sensitive for the degree of spatial discretization, they will certainly also depend on the type and degree of interpolation and also on the order of numerical integration. This seems a truism with regard to the finite element method, but here we mean that the choice of interpolation polynomials and of quadrature rules not only affects the accuracy of the results, but may even dominate the computational results to such an extent that an improper choice may entail solutions which are fundamentally different from the actual response of the underlying structure.

An example of the impact of spatial discretization and the order of numerical integration on the computational results is given below. It concerns the axisymmetric slab of Figures 3.6 and 3.7. The properties of the concrete are assumed to be: Young's modulus  $E_c = 28000 \text{ N/mm}^2$ , Poisson's ratio  $\nu=0.2$ , tensile strength  $f_{ct}=2.6 \text{ N/mm}^2$ , shear reduction factor  $\beta^*=0.25$  and fracture energy  $G_f=0.06 \text{ N/mm}$ . The simplified







Figure 3.7. Refined mesh for axisymmetric slab.

elastic-perfectly plastic version of the constitutive model (see Section 4.2) was employed with a cohesion  $c = 9.6 \text{ N/mm}^2$  and a friction angle  $\varphi = 30^\circ$ . The slab is reinforced isotropically with a reinforcement ratio of 1% and the properties of the reinforcement are  $E_s = 205000 \text{ N/mm}^2$  and  $\sigma_y = 465 \text{ N/mm}^2$ . The experimental failure mechanism of the slab is ultimately due to punching shear<sup>122</sup>

The first analysis was carried out for the coarse mesh of Figure 3.6 and the displacement up to a deflection of 9.6 mm no convergence appeared possible although very small displacement increments were imposed (see Figure 3.8). However, when the analysis was repeated with exactly the same mesh and the same material parameters, but with 'full' 9-point integration, displacements could be imposed until and beyond a plateau in the load-displacement curve. The same trend was observed for computations with the refined mesh of Figure 3.7, i.e. the analysis with while the analysis with reduced integration diverged at some displacement level (Figure 3.8). Yet, the latter calculation with 'reduced' integration could be continued much further than the computation with 'reduced' integration for the coarser mesh. This indicates that spurious



Figure 3.8. Load-deflection curves for axisymmetric slab.

snap-backs and divergence gradually disappear with mesh refinement.

The poor behaviour of reduced integration in conjunction with crack formation has been explained to be caused by the introduction of spurious zero-energy modes upon the formation of cracks in 'reduced' integrated elements<sup>39</sup>. With the formation of a new crack, an extra spurious zero-energy mode is introduced, so that we have four additional zero-energy modes when all four integration points are cracked. Observations about incorrect predictions of structural behaviour when using reduced integration have also been reported by Crisfield<sup>35,36</sup>.

## 4. CONSTITUTIVE MODELS

In this chapter, we will describe the constitutive models for soil and concrete which have been adopted in the sample problems which we will discuss. Restriction is made to time-independent phenomena and in particular we will confine the treatment to cracking and the non-linear behaviour in triaxial compression. Cracking is described using a smeared approach and the behaviour under triaxial stress states is modelled using a non-associated plasticity model with hardening on the frictional and softening on the cohesive properties.

## 4.1. Fracture in cohesive granular materials

In the smeared crack concept which is utilised in this study, a crack is conceived to be distributed over the entire area belonging to an integration point. Indeed, we look upon the smeared crack concept as a genuine continuum approach in the sense that there is a representative domain for which we can define notions like 'stress', 'strain' and so on. We recognise that objections may be raised against such a conception, owing to the heterogeinity of concrete and the discontinuous nature of dominant cracks. Nevertheless, the examples of the final chapters indicate that concrete including phenomena like crack propagation can be described sufficiently accurately within the framework of continuum mechanics.

In this section, we will outline the fundamentals of a smeared crack model which is capable of properly combining crack formation and the non-linear behaviour of the concrete between the cracks and of handling secondary cracking owing to rotation of the principal stress axes after primary crack formation. In the present approach, a secondary crack is allowed if the major principal stress exceeds the tensile strength and if the angle between the primary crack and the secondary crack exceeds a threshold angle  $\alpha$ . This threshold angle need not be equal to 90°, so that the model permits non-orthogonal cracks.

## 4.1.1 Multiple cracking

The basic assumption of the smeared crack model is that the total strain rate  $\dot{\varepsilon}_{kl}$  is composed of a concrete strain rate  $\dot{\varepsilon}_{kl}^{co}$  and several crack strain rates which we will denote by  $\dot{\varepsilon}_{kl}^{I}$ ,  $\dot{\varepsilon}_{kl}^{II}$  etc., so that

$$\dot{\varepsilon}_{kl} = \dot{\varepsilon}_{kl}^{co} + \dot{\varepsilon}_{kl}^{I} + \dot{\varepsilon}_{kl}^{II} + \dots$$

$$\tag{4.1}$$

For the present, we will restrict ourselves to two active cracks, so that we

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have

$$\dot{\varepsilon}_{kl} = \dot{\varepsilon}_{kl}^{co} + \dot{\varepsilon}_{kl}^{I} + \dot{\varepsilon}_{kl}^{II} \tag{4.2}$$

This restriction is not essential and we will generalise to an arbitrary number of cracks in the next chapter, but it serves the purpose of simplifying the algebraic expressions. The concrete strain rate is assumed to be related to some objective stress rate  $\dot{\sigma}_{ij}$  (e.g. the Jaumann derivative of the Cauchy stress tensor) via

$$\dot{\sigma}_{ij} = D_{ijkl}^{co} \dot{\varepsilon}_{kl}^{co} \tag{4.3}$$

The fourth-order tensor  $D_{ijkl}^{co}$  contains the instantaneous moduli of the concrete. It is noted that such a formalism can not describe timedependent effects, nor can autogeneous strain rates be taken into account (e.g. thermal dilatation or shrinkage). However, generalisation of the constitutive law to include such effects is not difficult and a successful implementation has already been achieved<sup>23</sup>. It is noted that the concrete strain rate itself may also be conceived as a summation of several components, for instance of an elastic and a plastic part.

The relation between the stress rate in the crack  $\dot{\sigma}'_{ij}$  and the crack strain rate  $\dot{\varepsilon}'_{kl}$  of the primary crack is assumed to be given by

$$\dot{\sigma}'_{ij} = D'_{ijkl} \dot{\varepsilon}'_{kl} \tag{4.4}$$

where the primes signify that the stress rate respectively the crack strain rate components of the primary crack are taken with respect to the coordinate system of the crack. The fourth-order tensor  $D'_{ijkl}$  represents the stress-strain relation within the primary crack. Analogously, we have for a secondary crack

$$\dot{\sigma}''_{ij} = D''_{ijkl} \dot{\varepsilon}''_{kl} \tag{4.5}$$

The double primes mean that the stress rate respectively crack strain rate components of the secondary crack are taken with respect to the coordinate system of the secondary crack.

The strain rate tensor is a second order tensor. If, now,  $\alpha_{ik}$  are the direction cosines of the global coordinate system with respect to the coordinate system of the primary crack, and if  $\beta_{ik}$  are the direction cosines of the global coordinate system with respect to that of the secondary crack, we have the identities

$$\dot{\varepsilon}_{ij}^{I} = \alpha_{ik} \, \alpha_{jl} \, \dot{\varepsilon}'_{kl} \tag{4.6}$$

$$\dot{\varepsilon}_{ij}^{II} = \beta_{ik} \beta_{jl} \dot{\varepsilon}^{\prime\prime}{}_{kl} \tag{4.7}$$

Moreover, as we restrict our considerations to Cartesian tensors we also have

$$\dot{\sigma}'_{ij} = \alpha_{ki} \alpha_{lj} \dot{\sigma}_{kl} \tag{4.8}$$

$$\sigma''_{ij} = \beta_{ki} \beta_{lj} \dot{\sigma}_{kl} \tag{4.9}$$

To derive the final stress-strain law of the cracked concrete, we proceed as follows. First substitute the fundamental decomposition (4.1) in the constitutive law for the concrete and transform the crack strain rates in global coordinates  $\dot{\epsilon}_{ij}^{I}$  and  $\dot{\epsilon}_{ij}^{II}$  to local coordinates according to equations (4.6) and (4.7). This results in

$$\dot{\sigma}_{kl} = D_{klmn}^{co}(\dot{\varepsilon}_{mn} - \alpha_{mo} \,\alpha_{np} \,\dot{\varepsilon}'_{op} - \beta_{mo} \,\beta_{np} \,\dot{\varepsilon}''_{op}) \tag{4.10}$$

Transforming this expression for the stress rate to local coordinates according to the identities (4.8) and (4.9), equating the resulting expressions to the right-hand sides of the crack stress-strain relations (4.4) and (4.5) and rearranging gives respectively

$$A_{ijop}\dot{\varepsilon}'_{op} + B_{ijop}\dot{\varepsilon}''_{op} = \alpha_{ki}\alpha_{lj}D_{klmn}^{co}\dot{\varepsilon}_{mn}$$

$$\tag{4.11}$$

$$C_{iion}\dot{\varepsilon}'_{on} + E_{ijon}\dot{\varepsilon}''_{op} = \beta_{ki}\beta_{lj}D_{klmn}^{co}\dot{\varepsilon}_{mn}$$

$$\tag{4.12}$$

wherein we have put

$$A_{iion} = D'_{iion} + \alpha_{ki} \alpha_{lj} \alpha_{mo} \alpha_{np} D_{klmn}^{co}$$

$$\tag{4.13}$$

$$B_{ijop} = \alpha_{ki} \alpha_{lj} \beta_{mo} \beta_{np} D_{klmn}^{co} \tag{4.14}$$

1 . . .

$$C_{ijop} = \beta_{ki} \beta_{lj} \alpha_{mo} \alpha_{np} D_{klmn}^{co}$$
(4.15)

$$E_{ijon} = D''_{ijon} + \beta_{ki} \beta_{lj} \beta_{mo} \beta_{np} D_{klmn}^{co}$$

$$\tag{4.16}$$

Solving for  $\dot{\epsilon}'_{mn}$  and  $\dot{\epsilon}''_{mn}$  and substituting these expressions in equation (4.10) finally gives

$$\begin{split} \dot{\sigma}_{ij} &= \left[ D_{ijkl}^{co} - D_{ijab}^{co} \alpha_{ac} \alpha_{bd} (A_{cdef} - B_{cduv} E_{urwz}^{-1} C_{wzef})^{-1} \alpha_{se} \alpha_{tf} \right. \\ &+ D_{ijab}^{co} \alpha_{ac} \alpha_{bd} (A_{cdef} - B_{cduv} E_{urwz}^{-1} C_{wzef})^{-1} B_{efgh} E_{ghmn}^{-1} \beta_{sm} \beta_{tn} \\ &+ D_{ijab}^{co} \beta_{ac} \beta_{bd} E_{cdef}^{-1} C_{efgh} (A_{ghmn} - B_{ghuv} E_{urwz}^{-1} C_{wzmn})^{-1} \alpha_{sm} \alpha_{tn} \\ &- D_{ijab}^{co} \beta_{ac} \beta_{bd} [E_{cdqr}^{-1} + E_{cdef}^{-1} C_{efgh} (A_{ghmn} - B_{ghuv} E_{urwz}^{-1} C_{wzmn})^{-1} \\ &- B_{mnop} E_{opqr}^{-1}] \beta_{sq} \beta_{tr} D_{stkl}^{co} ] \dot{\varepsilon}_{kl} \end{split}$$

It is noted that as long as the constitutive tensors  $D^{eq}_{ijkl}$ ,  $D'_{ijkl}$  and  $D''_{ijkl}$ 

remain symmetric with respect to the interchange of ij and kl, also the symmetry of the total stress-strain law (4.17) with respect to ij and kl is preserved.

The constitutive law (4.17) of the cracked concrete obeys the principle of material frame indifference (objectivity) if the constitutive law for the intact concrete (4.3) obeys this principle. This follows immediately from the objectivity of stress and strain rates, from the objectivity of the stress-strain relation for the cracks as expressed through equations (4.4) to (4.9), and from the assumed objectivity of equation (4.3). Hence, all quantities and constitutive assumptions which we use in deriving equation (4.17) are objective, and so equations (4.17) is objective as this equation is simply the result of algebraic manipulations with objective quantities. It is obvious that a generalisation to an arbitrary number of cracks also results in an objective stress-strain relationship for the cracked concrete since an extra term in equation (4.1) does not affect the objectivity of this equation, and since the definition of the stress-strain law in subsequent cracks is essentially similar to the definitions (4.4) to (4.9) for the first two cracks.

In fact, the structure of equation (4.17) is quite similar to the structure of an elastoplastic stiffness tensor at a yield vertex. Indeed, any constitutive law in which a decomposition in the sense of equation (4.1) is assumed, will lead to an equation with a similar structure. This holds true for a yield vertex in which two yield surfaces are active, but for instance also for the intersection of a yield surface and a fracture surface, issues to which we will return in subsequent sections.

## 4.1.2 Crack stress-strain relation

When we consider the physical phenomenon of crack formation, we observe that the behaviour of a concrete element is approximately isotropic and linearly elastic until a crack arises for some combination of tensile and compressive stresses. Below this threshold stress level, no crack strains exist ( $\varepsilon_{ij}^{I}=0$  etc.), and all strain rates are concrete strain rates. Above the critical stress level, i.e. after crack formation, the strain rates are decomposed according to equation (4.1). As a criterion for crack formation we can analogous to plasticity introduce a scalar function of the stresses (and possibly of the strain history) which may be called a fracture function and which distinguishes between cracked and uncracked states. In this study it is superfluous to introduce such a formalism because we will always assume that a crack is initiated if the major principal stress  $\sigma_1$  exceeds the tensile strength  $f_{ct}$ . Hence, a crack will arise

 $\sigma_1 = f_{ct}$ 

if

and the direction of the crack is assumed to be orthogonal to the principal major stress. Moreover, the crack direction is assumed not to change with (possible) rotation of principal stresses in subsequent loading steps, or put differently, the crack has a memory for the damage direction.

There is some argument in literature whether such a simple criterion complies with experimental data or more precisely, how compressive stresses orthogonal to the major principal tensile stress decrease the apparent tensile strength<sup>69</sup>. Most researchers agree that the existence of such compressive stresses decreases the tensile stress that is needed for crack formation, but there is no communis opinio about the relationship, and the simple relation (4.18) has been adopted because of the experimental scatter. It is noted that this effect is also partially obtained when we limit the compressive stresses by a yield function as will be discussed in a subsequent section.



Figure 4.1. Local coordinate system of a crack and sign convention.

Another salient characteristic of crack formation concerns the fact that in the most general case of a three-dimensional solid, only 3 out of 6 components of the crack strain-rate vector are possibly non-zero, viz. the normal strain rate and two shear strain rates. We therefore assume that the stress-strain law for the crack has a structure such that the other strain-rate components vanish. Moreover, we assume that the nonvanishing strain-rate components are related to the components of the stress-rate vector via

$$\dot{\sigma}'_{xx} \\ \dot{\sigma}'_{xy} \\ \dot{\sigma}'_{xz} \end{bmatrix} = \begin{bmatrix} D'_{11} & D'_{12} & D'_{13} \\ D'_{21} & D'_{22} & D'_{23} \\ D'_{31} & D'_{32} & D'_{33} \end{bmatrix} \begin{bmatrix} \dot{\varepsilon}'_{xx} \\ \dot{\varepsilon}'_{xy} \\ \dot{\varepsilon}'_{xz} \end{bmatrix}$$
(4.19)

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(4.18)

the meaning of the subscripts being explained in Figure 4.1.

Equation (4.19) is a very general constitutive law for a crack as it allows for coupling effects in the sense that e.g. the normal stress rate in the crack not only depends on the normal crack strain rate but also on both shear crack strain rates. Similarly, any one of the shear stress rates may depend on all non-vanishing crack strain-rate components. Such coupling effects for instance occur in crack-dilatancy theories<sup>91</sup>. Most of our applications are however restricted to small crack strains and then the off-diagonal terms in (4.19) are less important. Consequently, we have set the off-diagonal terms equal to zero in the sample problems, so that (4.19) reduces to:

$$\begin{vmatrix} \dot{\sigma}'_{xx} \\ \dot{\sigma}'_{xy} \\ \dot{\sigma}'_{xz} \end{vmatrix} = \begin{vmatrix} C & 0 & 0 \\ 0 & \beta^* \mu & 0 \\ 0 & 0 & \beta^* \mu \end{vmatrix} \begin{vmatrix} \dot{\varepsilon}'_{xx} \\ \dot{\varepsilon}'_{xy} \\ \dot{\varepsilon}'_{xz} \end{vmatrix}$$
(4.20)

Herein, the tangent modulus C represents the relation between the normal crack strain increment and the normal stress increment (see Figure 4.2),  $\mu$  is the elastic shear modulus and  $\beta^*$  is a shear stiffness reduction factor.



Figure 4.2. Normal stress versus normal crack strain.

In practice, the modulus C will be negative as we will normally have a descending relation between the stress rate normal to a crack and the normal crack strain rate. Various possibilities exist for the shape of the softening curve. In the examples of the last chapters a linear as well as a non-linear curve<sup>31</sup> have been employed, the latter curve being more

ecr

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consistent with experimental evidence. However, recent research? indicates that a straightforward translation from experimental data in a value for C leads to results which are not objective with regard to mesh refinement as we will discuss in somewhat greater detail in Section 4.4. To overcome this problem, it has been proposed to consider the fracture energy  $G_f$  which is defined as the amount of energy needed to create one unit of area of a continuous crack<sup>7,58</sup> as the fundamental parameter which governs crack propagation. This so-called 'fictitious crack' or 'tension-softening' model has also been adopted in this study, although it is beginning to emerge that the concept is not entirely free from deficiencies. This is particularly so when we allow for the possibility of multiple cracks. Suppose that a primary crack has been created with a softening modulus C determined from the fracture energy  $G_f$ . If upon formation of a secondary crack the same crack stress-strain relation is adopted for the second crack, the fracture energy will be consumed twice. If both cracks are orthogonal to each other, this seems not unrealistic, but for any other inclination angle it seems incorrect. A solution to this problem seems only possible if a comprehensive stress-strain relation within a crack has been developed which incorporates at least an objective (with regard to mesh refinement) relation for shear softening and possibly also some theory for normal-shear coupling. Hence, the concept of a fracture energy as outlined above seems not to suffice for multiple crack formation. Indeed, a solution in which the fracture energy is distributed over both cracks is not correct as the fracture energy  $G_f$ is not a scalar, but a vector although this seems not to have been recognised widely. In this respect, use of the term fracture energy for  $G_f$  is perhaps somewhat misleading.

Other problems with the application of fracture energy concepts in smeared crack analysis relate to axisymmetric configurations where the integration of the strain over the crack band width entails complications owing to the 1/r term, and to the shape of the softening branch which may influence the results significantly<sup>96</sup>.

The term  $\beta^* \mu$  gives the relation between the shear stress increment in a crack and the shear crack strain increment and accounts for effects like aggregate interlock. The meaning of the reduction factor  $\beta^*$  differs from the classical shear retention factor  $\beta$  as introduced by Suidan and Schnobrich<sup>109</sup>. A relation between  $\beta$  and  $\beta^*$  can be derived from the consideration that the total shear strain increment is resolved in a crack strain increment and a concrete strain increment. Assuming that the concrete behaves in a linearly elastic manner, we may then derive that  $\beta^* = \beta / (1-\beta)$  if we have only one crack and  $\beta^* = 2\beta / (1-\beta)$  if we have two orthogonal cracks. For multiple non-orthogonal cracks or for inelastic concrete behaviour, more complicated relations ensue. It should be emphasised that in the present approach, the relation between the shear stress and the shear crack strain is considered as being fundamental. Hence,  $\beta^{\bullet}$  is conceived as a material parameter and the relation with the more familiar  $\beta$  has only been derived to give the reader an idea of the range of values which can be used for  $\beta^{\bullet}$ . In the sample problems which we will present,  $\beta^{\bullet}$  has been assumed to be a constant both for loading and unloading, but a more realistic approach would be to make  $\beta^{\bullet}$  a function at least of the crack strain<sup>64</sup>. Unfortunately, few experimental data exist to support a particular expression for  $\beta^{\bullet}$ .

Especially when we allow the formation of non-orthogonal multiple cracks, we face the problem of crack arrest, unloading and even closing of existing cracks. This occurs for instance when a new crack arises in an integration point. It is therefore important to carefully handle closing and eventually reopening of cracks. For the unloading branch we have adopted a secant approach as is shown in Figure 4.2. So when a crack starts unloading, expression (4.20) is replaced by:

$$\begin{vmatrix} \dot{\sigma}'_{xx} \\ \dot{\sigma}'_{xy} \\ \dot{\sigma}'_{xz} \end{vmatrix} = \begin{bmatrix} S & 0 & 0 \\ 0 & \beta^* \mu & 0 \\ 0 & 0 & \beta^* \mu \end{bmatrix} \begin{vmatrix} \dot{\varepsilon}'_{xx} \\ \dot{\varepsilon}'_{xy} \\ \dot{\varepsilon}'_{xz} \end{vmatrix}$$
(4.21)

where S is the secant modulus of the unloading branch. This assumption is too simple as in reality we may expect some residual strain upon closing of a crack<sup>92</sup>, but current experiences indicate that this procedure is numerically stable.

When a crack fully closes, i.e. when the normal stress in the crack changes from tension into compression, the stiffness of the uncracked concrete is inserted again, so that  $\dot{\sigma}_{ij} = D_{ijkl}^{co} \dot{\varepsilon}_{kl}$ . Although upon closing of a crack, the normal strain and the normal stress both vanish in the present conception, this need not be true for the shear stress and the shear strain. Consequently, these stresses and strains are considered as initial stresses and initial strains upon closing of the crack and the stress after closing formally follows from:

$$\sigma_{ij} = \sigma_{ij}^0 + \int D_{ijkl}^{co} \dot{\varepsilon}_{kl} \, d\tau \tag{4.22}$$

where  $\sigma_{ij}^0$  is the stress state that exists when the crack closes. If we have linear-elastic material behaviour in compression  $(D_{ijkl}^{co}=D_{ijkl}^{e})$ , equation (4.22) reduces to:

$$\sigma_{ij} = \sigma_{ij}^0 + D_{ijkl}^e (\varepsilon_{kl} - \varepsilon_{kl}^0) \tag{4.23}$$

with  $\varepsilon_{kl}^{0}$  the strain that exists in the sampling point upon closing of the crack. When a crack opens again the shear crack strain is initialized with the value which follows from the crack strain which existed when the crack closed. During reloading equation (4.21) is used for the incremental relation between the crack stresses and the crack strains. If the normal strain in the crack exceeds the previously reached maximum strain, relation (4.21) is replaced by equation (4.20) for the softening branch.

#### 4.1.3 Threshold angle for non-orthogonal cracks

With the crack model as outlined in the preceding sub-sections, it is possible to permit an arbitrary number of cracks in an integration point. Yet, allowing new cracks to form every time that the stresses rotate slightly and violate the tensile strength in the new principal direction, leads to excessive cracking and closing of existing cracks. Moreover, the existence of multiple cracks in an integration point results in a rapid decrease of the shear stiffness of the total system of concrete and cracks, which may lead to premature ill-conditioning of the stiffness matrix. For this reason, a threshold angle  $\alpha$  has been introduced which allows new cracks to form only when the angle between the current direction of the major principal stress and the normal to the existing cracks has exceeded  $\alpha$ .

The threshold angle  $\alpha$  has a similar influence on the results as the shear reduction factor  $\beta^*$  since a low value for  $\alpha$  also decreases the shear capacity of the system. For the particular case of two orthogonal cracks ( $\alpha$ =90°), an expression between the shear stiffness of the system of concrete and cracks and the shear reduction factor  $\beta^*$  as used here can be derived analytically<sup>22</sup>. For the more general case of non-orthogonal cracks, such a derivation is very cumbersome and the exact way in which the values of  $\alpha$  and  $\beta^*$  interact, is difficult to assess. We will therefore demonstrate the possible impact of the value of  $\alpha$  on the computational results by means of a moderately deep shear-critical beam<sup>121</sup>.

The beam which we will consider is shown in Figure 4.3. The analysis has been carried out using 8-noded plane stress elements with 9-point Gaussian integration to reduce the possibility of the occurrence of spurious zero-energy modes. The properties of the concrete were assumed to be: Young's modulus  $E_c$ =28000 N/mm<sup>2</sup>, Poisson's ratio  $\nu$ =0.2, tensile strength  $f_{ct}$ =2.5 N/mm<sup>2</sup>, shear reduction factor  $\beta^*$ =0.087 and fracture energy  $G_f$ =0.06 N/mm. The beam has a thickness of 200 mm and has no shear reinforcement. The reinforcement at the bottom of the beam has a Young's modulus  $E_s$ =210000 N/mm<sup>2</sup> and a yield stress  $\sigma_y$ =440 N/mm<sup>2</sup>.



Figure 4.3. Finite element mesh for moderately deep beam.

Numerical analyses have been carried out for two different threshold angles, namely for  $\alpha=60^{\circ}$  and  $\alpha=30^{\circ}$ . In both analyses we observe that vertical cracks due to bending arise first. On subsequent loading the stresses rotate and new, non-vertical crack form in the region between the point load and the support (see Figure 4.4). Now we observe a marked difference between the results for both threshold angles. The diagonal cracks in the analysis for the high threshold angle tend to run rather steep (Figure 4.5). For the lower threshold angle ( $\alpha=30^{\circ}$ ), the shear capacity at a cross-section is exhausted more quickly. The formation of cracks between the load and the support remains more or less near the reinforcement and the initiation of diagonal cracking is forced to move to the centre of the beam (Figure 4.6). A similar trend has been observed for another beam upon reduction of the shear reduction factor  $\beta^*$ , as in that case inclined cracks tended to propagate along the reinforcement for very low values of  $\beta^*$  <sup>95</sup>.

The difference between the results for both threshold angle becomes even more apparent when only those cracks are plotted which are out of the softening branch, which may be interpreted as micro-cracks which have coalesced into one macro-crack (Figures 4.7 and 4.8). These figures also serve the purpose of demonstrating that the somewhat diffuse crack pattern which is observed in most smeared crack analyses largely disappears when we plot only those cracks which really open. Indeed, the crack patterns of Figures 4.7 and 4.8 clearly show strain-localisation including the initiation of a diagonal crack<sup>20,22,23</sup>.

# 4.1.4 Discussion

In the past, various other smeared crack models have been developed and definite relations can be elaborated with some of these models. The first smeared crack analysis has probably been performed by Rashid<sup>89</sup>. In



Figure 4.4. Experimentally recorded crack pattern at impending failure.



Figure 4.5. Calculated crack pattern at impending failure for high threshold angle (all cracks plotted).



Figure 4.6. Calculated crack pattern at impending failure for lower threshold angle (all cracks plotted).

his approach the normal stress was set to zero immediately upon crack formation. Moreover, no shear resistance across a crack was permitted as the shear modulus  $\mu$  was also assumed to vanish upon crack formation. Later, it was recognised that such an approach grossly underestimates the stiffness of a structure after crack formation and a reduced shear stiffness was inserted in the stiffness matrix after crack formation<sup>109</sup>. Also, the sudden drop in normal tensile stress across a crack was replaced by a gradual softening branch. Initially, this softening branch was attributed to the contribution of the stiffness of the concrete between the cracks to the stiffness of the reinforcement, but later<sup>7,58</sup>, it was also related to the softening of the concrete itself. Further, Bažant



Figure 4.7. Cracks that transfer no normal stress at impending failure for high threshold angle.



Figure 4.8. Cracks that transfer no normal stress at impending failure for lower threshold angle.

and  $Oh^7$  proposed to start from a compliance approach instead of a stiffness approach, so that off-diagonal terms in the orthotropic stiffness matrix of the cracked concrete were retained. The present approach relates to their approach because for the case of only one crack and elastic concrete behaviour (to which their treatment was restricted), their model is recovered as a special case of the present approach<sup>22</sup>.

The introduction of a reduced shear stiffness to represent aggregate interlock and possibly dowel action, and the gradual decrease of the normal tensile stress after crack formation introduced the problem that on subsequent loading, the direction of the current major principal stress may deviate from the normal to the crack and moreover, the current major principal stress may exceed the tensile strength. An approach which is often adopted is to allow a secondary crack to form only orthogonal to a primary crack<sup>79</sup>. Again this approach is recovered as a particular case of the model adopted here if the threshold angle is set equal to 90° <sup>22</sup>. The problem of rotation of principal stress axes after primary crack formation was also recognised by Cope et al.<sup>28,29</sup>, who proposed to corotate the material axes with the rotation of principal stress axes, either without<sup>28</sup>, or with a threshold value for the angle between the current direction of the major principal stress and the original crack direction after which the material axes are rotated<sup>29</sup>. Although this approach was criticised by Bažant because it violates the principle of material frame indifference<sup>6,10,111</sup>, recent research has indicated that results obtained using such a 'rotating crack model' are often superior to results obtained using the traditional fixed crack models<sup>74</sup>. Other models which allow for the possibility of multiple crack formation to solve this problem have been proposed by Ebbinghaus<sup>42</sup>, Kristjansson<sup>66</sup> and Litton<sup>68</sup>. Nevertheless, these models have limited capabilities in describing the stress-strain relation within a crack, and à priori assume linear-elastic material behaviour of the concrete between the cracks. Owing to the rigorous decomposition of strain increments, such limitations do not have to be imposed in the approach outlined here. Moreover, the model complies with the principle of material frame indifference.

#### 4.2. Granular materials in triaxial compression

In this section we shall outline a plasticity model which is able to describe some of the basic characteristics which concrete displays in triaxial compression. The treatment involves as salient characteristics hardening on the frictional and softening on the cohesive properties of concrete as originally conceived by Vermeer<sup>119</sup>, while a non-associated flow rule is employed to control the inelastic volume changes.

## 4.2.1 The yield function

A first observation with regard to the strength properties of materials like concrete, rock and soils is that their strength depends on the first stress invariant. The oldest criterion which implies a dependence of the strength on the stress level, is the Mohr-Coulomb criterion. Let f be the yield function,  $\sigma_1$  be the minor and  $\sigma_3$  be the major principal stress (tension being taken as positive in accordance with the sign convention in continuum mechanics). Then, this criterion reads:

$$f = \frac{1}{2} (\sigma_3 - \sigma_1) + \frac{1}{2} (\sigma_3 + \sigma_1) \sin \varphi^* - c^*$$
(4.24)

The Mohr-Coulomb criterion has two strength parameters, namely the so-called mobilised friction angle  $\varphi^*$  and a mobilised cohesion  $c^*$  (which has a slightly different definition than is normally given to a cohesion as we observe from equation (4.24)). We have introduced the notation \* to indicate that  $\varphi^*$  and  $c^*$  are no constants, but depend on the plastic strain history through a hardening parameter  $\kappa$ .

The Mohr-Coulomb criterion is simple and has a clear physical meaning. Moreover, extensive testing on sand has shown that it matches test data quite reasonably<sup>49</sup>. For materials which in addition to frictional strength also possess a cohesive strength, deviations occur and the Mohr-Coulomb criterion appears to be conservative, see for instance test data by van Mier<sup>73</sup> and Gerstle et al.<sup>46,47</sup>. Although more sophisticated criteria exist<sup>26,67,123</sup>, we will adhere to the Mohr-Coulomb idea in order to preserve a transparency in the mathematical model which we will develop to describe the inelastic behaviour of concrete in triaxial compression. Another reason is that in actual computations in which triaxial stress states occur in concrete structures, other criteria are seldom employed. Here, we see an example of the discrepancy between the levels of sophistication of constitutive modelling and of numerical modelling, which was discussed in the introduction.

The strength parameters  $\varphi^{\bullet}$  and  $c^{\bullet}$  can both depend on the strain history through the hardening parameter  $\kappa$ . The choice of a correct definition for  $\kappa$  is difficult as its definition should be such that it attains the same value for all points on the yield surface f. Schreyer<sup>101,103</sup> for instance uses a linear combination of all three inelastic strain invariants weighted by the stress level for  $\kappa$  and found that for a low strength concrete, only the first and the third inelastic strain invariants are of importance. Vermeer<sup>116,117</sup> on the other hand only uses the second inelastic strain invariant in describing another granular material, namely sand. In this study, the definition

$$\kappa = \int \sqrt{\frac{2}{3}} \dot{\varepsilon}_{ij}^{p} \dot{\varepsilon}_{ij}^{p} dt \tag{4.25}$$

has been employed with  $\dot{\epsilon}_{ij}^{p}$  the plastic strain-rate tensor, but it seems that a proper choice for  $\kappa$  is still an open question which has to be answered by considering a great deal of experimental data.

In spite of the paucity of experimental data to support a definition of  $\kappa$ , we are able to say something about the dependence of  $\sin \varphi^*$  and  $c^*$  on  $\kappa$ , for  $\sin \varphi^*$  should generally be an ascending function of  $\kappa$ , while  $c^*$  may expected to be a descending function of  $\kappa$ . The rationale behind the latter assumption is that during loading of a specimen of originally intact concrete, micro-cracks gradually develop. So, the cementation gradually vanishes at continued loading which leads to a degradation of the cohesive strength. A possible choice which expresses such a softening on the cohesive strength mathematically, is:

$$c^* = c \exp\left[-\left(\frac{\kappa}{\varepsilon_c}\right)^2\right] \tag{4.26}$$



Figure 4.9. Possible relations for the hardening/softening model. (a)  $c^* - \kappa$  relation (b)  $\sin \varphi^* - \kappa$  relation.

with c the cohesion of the intact material. The meaning of  $\varepsilon_c$  is shown in Figure 4.9a, in which relation (4.26) has been plotted. It is noted that the softening on the cohesion as formulated here implies that softening is conceived as a material property. Lately, there has been much debate whether softening is indeed a material property or whether it is a structural property<sup>90</sup> and we will return to this issue in a subsequent section.

At continued loading, concrete becomes more and more cracked and the initially cemented material very much begins to resemble a particulate material with only frictional resistance. The frictional resistance gradually increases in the loading process as the asperities protruding from the faces of the micro-cracks offer more and more resistance to subsequent sliding. This phenomenon may be modelled by applying hardening on the friction angle in addition to the softening on the cohesion. A possible  $\sin \varphi^* - \kappa$  relation is:

$$\begin{cases} \sin\varphi^* = 2\frac{\sqrt{\kappa\varepsilon_f}}{\kappa + \varepsilon_f} \sin\varphi & \kappa \le \varepsilon_f \\ \sin\varphi^* = \sin\varphi & \kappa > \varepsilon_f \end{cases}$$
(4.27)

which has been plotted in 4.9b. In it,  $\varepsilon_f$  is the value which  $\kappa$  attains when the frictional strength has been mobilised fully  $(\varphi^* = \varphi)$ .

Let us now explore the implications of the combination of friction hardening and cohesive softening. This is done most easily by simulating a triaxial test for different levels of confining pressure, the results of which have been plotted in Figure 4.10. Two important observations can be made from these results. First, the model predicts an increased ductility with higher confining pressure, and secondly, the amount of postpeak softening gradually vanishes at higher load levels. At a first sight,

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especially the latter result seems peculiar, because the cohesion is assumed to vanish entirely for all stress levels. It is however not correct to identify the entire degradation of the cohesion at all stress levels with a vertical shift of the limit surface<sup>126</sup>. Indeed, the softening on the cohesion corresponds to a vertical shift, but the hardening on the friction angle corresponds to a rotation of the yield surface. These effects interact and cause the amount of softening to vanish with increasing stress level, so that for high stress levels we end up with a ductile, nonsoftening material.

An inherent drawback in adopting a Mohr-Coulomb yield function is that it possesses straight meridians. Hence, the ultimate yield surface (that is the yield surface which is reached when all cohesion has vanished and when the frictional resistance has been mobilised fully), also has straight meridians. For high stress levels, it was argued that the difference between the failure surface (which gives the peak stress states) and the ultimate yield surface vanishes asymptotically, but for low stress levels the ultimate yield surface is clearly inside the failure surface. As the ultimate yield surface is straight, these observations imply that the failure surface is not convex. A possible solution is to slightly modify the yield function as follows:

$$f = \frac{1}{2} (\sigma_3 - \sigma_1) - [c^* - \frac{1}{2} (\sigma_3 + \sigma_1) \sin \varphi^*]^m$$
(4.28)

with m an additional parameter. For m=1 we recover the linear relationship (4.24) and for  $m=\frac{1}{2}$  we obtain a parabolic relationship. Yet, equation (4.28) is still formulated in the spirit of the Mohr-Coulomb idea as it neglects any possible dependence of the yield function on the intermediate stress.

# 4.2.2 Plastic potential and flow rule

Another salient characteristic of granular materials including concrete is the observed shear-enhanced inelastic volume changes. When a concrete specimen is sheared, we not only measure a shear strain, but also a volumetric strain. In the early stages of the loading process, this volumetric strain appears to be negative (so that we have contraction), but at continued loading the volumetric strain rate becomes positive (see Figure 4.11).



Figure 4.11. Axial strain versus volumetric strain for triaxial test results on a high-strength concrete<sup>51</sup> and model simulation using equation (4.34). Confining pressure  $\sigma_3$ =-13.78 N/mm<sup>2</sup>.

This phenomenon is not captured within classical plasticity theory, as the associated flow rule which is based on normality of plastic strain rate and yield surface predicts positive volumetric strain rates from the onset of inelastic behaviour. Moreover, the predicted rate of volume production greatly exceeds the experimental data, as application of an associated flow rule

$$\dot{\varepsilon}_{ij}^{p} = \lambda \frac{\partial f}{\partial \sigma_{ij}} \tag{4.29}$$

with  $\lambda$  a non-negative multiplier, gives in conjunction with a Mohr-Coulomb yield function the relation

$$\dot{\varepsilon}_{v}^{p} = \frac{-2\sin\varphi^{*}}{1-\sin\varphi^{*}}\dot{\varepsilon}_{1}^{p} \tag{4.30}$$

between the inelastic volumetric strain rate and the inelastic axial strain rate. For realistic values of  $\varphi^*$ , however, this leads to a large deviation from the experimental curve of Figure 4.11. Therefore,  $\varphi^*$  is replaced by the so-called mobilised dilatancy angle  $\psi^*$ . Rewriting equation (4.30) with  $\psi^*$  in place of  $\varphi^*$ , gives

$$\sin\psi^* = \frac{\dot{\varepsilon}_v^p}{-2\dot{\varepsilon}_v^p + \dot{\varepsilon}_v^p} \tag{4.31}$$

which can be regarded as the definition for the mobilised dilatancy angle  $\psi^*$  and which is valid for general triaxial stress states<sup>119</sup>. Mathematically, this phenomenon can be incorporated within plasticity theory via a non-associated flow rule. Then we must define an independent function g

$$g = \frac{1}{2} (\sigma_3 - \sigma_1) + \frac{1}{2} (\sigma_3 + \sigma_1) \sin \psi^* - c^*$$
(4.32)

which differs from the yield function f as the mobilised friction angle  $\varphi^*$  is replaced by the mobilised dilatancy angle  $\psi^*$ . Replacing the yield function f in equation (4.29) by the plastic potential function g, we precisely obtain expression (4.31) for  $\sin\psi^*$ . It is noted that for  $\varphi^* = \psi^*$ , we have f = g and the classical associated flow rule is recovered.

From Figure 4.11, we observe that  $\dot{\varepsilon}_v/\dot{\varepsilon}_1$  is initially positive (contraction) and then becomes more and more negative (dilatation) until a limiting value is attained at and beyond peak stress level. Noting that for compressive loadings, the elastic volumetric strain is negative, the gradual decrease of the dilatation rate  $\dot{\varepsilon}_v/\dot{\varepsilon}_1$  can only be modelled by making the plastic dilatation rate  $\dot{\varepsilon}_v^p/\dot{\varepsilon}_1^p$  a function of some stress measure. An example thereof is Rowe's<sup>98</sup> stress-dilatancy theory which states that:

$$1 - \dot{\varepsilon}_{v}^{p} / \dot{\varepsilon}_{1}^{p} = \left\{ \frac{\tan(45^{\circ} + \varphi^{*} / 2)}{\tan(45^{\circ} + \varphi_{cv} / 2)} \right\}^{2}$$
(4.33)

with  $\varphi_{cv}$  a constant, which is usually referred to as the 'friction angle of constant volume', which stems from the fact that equation (4.33) gives  $\dot{\varepsilon}_v^p = 0$  for  $\varphi^* = \varphi_{cv}$ . As a matter of fact, the equation predicts negative plastic dilatation for small friction angles ( $\varphi^* < \varphi_{cv}$ ) and positive plastic dilatation for larger values ( $\varphi^* > \varphi_{cv}$ ). Hence,  $\varphi_{cv}$  marks a turning point at which plastic contraction stops and dilatation starts. Rowe's stress-dilatancy equation (4.33) has originally been proposed to describe the dilatant behaviour of soils and it has been proved to be accurate for sand<sup>116</sup>. Figure 4.11 shows a model simulation for concrete and

comparison with the experimental data shows that equation (4.33) may also be useful for concrete and probably also for other cemented granular materials like rock, although it should be added that the precise outcome of the model depends on the values of the parameters, especially those for  $\varepsilon_f$  and  $\varepsilon_c$ . Furthermore, preliminary results seem to indicate that equation (4.33) should be amended for the behaviour of concrete at lower stress levels because of the increased brittleness.

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It is convenient to use equation (4.33) in a somewhat different form. Eliminating  $\dot{\varepsilon}_{v}^{p}/\dot{\varepsilon}_{1}^{p}$  from (4.31) and (4.33), we obtain the more suitable form

$$\sin\psi^{\bullet} = \frac{\sin\varphi^{\bullet} - \sin\varphi_{cv}}{1 - \sin\varphi^{\bullet}\sin\varphi_{cv}}$$
(4.34)

This is a useful relationship between the mobilised dilatancy angle and the effective strain as  $\varphi^*$  is a function of  $\kappa$ . The constant  $\varphi_{cv}$  is readily calculated from the limit dilatancy angle  $\psi$  and the limit friction angle  $\varphi$ . Substituting  $\psi^* = \psi$  and  $\varphi^* = \varphi$  and rearranging equation (4.34) we obtain

$$\sin\varphi_{cv} = \frac{\sin\varphi - \sin\psi}{1 - \sin\varphi \sin\psi} \tag{4.35}$$

For the concrete of Figure 4.11<sup>51</sup> we have  $\varphi=37^{\circ}$ ,  $\psi=12.5^{\circ}$  and consequently  $\varphi_{cv}=26^{\circ}$ , which agrees remarkably well with values which are reported for sands  $(25^{\circ} < \varphi_{cv} < 33^{\circ})$ .





The above formulated constitutive model is not able to represent all characteristics of concrete. An example thereof is the inelastic behaviour of concrete in pure hydrostatic loading (see Figure 4.12). This behaviour may be explained from the consideration that above some threshold stress level, the pores collapse and a softer response to subsequent load increments is obtained. At a later stage in the loading process, a stiffening is again observed. However, the Mohr-Coulomb yield function is open in the hydrostatic direction, so that the response of a concrete specimen will be purely elastic when it is loaded hydrostatically. In order to capture plastic volume changes under hydrostatic stressing, we need a yield cap that closes the Mohr-Coulomb surface<sup>116,117</sup>.

## 4.2.3. A simplified model

The constitutive model for granular materials as formulated in the preceding attempts to strike a balance between accuracy in describing the basic characteristics of granular materials in triaxial compression and simplicity which is necessary for successful use in numerical calculations. In a considerable number of problems, however, we can suffice by using a simpler model.

In the hardening-softening model  $c^*$ ,  $\varphi^*$  and  $\psi^*$  all depend on the plastic strain history through the hardening parameter  $\kappa$ . A considerable simplification occurs if we neglect this dependence, i.e. we assume  $c^*$ ,  $\varphi^*$  and  $\psi^*$  to be constants:

$$c^* = c \quad \varphi^* = \varphi \quad \psi^* = \psi$$

This elastic-perfectly plastic model is especially useful for limit load calculations. For accurate predictions of stresses and strain under working loads the simplified model is less suitable, but as in this study attention is primarily concentrated on calculating limit points and post-peak behaviour, there is little restriction employing such a relatively simple model.

The determination of the parameters  $\varphi$  and  $\psi$  for such a model is straightforward, as  $\varphi$  simply is the limit friction angle, i.e. the value which  $\varphi$  attains when the frictional resistance has been mobilised fully.  $\psi$ is the value for the dilatancy angle which belongs to the straight part of the curves of Figure 4.11 and which can e.g. be calculated from

$$\sin\psi = \frac{\sin\varphi - \sin\varphi_{cv}}{1 - \sin\varphi \sin\varphi_{cv}} \tag{4.36}$$

The value for c is more difficult to estimate, as for cemented granular materials, the cohesion gradually vanishes. Depending on the amount of plastic straining in the limit state, an estimate for c can be used, but if the total structural response depends significantly on the compressive strength of the material, softening under compressive stresses cannot be

disregarded and the simplified model cannot be used for accurate predictions of the limit load and the post-peak behaviour.

#### 4.3. A plastic-fracture model

A merit of the smeared crack model described in this chapter is that it can be combined easily with models which describe the non-linear behaviour of the concrete between the cracks. We will now outline the mathematical structure of a model which permits crack formation and plasticity to occur simultaneously in a representative volume.

The starting point of the model is again the decomposition of strain rates. Recalling that application of a plasticity model to the concrete implies that the concrete strain rate is divided into an elastic strain rate  $\dot{\varepsilon}_{kl}^{p}$ , we have the following decomposition of the total strain rate

$$\dot{\varepsilon}_{kl} = \dot{\varepsilon}_{kl}^{e} + \dot{\varepsilon}_{kl}^{p} + \dot{\varepsilon}_{kl}^{cr} \tag{4.37}$$

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where the notation  $\dot{\varepsilon}_{kl}^{cr}$  has been introduced as the sum of the individual crack strain rates  $\dot{\varepsilon}_{kl}^{II}\dot{\varepsilon}_{kl}^{II}$  etc. This decomposition may be interpreted as a vertex in which a yield surface and a fracture surface intersect (see Figure 4.13).



Figure 4.13. Fan of possible strain rates at the intersection of a yield and a fracture surface.

In plasticity theory, the elastic strain rate tensor  $\dot{\varepsilon}_{kl}^{e}$  is assumed to be related to the stress rate tensor via the elasticity tensor  $D_{ijkl}^{e}$ 

$$\dot{\sigma}_{ij} = D^e_{ijkl} \dot{\varepsilon}^e_{kl} \tag{4.38}$$

while the plastic strain rate tensor is assumed to be derivable from the plastic potential g

$$\dot{\varepsilon}_{kl}^{p} = \lambda \frac{\partial g}{\partial \sigma_{kl}} \tag{4.39}$$

with  $\lambda$  some non-negative multiplier which can be determined from the condition that during loading, the stress tensor must satisfy the consistency condition  $\dot{f} = 0$ . In the preceding, f has been assumed to be a function of  $\sigma_{ij}$  and of the plastic strain history through the hardening parameter  $\kappa$ :  $f = f(\sigma_{ij}, \kappa)$ . Consequently, the consistency condition  $\dot{f} = 0$  can be elaborated as

$$\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \lambda \frac{\partial f}{\partial \kappa} \frac{\partial \kappa}{\partial \varepsilon_{ij}} \frac{\partial g}{\partial \sigma_{ij}} = 0$$
(4.40)

We can write this equation in a more compact form. If we define

$$h = -\frac{\partial f}{\partial \kappa} \frac{\partial \kappa}{\partial \varepsilon_{ij}} \frac{\partial g}{\partial \sigma_{ij}}$$
(4.41)

as the hardening modulus, we get

$$\frac{\partial f}{\partial \sigma_{kl}} \dot{\sigma}_{kl} - \lambda h = 0 \tag{4.42}$$

For the hardening/softening model of the preceding section,  $\boldsymbol{h}$  can be elaborated to be

$$h = -\sqrt{\frac{1}{3}(1 + \sin^2\psi^*)} \left\{ \frac{1}{2} (\sigma_3 + \sigma_1) \frac{\partial \sin\varphi^*}{\partial\kappa} - \frac{\partial c^*}{\partial\kappa} \right\}$$
(4.43)

We now proceed in a way which is essentially similar to the derivation for the elastic-fracturing material and we first substitute the decomposition (4.37) into equation (4.38). With the relations (4.6), (4.7) and (4.39) for  $\dot{\varepsilon}_{kl}^{p}$  and  $\dot{\varepsilon}_{kl}^{p}$  we obtain

$$\dot{\sigma}_{kl} = D^{e}_{klmn} \left[ \dot{\varepsilon}_{mn} - \alpha_{mo} \,\alpha_{np} \,\dot{\varepsilon}'_{op} - \beta_{mo} \,\beta_{np} \,\dot{\varepsilon}''_{op} - \lambda \frac{\partial g}{\partial \sigma_{mn}} \right] \tag{4.44}$$

Multiplying this equation with  $\frac{\partial f}{\partial \sigma_{\mu}}$  and invoking equation (4.42) gives

$$\left[ \frac{\partial f}{\partial \sigma_{kl}} D^{e}_{klmn} \frac{\partial g}{\partial \sigma_{mn}} + h \right] \lambda + \frac{\partial f}{\partial \sigma_{kl}} D^{e}_{klmn} \alpha_{mo} \alpha_{np} \dot{\varepsilon}'_{op}$$

$$+ \frac{\partial f}{\partial \sigma_{kl}} D^{e}_{klmn} \beta_{mo} \beta_{np} \dot{\varepsilon}''_{op} = \frac{\partial f}{\partial \sigma_{kl}} D^{e}_{klmn} \dot{\varepsilon}_{mn}$$

$$(4.45)$$

Similarly, multiplying equation (4.44) with  $\alpha_{ki}\alpha_{lj}$ , invoking (4.8) and the stress-strain law for the crack (4.4), we get

$$\begin{bmatrix} D'_{ijop} + \alpha_{ki} \alpha_{lj} \alpha_{mo} \alpha_{np} D^{e}_{klmn} \end{bmatrix} \dot{\varepsilon}'_{op} + \alpha_{ki} \alpha_{lj} \beta_{mo} \beta_{np} D^{e}_{klmn} \dot{\varepsilon}''_{op} + \lambda \alpha_{ki} \alpha_{lj} D^{e}_{klmn} \frac{\partial g}{\partial \sigma_{mn}} = \alpha_{ki} \alpha_{lj} D^{e}_{klmn} \dot{\varepsilon}_{mn}$$
(4.46)

while multiplying with  $\beta_{ki}\beta_{lj}$  gives with equations (4.5) and (4.9)

$$\beta_{ki}\beta_{lj}\alpha_{mo}\alpha_{np}D^{e}_{klmn}\dot{\varepsilon}'_{op} + [D''_{ijop} + \beta_{ki}\beta_{lj}\beta_{mo}\beta_{np}D^{e}_{klmn}]\dot{\varepsilon}''_{op} + \lambda\beta_{ki}\beta_{lj}D^{e}_{klmn}\frac{\partial g}{\partial\sigma_{mn}} = \beta_{ki}\beta_{lj}D^{e}_{klmn}\dot{\varepsilon}_{mn}$$

$$(4.47)$$

Solving for  $\lambda$ ,  $\dot{\varepsilon}'_{op}$  and  $\dot{\varepsilon}''_{op}$  and substituting these expressions in equation (4.44) finally yields

$$\dot{\sigma}_{ij} = \left\{ D_{ijkl}^{ef} - \frac{D_{ijmn}^{ef} \frac{\partial g}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{op}} D_{opkl}^{ef}}{h + \frac{\partial f}{\partial \sigma_{qr}} D_{qrst}^{ef} \frac{\partial g}{\partial \sigma_{st}}} \right\} \dot{\varepsilon}_{kl}$$
(4.48)

1

with  $D_{ijkl}^{ef}$  the elastic-fracture tensor,

$$D_{ijkl}^{ef} = D_{ijkl}^{e} - D_{ijab}^{e} \alpha_{ac} \alpha_{bd} [A_{cdef} - B_{cduv} E_{uvvz}^{-1} C_{wzef}]^{-1} \alpha_{se} \alpha_{tf}$$

$$+ D_{ijab}^{e} \alpha_{ac} \alpha_{bd} [A_{cdef} - B_{cduv} E_{uvvz}^{-1} C_{wzef}]^{-1} B_{efgh} E_{ghmn}^{-1} \beta_{sm} \beta_{tn}$$

$$+ D_{ijab}^{e} \beta_{ac} \beta_{bd} E_{cdef}^{-1} C_{efgh} [A_{ghmn} - B_{ghuv} E_{uvvz}^{-1} C_{wzmn}]^{-1} \alpha_{sm} \alpha_{tm}$$

$$- D_{ijab}^{e} \beta_{ac} \beta_{bd} \left[ E_{cdqr}^{-1} + E_{cdef}^{-1} C_{efgh} [A_{ghmn} - B_{ghuv} E_{uvvz}^{-1} C_{wzmn}]^{-1} \alpha_{sm} \alpha_{tm}$$

$$- D_{ijab}^{e} \beta_{ac} \beta_{bd} \left[ E_{cdqr}^{-1} + E_{cdef}^{-1} C_{efgh} [A_{ghmn} - B_{ghuv} E_{uvvz}^{-1} C_{wzmn}]^{-1} \beta_{sq} \beta_{tr} D_{stkl}^{e} \right]$$

$$(4.49)$$

wherein  $A_{ijkl}$ ,  $B_{ijkl}$ ,  $C_{ijkl}$  and  $E_{ijkl}$  are given by (4.13) to (4.16).

In fact, expression (4.48) for the plastic-fracturing material is insensitive to the number of cracks as only the tensor  $D_{klmn}^{ef}$  is affected by the number of cracks. For one crack for instance,  $D_{klmn}^{ef}$  reduces to

$$D_{ijkl}^{ef} = D_{ijkl}^{e} - D_{ijst}^{e} \alpha_{sm} \alpha_{tn} [D_{mnop}^{cr} + \alpha_{qm} \alpha_{rn} D_{qrst}^{e} \alpha_{so} \alpha_{tp}]^{-1} D_{opuv}^{e} \alpha_{uk} \alpha_{vl} (4.50)$$

and for the case that we have no cracks,  $D_{ijkl}^{ef}$  simply reduces to  $D_{ijkl}^{e}$ .

One might wonder why the cumbersome derivation from equation (4.37) to equation (4.49) has been presented, because a constitutive equation for a plastic-fracturing material can also be obtained by replacing the stiffness tensor of the concrete  $D_{ijkl}^{co}$  in equation (4.17) by the elastoplastic stiffness tensor  $D_{ijkl}^{ep}$ , so that

$$D_{ijkl}^{co} = D_{ijkl}^{ep} = D_{ijkl}^{e} - \frac{D_{ijmn}^{e} \frac{\partial g}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{op}} D_{opkl}^{e}}{h + \frac{\partial f}{\partial \sigma_{qr}} D_{qrst}^{e} \frac{\partial g}{\partial \sigma_{st}}}$$
(4.51)

Indeed, for infinitesimally small increments, both expressions for the tangent stiffness tensor of the plastic-fracturing material are entirely equivalent. However, equation (4.48) with  $D_{ijkl}^{ef}$  given by (4.49) is more suitable for integration to finite load increments than equation (4.17) with  $D_{ijkl}^{eo}$  given by (4.51), because under certain condition which will be discussed in the next chapter, equation (4.48) can be integrated numerically such that the stress at the end of the loading step exactly satisfies the yield function as well as the fracture function.

## 4.4. Consequences of strain-softening and non-associated plasticity

In Section 2.3 it was argued, that for small displacement gradients, stability of a structure under dead loading was ensured as long as

$$\int_{V} \dot{\sigma}_{ij} \dot{\varepsilon}_{ij} dV > 0 \tag{4.52}$$

Thus, a sufficient condition stability of a structure under dead loading is that everywhere in the structure, we have

$$\dot{\sigma}_{ij}\dot{\varepsilon}_{ij} \ge 0 \tag{4.53}$$

while in a part of the structure we have

$$\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}>0$$
 (4.54

Consequently, structural stability is guaranteed if the constitutive law meets this requirement, but the equilibrium may be potentially unstable if the stress-strain law is such that strain paths exist for which

$$\dot{\sigma}_{ij}\dot{\varepsilon}_{ij} < 0 \tag{4.55}$$

With a local, incrementally-linear constitutive model

$$\dot{\sigma}_{ij} = D_{ijkl} \dot{\varepsilon}_{kl} \tag{4.56}$$

as we employ throughout this study, we find that the equilibrium is potentially unstable if

 $D_{ijkl}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{kl} < 0 \tag{4.57}$ 

We will show in the next sub-sections that strain-softening, either under compressive or under tensile loadings, and non-associated flow rules may both give rise to unstable material behaviour in the sense of equations (4.55) or (4.57). To this end, we will restrict the treatment to homogeneous deformations prior to the occurrence of instability, so that we can replace the stability condition for a body (4.52) by the local stability condition (4.54). Instability then occurs if  $\dot{\sigma}_{ij}\dot{arepsilon}_{ij}$  vanishes pointwise.

4.4.1 Non-associated plasticity

In classical associated small deformation plasticity, material stability is ensured by Drucker's postulate<sup>41,62</sup>:

Consider a material point of a body in some initial state of stress  $\sigma_{ij}^0$ , and let an external agency slowly apply and remove additional stresses. During a complete cycle of application and removal of the added stresses, the work performed by the external agency is then non-negative.

Mathematically, this is expressed by

$$\int (\sigma_{ij} - \sigma_{ij}^0) \dot{\varepsilon}_{ij}^p dt \ge 0 \qquad f(\sigma_{ij}^0) \le 0 \tag{4.58}$$

A sufficient condition for this inequality to hold, is that

$$(\sigma_{ij} - \sigma_{ij}^0) \dot{\varepsilon}_{ij}^p \ge 0 \qquad f(\sigma_{ij}^0) \le 0 \tag{4.59}$$

When we choose  $\sigma_{ij} = \sigma_{ij}^0 + \dot{\sigma}_{ij} dt$ , we get

$$\dot{\sigma}_{ij}\dot{\varepsilon}^{p}_{ij} \ge 0 \tag{4.60}$$

For the elastic strain rates, we have

$$\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^{e} \ge 0 \tag{4.61}$$

so that we obtain equation (4.53) when adding equations (4.60) and (4.61). Hence, material stability is ensured if we take Drucker's Postulate as starting point for the formulation of the constitutive law.



Figure 4.14. Computed responses of sand in simple shear using the nonassociated perfect-plasticity model of the preceding section<sup>119</sup>.

Adoption of this postulate implies that the plastic potential g is a function of the yield function f. This situation is also referred to by the

statements that the flow rule is associated, or that the plastic potential gand the yield function f coincide (f=g). For  $f \neq g$  the inner product  $\dot{\sigma}_{ij}\dot{\varepsilon}^{p}_{ij}$  becomes negative for certain stress paths, thus violating Drucker's Postulate. As a consequence, the inner product  $\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}$  may also become negative for particular stress paths, so that we have unstable material behaviour. An example is given in Figure 4.14, which shows a granular material with a non-associated flow rule subjected to an isochoric deformation in a shear box. Depending on the initial stress conditions, the stress path may be such that we encounter softening as a consequence of the negativeness of the inner product  $\dot{\sigma}_{ij} \dot{\varepsilon}_{ij}^p$ .

We will now become more precise and investigate under which conditions stability fails. To this end we will consider a Mohr-Coulomb frictionhardening solid without cohesion, i.e.

$$f = \frac{1}{2} (\sigma_3 - \sigma_1) + \frac{1}{2} (\sigma_3 + \sigma_1) \sin \varphi^*$$
(4.62)

and the functional relation between the mobilised friction angle  $\varphi^*$  and the hardening parameter  $\kappa$  is assumed to be given by the relationship (4.27). For planar deformations, the relation between the stress-rate vector  $\dot{\boldsymbol{\sigma}}$  and the strain-rate vector  $\dot{\boldsymbol{\varepsilon}}$  formally reads

$$\begin{vmatrix} \dot{\sigma}_{xx} \\ \dot{\sigma}_{yy} \\ \dot{\sigma}_{xy} \end{vmatrix} = \begin{bmatrix} D_{1111} & D_{1122} & D_{1112} \\ D_{2211} & D_{2222} & D_{2212} \\ D_{1211} & D_{1222} & D_{1212} \end{bmatrix} \begin{vmatrix} \dot{\varepsilon}_{xx} \\ \dot{\varepsilon}_{yy} \\ \dot{\varepsilon}_{xy} \end{vmatrix}$$
(4.63)

 $D_{1111}$  etc. are the components of the stiffness tensor  $D_{ijkl}$ . With aid of equations (4.32), (4.48) and (4.62), we can derive that

1

$$D_{1111} = \mu \left\{ \frac{2(1-\nu)}{1-2\nu} - \frac{\left[\frac{s}{r} + \frac{\sin\varphi^*}{1-2\nu}\right] \left[\frac{s}{r} + \frac{\sin\psi^*}{1-2\nu}\right]}{\frac{h}{\mu} + 1 + \frac{\sin\varphi^* \sin\psi^*}{1-2\nu}} \right\}$$
(4.64)

1

$$D_{2211} = \mu \left\{ \frac{2\nu}{1 - 2\nu} + \frac{\left[\frac{s}{r} + \frac{\sin\varphi^*}{1 - 2\nu}\right] \left[\frac{s}{r} - \frac{\sin\psi^*}{1 - 2\nu}\right]}{\frac{h}{\mu} + 1 + \frac{\sin\varphi^* \sin\psi^*}{1 - 2\nu}} \right\}$$
(4.65)

$$D_{1211} = \frac{-\mu \left[ \frac{s}{r} + \frac{\sin \varphi^*}{1 - 2\nu} \right] \frac{\sigma_{xy}}{r}}{\frac{h}{\mu} + 1 + \frac{\sin \varphi^* \sin \psi^*}{1 - 2\nu}}$$
(4.66)

$$D_{1122} = \mu \left\{ \frac{2\nu}{1 - 2\nu} + \frac{\left[\frac{s}{r} - \frac{\sin\varphi^*}{1 - 2\nu}\right] \left[\frac{s}{r} + \frac{\sin\psi^*}{1 - 2\nu}\right]}{\frac{h}{\mu} + 1 + \frac{\sin\varphi^* \sin\psi^*}{1 - 2\nu}} \right\}$$
(4.67)

$$D_{2222} = \mu \left\{ \frac{2(1-\nu)}{1-2\nu} + \frac{\left(\frac{s}{r} - \frac{\sin\varphi^*}{1-2\nu}\right) \left(\frac{s}{r} - \frac{\sin\psi^*}{1-2\nu}\right)}{\frac{h}{\mu} + 1 + \frac{\sin\varphi^* \sin\psi^*}{1-2\nu}} \right\}$$
(4.68)

$$D_{1222} = \frac{\mu \left[ \frac{s}{r} - \frac{\sin \varphi^*}{1 - 2\nu} \right] \frac{\sigma_{xy}}{r}}{\frac{h}{\mu} + 1 + \frac{\sin \varphi^* \sin \psi^*}{1 - 2\nu}}$$
(4.69)

$$D_{1112} = \frac{-\mu \left[ \frac{s}{r} + \frac{\sin \psi^*}{1 - 2\nu} \right] \frac{\sigma_{xy}}{r}}{\frac{h}{\mu} + 1 + \frac{\sin \varphi^* \sin \psi^*}{1 - 2\nu}}$$
(4.70)

$$D_{2212} = \frac{\mu \left[ \frac{s}{r} - \frac{\sin \psi^*}{1 - 2\nu} \right] \frac{\sigma_{xy}}{r}}{\frac{h}{\mu} + 1 + \frac{\sin \psi^* \sin \psi^*}{1 - 2\nu}}$$
(4.71)

$$D_{1212} = \mu \left\{ 1 - \frac{\left(\frac{\sigma_{xy}}{r}\right)^2}{\frac{h}{\mu} + 1 + \frac{\sin\varphi^* \sin\psi^*}{1 - 2\nu}} \right\}$$
(4.72)

with  $\mu$  the elastic shear modulus and  $\nu$  Poisson's ratio. Further, the auxiliary stress measure s

$$s = \frac{1}{2} \left( \sigma_{xx} - \sigma_{yy} \right) \tag{4.73}$$

and the radius of Mohr's circle r

1

$$r = \sqrt{\left(\frac{1}{2}(\sigma_{xx} - \sigma_{yy})\right)^2 + \sigma_{xy}^2}$$
(4.74)

have been used.

Without loss of generality, we may evaluate the stability condition U>0 with  $U=\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}$ , in terms of principal stresses and principal strains. When the subscripts 1 and 3 refer to principal directions, equation (4.63) can be replaced by

$$\begin{bmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_3 \end{bmatrix} = \begin{bmatrix} D_{1111} & D_{1122} \\ D_{2211} & D_{2222} \end{bmatrix} \begin{bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_3 \end{bmatrix}$$
(4.75)

where the coefficients  $D_{1111}$  etc. are given by equations (4.64) to (4.72) with the simplification that s/r=1 because  $\sigma_{xy}=0$ . In terms of principal stresses and principal strains, U can be written as

$$U = \dot{\sigma}_1 \dot{\varepsilon}_1 + \dot{\sigma}_3 \dot{\varepsilon}_3 \tag{4.76}$$

or upon substitution of equation (4.75)

$$U = D_{1111} \dot{\varepsilon}_1^2 + (D_{1122} + D_{2211}) \dot{\varepsilon}_1 \dot{\varepsilon}_3 + D_{2222} \dot{\varepsilon}_3^2 \tag{4.77}$$

U becomes stationary when

$$\frac{\partial U}{\partial \dot{\varepsilon}_1} = 0$$
 and  $\frac{\partial U}{\partial \dot{\varepsilon}_3} = 0$  (4.78)

Without further proof, we will consider this stationary value of U to be a minimum. Elaboration of conditions (4.78) gives with (4.77)

$$2D_{1111}\dot{\varepsilon}_1 + (D_{1122} + D_{2211})\dot{\varepsilon}_3 = 0$$

$$(D_{1122} + D_{2211})\dot{\varepsilon}_1 + 2D_{2222}\dot{\varepsilon}_3 = 0$$

$$(4.79)$$

This set of equations has a non-trivial solution if and only if the determinant vanishes:

$$4D_{1111}D_{2222} - (D_{1122} + D_{2211})^2 = 0 (4.80)$$

Inserting the expressions for  $D_{1111}$  etc. in (4.80) yields after some algebraic manipulations

$$\left(\frac{h}{\mu} + 1 + \frac{\sin\varphi^* \sin\psi^*}{1 - 2\nu}\right)^2 - \left(1 + \frac{\sin\varphi^* \sin\psi^*}{1 - 2\nu}\right) \left(\frac{h}{\mu} + 1 + \frac{\sin\varphi^* \sin\psi^*}{1 - 2\nu}\right) - \frac{(\sin\varphi^* - \sin\psi^*)^2}{4(1 - 2\nu)} = 0$$
(4.81)

Solving this quadratic equation in h yields after some minor rearrangements

$$\frac{2h_f}{\mu} = \sqrt{\left[1 + \frac{\sin^2 \varphi^*}{1 - 2\nu}\right] \left[1 + \frac{\sin^2 \psi^*}{1 - 2\nu}\right]} - \left[1 + \frac{\sin \varphi^* \sin \psi^*}{1 - 2\nu}\right]$$
(4.82)

as expression for the critical hardening modulus  $h_f$  at which U becomes stationary. It is noted that the other solution for h is negative and therefore of no interest.

Expression (4.82) merits some further discussion. First, we observe that for associated plasticity, i.e.  $\varphi^* = \psi^*$ , the expression for  $h_f$  reduces to

$$a_f = 0$$
 (4.83)

so that instability is not possible as long as h>0. This is not surprising as the material then obeys Drucker's Postulate, but it is satisfying that it is obtained as a limiting case of expression (4.82). Secondly, the stability criterion derived here coincides with the expression which is obtained when the general stability criterion for elastic-plastic solids with a nonassociated flow law

$$2h_{f} = \sqrt{\frac{\partial g}{\partial \sigma_{ij}} D^{e}_{ijkl} \frac{\partial g}{\partial \sigma_{kl}} \frac{\partial f}{\partial \sigma_{mn}} D^{e}_{mnop} \frac{\partial f}{\partial \sigma_{op}}} - \frac{\partial g}{\partial \sigma_{ij}} D^{e}_{ijkl} \frac{\partial f}{\partial \sigma_{kl}}$$
(4.84)

as formulated by Vermeer<sup>117</sup> and Maier & Hueckel<sup>70</sup>, is applied to a Mohr-Coulomb friction hardening model, although all derivations run along different lines.

Equation (4.82) gives the expression for h for which U becomes stationary, but U does not vanishes for all strain-rate directions. To investigate for which strain-rate direction U vanishes, we again consider equa-

tion (4.77) and require that U=0. Solving for  $\frac{\dot{\varepsilon}_1}{\dot{\varepsilon}_2}$  gives

$$\frac{\dot{\varepsilon}_1}{\dot{\varepsilon}_3} = \frac{-(D_{1122} + D_{2211}) \pm \sqrt{(D_{1122} + D_{2211})^2 - 4D_{1111}D_{2222}}}{2D_{1111}}$$
(4.85)

or invoking equation (4.80)

$$\frac{\dot{\varepsilon}_1}{\dot{\varepsilon}_3} = \frac{-(D_{1122} + D_{2211})}{2D_{1111}} \tag{4.86}$$

where  $D_{1111}$  etc. are given by equations (4.64) to (4.72).

To illustrate the significance of the above equations, we will apply them to a particular granular material with  $\nu=0.2$ ,  $\varphi=40^{\circ}$  and  $\varepsilon_{f}=0.02$ , being typical for a sand. For sake of simplicity, expression (4.34) for the mobilised dilatancy angle  $\psi^{*}$  has been replaced by the simple relation  $\psi^{*}=0^{\circ}$ . With these data we can derive

$$\frac{h_f}{\mu}$$
=0.1359

$$\varphi_f = 37.49^{\circ}$$

for the corresponding friction angle  $\varphi_f$ . We observe that  $\varphi_f$  is appreciably lower than the limit friction angle  $\varphi$ . With aid of equation (4.86) we can derive that

$$\frac{\dot{\varepsilon}_1}{\dot{\varepsilon}_3} = -1.205$$

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for the critical strain-rate direction for which U vanishes. This strainrate direction and the corresponding stress rate direction have been plotted in Figure 4.15.



Figure 4.15. Strain-rate and corresponding stress-rate direction for which  $\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}$  vanishes at the critical load in the hardening regime.

Having determined a critical value of the hardening modulus for which stability is no longer assured, i.e. a value of the rate of hardening for which there exists a particular strain-rate direction for which  $\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}$ vanishes, we will investigate the nature of this instability. It was pointed out in Section 3.2 that for non-symmetric systems, stability may fail prior to the vanishing of the determinant of the stiffness matrix, or equivalently before the lowest eigenvalue  $\lambda_1$  of the stiffness matrix vanishes. We will therefore derive the critical hardening modulus  $h_c$  for which, again under the assumption of homogeneous deformations, the determinant of the stiffness matrix **D** first vanishes. When the critical hardening modulus  $h_c$  equals the hardening modulus  $h_f$  for which  $\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}$ vanishes in some strain-rate direction, the instability can be associated with a bifurcation or a limit point. If  $h_f > h_c$ , we apparently have a 'flutter'-type instability prior to the occurrence of a limit or bifurcation type instability.

At a first sight, the evaluation of det(D)

$$\det(\mathbf{D}) = \begin{bmatrix} D_{1111} & D_{1122} & D_{1112} \\ D_{2211} & D_{2222} & D_{2212} \\ D_{1211} & D_{1222} & D_{1212} \end{bmatrix}$$
(4.87)

with the coefficients  $D_{1111}$  etc. again given by equations (4.64) to (4.72), seems rather intractable, but a straightforward although somewhat long derivation shows that most terms cancel and that it can be simplified to

$$\det(\boldsymbol{D}) = \mu^3 \frac{\frac{h}{\mu}}{\frac{h}{\mu} + 1 + \frac{\sin\varphi^* \sin\psi^*}{1 - 2\nu}}$$
(4.88)

This relation between det(D) and  $h/\mu$  is plotted in Figure 4.16. We observe that the first value of h for which det(D) vanishes is zero, either when the flow is associated or non-associated. This implies that a bifurcation or a limit type instability for which the lowest eigenvalue must vanish, can only occur at the peak of the stress-strain curve and not already in the hardening regime. Consequently, the stability criterion (4.83) and also the more general criterion by Vermeer<sup>117</sup> and Maier & Hueckel<sup>70</sup> generally define 'flutter' type instabilities, and not necessarily bifurcation type instabilities, as the lowest eigenvalue  $\lambda_1$  does not vanish for  $h=h_f$ , but first vanishes for h=0.



Figure 4.16. det(**D**) as a function of  $h/\mu$ .



Figure 4.17. Coordinate system and discontinuity in the velocity field.

Nevertheless, the observation that bifurcation instabilities can not occur prior to peak seems to be in contradiction with the literature on shear-band bifurcations in soil bodies. Indeed, bifurcations of originally homogeneously deformed bodies are possible in the hardening regime. but these bifurcations at most lead to piecewise homogeneous deformations in the post-bifurcation regime. Hence, the assumption of continuing homogeneous deformations precludes bifurcation in the hardening regime. In other words, the constitutive law is such that bifurcations with homogeneous deformations in the post-bifurcation regime are not possible. It is noted that for rubber-like materials the inherent geometrical nonlinearity in combination with some constitutive laws also admits bifurcations with homogeneous deformations in the post-bifurcation regime. An example thereof has been given by Rivlin<sup>94</sup> who considers a cube of rubber-like material under all-round dead loading. At some critical stress level, a bifurcation point is encountered and the cube can attain the shape of a box with unequal sides.

Considering velocity fields which are only piecewise homogeneous, we find that bifurcations are already possible in the hardening regime for a critical hardening modulus  $h = h_s$ . The subscript s is employed to denote that these bifurcations are associated with discontinuities in the velocity gradient, so-called shear bands. A homogeneous velocity field  $\dot{u}_i$  is characterised by

 $\dot{u}_i = F_{ij} x_j$ 

with  $x_i$  the spatial coordinates and  $F_{ij}$  a second-order tensor which does not depend upon  $x_i$ . Alternatively, (4.89) can be written as

$$\dot{u}_i = f_{ij} \nu_j \tag{4.90}$$

with  $f_{ij}$  a second-order tensor function

$$\boldsymbol{f}_{ij} = F_{ij} \boldsymbol{x}_k \boldsymbol{\nu}_k \tag{4.91}$$

and  $\boldsymbol{\nu}$  a unit vector with the same direction as  $\boldsymbol{z}$ .

For a piecewise homogeneous velocity field with a discontinuity in the velocity gradient with a unit normal vector  $\boldsymbol{n}$  pointing in an arbitrary but fixed half-space (see Figure 4.17), we have

$$\dot{u}_i = f_{ij}(x_k n_k) \nu_j \tag{4.92}$$

where  $f_{ij}(x_k n_k)$  is defined as

$$f_{ij}(x_k n_k) = \begin{cases} F_{ij}^+ x_k \nu_k, & x_k n_k > 0 \\ F_{ij}^- x_k \nu_k, & x_k n_k < 0 \end{cases}$$
(4.93)

with  $F_{ij}^+$  and  $F_{ij}^-$  constant tensors which do not depend on the place, the generalisation to several parallel surfaces of a discontinous velocity gradient being obvious. Because of continuity in the velocity field, the discontinuity in the velocity gradient must be such that

$$F_{ij}^+ l_j = F_{ij}^- l_j \tag{4.94}$$

with **l** an arbitrary vector in the discontinuity plane.

Considering a linear comparison solid, i.e. a solid for which all moduli are plastic for points which are in a plastic state, the condition for bifurcation becomes equivalent to the condition that stability fails, i.e.

 $\dot{\varepsilon}_{ij} D_{ijkl} \dot{\varepsilon}_{kl} = 0 \tag{4.95}$ 

Differentiating (4.92) yields for the strain-rate tensor  $\dot{\varepsilon}_{ij}$ 

$$\dot{\varepsilon}_{ij} = \frac{1}{2} \left( f'_{ik} n_j + f'_{jk} n_i \right) \nu_k \tag{4.96}$$

where the primes signify differentiation with respect to  $x_k n_k \ddagger$ . Inserting this identity in (4.95) results in

$$f'_{io}\nu_{o}n_{j}D_{ijkl}f'_{kp}\nu_{p}n_{l}=0$$
(4.97)

<sup>‡</sup> Evaluation of  $f'_{ij}$  for  $x_k n_k > 0$  gives  $f'_{ij} = \nu_k n_k F_{ij}^+$ For the strain-rate tensor  $\dot{\varepsilon}_{ij}$ , this results in  $\dot{\varepsilon}_{ij} = \frac{1}{2} (F_{ij}^+ + F_{ji}^+)$  A sufficient condition for this equation to be satisfied for arbitrary vectors with components  $\nu_i$  is that

$$\det(n_j D_{ijkl} n_l) = 0 \tag{4.98}$$

which is Hill's<sup>57</sup> bifurcation criterion for a velocity field with a discontinous velocity gradient, although Hill's derivation includes large velocity gradients. Without loss of generality, we can elect a coordinate system  $x'_{1,x}'_{2,x}x'_{3}$  which is aligned with the discontinuity surface (see Figure 4.17). Then

$$n_i = \delta_{i2} \tag{4.99}$$

so that the bifurcation criterion (4.98) reduces to

 $\det(D_{i2k2})=0$ (4.100)

which is in fact the bifurcation criterion given by Rudnicki and Rice<sup>99</sup>, but for the fact that large velocity gradients have not been taken into account in the present analysis. As indicated in Chapter 2, such terms could have been included by a suitable redefinition of the the constitutive tensor  $D_{ijkl}$ . Again, equations (4.98) and (4.100) are sufficient conditions for instability of a velocity field with discontinous velocity gradients, but only necessary conditions in case of a symmetric tensor D<sub>iikl</sub>.

For planar deformations (4.100) reduces to

$$\begin{vmatrix} D_{2222} & D_{2212} \\ D_{1222} & D_{1212} \end{vmatrix} = 0$$
 (4.101)

As Vermeer<sup>118</sup> observed, this is equivalent to

.

 $C_{1111}=0$ (4.102)

since  $\det(\mathbf{C})$  does not vanish prior to peak, and  $C_{1111}$  is a coefficient of the compliance relation

$$\begin{vmatrix} \dot{\varepsilon}_{xx} \\ \dot{\varepsilon}_{yy} \\ \dot{\varepsilon}_{xy} \end{vmatrix} = \begin{vmatrix} C_{1111} & C_{2211} & C_{1211} \\ C_{1122} & C_{2222} & C_{1222} \\ C_{1112} & C_{2212} & C_{1212} \end{vmatrix} \begin{vmatrix} \dot{\sigma}_{xx} \\ \dot{\sigma}_{yy} \\ \dot{\sigma}_{xy} \end{vmatrix}$$
(4.103)

Elaboration of this condition for a Mohr-Coulomb friction hardening solid yields the following expression for the critical hardening modulus for shear-band bifurcation<sup>71,118</sup>

$$\frac{h_s}{\mu} = \frac{(\sin\varphi^* - \sin\psi^*)^2}{8(1-\nu)}$$
(4.104)

For the data considered above, we can calculate a critical hardening modulus

$$h_{s} = 0.0547$$

and a corresponding mobilised friction angle

 $\varphi_{s} = 39.23^{\circ}$ 

Comparing these data with the data derived for  $h_f$  and  $\varphi_f$ , we observe that for this particular material, stability fails prior to the possibility of development of shear bands.

#### 4.4.2 Strain-softening

In stress-strain relations where we have a negative modulus for the relation between the normal stress and the normal strain or between the shear stress and the shear strain (which we shall here refer to as softening stress-strain relations), we either have  $\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^p<0$  or  $\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^{cr}<0$ , so that again we have the possibility of  $\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}$  <0 , implying material instability. In this study we will consider strain-softening as a material property. Gradually, however, it begins to emerge that this is generally not true as most and perhaps all softening which is observed in experiments, is due to non-homogeneous deformations and the stiffness of the test equipment<sup>4,54,73,90,119</sup>, being only triggered by material or geometric instabilities. Hence, the observed softening is a structural effect rather than a material property.



Figure 4.18. Stress-strain law for concrete in tension. The stress is plotted against the total strain, i.e. the sum of crack strain and concrete strain.



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Nevertheless, it is interesting to investigate whether strain-softening can still be used as a working hypothesis for the formulation of constitutive laws in finite element programs. To this end, we consider an unreinforced bar which is subjected to pure tension<sup>34</sup>. We model the material of the bar as elastic-softening with an ultimate strain  $\varepsilon_{u}$  at which the tensile strength has vanished completely (see Figure 7.18). We assume that  $\varepsilon_n$  is equal to n times the strain at the tensile strength. The bar is modelled with m elements (see Figure 4.19). If we have a perfect bar, so that all elements have exactly the same tensile strength and so on, the bar deforms uniformly throughout the loading process and the loaddeflection curve is simply a copy of the imposed stress-strain law. However, if one element has a slight imperfection, only this element will show loading while the other elements will show unloading. In this situation, the imposed stress-strain law at local level is not reproduced. Instead, an average strain is calculated in the post-peak regime which is smaller than the strain of the stress-strain law. This may be explained as follows. The element which shows loading, will follow the path A-B in Figure 4.18, while the other elements will follow the path A-C. This implies that when all elements have the same dimensions, we have for the average strain increment  $\Delta \overline{\epsilon}$ 

$$\Delta \overline{\varepsilon} = \frac{1}{m} \left[ -(m-1)\frac{\Delta \sigma}{E} + \frac{\Delta \sigma}{E \neq (n-1)} \right] = \left[ \frac{n}{m} - 1 \right] \frac{\Delta \sigma}{E}$$
(4.105)

Consequently, if we increase the number of elements while keeping the length of the bar fixed, the average strain in the post-peak regime gradually becomes smaller and for m > n the average strain in the post-peak regime even becomes smaller than the strain at peak load (Figure 4.20). This implies that for m > n, the load-deflection curve shows a 'snapback'<sup>34</sup>.

The above result implies that computational results for materials with a local softening constitutive law are not objective upon mesh refinement. To remedy this disease, it has been proposed to make the softening modulus dependent on the element size<sup>7,58</sup>. Numerical experiments for tensile loadings have confirmed that numerical results are



Figure 4.20. Average strain for different numbers of elements.

then objective with regard to mesh refinement<sup>7,95</sup>. For softening under compressive stresses, numerical studies are not decisive as contradictory results have been reported<sup>84,125</sup>. The problem of making the softening modulus dependent on the element size is that for an arbitrary concrete structure, the spread of the softening region is not known in advance. Consequently, the observation that use of a local softening law may involve snap-back behaviour on structural level also holds when we use a model in which the softening modulus has been adapted to some structural size.

The fact that local softening laws may involve snap-back phenomena and other numerical instabilities, makes it questionable whether such a constitutive law in which the stress increment only depends on the strain increment and the strain history of the same material point, is suitable for describing structural softening. Indeed, some attempts have recently been published<sup>8,102</sup> which aim at describing the behaviour of concrete by a non-local constitutive law, in which the stress increment in a material point also depends on the neighbourhood, either via the strain gradient or via the strain increment and strain history of nearby material points. The difficulty of this approach is that it does not fit the nature of the finite element method.
# 5. SOLUTION OF THE BOUNDARY VALUE PROBLEM

In this chapter, an accurate yet simple procedure for the integration of stress-strain laws which were derived in the preceding, is presented. Special attention is devoted to singularities in the stress-strain law which are repeatedly reported to lead to numerical instabilities and nonconvergence. Another aspect which is important for a successful solution of the boundary value problem especially near limit states, is the iterative solution procedure to solve the set of algebraic equations on structural level. This matter will be discussed in the second part of the chapter.

# 5.1. Numerical integration of the stress-strain law

For computational purposes, matrix-vector notation is usually preferable over tensor notation which was employed in the preceding chapter. We will therefore restate the constitutive equation (4.48) for the plasticfracture model outlined in the preceding chapter in matrix-vector notation. This gives

$$\dot{\boldsymbol{\sigma}} = \left\{ \boldsymbol{D}^{ef} - \frac{\boldsymbol{D}^{ef} \frac{\partial g}{\partial \boldsymbol{\sigma}} \frac{\partial f}{\partial \boldsymbol{\sigma}}^{T} \boldsymbol{D}^{ef}}{h + \frac{\partial f}{\partial \boldsymbol{\sigma}}^{T} \boldsymbol{D}^{ef} \frac{\partial g}{\partial \boldsymbol{\sigma}}} \right\} \dot{\boldsymbol{\varepsilon}}$$
(5.1)

where the symbol T denotes a transpose, and the other symbols are defined in accordance with the definitions of the preceding chapter.

A further advantage of matrix-vector notation is that we can write the expression for the elastic-fracture matrix  $D^{ef}$ 

$$\boldsymbol{D}^{ef} = \boldsymbol{D}^{e} - \boldsymbol{D}^{e} \boldsymbol{N}_{I} \boldsymbol{M} \boldsymbol{N}_{I}^{T} \boldsymbol{D}^{e} + \boldsymbol{D}^{e} \boldsymbol{N}_{I} \boldsymbol{M} \boldsymbol{B} \boldsymbol{E}^{-1} \boldsymbol{N}_{II}^{T} \boldsymbol{D}^{e}$$
$$+ \boldsymbol{D}^{e} \boldsymbol{N}_{II} \boldsymbol{E}^{-1} \boldsymbol{C} \boldsymbol{M} \boldsymbol{N}_{I}^{T} \boldsymbol{D}^{e} + \boldsymbol{D}^{e} \boldsymbol{N}_{II} \left[ \boldsymbol{E}^{-1} + \boldsymbol{E}^{-1} \boldsymbol{C} \boldsymbol{M} \boldsymbol{B} \boldsymbol{E}^{-1} \right] \boldsymbol{N}_{II}^{T} \boldsymbol{D}^{e} \qquad (5.2)$$

in a more compact form and that we can generalise to in infinite number of cracks. For convenience we will first define the matrices A, B, C, E and M:

$$\boldsymbol{A} = \boldsymbol{D}' + \boldsymbol{N}_{\boldsymbol{f}}^{T} \boldsymbol{D}^{\boldsymbol{e}} \, \boldsymbol{N}_{\boldsymbol{I}} \tag{5.3}$$

 $\boldsymbol{B} = \boldsymbol{N}_I^T \boldsymbol{D}^e \, \boldsymbol{N}_{II} \tag{5.4}$ 

$$\boldsymbol{C} = \boldsymbol{N}_{II}^T \boldsymbol{D}^e \, \boldsymbol{N}_I \tag{5.5}$$

$$\boldsymbol{E} = \boldsymbol{D}^{\prime\prime} + \boldsymbol{N}_{II}^T \boldsymbol{D}^e \, \boldsymbol{N}_{II} \tag{5.6}$$

$$\boldsymbol{M} = \left(\boldsymbol{A} - \boldsymbol{B} \, \boldsymbol{E}^{-1} \, \boldsymbol{C}\right)^{-1} \tag{5.7}$$

and the transformation matrices  $N_I$ ,  $N_{II}$  for the stress or strain vectors from the coordinate system of the first resp. the second crack to the global coordinate system:

$$\boldsymbol{N}_{I} = \begin{bmatrix} l_{x}^{2} & l_{x}m_{x} & l_{x}n_{x} \\ l_{y}^{2} & l_{y}m_{y} & l_{y}n_{y} \\ l_{z}^{2} & l_{z}m_{z} & l_{z}n_{z} \\ 2l_{x}l_{y} & l_{x}m_{y}+l_{y}m_{x} & l_{x}n_{y}+l_{y}n_{x} \\ 2l_{y}l_{z} & l_{y}m_{z}+l_{z}m_{y} & l_{y}n_{z}+l_{z}n_{y} \\ 2l_{z}l_{x} & l_{z}m_{x}+l_{x}m_{z} & l_{z}n_{x}+l_{x}n_{z} \end{bmatrix}$$
(5.8)

the matrices  $N_{II}$ ,  $N_{III}$  etc. being defined similarly.  $l_x$ ,  $l_y$  and  $l_z$  are the cosines of the angle between the x'-axis and the x-axis, resp. the y-axis and the z-axis, and the other direction cosines are defined in accordance with this convention. As the stress-rate vector and the strain-rate vectors normally have six independent components, we would expect  $N_I$  etc. to be 6\*6 matrices and not 6\*3 matrices. However, it is recalled from the preceding chapter that the only non-vanishing crack strain components are the strain component normal to the crack and two shear crack strain components. If we also assume that the non-vanishing components of the crack strain rate are only related to the corresponding components (that is the normal and the two shear stress rates) of the stress-rate vector in the coordinate system of the crack (see equation (4.19)), we may delete the appropriate columns from the transformation matrices  $N_I$ ,  $N_{II}$ , etc., so that we end up with the 6\*3 matrix (5.8).

As a first step to express  $D^{ef}$  in a more compact form, we note that equation (5.2) is equivalent to

$$\boldsymbol{D}^{ef} = \boldsymbol{D}^{e} - \boldsymbol{D}^{e} \begin{bmatrix} \boldsymbol{N}_{I} & \boldsymbol{N}_{II} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{E} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{N}_{I}^{T} \\ \boldsymbol{N}_{II}^{T} \end{bmatrix} \boldsymbol{D}^{e}$$
(5.9)

Next define

$$\boldsymbol{D}^{cr} = \begin{bmatrix} \boldsymbol{D}^{\prime} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}^{\prime\prime} \end{bmatrix}$$
(5.10)

so that the stress-rate vector  $\dot{\mathbf{s}}$ ,

$$\dot{\mathbf{s}} = \begin{bmatrix} \dot{\boldsymbol{\sigma}}' \\ \dot{\boldsymbol{\sigma}}'' \end{bmatrix} \tag{5.11}$$

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which assembles the stress-rate components in the local coordinate systems of the cracks and the crack strain-rate vector  $\dot{\boldsymbol{e}}^{cr}$ ,

$$\dot{\boldsymbol{e}}^{cr} = \begin{bmatrix} \dot{\boldsymbol{\varepsilon}}^{\prime} \\ \dot{\boldsymbol{\varepsilon}}^{\prime\prime} \end{bmatrix} \tag{5.12}$$

which contains the crack strain-rate components in the local coordinate systems, are related by

$$\dot{\mathbf{s}} = \mathbf{D}^{cr} \dot{\mathbf{e}}^{cr} \tag{5.13}$$

With the additional definition

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{N}_I & \boldsymbol{N}_{II} \end{bmatrix}$$
(5.14)

we can derive that

r 1

$$\boldsymbol{D}^{ef} = \boldsymbol{D}^{e} - \boldsymbol{D}^{e} \, \boldsymbol{N} \, (\boldsymbol{D}^{cr} + \boldsymbol{N}^{T} \boldsymbol{D}^{e} \, \boldsymbol{N})^{-1} \, \boldsymbol{N}^{T} \, \boldsymbol{D}^{e} \tag{5.15}$$

It is important to distinguish  $\dot{\boldsymbol{e}}^{cr}$  which assembles all individual crack strain rates with respect to their own coordinate system from  $\dot{\boldsymbol{\varepsilon}}^{cr}$  which is the sum of all crack strain rates defined in the global xyz-coordinate system. With the definition of the composite transformation matrix  $\boldsymbol{N}$ , we observe that  $\dot{\boldsymbol{\varepsilon}}^{cr}$  and  $\dot{\boldsymbol{e}}^{cr}$  are related through (see equations (4.6) and (4.7)):

$$\dot{\boldsymbol{\varepsilon}}^{cr} = \boldsymbol{N}\boldsymbol{e}^{cr} \tag{5.16}$$

Similarly, the vector  $\dot{s}$  which assembles the stress rates in the individual cracks with respect to their own coordinate system is related to the global stress rate  $\dot{\sigma}$  by

$$\dot{\mathbf{s}} = \mathbf{N}^T \dot{\boldsymbol{\sigma}}$$
 (5.17)

The generalisation to more than two cracks in the same integration point is now straightforward, as we only have to expand the vectors  $\dot{\mathbf{s}}$ :

$$\dot{\mathbf{s}} = \begin{vmatrix} \dot{\mathbf{o}}' \\ \dot{\mathbf{o}}'' \\ \dot{\mathbf{o}}'' \\ \dot{\mathbf{o}}'' \\ \vdots'' \end{vmatrix}$$
(5.18)  
and  $\dot{\mathbf{e}}^{cr}$ :  
$$\dot{\mathbf{e}}^{cr} = \begin{vmatrix} \dot{\mathbf{e}}' \\ \dot{\mathbf{e}}'' \\ \dot{\mathbf{e}}'' \\ \vdots'' \\ \vdots'' \end{vmatrix}$$
(5.19)

and the matrices  ${\it D}^{cr}$  and  ${\it N}$  to

$$\boldsymbol{D}^{cr} = \begin{bmatrix} \boldsymbol{D}' & \mathbf{0} & \mathbf{0} & \cdot \\ \mathbf{0} & \boldsymbol{D}'' & \mathbf{0} & \cdot \\ \mathbf{0} & \mathbf{0} & \boldsymbol{D}''' & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$
(5.20)

and

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{N}_{I} & \boldsymbol{N}_{II} & \boldsymbol{N}_{III} \end{bmatrix}$$
(5.21)

while equation (5.15) is unaffected.

We can now integrate equation (5.1) for a finite stress increment:

$$\boldsymbol{\Delta \sigma} = \int_{t-\Delta t}^{t} \left\{ \boldsymbol{D}^{ef} - \frac{\boldsymbol{D}^{ef} \frac{\partial g}{\partial \boldsymbol{\sigma}} \frac{\partial f}{\partial \boldsymbol{\sigma}}^{T} \boldsymbol{D}^{ef}}{h + \frac{\partial f}{\partial \boldsymbol{\sigma}}^{T} \boldsymbol{D}^{ef} \frac{\partial g}{\partial \boldsymbol{\sigma}}} \right\} \boldsymbol{\dot{\varepsilon}} d\tau$$
(5.22)

During the calculation of the test stress increment  $\pmb{\Delta\sigma^{t}}$  ,

 $\Delta \sigma^t = D^{ef} \Delta \varepsilon \tag{5.23}$ 

no plasticity is assumed to occur, but only the possibility of cracking is considered. This implies that during this predictor phase, we have the identities

$$D^{ef} \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\sigma}}^t \tag{5.24}$$

$$\frac{\partial \boldsymbol{f}^{T}}{\partial \boldsymbol{\sigma}} \dot{\boldsymbol{\sigma}}^{t} = \dot{\boldsymbol{f}}$$
(5.25)

so that we can rewrite equation (5.22) as

$$\boldsymbol{\Delta \sigma} = \int_{t-\Delta t}^{t} \left\{ \dot{\boldsymbol{\sigma}}^{t} - \frac{\dot{f} \boldsymbol{D}^{ef} \frac{\partial g}{\partial \boldsymbol{\sigma}}}{h + \frac{\partial f}{\partial \boldsymbol{\sigma}}^{T} \boldsymbol{D}^{ef} \frac{\partial g}{\partial \boldsymbol{\sigma}}} \right\} d\tau$$
(5.26)

Introducing the notation

 $\boldsymbol{\sigma}^t = \boldsymbol{\sigma}^0 + \boldsymbol{\Delta} \boldsymbol{\sigma}^t \tag{5.27}$ 

with  $\sigma^0$  either the contact stress at the intersection of the stress path and the yield surface or the stress at the beginning of the loading step (see Figure 5.1), we get with a single-point numerical integration rule

$$\boldsymbol{\Delta \sigma} = \boldsymbol{\Delta \sigma}^{t} - \frac{f\left(\boldsymbol{\sigma}^{t},\boldsymbol{\kappa}\right)}{h + \frac{\partial f}{\partial \boldsymbol{\sigma}}^{T}} \boldsymbol{D}^{ef} \frac{\partial g}{\partial \boldsymbol{\sigma}}} \boldsymbol{D}^{ef} \frac{\partial g}{\partial \boldsymbol{\sigma}}$$
(5.28)

5.16)

as by definition, we have  $f(\sigma^0,\kappa)=0$ , Numerically, this condition need not be satisfied as the stresses resulting from the previous step may violate the yield criterion slightly. However, by putting  $f(\sigma^0,\kappa)=0$  we strive to satisfy the yield criterion at any stage of the loading process, rather than to satisfy the consistency condition  $\dot{f}=0$ , so that inaccuracies from previous loading steps are not carried along.





The approach becomes very simple if the gradients to the yield function f and the plastic potential g are evaluated for  $\sigma = \sigma^t$ . In this approach, there is no need to determine the intersection point of the stress path with the yield function if the response is partly elastic and partly plastic within the loading step, which may simplify the computer code significantly. In fact, the algorithm then may be conceived as a single step Euler backward integration method.

The algorithm for handling plasticity and fracture is not only relatively simple, but it is also quite accurate. Indeed, if we have linear hardening or softening for the yield function and for the fracture function, if we have a constant shear reduction factor in the crack  $\beta^*$  and if we have no physical changes during the loading step (e.g. crack closing), we can prove that the algorithm guarantees a rigorous return to the fracture surface as well as to the yield surface for linear yield and fracture surfaces. Assume for this matter that some test stress  $\boldsymbol{\sigma}^t$  has been computed:

$$\boldsymbol{\sigma}^{t} = \boldsymbol{\sigma}^{0} + \boldsymbol{D}^{ef} \, \boldsymbol{\Delta} \boldsymbol{\varepsilon} \tag{5.29}$$

If  $\boldsymbol{\sigma}^t$  subsequently appears to lie outside the yield surface, a correction must be applied so that the final stress will be on the yield surface. The plastic part of the strain increment follows from,

$$\mathbf{\Delta}\boldsymbol{\varepsilon}^{p} = \mathbf{\Delta}\boldsymbol{\varepsilon} - \mathbf{\Delta}\boldsymbol{\varepsilon}^{e} - \mathbf{\Delta}\boldsymbol{\varepsilon}^{cr} \tag{5.30}$$

but we must also require:

$$\Delta \boldsymbol{\varepsilon}^{p} = \lambda \frac{\partial g}{\partial \boldsymbol{\sigma}} \tag{5.31}$$

By virtue of equations (5.29) to (5.31) and the identity

$$\boldsymbol{\Delta \varepsilon}^{e} + \boldsymbol{\Delta \varepsilon}^{cr} = \left[ \boldsymbol{D}^{ef} \right]^{-1} \left[ \boldsymbol{\sigma}^{1} - \boldsymbol{\sigma}^{0} \right]$$
(5.32)

we obtain for the final stress state  $\sigma^1$ :

$$\boldsymbol{\sigma}^{1} = \boldsymbol{\sigma}^{t} - \lambda \boldsymbol{D}^{ef} \frac{\partial g}{\partial \boldsymbol{\sigma}}$$
(5.33)

The multiplier  $\lambda$  is determined implicitly by the condition that the final stress be on the yield surface:

$$\boldsymbol{f}\left(\boldsymbol{\sigma}^{1},\boldsymbol{\kappa}^{1}\right)=0\tag{5.34}$$

and must generally be determined by an iterative procedure. Alternatively,  $\lambda$  may be determined by expanding  $f(\boldsymbol{\sigma},\kappa)$  in a Taylor series around  $\boldsymbol{\sigma} = \boldsymbol{\sigma}^t$ ,  $\kappa = \kappa^0$ . Omitting second and higher order terms, this yields:

$$f(\boldsymbol{\sigma}^{1},\kappa^{0}) - \lambda \left[ h + \frac{\partial f^{T}}{\partial \boldsymbol{\sigma}} \boldsymbol{D}^{ef} \frac{\partial g}{\partial \boldsymbol{\sigma}} \right] = 0$$
(5.35)

so that the following stress-strain relation is obtained:

$$\boldsymbol{\sigma}^{1} = \boldsymbol{\sigma}^{t} - \frac{f\left(\boldsymbol{\sigma}^{t}, \boldsymbol{\kappa}^{0}\right)}{h + \frac{\partial f}{\partial \boldsymbol{\sigma}}^{T} \boldsymbol{D}^{ef} \frac{\partial g}{\partial \boldsymbol{\sigma}}} \boldsymbol{D}^{ef} \frac{\partial g}{\partial \boldsymbol{\sigma}}$$
(5.36)

Comparing equations (5.27), (5.28) and (5.36), we observe that this approach results in the same integration scheme as derived in equations (5.22) to (5.28). Hence, we can conclude that for all yield functions which are linear in the principal stress space (such as the Mohr-Coulomb and Tresca criteria), the present procedure offers a rigorous return to the yield surface. No drifting error is committed as may be the case with some other integration schemes. Especially when the stresses rotate strongly, these drifting errors may be considerable. Then, a correction procedure should be applied to bring the stresses back to the yield locus<sup>76</sup>. With the present scheme, such a correction procedure is not needed.

It is noted that several restrictions have been imposed in proving the rigorous return to the yield surface. When these restrictions are violated, the rigorous return is not obtained, although in these cases the algorithm is still competitive. The restriction which entails the most serious errors is the assumption that no physical changes may occur during the loading step. If the errors caused by this assumption cannot be tolerated, an inner iteration loop must be applied, or we must divide the strain path in several parts which are bounded by physical changes (e.g. crack formation).

Algorithms for the combination of cracking and plasticity in smeared models are not often described in literature, but an example thereof has been discussed by Owen et al.<sup>83</sup> for the combination of cracking and visco-plasticity. Their algorithm bears some resemblance to the treatment given here, as they employ a decomposition of the concrete strain rate into several components, but a rigorous decomposition of the total strain rate into several crack strain-rate and into several concrete strain-rate components in the sense of equation (4.32) is not utilised. Such a decomposition would however have been implied in their equations if they had adopted a compliance formulation as given by Bažant and Oh for their elastic-fracture matrix instead of a stiffness formulation<sup>7,22</sup>. Then, the explicit formulation of their algorithm would have been the only difference with the algorithm presented here, at least for only one active crack.

# 5.1.1 Relation with the radial return scheme

The present algorithm in fact constitutes a far-reaching generalisation of the elastic predictor-radial return scheme used in metal plasticity<sup>59,65,100,104</sup>. Leaving out cracking  $(D^{ef} = D^e)$  and adopting an associated flow rule (f = g), equation (5.36) reduces to

$$\boldsymbol{\sigma}^{1} = \boldsymbol{s}^{t} + \boldsymbol{p} - \frac{f(\boldsymbol{\sigma}^{t}, \kappa^{0})}{h + \frac{\partial f}{\partial \boldsymbol{\sigma}}^{T}} \boldsymbol{D}^{e} \frac{\partial f}{\partial \boldsymbol{\sigma}} \qquad (5.37)$$

where the trial stress vector  $\boldsymbol{\sigma}^t$  has been separated into a volumetric part  $\boldsymbol{p}$ ,

$$\mathbf{p} = p [1 \ 1 \ 1 \ 0 \ 0 \ 0]^T \tag{5.38}$$

with p the hydrostatic pressure and  $\mathbf{s}^t$ , a deviatoric stress vector. For a von Mises yield criterion

$$f = \sqrt{3J_2} - \sigma_y(\kappa) \tag{5.39}$$

with  $J_2$  the second invariant of the deviatoric stress tensor and  $\sigma_{\pmb{y}}$  the yield stress, we can work out that

$$\boldsymbol{D}^{\boldsymbol{e}} \, \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\sigma}} = \frac{3\mu}{\sqrt{3J_2}} \, \mathbf{s}^{\,\boldsymbol{t}} \tag{5.40}$$

where it is recalled that  $\mu$  is the shear modulus. Premultiplying with  $\frac{\partial f}{\partial \sigma}$ 

$$\frac{\partial f}{\partial \sigma}^{T} \boldsymbol{D}^{e} \frac{\partial f}{\partial \sigma} = 3\mu \tag{5.41}$$

so that we get for the final stress  $\boldsymbol{\sigma}^1$  after some algebraic manipulations

$$\boldsymbol{\sigma}^{1} = \boldsymbol{p} + \left\{ 1 - \left[ 1 - \frac{\sigma_{\boldsymbol{y}}(\kappa^{0})}{\sqrt{3J_{2}^{t}}} \right] \frac{3\mu}{h+3\mu} \right\} \boldsymbol{s}^{t}$$
(5.42)

where it is recalled that the superscripts 0 and t refer to the initial state and the trial state respectively. It is observed that equation (5.42) indeed gives a radial correction in the deviatoric stress space. For a nonhardening material, expression (5.42) reduces to

$$\boldsymbol{\sigma}^{1} = \boldsymbol{p} + \frac{\sigma_{\boldsymbol{y}}(\kappa^{0})}{\sqrt{3J_{2}^{t}}} \boldsymbol{s}^{t}$$
(5.43)

which is exactly the elastic predictor-radial return scheme for nonhardening  $J_2$ -plasticity, which has been shown to be very competitive amongst the single step methods<sup>65</sup>.

Equation (5.42) slightly differs from Schreyer's<sup>100</sup> expression for use of the radial corrector scheme for a hardening solid because he used

$$f = 2J_2 - \frac{2}{3}\sigma_y^2(\kappa) \tag{5.44}$$

as definition of the yield function. We prefer definition (5.39) as it can be proved that for this particular choice of f, we have no drifting errors for a linear-hardening solid, which is not the case for definition (5.44) where this property can only be proved for a non-hardening solid. In fact, it is a peculiarity that the resulting stress exactly complies with the yield function because the von Mises function is not a linear yield function in the principal stress space, while the rigorous return to the yield surface was only demonstrated for such yield functions. For the particular choice (5.39) however, the higher order terms happen to vanish.

## 5.1.2 Singularities in the yield surface

A major advantage of the scheme discussed in the preceding chapter is the easy manner in which singularities which occur in Mohr-Coulomb type yield functions can be dealt with<sup>15,16</sup>. These singularities occur if two principal stresses are equal, either  $\sigma_1$  and  $\sigma_2$ , or  $\sigma_2$  and  $\sigma_3$ . Koiter<sup>61,62</sup> has shown that in such cases in which in fact two yield functions are active, the plastic strain rate can be derived as follows:

$$\tilde{\boldsymbol{\varepsilon}}^{p} = \lambda_{1} \frac{\partial g_{1}}{\partial \boldsymbol{\sigma}} + \lambda_{2} \frac{\partial g_{2}}{\partial \boldsymbol{\sigma}}$$
(5.45)

Here  $g_1$  and  $g_2$  are the plastic potential functions which belong to the yield functions which are active ( $f_1$  and  $f_2$ , see Figure 5.2). We observe that we now have two non-negative multipliers ( $\lambda_1$  and  $\lambda_2$ ) instead of one.

As we must require that at the end of the loading step, that is after stress correction, neither of both yield functions is violated, we can determine these multipliers from the conditions  $f_1(\sigma^1)=0$  and  $f_2(\sigma^1)=0$ . Noting that by virtue of equation (5.45), we have at a singularity:

$$\boldsymbol{\sigma}^{1} = \boldsymbol{\sigma}^{t} - \lambda_{1} \boldsymbol{D}^{ef} \frac{\partial g_{1}}{\partial \boldsymbol{\sigma}} - \lambda_{2} \boldsymbol{D}^{ef} \frac{\partial g_{2}}{\partial \boldsymbol{\sigma}}$$
(5.46)

we may elaborate these conditions to yield the following equations:

$$f_{1}(\boldsymbol{\sigma}^{t} - \lambda_{1}\boldsymbol{D}^{ef} \frac{\partial g_{1}}{\partial \boldsymbol{\sigma}} - \lambda_{2}\boldsymbol{D}^{ef} \frac{\partial g_{2}}{\partial \boldsymbol{\sigma}}, \kappa) = 0$$

$$(5.47)$$

$$f_{2}(\boldsymbol{\sigma}^{t} - \lambda_{1}\boldsymbol{D}^{ef} \frac{\partial g_{1}}{\partial \boldsymbol{\sigma}} - \lambda_{2}\boldsymbol{D}^{ef} \frac{\partial g_{2}}{\partial \boldsymbol{\sigma}}, \kappa) = 0$$
(5.48)

When we develop these equations in a Taylor series around  $\boldsymbol{\sigma} = \boldsymbol{\sigma}^t$ ,  $\kappa = \kappa^0$ , we get:

$$f_{1}(\boldsymbol{\sigma}^{1},\kappa^{0}) = \lambda_{1} \left\{ h_{1} + \left( \frac{\partial f_{1}}{\partial \boldsymbol{\sigma}} \right)^{T} \boldsymbol{D}^{ef} \frac{\partial g_{1}}{\partial \boldsymbol{\sigma}} \right\}$$

$$+ \lambda_{2} \left\{ h_{2} + \left( \frac{\partial f_{1}}{\partial \boldsymbol{\sigma}} \right)^{T} \boldsymbol{D}^{ef} \frac{\partial g_{2}}{\partial \boldsymbol{\sigma}} \right\}$$

$$f_{2}(\boldsymbol{\sigma}^{1},\kappa^{0}) = \lambda_{1} \left\{ h_{3} + \left( \frac{\partial f_{2}}{\partial \boldsymbol{\sigma}} \right)^{T} \boldsymbol{D}^{ef} \frac{\partial g_{1}}{\partial \boldsymbol{\sigma}} \right\}$$

$$+ \lambda_{2} \left\{ h_{4} + \left( \frac{\partial f_{2}}{\partial \boldsymbol{\sigma}} \right)^{T} \boldsymbol{D}^{ef} \frac{\partial g_{2}}{\partial \boldsymbol{\sigma}} \right\}$$

$$(5.49)$$

$$(5.49)$$

with  $h_1$ ,  $h_2$ ,  $h_3$  and  $h_3$  defined as

$$h_1 = -\frac{\partial f_1}{\partial \kappa} \left( \frac{\partial \kappa}{\partial \boldsymbol{z}^p} \right)^T \frac{\partial g_1}{\partial \boldsymbol{\sigma}}$$
(5.51)

(5.52)

$$h_{3} = -\frac{\partial f_{2}}{\partial \kappa} \left[ \frac{\partial \kappa}{\partial \boldsymbol{\varepsilon}^{p}} \right]^{T} \frac{\partial g_{1}}{\partial \boldsymbol{\sigma}}$$
(5.53)

$$h_4 = -\frac{\partial f_2}{\partial \kappa} \left[ \frac{\partial \kappa}{\partial \boldsymbol{z}^p} \right]^T \frac{\partial g_2}{\partial \boldsymbol{\sigma}}$$
(5.54)



 $h_2 = -\frac{\partial f_1}{\partial \kappa} \left( \frac{\partial \kappa}{\partial \boldsymbol{\varepsilon}^p} \right)^T \frac{\partial g_2}{\partial \boldsymbol{\sigma}}$ 

Figure 5.2. Plastic flow at a singularity in the yield surface.

Again only the linear terms have been retained. With respect to the yield surface this is no limitation, as any yield function may be linearized around a singularity without loss of generality (see Figure 5.2). With respect to the dependence of the yield function on the hardening parameter  $\kappa$ , neglecting higher order terms means that the treatment is only exact for linear-hardening solids. Furthermore, the assumption that  $\dot{\kappa}$  is a linear function of  $\dot{\boldsymbol{\varepsilon}}^p$  which has been made implicitly, is not valid for important hardening hypotheses like the strain-hardening assumption (equation (4.25)). For regular parts of the yield surface, the non-linear dependence of  $\dot{\kappa}$  on  $\dot{\boldsymbol{\varepsilon}}^p$  does not entail errors, but for the corner regimes an additional error is introduced.

With aid of Cramer's rule, we can obtain explicit expressions for the scalars  $\lambda_1$  and  $\lambda_2$ . Introducing the auxiliary variables

$$\mu_1 = \left\{ h_1 + \left( \frac{\partial f_1}{\partial \boldsymbol{\sigma}} \right)^T \boldsymbol{D}^{\boldsymbol{e}f} \, \frac{\partial g_1}{\partial \boldsymbol{\sigma}} \right\} \tag{5.55}$$

$$\mu_{2} = \left\{ h_{2} + \left( \frac{\partial f_{1}}{\partial \boldsymbol{\sigma}} \right)^{T} \boldsymbol{D}^{ef} \frac{\partial g_{2}}{\partial \boldsymbol{\sigma}} \right\}$$
(5.56)

$$\mu_{3} = \left\{ h_{3} + \left( \frac{\partial f_{2}}{\partial \boldsymbol{\sigma}} \right)^{T} \boldsymbol{D}^{ef} \frac{\partial g_{1}}{\partial \boldsymbol{\sigma}} \right\}$$
(5.57)

$$\mu_{4} = \left\{ h_{4} + \left( \frac{\partial f_{2}}{\partial \boldsymbol{\sigma}} \right)^{T} \boldsymbol{D}^{ef} \frac{\partial g_{2}}{\partial \boldsymbol{\sigma}} \right\}$$
(5.58)

we get for  $\lambda_1$  and  $\lambda_2$ :

$$\lambda_1 = \frac{\mu_4 f_1(\boldsymbol{\sigma}^t) - \mu_2 f_2(\boldsymbol{\sigma}^t)}{\mu_1 \mu_4 - \mu_2 \mu_3}$$
(5.59)

$$\lambda_{2} = \frac{\mu_{1} f_{2}(\boldsymbol{\sigma}^{t}) - \mu_{3} f_{1}(\boldsymbol{\sigma}^{t})}{\mu_{1} \mu_{4} - \mu_{2} \mu_{3}}$$
(5.60)

It is finally noted that the procedure as described in this section for handling corner points in a yield surface, is in full agreement with the requirements as formulated by  $Koiter^{61,62}$ . It differs from some previous studies<sup>76</sup>, in that corners are not 'rounded-off', but are treated in a more exact manner.

## 5.1.3 Mohr-Coulomb type yield surfaces

Recalling the definition of the Mohr-Coulomb yield function

$$f = \frac{1}{2}(\sigma_3 - \sigma_1) + \frac{1}{2}(\sigma_3 + \sigma_1)\sin\varphi^* - c^*$$
(5.61)

where

$$\sigma_3 \ge \sigma_2 \ge \sigma_1 \tag{5.62}$$

we observe that we have to determine the principal stresses if we want to employ this criterion. For a general three-dimensional stress state, the principal stresses can be found as the roots of the cubic equation

$$\sigma^3 - I_1 \sigma_2 + I_2 \sigma - I_3 = 0 \tag{5.63}$$

with  $I_1$ ,  $I_2$  and  $I_3$  the stress invariants, see for instance Fung<sup>44</sup>. Using the deviatoric stress invariants  $J_2$  and  $J_3$ ,

$$J_{2} = \frac{1}{6} \left[ (\sigma_{xx} - \sigma_{yy})^{2} + (\sigma_{yy} - \sigma_{zz})^{2} + (\sigma_{zz} - \sigma_{xx})^{2} \right] + \sigma_{xy}^{2} + \sigma_{yz}^{2} + \sigma_{zx}^{2} \quad (5.64)$$

$$J_{3} = (\sigma_{xx} - p)(\sigma_{yy} - p)(\sigma_{zz} - p) + 2\sigma_{xy}\sigma_{yz}\sigma_{zx}$$

$$- (\sigma_{xx} - p)\sigma_{yz}^{2} - (\sigma_{yy} - p)\sigma_{zx}^{2} - (\sigma_{zz} - p)\sigma_{xy}^{2} \quad (5.65)$$

with

 $p = \frac{1}{3} \left( \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \right) \tag{5.66}$ 

we may replace equation (5.63) by:

$$s^3 - J_2 s - J_3 = 0$$
 (5.67)

as the first deviatoric stress invariant vanishes by definition. The notation s is again employed to denote deviatoric stresses. Equation (5.67) can be solved using Cardano's formulas. For the case of three real roots (which holds true because of the symmetry of the stress tensor), they read:

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = 2\sqrt{\frac{1}{3}J_2} \begin{bmatrix} \sin(\alpha - \frac{2}{3}\pi) \\ \sin\alpha \\ \sin(\alpha + \frac{2}{3}\pi) \end{bmatrix}$$
(5.68)

where  $\alpha$ , which can be interpreted as an angle in the  $\pi$ -plane, follows from

$$\sin(3\alpha) = -\frac{3}{2}\sqrt{3}\frac{J_3}{J_2\sqrt{J_2}}$$
(5.69)

Hence, we obtain for the total principal stresses:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = 2\sqrt{\frac{1}{3}J_2} \begin{bmatrix} \sin(\alpha - \frac{2}{3}\pi) \\ \sin\alpha \\ \sin(\alpha + \frac{2}{3}\pi) \end{bmatrix} + p \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
(5.70)

We next assume that we have the situation in which the strict inequality signs of equations (5.62) hold. Then, we may substitute the expressions (5.70) for  $\sigma_1$  and  $\sigma_3$  in equation (5.61). This yields:

$$f = \sqrt{J_2} \cos\alpha - [2\sqrt{\frac{1}{3}J_2} - p] \sin\varphi^* - c^*$$
 (5.71)

so that we have for the gradient to the yield surface:

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \sin\varphi^* \frac{\partial p}{\partial \boldsymbol{\sigma}} + \left(\alpha + b \frac{\partial \alpha}{\partial J_2}\right) \frac{\partial J_2}{\partial \boldsymbol{\sigma}} + b \frac{\partial \alpha}{\partial J_3} \frac{\partial J_3}{\partial \boldsymbol{\sigma}}$$
(5.72)

with the scalars a and b given by

$$a = \frac{1}{2\sqrt{J_2}} \left[ \cos\alpha - \sqrt{\frac{1}{3}} \sin\alpha \sin\varphi^* \right]$$
(5.73)

and

$$b = -\sqrt{J_2} \left[ \sin\alpha + \sqrt{\frac{1}{3}} \cos\alpha \sin\varphi^* \right]$$
(5.74)

The derivatives which occur in expression (5.72) can be determined by differentiation of equation (5.69) and the expressions for  $J_2$  and  $J_3$ <sup>18</sup>. It is further noted that the manner in which the the gradient to the plastic

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potential function g is computed, is essentially similar to the computation of the gradient to the yield function f, but for the fact that the friction angle  $\varphi^*$  in formula (5.72) is replaced by the dilatancy angle  $\psi^*$ .



Figure 5.3. Mohr-Coulomb surface in three-dimensional stress space.



Figure 5.4. Active part of Mohr-Coulomb surface in the  $\pi$ -plane.

When the strict inequality signs of (5.62) do not hold, i.e. if two principal stresses become equal, the stress point is in a ridge of the Mohr-Coulomb yield surface (see Figure 5.3). In such a point, the yield function f is continuous, but not continuous differentiable and the plastic strain rate is determined via Koiter's generalisation (5.45). For the Mohr-Coulomb surface, Figure 5.4 shows that we essentially have two yield

corners when we order the principal stresses according to equation (5.62)‡. Furthermore, the yield function (5.61) is active for all cases, either at a smooth part of the yield surface, or at a singularity. Hence, one of the two required gradients to the yield surface is given by equations (5.72) to (5.74).

First suppose that we have the case in which  $\sigma_1{=}\sigma_2$ , so that the yield function

$$f = \frac{1}{2} (\sigma_3 - \sigma_2) + \frac{1}{2} (\sigma_3 + \sigma_2) \sin \varphi^* - c^*$$
(5.75)

is also active. Substituting equation (5.70) for the principal stresses, and differentiating again results in equation (5.72) for the gradient  $\frac{\partial f}{\partial \sigma}$ , but now the scalars  $\alpha$  and b are given by:

$$a = \frac{1}{4\sqrt{J_2}} \left\{ \cos\alpha - \sqrt{3}\sin\alpha + \left[\sqrt{\frac{1}{3}}\sin\alpha + \cos\alpha\right]\sin\varphi^* \right\}$$
(5.76)

and

$$b = \frac{1}{2}\sqrt{J_2} \left\{ -\sin\alpha - \sqrt{3}\cos\alpha + \left[\sqrt{\frac{1}{3}}\cos\alpha - \sin\alpha\right]\sin\varphi^* \right\}$$
(5.77)

Similarly, for  $\sigma_2 = \sigma_3$ , we obtain

$$a = \frac{1}{4\sqrt{J_2}} \left\{ \sin\alpha + \sqrt{3}\cos\alpha + \left[\sqrt{\frac{1}{3}}\sin\alpha - \cos\alpha\right]\sin\varphi^* \right\}$$
(5.78)

and

$$b = \frac{1}{2}\sqrt{J_2} \left\{ -\sin\alpha + \sqrt{3}\cos\alpha + \left[\sqrt{\frac{1}{3}}\cos\alpha + \sin\alpha\right]\sin\varphi^* \right\}$$
(5.79)

The gradients to the plastic potential are determined in a similar manner, but for the fact that the angle of internal friction  $\varphi^*$  is replaced by the dilatancy angle  $\psi^*$ .

Finally, it has to be determined whether the stress is such that we have a corner regime or whether the stress correction can be calculated for a regular part of the yield surface. Figure 5.4 shows two test stress

<sup>&</sup>lt;sup>‡</sup> Another singularity in the Mohr-Coulomb yield surface occurs at the apex of the yield cone. In fact, the numerical algorithm should check whether the stress is beyond the apex. If this happens to be the case, an additional correction should be applied as to bring the stress point to the apex of the yield cone [16]. This problem of course does only arise for cohesionless materials such as sand, because for cohesive materials, the fracture model bounds the allowable tensile stresses.

points, A and B. It is evident that for point A a flow rule in the sense of equation (4.39) can be used, but that for point B, application of such a rule would lead to a final stress lying outside of the yield surface, so that in the latter case we have to apply Koiter's generalisation for the flow rule. The selection of the precise flow equation might be made on basis of trial and error, but in the calculations reported here, we have adopted a more simple and elegant procedure. If we have perfect plasticity, the yield surface remains fixed, so that the position of the corner point is uniquely determined in stress space. It is then easy to analytically determine a plane which passes through the corner points and which has the direction

$$D^{ef} \frac{\partial g}{\partial \sigma}$$

The projections of these lines on the  $\pi$ -plane are plotted in Figure 5.4. The derivation of an analytical expression for these planes is straightforward, and it can be derived that if

$$h_1 < 0 \quad \text{and} \quad h_2 < 0 \tag{5.80}$$

we are on a regular part of the yield surface. In it,  $h_1$  and  $h_2$  are defined as

$$h_1 = f(\boldsymbol{\sigma}^t) + \frac{1 - 2\nu + \sin\varphi \sin\psi}{1 - 2\nu - (1 + 2\nu)\sin\psi} (\sigma_2 - \sigma_3)$$
(5.81)

and

$$h_2 = f(\boldsymbol{\sigma}^t) - \frac{1 - 2\nu + \sin\varphi \sin\psi}{1 - 2\nu - (1 + 2\nu)\sin\psi} (\sigma_2 - \sigma_1)$$
(5.82)

If we have the situation

 $h_1 > 0$  and  $h_2 < 0$  (5.83)

we are in a corner regime for  $\sigma_2 = \sigma_3$ , whereas for

$$h_1 < 0 \text{ and } h_2 > 0$$
 (5.84)

we have a corner regime for  $\sigma_2 = \sigma_1$ . For hardening plasticity, this procedure can not rigorously predict the correct regime, because the position of the yield surface is unknown. In particular, we may erroneously arrive at the conclusion that we have a corner regime for hardening plasticity or that we have a smooth regime when we have softening. Mostly, the prediction on basis of equations (5.80) to (5.84) will be adequate, but if necessary, an a posteriori check may be performed so that the erroneous assumption can be corrected.

A particular problem is sometimes said to arise if two principal stresses of the test stress become exactly equal, as then the derivatives  $\frac{\partial J_2}{\partial \sigma}$  and  $\frac{\partial J_3}{\partial \sigma}$  would become indeterminate. However, in practice no difficulties are encountered, as even when two stresses are exactly equal, the entire expression (5.72) for the gradient to the yield surface remains determinate.

# 5.2. Solution of the non-linear algebraic equations

Having discussed the evaluation of the stress-strain law on integration point level, we will now turn our attention to the solution of the algebraic equations on structural level. These equations may be highly non-linear, and an incremental-iterative solution procedure is usually needed for an accurate solution. Various procedures exist for controlling this process. Analogous to experiments, we have load control and (direct) displacement control. However, either of these procedures may fail in particular circumstances. With load control, we are not able to overcome limit points at all, and with direct displacement control it is not possible to properly analyse 'snap-back' behaviour (see for instance Figure 4.20). Fortunately, a very general and powerful method has been developed within the realm of geometrically non-linear analysis. In this method, the incremental-iterative process is controlled indirectly using a norm of incremental displacements<sup>33,88,93</sup>. For this reason, the name 'arc-length method' has been coined for the procedure. For materially non-linear analysis, a global norm on incremental displacements is often less successful, due to localisation effects and it may be more efficient to employ only one dominant degree of freedom or omitting some degrees of freedom from the norm of incremental displacements. The name arc-length control then no longer seems very appropriate, and instead we will use the term indirect displacement control.

# 5.2.1 Indirect displacement control

It is recalled from Chapter 3, that the iterative improvement  $\delta a_i$  to the displacement increment  $\Delta a_{i-1}$  is given by

$$\boldsymbol{\delta a}_{i} = \boldsymbol{K}_{i-1}^{-1} \left[ \boldsymbol{p}_{i-1} + \Delta \boldsymbol{\mu}_{i} \boldsymbol{q}^{*} \right]$$
(5.85)

The essence of controlling the iterative solution procedure indirectly by displacements, is that  $\delta a_i$  is conceived to be composed of 2 contributions

δ
$$\boldsymbol{a}_i = \boldsymbol{\delta} \boldsymbol{a}_i^I + \Delta \mu_i \, \boldsymbol{\delta} \boldsymbol{a}_i^{II}$$

with

$$\delta a_i = K_{i-1} p_{i-1}$$

and

 $\delta \boldsymbol{a}_{i}^{II} = \boldsymbol{K}_{i-1}^{-1} \boldsymbol{q}^{*}$ 

After calculating the displacement vectors  $\delta a_i^I$  and  $\delta a_i^{II}$ , the value for  $\Delta \mu_i$  is determined from some constraint equation on the displacement increments. Crisfield<sup>33</sup> for instance uses the norm of the incremental displacements as constraint equation

$$\mathbf{\Delta a}_{i}^{T} \mathbf{\Delta a}_{i} = \Delta l^{2} \tag{5.89}$$

where  $\Delta l$  is the arc-length of the equilibrium path in the *n*-dimensional displacement space. The draw-back of this so-called spherical arc-length method is that it yields a quadratic equation for the load increment. To circumvent this problem, one may linearize equation (5.89), yielding<sup>68</sup>:

$$\mathbf{\Delta a}_{i}^{T} \mathbf{\Delta a}_{i-1} = \Delta l^{2} \tag{5.90}$$

This method, known as the updated normal path method, results in a linear equation for the load increment.

Equation (5.90) may be simplified by subtracting the constraint equation of the previous iteration. This gives

$$\boldsymbol{\Delta \boldsymbol{a}_{i-1}^{T}} (\boldsymbol{\Delta \boldsymbol{a}_{i}} - \boldsymbol{\Delta \boldsymbol{a}_{i-2}}) = 0 \tag{5.91}$$

When we furthermore make the approximation

$$\delta \boldsymbol{a}_i \approx 2(\boldsymbol{\Delta} \boldsymbol{a}_i - \boldsymbol{\Delta} \boldsymbol{a}_{i-2}) \tag{5.92}$$

we obtain

 $\mathbf{\Delta a}_{i-1}^{T} \mathbf{\delta a}_{i} = 0 \tag{5.93}$ 

Substituting equation (5.86) then gives for  $\Delta \mu_i$ :

$$\Delta \mu_i = -\frac{\mathbf{\Delta} \mathbf{a}_{i-1}^T \mathbf{\delta} \mathbf{a}_i^I}{\mathbf{\Delta} \mathbf{a}_i^{T-1} \mathbf{\delta} \mathbf{a}_i^{II}}$$
(5.94)

Both equations (5.89) and (5.90) have been employed very successfully within the realm of geometrically non-linear problems, where snapping and buckling of thin shells can be traced very elegantly. Nevertheless, for physically non-linear problems the method sometimes fails, which may be explained by considering that for physically non-linear problems, failure or bifurcation modes are often highly localised. Hence, only a few nodes contribute to the norm of displacement increments, and failure is not sensed accurately by such a global norm. As straightforward application of equations (5.89) or (5.90) is not always successful, we may amend these constraint equations by applying weights to the different degrees of freedom or omitting some of them from the constraint equation. Examples thereof are given in the next chapters. The disadvantage of modifying the constraint equation is that the constraint equation becomes problem dependent. As a consequence, the method loses some of its generality and elegance.

## 5.2.2 Continuation beyond bifurcation points

The procedure of indirect displacement control discussed in the preceding sub-section in principle allows for overcoming limit points and tracing post-peak behaviour. If we have the rare case of a genuine bifurcation point, we will generally continue on the fundamental path, but this is an unstable equilibrium path after the bifurcation point and it is desirable to have a procedure to continue on the lowest bifurcation path after passing a bifurcation point. It has been demonstrated in Section 3.2, that for infinitesimally small increments, the velocity vector of the bifurcation path can be written as a linear combination of the velocity field belonging to the fundamental path  $\boldsymbol{a}^*$  and the eigenmode  $\boldsymbol{v}_1$ . For finite increments this may be integrated to yield<sup>93</sup>

$$\Delta \boldsymbol{a} = \alpha (\Delta \boldsymbol{a}^* + \beta \boldsymbol{v}_1) \tag{5.95}$$

1- -->

with  $\alpha$  and  $\beta$  scalars. The magnitude of these scalars is fixed by secondorder terms or by switch conditions for elastoplasticity or for plasticfracturing materials. The most simple way to determine  $\beta$  numerically is to construct a trial displacement increment  $\Delta \alpha$  such that it is orthogonal to the fundamental path:

$$\Delta \boldsymbol{a}^T \boldsymbol{\Delta} \boldsymbol{a}^* = 0 \tag{5.96}$$

Substituting equation (5.95) in this expression yields for  $\beta$ 

$$\beta = -\frac{(\Delta \boldsymbol{a}^*)^T \Delta \boldsymbol{a}^*}{(\Delta \boldsymbol{a}^*)^T \boldsymbol{v}_1}$$
(5.97)

so that we obtain for **Aa** 

$$\boldsymbol{\Delta \boldsymbol{a}} = \alpha \left\{ \boldsymbol{\Delta \boldsymbol{a}}^* - \frac{(\boldsymbol{\Delta \boldsymbol{a}}^*)^T \boldsymbol{\Delta \boldsymbol{a}}^*}{(\boldsymbol{\Delta \boldsymbol{a}}^*)^T \boldsymbol{v}_1} \boldsymbol{v}_1 \right\}$$
(5.98)

(5.86)

(5.87)

(5.88)

Equation (5.98) fails when  $(\mathbf{\Delta a}^*)^T \boldsymbol{v}_1 = 0$ , i.e. if the bifurcation mode is orthogonal to the basic path. A simple remedy is to normalize  $\mathbf{\Delta a}$  such that

$$(\Delta a^*)^T \Delta a^* = \Delta a^T \Delta a$$

This results in:

$$\boldsymbol{\Delta \boldsymbol{a}} = \frac{1}{\sqrt{(\boldsymbol{\Delta \boldsymbol{a}}^{*})^{T} \boldsymbol{\Delta \boldsymbol{a}}^{*} - [(\boldsymbol{\Delta \boldsymbol{a}}^{*})^{T} \boldsymbol{v}_{1}]^{2}}} \left\{ (\boldsymbol{\Delta \boldsymbol{a}}^{*})^{T} \boldsymbol{v}_{1} \boldsymbol{\Delta \boldsymbol{a}}^{*} - (\boldsymbol{\Delta \boldsymbol{a}}^{*})^{T} \boldsymbol{\Delta \boldsymbol{a}}^{*} \boldsymbol{v}_{1} \right\}$$
(5.100)

(5.99)

The denominator of this expression never vanishes, since this would imply that the eigenmode is identical with the fundamental path.

In general the bifurcation path will not be orthogonal to the fundamental path, but when we add equilibrium iterations, the orthogonality condition (5.96) will maximise the possibility that we converge on a bifurcation branch and not on the fundamental path, although this is not necessarily the lowest bifurcation path when there emanate more equilibrium branches from the bifurcation point. When we do not converge on the lowest bifurcation path, this will be revealed by negative eigenvalues of the bifurcated solution. The above described procedure can then be repeated until we ultimately arrive at the lowest bifurcation path.

# 5.2.3 Jumping over spurious snap-behaviour

Closely related to the determination of bifurcation and limit points and the tracing of snap-behaviour is the issue of avoiding 'spurious' snapbehaviour. It was argued in Chapter 3 that spatial discretization may introduce spurious, non-physical limit points and snap-behaviour. Also, it was noted that owing to temporal discretization, we deal with equilibrium states rather than equilibrium paths. The question therefore arises whether other equilibrium states can be reached via non-equilibrium paths. If there exists another equilibrium state which is "not too far away" from the current state, the examples of the subsequent chapters indicate that this is often possible when we adopt direct displacement control, i.e. we prescribe one or more displacements while the resulting nodal forces provide the applied load. When we obtain a converged state after a number of non-equilibrium states, i.e. non-converged states, we can generally conclude that we indeed have arrived on a new equilibrium path, although such a procedure may be quite dangerous and it can only be successful if the new equilibrium state is located "sufficiently close" to the old equilibrium state.

Under load control, such "jumping" is much more difficult and the danger of divergence is appreciably greater. An idea, not yet tried out, to enhance the possibility of traversing such a non-equilibrium path under load control, is to suppress the failure or the bifurcation mode. Effectively, this means that we remove the contribution of the eigenvector  $\boldsymbol{v}_1$  from the incremental displacement vector  $\boldsymbol{\Delta a}^*$ 

$$\Delta \boldsymbol{a} = \Delta \boldsymbol{a}^* - \alpha \boldsymbol{v}_1 \tag{5.101}$$

with  $\alpha$  some scalar which is found by premultiplying  $\Delta \alpha^*$  with the left eigenvector  $\boldsymbol{w}_1$  (see Section 3.2):

$$\alpha = \mathbf{w}_{1}^{T} \Delta \boldsymbol{a}^{*} \tag{5.102}$$

# 5.2.4 Quasi-Newton methods

Equation (5.85) not only implies that the internal force vector is updated every iteration, but also that the tangent stiffness matrix  $K_i$  is recomputed and factorised at each iteration. Although this Newton-Raphson method<sup>81</sup> is very powerful, it is expensive and therefore, modified procedures are often applied in structural analysis. The oldest modification is to update the stiffness matrix only at the beginning of a loading step, or only once every say 5 loading steps. Indeed, sometimes no update at all is done in the loading process and all equilibrium iterations are carried out with the initial, elastic stiffness matrix. Although convergence may be rather slow, especially in the vicinity of limit points, this method may still be competing when non-symmetric systems are considered.

In the last few years, there has been a search for iterative solution procedures which are faster than modified Newton-Raphson methods, but which avoid the costly calculating and factorising of the tangent stiffness matrix at every iteration. In particular, there has been a proliferation of so-called Quasi-Newton methods <sup>38</sup>. Amongst this class, especially the BFGS-update<sup>72</sup> has gained much popularity, although the author has found that the simpler Broyden formula may also be very efficient, especially for non-symmetric systems<sup>19,43</sup>. In the first applications of Quasi-Newton methods<sup>45,72</sup>, the analyses were carried out under load control or under direct displacement control. In this section, a derivation is presented for the application of Quasi-Newton methods in conjunction with indirect displacement control, so that the load level is variable





during the loading step<sup>19</sup>.

The essence of the difference between the various iteration procedures lies in the manner in which the stiffness matrix  $K_i$  is computed. When applying a Quasi-Newton method, this matrix is calculated from the previous stiffness matrix  $K_{i-1}$  in such a way, that it satisfies the condition

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$$\mathbf{K}_i \, \boldsymbol{\delta a}_i = \boldsymbol{\delta p}_i \tag{5.103}$$

which is known as the Quasi-Newton equation<sup>38,81</sup>.  $\delta p_i$  represents the variation in the internal force field and is defined as (see also Figure 5.5)

$$\delta \boldsymbol{p}_{i} = \int_{V} \boldsymbol{B}^{T} \boldsymbol{\sigma}_{i} dV - \int_{V} \boldsymbol{B}^{T} \boldsymbol{\sigma}_{i-1} dV$$
(5.104)

One of the simplest update formulae satisfying the Quasi-Newton equation is Broyden's update<sup>19,38,43</sup>, for which the new stiffness matrix  $K_i$  is obtained from the previous matrix  $K_{i-1}$  by the formula:

$$\boldsymbol{K}_{i} = \boldsymbol{K}_{i-1} \left[ \boldsymbol{I} - \frac{\boldsymbol{u}_{i} \, \boldsymbol{\delta} \boldsymbol{a}_{i}^{T}}{\boldsymbol{\delta} \boldsymbol{a}_{i}^{T} \boldsymbol{\delta} \boldsymbol{a}_{i}} \right]$$
(5.105)

where the auxiliary vector  $\boldsymbol{u}_i$  has been introduced:

$$\boldsymbol{u}_i = \boldsymbol{\delta} \boldsymbol{a}_i - \boldsymbol{K}_{i-1}^{-1} \boldsymbol{\delta} \boldsymbol{p}_i \tag{5.106}$$

Broyden's first order update formula satisfies the Quasi-Newton formula as can be verified by simple substitution. Approximating the true tangent stiffness matrix by a secant formulation in the sense of equation (5.105) avoids the expensive calculation of the tangent element stiffness matrices. However, the cost of factorising the updated stiffness matrix remains, so that when the updating is applied to the tangent stiffness matrix, the gain in computational effort remains moderate, especially when we consider that the number of iterations generally increases compared to a full tangent stiffness approach. To overcome this difficulty, we can make use of the Sherman-Morrison formula<sup>38,81.</sup>

$$(\boldsymbol{A} + \alpha \boldsymbol{u} \boldsymbol{v}^{T})^{-1} = \boldsymbol{A}^{-1} - \alpha \frac{\boldsymbol{A}^{-1} \boldsymbol{u} \boldsymbol{v}^{T} \boldsymbol{A}^{-1}}{1 + \alpha \boldsymbol{v}^{T} \boldsymbol{A}^{-1} \boldsymbol{u}}$$
(5.107)

with **A** a matrix, **u** and **v** vectors and  $\alpha$  a scalar. Application of this identity to Broyden's update formula gives

$$\boldsymbol{K}_{i}^{-1} = \boldsymbol{K}_{i-1}^{-1} + \frac{\boldsymbol{u}_{i} \, \delta \boldsymbol{a}_{i}^{T} \boldsymbol{K}_{i-1}^{-1}}{\delta \boldsymbol{a}_{i}^{T} \boldsymbol{K}_{i-1}^{-1} \, \delta \boldsymbol{p}_{i}}$$
(5.108)

This result implies that the factorisation of the updated stiffness matrix can be avoided as we can apply the updating directly to the inverse of the stiffness matrix. The added cost if compared to a Modified Newton-Raphson iteration without updating, is only a matter of a few matrixvector multiplications. As we will derive in the sequel, even such matrixvector multiplications need not performed in the actual algorithm, as the algorithmic implementation merely requires the calculation of a few inner products and multiplications of vectors by scalars.

Using equations (5.86) to (5.88), we can obtain an alternative expression for  $\boldsymbol{u}_i$ :

$$\boldsymbol{u}_{i} = \Delta \boldsymbol{\mu}_{i} \, \boldsymbol{\delta} \boldsymbol{a}_{i}^{II} + \boldsymbol{K}_{i-1}^{-1} \boldsymbol{p}_{i} \tag{5.109}$$

Applying Broyden's formula repeatedly to the last term gives:

$$\boldsymbol{u}_{i} = \Delta \mu_{i} \boldsymbol{\delta} \boldsymbol{a}_{i}^{II} + \prod_{j=1}^{i-1} [\boldsymbol{I} + \alpha_{j} \boldsymbol{u}_{j} \boldsymbol{\delta} \boldsymbol{a}_{j}^{T}] \boldsymbol{K}_{0}^{-1} \boldsymbol{p}_{i}$$
(5.110)

with

$$\alpha_j = [\boldsymbol{\delta a_j}^T (\boldsymbol{\delta a_j} - \boldsymbol{u}_j)]^{-1}$$
(5.111)

It is noted that equation (5.110) can be used directly to calculate  $\boldsymbol{u}_i$ . Next, the vectors  $\delta \boldsymbol{a}_i^I$  and  $\delta \boldsymbol{a}_i^{II}$  can be computed by inserting the update formula (5.108) in equations (5.87) and (5.88). This gives:

$$\boldsymbol{\delta a_i^I} = [\boldsymbol{I} + \boldsymbol{\alpha}_i \boldsymbol{u}_i \boldsymbol{\delta a}_i^T] (\boldsymbol{u}_i - \Delta \boldsymbol{\mu}_i \boldsymbol{\delta a}_i^{II})$$
(5.112)

$$\delta \boldsymbol{a}_{i}^{II} = \delta \boldsymbol{a}_{i-1}^{II} + \alpha_{i} (\delta \boldsymbol{a}_{i}^{T} \delta \boldsymbol{a}_{i}^{II}) \boldsymbol{u}_{i}$$
(5.113)

Thereafter, the load increment  $\Delta \mu_i$  is calculated in the usual way, either via the normal, via the spherical path method or via some other displacement parameter. Then, the new incremental displacement vector is calculated via equations (5.86) and (3.18). The construction of formulae of other first order Quasi-Newton updates with indirect displacement control runs along the same lines.

Construction of formulae for second order update methods like the BFGS method is essentially similar, although the derivation is more cumbersome. A number of equivalent formulations exist for the BFGSupdate, but here

$$\boldsymbol{K}_{i}^{-1} = \boldsymbol{K}_{i-1}^{-1} + \frac{\boldsymbol{u}_{i} \boldsymbol{\delta} \boldsymbol{a}_{i}^{T} + \boldsymbol{\delta} \boldsymbol{a}_{i} \boldsymbol{u}_{i}^{T}}{\boldsymbol{\delta} \boldsymbol{a}_{i}^{T} \boldsymbol{\delta} \boldsymbol{p}_{i}} - \frac{\boldsymbol{u}_{i}^{T} \boldsymbol{\delta} \boldsymbol{p}_{i}}{(\boldsymbol{\delta} \boldsymbol{a}_{i}^{T} \boldsymbol{\delta} \boldsymbol{p}_{i})^{2}} \boldsymbol{\delta} \boldsymbol{a}_{i} \boldsymbol{\delta} \boldsymbol{a}_{i}^{T}$$
(5.114)

is employed as update formula for the inverse of the stiffness matrix, where  $\boldsymbol{u}_i$  is again defined according to equation (5.106). Repeated application of the BFGS-formula on  $\boldsymbol{u}_i$  now results in

$$\boldsymbol{u}_{i} = \Delta \mu_{i} \boldsymbol{\delta} \boldsymbol{a}_{i}^{H} + \boldsymbol{K}_{0}^{-1} \boldsymbol{p}_{i} + \sum_{j=1}^{i-1} \left[ \alpha_{j} \gamma_{j} \boldsymbol{u}_{j} + \alpha_{j} (\kappa_{j} + \alpha_{j} \beta_{j} \gamma_{j}) \boldsymbol{\delta} \boldsymbol{a}_{j} \right]$$
(5.115)

in which the auxiliary scalars  $\alpha_i,\ \beta_i,\ \gamma_j$  and  $\kappa_j$  have been introduced which are defined as

$$\alpha_i = (\delta \boldsymbol{a}_i^T \delta \boldsymbol{a}_i)^{-1} \tag{5.116}$$

$$\boldsymbol{\beta}_i = -\boldsymbol{u}_i^T \boldsymbol{\delta} \boldsymbol{a}_i \tag{5.117}$$

$$\gamma_j = \boldsymbol{\delta a_j}^T \boldsymbol{p_i} \tag{5.118}$$

$$\kappa_j = \boldsymbol{u}_j^T \boldsymbol{p}_i \tag{5.119}$$

Again, equation (5.115) can be used directly to calculate  $\boldsymbol{u}_i$ . The new displacement vectors  $\boldsymbol{\delta a}_i^I$  and  $\boldsymbol{\delta a}_i^{II}$  can subsequently be calculated from

$$\boldsymbol{\delta a_{i+1}^{I} =} (1 + \alpha_i \gamma_i) \boldsymbol{u}_i + \alpha_i (\kappa_i + \alpha_i \beta_i \gamma_i) \boldsymbol{\delta a_i} - \Delta \mu_i \boldsymbol{\delta a_i^{II}}$$
(5.120)

and

$$\boldsymbol{\delta a}_{i+1}^{II} = \boldsymbol{\delta a}_{i}^{II} + \alpha_{i} (\boldsymbol{\delta a}_{i}^{T} \boldsymbol{q}^{*}) \boldsymbol{u}_{i} + \alpha_{i} [\boldsymbol{u}_{i}^{T} \boldsymbol{q}^{*} + \alpha_{i} \beta_{i} \boldsymbol{\delta a}_{i}^{T} \boldsymbol{q}^{*}] \boldsymbol{\delta a}_{i} \qquad (5.121)$$

In fact, the above described procedures may be conceived as accelerated Modified Newton-Raphson procedures, where the acceleration factors are calculated automatically. However, the methods become less efficient when the number of updates increases, because the evaluation of  $\mathbf{u}_i$  takes more and more computational effort. In structural analysis, this generally constitutes no problem as the load is applied in small

increments. In an incremental-iterative procedure, a new tangent stiffness can then be formed at the beginning of the loading step, and the acceleration via the Quasi-Newton method can be applied in the subsequent equilibrium iterations. Yet, if we allow for instance 20 equilibrium iterations, this may still constitute a problem when we consider larger systems, because this may require storage of 20 vectors with a length equal to the number of degrees of freedom for Broyden's method and of 40 vectors for the BFGS method. To avoid memorising these vectors, Crisfield<sup>32,33</sup> has proposed to apply a single cycle update which is related to the BFGS-formula and which is given by:

$$\boldsymbol{\delta a}_{i+1}^{I} = (1 + \alpha_i \gamma_i) \boldsymbol{K}_0^{-1} \boldsymbol{p}_i + \alpha_i (\beta_i + \gamma_i + \alpha_i \beta_i \gamma_i) \boldsymbol{\delta a}_i$$
(5.122)

and

$$\boldsymbol{\delta a}_{i+1}^{II} = \alpha_i (\boldsymbol{\delta a}_i^T \boldsymbol{q}^*) \boldsymbol{K}_0^{-1} \boldsymbol{q}^* + \alpha_i [\boldsymbol{\delta a}_i^T \boldsymbol{q}^* - \alpha_i \beta_i (\boldsymbol{\delta a}_i^T \boldsymbol{q}^*) \boldsymbol{\delta p}_i^T \boldsymbol{K}_0^{-1} \boldsymbol{q}^*] \boldsymbol{\delta a}_i$$
(5.123)

but  $\beta_i$  is now given by

$$\boldsymbol{\beta}_i = -\boldsymbol{\delta} \boldsymbol{p}_i^T \boldsymbol{K}_0^{-1} \boldsymbol{p}_i \tag{5.124}$$

It is the author's experience that this update formula is nearly as efficient as the original Quasi-Newton methods, but it has the advantage that only 2 vectors of n degrees of freedom need to be stored. However, this update formula no longer satisfies the Quasi-Newton equation (5.103), but merely satisfies the n-dimensional Secant-formula

$$\delta \boldsymbol{p}_i^T \boldsymbol{a}_{i+1} = \boldsymbol{a}_i^T (\boldsymbol{p}_i + \Delta \mu_i \boldsymbol{q}^*) \tag{5.125}$$

Therefore, the name Secant-Newton methods has been coined for this class of methods<sup>32,33</sup>.

#### 5.2.5 Examples

We will demonstrate the efficiency of Quasi and Secant-Newton methods by means of two examples. The first example concerns a crack propagation problem in a notched, unreinforced concrete beam (Figure 5.6). The loading is applied symmetrically, so that the crack only shows opening, but no sliding. As attention was focussed on the tensile behaviour of the concrete, the simple elastic-fracture model with a linear softening branch was adopted.

Several Newton-Raphson type iterative procedures and Quasi and Secant-Newton updates were tested for the same energy criterion





Figure 5.6. Load-displacement curve, geometry and material data for crack propagation problem.

Table 5.1. Number of iterations and C	PU-times for crack propagation
problem	m.

Method	Number of iterations	CPU-time in seconds
Newton Raphson	357	7836
Modified Newton	384	5490
BFGS	159	2776
Broyden	174	2981
Davidon	407	6088
Secant-Newton	167	2907

Table 5.2. Number of iterations and CPU-times for slope stability problem.

Method	Restart	Iterations	CPU-time (sec.)
Constant stiffness	elastic stiffness	563	12165
BFGS	elastic stiffness	294	6959
Broyden	elastic stiffness	190	4490
Secant-Newton	elastic stiffness	257	6156
Constant stiffness	equation (5.126)	300	6274
BFGS	equation (5.126)	179	4430
Broyden	equation (5.126)	on (5.126) 151 3605	
Secant-Newton	equation (5.126)	158	3771

 $(\varepsilon = 0.000001)$ . All Quasi and Secant-Newton updates were obtained using a tangential restart in the first iteration. The results which have been summarised in Table 1, reveal that Quasi and Secant-Newton methods indeed yield a considerable savings in computer time for this particular problem.

An important observation is that Quasi and Secant Newton methods sometimes produce worse solutions for the contribution to the incremental displacement vector than would have been obtained using a Newtontype method. This is in accordance with observations of other researchers<sup>32,72</sup>, who note that these especially seem to occur if the ratio between the factors in equations (5.112) and (5.113) or (5.120) and (5.121) for the contributions to the incremental displacement vectors become disproportional. It has been proposed to define bounds between which the acceleration factors  $\alpha_i$  etc. should remain<sup>32</sup>. If these bounds, also called "cut-out criteria", are violated, the updating according to a Quasi-Newton formula is omitted and a conventional (Modified) Newton-Raphson method is applied. For the problem considered here, it appeared that all Quasi and Secant-Newton methods performed optimal when rather loose "cut-







Figure 5.8. Incremental deformations of slope stability problem at failure.

out criteria" were employed.

The other example problem concerns a slope stability analysis for which the non-associated perfectly-plastic model was used. In this example, the loading was applied by incrementing the self-weight of the soil, while a global updated normal path method was used as control parameter of the solution process. The results of the various calculations are given in Table 5.2. As can be observed, Broyden's method and particularly the variant in which the incremental displacements in the first iteration  $\mathbf{Aa}_1$  were estimated from the result  $\mathbf{Aa}_0$  of the preceding loading step,

$$\boldsymbol{\Delta \boldsymbol{a}}_{1} = \frac{\Delta \mu_{1}}{\Delta \mu_{0}} \boldsymbol{\Delta \boldsymbol{a}}_{0}$$

(5.126)

with  $\Delta\mu_0$  and  $\Delta\mu_1$  the load increment in the preceding step and in the first iteration of the current step respectively, appeared to be the fastest, and performed better than the BFGS-formula which appeared to be superior in almost all problems with a symmetric tangent matrix which have been tested by the Author. Apparently, the non-symmetric character of Broyden's update formula is indeed beneficial for problems which generate non-symmetric tangent stiffness matrices.

Just as with the concrete beam, the results depend on the particular cut-out criteria which were employed. Again, it appeared that the best results were obtained with rather loose cut-out criteria. Nevertheless, a general recommendation to use loose cut-out criteria does not seem to be appropriate as this may lead to poor results in some particular cases.

Another problem with Quasi and Secant Newton methods is that they sometimes lead to what might be named as 'rogue solutions'. This is illustrated in Figure 5.7, where solutions obtained with Broyden's method are shown. In the vicinity of the limit load, and especially when the shear band leading to failure begins to develop (see Figure 5.8), the Quasi-Newton method gives an oscillatory load-deflection curve. However, when the incremental displacements in the first iteration were determined according to equation (5.126), the Quasi-Newton method yielded a smooth load-deflection curve.

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# 6. BIFURCATION POINTS AND OF POST-BIFURCATION BEHAVIOUR

In this chapter, we shall apply the models and the techniques discussed in the preceding chapters to some typical bifurcation problems in soil and concrete mechanics. In contrast to many bifurcation problems, the bifurcations considered here are solely caused by material behaviour and not by geometrically non-linear effects.

# 6.1. Strain localisation in concrete members

The first bifurcation problem which we will consider is a perfect bar of elastic-softening material which is subjected to a uniformly distributed (tensile) load. The associated limit problem has already been introduced in Section 4.4.2, where it has been explained that the response of an imperfect bar in the post-failure regime will depend upon the number of elements and the degree of interpolation within the elements. For sake of simplicity, the latter variable is eliminated by electing 4-noded elements with a bilinear displacement interpolation, so that we have a constant strain in the axial direction for each element. Then, the response in the post-peak regime only depends upon the number of elements.



Figure 6.1. Possible post-bifurcation behaviour for a bar loaded in tension. Which equilibrium path is followed depends on the number of elements in which the crack localises.

In the spirit of Section 4.2.2 we suppose that the bar is modelled by m elements. Then, the limit point is an m-1 fold bifurcation point in the sense that m-1 alternative equilibrium branches emanate from this point apart from the fundamental mode which continuous to deform homogeneously. The other bifurcation branches are associated with localisation modes in one or more elements, whereby the other elements

unload. This results in a fan of possible bifurcation branches which emanate from the bifurcation point. Which equilibrium path will be traversed depends on the number of elements in which the crack localises (see Figure 6.1).

When we introduce no imperfections, so that we do not transfer the bifurcation problem into a pure limit problem, a 64-bit processor is usually sufficient to guarantee that the bar deforms homogeneously also after the bifurcation point has been passed. Continuation on an equilibrium path which shows strain localisation is then possible by adding a part of the eigenvector which corresponds to a zero eigenvalue to the fundamental solution (see the preceding chapter). In practice, a bifurcation point cannot be isolated exactly since we work with finite arithmetic. Consequently, we load the bar slightly, say 1 percent, beyond the bifurcation point and negative rather than zero eigenvalues are obtained. Nevertheless, this does not affects the essentials of the procedure.



Figure 6.2. Eigenmodes for two-element bar.



Figure 6.3. Final displacements when the load has come down to zero.

In the example, we consider a material with a linear softening branch for which the ultimate strain  $\varepsilon_u$  at which the crack transfers no more normal stress, equals 10 times the strain at peak load. Let us first assume that the bar is modelled by only two elements in the axial direction (Figure 6.2). The bar is loaded by a uniformly distributed traction slightly beyond the peak load, using indirect displacement control to overcome the limit point properly. Next, the tangent stiffness matrix is reformed and two negative eigenvalues are calculated, the corresponding eigenmodes being plotted in Figure 6.2. Adding a part of the latter eigenmode to the fundamental solution resulted in continuation on the localisation path (A-C in Figure 4.20). The resulting displacements when the load has become zero have been plotted in Figure 6.3. - 100 -



Figure 6.4. Eigenvector for bar when divided in 10 elements when all 10 elements show loading.



Figure 6.5. Eigenvector for bar when divided in 10 elements when 5 elements show loading and the other 5 show unloading.



Figure 6.6. Eigenvector for bar when divided in 10 elements when 3 elements show loading.



Figure 6.7. Eigenvector for bar when divided in 10 elements when only 2 elements still show loading.



Figure 6.8. Final displacements for bar composed of 10 elements showing localisation in only one element.

As explained in Section 4.4.2, we obtain a multiple bifurcation point when we model the bar by more than 2 elements. Indeed, we calculate mnegative eigenvalues beyond peak load when we divide the bar in m elements. The selection of an appropriate eigenvector which gives localisation in only one element then becomes a somewhat tedious task. As indicated in the preceding chapter, the most simple way to solve this difficulty in practice, is to take the eigenvector corresponding to the lowest eigenvalue and add it to the fundamental solution. We will then converge on a localisation branch which is not necessarily the lowest bifurcation branch. However, when we extract the lowest eigenvalue and the corresponding eigenvector for the new state and add them to the current displacement increment, we generally arrive at a lower bifurcation branch, whereby it seems superfluous to remark that an extremely small increment must be employed. Repeating this process several times finally results in convergence on the lowest bifurcation branch. For the present example, this implies that we finally converge on a branch in which the softening localises in only one element while the other elements show unloading.

The above procedure is illustrated by an analysis in which the bar is divided in 10 elements. An eigenvalue analysis slightly beyond peak strength with a Jacobi-subspace method<sup>87</sup> resulted in 10 negative eigenvalues, the eigenvector belonging to the lowest eigenvalue having been plotted in Figure 6.4. Adding this eigenvector to the fundamental mode did not result in localisation in one element, but in 5 elements, which is not surprising in view of Figure 6.4. Performing an eigenvalue analysis for this tangent stiffness, in which the moduli of 5 elements are softening and the moduli of the remaining 5 elements unload via the secant branch of equation (4.21), resulted in 5 negative eigenvalues, the eigenvector corresponding to the lowest eigenvalue being plotted in Figure 6.5. Adding this eigenvector to the current (small) displacement increment with localisation in 5 elements, resulted in a new equilibrium state with localisation in 3 elements. A new eigenvalue analysis yielded 3 negative eigenvalues and addition of the eigenvector (Figure 6.6) corresponding to the lowest eigenvector to the current displacement increment resulted in localisation in 2 elements. A final loop with the eigenvector of Figure 6.7 yielded localisation in 1 element (Figure 6.8).

The case when the bar is modelled with 10 elements represents a critical case when the load falls down on the localisation path without any additional displacement of the end of the bar (line A-D in Figure 4.20), i.e. the strain increment in the element in which the deformation has localised together with the strain increments of the unloading elements is exactly zero. For a smaller number of elements the additional displacement at the end of the bar is positive, but for a greater number of elements, the additional displacement is negative, so that the total displacement at the end of the bar becomes *smaller* after the peak has been passed (line A-F in Figure 4.20 which is for 20 elements). Obviously, such a 'snap-back' behaviour cannot be analysed under direct displacement control, but only with indirect displacement control. Yet, many analysts have ignored the possibility of this phenomenon in the past, and many analyses have been terminated at such a point because of divergence of the iterative procedure. A further parallel can be drawn with experiments

which can not be performed properly under displacement control, e.g. with shear or other brittle failures. The observed explosive failure is then simply the result of an attempt to traverse an equilibrium path under improper static loading conditions.

# 6.2. Shear-band formation in the biaxial test

The examples in the preceding section are interesting as a thorough insight can be gained in bifurcation and localisation phenomena in softening media and as they present a nice illustration that the procedure for continuation on alternative equilibrium branches is versatile also for non-linear analyses of materials with a non-unique stress-strain law. In the present section a more challenging problem will be analysed, namely plane strain compression of dry sand in a biaxial testing device. In recent years, this problem has received much attention from a theoretical side as well as from an experimental side<sup>2,113,114,118</sup>, but it seems that a proper numerical bifurcation analysis has not yet been published. Only some numerical approaches  $exist^{86,119}$  in which the bifurcation problem is transferred into a limit problem by introducing a small imperfection, either material or geometrical.

It was noted in Chapter 4 that non-associated flow laws may cause unstable material behaviour and non-unique solutions before peak strength has been reached. Considering a plane strain compression test on dry sand in a biaxial device, this implies that the sample can bifurcate before the limit friction angle  $\varphi$  has been attained, i.e. when the hardening modulus is still positive. Neglecting the possible impact of large deformation gradients, it was derived in Chapter 4 that for the friction-hardening Mohr-Coulomb elasto-plastic model, the critical hardening modulus  $h_s$  for which shear-band bifurcation is first possible, is given by the expression<sup>71,118</sup>

$$\frac{h_s}{\mu} = \frac{(\sin\varphi^* - \sin\psi^*)^2}{8(1-\nu)}$$
(6.1)

After the cricital hardening modulus has been reached or alternatively if the mobilised friction angle  $\varphi^{\bullet}$  has attained some critical value, all further deformation is localised in a thin layer, while the remainder of the sample experiences no additional straining. In the classical view, the inclination angle  $\vartheta$  of such a shear band with the axis of minor principal stress is given by  $\vartheta = 45^{\circ} + \frac{1}{2} \varphi^{\bullet}$ , but careful experiments by Vardoulakis et al.<sup>113</sup> and Arthur et al.<sup>2</sup> have revealed that the expression

$$\vartheta = 45^{\circ} + \frac{1}{4} \left( \varphi^* + \psi^* \right) \tag{6.2}$$

better matches experimental data. Indeed, for a material model which slightly differs from the model employed here and including large displacement gradients, Vardoulakis<sup>114</sup> showed that a proper bifurcation analysis approximately predicts such an inclination angle for a shear band. Later, Vermeer<sup>118</sup> showed that the inclination angle is relatively insensitive to the employed material model and also holds for the friction-hardening Mohr-Coulomb model. Moreover, Vermeer<sup>118</sup> showed that the impact of the large displacement gradients on the bifurcation load is not very significant and that non-normality is the governing factor which causes shear-band bifurcation in sand samples.



Figure 6.9. Eigen-displacement field at bifurcation.

In the numerical analyses, a cohesionless sand has been considered with  $E=100 \text{ N/mm}^2$ ,  $\nu=0.2$ ,  $\varphi=40^\circ$ ,  $\varepsilon_f=0.02$ ,  $\psi^*=0^\circ$  and the confining pressure has been taken equal to -0.4 N/mm<sup>2</sup>. The value for Poisson's ratio is perhaps somewhat high considering the fact that no yield cap has been used to close the Mohr-Coulomb surface in the hydrostatic direction. Further, the assumption of a non-dilatant material is not very realistic near peak strength, but has been adopted for sake of simplicity.

Load incrementation was started from a strain free initial stress state of  $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -0.4 \text{ N/mm}^2$ . The load was applied to the top of the sample using indirect displacement control of the top of the sample (see Chapter 5). Dependence relations have been employed to ensure that all nodal points at the top displaced the same amount. Perfect



Figure 6.10. Load-displacement curves for biaxial test on dry sand.



Figure 6.11. Velocity field at limit point of the localisation branch.

lubrication was assumed between the platens and the sample, so that the sample could deform homogeneously.

The sample was loaded until  $\sigma_{yy}$  =-1.802 N/mm<sup>2</sup>, which corresponds to a mobilised friction angle  $\varphi^{\bullet}$  =39.55°. Then, a negative eigenvalue was calculated after assembling the tangent stiffness matrix. Since load increments of 0.001 N/mm<sup>2</sup> have been used near bifurcation, the bifurcation load is actually between -1.801 N/mm<sup>2</sup> and -1.802 N/mm<sup>2</sup>. The



Figure 6.12. Elastic and plastic zones at limit point of localisation branch. The shaded area is still plastic.

corresponding eigenmode is plotted in Figure 6.9. For a proper interpretation of this figure, it is necessary to look at it from the bottom lefthand corner. Then, we observe a number of 'waves', quite similar to the wave pattern which we observe for buckled shells or plates. The 'wave' pattern of Figure 6.9 has an inclination angle which reasonably corresponds with formula (6.2), but the shear-band mode itself is not an eigenmode. This might be due to the spatial discretization so that we would obtain a shear-band mode as eigenmode for a finer element mesh.

It is well possible that more negative eigenvalues exist at the numerical bifurcation point, but because the power method has been employed, only the lowest eigenvalue could be extracted. It is noted that the power method generally converges to the absolutely smallest eigenvalue, but convergence to the lowest eigenvalue was obtained because a shift was applied to the stiffness and identity matrices.

After locating the bifurcation point, the analysis proceeds in the same manner as described in the preceding section but for the added complexity that we now have a non-symmetric tangent stiffness matrix. The resulting load-displacement diagram is given in Figure 6.10 in which we have also plotted the solution which has been obtained for continued homogeneous deformations, i.e. when the solution is obtained without perturbation. Figure 6.11 shows the velocity field after bifurcation. It is noted that the inclination angle of shear band approximately equals 53° which is in reasonable agreement with equation (6.2) which would predict  $\vartheta \approx 55^\circ$ .





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Let us consider the post-bifurcation behaviour of the localised solution in somewhat greater detail (Figure 6.13). We observe that we obtain a stable solution for the localisation path after bifurcation. Indeed, no negative eigenvalues were calculated for the tangent stiffness matrices in this regime. Subsequent states of stable equilibrium were computed until the load was incremented to  $\sigma_{yy}$ =-1.8128 N/mm<sup>2</sup>. After this point, a negative eigenvalue was extracted after assembling the tangent stiffness matrix. A converged solution could not be obtained by incrementing the axial load any further, and use of indirect displacement control resulted in a converged solution has a limit point for  $\sigma_{yy}$ =-1.8128 N/mm<sup>2</sup>. Continuation of the solution resulted in the descending branch of Figure 6.13 and ultimately resulted in a residual load of  $\sigma_{yy}$ =-1.7594 N/mm<sup>2</sup>.

At a first sight, the initial rise of the localisation path before descending seems somewhat peculiar. A possible explanation is that the shear band needs some time to develop. Especially at the ends of the shear band, stresses must be mobilised which can support the shearband mechanism. Indeed, immediately after bifurcation, the incremental displacement or velocity field is still very much alike the incremental displacement field of the homogeneous solution. It is between the bifurcation point and the limit point of the localisation path that the shear band gradually develops culminating in the velocity field at the limit point (Figure 6.11). A similar observation holds for the plastic regions. Here, unloading of plastic to elastic states progresses gradually between the bifurcation and the limit point, until only the band of intensively sheared elements is still plastic (Figure 6.12). It is noted that similar results, i.e. the bifurcation path showed an initial rise before descending, have been reported by Hutchinson<sup>60</sup> in connection with plastic buckling of Shanley type columns. It is furthermore noted that the observed initial post-bifurcation behaviour will probably significantly depend on the adopted mechanical model. Another material model or inclusion of large displacement effects may well change the precise outcome of the analysis.



Figure 6.14. Finite element mesh composed of 8-noded quadrilateral elements.

The above results in which we computed a bifurcation point for a mobilised friction angle  $\varphi^*=39.55^{\circ}$ , were obtained for the mesh of Figure 6.14, which is composed of 8-noded quadrilateral elements with 'full' 9-point Gaussian integration. This result is significantly higher than the critical friction angle  $\varphi_c = 39.23^{\circ}$  which can be computed on basis of equations (6.1) and (4.27). In fact, the too 'stiff' behaviour of the numerical analysis is not so surprising in the light of the observations of Nagtegaal et al.<sup>75</sup>, who demonstrated that the kinematical constraints which are

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Figure 6.15. Finite element meshes composed of 15-noded triangular elements. Left: fine mesh. Right: coarse mesh.

imposed by the incompressibility constraint for a fully plastic solution, a situation which occurs at complete collapse, may cause 'locking' of elements, thus postponing or even avoiding failure. Eight-noded elements with 9-point integration represent a critical case for planar deformations, in the sense that failure loads can be computed with such an assembly, but that the limit load is usually overestimated unless very fine element divisions are employed. Because of the overstiff behaviour of the displacement based version of the finite element method, the bifurcation load is also overestimated. To alleviate this problem, Zienkiewicz et al.<sup>127</sup> advocated the use of so-called 'reduced' 4-point integration which technique has been widely employed in soil mechanics<sup>17,52,106</sup>. Use of such an integration rule resulted in a critical friction angle of  $\varphi_c$ =39.41° ( $\sigma_{yy}$ =-1.791 N/mm<sup>2</sup>), which is appreciably lower than the critical friction angle which was obtained in the analysis with full integration.

Nevertheless, it has been shown that especially in non-linear analysis, the use of reduced integration may be dangerous, see for instance Chapter 3 for use in conjunction with cracking and de Borst<sup>16,21</sup> for plasticity calculations. For cracking analyses it was concluded in Chapter 3, that 'full' 9-point integration largely avoids difficulties, but as argued, such a rule results in a too stiff behaviour for plasticity calculations. At



Figure 6.16. Bifurcation points for different discretizations.

present, the most reliable technique seems to be the use of 15-noded displacement based triangular elements with a 12-point integration rule<sup>16,21,105</sup>. Repeating the analysis with such an assembly (Figure 6.15a which has approximately the same number of nodes as the mesh of Figure 6.14) resulted in a critical friction angle  $\varphi_c = 39.31^\circ$  ( $\sigma_{yy} = -1.783$ N/mm<sup>2</sup>), while an analysis with a coarser mesh (Figure 6.15b) resulted in  $\varphi_c = 39.48^\circ$  ( $\sigma_{yy} = -1.797$  N/mm<sup>2</sup>), which is still lower than the results with 9-point integration for the much finer mesh of Figure 6.14. The fact that for the triangles a lower bifurcation load was obtained for the finer mesh confirms the well-known fact that the numerical solution converges to the 'true' solution upon mesh refinement. The results for the bifurcation load for the different assemblies are summarised in Figure 6.16.

A final remark concerns the relatively small differences between the various friction angles, e.g. the difference between the friction angle at which bifurcation is theoretically possible and the limit friction angle amounts less than 2 percent. This is caused by the convex relationship (4.27) between  $\sin\varphi^*$  and the hardening parameter  $\kappa$ . Indeed, when the rate of hardening equals the critical hardening modulus  $h_s$ , the mobilised friction angle is less than 2 percent below the limit friction angle, but the hardening parameter  $\kappa$  is only about 75 percent of  $\varepsilon_f$ , i.e. the value which  $\kappa$  attains when  $\varphi^* = \varphi$ .

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# 7. CALCULATION OF LIMIT POINTS AND OF POST-PEAK BEHAVIOUR

In the preceding chapter, we demonstrated how solutions can be obtained in the post-bifurcation regime. Bifurcations however are rather rare in normal structures owing to imperfections, and even if a bifurcation point exists in a structure, numerical round-off errors and spatial discretization usually transfer the bifurcation point into a limit point unless we have a homogeneous stress field as was the case in the examples of the preceding chapter. This observation does not render the approach pursued in the preceding chapter worthless as it provides a thorough insight which is of importance for the associated limit problems, but it is obvious that numerical procedures must also be capable of locating limit points and tracing post-limit behaviour. In the present chapter, we will show that the models and techniques developed in the preceding allow for tracing limit and post-limit behaviour. Furthermore, we will show that some theses postulated in the preceding concerning the consequences of strain-softening and non-associated plasticity are not merely academic. but that they are encountered in realistic soil and concrete structures.

#### 7.1. Mixed-mode fracture in a notched specimen

The first example which we consider, is the notched unreinforced beam of Figure 7.1. The beam has been analysed using 8-noded plane stress elements and 6-noded triangles have been used in the transition region between the coarse part and the fine part of the mesh. Nine-point Gauss quadrature was applied for the quadrilateral elements. The concrete has been modelled as linearly elastic in compression with a Young's modulus  $E_c = 24800 \text{ N/mm}^2$  and a Poisson's ratio  $\nu = 0.18$ . This approach is justified in this case, because the compressive stresses remain low enough to avoid yielding in compression. In tension, the crack model as outlined in the preceding has been employed. The crack parameters have been taken as: tensile strength  $f_{ct} = 2.8 \text{ N/mm}^2$  and fracture energy  $G_f = 0.055 \text{ N/mm}$ . The width of the crack band was assumed to be h = 10.167 mm.

Figure 7.1 also gives the loading configuration which shows that the beam is loaded asymmetrically so that the crack propagating from the notch shows opening as well as sliding. In the experiment<sup>1</sup> the load was applied cyclically at point C of the steel beam AB and was controlled by a feed-back mechanism with the so-called Crack Mouth Sliding Displacement (CMSD) as control parameter. After peak, the envelope of the load cycles falls down sharply (see Figure 7.2), which is particularly challenging for a numerical simulation.





A number of researchers have endeavoured a numerical analysis of this beam<sup>1,14,37,48,80,95,124</sup>, either with the smeared crack approach or with the discrete crack approach. Invariably, however, they adopted displacement control to the point of load application (point C in Figure 7.1) and all the calculated load-CMSD curves showed far too much ductility in the post-peak regime. Moreover, the post-peak regime was repeatedly reported to be highly unstable and converged equilibrium states could not be obtained. Indeed, we will show that such attempts are deemed to fail.



Figure 7.2. Load versus CMSD computed with CMSD-control. The shaded area denotes the range of experimental results.





Figure 7.3. Eigen-displacement field at peak load.

In the present study, the loading process has been controlled indirectly by a displacement parameter and the results of Figure 7.2 show that the computational results for the load-CMSD curves nicely fall within the experimental bounds, which is in sharp contrast with previous results which showed too much ductility after peak load. Initially, a global norm of displacements was employed, but this analysis was not successful as this constraint equation failed near peak load, probably for reasons as stated in Section 5.2. Indeed, we observe a strong localisation if we plot the eigen displacement mode at peak load (see Figure 7.3). However, for the present problem, a displacement parameter which can be used to control the loading process is naturally available, namely the CMSD itself. In this way the numerical analysis entirely parallels the experiment. With CMSD-control, the load-increment is in the linearised version determined from the condition

$$\Delta a_j^T \Delta a_{j-1} = \Delta l^2 \tag{7.1}$$

with  $\Delta a_j$  the increment in CMSD at step j. Using the constraint equation (7.1), the limit point could be overcome without problem with the same step size where the constraint formulae (5.89) and (5.90) failed.

Near peak load, an attempt to increment the load without CMSD control resulted in a divergence of the iterative procedure, but with CMSDcontrol and a full Newton-Raphson procedure a converged solution was obtained. Moreover, a negative pivot was encountered upon factorising the tangent stiffness matrix after some iterations. An eigenvalue analysis was performed subsequently and this resulted in one negative eigenvalue with the eigenmode of Figure 7.3 which is identical with the incremental displacement field at peak load. Hence, the peak load is a limit point and not a bifurcation point. After peak, the load was decremented and a genuine equilibrium path could be obtained.

If we assume that the beam of the test rig is infinitely stiff, we can calculate the vertical displacement of the point of load application C from



the calculated displacements of points A and B (Figures 7.4 and 7.5). For

the calculated displacements of points A and B (right of the second of 0.05, the analysis with non-linear softening and a shear retention factor of 0.05, this results in the load-deflection curve of Figure 7.6, which shows a violent snap-back behaviour.

The snap-back after peak load entirely explains why previous solutions which adopted displacement control with respect to point C did not result in a stable post-peak response, as such a solution simply does not exist, at least not in the vicinity of the limit point. In fact, the situation is

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Figure 7.6. Load-deflection curve of point C.

0.13 F	NI	7.F		
	-KH	<b>MAK</b>		
	- AT	NHK		
	-	<b>VIIK</b>	1	
		1 / HIK	1	

Figure 7.7. Incremental displacements at ultimate load.

even worse when the analysis is performed under displacement control of point C. This is because the beam of the test rig cannot be modelled as infinitely stiff, so that after peak the elastically stored energy of this beam is partly released, thus making the snap-behaviour even more violent.

The load at point B versus the computed CMSD is plotted in Figure 7.2. The most ductile curve corresponds to  $\beta^*=0.1111$  (so that we have for the shear retention factor  $\beta$ =0.1) and a linear softening curve. We observe that the computed ultimate load overestimates the experimental values. Figure 7.7, which gives the incremental displacements and Figures 7.8 which show the crack pattern at ultimate load, reveal that the crack arising from the notch has developed fully. Consequently, all stresses which are transferred in this crack, are shear stresses, and these stresses cannot decrease because of the constant shear retention factor and because of the relatively high threshold angle for the formation of secondary cracks ( $\alpha$ =60°). Indeed, beyond this point, the load

Figure 7.8. Crack pattern at ultimate load level. (a) All cracks. (b) Cracks which transfer no normal stress.

again falls down sharply, which appeared to be due to formation of cracks elsewhere in the beam. Because the solution then becomes physically meaningless, this part of the curve has not been plotted.

At ultimate load, the beam is fully cracked and only shear stresses can be transferred across the cracks. As the magnitude of the shear stresses is determined by the shear reduction factor  $\beta^*$  (or alternatively the shear retention factor  $\beta$ ), the ultimate load is primarily a function of  $\beta$ . The correctness of this hypothesis was confirmed in subsequent calculations with  $\beta$ =0.05. These calculations were performed with a linear as well as with a non-linear softening curve and yielded a significantly lower ultimate load. It is interesting to note that use of a non-linear softening curve<sup>31</sup> instead of a linear softening curve yielded almost the same ultimate load (which strengthened the above hypothesis), but resulted in a significantly lower peak load. Hence, the shape of the softening curve also strongly influences the computational results<sup>96</sup>.

We finally note that the excellent match of the computational results with an experiment in which we have crack sliding as well as crack opening (mixed-mode fracture) casts some doubts on the necessity of refining the stress-strain relation in the crack very much (see Section 4.1). Replacement of the constant shear retention factor by an expression which makes it a function of the crack strain seems essential, but it is doubted whether the off-diagonal terms in the crack stress-strain matrix have to be made non-zero unless relatively large crack strains are considered.



# 7.2. Tension-pull specimen

It is a widespread belief that reinforcement stabilises the numerical process. However, this is not generally true, as addition of reinforcement not only gives rise to stiffness differences in the structure, thus leading a deterioration of the condition of the stiffness matrix, but it also adds to the possibility of the occurrence of spurious alternative equilibrium states and of snap-back behaviour<sup>35,36</sup>. We will demonstrate this by means of perhaps the most simple reinforced structure, namely an axisymmetric specimen with an axial reinforcing bar.

Specifically, we will consider the tension-pull specimen<sup>40</sup> which is shown in Figure 7.9. The reinforcing bar is given by the line AB and a linear bond-slip law is assumed between the concrete and the reinforcement, i.e. the relation between the slip and the shear stress between concrete and steel has been assumed to be linear. In fact, the element which is employed for the reinforcement is a combined steel-bond slip element<sup>97</sup>. The concrete has been modelled as linearly elastic in compression just as in the preceding example with a Young's modulus  $E_c = 25000$ N/mm<sup>2</sup> and a Poisson's ratio  $\nu=0.2$ . Also in this case the approach is justified because of the relatively low compressive stresses. The tensile strength has been assumed as  $f_{ct}=2.1$  N/mm<sup>2</sup> and the non-linear softening curve has been employed after crack formation with a fracture energy  $G_f = 0.06$  N/mm. The shear reduction factor  $\beta^*$  was taken equal to 0.1111. The reinforcing bar was assigned a Young's modulus  $E_s = 177000$ N/mm<sup>2</sup> and a yield strength  $\sigma_y = 210$  N/mm<sup>2</sup>.



Figure 7.9. Tension-pull specimen of Dörr<sup>40</sup>.



Figure 7.10. Load versus displacement of the end of the bar.

The loading is applied to point A of Figure 7.9 in the form of a concentrated load and the ensuing load-displacement diagram is given in Figure 7.10. The present problem has much in common with the preceding example as also in this case straightforward application of a norm of incremental displacement to control the solution process did not work effectively. This can again be understood if we consider the incremental displacement fields just prior to and just beyond the limit point (Figure 7.11 and 7.12). Prior to the limit point, the elastic deformations of the bar are relatively so great, that they dominate the norm of incremental displacements. Just beyond the peak, when the crack near the centreline has localised, the incremental deformations of the reinforcing bar nearly vanish (they even change sign, so that we again have a 'snap-back') and the concrete is the prime contributor to the total norm of incremental displacements. However, because of the relatively great values of the steel deformations just prior to the limit point, the arc-length in the displacement space is not influenced significantly. In this case, the degrees of freedom belonging to the steel have therefore been omitted from the norm of incremental displacements for overcoming the limit point. For traversing the valley in the load-displacement curve of Figure 7.10 on the other hand, the solution process has been controlled by the displacements of the steel, as then these displacements increase monotonically.

In the preceding, the question was raised whether an equilibrium state could be reached via a non-equilibrium path. The present example





Figure 7.11. Incremental displacement field just prior to the limit point.





is well suited for demonstrating that this may be possible for analyses under displacement control. Indeed, when we attempted to analyse the structure by prescribing the displacement of point A, we obtained a number of non-converged states just after the limit point. This nonequilibrium path is indicated by the dotted line in Figure 7.10. However, after the crack had localised, we again obtained converged equilibrium states (dashed line in Figure 7.10). This illustrates that reaching another part of the equilibrium path is sometimes possible, provided that there exists a new equilibrium state which is "sufficiently close" to the previous equilibrium state.

It is interesting to note that during the drop of the load no new cracks arise, so that the crack pattern of Figure 7.13 remains unchanged. When the load is increased again, new 'cone-shaped' cracks<sup>50</sup> arise which are a consequence of the reversal of the direction of the shear stresses along the bar near the centre of the specimen (see Figure 7.14).







Figure 7.14. Crack pattern at ultimate failure.

## 7.3. A trap-door problem

The two preceding examples nicely illustrated the possible consequences of strain-softening on the structural response. As crack formation was the principal cause of non-linearity in these examples, the compressive stresses remained relatively low, and the examples could not be employed to demonstrate possible consequences of non-associated plasticity.

It was argued in the preceding that non-associated flow rules may cause non-uniqueness of the limit load and post-peak softening even if no softening is assumed directly in the relation between the normal stress and the normal strain, or alternatively, between the shear stress and the shear strain. Investigating this for strip as well as for circular footing problems, the Author found that the limit load was practically insensitive with regard to the adopted flow rule<sup>16,21</sup>, the elasticity parameters or the initial stress conditions. Examples thereof are shown in Figure 7.15 which gives results for a strip footing problem and Figure 7.16 which gives results for a circular footing problem.

A more interesting problem from this point of view is a trap-door problem for a cohesionless, ponderable soil (see Figure 7.17). For this problem which is more confined than the footing problems, different limit



Figure 7.15. Results for strip footing problem on cohesive-frictional soil. The friction angle  $\varphi$  has been taken equal to 40°, while two different values for the dilatancy angle have been adopted, viz.  $\psi = 20^{\circ}$  and  $\psi = 40^{\circ}$  (associated). Analytical failure load after Prandtl<sup>85</sup>.



Figure 7.16. Results for circular footing problem on cohesive-frictional soil. The friction angle  $\varphi$  has been taken equal to 20°, while two different values for the dilatancy angle have been adopted, viz.  $\psi = 0^{\circ}$  and  $\psi = 20^{\circ}$  (associated). An additional calculation for an initial stress field with  $\sigma_{xx} = \sigma_{yy} = -2 \text{ N/mm}^2$  also yielded the semi-analytical slip-line solution<sup>31</sup>.



Figure 7.17. Finite element discretization for trap-door problem.



Figure 7.18. Load-displacement curves for trap-door problem.

loads are calculated when varying the dilatancy angle or the initial stress field as previously shown by de Borst & Vermeer<sup>21</sup> and Vermeer & Sutjiadi<sup>120</sup> for a perfectly plastic material.

The load has been applied to the mid-node of the trapdoor (passive mode) and the other nodes of the trapdoor have been forced to displace as much using dependence relations. For the nodes at the bottom of the

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element next to the trapdoor, dependence relations have also been employed in the sense that a linear displacement distribution has been enforced with the leftmost node at the bottom remaining fixed and the rightmost node attached to the trapdoor.

Figure 7.18 shows results of various calculations for a trap-door problem with an embedment ratio of h/D=4. It is instructive to compare the results of the simplified ideally-plastic model for an associated flow rule and a non-associated, plastically volume-preserving flow rule ‡. We obtain different values for the limit load, the difference even being more pronounced for the residual load. This is because the computation with the non-associated flow rule shows post-peak softening whereas the results for the associated flow rule do not give softening after the limit load has been reached. The fact that a calculation with a non-associated flow rule may involve softening has been explained in Chapter 4, where it has also been argued that associated flow rules cannot give softening. Nevertheless, turning our attention again to the results of Figure 7.15 and 7.16 for the footing problems, we observe some post-peak softening also for the computations with the associated flow rule. Apparently, this effect is attributable to the numerical procedure and a small overshoot of the failure load before levelling out towards a final value has also been observed in other problems<sup>16,17,21,53</sup>. It has been argued that this effects is due to the convergence tolerance, i.e. the overshoot gradually disappears when we tighten the convergence tolerance. Nevertheless, the fact that the associated flow rule may involve numerical softening when frictional materials are considered, casts some doubt whether the calculated post-peak softening for the non-associated flow rule is indeed physical. Perhaps it is more accurate to question which part of the softening for the calculation with the non-associated flow rule is caused by the numerical approximation and which part is physical, i.e. caused by the mathematical model. The fact that with the same convergence criterion, the associated flow rule did not produce softening, supports the assertion that the amount of softening of Figure 7.18 is largely physical. Yet, it should be mentioned that comparison with previous calculations<sup>53</sup> showed that the computed softening of those calculations was partly numerical, because they yielded a more pronounced and higher limit load. Nevertheless, the same residual load was obtained.

More evidence for the non-uniqueness of the limit load is given by another calculation in which the friction hardening model of Section 4.2 with Rowe's stress-dilatancy theory (with  $\psi=5^{\circ}$  so that for  $\varphi=30^{\circ}$  we obtain  $\varphi_{cv}=25.57^{\circ}$ ) was utilised. The ultimate load nicely lies within the bounds of the calculations with the non-dilatant and the associated flow rule. Moreover, the post-peak softening is still present, but less marked than for the calculation with the non-dilatant soil. Unfortunately, this softening could not be plotted in Figure 7.18, because the peak load and the post-peak softening occurred at very large displacements which exceeded the scale of Figure 7.18.

Summarising, we can state that for confined configurations, a nonassociated flow rule will generally result in a non-unique limit load and a non-unique residual load. Moreover, non-associated flow rules may cause post-peak softening behaviour although this is less pronounced and less important than the non-uniqueness of the limit load. The observed postpeak softening and the non-uniqueness gradually disappear when the degree of non-normality vanishes.

The calculations of the trap-door problems show a non-uniqueness of the limit load which has not been found for less confined configurations (Figure 7.15 and 7.16). Similar to the footing problems however, we observe that the failure mechanism heavily depends on the adopted flow rule. This is shown in Figure 7.19 in which the velocity fields at failure have been plotted for all three calculations. We observe a clear localised failure mechanism for all cases which confirms the capability of numerical methods to simulate highly localised failure modes.

A question to which the present calculations do not present a conclusive answer is whether the calculations with the non-associated flow rules which are reported in this section are really limit problems or whether they are bifurcation problems. When solving this problem with a tangent stiffness method, convergence was obtained until a load factor of approximately 2.65, depending on the flow parameters. This is significantly below the limit load of 2.92 which was calculated when the analysis was continued with the elastic stiffness method including the modification of equation (5.126). Indeed, after the load level of 2.65 negative eigenvalues were calculated for the tangent stiffness matrix, indicating the presence of alternative equilibrium states. Nevertheless, it can not be concluded that such states indeed exist because the negative eigenvalues were calculated for a tangent stiffness matrix which was based on a non-equilibrium state.

The convergence difficulties for a tangent stiffness method when the load exceeds a value of 2.65 are probably caused by the fact that no

<sup>&</sup>lt;sup>‡</sup> The calculation for the associated flow rule is strictly speaking somewhat suspect for a cohesionless material as explained by Vermeer & de Borst [119]. Inclusion of some cohesion leads to a theoretically consistent model, but does not basically affect the observations.

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Figure 7.19. Velocity field at failure for an associated flow rule (upper,  $\varphi = \psi = 30^{\circ}$ ), Rowe's stress-dilatancy theory (left,  $\psi = 5^{\circ}$ ) and for a plastically volume-preserving flow rule (right,  $\psi = 0^{\circ}$ ).

pivoting procedure has been used when factorising the tangent stiffness matrix. It is known from numerical analysis that such a pivoting procedure is necessary for non-symmetric matrices especially when the degree of non-symmetry is strong. However, very few calculations have been reported for non-symmetric systems in structural mechanics and the computations which have been reported mostly employ a symmetrised stiffness matrix. Moreover, whether or not omission of a pivoting procedure for a non-symmetric matrix leads to non-convergence seems to be problem-dependent, since no difficulties were experienced in the calculations for the biaxial test of the preceding chapter.

7.4. Pull-out test

# A problem which bears some similarity to the trap-door problem is a pull-out test of a steel disc out of a mass of concrete. Such a test has been proposed to test the strength of concrete, but there is much debate<sup>82,107</sup> whether such a test measures the tensile strength or the compressive strength. Here, we will not pursue this issue and we will consider the problem primarily because, as the argument already reveals, tensile as well as compressive stresses contribute to the non-linear response, so that fracture as well as plasticity are of importance. Further, we will again demonstrate that a local softening law may induce 'snap-back' behaviour on global level depending upon the stiffness of the steel disc.



Figure 7.20. Finite element discretization for pull-out test.

The dimensions and the discretization of the particular problem which we consider are shown in Figure 7.21. The dimensions are similar to those considered by Ottosen<sup>82</sup> and correspond to the so-called Lok-test. Eight-noded quadrilateral axisymmetric elements have been used with a 9-point Gauss quadrature rule. The elasticity parameters of the concrete have been taken as  $E_c = 32400 \text{ N/mm}^2$  and  $\nu = 0.2$  in accordance with Ottosen's data<sup>62</sup>. A value of  $f_{ct} = 3.18 \text{ N/mm}^2$ has been assumed for the tensile strength, while after crack formation a shear reduction factor  $\beta^* = 0.1111$  and a value  $G_f = 0.055 \text{ N/mm}$  have been used, while the width of the crack band has been assumed to be h = 3.0 mm. The steel of the disc has been assigned a Young's modulus  $E_s = 205000 \text{ N/mm}^2$  and a



Figure 7.21. Load-deflection curve for Lok-test with thick steel disc. The limit load for the thin steel disc (thickness 1/5 of thick disc) is indicated by the dashed line.



Figure 7.22. Experimental and numerical results for Lok-test on a scale  $12:1^{107}$ .

Poisson's ratio  $\nu$ =0.3.

The resulting load-deflection curve of this calculation is given in Figure 7.21, which shows a clear post-peak softening behaviour. However, there is a significant discrepancy between the computed limit load and





Figure 7.23. Incremental displacements before failure for the thin steel disc.



Figure 7.24. Incremental displacements at failure for the thin steel disc.



Figure 7.25. Incremental displacements beyond failure for the thin steel disc.

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numerical results reported elsewhere<sup>82</sup>. Also, the present results are lower than semi-analytical formulae which predict a limit load which is at least twice as high. The analysis was therefore repeated on a scale 12:1, which corresponds to recent experimental work<sup>107</sup>. It appeared that especially the value for the fracture energy, and consequently also the estimate for the crackband width h, have a significant influence upon the computed limit load. Indeed, for  $G_f = 0.165$  N/mm and h=4 mm., the curve of Figure 7.22 was obtained, which shows a reasonable agreement between experiment and analysis. A calculation with such fracture parameters however, results in a very ductile post-peak response with hardly any softening. As such a behaviour is less interesting from the point of view of this study, analyses with similar fracture parameters have not been undertaken.

The present example also serves the purpose of clearifying the term "unstable softening". It is known that for less stiff testing devices. difficulties arise in keeping the loading process stable even under displacement control<sup>5,108</sup>, a phenomenon which is usually referred to as "unstable softening". Such a terminology is rather misleading and it is really meant that such loading paths cannot be traced quasi-statically under displacement control as an equilibrium path does not exist under quasi-static loading conditions. Hence, the process becomes dynamic when it attempted to traverse such a path under displacement control. In fact, such a situation arises for the present problem when the steel disc is made less thick (ratio 1:5). We then get the incremental deformation patterns of Figures 7.23 to 7.25, which give the situation before failure, at failure and slightly beyond failure. We observe that the energy which is stored in the steel disc causes explosive crack propagation once the crack has reached the surface. The observed explosive crack propagation also explains the fact that a limit load has been obtained for the thin steel disc which is significantly lower than the limit load for the thick steel disc (see Figure 7.21).

It is finally instructive to compare the crack patterns at the limit load for the thick and the thin steel disc (Figures 7.26 and 7.27). It appears that the more flexible disc causes a quite different crack pattern.



Figure 7.26. Crack pattern at limit load for thick steel disc.



Figure 7.27. Crack patterns at limit load for thin steel disc.

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#### SAMENVATTING

#### NIET-LINEAIRE BEREKENINGEN AAN WRIJVINGSMATERIALEN

Dit proefschrift behandelt numerieke technieken om tot een geconvergeerde oplossing te komen voor niet-lineaire mechanicamodellen die ontstaan uit een ruimtelijke discretisatie van continua die opgebouwd zijn uit materialen waarvan de sterkte significant afhangt van het spanningsniveau. Met name worden methodes behandeld om voor dergelijke modellen bifurcatie en bezwijkpunten in de last-verplaatsingscurve correct te voorspellen en het gedrag na bezwijken of na bifurcatie te berekenen.

In Hoofdstuk 2 wordt eerst een algemene inleiding gegeven over de spannings- en deformatietensoren die we zullen gebruiken. Na een korte excursie in het gebied van grote deformaties, wordt de behandeling verder beperkt tot kleine verplaatsingsgradiënten. Evenwicht wordt geformuleerd met behulp van de door Piola voorgestelde variant van het principe van virtuele arbeid, en de klasse van constitutieve modellen tot welke de behandeling beperkt zal blijven, wordt besproken. Tot slot van dit hoofdstuk wordt een behandeling gegeven van criteria voor stabiliteit en eenduidigheid van de oplossing van een randvoorwaardeprobleem.

Waar in Hoofdstuk 2 de behandeling is gegeven voor continue mechanische systemen, worden in Hoofdstuk 3 discrete mechanicamodellen besproken. De in het voorafgaande hoofdstuk gegeven definities van stabiliteit en eenduidigheid worden nu uitgediept voor discrete stelsels. Dit leidt tot een noodzakelijke voorwaarde voor stabiliteit. Dit stabiliteitscriterium is slechts ook een voldoende voorwaarde voor stabiliteit als de spannings-rek relatie symmetrisch is. Voor materialen waarvan de sterkte van het spanningsniveau afhangt zoals beton en grond, is dit echter niet het geval. Tot slot van Hoofdstuk 3 worden de consequenties van de ruimtelijke discretisatie van het onderliggende continuum onderzocht.

Hoofdstuk 4 behandelt de constitutieve modellen die in dit proefschrift gebruikt zijn. Een fundamentele aanname waarbinnen alle gebruikte modellen ingebed zijn, is dat de total reksnelheid gesplitst kan worden in een aantal bijdragen, één voor elke (uitgesmeerde) scheur, één voor het elastische aandeel en één voor het plastische aandeel van het materiaal tussen de scheuren. Op deze manier wordt het mogelijk om binnen het raamwerk van het uitgesmeerde scheurconcept, nietorthogonale scheuren en de combinatie van scheurvorming en plasticiteit op een correcte manier te beschrijven. Het hoofdstuk besluit met een onderzoek naar de consequenties van het gebruik van 'strain-softening' en niet-geassocieerde plasticiteit, welke beide gebruikt worden in de constitutieve relaties.

Het eerste deel van Hoofdstuk 5 is gewijd aan de numerieke integratie van de differentiaalvergelijking voor de spannings-rek relatie over een eindige belastingsstap, waarbij bijzondere aandacht wordt geschonken aan singulariteiten die in breuk- en vloeifuncties kunnen optreden. In het tweede deel van het hoofdstuk wordt ingegaan op methodes om het na discretisatie resulterende stelsel niet-lineaire vergelijkingen op een efficiënte en nauwkeurige wijze op te lossen. Hiertoe worden Quasi-Newton methodes gecombineerd met indirecte verplaatsingscontrole. Speciale aandacht wordt besteed aan het continueren van de oplossing na bifurcatie- en bezwijkpunten.

Hoofdstuk 6 geeft twee typische voorbeelden van de numerieke behandeling van bifurcatieproblemen uit de grond- en betonmechanica. Het eerste voorbeeld betreft een ongewapend betonnen staaf die op trek wordt belast. De numerieke breedte van de scheurvoortplantingszone geeft dan aanleiding tot verschillend post-bifurcatiegedrag. Het tweede voorbeeld is een simulatie van schuifvlakvorming in een biaxiaaltest op droog zand. Deze voorbeelden onderscheiden zich met name daarin, dat de niet-eenduidigheid en instabiliteit veroorzaakt wordt door de constitutieve relatie en niet door geometrische effecten.

In het slothoofdstuk wordt een viertal voorbeelden gegeven van berekeningen van bezwijkgedrag en het gedrag na bezwijken van gronden betonconstructies. Het blijkt dat de toegepaste materiaalwetten, aanleiding kunnen geven tot spectaculaire en onverwachte lastverplaatsingsdiagrammen. Het niet onderkennen van de mogelijkheid van dergelijk constructiegedrag leidt meestal tot divergentie van het iteratieproces, doch tenminste tot een niet-geconvergeerde oplossing.

# STELLINGEN

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# CURRICULUM VITAE

22 februari 1958	Geboren te 's-Gravenhage.
augustus 1970 - juni 1976	Scholengemeenschap ''Simon Stevin'' te 's-Gravenhage afgesloten met gymnasium- $meta$ diploma.
september 1976	Aanvang studie aan de Afdeling der Civiele Techniek, Technische Hogeschool Delft.
juni 1979 - augustus 1979	Stage aan de University of Minnesota, Depart- ment of Civil and Mineral Engineering waar onderzoek verricht werd op het gebied van de grondwatermechanica.
juni 1982	Behalen van het diploma van civiel ingenieur (met lof) aan de Technische Hogeschool Delft. Het afstudeerwerk lag op het gebied van de theoretische en numerieke grondmechanica.
augustus 1982 - nu	Werkzaam bij het Instituut TNO voor Bouw- materialen en Bouwconstructies als weten- schappelijk medewerker. Hier wordt onder- zoek verricht op het gebied van de numerieke mechanica en aan constitutieve modellen voor grond, beton, staal, rubbers en kunststoffen. De auteur is tevens opgetreden als docent van post-academiale cursussen en als begeleider van een aantal afstudeerontwerpen aan de Afdeling der Civiele Techniek, TH Delft.

# BEHORENDE BIJ HET PROEFSCHRIFT VAN R. DE BORST

- 1. Het is onjuist om divergentie van het iteratieproces in een nietlineaire numerieke berekening te identificeren met bezwijken van de constructie.
- 2. De bij het representeren van triaxiaalproeven op beton gebruikte methode waarbij één hoofdspanning ( $\sigma_3$ ) wordt gekoppeld aan een andere hoofdspanning, leidt tot asymmetrische bezwijkomhullenden in het  $\sigma_1, \sigma_2$  -vlak. Desalniettemin vertoont het plastic-fracturing model van Bažant en Kim symmetrie in dergelijke vlakken. Dit impliceert dat dit model het gedrag van beton onder meerassige drukspanningen niet goed kan beschrijven.

3. Het gebruik van 'reduced integration' dient in fysisch niet-lineaire berekeningen vermeden te worden.

R. de Borst and P. Nauta, Non-orthogonal cracks in a smeared fintie element model. Engineering Computations, Vol. 2, 35-46 (1985).

- 4. Het aanbrengen van een zogenaamde 'slurry-wall' achter een drainagebuis in een watervoerende laag verhoogt de effectiviteit van de drainage slechts marginaal.
- 5. Voor normaal-geconsolideerde klei wordt de conusweerstand geheel bepaald door de ongedraineerde schuifsterkte.

R. de Borst and P.A. Vermeer, Possibilities and limitations of finite element for limit analysis. Géotechnique, Vol. 34, 199-210 (1984).

6. De aanhechting tussen staal en rubber in rubber-staal oplegpakketten hangt in hoge mate af van de detaillering nabij de uiteindes.

Z.P. Bažant and S.-S. Kim, Plastic-fracturing theory for concrete. ASCE J. Eng. Mech., Vol. 105, 407-428 (1979).
- 7. De termen 'stable softening' en 'unstable softening' zijn onzinnig daar softening altijd aanleiding geeft tot instabiel constructiegedrag.
- 8. Het in dit proefschrift voorgestelde scheurvormingsmodel kan op eenvoudige wijze uitgebreid worden voor tijdsafhankelijk materiaalgedrag. Het zo verkregen model draagt het recentelijk door Bažant en Chern voorgestelde model voor combinatie van kruip en strainsoftening als een bijzonder geval in zich.

Z.P. Bažant and J.C. Chern, Strain softening with creep and exponential algorithm, ASCE J. Eng. Mech., Vol. 111, 391-415 (1985).

- 9. De wisselwerking tussen numeriek en experimenteel onderzoek op het gebied van de mechanica dient versterkt te worden.
- 10. In de opleiding tot civiel-ingenieur dient meer aandacht besteed te worden aan onderwijs in numerieke niet-lineaire mechanica en in informatica.