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Optimal Crop Rotations subject to Weed Dynamics: Exponential Stability and Nonlinear Programming

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***Abstract*— Agricultural production of annual crops is often hampered by annual weeds, which compete with planted crops and persist through the collection of dormant seeds in the soil called the *weed seed bank*. Conventional weed management relies heavily on chemical herbicides, which are not sustainable. A complementary method that reduces the need for herbicides is 'cultural control', in which the *crop rotation* is designed in part to manage the weed population. We propose a methodology that optimizes the crop rotation, here defined as periodic crop planting densities, subject to periodic weed dynamics. We adopt a well-established model of discrete-time annual weed seed bank dynamics with crop planting density inputs, and show that any periodic weed seed bank trajectory corresponding to a periodic crop rotation is globally exponentially stable. This guarantees convergence to the optimal periodic trajectory obtained by solving a nonlinear optimal control problem with periodic constraints, which we formulate as a nonlinear program.**

I. Introduction

Weed management in agricultural systems is a persistent challenge, and the potential loss of agricultural production due to weeds exceeds 30%. This loss is caused mainly by competition for resources like light, water and nutrients [1], [2]. Annual weeds, which complete their life cycle within a single year, are particularly problematic due to rapid growth and prolific seed production [3]. Throughout most of the year, the annual weed population exists merely as dormant seeds in the *weed seed bank* (WSB), i.e., the reservoir of weed seeds in the soil. At the start of the respective growth season (e.g., summer or winter), some of the seeds emerge and start to compete with planted annual crops. Mature weeds produce new seeds, of which some are incorporated in the WSB. Seeds in the WSB can survive from a few years to several decades [4], [5]. This persistence necessitates a long-term approach to annual weed management. Conventional weed management practices often rely heavily on chemical herbicides, which have adverse effects on the environment and sustainability, and lead to resistance buildup [1], [6]; optimal control for yearly herbicide treatment subject to an evolving WSB has been studied e.g. in [7], [8]. An alternative or complementary approach is *cultural control*, in which the *crop rotation* is designed in part to manage the weed population. A major element in cultural control is the natural competitive ability of crop species; weed-suppressive crops may be cultivated simultaneously with (*intercropping*) or subsequent to (*crop rotation*) less weed-suppressive crops [1], [3]. The planting density of crops (plants/area) has a large effect on the suppression of weeds [3]. Therefore, one may look for optimal crop rotations in terms of yearly crop planting densities for each available crop species.

Optimization of crop rotations has been explored in the form of linear programs [9], [10] and exhaustive approaches [11]–[15], which are restricted to monocultures or bicultures with fixed planting densities. Moreover, many of them optimise across a finite horizon [9]–[12], which may lead to myopic solutions that allow the WSB to significantly increase towards the end of the horizon. Some studies simulate WSB trajectories with candidate crop rotations until periodic WSB trajectories are observed, evaluating them across a single period [14], [15], but do not address uniqueness and stability of the periodic patterns; also, such a methodology does not lend itself to a continuous action space.

Here, we adopt a new long-term optimization approach where the annual planting densities of available crops are nonnegative real optimization variables and we minimize an economic cost subject to the dynamics of the annual WSB, considered as the state variable. To this aim, we rely on a well-established model of WSB dynamics, as well as crop-weed and crop-crop competition [1], [11], [12], [15]–[17]. We formulate the problem as an economic nonlinear optimal control problem with periodic constraints [18]–[20], which can be cast as a nonlinear program. The parameters within our cost function are related to economic and ecological quantities, and are therefore interpretable and can be estimated in real situations. Our periodic constraints require that the optimal solution must start and end with the same WSB, making our optimal strategy sustainable in the long term.

This approach is similar in spirit to economic model predictive control (EMPC) [20], but with key distinctions. WSBs are often difficult to measure and therefore deploying EMPC in a moving horizon fashion with feedback is not practical. Instead, we demonstrate that any periodic trajectory of planting densities results in a unique, exponentially stable WSB trajectory. Thus, we can solve for an optimal periodic trajectory without any knowledge of the current WSB. The optimal crop rotation can then be implemented (repetitively), so that we obtain a sustainable infinite horizon crop rotation trajectory with an optimal average cost.

Moreover, the proposed optimization problem determines

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both which crops to sow and their corresponding planting densities. Thus, we do not a-priori enforce yearly monocultures, or bicultures, or polycultures: These are just possible solutions within a more general framework.

Our main contributions are the following:

- we consider a weed dynamics model with crop planting densities as inputs (Section II) and we analyse its qualitative properties (Section III);
- we prove existence, uniqueness and global exponential stability (i) of its positive steady state for constant crop planting densities (Section III-A), and then, building upon this result, (ii) of periodic positive trajectories for periodic crop planting densities (Section III-B);
- we formulate, and solve with gradient-based methods, a nonlinear optimal control problem with periodic constraints to find the optimal crop rotation and WSB trajectory for a given economic cost (Section IV); the model properties guarantee that, under periodic inputs, the system trajectory converges to the optimal periodic orbit, as confirmed by our numerical tests (Section V).

II. Weed Seed Bank Dynamic Model

Both the crops and the weeds we consider are annual. Hence, it is natural to resort to a discrete-time dynamic model, with an annual time step. The WSB in year t is represented by $x_t \in [0, \infty)$, which is the number of seeds per unit area, at the start of a new season. In the integration from one year to the next, we consider three parallel contributions to the WSB: *surviving* non-germinated seeds of the current year that survive into the next year; *incorporated* seeds that are newly produced by weed plants, are incorporated in the soil, and survive to the next year [1]; and *inflow* seeds that are coming in from outside the field, such as seeds carried by the wind, or excreted by birds or other animals [21]. Seed survival is generally assumed to be an exponential decay process [22]: we denote by $\delta \in (0, 1)$ the constant fraction of established seeds that do not germinate and survive to the next year. The second contribution embodies seed germination, weed development, reproduction, and incorporation into the soil of the new seeds, and is generally described by a saturation curve, reflecting approximately exponential growth under low WSB densities, and saturation at high WSB densities due to intra-specific competition [1], [17], [23]. We denote by $\lambda \in (0, \infty)$ the seed incorporation per seed in the absence of intra-specific competition, and by $\alpha \in (0, \infty)$ the density-based intraspecific competition constant. For the n available crops that can be sown in the field, we denote crop planting densities in year t by $u_t \in [0, \infty)^n$ (seeds/area); the resulting competition effect is modelled by an additional weighted sum in the denominator, with competition constants captured by $b \in [0, \infty)^n$. Lastly, the annual seed inflow is modelled as a positive constant $\mu \in (0, \infty)$. The three contributions are added up in the transition function $f(\cdot)$, resulting in the difference equation:

$$x_{t+1} = f(x_t, u_t) \doteq \delta x_t + \frac{\lambda x_t}{1 + \alpha x_t + b^\top u_t} + \mu \qquad (1)$$

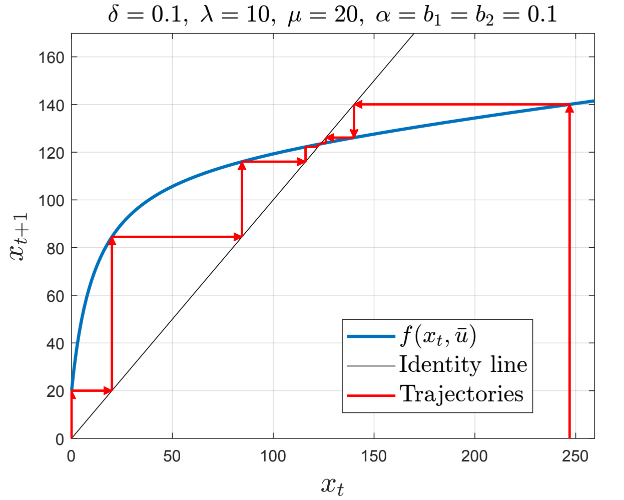

Fig. 1. Cobweb map of the weed dynamics in equation (1) with a constant input $u_t \equiv \bar{u}$. The steady state $\bar{x} \approx 125$ resides at the intersection of $f(x_t, \bar{u})$ and the identity line. Both $x_0 = 0 < \bar{x}$ and $x_0 = 248 > \bar{x}$ eventually result in convergence to $\bar{x}$.

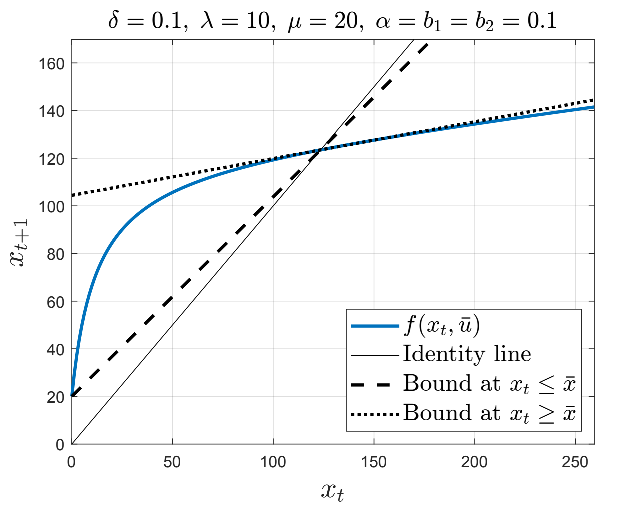

Fig. 2. Exponential convergence bounds of system (1) when $x \leq \bar{x}$ and $x \geq \bar{x}$, with a constant input $u_t \equiv \bar{u}$. The system is exponentially stable under all non-negative inputs, see Proposition 5.

where $b^\top$ is a row vector. Given a constant input $u_t \equiv \bar{u} \in [0, \infty)^n$, the system dynamics can be visualized by a cobweb map, such as the one shown in Figure 1.

In the following, we say that a property is structural [24] if it holds for all admissible values of the parameters (i.e., $\delta \in (0, 1)$, while all the other parameters are positive).

III. Exponential Stability

In this section, we outline key properties of system (1), which stem from the qualitative (i.e., structural) properties of the functional expression of $f(x, u)$.

Proposition 1 (Qualitative properties of $f(x, u)$)**.** *The function $f(x, u) \colon [0, \infty) \times [0, \infty)^n \to (0, \infty)$ in equation* (1) *has the following properties: given a fixed $u \in [0, \infty)^n$,*

(i) the function $f(x, u)$ is monotonically increasing in x: $\frac{\partial}{\partial x} f(x, u) > 0$;
(ii) $\lim_{x \to \infty} \frac{\partial}{\partial x} f(x, u) = \delta \in (0, 1)$;

(iii) $f(x,u) \geq \mu > 0$;
(iv) $f(x,u)$ *is concave in* x: $\frac{\partial^2}{\partial x^2} f(x,u) < 0$.

Proof. *(i)* Given a fixed $u \in [0,\infty)^n$, the derivative

$$\frac{\partial}{\partial x} f(x,u) = \delta + \frac{\lambda(1+b^\top u)}{(1+\alpha x+b^\top u)^2}$$

is positive for all $x \geq 0$.

(ii) As $x \to \infty$, the second term of $\frac{\partial}{\partial x} f(x,u)$ tends to zero, thus $\lim_{x\to\infty} \frac{\partial}{\partial x} f(x,u) = \delta \in (0,1)$.

(iii) Since $f(0,u) = \mu > 0$ and $f(x,u)$ is monotonically increasing in x, it must be $f(x,u) \geq \mu$ for all $x \geq 0$.

(iv) The second derivative

$$\frac{\partial^2}{\partial x^2} f(x,u) = -2\frac{\alpha\lambda(1+b^\top u)}{(1+\alpha x+b^\top u)^3}$$

is negative for all $x \geq 0$. ∎

The fundamental properties of $f(x,u)$ allow us to draw conclusions on the qualitative (i.e., structural) dynamic behaviour of the system $x_{t+1} = f(x_t, u_t)$.

Proposition 2 (Positivity)**.** *The dynamical system* $x_{t+1} = f(x_t, u_t)$ *in* (1) *is positive: if* $x_0 \geq 0$ *and* $u_t \in [0,\infty)^n$ *for all* t*, then* $x_t \geq 0$ *for all* t*.*

Proof. Positivity of system (1) can be shown by noticing that $x_{t+1} = f(x_t, u_t) > 0$ for any $x_t \geq 0$ and $u_t \in [0,\infty)^n$, also in view of Proposition 1, *(iii)*. ∎

Since the system is positive, we refer to properties of the system (such as exponential stability) as *global* if they hold for all $x_0 \geq 0$.

A. Constant planting density

We first analyze the system $x_{t+1} = f(x_t, u_t)$ in equation (1) when the planting density is constant, $u_t \equiv \bar{u} \in [0,\infty)^n$, and we prove the existence, uniqueness, and exponential stability of the steady state $\bar{x} = f(\bar{x}, \bar{u})$ for this system.

Proposition 3 (Steady state: existence and uniqueness)**.** *Given any constant input* $\bar{u} \in [0,\infty)^n$*, the dynamical system* $x_{t+1} = f(x_t, \bar{u})$ *as per equation* (1) *admits a unique non-negative steady state* $\bar{x} = f(\bar{x}, \bar{u})$*, which is strictly positive.*

Proof. Since, as shown in Proposition 1, $f(x, \bar{u})$ is strictly positive for $x = 0$, monotonically increasing and concave, and $\frac{\partial}{\partial x} f(x,u)$ has a limit $\delta \in (0,1)$ for $x \to \infty$, the function must have a unique intersection with the identity line, which occurs at strictly positive values. ∎

Proposition 4 (Local asymptotic stability)**.** *The steady state* $\bar{x}$ *of the dynamical system* $x_{t+1} = f(x_t, u_t)$ *in* (1) *is structurally locally asymptotically stable.*

Proof. The system Jacobian computed at the steady state, $f'(\bar{x})$, is Schur: $0 < f'(\bar{x}) < 1$. In fact, the system Jacobian

$$f'(x) = \frac{\partial}{\partial x} f(x,u) = \delta + \frac{\lambda(1+b^\top u)}{(1+\alpha x+b^\top u)^2} > 0 \qquad (2)$$

is always positive, as shown in the proof of Proposition 1. From the steady-state conditions, we have that $\lambda = \frac{1+\alpha\bar{x}+b^\top\bar{u}}{\bar{x}}[(1-\delta)\bar{x} - \mu]$. Substituting this expression of λ into (2) yields

$$f'(\bar{x}) = \frac{\partial}{\partial x} f(x,u)|_{x=\bar{x}} = \delta + \frac{[(1-\delta)\bar{x} - \mu](1+b^\top\bar{u})}{\bar{x}(1+\alpha\bar{x}+b^\top\bar{u})}.$$

We want to show that $f'(\bar{x}) < 1$. Rearranging and simplifying gives $-\mu(1+b^\top\bar{u}) < \alpha x^2(1-\delta)$, which is always true when $\delta \in (0,1)$ and the other parameters are positive. ∎

Remark 1. *Any strictly positive, monotonically increasing and concave function* $\rho(x)$ *for which* $\lim_{x\to\infty} \rho'(x) < 1$ *satisfies* $\rho'(\bar{x}) < 1$ *for the unique intersection point with the identity,* $\bar{x} = \rho(\bar{x})$*; see e.g. Figure 2.*

Proposition 5 (Global exponential stability)**.** *The steady state* $\bar{x}$ *of system* (1) *is structurally globally exponentially stable: there exists* $\nu \in (0,1)$ *such that the solution* x_t *to the equation* $x_{t+1} = f(x_t, \bar{u})$ *satisfies*

$$|x_t - \bar{x}| \leq \nu^t |x_0 - \bar{x}| \quad \forall x_0 \geq 0,\ t \in \{0,1,\dots\}.$$

Proof. Given $\bar{u} \in (0,\infty)^n$, we denote $f(x) \doteq f(x,\bar{u})$. In view of the steady-state condition, $\mu < \bar{x}$ and therefore

$$\psi \doteq \frac{\bar{x}-\mu}{\bar{x}} \in (0,1). \qquad (3)$$

When $x \leq \bar{x}$, monotonicity of f implies that $f(x) \leq f(\bar{x}) = \bar{x}$, where the equality expresses the steady-state condition. Concavity of f implies that the curve f lies above the line passing through the points $(0,\mu)$ and $(\bar{x}, f(\bar{x}))$, where $f(\bar{x}) = \bar{x}$; see Figure 2. Therefore, we can write $f(x) \geq \psi x + \mu$, where ψ is defined as in (3), and

$$|\bar{x} - f(x)| = \bar{x} - f(x) \leq \psi(\bar{x} - x) = \psi|\bar{x} - x|,$$

where the equalities hold since $\bar{x} - x \geq 0$ and $\bar{x} - f(x) \geq 0$. If $x_0 \leq \bar{x}$, then $x_t \leq \bar{x}$ for all t and we obtain the bound

$$|x_t - \bar{x}| \leq \psi^t |x_0 - \bar{x}|.$$

If $x_t \geq \bar{x}$, concavity of f implies that the derivative of f computed at $\bar{x}$, $f'(\bar{x})$, is larger than the derivative computed at any point in $(\bar{x}, x_t)$; see also Figure 2. In particular, consider the value $\theta \in (\bar{x}, x_t)$ such that $f'(\theta) = \frac{f(x_t)-\bar{x}}{x_t-\bar{x}}$, which exists in view of Lagrange's mean value theorem. Then, $f'(\theta) \leq f'(\bar{x})$ and therefore

$$|f(x_t) - \bar{x}| = f(x_t) - \bar{x} \leq f'(\bar{x})(x_t - \bar{x}) = f'(\bar{x})|x_t - \bar{x}|,$$

where the equalities hold because $x_t - \bar{x} \geq 0$ and thus $f(x_t) - \bar{x} \geq 0$ due to the monotonicity of f. Note that $f'(\bar{x}) \in (0,1)$, as shown in the proof of Proposition 4. If $x_0 \geq \bar{x}$, then $x_t \geq \bar{x}$ for all t and we have the bound

$$|x_t - \bar{x}| \leq (f'(\bar{x}))^t |x_0 - \bar{x}|.$$

For all $x \geq 0$, we have

$$|x_t - \bar{x}| \leq \nu^t |x_0 - \bar{x}|$$

where $\nu = \max\{\frac{\bar{x}-\mu}{\bar{x}}, f'(\bar{x})\} \in (0,1)$. Hence, the unique positive steady state is exponentially stable for all $x \geq 0$. ∎

Corollary 1 (Global asymptotic stability)**.** *The steady state* $\bar{x}$ *of system* (1) *is structurally globally asymptotically stable.*

B. Periodic planting densities

For crop rotations, we have a periodic planting density trajectory denoted $\bar{\mathbf{u}} = \{\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1}\}$ with period $N \in \mathbb{N}$ instead of a constant planting density. We have proven the existence, uniqueness and exponential stability of the steady state for system (1) in the case of a constant input $\bar{u}$. We now show that analogous properties hold in the case of a periodic input: for any N-periodic input trajectory $\bar{\mathbf{u}}$, there exists a unique N-periodic trajectory $\bar{\mathbf{x}} = \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{N-1}\}$ that is globally exponentially stable for any initial condition $x_0 \geq 0$.

Proposition 6 (Existence and uniqueness of periodic weed seed bank)**.** *Given the N-periodic input trajectory $\bar{\mathbf{u}} = \{\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1}\}$, the system* (1) *admits a unique N-periodic trajectory of non-negative states $\bar{\mathbf{x}} = \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{N-1}\}$, such that*

$$\begin{aligned} \bar{x}_{t+1} &= f(\bar{x}_t, \bar{u}_t) \qquad \forall t \in \{0, 1, \ldots, N-2\}, \\ \bar{x}_0 &= f(\bar{x}_{N-1}, \bar{u}_{N-1}). \end{aligned} \tag{4}$$

Proof. For $j \in \{0, 1, \ldots, N-1\}$, define $f_{\bar{u}_j}(x) := f(x, \bar{u}_j)$ and, denoting by $\circ$ function composition,

$$\phi(x) := f_{\bar{u}_{N-1}} \circ f_{\bar{u}_{N-2}} \circ \cdots \circ f_{\bar{u}_1} \circ f_{\bar{u}_0}(x).$$

Then, given $k \in \mathbb{N}$, the dynamics satisfies

$$x_{kN+j} = \underbrace{\phi \circ \cdots \circ \phi}_{k \text{ times}}(x_j) := \phi^k(x_j). \tag{5}$$

Function ϕ inherits key properties from f. In particular: $\phi(0) > 0$, because $f_{\bar{u}_0}(0) = \mu > 0$ and $f(\cdot, w)$ maps $(0, \infty)$ into $(0, \infty)$ for every w; ϕ is monotonically increasing and concave, because it is the composition of monotonically increasing and concave functions; moreover, $\lim_{x\to\infty} \phi'(x) = \delta^N \in (0, 1)$ by application of the chain rule to ϕ and the fact that $f(\cdot)$ is monotonically increasing with $\lim_{x\to\infty} f(x) = \infty$. Thus, ϕ admits a unique non-negative fixed point $\bar{x} = \phi(\bar{x})$, which is strictly positive.

We now initialize the dynamics at $\bar{x}_0 = \bar{x}$. By equation (5), we have $\bar{x}_{kN} = \phi^k(\bar{x}_0) = \bar{x}_0$. By applying the periodic input, for any $1 \leq j \leq N-1$, we have

$$\begin{aligned} \bar{x}_{kN+j} &= f_{\bar{u}_j} \circ \cdots \circ f_{\bar{u}_1} \circ f_{\bar{u}_0}(\bar{x}_{kN}) \\ &= f_{\bar{u}_j} \circ \cdots \circ f_{\bar{u}_1} \circ f_{\bar{u}_0}(\bar{x}_0) = \bar{x}_j. \end{aligned}$$

Hence, initialising system (1) at $\bar{x}_0$ and applying the N-periodic input $\bar{\mathbf{u}} = \{\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1}\}$ generates the unique N-periodic trajectory of non-negative states $\bar{\mathbf{x}} = \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{N-1}\}$ that satisfies the requirements in (4). ■

Proposition 7 (Global exponential stability of periodic weed seed bank)**.** *The N-periodic trajectory for system* (1) *discussed in Proposition 6 is exponentially stable, namely, there exist $\nu \in (0, 1)$ and $M > 0$ such that the solution x_t to the equation $x_{t+1} = f(x_t, \bar{u}_{t \bmod N})$ satisfies*

$$|x_t - \bar{x}_{t \bmod N}| \leq M\nu^t |x_0 - \bar{x}_0| \quad \forall x_0 \geq 0,\ t \in \{0, 1, \ldots\}.$$

Proof. Due to the properties of function ϕ outlined in the proof of Proposition 6, as discussed in Remark 1, we have

$$0 < \phi'(\bar{x}) < 1.$$

Also, since the functions ϕ and f share the same qualitative properties, by following the same reasoning as in the proof of Proposition 5 with f now replaced by ϕ, we can show that there exists $\nu \in (0, 1)$ such that, for any $t = kN$, $k \in \mathbb{N}$,

$$|x_t - \bar{x}_0| = |\phi^k(x_0) - \bar{x}_0| \leq \nu^k |x_0 - \bar{x}_0| = (\nu^{\frac{1}{N}})^t |x_0 - \bar{x}_0|.$$

Denote $\tilde{\nu} = \nu^{\frac{1}{N}} \in (0, 1)$. Due to the expression of f' shown in equation (2), there exists $0 < M_{\bar{u}_0} < \infty$ such that $0 < f'_{\bar{u}_0}(x) < M_{\bar{u}_0}$ for all x. Now, let $t = kN + 1$, $k \in \mathbb{N}$. Then, applying Lagrange's mean value theorem yields

$$\begin{aligned} |x_t - \bar{x}_1| &= |f_{\bar{u}_0}(\phi^k(x_0)) - f_{\bar{u}_0}(\bar{x}_0)| \\ &\leq M_{\bar{u}_0}\nu^k |x_0 - \bar{x}_0| = M_{\bar{u}_0}\tilde{\nu}^{-1}\tilde{\nu}^t |x_0 - \bar{x}_0|. \end{aligned}$$

Continuing by induction for $t = kN + j$, with $k \in \mathbb{N}$ and $j = 2, \ldots, N-1$, we can finally obtain a constant $\bar{M}$ such that

$$|x_{kN+j} - \bar{x}_j| \leq \bar{M}\tilde{\nu}^{kN+j} |x_0 - \bar{x}_0|,$$

for all $k \in \mathbb{N}$ and all $j \in \{0, 1, \ldots, N-1\}$. ■

Therefore, we can guarantee that for any periodic planting density trajectory $\bar{\mathbf{u}}$, the WSB converges to a unique periodic trajectory $\bar{\mathbf{x}}$ for any initial WSB $x_0 \geq 0$. In practice, this means that measurements of the WSB are not required to drive the system to this desired periodic trajectory, i.e., we do not need feedback control to reach this desired periodic trajectory. Thus, we can now focus on determining an optimal rotation and corresponding periodic WSB trajectory for the system, with the guarantee that any periodic trajectory is globally exponentially stable.

IV. Periodic Optimal Control Problem

Given the existence and stability guarantees outlined in the previous section, the optimal control problem consists in finding a sequence of inputs that minimizes the desired cost function, subject to periodic WSB dynamics.

Crop yield, in our case interpreted as economic gain, can be expressed in terms of crop and weed densities as a saturating function, which has the same form as the weed reproductive function [17], [23]. The maximum yield per plant is given by the constant vector $h \in (0, \infty)^n$, yield reduction by weeds constants $c \in (0, \infty)^n$, and crop-crop competition matrix $D \in \mathbb{R}^{n\times n}_{\geq 0}$. The cost per seed sown is associated with cost vector $p \in (0, \infty)^n$. At each time step t the cost can be written as total cost minus total yield:

$$g(x_t, u_t) = \sum_{i=1}^{n} \left(p_i u_{i,t} - \frac{h_i u_{i,t}}{1 + c_i x_t + D_i u_t} \right), \tag{6}$$

where we denote by v_i the ith element of a vector v, while D_i is the ith row of matrix D and $u_{i,t}$ is the ith element of the input vector u_t at the time step t. The cost function over a period of N time steps is therefore:

$$J(\mathbf{x}, \mathbf{u}) = \sum_{t=0}^{N-1} g(x_t, u_t). \tag{7}$$

The optimization problem is constrained by the WSB dynamics in system (1) and includes periodic constraints on the

WSB. The optimization problem can therefore be expressed as:

$$
\begin{aligned}
\min_{\mathbf{x},\mathbf{u}} \ & J(\mathbf{x},\mathbf{u}) \\
\text{s.t. } & x_{t+1} = f(x_t, u_t) && t \in \{0,1,\ldots,N-1\} \\
& x_t \geq 0,\ u_t \geq 0 && t \in \{0,1,\ldots,N-1\} \\
& x_0 = x_N
\end{aligned}
\tag{8}
$$

This optimization problem is a nonconvex nonlinear program (NLP), in which both the cost function J and dynamics f are nonconvex. In fact, the functional form of both J and f permits a reformulation of (8) as a nonconvex quadratically constrained quadratic program (QCQP). We solve this NLP or QCQP with gradient-based methods via IPOPT [25] with CasADi for algorithmic differentiation [26]. Since the problem is nonconvex, we cannot guarantee that a global minimum of (8) is found and we observe that the solution reported by IPOPT is sensitive to the initial guess for both NLP and QCQP formulations.

V. Numerical Examples

In this section, we explore possible solutions of the NLP, given different numerical parameters. The parameters are not meant to be biologically representative, instead, we aim to show how our methodology has monocultures and bicultures as possible solutions in a more general framework, and how the solutions correspond to exponentially stable periodic trajectories of the WSB. We consider two hypothetical crop plants and a WSB, characterized by the parameters displayed in Figure 3. The two crops have the same cost. Crop 1 is highly valuable, but it is strongly affected by weeds, and it does not suppress weed seed production. Crop 2 has no economic value, but it is not affected by weeds, whereas it does suppress weed seed production. We look for a rotation of (at most) six years that minimizes the cost function subject to the dynamics, resulting in the NLP formulated in (8). The optimal rotation critically depends on the interspecific crop-competition constants D_{12} and D_{21}. We consider three cases in Figure 3 (code available at: https://gitlab.tudelft.nl/mndejong/cdc-2024-optimal-crop-rotation-st-weed-dynamics). With no interspecific competition ($D_{12} = D_{21} = 0$, Figure 3 left), the optimal rotation $\bar{\mathbf{u}}^*$ is constant intercropping: the same positive crop planting densities are applied each year, allowing for simultaneous cash crop cultivation and weed suppression. The resulting WSB trajectory $\bar{\mathbf{x}}^*$ is also constant. Hence, the optimal period-6 solution is (also) the optimal steady-state solution. In general, an optimal crop rotation of period N may be composed of shorter period solutions by which N is divisible (including period 1 for steady-state). With strong interspecific crop-crop competition ($D_{12} = D_{21} = 10$, Figure 3 center), the optimal crop rotation $\bar{\mathbf{u}}^*$ alternates monocultures of weed-suppressive crop 2 with the intercropping of crops 1 and 2. The corresponding optimal WSB trajectory $\bar{\mathbf{x}}^*$ is now larger, and varies within the cycle. This solution is not composed of repetitions of shorter crop rotations. For even stronger interspecific crop-crop competition ($D_{12} = D_{21} = 20$, Figure 3 right), the optimal rotation $\bar{\mathbf{u}}^*$ consists of alternating monocultures: crop 2 is sown to manage the WSB, and crop 1 is sown for revenue. The corresponding WSB trajectory $\bar{\mathbf{x}}^*$ declines following cultivation of crop 2, and increases again following cultivation of crop 1. Here, the period-6 optimal crop rotation is composed of two period-3 rotations. The global exponential stability of the optimal WSB trajectory $\bar{\mathbf{x}}^*$ corresponding to the optimal crop rotation $\bar{\mathbf{u}}^*$, which was proven in Proposition 7, implies the convergence of any WSB trajectory to $\bar{\mathbf{x}}^*$; see e.g. Figure 4.

VI. Concluding Remarks

We presented a novel approach to find crop rotations that are optimal in the long run, in the presence of an annual WSB. We show that any feasible (periodic) crop rotation results in a periodic trajectory for the WSB population. We also show how to formulate and solve the problem as a NLP. Interestingly, monoculture solutions turn out to be optimal only in the presence of very strong interspecific competition between the crops. Clearly, other reasons in favour of monoculture cultivation are not embedded in our optimization problem, such as the fact that most operations are cheaper for monocultures. Moreover, competition is not the only way that a crop rotation affects the WSB; agricultural operations associated with each crop type typically correspond to unique parameters in the transition function (as is the basis of matrix models, e.g. [13], [14]). A more involved mixed-integer NLP formulation would be required to both capture the continuous aspects of density-dependent plant-plant competition, and the categorical aspects of costs and WSB dynamics associated with each crop type. Crucially, the stability guarantees outlined in Section III still hold for such a formulation.

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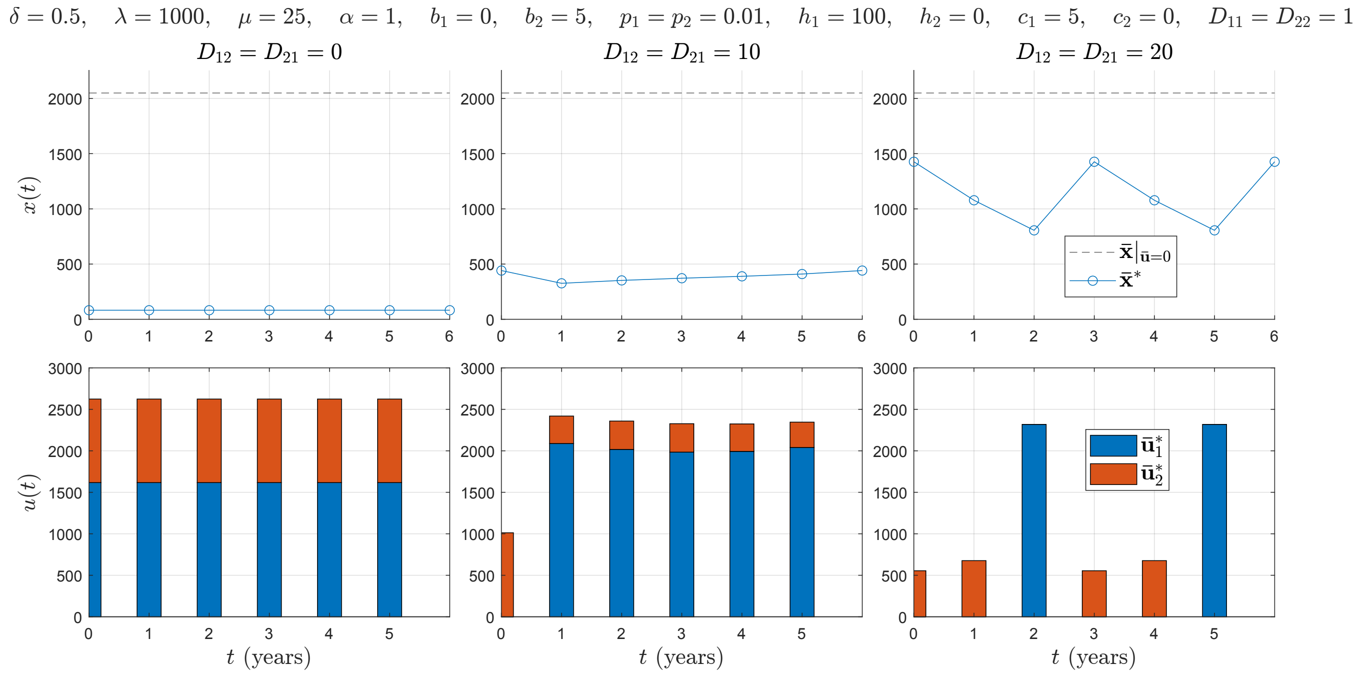

Fig. 3. Optimal weed orbit $\bar{\mathbf{x}}^*$ (top) and crop rotation cycle $\bar{\mathbf{u}}^*$ (bottom) for the system in Section V with varying interspecific crop-crop competition constants D_{12} and D_{21}. Crop 1 is valuable but weed-sensitive, crop 2 is worthless but weed suppressive. Low crop-crop competition results in steady-state intercropping (left), whereas high crop-crop competition results in rotations of monocultures (right), and intermediate values result in alternated intercropping and monocultures (middle). The code for all simulations is available at: https://gitlab.tudelft.nl/mndejong/cdc-2024-optimal-crop-rotation-st-weed-dynamics.

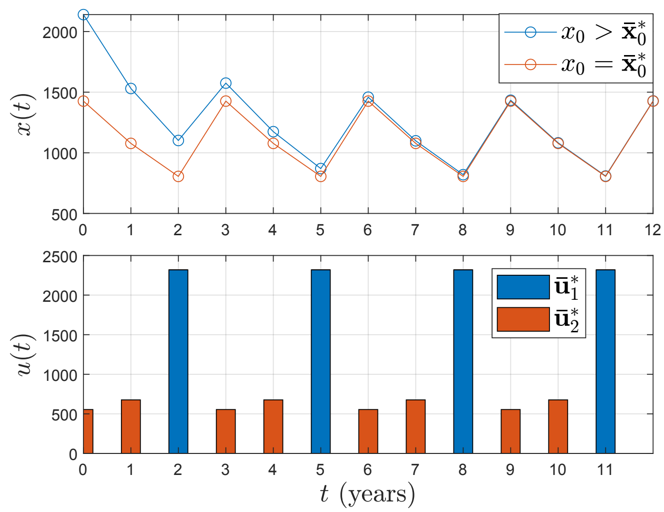

Fig. 4. When repeating the optimal crop rotation $\bar{\mathbf{u}}^*$ corresponding to Figure 3, right (bottom), the optimal WSB trajectory $\bar{\mathbf{x}}^*$ displays the same periodic trajectory (top, red). Any WSB trajectory starting outside of that periodic orbit (top, blue) converges to it. This result is true in general, for any periodic set of inputs (see Proposition 7). Code available at: https://gitlab.tudelft.nl/mndejong/cdc-2024-optimal-crop-rotation-st-weed-dynamics.

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