The Limit Random Variable W of a Branching Process

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Februari 6, 2005

Abstract

A formal Taylor series approach for the generating function of the limit random variable W of a branching process is presented. The framework is applicable to any production distribution function for which all moments exist. The Taylor coefficients show an interesting relation to Gaussian polynomials. The application of the formal series approach to the Poisson production function leads to (a) a modular-like functional equation for the moment generating function of W and (b) two different series for the probability distribution function of W.

1 The Limit Random Variable W

We consider a branching process in which each offspring produces a number of items independent from the others but with same distribution. Let X_k denote the total number of items produced in generation k and let the i.i.d. production in any generation be specified by the non-negative discrete random variable Y with $E[Y] = \mu > 1$. The set X_k describes the evolution of a branching process over the generations k. The scaled random variables $\{W_k\}_{k\geq 1}$ defined by $W_k = \frac{X_k}{\mu^k}$ constitute a martingale process with characteristic property that $E[W_k] = E[X_0]$ for all k. It is known [6] that the limit variable $W = \lim_{k\to\infty} W_k$ exists if $\mu > 1$. In the sequel, we confine to the case where $X_0 = 1$ and, mostly, $\mu > 1$. The moment generating function (mgf) $\chi_W(t) = E\left[e^{-tW}\right]$ obeys the functional equation for $\mathrm{Re}(t) \geq 0$,

$$\chi_W(t) = \varphi_Y\left(\chi_W\left(\frac{t}{\mu}\right)\right) \tag{1}$$

where $\varphi_Y(z) = E\left[z^Y\right]$ is production generating function. The limit $t \to \infty$ exists and $\lim_{t\to\infty} \chi_W(t) = \Pr[W = 0] = \pi_0$ is the extinction probability which obeys, as follows from (1), the well-known equation $\pi_0 = \varphi_Y(\pi_0)$.

The main motivation for this study was the computation of the limit random variable W that appeared in the distribution of the hopcount or distance between two arbitrary nodes in graphs with finite variance degree distributions [7, 12]. Although many theoretical results are available (see e.g. [6],[10],[3]), less effort has been devoted to compute the mgf $\chi_W(t)$ and the probability density function $f_W(x)$ of W.

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This paper presents a formal Taylor series approach for the mgf $\chi_W(t)$ of branching processes with production generating functions $\varphi_Y(z)$ that possess a Taylor series around $z = 1$; i.e. all moments of Y exist. A recursion relation for the Taylor coefficients is given that, on modern computers, allows the computation to any desired order. By computing the Taylor coefficients of the mgf $\chi_W(t)$ explicitly, we found a remarkable appearance of Gaussian-like polynomials, also called q-binomials [5, 8].

The major part of the article is devoted to a Poisson production function for which we present several results that culminate in two exact series for the probability density function $f_{W;Po}(x)$ of W_{Po}. In order to make the dependence on the Poisson production rate μ explicit, we sometimes denote the corresponding mgf by $\chi_{W;Po}(\mu|t)$. Apart from the series for $f_{W;Po}(x)$ and from the well-known functional equation

$$\chi_{W;Po}(\mu|t) = \exp\left(\mu\left(\chi_{W;Po}\left(\mu\left|\frac{t}{\mu}\right.\right) - 1\right)\right) \tag{2}$$

we found an intriguing *theta-function or modular*-like functional equation

$$\chi_{W;Po}(\mu|t) = \pi_0 \chi_{W;Po}\left(\mu\pi_0\left|-\frac{F_\mu}{\pi_0 \beta t^\beta}\right.\right) \tag{3}$$

where $\beta = \frac{(1-\pi_0)\mu}{\log \mu} - 1$ and F_μ is a newly appearing parameter that can be solved from (3) as illustrated in Section 4. The methods presented indicate that, for entire generating functions $\chi_W(t)$, a same type of exact series for $f_W(x)$ may exist. The duality principle [2, pp. 164] states that a Poisson branching process with mean μ has, conditional on extinction, the same distribution of the Poisson branching process with mean $\mu\pi_0$. This duality principle seems related to our *modular*-like functional equation (3), however, not in any obvious way. It would be of interest to find a physical or probabilitistic interpretation of (3) and F_μ.

The confinement to a Poisson production function is not that narrow as it first appears. The geometric distribution function has as Taylor coefficients around $z = 1$, $u_k = \mu^k$, while a Poisson distribution possesses as Taylor coefficients around $z = 1$, $u_k = \frac{\mu^k}{k!}$, which are, from an analytic point of view, two basic types of series coefficients. Moreover, as illustrated in Figure 3, the geometric distribution leads to a definitely distinct mgf $\chi_W(t)$ and pdf $f_W(x)$ than the Poisson distribution.

The next subsection briefly reviews the well-known functional equation for W. Section 2 presents the Taylor series for $\chi_W(t)$ while Section 3 gives the asymptotic series for $\chi_W(t)$. Section 4 applies the framework to a Poisson production.

2 Taylor Expansions of the Generating Functions

If $f(z)$ has a Taylor series around z_0,

$$f(z) = \sum_{k=0}^{\infty} f_k(z_0)(z - z_0)^k \qquad \text{with } f_k(z_0) = \frac{1}{k!} \left.\frac{d^k f(z)}{dz^k}\right|_{z=z_0}$$

then the general relation where $G(z)$ is analytic around $f(z_0)$ is

$$G(f(z)) = G(f(z_0)) + \sum_{m=1}^{\infty} \left(\sum_{k=1}^{m} \frac{1}{k!} \left.\frac{d^k G(p)}{dp^k}\right|_{p=f(z_0)} s[k,m]_{f(z)}(z_0) \right) (z - z_0)^m \tag{4}$$

where the characteristic coefficient [13] of a complex function $f(z)$ is

$$s[k,m]_{f(z)}(z_0) = \sum_{\sum_{i=1}^{k} j_i = m; j_i > 0} \prod_{i=1}^{k} f_{j_i}(z_0)$$

which obeys the recursion relation

$$\begin{aligned} s[1,m]_{f(z)}(z_0) &= f_m(z_0) \\ s[k,m]_{f(z)}(z_0) &= \sum_{j=1}^{m-k+1} f_j\ (z_0)\, s[k-1,m-j]_{f(z)}(z_0) \qquad (k>1) \end{aligned} \tag{5}$$

2.1 Expansion of $\chi_W(t)$ around $t=0$

If $\varphi_Y(z)$ is analytic inside a circle with radius $R_Y > 0$ centered at $z=1$, then the Taylor series around $z_0 = 1$,

$$\varphi_Y(z) = 1 + \sum_{k=0}^{\infty} u_k (z-1)^k$$

with $u_1 = \mu$ and for $k > 1$,

$$u_k = \frac{1}{k!} \left. \frac{d^k \varphi_Y(z)}{dz^k} \right|_{z=1} \tag{6}$$

converges for all $|z-1| < R_Y$. The definition $\chi_W(t) = E\left[e^{-tW}\right]$ implies that the maximum value of $|\chi_W(t)|$ inside and on a circle with radius r around the origin is attained at $\chi_W(-r)$. The functional equation (1) then shows that $\chi_W(t)$ is analytic inside a circle around $t=0$ with radius R_W for which $\chi_W\left(-\frac{R_W}{\mu}\right) < 1 + R_Y$. Since $\chi_W(0) = 1$, $\chi_W(t)$ is convex and decreasing for real t and $R_Y > 0$, there exists such a non-zero value of R_W. This implies that the Taylor series

$$\chi_W(t) = 1 + \sum_{k=1}^{\infty} \omega_k t^k \tag{7}$$

converges around $t=0$ for $|t| < R_W$. Since $\chi_W(t) = E\left[e^{-tW}\right] = \sum_{k=0}^{\infty} (-1)^k E\left[W^k\right] \frac{t^k}{k!}$, there holds for $k > 0$ that

$$\omega_k = \frac{1}{k!} \left. \frac{d^k \chi_W(t)}{dt^k} \right|_{t=0} = \frac{(-1)^k E\left[W^k\right]}{k!} \tag{8}$$

and $\omega_1 = -E[W] = -1$. In other words, the coefficients ω_k are alternating because $E\left[W^k\right] > 0$.

We will expand the functional equation (1) around $t_0 = 0$. Using (4), the right hand side of (1) has the Taylor series

$$\varphi_Y\left(\chi_W\left(\frac{t}{\mu}\right)\right) = 1 + \sum_{m=1}^{\infty} \left(\sum_{k=1}^{m} \frac{1}{k!} \left. \frac{d^k \varphi_Y(p)}{dp^k} \right|_{p=1} s[k,m]_{\chi_W}(0) \right) \left(\frac{t}{\mu}\right)^m$$

Equating corresponding powers in t yields for $m > 0$,

$$\omega_m = \frac{1}{\mu^m} \sum_{k=1}^{m} u_k\, s[k,m]_{\chi_W}(0)$$

from which the recursion for ω_m follows because $s[1,m] = \omega_m$ by (5),

$$\omega_m = \frac{1}{\mu^m - \mu} \sum_{k=2}^{m} u_k \, s[k,m]_{\chi_W}(0) \tag{9}$$

In summary, only if the probability generating function of the production process $\varphi_Y(z)$ is analytic in some region around $z = 1$ (which implies that all derivatives u_k at $z = 1$ exist), the recursion relation (9) determines all derivatives ω_m of $\chi_W(t)$ around $t = 0$, for $\mu \neq 1$. For $\mu = 1$, we obtain with $s[2,m]_{\chi_W}(0) = \sum_{j=1}^{m-1} \omega_j \omega_{m-j}$ for $m > 1$,

$$\omega_{m-1} = -\frac{1}{u_2} \left(\sum_{j=2}^{m-1} \omega_j \omega_{m-j} + \sum_{k=3}^{m} u_k \, s[k,m]_{\chi_W}(0) \right)$$

For $\mu \neq 1$, the first few values are, with $\omega_1 = -1$,

$$\begin{aligned}
\omega_2 &= \frac{u_2}{\mu(\mu-1)} \\
\omega_3 &= \frac{2\,u_2^2 + \mu(\mu-1)u_3}{\mu^2\,(\mu-1)\,(\mu^2-1)} \\
\omega_4 &= \frac{(\mu+5)\;u_2^3 + \mu\,(\mu-1)\,(3\mu+5)\;\;u_2\,u_3 + \;\mu^2\;(\mu-1)\,\left(\mu^2-1\right)\,u_4}{\mu^3\;(\mu-1)\,(\mu^2-1)\,(\mu^3-1)} \\
\omega_5 &= \frac{2\left(2\mu^2+3\mu+7\right)\;u_2^4 + \mu\,(\mu-1)\left(3\mu^3+14\mu^2+20\mu+21\right)\;u_2^2\,u_3}{\mu^4\;(\mu-1)\,(\mu^2-1)\,(\mu^3-1)\,(\mu^4-1)} \\
&\quad + \frac{2\mu\,(\mu-1)\left(\mu^2-1\right)\left(2\mu^2+2\mu+3\right)\;u_2\,u_4 + 3\mu^2\;(\mu-1)\left(\mu^3-1\right)\;u_3^2}{\mu^4\;(\mu-1)\,(\mu^2-1)\,(\mu^3-1)\,(\mu^4-1)} \\
&\quad + \frac{\mu^3\;(\mu-1)\left(\mu^2-1\right)\left(\mu^3-1\right)\;u_5}{\mu^4\;(\mu-1)\,(\mu^2-1)\,(\mu^3-1)\,(\mu^4-1)}
\end{aligned}$$

from which, by inspection, the general structure arises

$$\omega_m(\mu) = \frac{(-1)^m \sum_{k=0}^{\binom{m}{2}-1} a_k(m)\,\mu^k}{\mu^{m-1} \prod_{j=1}^{m-1} (\mu^j - 1)} \tag{10}$$

where $a_k(m)$ are, in general, rather cumbersome coefficients in the u_k with $a_{\binom{m}{2}-1}(m) = -a_{\binom{m}{2}-2}(m) = u_m$. This form bears resemblance to Gaussian polynomials [5, 8].

2.2 Computation of $\chi_W(t)$

If $\chi_W(t)$ is not known in closed form, the interest of the Taylor series (7) of $\chi_W(t)$ around $t = 0$ lies in the fast convergence for small values of $|t| < 1$. The recursion (9) for the Taylor coefficients ω_k enables the computation of $\chi_W(t)$ for $|t| < 1$ to any desired degree of accuracy. The functional equation $\chi_W(t) = \varphi_Y\left(\chi_W\left(\frac{t}{\mu}\right)\right)$ extends the t-range to the entire complex plane. For large values of t and in particular for negative real t, $\chi_W(t)$ is best computed from $\chi_W\left(\frac{t}{\mu^{[\log_\mu |t|]+1}}\right)$ after $[\log_\mu |t|] + 1$ functional iteratives of (1). Indeed, since $\mu > 1$ such that $\frac{t}{\mu^{[\log_\mu |t|]+1}} < 1$, the Taylor series (7) provides an accurate start value $\chi_W\left(\frac{t}{\mu^{[\log_\mu |t|]+1}}\right)$ for this iterative scheme.

3 The Asymptotic Behavior of $\chi_W(t)$

The convexity of $\chi_W(t) = E\left[e^{-tW}\right]$ implies that $\chi'_W(t) \le 0$ for all real t and that $\chi'_W(t)$ is increasing in t. We know that $\chi'_W(0) = -E[W] = -1$. Since $\lim_{t\to\infty} \chi_W(t) = \pi_0$, it follows that $\lim_{t\to\infty} \chi'_W(t) = 0$. The following Lemma 1 is a little more precise.

Lemma 1 *$\chi'_W(t) = o(t^{-1})$ for $t \to \infty$.*

Proof: The derivative of the functional equation (1) is $\mu\chi'_W(\mu t) = \varphi'_Y(\chi_W(t))\,\chi'_W(t)$. By iteration, we have

$$\mu^K \chi'_W\left(\mu^K t\right) = \chi'_W(t) \prod_{j=0}^{K-1} \varphi'_Y\left(\chi_W\left(\mu^j t\right)\right)$$

Since $\chi_W(t) \in [\pi_0, 1]$ for real $t \ge 0$, then $\varphi'_Y(\pi_0) \le \varphi'_Y\left(\chi_W\left(\mu^j t\right)\right) \le \mu$ for any j. In addition, it is well-known in the theory of branching processes that, if $\mu = \varphi'_Y(1) > 1$, then there are two zeros π_0 and 1 of $f(z) = \varphi_Y(z) - z$ in $z \in [0,1]$. By Rolle's Theorem applied to the continuous function $f(z) = \varphi_Y(z) - z$, there exist an $\xi \in (\pi_0, 1)$ for which $f'(\xi) = 0$. Equivalently, $\varphi'_Y(\xi) = 1$ and $\xi > \pi_0$. Since $\varphi'_Y(z)$ is monotonously increasing in $z \in [0,1]$, we have that $\varphi'_Y(0) = \Pr[Y=1] \le \varphi'_Y(\pi_0) < 1$. Since $\chi_W(t)$ is continuous and monotone decreasing, there exists an integer K_0 such that $\varphi'_Y\left(\chi_W\left(\mu^j t\right)\right) < 1$ for $j > K_0$ and any $t > 0$. Hence,

$$\lim_{K\to\infty} \prod_{j=0}^{K-1} \varphi'_Y\left(\chi_W\left(\mu^j t\right)\right) = \prod_{j=0}^{K_0-1} \varphi'_Y\left(\chi_W\left(\mu^j t\right)\right) \prod_{j=K_0}^{\infty} \varphi'_Y\left(\chi_W\left(\mu^j t\right)\right) \to 0$$

and, for any finite $t > 0$, $\mu^K \chi'_W\left(\mu^K t\right) \to 0$ for $K \to \infty$ which implies the lemma. $\square$

Lemma 1 is, for large t, equivalent to $|\chi'_W(t)| \le Ct^{-1-\beta}$ for some real $\beta > 0$ and where C is a finite positive real number. Lemma 1 thus suggests to consider

$$\chi'_W(t) = -g_\mu(t)\, t^{-\beta-1} \tag{11}$$

where $0 < g_\mu(t) \le C$ on the real positive $t-$axis.

Lemma 2 *If $\varphi'_Y(\pi_0) > 0$ and $\mu > 1$, then*

$$F_\mu = \lim_{t\to\infty} g_\mu(t) \tag{12}$$

exists, is finite and strict positive.

Proof: We first use **(a)** the convexity of any pgf $\chi_W(t)$ implying that $\chi''_W(t) \ge 0$ for all t and we then invoke **(b)** the functional equation (1) of $\chi_W(t)$.

(a) The function $g_\mu(t) = -\chi'_W(t)\, t^{\beta+1}$ is differentiable, thus continuous, and has for real $t > 0$ only one extremum at $t = \tau$ obeying $\tau = \frac{-\chi'_W(\tau)}{\chi''_W(\tau)}(\beta+1) > 0$. Since $\chi'_W(0) = -1$ implying that $g_\mu(t) = t^{\beta+1}(1 + o(t)$ as $t \downarrow 0$ or that $g_\mu(t)$ is initially monotone increasing in t, the extremum at $t = \tau$ is a maximum. The derivative of $g_\mu(t) = -\chi'_W(t)\, t^{\beta+1}$ is with (11)

$$g'_\mu(t) = \frac{\beta+1}{t} g_\mu(t) - \chi''_W(t)\, t^{\beta+1}$$

such that, for τ finite, $\max g_\mu(t) = \frac{\tau^{\beta+2}}{\beta+1}\chi_W''(\tau)$. Since $\chi_W''(t) \geq 0$ for all t, we also obtain the inequality

$$g_\mu'(t) \leq \frac{\beta+1}{t} g_\mu(t) \leq \frac{\beta+1}{t} C$$

from which $\lim_{t\to\infty} g_\mu'(t) \leq 0$. Hence, $g_\mu(t)$ is not increasing for $t \to \infty$.

(b) Substitution of (11) in the derivative of the functional equation (1) yields

$$g_\mu(t) = \varphi_Y'\left(\chi_W\left(\frac{t}{\mu}\right)\right) g_\mu\left(\frac{t}{\mu}\right)\mu^\beta \tag{13}$$

Since $\varphi_Y'\left(\chi_W\left(\frac{t}{\mu}\right)\right) \geq \varphi_Y'(\pi_0) > 0$ (restriction of this Lemma), there holds with $A = \varphi_Y'(\pi_0)\mu^\beta > 0$ for all $t > 0$ that

$$g_\mu(t) \geq A g_\mu\left(\frac{t}{\mu}\right)$$

For $t < \tau$, $g_\mu(t)$ is shown in **(a)** to be monotone increasing which requires that $A \geq 1$ for $\mu > 1$. But, since the inequality with $A \geq 1$ holds for all $t > 0$, we must have that $\tau \to \infty$. Hence, $g_\mu(t)$ is continuous and strict increasing for all $t \geq 0$ with a maximum at infinity which proves the existence of a unique limit $F_\mu \leq C$.

If $F_\mu = 0$, the suggestion (11) is not correct implying that $\chi_W'(t)$ decreases faster than any power of t^{-1}. The proof of Lemma 1 indicate that his case can occur if $\varphi_Y'(\pi_0) = 0$. $\square$

In fact, $A = 1$. For, when passing to the limit $t \to \infty$ in (13) using Lemma 2, we obtain

$$\mu^{-\beta} = \varphi_Y'(\pi_0)$$

which determines the exponent $\beta \geq 1$ as

$$\beta = -\frac{\log \varphi_Y'(\pi_0)}{\log \mu} \tag{14}$$

Dubuc [3, Theorem 1.1] has derived (14) earlier based on an entirely different method. Applying (4) to $G(f^{-1}(z))$, the exact series of $\varphi_Y'(\pi_0)$ in terms of μ can be derived. For small $\mu \to 1$, we obtain

$$\varphi_Y'(\pi_0) = 2 - \mu + \frac{u_3}{u_2^2}(\mu-1)^2 + 2\frac{u_3^2 - u_2 u_4}{u_2^4}(\mu-1)^3 + O\left((\mu-1)^4\right)$$

which shows that $\beta \to 1$ if $\mu \to 1$.

3.1 Asymptotic series for $g_\mu(t)$

We now give the precise asymptotic series for $g_\mu(t)$ in case $\varphi_Y'(\pi_0) > 0$ and $\mu > 1$. Integrating both sides of (11) gives

$$\chi_W(t) = \pi_0 + \int_t^\infty g_\mu(u)\, u^{-\beta-1} du \tag{15}$$

Iterating (13) yields

$$g_\mu(t) = \frac{g_\mu\left(t\mu^K\right)\mu^{-K\beta}}{\prod_{j=0}^{K-1} \varphi_Y'\left(\chi_W\left(t\mu^j\right)\right)} = g_\mu\left(t\mu^K\right) \prod_{j=0}^{K-1} \frac{\varphi_Y'(\pi_0)}{\varphi_Y'\left(\pi_0 + \int_{t\mu^j}^\infty g_\mu(u)\, u^{-\beta-1} du\right)}$$

where (15) is used. The limit $K \to \infty$ gives with Lemma 2,

$$g_\mu(t) = F_\mu \prod_{j=0}^{\infty} \frac{\varphi_Y'(\pi_0)}{\varphi_Y'\left(\pi_0 + \int_{t\mu^j}^{\infty} g_\mu(u)\, u^{-\beta-1} du\right)} \tag{16}$$

Since $0 \le g_\mu(t) \le F_\mu$ for $t \ge 0$, we have

$$\int_{\mu^j t}^{\infty} g_\mu(u)\, u^{-\beta-1} du \le F_\mu \frac{\left(\mu^{-\beta}\right)^j}{t^\beta \beta} = \frac{F_\mu}{\beta} \frac{\left(\varphi_Y'(\pi_0)\right)^j}{t^\beta}$$

For large t, expanding (16) gives

$$\begin{aligned} g_\mu(t) &= F_\mu \exp\left(-\sum_{j=0}^{\infty} \log \frac{\varphi_Y'\left(\pi_0 + \int_{t\mu^j}^{\infty} g_\mu(u)\, u^{-\beta-1} du\right)}{\varphi_Y'(\pi_0)} \right) \\ &\sim F_\mu \left(1 - \frac{\varphi_Y''(\pi_0)\, F_\mu}{t^\beta \beta} \frac{\mu^{2\beta}}{\mu^\beta - 1} + O\left(t^{-2\beta}\right) \right) \end{aligned}$$

which suggests that $g_\mu(t)$ has an asymptotic series of the form

$$g_\mu(t) = \sum_{k=0}^{N-1} g_k t^{-k\beta} + O\left(t^{-N\beta}\right) \tag{17}$$

with $g_0 = F_\mu$. Using (15) with (17) gives

$$\chi_W(t) = \pi_0 + \sum_{k=1}^{N} \frac{g_{k-1}}{\beta k} \left(t^{-\beta}\right)^k + O\left(t^{-(N+1)\beta}\right) \equiv \sum_{k=0}^{N} c_k z^k + O\left(z^{N+1}\right) \tag{18}$$

where $c_k = \frac{g_{k-1}}{\beta k}$ and $c_0 = \pi_0$ and $z = t^{-\beta}$. This series for $\chi_W(t)$ can be regarded as a truncated Taylor series at $t \to \infty$. Earlier, Dubuc [3, Theorem 2] has shown that the term by term (inverse) Laplace transform of the series (18) exists. Expanding the functional equation (1) in a Taylor series around $t \to \infty$ using (4) with $z - z_0 = t^{-\beta}$ and $z_0 \to \infty$, gives

$$\varphi_Y\left(\chi_W\left(\frac{t}{\mu}\right)\right) = \varphi_Y(\pi_0) + \sum_{m=1}^{\infty} \left(\sum_{k=1}^{m} \frac{1}{k!} \left.\frac{d^k \varphi_Y(p)}{dp^k}\right|_{p=\pi_0} s[k,m]_{\chi_W}(\infty) \right) \left(\mu^\beta t^{-\beta}\right)^m$$

We denote the Taylor coefficient of the production generating function $\varphi_Y(z)$ around $z = \pi_0$ with $v_0 = \varphi_Y(\pi_0) = \pi_0$ and $v_1 = \varphi_Y'(\pi_0) = \mu^{-\beta} < 1$ by (14). After equating corresponding powers of $t^{-\beta}$ in (1) leads for $m > 0$ to

$$c_m = \mu^{m\beta} \sum_{k=1}^{m} v_k\, s[k,m]_{\chi_W}(\infty)$$

from which the recursion for c_m and $m > 1$ follows using $v_1 = \mu^{-\beta}$ as

$$c_m = \frac{1}{\mu^{-m\beta} - \mu^{-\beta}} \sum_{k=2}^{m} v_k\, s[k,m]_{\chi_W}(\infty)$$

For $m = 1$ for which $c_1 = \mu^\beta v_1\, s[1,1]_{\chi_W}(\infty) = c_1$, we obtain an identity for $c_1 = \frac{F_\mu}{\beta}$ where the parameter F_μ appears as a natural measure or property for $\chi_W(t)$. This recursion for c_m ($\chi_W(t)$

expanded around $t_0 = \infty$) is formally the same as (9) for ω_m ($\chi_W(t)$ expanded around $t_0 = 0$) if we replace $u_k \to v_k$, $\mu \to \mu^{-\beta}$ and the initial start value $\omega_1 = -1$ by $c_1 = \frac{F_\mu}{\beta}$. Hence, the Gaussian polynomial-like form (10) for ω_m translates to

$$c_m = \frac{\sum_{k=0}^{\binom{m}{2}-1} \widetilde{a_k} \left(\mu^{-\beta}\right)^k}{\left(\mu^{-\beta}\right)^{m-1} \prod_{j=1}^{m-1} \left(\left(\mu^{-\beta}\right)^j - 1\right)} \left(\frac{F_\mu}{\beta}\right)^m$$

where the coefficients $\widetilde{a_k}$ are functions of v_k. Provided the set of u_k and the set of v_k together with F_μ is known, the computation of the Taylor coefficients of $\chi_W(t)$ around $t = 0$ and $t = \infty$ is equally intensive.

The analysis shows first of all that all Taylor coefficients around $t \to \infty$ exist and that $\chi_W(t)$ is analytic around $t \to \infty$ such that the asymptotic series (18) exists for all N. Hence, we may termwise integrate that series within the radius of convergence. If that radius is infinite, a series for probability density function $f_W(x)$ is obtained from

$$f_W(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \chi_W(t)\, e^{xt} dt \tag{19}$$

An example is the branching process for a Poisson production function given in Section 4.6. Second, if there is a simple relation (independent of k) between $u_k = \frac{1}{k!} \left.\frac{d^k \varphi_Y(z)}{dz^k}\right|_{z=1}$ and $v_k = \frac{1}{k!} \left.\frac{d^k \varphi_Y(z)}{dz^k}\right|_{z=\pi_0}$ such as in the case of a Poisson production function $\varphi_{Y;Po}(z) = e^{\mu(z-1)}$ where $u_{k;Po} = \frac{\mu^k}{k!}$ and $v_{k;Po} = \pi_0 \frac{\mu^k}{k!} = \pi_0 u_{k;Po}$, a new, modular-like functional equation (3) is obtained as shown in Section 4.4. Finally, if $\mu > 1$, $\varphi_Y(z)$ has two fixed points $z = 1$ and $z = \pi_0$. The theory of automorphisms [9] illustrates the importance of fixed points. Hence, it is not surprising that the two expansions around the two fixed points of $\varphi_Y(z)$ play a characteristic role.

4 The Poisson Distribution

4.1 The Functional Equation of $\chi_{W;Po}(t)$

For a Poisson production function with generating function $\varphi_{Y;Po}(z) = e^{\mu(z-1)}$, the functional equation (1) of $\chi_{W;Po}(t)$ is given in (2). By taking the logarithm of the functional equation (2), we obtain

$$\log\left(\chi_{W;Po}(t)\right) = \mu\left(\chi_{W;Po}\left(\frac{t}{\mu}\right) - 1\right) \tag{20}$$

while the logarithmic derivative of the functional equation (2) is

$$\chi'_{W;Po}(t) = \chi_{W;Po}(t)\, \chi'_{W;Po}\left(\frac{t}{\mu}\right) \tag{21}$$

4.2 Properties of $\chi_{W;Po}(t)$

Theorem 3 $\chi_{W;Po}(t)$ *is an entire function.*

An entire function has no singularities in the finite complex plane. Consequently, any Taylor series around a finite point has an infinite radius of convergence implying that the Taylor series

$$\chi_{W;Po}(t) = 1 + \sum_{m=1}^{\infty} \omega_{m;Po}(\mu)\, t^m \tag{22}$$

converges for all t.

Proof: Suppose that $\chi_{W;Po}(t)$ possesses a singularity at $t = t_s$ and that t_s is the singularity with the smallest modulus. Moreover, $t_s > 0$ because $\chi_{W;Po}(t)$ is analytic in a region around $t = 0$ as $\varphi_Y(z) = e^{\mu(z-1)}$ is an entire function. Since the precise type of singularity is not relevant, but only its position matters, we confine ourselves here to a simple pole. Then, we may write

$$\chi_{W;Po}(t) = \frac{\alpha}{t - t_s} + g(t)$$

where $g(t)$ is analytic for $|t| \leq |t_s|$ and where α is a complex number. Introduced into the functional equation (2),

$$\chi_{W;Po}(t) = \exp\left(\mu\left(\frac{\alpha}{\frac{t}{\mu} - t_s} + g\left(\frac{t}{\mu}\right) - 1\right)\right)$$

shows that $\chi_{W;Po}(t)$ is analytic for $|t| < \mu|t_s|$. Since $\mu > 1$, the initial assumption that $\chi_{W;Po}(t)$ has a singularity at t_s leads to a contradiction. Hence, $\chi_{W;Po}(t)$ cannot have singularities in the finite complex plane. $\square$

Corollary 1 $\chi_{W;Po}(t)$ *does not possess zeros in the finite complex plane.*

Theorem 3 applied to the logarithm (20) of the functional equation indicates that also $\log\left(\chi_{W;Po}(t)\right)$ is an entire function which immediately implies Corollary 1. The proof also follows from Jensen's theorem [11, sec. 3.61]. This type of entire function can be written as $\chi_{W;Po}(t) = \exp(h(t))$, where $h(t)$ is also an integral function. An entire function $h(t)$ is of finite order ρ [11, pp. 248] if $h(t) = O\left(e^{t^{\rho+\epsilon}}\right)$ for any arbitrary small, but positive ϵ.

Corollary 2 $\chi_{W;Po}(t)$ *is an entire function of infinite order.*

It is readily verified from (20) that, for large real t, $\chi_{W;Po}(-t)$ cannot be of the form $\chi_{W;Po}(-t) \sim Ke^{t^\rho}$ for finite ρ.

The order ρ of the entire function $\chi_{W;Po}(t)$ is determined [11, pp. 253] by $\frac{1}{\rho} = \lim_{m\to\infty} \frac{-\log \omega_{m;Po}(\mu)}{m \log m}$. Corollary 2 shows that, for all finite μ and any finite, positive a, the coefficients $\omega_{m;Po}(\mu)$ tend slower than $\frac{a}{m!}$ to zero for large m, but since $\chi_{W;Po}(t)$ has infinite radius, the $\omega_{m;Po}(\mu)$ tend faster than a^{-m} to zero.

4.3 The Taylor series of $\chi_{W;Po}(t)$

The Taylor coefficients of $\chi_{W;Po}(t)$ in (22) for a Poisson production function follow from (9) with $u_k = \frac{\mu^k}{k!}$ as

$$\omega_{m;Po}(\mu) = \frac{(-1)^m \mu^{m-1} \Omega_m(\mu)}{m! \prod_{j=1}^{m-1} (\mu^j - 1)} \tag{23}$$

with, for $m \geq 1$, the polynomial $\Omega_m(\mu) = \sum_{k=0}^{\binom{m-1}{2}} b_k(m)\mu^k$. Specifically,

$$
\begin{aligned}
\Omega_1(\mu) &= 1 \\
\Omega_2(\mu) &= 1 \\
\Omega_3(\mu) &= 2+\mu \\
\Omega_4(\mu) &= 6+6\,\mu+5\,\mu^2+\mu^3 \\
\Omega_5(\mu) &= 24+36\,\mu+46\,\mu^2+40\,\mu^3+24\mu^4+9\,\mu^5+\mu^6 \\
\Omega_6(\mu) &= 120+240\mu+390\,\mu^2+480\,\mu^3+514\,\mu^4+416\,\mu^5+301\,\mu^6+160\mu^7+64\,\mu^8+14\,\mu^9+\mu^{10} \\
\Omega_7(\mu) &= 720+1800\,\mu+3480\,\mu^2+5250\,\mu^3+7028\,\mu^4+8056\,\mu^5+8252\,\mu^6+7426\,\mu^7+5979\,\mu^8 \\
&\quad +4208\,\mu^9+2542\,\mu^{10}+1295\,\mu^{11}+504\,\mu^{12}+139\,\mu^{13}+20\,\mu^{14}+\mu^{15}
\end{aligned}
$$

Further, by inspection, some coefficients are found explicitly,

$$
\begin{aligned}
b_{\binom{m-1}{2}}(m) &= 1 \\
b_{\binom{m-1}{2}-1}(m) &= \binom{m}{2}-1 = \mathcal{S}_m^{(m-1)}-1 \\
b_{\binom{m-1}{2}-2}(m) &= \mathcal{S}_m^{(m-2)}-1 \\
b_{\binom{m-1}{2}-3}(m) &= \mathcal{S}_m^{(m-3)}+\mathcal{S}_m^{(m-2)}+\binom{m-2}{3}+\binom{m-2}{2}-\binom{m-2}{1}-1 \\
&\vdots \\
b_2(m) &= (m-3)!\left(\mathcal{S}_{m-1}^{(m-3)}+4\binom{m-1}{3}\right) \\
b_1(m) &= (m-2)!\binom{m-1}{2} = (m-2)!\mathcal{S}_{m-1}^{(m-2)} \\
b_0(m) &= (m-1)!
\end{aligned}
$$

where $\mathcal{S}_m^{(k)}$ are the Stirling Numbers of the Second Kind [1, Sec. 24.1.4] and

$$
\begin{aligned}
\Omega_m(1) &= \prod_{j=2}^{m}\binom{j}{2} = \frac{m!(m-1)!}{2^{m-1}} \\
\Omega_m(-1) &= \prod_{j=0}^{\frac{m-2}{3}}\binom{m-2j}{3}
\end{aligned}
$$

The positive integers $\Omega_m(2), \Omega_m(3), \ldots$ that rapidly increase in m contain in their factorization relatively large prime numbers different for different m which quite likely excludes the existence of simple expressions as for $\Omega_m(1)$ and $\Omega_m(-1)$. Also, we found that the polynomials $\Omega_m(\mu)$ are not divisible by polynomials in the numerator of $\omega_{m;Po}$, in contrast to the geometric distribution where the presence of Gaussian-like polynomials is eliminated in this way. Perhaps, here lies the basic difference in the properties of W between a geometric and a Poisson production function.

We can rewrite the series as

$$
\chi_{W;Po}(t) = 1+\sum_{m=1}^{\infty}\frac{1+\sum_{j=1}^{\binom{m-1}{2}} b_{\binom{m-1}{2}-j}(m)\mu^{-j}}{\prod_{j=1}^{m-1}\left(1-\frac{1}{\mu^j}\right)}\frac{(-t)^m}{m!}
$$

The theory of partitions [8, pp. 222] states that the series

$$\frac{1}{\prod_{j=1}^{m}\left(1-x^{j}\right)}=\sum_{n=0}^{\infty}p_{m}\left(n\right)x^{n} \tag{24}$$

where $p_m(n)$ is the number of partitions of n into parts not exceeding m, converges for $|x|<1$. Introduced yields the expansion of $\chi_{W;Po}(\mu|t)$ in powers of $\frac{1}{\mu}$,

$$\chi_{W;Po}\left(\mu|t\right)=e^{-t}+\frac{t^{2}e^{-t}}{2\mu}+\frac{e^{-t}}{\mu^{2}}\left(\frac{t^{4}}{8}-\frac{t^{3}}{6}+\frac{t^{2}}{2}\right)+\sum_{n=2}^{\infty}\left(\sum_{m=0}^{\infty}\frac{(-t)^{m}}{m!}\sum_{j=0}^{n}b_{\binom{m-1}{2}-j}(m)p_{m-1}\left(n-j\right)\right)\mu^{-n}$$

For large μ, $\chi_{W;Po}(t)\sim e^{-t}$ which implies[1] that $f_W(t)=\delta(t-1)$, an atom at $t=1$ or $W=1$ for $\mu\rightarrow\infty$.

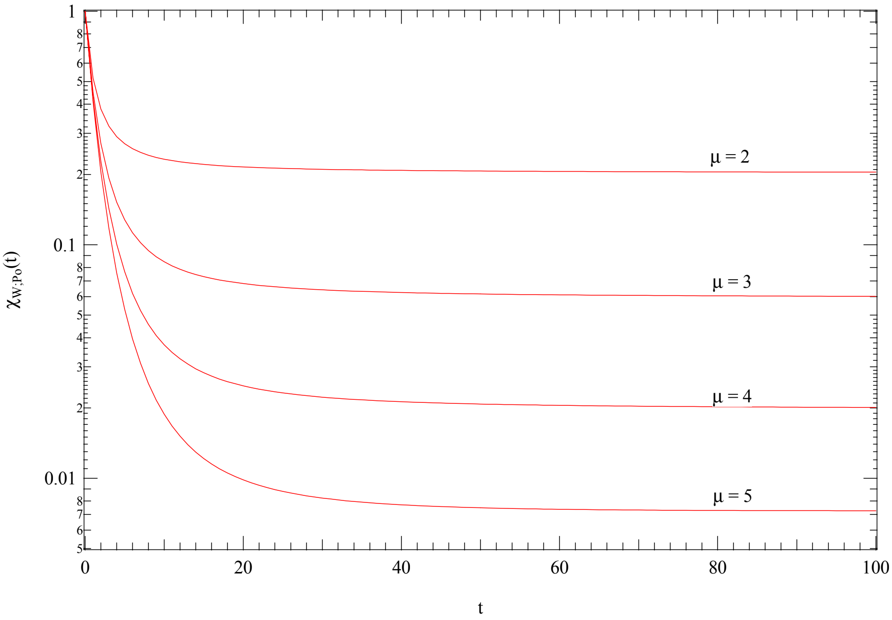

Figure 1: The generating function $\chi_{W;Po}(t)$ as a function of t for various values of μ. Observe that $\chi_{W;Po}(t)$ rapidly converges to π_0.

The generating function $\chi_{W;Po}(t)$ is efficiently computed by using the well-known Euler transformation because the coefficients $\omega_k(\mu)$ are alternating

$$\chi_{W;Po}\left(t\right)=1+\sum_{m=1}^{\infty}\left[\sum_{k=1}^{m}\binom{m-1}{k-1}\omega_{k;Po}\left(\mu\right)q^{k-m}\right]\left(\frac{t}{1+qt}\right)^{m} \tag{25}$$

[1] It also follows from the functional equation (2) and the expansion $\chi_W(t)=1-t+O\left(t^2\right)$ that

$$\chi_{W;Po}\left(t\right)=e^{-t}\left(1+O\left(\frac{t^{2}}{\mu}\right)\right)$$

The fact that $E\left[W_{Po}\right]=1$ and $var\left[W_{Po}\right]=\frac{1}{\mu-1}$ gives additional support.

Since $\chi_{W;Po}(t)$ is an entire function, the Euler transform converges for all t with $\mathrm{Re}(t) > 0$ provided $q > 0$ and for all t with $\mathrm{Re}(t) < 0$ provided $q < 0$. We have used for the computations in Figure 1 and Figure 2 the value $q = 1$ for $t > 0$ and $q = -1$ for $t < 0$. Since $\lim_{t\to\infty} \chi_{W;Po}(t) = \pi_0$, it follows from the Euler transformation that the extinction probability obeys

$$\pi_0 = 1 + \sum_{m=1}^{\infty} \left[\sum_{k=1}^{m} \binom{m-1}{k-1} \omega_{k;Po}(\mu)\, q^{k-m} \right] \frac{1}{q^m} \tag{26}$$

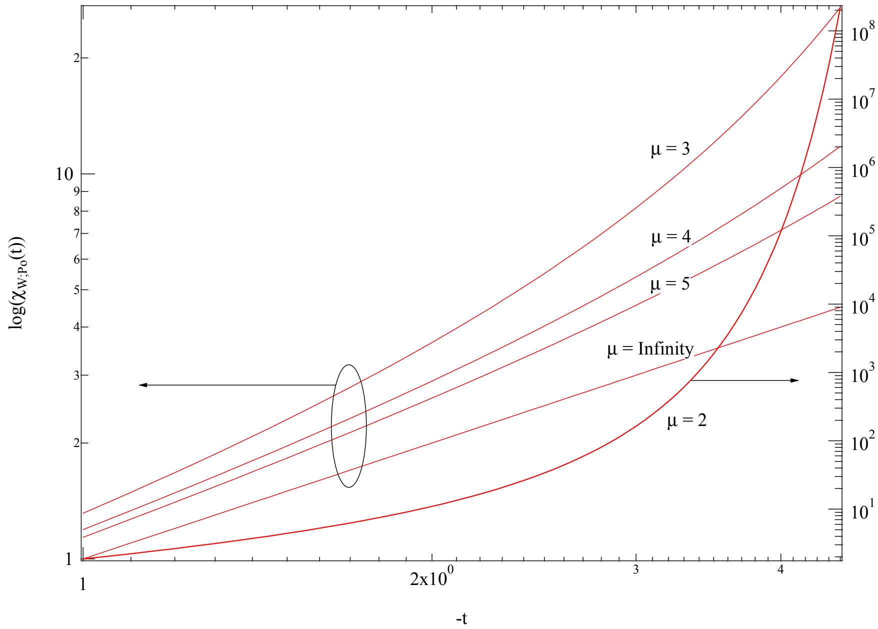

Figure 2: The logarithm of generating function $\chi_{W;Po}(t)$ as a function of real negative values of t for several values of μ on a log-log plot.

4.4 Second Recursion for $\omega_{k;Po}$

By equating in (21) corresponding powers in t and using $\omega_{0;Po}(\mu) = 1$, we obtain a new recursion

$$\omega_{k+1;Po}(\mu) = \frac{1}{\left(1 - \frac{1}{\mu^k}\right)(k+1)} \sum_{m=0}^{k-1} \frac{(m+1)\omega_{m+1;Po}(\mu)}{\mu^m} \omega_{k-m;Po}(\mu) \tag{27}$$

which is computationally more attractive than the general recursion (9).

The recursion (27) can be used to obtain an additional relation between the polynomials $\Omega_m(\mu)$. Substitution of (23) into (27) and invoking Gaussian polynomials defined [8, pp. 250] as

$$\begin{bmatrix} k \\ l \end{bmatrix}(q) = \frac{\prod_{j=1}^{k}(1-q^j)}{\prod_{j=1}^{l}(1-q^j)\ \prod_{j=1}^{k-l}(1-q^j)} \qquad k \geq 0, l > 0$$

yields

$$\Omega_{k+1}(\mu) = \sum_{m=0}^{k-1} \begin{bmatrix} k-1 \\ m \end{bmatrix}(\mu)\, \Omega_{k-m}(\mu)\, \Omega_{m+1}(\mu) \binom{k}{m} \mu^{k-1-m}$$

Since $\Omega_1(\mu) = 1$ and $\Omega_2(\mu) = 1$, the relation shows that all $\Omega_m(\mu)$ are polynomials with positive coefficients $b_k(m)$ because all appearing terms in the sum are positive for $\mu > 1$ and the Gaussian polynomials have positive coefficients. The argument also shows that $b_k(m+1) > b_k(m)$.

4.5 Second Functional Equation

Lemma 4 *For a Poisson production function, the coefficients of the asymptotic series (17) are*

$$g_m = F_\mu(-1)^{m+1}(m+1)\,\omega_{m+1;Po}(\mu\pi_0)\left(\frac{F_\mu}{\pi_0\beta}\right)^m \tag{28}$$

Proof: Change in the recursion (27) for $\omega_{k+1;Po}$ the summation index to $j = k-1-m$ and rewrite that recursion as

$$(k+1)\,\omega_{k+1;Po}(\mu) = -\frac{1}{\mu(1-\mu^k)}\sum_{j=1}^{k}(k+1-j)\omega_{k+1-j;Po}(\mu)\,\omega_{j;Po}(\mu)\,\mu^j$$

Substitution of $\chi'_{W;Po}(t) = -g_\mu(t)\,t^{-\beta-1}$ in the functional equation (21) yields

$$g_\mu(t) = g_\mu\left(\frac{t}{\mu}\right)\left(1+\frac{1}{\pi_0}\int_t^\infty g_\mu(u)\,u^{-\beta-1}du\right)$$

Using the asymptotic expansion (17) of $g_\mu(t)$ leads after equating corresponding powers in $t^{-\beta}$ to the recursion for $k > 0$

$$g_k = -\frac{1}{\beta\pi_0\left(1-(\mu\pi_0)^k\right)}\sum_{m=1}^{k}\frac{g_{m-1}g_{k-m}}{m}(\mu\pi_0)^m$$

Comparison shows that g_k possesses a same recursion as $(k+1)\,\omega_{k+1;Po}$ with $\mu \to \mu\pi_0$, apart from the scaling by a factor $\frac{\mu}{\beta}$ and the initial value $g_0 = F_\mu$ while $\omega_{1;Po}(\mu) = -1$. This correspondence proves the expression (28). $\square$

By using (28) into (15), we obtain the asymptotic series for

$$\frac{\chi_{W;Po}(t)}{\pi_0} = 1 + \sum_{m=1}^{N}\omega_{m;Po}(\mu\pi_0)\left(-\frac{F_\mu}{\pi_0\beta t^\beta}\right)^m + O\left(t^{-(N+1)\beta}\right)$$

This sum converges for all N provided $\left|\frac{F_\mu}{\pi_0\beta t^\beta}\right| < 1$. For these values of t, comparison with the Taylor series (22) leads to a second, modular-like functional equation (3). By analytic continuation because $\chi_{W;Po}(\mu|t)$ is an entire function in t, the second, modular-like functional equation (3) for $\chi_{W;Po}(\mu|t)$ holds for all t and μ, except for negative real t where the right hand side has a branch cut. This modular-like functional equation (3) extends the μ-range to values smaller than 1 since $\mu\pi_0 < 1$. Since $\Omega_m(\mu)$ is a polynomial and with (24), the definition (23) of $\omega_{m;Po}(\mu)$ demonstrates that $\omega_{m;Po}(\mu)$ is an analytic function for $\mu < 1$, and, hence, so is $\chi_{W;Po}(\mu|t)$ for $\mu < 1$.

It follows from the functional equation (3) that $\lim_{t\to-\infty}\chi_{W;Po}(\mu\pi_0|t) = \frac{1}{\pi_0}$.

4.5.1 The Parameter F_μ

The series for $g_\mu(t)$ and for $\log g_\mu(t)$,

$$\log g_\mu(t) = \log F_\mu - \mu\pi_0 \sum_{m=1}^{\infty} \frac{\omega_{m;Po}(\mu\pi_0)}{1-(\mu\pi_0)^m} \left(-\frac{F_\mu}{\pi_0\beta t^\beta}\right)^m$$

contain F_μ as a natural parameter which seems equally important as π_0. We have determined F_μ numerically from the second, modular-like functional equation (3) after rescaling that equation as $\chi_{W;Po}(\mu|a_\mu y) = \pi_0\chi_{W;Po}\left(\mu\pi_0\left|-\frac{1}{y^\beta}\right.\right)$ with $a_\mu = \left(\frac{F_\mu}{\pi_0\beta}\right)^{\frac{1}{\beta}}$ and solving for a_μ with $y=1$. For a few values for μ, we have listed the relevant parameters in the table below:

μ	π_0	β	a_μ	F_μ
2	0.203188	1.29910	2.35150	0.80161
3	0.0595202	1.56818	5.94034	1.55259
4	0.0198274	1.82818	9.53381	2.23646
5	$6.97715\ 10^{-3}$	2.08500	13.0823	3.09793
6	$2.51646\ 10^{-3}$	2.34024	16.6420	4.24581
7	$9.17759\ 10^{-4}$	2.59399	20.2566	5.83328
8	$3.36367\ 10^{-4}$	2.84580	23.9487	8.05973
9	$1.23547\ 10^{-4}$	3.09557	27.7262	11.1980
10	$4.54206\ 10^{-5}$	3.34275	31.5893	15.6293

Although for $\mu \in [2,5]$, the linear approximation $F_\mu \approx 0.75(\mu-1)$ seems good, for larger values of μ, F_μ exhibits a faster than linear growth in μ. A good fit for $\mu \in [4,10]$ is $\log F_\mu = 0.3229\mu - 0.4899$.

4.6 Series for the Probability Density Function $f_{W;Po}(x)$

In this section, we present two different series for the probability density function $f_W(x)$ for a Poisson production function.

The probability density function $f_W(x)$ can be obtained from (19) by termwise integration of the series for $\pi_0\chi_{W;Po}\left(\mu\pi_0\left|-\frac{F_\mu}{\pi_0\beta t^\beta}\right.\right)$ in (3) for sufficiently large t. Indeed, since $\chi_{W;Po}(t)$ is an entire function and $\left|\chi_{W;Po}(t)\right| \leq \chi_{W;Po}(\mathrm{Re}(t))$, the line of integration in (19) can be shifted to any arbitrary real number c. With $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{xt}}{t^B}dt = \frac{x^{B-1}}{\Gamma(B)}$ for $B>0$, the probability density function $f_W(x)$ is

$$f_{W;Po}(\mu|x) = \pi_0\delta(x) + \frac{\pi_0}{x}\sum_{m=1}^{\infty}\frac{\omega_{m;Po}(\mu\pi_0)}{\Gamma(m\beta)}\left(-\frac{F_\mu x^\beta}{\pi_0\beta}\right)^m \tag{29}$$

Alternatively, the series (25) obtained by the Euler transform can be termwise integrated provided the contour stays in one half-plane. Formally, we apply (19),

$$f_{W;Po}(x) = \delta(x) + \sum_{m=1}^{\infty}\left[\sum_{k=1}^{m}\binom{m-1}{k-1}\omega_{k;Po}(\mu)\,q^{k-m}\right]\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{xt}\frac{t^m}{(1+qt)^m}dt$$

For $x > 0$ and $q > 0$, the contour in the integral can be closed over the negative $\mathrm{Re}\,(t)$-plane. Cauchy integral theorem applied to the $(m-1)$-th derivative gives

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{xt}\frac{t^m}{(1+qt)^m}dt &= \frac{1}{q^m(m-1)!}\frac{d^{m-1}}{dt^{m-1}}\left(t^m e^{xt}\right)\Big|_{t=-\frac{1}{q}} = -\frac{1}{q^{m+1}}e^{-\frac{x}{q}}\sum_{j=0}^{m-1}\binom{m}{j+1}\frac{1}{j!}\left(-\frac{x}{q}\right)^j \\ &= -\frac{1}{q^{m+1}}e^{-\frac{x}{q}}m\,M\left(1-m,2,\frac{x}{q}\right)\end{aligned}$$

where $M(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(b+k)}\frac{x^k}{k!}$ is Kummer's confluent hypergeometric function [1]. For $x = 0$, the integral diverges which we represent by $\frac{1}{q^m}\delta(x)$. Then, taking (26) into account,

$$f_{W;Po}(x) = \pi_0\delta(x) + \frac{e^{-\frac{x}{q}}}{x}\sum_{m=1}^{\infty}\left[\sum_{k=1}^{m}\binom{m-1}{k-1}\omega_{k;Po}(\mu)\,q^k\right]\left[\sum_{j=1}^{m}\binom{m}{j}\frac{1}{(j-1)!}\left(-\frac{x}{q}\right)^j\right]\frac{1}{q^{2m}}$$

Although q may be chosen in a special way, we just choose $q = 1$ and obtain the series

$$f_{W;Po}(x) = \pi_0\delta(x) - e^{-x}\sum_{m=1}^{\infty}\left[\sum_{k=1}^{m}\binom{m-1}{k-1}\omega_{k;Po}(\mu)\right]\left[\sum_{j=1}^{m}\binom{m}{j}\frac{1}{(j-1)!}(-x)^{j-1}\right] \tag{30}$$

Finally, Figure 3 shows the distinct difference between a Poisson and Geometric production function, where [4]

$$f_{W;Geo}(x) = \begin{cases}\left(\frac{1}{\mu}-1\right)^2\exp\left(-x\left(1-\frac{1}{\mu}\right)\right) & x > 0\\ \frac{1}{\mu}\delta(x) & x = 0\\ 0 & x < 0\end{cases} \tag{31}$$

5 Conclusion

The presented formal Taylor series approach enables numerical computations of the generating functions of the limit random variable W of a branching process produced by a generation distribution function for which all moments exist. Applications to a Poisson production function illustrates the power of the method and suggests that, via the Euler transform, a series for the probability density function of W may be obtained for other production functions as well.

Apart from the computational aspect, we discovered an interesting relation to Gaussian polynomials and the theory of partitions or modular forms. For a Poisson branching process, a second, *modular*-like functional equation (3) is presented.

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions.* Dover Publications, Inc., N.Y., 1968.

[2] N. Alon and J. H. Spencer. *The Probabilistic Method.* Discrete Mathematics and Optimization. Wiley, 2000.

[3] M. S. Dubuc. La densite de la loi-limite d un processus en cascade expansif. *Z. Wahrscheinlichkeitstheorie*, 19:281–290, 1971.

[4] W. Feller. *An Introduction to Probability Theory and Its Applications*, volume 1. Wiley, New York, 3 edition, 1970.

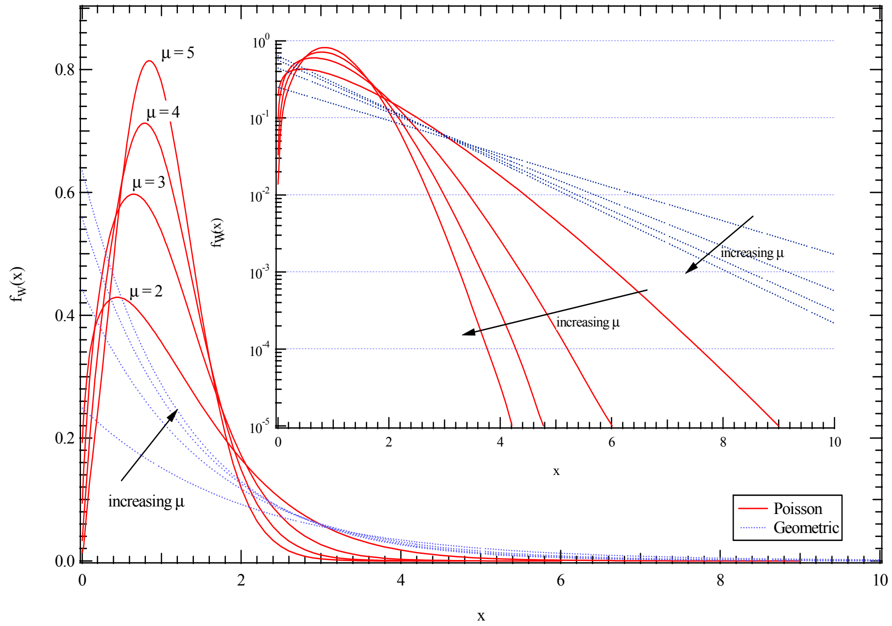

Figure 3: The probability density function $f_W(x)$ for both a Poisson and Geometric production function. For each production function the same values of $\mu = 2, 3, 4$ and 5 have been computed. The insert shows $f_W(x)$ on a log-lin scale.

[5] I. P. Goulden and D. M. Jackson. *Combinatorial Enumeration*. Wiley, New York, 1983.

[6] T. E. Harris. *The Theory of Branching Processes*. Springer-Verlag, Berlin, 1963.

[7] G. Hooghiemstra and P. Van Mieghem. On the mean distance in scale free graphs. *Methodology and Computing in Applied Probability (MCAP)*, submitted 2004.

[8] H. Rademacher. *Topics in Analytic Number Theory*. Springer-Verlag, Berlin, 1973.

[9] G. Sansone and J. Gerretsen. *Lectures on the Theory of Functions of a Complex Variable*, volume 1 and 2. P. Noordhoff, Groningen, 1960.

[10] E. Seneta. Functional equations and the Galton-Watson process. *Adv. Appl. Prob.*, 1:1–42, 1969.

[11] E. C. Titchmarsh. *The Theory of Functions*. Oxford University Press, Amen House, London E. C. 4, 1964.

[12] R. van der Hofstad, G. Hooghiemstra, and P. Van Mieghem. Distances in random graphs with finite variance degree. *Random Structures and Algorithms*, accepted 2004.

[13] P. Van Mieghem. The asymptotic behaviour of queueing systems: Large deviations theory and dominant pole approximation. *Queueing Systems*, 23:27–55, 1996.