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Kalman Filtering for Simplicial Processes

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***Abstract*—In this paper, we propose a topology-aware Kalman filter for hidden dynamics over simplicial complex. Specifically, we consider that the hidden dynamics of a system can be expressed as a simplicial process that respects the structure of the underlying network. And these dynamics are observed through an observation matrix, which can be represented using simplicial convolution filters. This combination allows us to model effectively a broader spectrum of network dynamics than graph-based alternatives, such as edge flow evolution. Additionally, we propose a parametric, structure-aware noise covariance model for the system dynamics. We alternate between estimating the process state using the Kalman filter and updating the parameters through maximum likelihood estimation. The efficacy of the proposed approach is demonstrated through experiments on both real-world and synthetic datasets.**

***Index Terms*—Simplicial Complex, Dynamic system, SPDE**

I. Introduction

Time-series measured over a network often reflect observations of an underlying hidden process. Effective modeling of this hidden process is crucial for various inference and decision making tasks [1]. For instance, in communication networks, interference caused by other devices represents the hidden process, while measurable observations at network nodes, such as packet loss at routers, capture its effects. By effectively modeling interference dynamics, one can detect abrupt shifts in network conditions, predict future packet loss trends, and optimize resource allocation strategies. Such real-world systems often involve a large number of variables, and as dimensionality increases, the data complexity grows exponentially. This high dimensionality can cause poor generalization, making it challenging to develop models that accurately capture and generalize underlying dynamics. Incorporating structural information from the network as an inductive bias is a powerful approach to improve generalization of the model. We seek to utilize the topological structure of the network to model the hidden dynamics of the process within a stochastic partial differential equation (SPDE) framework, utilizing Kalman filter-based state estimation.

SPDE is a well-established mathematical framework used to model real-world spatio-temporal processes and have recently been explored for graph processes [2], [3]. In [2] and [3], the solution of SPDE is a Gaussian process, with the mean and covariance function depends on the graph Laplacian. But estimating the covariance function is a daunting task in [2], especially in online setting, which limits its applications to small networks. To address this issue, [3] proposes a graph-based dispersion, which intern leads to a covariance matrix respecting the graph structure and with a significantly smaller number of parameters to estimate.

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While these models reflect the underlying process, graph representations often lack the complexity needed to accurately capture the data structure, primarily due to their inability to represent higher-order relationships [4]. The higher-order structures are prevalent in many real-world networks. For instance, in brain functional analysis, while we observe EEG measurements, the actual underlying process involves higher-order interactions between brain regions [5]. Similarly, such higher-order relationships can be found in a wide range of networks, including social networks, communication networks, and power systems [4]. In this regard, simplicial complexes (SC) are becoming increasingly popular given their ability to model higher-order structures in a mathematically tractable way [4], [6]. We model the process as an SPDE, where the diffusion respects the simplicial structure of the higher-order network via Laplacian and uncertainty is entered in the system via boundary maps. First, we convert the proposed SPDE-based model to a discrete state-space model, which consists of a noisy model and a noisy observation. We use simplcial convolution filters to approximate the observation matrix [7], [8]. Then the proposed noisy model and noisy observation are combined using a Kalman filter to get the best linear unbiased estimate of the process. Unlike previous works that use Kalman filter for state estimation [3], [9], [10], the noise covariance matrix of the proposed model is simplicial-aware. The noise covariance matrix and the filter parameters that characterize the simplicial convolution filter are then estimated using maximum likelihood estimation. We alternate between updating the states with the Kalman filter and estimating the parameters using maximum likelihood. Our main contribution is three-fold:

1) We propose a model that incorporates the higher-order structures present in the network to effectively represent topology-based dynamical processes. The dynamics of the states are modeled using a Hodge Laplacian-based SPDE, and, the observation matrix is constructed using simplicial convolution filter. To the best of our knowledge, this is the first work to model hidden higher-order dynamics as a simplicial process based on observed data.
2) We derive an expression for the noise covariance matrix in the model. By incorporating the structural information as an inductive bias, the number of parameters required

to represent the noise covariance matrix reduces to linear in the number of states.

3) The Kalman filter formulation relies on the noise covariance of the model, which is often challenging to estimate. We mitigate this challenge by leveraging a parametric, structure-aware covariance matrix. The structural information incorporated into the model facilitates the efficient estimation of parameters characterizing the covariance matrix using the maximum likelihood estimation.

II. Preliminaries

A. Simplicial Complex

Consider a set of vertices $\mathcal{V} = 1, \dots, N_0$. A k-simplex, denoted as $\mathcal{S}^k$, is a subset of $\mathcal{V}$ containing $k+1$ unique elements. A simplicial complex (SC) $\mathcal{X}^K$ of order K is a collection of simplices, where at least one K-simplex exists. Moreover, a simplex $\mathcal{S}^k$ is included in $\mathcal{X}^K$ if and only if all its subsets are also contained within. To ease exposition, we will focus on low-order simplices in a SC; which can be associated to geometric shapes, as nodes (0-simplices), edges formed by two nodes (1-simplices), and triangles formed by three nodes (2-simplices). Let N_k be the number of k-simplices in $\mathcal{X}^K$. The structure of a SC is described by the Hodge Laplacians, which are high-order combinatorial Laplacian matrices crafted from the incidence matrices. Laplcians representing the structure of $\mathcal{X}^2$ can be constructed using the node to edge incidence matrix $\mathbf{B}_1 \in \mathbb{R}^{N_0 \times N_1}$ and the edge to triangle incidence matrix $\mathbf{B}_2 \in \mathbb{R}^{N_1 \times N_2}$ as,

$$\begin{aligned} &\mathbf{L}_0 = \mathbf{B}_1\mathbf{B}_1^\top \in \mathbb{R}^{N_0 \times N_0}, \\ &\mathbf{L}_1 = \mathbf{L}_{1,d} + \mathbf{L}_{1,u} := \mathbf{B}_1^\top\mathbf{B}_1 + \mathbf{B}_2\mathbf{B}_2^\top \in \mathbb{R}^{N_1 \times N_1}, \\ &\mathbf{L}_2 = \mathbf{B}_2^\top\mathbf{B}_2 \in \mathbb{R}^{N_2 \times N_2}. \end{aligned} \tag{1}$$

The zeroth Hodge Laplacian, $\mathbf{L}_0$, corresponds to the well-known graph Laplacian [11]; which captures adjacencies between vertices through shared edges. The first Hodge Laplacian, $\mathbf{L}_1$, represents adjacencies between edges, defined by two vertical relations: (i) via shared vertices, captured by the lower Laplacian $\mathbf{L}_{1,d} = \mathbf{B}_1^\top\mathbf{B}_1$, and (ii) via shared triangles, expressed through the upper Laplacian $\mathbf{L}_{1,u} = \mathbf{B}_2\mathbf{B}_2^\top$. Similarly, the second Hodge Laplacian, $\mathbf{L}_2$, captures relationship between triangles through shared edges [4], [6]. To ease notation, we represent the structure of simplical complex through a combined Laplacian $\mathbf{L} = \text{blkdiag}(\mathbf{L}_0, \mathbf{L}_1, \mathbf{L}_2) \in \mathbb{R}^{N \times N}$, where $N = N_0 + N_1 + N_2$.

B. Simplicial signals and transformations via boundary maps

We can define simplicial signals as functions mapping k-simplices to real numbers. Take the case of $\mathcal{S}^1$, the signals can be defined over edges. For example in an electrical grid, edges represent power lines, and the signal associated can be current flow. We denote time series value of edge signlas at time t as $\mathbf{x}_t^1 \in \mathbb{R}^{N_1}$. Similarly, node signals over $\mathcal{S}^0$ are denoted as $\mathbf{x}_t^0 \in \mathbb{R}^{N_0}$, and triangle signals over $\mathcal{S}^2$ as $\mathbf{x}_t^2 \in \mathbb{R}^{N_2}$. These can be combined into a simplicial complex signal $\mathbf{x}_t = [\mathbf{x}_t^{0\top}, \mathbf{x}_t^{1\top}, \mathbf{x}_t^{2\top}]^\top \in \mathbb{R}^N$, encompassing node, edge, and triangle signals. The boundary maps $\mathbf{B}_1$ and $\mathbf{B}_2$ allow us to transform signals from one simplical level to other, and in many cases these transformations are intuitive and informative. For example in water distribution networks, transformation of edge signal to node signal via $\mathbf{B}_1$ can be considered as sum of flows at a node. Similarly the transformation of node signals to edge signals via $\mathbf{B}_1^\top$ can be considered as head difference across the pipe (edge). In the case of a second order simplicial complex $\mathcal{X}^2$, the node signal can be transformed to an edge signal, the edge signal into node and triangle signals; and triangle signal into edge signal. These transformations can be combined using the Dirac operator [12],

$$\mathbf{D} = \begin{pmatrix} \mathbf{0}, & \mathbf{B}_1, & \mathbf{0} \\ \mathbf{B}_1^\top, & \mathbf{0}, & \mathbf{B}_2 \\ \mathbf{0}, & \mathbf{B}_2^\top, & \mathbf{0} \end{pmatrix} \in \mathbb{R}^{N \times N}. \tag{2}$$

As $\mathbf{L} = \mathbf{D}^2$, notably, the combined Laplacian can be derived from the Dirac operator.

C. Simplicial convolutions filters

Drawing from the graph signal processing literature [13], the Hodge Laplacian serves as a natural shift operator for signals defined on simplicial complexes, as explored in [7], [8], [14]. Simplical convolution filter can be defined as,

$$\mathbf{H}^k(\mathbf{L}_k) = \begin{cases} \sum_{n=0}^{L-1} \beta_n \mathbf{L}_k^n, & \text{for } k = 0, 2, \\ \underbrace{\sum_{n=0}^{L_d-1} \beta_n^d \mathbf{L}_{1,d}^n}_{\mathbf{H}^{1,d}(\mathbf{L}_{1,d})} + \underbrace{\sum_{n=0}^{L-L_d-1} \beta_n^u \mathbf{L}_{1,u}^n}_{\mathbf{H}^{1,u}(\mathbf{L}_{1,u})}, & \text{for } k = 1. \end{cases} \tag{3}$$

Here L is the filter order, and $\{\beta_n\}_{n=0}^{L-1}$ are the filter coefficients that weigh $\mathbf{x}_t^k$ after being shifted n times by $\mathbf{L}_k$. Now a signal $\mathbf{y}_t^k$ defined over k-simplices can be expressed as sum of filtered versions of signals defined over k-simplicies and signals defined over its lower and upper adjacent simplcies as,

$$\mathbf{y}_t^k = \mathbf{H}(\mathbf{L}_k)\mathbf{x}_t^k + \mathbf{B}_k^\top\mathbf{H}(\mathbf{L}_{k-1})\mathbf{x}_t^{k-1} + \mathbf{B}_{k+1}\mathbf{H}(\mathbf{L}_{k+1})\mathbf{x}_t^{k+1} + \mathbf{v}_t^k, \tag{4}$$

where $\mathbf{v}_t^k \in \mathbb{R}^{N_k}$ is the noise vector. The convolve-transform operations allow us incorporate information from upper and lower simplicials in modeling a signal defined over a k-simplex. For the output signal $\mathbf{y}_t = [\mathbf{y}_t^{0\top}, \mathbf{y}_t^{1\top}, \mathbf{y}_t^{2\top}]^\top \in \mathbb{R}^N$, defined over node, edge and triangle, the combined filtering operation can be represented as,

$$\mathbf{y}_t = \mathbf{H}(\mathbf{L})\mathbf{x}_t + \mathbf{v}_t, \tag{5}$$

where

$$\mathbf{H}(\mathbf{L}) = \begin{pmatrix} \mathbf{H}(\mathbf{L}_0), & \mathbf{B}_1\mathbf{H}(\mathbf{L}_1), & \mathbf{0} \\ \mathbf{B}_1^\top\mathbf{H}(\mathbf{L}_0), & \mathbf{H}(\mathbf{L}_1), & \mathbf{B}_2\mathbf{H}(\mathbf{L}_2) \\ \mathbf{0}, & \mathbf{B}_2^\top\mathbf{H}(\mathbf{L}_1), & \mathbf{H}(\mathbf{L}_2) \end{pmatrix} \in \mathbb{R}^{N \times N}, \tag{6}$$

and, $\mathbf{v}_t = [\mathbf{v}_t^{0\top}, \mathbf{v}_t^{1\top}, \mathbf{v}_t^{2\top}]^\top \in \mathbb{R}^N$.

III. System model respecting structure of the network

In this section, we model the hidden dynamics of the system using a structure-aware state-space framework based on SPDE; that can effectively represent many time-varying processes across diverse domains [15]. The stochastic heat equation is one of the most widely studied SPDE due to its wide applicability and universal nature [15]. The stochastic heat equation can be defined as,

$$\frac{d\mathbf{x}_t}{d\mathbf{t}} = c\nabla^2\mathbf{x}_t + \boldsymbol{\zeta}_t, \tag{7}$$

where the evolution of the process $\mathbf{x}_t$ is governed by the Laplacian operator ∇^2, the stochastic white noise $\boldsymbol{\zeta}_t \in \mathbb{R}^V$ is acting as a forcing function and c is the diffusion coefficient. To enforce structure into the evolution of the process, we replace the Laplace operator ∇^2 with its discrete analogue $\mathbf{L}$ [12] resulting in:

$$d\mathbf{x}_t = -c\mathbf{L}\mathbf{x}_t dt + \mathbf{S}d\boldsymbol{\beta}_t, \tag{8}$$

where $\boldsymbol{\beta}_t \in \mathbb{R}^V$ is a standard Brownian motion that acts as a forcing function through the dispersion matrix $\mathbf{S} \in \mathbb{R}^{N\times V}$. Here $\mathbf{S}$ encodes the information how the uncertainty enters the system. Equation (8) shows that the dynamics over each simplicial level is governed by the corresponding Hodge Laplacian with some uncertainty.

If the initial conditions are Gaussian $\mathbf{x}_0 \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$, the solution of (8) is a Gaussian process $\mathbf{x}_t \sim \mathcal{N}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_{t,s}(\mathbf{L}))$, with the mean function $\boldsymbol{\mu}_t = \exp(-c\mathbf{L}t)$ and covariance function $\boldsymbol{\Sigma}_{t,s}(\mathbf{L}) = \mathbf{V}\mathbf{C}_{t,s}\mathbf{V}^\top$, where $\exp(\cdot)$ is a matrix exponential, $\mathbf{V}$ is the matrix of eigenvectors of $\mathbf{L}$ and matrix $\mathbf{C}_{t,s}$ has individual entries (i, j) as,

$$[\mathbf{C}_{t,s}]_{ij} = \frac{[\mathbf{V}^\top\mathbf{S}\mathbf{S}^\top\mathbf{V}]_{ij}}{c(\lambda_i + \lambda_j)}(e^{-c\lambda_i|t-s|} - e^{-c(\lambda_i t+\lambda_j s)}), \tag{9}$$

where λ_i and λ_j are i-th and j-th eigenvalues of $\mathbf{L}$. Modeling the process as (9), posses three main short comings. First, the computational complexity of the covariance function is of order $O(N^3T^3)$; second, the Brownian motion diffuses through the matrix $\mathbf{S}$ without respecting the topological structure, resulting in poor generalization. Third, the solution is limited to a low-pass process over SC. In our work, the first two challenges are addressed by introducing a structure-aware dispersion matrix, while the third issue of the low-pass nature of the solution is resolved through the incorporation of a state-space model, which provides a more generalized framework.

A. Structure aware dispersion matrix

To enforce a structure into the dispersion matrix $\mathbf{S}$, we reformulate (8) by assuming that the uncertainty affects simplicial signals via the boundary map. For the node signal $\mathbf{x}_t^0$, the dynamics is governed by the graph Laplacian $\mathbf{L}_0$, which encodes adjacency relationships between nodes through connected edges. Therefore, we can assume that any uncertainty in the dynamics is caused by the edges. Similarly, uncertainty in the dynamics of edge signals is caused by nodes and triangles, while uncertainty in the dynamics of triangle signals is caused by edges. Based on this premise, we reformulate (8) as,

$$d\mathbf{x}_t = -c\mathbf{L}\mathbf{x}_t dt + \mathbf{D}\text{diag}(\boldsymbol{\alpha})d\boldsymbol{\beta}_t, \tag{10}$$

where $\mathbf{S}$ is replaced with $\mathbf{D}\text{diag}(\boldsymbol{\alpha})$. The Dirac operator $\mathbf{D}$ (2) helps us to model the uncertainty in node dynamics via edges, the edge dynamics via nodes and triangles, and the triangle dynamics via edges. The vector $\boldsymbol{\alpha} \in \mathbb{R}^N$ encodes the information about the level of uncertainty in each of the N simplicial signals. In (10), we described the structure-aware dynamics of the hidden process. Next, we introduce the relationship between the hidden process and the observations using an observation matrix and formulate a state-space model to describe the complete dynamics of the system.

B. State-space model

Let us consider a linear relation between process and observation. At every discrete time instants i, we are provided with a noisy observation of the simplicial signal $\mathbf{x}_i$ as,

$$\mathbf{y}_i = \mathbf{M}\mathbf{x}_i + \mathbf{v}_i, \tag{11}$$

where $\mathbf{y}_i \in \mathbb{R}^N$ is the available observation and $\mathbf{M} \in \mathbb{R}^{N\times N}$ is the unknown observation matrix. We propose to learn matrix $\mathbf{M}$ from data using simplicial convolution filters (5), (6). This allow us to rewrite (11) as,

$$\mathbf{y}_i = \mathbf{H}(\mathbf{L})\mathbf{x}_i + \mathbf{v}_i. \tag{12}$$

Combining (12) and (10), the complete system model can be described as a continuous-discrete state-space model:

$$\begin{aligned} d\mathbf{x}_t &= -c\mathbf{L}\mathbf{x}_t dt + \mathbf{D}\text{diag}(\boldsymbol{\alpha})d\boldsymbol{\beta}_t, \\ \mathbf{y}_i &= \mathbf{H}(\mathbf{L})\mathbf{x}_i + \mathbf{v}_i. \end{aligned} \tag{13}$$

Although we give a general model in (13), in many cases, signals might not be observed over all the simplicial levels. For instance, in a traffic network, the only available observation might be the traffic flow across edges. In such cases, we can utilize only the informative part of the filter $\mathbf{H}(\mathbf{L})$ for modeling; essentially treating (4) as the observation model. In the context of a traffic network, even though observations are limited to edges, our model can still leverage hidden dynamics that may exist across nodes, edges, and triangles, as these are considered part of the state-space. This also highlights a unique advantage of our approach: even when signal are observed only on edges, our model posits that the signals defined over triangles are hidden, and, it is possible to extract meaningful triangle signals from the observed edge signals. To simplify notation, we represent the filter $\mathbf{H}(\mathbf{L})$ as $\mathbf{H}$ and its parameters as $\mathbf{h}$. Next, we propose an efficient optimization framework to estimate the evolving states and the unknown parameters in the state-space model presented in (13).

IV. Joint estimation of states and parameters

The proposed approach consists of first discretizing the SPDE and then applying Kalman filtering with online parameter estimation through maximum likelihood.

A. Discretize the continues state

Assume that the data has high resolution, uniform sampling interval δt, and Gaussian initial conditions $\mathbf{x}_0 \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$. The discrete state equation is obtained as the first order approximation of the continuous analogue [3]:

$$\mathbf{x}_{t_i+\delta t} = \tilde{\mathbf{L}}\mathbf{x}_{t_i} + \mathbf{q}_i, \quad \mathbf{q}_i \sim \mathcal{N}(0, \mathbf{Q}), \tag{14}$$

with transition operators $\tilde{\mathbf{L}}$ and $\mathbf{Q}$, derived as [3],

$$\tilde{\mathbf{L}} \simeq \mathbf{I} - c\delta t\mathbf{L}, \tag{15}$$

$$\mathbf{Q} \simeq \delta t\mathbf{D}\mathrm{diag}^2(\boldsymbol{\alpha})\mathbf{D}. \tag{16}$$

Interestingly, from (16) we obtain a noise covariance matrix of the model, which is parametrized by the vector $\boldsymbol{\alpha}$ and also accounts for the topological structure through the Dirac operator matrix $\mathbf{D}$. Here, we can also see that the number of parameters needed to represent the covariance matrix is reduced to be linear in the number of states, as opposed to (9). Now, combining (12) and (14) we get discrete state-space model as,

$$\begin{aligned} \mathbf{x}_i &= \tilde{\mathbf{L}}\mathbf{x}_{i-1} + \mathbf{q}_i, \\ \mathbf{y}_i &= \mathbf{H}\mathbf{x}_i + \mathbf{v}_i. \end{aligned} \tag{17}$$

Interested readers about the the details of this discretization procedure and similar ones, are referred to [3, subsection 3.1].

B. State estimation

Given the uncertainty vector $\boldsymbol{\alpha}$ and the filter $\mathbf{H}$, the solution of (17) can be obtained using a Kalman Filter. Given the updates up to $i-1$ iterations of the state $\mathbf{x}_{i-1|i-1}$ and the covariance matrix $\mathbf{P}_{i-1|i-1}$, the Kalman filter estimates the next state in two steps: first, it predicts the state based on the model, and then, it corrects the prediction using the observations.

1) Prediction:

$$\mathbf{x}_{i|i-1} = \tilde{\mathbf{L}}\mathbf{x}_{i-1|i-1}, \tag{18}$$

$$\mathbf{P}_{i|i-1} = \tilde{\mathbf{L}}\mathbf{P}_{i-1|i-1}\tilde{\mathbf{L}} + \mathbf{Q}; \tag{19}$$

2) Correction:

$$\mathbf{K}_i = \mathbf{P}_{i|i-1}\mathbf{H}^\top(\mathbf{H}\mathbf{P}_{i|i-1}\mathbf{H}^\top + \sigma^2\mathbf{I})^{-1}, \tag{20}$$

$$\mathbf{x}_{i|i} = \mathbf{x}_{i|i-1} + \mathbf{K}_i(\mathbf{y}_i - \mathbf{H}\mathbf{x}_{i|i-1}), \tag{21}$$

$$\mathbf{P}_{i|i} = \mathbf{P}_{i|i-1} - \mathbf{K}_i\mathbf{P}_{i|i-1}\mathbf{K}_i^\top; \tag{22}$$

where $\mathbf{K}_i$ is the Kalman gain and we obtain the required estimate $\mathbf{x}_{i|i}$ after i iterations as in (21).

C. Parameter estimation

Next, we estimate the unknown parameters $\boldsymbol{\alpha}$ and $\mathbf{h}$ through maximum likelihood estimation (MLE). From the fixed parameter Kalman filtering, we obtain the distribution of $\mathbf{y}_i$ as,

$$p(\mathbf{y}_i|\mathbf{y}_1, \ldots, \mathbf{y}_{i-1}) = \mathcal{N}(\mathbf{H}\mathbf{x}_{i|i-1}, \mathbf{H}\mathbf{P}_{i|i-1}\mathbf{H}^\top + \sigma^2\mathbf{I}). \tag{23}$$

Now, the negative log-likelihood function can be computed recursively as,

$$\begin{aligned} \mathcal{L}_i(\boldsymbol{\alpha}, \mathbf{h}) &= \mathcal{L}_{i-1}(\boldsymbol{\alpha}, \mathbf{h}) + \frac{1}{2}\log|\mathbf{S}_i| \\ &+ \frac{1}{2}(\mathbf{y}_i - \mathbf{H}\mathbf{x}_{i|i-1})^\top\mathbf{S}_i^{-1}(\mathbf{y}_i - \mathbf{H}\mathbf{x}_{i|i-1}), \end{aligned} \tag{24}$$

Algorithm 1: SC-SPDE

Result: $\mathbf{x}_{i|i}$, $\mathbf{h}_i, \boldsymbol{\alpha}_i$
Given: $\mathbf{L}$
Initialize: a_1, a_2, c, δt, $\mathbf{x}_{0|0}$, $\mathbf{P}_{0|0}$, $\mathbf{h}_0$ and $\boldsymbol{\alpha}_0$
Compute $\tilde{\mathbf{L}}$ via (15), $\mathbf{Q}$ via (16)
for $i = 1, 2, \ldots$ **do**
  Obtain the observation $\mathbf{y}_i$
  Kalman Filtering: to obtain $\mathbf{x}_{i|i}$
  Prediction: (18), (19)
  Correction: (20), (21), (22)
  MLE: Compute $\boldsymbol{\alpha}_i$ via (25), $\mathbf{h}_i$ via (26)
end

where $\mathbf{S}_i = \mathbf{H}\mathbf{P}_{i|i-1}\mathbf{H}^\top + \sigma\mathbf{I}$, and $|\cdot|$ is the matrix determinant. To estimate the uncertainty vector $\boldsymbol{\alpha}$ and filter parameters $\mathbf{h}$, we alternately minimize $\mathcal{L}_i(\boldsymbol{\alpha}, \mathbf{h})$; first with respect to $\boldsymbol{\alpha}$ and then with respect to $\mathbf{h}$. We employ recursive online gradient descent update as follows,

$$\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_{i-1} - a_1\nabla_{\boldsymbol{\alpha}}\mathcal{L}_i(\boldsymbol{\alpha}_{i-1}, \mathbf{h}_{i-1}), \tag{25}$$

$$\mathbf{h}_i = \mathbf{h}_{i-1} - a_2\nabla_{\mathbf{h}}\mathcal{L}_i(\boldsymbol{\alpha}_i, \mathbf{h}_{i-1}), \tag{26}$$

where, $\nabla_{\mathbf{h}}$ and $\nabla_{\boldsymbol{\alpha}}$ denote the gradient operators with respect to $\mathbf{h}$ and $\boldsymbol{\alpha}$, respectively. The parameters a_1 and a_2 represent the learning rates. Pseudo code for the proposed estimation framework (SC-SPDE) is given in **Algorithm 1**.

V. Experiments

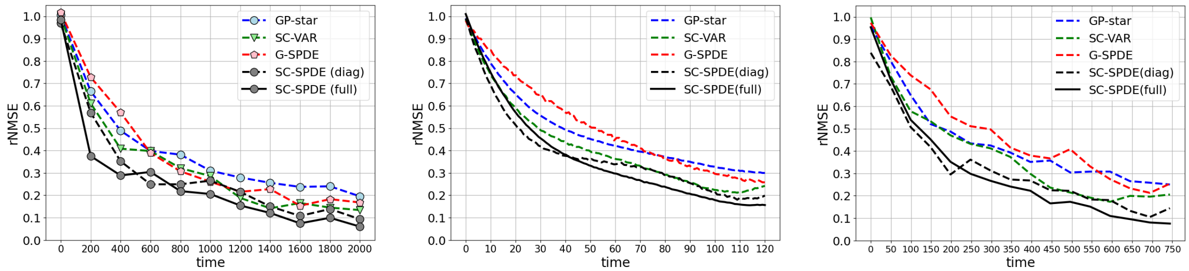
We test the proposed method (SC-SPDE) on nodal and edge data from three different datasets: *a)* synthetic dataset, *b)* water flow data from the Cherry Hills water network, and *c)* Molene weather temperature data. The performance is measured via the root normalized mean squared error (rNMSE), defined as

$$\mathrm{rNMSE}(t) = \left(\frac{1}{N}\sum_{e=1}^{N}\frac{\sum_{\tau=1}^{t}(f_e(\tau) - \hat{f}_e(\tau))^2}{\sum_{\tau=1}^{t} f(\tau)_e^2}\right), \tag{27}$$

where $\hat{f}_e(t)$ is the predicted data on index e at time t. We used GP-star [10], SC-VAR [8], G-SPDE [3] as the benchmark algorithms. We also consider a variation of the model where the observation matrix is diagonal to emphasize the importance of merging nodal and edge data to have a more expressive state variable. The initial state is initialized with a standard Gaussian, the uncertainty $\boldsymbol{\alpha}$ is initiated as zeros, and the graph filter is initiated with a K-th order Chebyshev approximation of a standard spectral heat kernel.

A. Synthetic data

The simplicial complex structure and experimental setup are similar to [8] with data over both nodes and edges, while the triangles are used as the source of uncertainty. This data contains 1000 distinct sequences of length 2000 each with a different filter and uncertainty values $\boldsymbol{\alpha}$, and, we run the algorithm for the batches of length 200. The model hyperparameters are obtained via grid search corresponding to the lowest rNMSE except for the filter order which aligns with the data generator. The resulting values are $(a_1, a_2, \sigma^2) = (0.05, 0.001, 0.01)$. Fig. 1a shows that SC-SPDE outperforms other models as the synthetic data aligns well with



(a) (b) (c)

Fig. 1: The performance of SC-SPDE over (a) Synthetic dataset (b) Cherry Hills water network (c) Molene weather temperature data.

its assumptions. However, importantly, the full observation matrix outperforms and converges faster than the diagonal version, highlighting the importance of combining nodal and edge data for state estimation.

B. Cherry Hills Water Network

This dataset consists of 36 nodes, 40 pipes, and 2 triangles. We assume a similar setup as in [8]. The flow signals are 112 hourly sampled volumes of water in m^3/h. All the filter orders are 3. We run the algorithm online without batching, and, with hyperparameters retrieved via grid search for the lowest rNMSE. The resulting values are $(a_1, a_2, \sigma^2) = (0.025, 0.05, 0.025)$. Fig. 1b indicates the superior performance of SC-SPDE compared to alternatives. The models leveraging the topological structure outperform the ones only inducing the graph structure as the water network data is highly dependent on the interaction of edge flows and nodal pressures. Moreover, SC-SPDE performs better than SC-VAR due to accounting for edge weight uncertainties. Finally, using the full matrix instead of a diagonal observation transformation enhances performance by capturing edge-node interactions, at the expense of more parameters and slower convergence.

C. Temperature data

The Molene dataset contains 744 hourly temperature measurements across 32 stations in a region of France and is used for temperature forecasting. There are no observed edge signals in this dataset and we generate them as the difference of true node signals sharing an edge. The practical factors such as temperature differences between nodes or circulating air in a region can influence the nodal time series. These effects can be captured in our model through hidden higher-order signals. We run the algorithm online with batches of length 24 (daily data) with grid-searched hyperparameters $(a_1, a_2, \sigma^2) = (0.075, 0.002, 0.01)$. Fig. 1c suggests that simplicial algorithms perform better even without observing the edge signals. A similar pattern is observed between full and diagonal observation matrices for SC-SPDE where better accuracy is achieved with more data and model complexity despite the edge flows being completely lifted from nodal temperatures.

VI. Conclusion

We introduce a novel framework for processing information in topological dynamical systems using a Kalman filter. In this approach, the system dynamics is modeled as a Hodge Laplacian-based SPDE, while the observation matrix is learned using simplicial convolution filters. To further enhance the model, we propose a structure-aware noise covariance for the dynamics using the Dirac operator. The topological information encoded in these matrices is combined with observed data through the Kalman filter, enabling scalable and efficient state estimation. Experiments demonstrate the effectiveness of the proposed method on both real and synthetic datasets. Furthermore, the results highlight the importance of uncovering hidden higher-order signals, paving the way for further explorations on higher-order dynamics.

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