Generalised Cross-Ratio of Projective Linear Spaces and Limits of Local Height Pairings

Author:
Hidde Lammert Bakker

Supervisors:
Dr. E. C. Sertöz
Dr. B. Janssens

Examination Committee:
Prof. Dr. J. M. A. M. van Neerven
Dr. E. C. Sertöz
Dr. B. Janssens
Dr. R. C. Kraaij

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Universiteit Leiden
The Netherlands

TU Delft
Delft University of Technology

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Abstract

This thesis generalises the classical cross-ratio of four projective points in $\mathbb{CP}^1$ *to a pairing of two* k*-dimensional and two* $(n-k-1)$*-dimensional linear subspaces of* $\mathbb{CP}^n$*. We prove that the absolute values of the generalised cross-ratio determine the local height pairings of the corresponding cycles. For the Archimedean height, the proof avoids integrating Green's forms over high-dimensional subspaces by relating this height, via an incidence correspondence, to an explicit height pairing of points and a principal divisor on a Grassmannian.*

We then study degenerating families of these linear subspaces and describe the asymptotic behaviour of all local height pairings as the cycles intersect. We show that the asymptotics are governed by the intersection degree of the families of cycles, in any degeneration.

Furthermore, using a global construction, we show that the generalized cross-ratio itself, rather than its norms, computes a generalised Hodge theoretic height. We do this by showing that the relative homology group, induced by the two k*-planes and two* $(n-k-1)$*-planes, is naturally an extension of mixed Hodge structures of* $\mathbb{Z}(0)$ *by* $\mathbb{Z}(1)$*, and that the height of this extension equals the generalised cross-ratio.*

Finally, using local methods, we can study the regularised limits of local heights, and interpret these limits using only the central degenerate geometry, together with small perturbations. This thesis provides an entirely new class of examples associating geometric shapes to limits of local heights.

Contents

Introduction

Height pairings are an important tool in modern algebraic geometry. For example, they are used as a powerful tool in the study of *diophantine equations*, the field of finding integer solutions to equations with integer coefficients. These diophantine equations have wide-ranging applications such as balancing chemical reactions [Cro68], describing the geometric structure of viruses [GG10] and elliptic curve cryptography [Kob87].

Recently, Gerd Faltings was awarded the Abel prize [Nor26], one of the most prestigious mathematical awards, for his work in arithmetic geometry and resolving the long-standing diophantine conjectures of Mordell and Lang [Fal83]. Mordell predicted that if a polynomial equation defines a sufficiently complicated curve, then it has only finitely many rational solutions. More specifically, any non-singular algebraic curve, defined over $\mathbb{Q}$, of genus $g \geq 2$, has at most finitely many rational points. Faltings' work is not only spectacular for the result he proved, which for example implies a weak version of Fermat's Last Theorem, namely that there are at most *finitely* many integer solutions to the equation

$$a^n + b^n = c^n,$$

for $n \geq 4$. His work is also important because it proves that the arithmetic (rational points) is controlled by topology (geometric complexity of the curve). The main idea of Faltings' proof is the comparison between a specific kind of height pairing, the *Faltings height*, and naive heights via Siegel modular varieties [Blo84].

Another motivation for studying heights is that they give rise to interesting numbers. Like familiar constants such as π, these numbers often reappear in different parts of mathematics, sometimes through constructions that look unrelated at first sight [KZ01, BD21]. In particular, one may encounter the same invariants in very analytic and purely arithmetic constructions. Such coincidences suggest that these theories are connected in a deeper way, even when the methods used to define the numbers are quite different. One way to investigate why the same numbers appear in different contexts is to study different height pairings.

In this thesis, we will study height pairings in a concrete and classical setting. We prove a formula that computes the *Archimedean height pairing* of projective linear subspaces of complex projective space, which is a complex manifold that we will introduce below. The Archimedean height will be entirely determined by the *generalised cross-ratio* which we define in Chapter 2.

Because this formula is so explicit, we are able to prove an instance of a conjecture by Z. Chen and R. de Jong [Che25, Conjecture 1.5]. This conjecture states that the rate of divergence of the Archimedean height of certain families of algebraic cycles is controlled by their intersection degree, in *proper* degenerations. In [HdJ15, Theorem 2.1] such a result has been proven for degenerations of curves. We have been able to prove this for degenerating *higher-dimensional* projective linear subspaces, not just for *proper*, but also for *general* degenerations. This suggests that the conjecture by Chen and de Jong, could hold in a more general setting.

Using this result, we can renormalize the Archimedean height by subtracting the correct divergent term and talk about the *regularized limit*. It is very difficult to study these numbers themselves, because they lack an obvious structure. An algebraic variety which gives rise to such numbers is far easier to understand. The *limit geometry* of curves [BdJS23], and later nodal degenerations of certain odd-dimensional varieties [Bei25] has been studied. We give a geometric interpretation of this number and show that it is closely related with the central geometry of the degeneration. This thesis provides an entirely new class of examples, in arbitrary dimensions and for all degenerations of linear projective subspaces.

Complex projective space. A complex manifold, of complex dimension n, can be thought of as a shape, which locally looks like an open subset of $\mathbb{C}^n$. The transition functions describing how to change between different local complex coordinates are required to be holomorphic. In particular, $\mathbb{C}^n$ is a complex manifold of complex dimension n, and so is any open subset $U \subset \mathbb{C}^n$. By virtue of being compact, *complex projective spaces* are important examples. They are the standard ambient spaces in algebraic geometry and defined by

$$\mathbb{CP}^n = \{L \subset \mathbb{C}^{n+1} \mid L \text{ is a complex line through the origin}\},$$

as discussed in Section 2.1. Similarly to complex space itself, they can be equipped with coordinates. An equivalent definition which makes this apparent is the following

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$$

where we declare $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$ if there exists a *single* scalar λ such that $y_i = \lambda x_i$ for all i. Such an equivalence class is denoted by $[x_0 : \cdots : x_n]$, and these x_i are the *homogeneous coordinates*. There is a natural action of invertible matrices $M \in \mathrm{GL}_{n+1}(\mathbb{C})$ on these elements, which is invariant under scalar multiples of these matrices. For this reason, we define *projective transformations* to be

$$\mathrm{PGL}_{n+1}(\mathbb{C}) = \mathrm{GL}_{n+1}(\mathbb{C})/\sim,$$

where $A \sim B$ if $B = \lambda A$ for some scalar λ.

Classical cross-ratio. Now, let us specialize to the case where $n = 1$, $\mathbb{CP}^1$, the *Riemann sphere*. There are three special points in $\mathbb{CP}^1$

$$0 = [1 : 0], \qquad 1 = [1 : 1], \qquad \infty = [0 : 1].$$

Now, given four distinct points P, Q, R and S in $\mathbb{CP}^1$, there exists a unique projective transformation M, which maps

$$P \longmapsto 0, \qquad Q \longmapsto 1, \qquad R \longmapsto \infty.$$

One classically defines the *cross-ratio* to be the unique complex number $\mathrm{CR}(P, Q; R, S)$ such that, under the projective transformation M,

$$S \longmapsto [1 : \mathrm{CR}(P, Q; R, S)].$$

If we assume that the original points P, Q, R and S can be written as

$$P = [1 : p], \qquad Q = [1 : q], \qquad R = [1 : r], \qquad S = [1 : s], \tag{0.1}$$

then the cross-ratio is given by

$$\mathrm{CR}(P, Q; R, S) = \frac{(p - r)(q - s)}{(q - r)(p - s)}.$$

The cross-ratio is an important concept, as any projective invariant of four projective points is a function of the cross-ratio, see [Olv99][Example 8.34]. Furthermore, the moduli space $\mathcal{M}_{0,4}$ parametrizes smooth genus 0 curves with four ordered marked points, up to isomorphism. Since every smooth genus 0 curve is isomorphic to $\mathbb{P}^1$, this is equivalently the moduli space of ordered configurations of four distinct points on $\mathbb{P}^1$, up to projective transformations. Thus, the cross-ratio is the natural parameter on this moduli space, see [Vak03, p. 5].

Generalised cross-ratio. Alternatively, the points P, Q, R and S correspond to lines L_P, L_Q, L_R and L_S in $\mathbb{C}^2$ that pass through the origin. Let v_P, v_Q, v_R and v_S be elements of $\mathbb{C}^2$ that generate the respective lines, i.e., any non-zero vector contained in the line. Because $\mathbb{C}^2$ is a complex vector space of dimension 2, there exists a *non-canonical* isomorphism of complex vector spaces between the two-fold exterior product $\Lambda^2\mathbb{C}^2$ and the complex numbers themselves. In Proposition 2.3 we show that regardless of the chosen isomorphism, the value

$$\frac{(v_P \wedge v_R)(v_Q \wedge v_S)}{(v_Q \wedge v_R)(v_P \wedge v_S)}$$

is well-defined, independent of the choice of vectors, and equals the classical cross-ratio. This definition can now be generalised from four distinct points P, Q, R and S in $\mathbb{CP}^1$, to two k-dimensional linear subspaces Y_P, Y_Q and two $(n-k-1)$-dimensional linear subspaces Y_R, Y_S in $\mathbb{CP}^n$, for which

$$Y_P \cap Y_R = Y_P \cap Y_S = Y_Q \cap Y_R = Y_Q \cap Y_S = \varnothing,$$

which we call a non-degenerate quadruple. These dimensions are chosen in such a way that

$$\dim(Y_P) + \dim(Y_R) = \dim(\mathbb{CP}^n) - 1,$$

similarly to the simpler case of points in $\mathbb{CP}^1$. We can choose a basis $v_0^P, \ldots, v_k^P$ for Y_P when viewed as a $(k+1)$-dimensional subspace of $\mathbb{C}^{n+1}$. Let $v_P = v_0^P \wedge \cdots \wedge v_k^P$ and similarly for v_Q, v_R and v_S. We then define, see Definition 2.3, the *generalised cross-ratio* to be

$$\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S) = \frac{(v_P \wedge v_R)(v_Q \wedge v_S)}{(v_Q \wedge v_R)(v_P \wedge v_S)}.$$

Archimedean heights. With this in mind, we now turn to the *Archimedean height pairing*. All these definitions are carefully introduced in Chapter 1. Suppose Z and W are formal sums of subspaces of a complex manifold X, such that

$$\dim(Z) + \dim(W) = \dim(X) - 1.$$

Then the Archimedean height, denoted by $\langle Z, W\rangle_\infty$, is a way of describing the *complexity* of the subspaces Z and W. For simplicity, assume that Z and W are homologically trivial analytic subspaces. Then the Archimedean height can be computed as follows. First, we associate to the subspace Z a Green's current G_Z satisfying

$$dd^c G_Z + \delta_Z = 0.$$

Then, we compute a Green's form for Z, i.e., a differential form g_Z such that the associated current $[\![g_Z]\!] = G_Z$. Lastly

$$\langle Z, W\rangle_\infty = -\int_W g_Z.$$

However, note that computing these Green's forms and then integrating over the possibly high-dimensional subspace W is hard to do explicitly in general.

Archimedean height in $\mathbb{CP}^1$. Since $\mathbb{CP}^1$ is a complex manifold of complex dimension 1, we can only pair points against each other. Because we assume the cycles to be homologically trivial, we will pair $P-Q$ against $R-S$, where P, Q, R and S are distinct points in $\mathbb{CP}^1$. Again, let us assume that P, Q, R and S can be written as in Equation 0.1. In Section 2.2, we will show that a Green's form for $P-Q$ is given by

$$g_{P-Q}(z) = -\log\left|\frac{z-p}{z-q}\right|.$$

Since integration over points is simply evaluation, we conclude that the Archimedean height is given by

$$\langle P - Q, R - S\rangle_\infty = \log |\mathrm{CR}(P, Q; R, S)| , \tag{0.2}$$

see Proposition 2.5. It is classical, yet remarkable, that the *arithmetic complexity* of differences of points, as captured by the Archimedean height, is completely governed by their cross-ratio.

Archimedean height in $\mathbb{CP}^n$. The first main result of this thesis is that we extend this relation to the case where Y_P, Y_Q are k-dimensional linear subspaces, and Y_R, Y_S are $(n-k-1)$-dimensional linear subspaces, non-degenerate in $\mathbb{CP}^n$.

Theorem A (Theorem 2.20)**.** Let Y_P, Y_Q, Y_R and Y_S be as above. Then the Archimedean height is given by

$$\langle Y_P - Y_Q, Y_R - Y_S\rangle_\infty = \log |\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S)|.$$

This formula allows one to easily compute and manipulate the Archimedean height, in higher-dimensional examples. A formula for a Green's current associated to a linear subspace of $\mathbb{CP}^n$ exists, it was first introduced by H. Levine, see [GS90b, Proposition 5.1]. However, it is very complicated and integrating it over a high-dimensional analytic subspace seems unpleasant. Instead, we compute it by relating it to the case of principal divisors and points, in another ambient space called a *Grassmannian*, see Lemma 1.23.

Asymptotic behaviour. Throughout this introduction, we have always assumed the objects of study to be disjoint. The cross-ratio is not well-defined when the points, or higher dimensional subspaces, intersect and neither is the Archimedean height associated to them. However, it is interesting to study the asymptotic behaviour of the Archimedean height pairing if the subspaces are *moving*, say holomorphically dependent on some complex parameter t.

Theorem B (Theorem 3.5 & Theorem 3.21)**.** Let $Y_P(t)$ and $Y_Q(t)$ be moving k-dimensional linear subspaces of $\mathbb{CP}^n$, and let $Y_R(t)$ and $Y_S(t)$ be moving $(n-k-1)$-dimensional linear subspaces of $\mathbb{CP}^n$, all holomorphically dependent on t. Assume that for all sufficiently small non-zero t, the linear subspaces are non-degenerate, then

$$\langle Y_P(t) - Y_Q(t), Y_R(t) - Y_S(t)\rangle_\infty = \deg\left((\overline{Y_P} - \overline{Y_Q}) \cdot (\overline{Y_R} - \overline{Y_S})\right) \log |t| + \log |u(t)|, \tag{0.3}$$

for some nowhere-vanishing holomorphic function $u(t)$.

We denote the families by $\overline{Y_P}, \overline{Y_Q}, \overline{Y_R}$ and $\overline{Y_S}$. Furthermore, $\cdot$ denotes the intersection product. Because the families are of complementary dimension, their intersection product will consist of a formal sum of points, the degree $\deg(-)$ simply sums the integral coefficients. In Section 3.3, we will introduce the necessary ingredients from *intersection theory*, to properly define these families and the intersection degree.

First, we assume that the intersection is *proper*, i.e., of expected dimension, when $t = 0$. For example, if $Y_P(t)$ is a line and $Y_R(t)$ is a plane, for all t, then $\dim(Y_P(t)) = 1$ and $\dim(Y_R(t)) = 2$ and as a family $\dim(\overline{Y_P}) = 2$ and $\dim(\overline{Y_R}) = 3$. These families are now not just subspaces of $\mathbb{CP}^4$ but of $\mathbb{CP}^4_{\mathbb{C}\{t\}}$, which allows them to be viewed as if they are moving, and is now a 5-dimensional space. All these concepts are introduced in Section 3.1. For a *proper* intersection, we expect

$$\dim(\overline{Y_P} \cap \overline{Y_R}) = \dim(\overline{Y_P}) + \dim(\overline{Y_R}) - \dim(\mathbb{CP}^4_{\mathbb{C}\{t\}}) = 2 + 3 - 5 = 0.$$

In the next figure we show an example of two different degenerations of $\overline{Y_P}$ and $\overline{Y_R}$, one that is proper and one that is not.

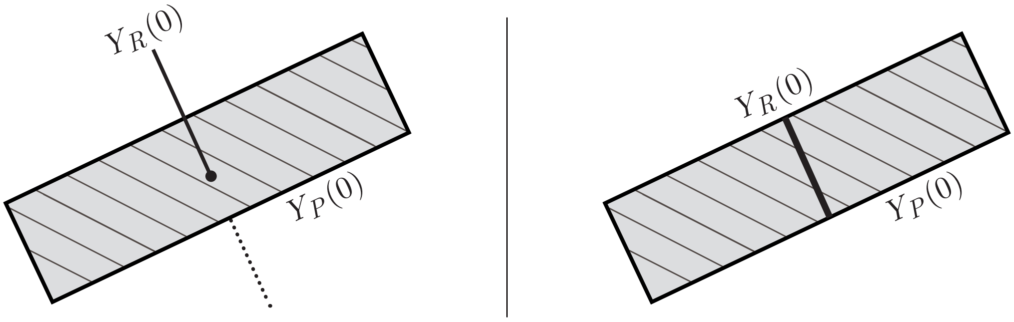


Figure 1: Left: Proper intersection. Right: Excess intersection.

When we restrict to proper degenerations, Theorem 3.5 proves a Conjecture by Z. Chen and R. de Jong [Che25, Conjecture 1.5] for the specific cycles $\overline{Y_P}-\overline{Y_Q}$ and $\overline{Y_R}-\overline{Y_S}$. In Theorem 3.21 we generalize this result to arbitrary degenerations. This provides evidence that their conjecture might hold in an even more general setting.

Non-Archimedean contributions. Number theory often studies geometric objects not only over the complex numbers, but also through their reductions modulo primes. Each prime p gives rise to a different local picture: even when planes are disjoint over the complex numbers, it can happen that they intersect non-trivially when reduced modulo some prime p. The corresponding local height at p measures precisely this p-adic contribution to the arithmetic complexity of the cycles. In Chapter 4 we apply the machinery from Chapter 3 to the ring $\mathbb{Z}_p$, for any prime p, to find the local heights whenever the planes Y_P, Y_Q, Y_R and Y_S are defined over $\mathbb{Z}$ or $\mathbb{Z}_p$.

Theorem C (Corollary 4.2)**.** Let p be a prime number. Let Y_P and Y_Q be k-dimensional linear subspaces of $\mathbb{CP}^n$ and let Y_R and Y_S be $(n-k-1)$-dimensional linear subspaces of $\mathbb{CP}^n$, non-degenerate and defined over $\mathbb{Z}_p$. Then the local height is given by

$$\langle Y_P - Y_Q, Y_R - Y_S\rangle_p = \log|\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S)|_p,$$

where $|\cdot|_p$ denotes the p-adic norm.

Here $\langle\cdot,\cdot\rangle_p$ denotes the local height at place p. Thus, not only the Archimedean height, but *all* local heights are completely determined by the corresponding norms of the cross-ratio.

Regularized cross-ratio. When the k-planes Y_P and Y_Q and the $(n-k-1)$-planes Y_R and Y_S are defined over the integers we can further extend the definition of the cross-ratio. We now allow Y_P and Y_Q to intersect Y_R and Y_S. Let V_P, V_Q, V_R and V_S denote the underlying lattices of the projective linear spaces. Then, when intersection occurs, choose normal tuples which consist of vectors

$$r^1_{P,R}, \dots, r^m_{P,R} \in \frac{\mathbb{Z}^{n+1}}{(V_P+V_R)^{\mathrm{sat}}}$$

such that the images of $r^1_{P,R}, \dots, r^m_{P,R}$ form a basis of the normal directions after tensoring with $\mathbb{Q}$. Let $r_{P,R} = (r^1_{P,R}, \dots, r^m_{P,R}$ and similarly define $r_{P,S}, r_{Q,R}$ and $r_{Q,S}$. Then we define the *regularized cross-ratio* of Y_P, Y_Q, Y_R and Y_S, together with these normal vectors $r = (r_{P,R}, r_{P,S}, r_{Q,R}, r_{Q,S})$ by

$$|\mathrm{CR}^n_r(Y_P, Y_Q; Y_R, Y_S)| = \frac{[\mathbb{Z}^{n+1} : (V_P + V_R + r_{P,R})]}{[\mathbb{Z}^{n+1} : (V_Q + V_R + r_{Q,R})]}\frac{[\mathbb{Z}^{n+1} : (V_Q + V_S + r_{Q,S})]}{[\mathbb{Z}^{n+1} : (V_P + V_S + r_{P,S})]},$$

which is well-defined up to sign, see Definition 4.9 for more details and Figure 2.

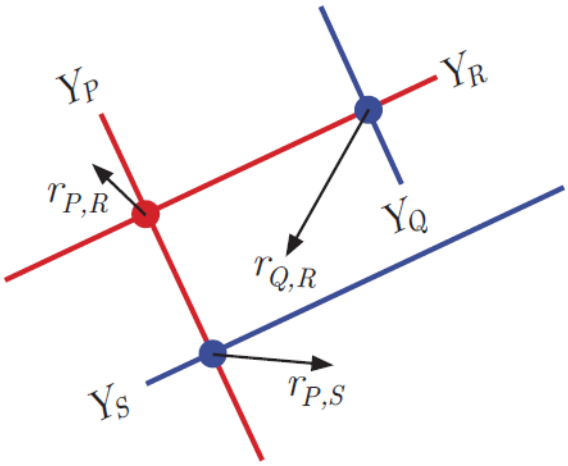


Figure 2: An example showing the vectors $r_{P,R}, r_{P,S}$ and $r_{Q,R}$ used to define the *regularized* cross-ratio for intersecting Y_P, Y_Q, Y_R and Y_S.

Regularized limit. In Chapter 4 we also find a *geometric* interpretation of the remaining constant $u(0)$ from Equation 3.2, which we call the regularized limit. Unlike the cross-ratio itself, this constant is well-defined up to sign for the individual intersections $\overline{Y_P}_s\cdot\overline{Y_R}_s, \overline{Y_P}_s\cdot\overline{Y_S}_s, \overline{Y_Q}_s\cdot\overline{Y_R}_s$ and $\overline{Y_Q}_s\cdot\overline{Y_S}_s$ so, for simplicity, we only focus on a single intersection, and denote the families by X and Y. When the families of planes X and Y are defined over $\mathbb{Z}$, we show that there is a canonical $\mathbb{Z}$-basis for the direction in which the planes collide. In the case of an excess intersection, there are multiple normal directions, see Figure 3.

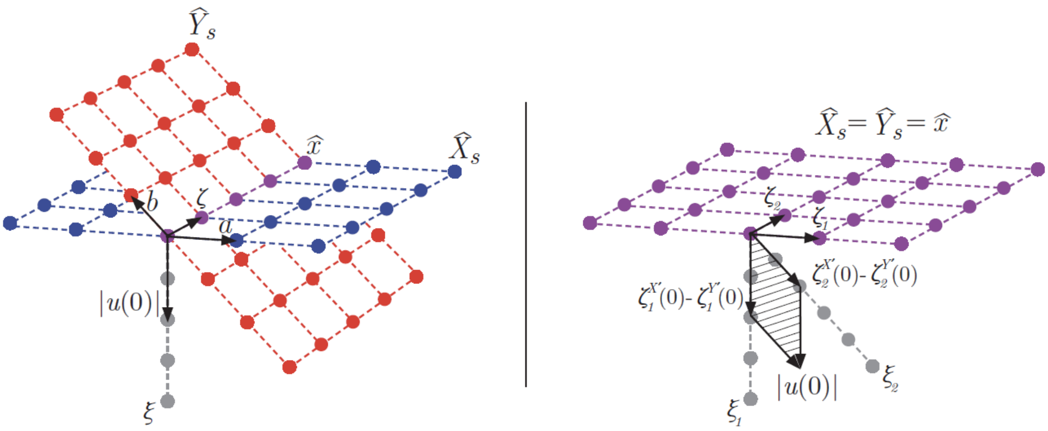


Figure 3: Left: Proper degeneration. Right: Excess degeneration.

On the left, the unique normal direction is denoted by ξ, and on the right there are two normal directions spanned by ξ_1 and ξ_2. The families X and Y induce vectors, one for each normal direction denoted by r (left) or r_1 and r_2 (right), which are determined by the rate at which these families move in these normal directions.

Theorem D (Corollary 4.10)**.** The absolute value of the constant $|u(0)|$, as in Equation 3.2, is given by the regularized cross-ratio of $\overline{Y_P}_s, \overline{Y_Q}_s, \overline{Y_R}_s$ and $\overline{Y_S}_s$ together with the induced rate of collision vectors $r_{P,R}, r_{P,S}, r_{Q,R}$ and $r_{Q,S}$.

For the precise statement of Theorem D we refer to Theorem 4.8. It is via this interpretation that we can associate a canonical geometric object to the regularized limit $u(0)$. Finding such a *regularized limit of shapes* is in general very hard, and an interesting open problem. The case of curves [BdJS23] and nodal degenerations of certain odd-dimensional varieties [Bei25] have been handled. This thesis shows the that the regularised limit is determined by the central geometry, also in the case of degenerations of projective linear subspaces in arbitrary dimension.

Augmented height pairing. In some sense, Theorems A and C are evidence that the proposed generalisation of the cross-ratio is a natural choice. We see that the *norms* of the generalised complex-valued cross-ratio share the same relation with local heights as the classical cross-ratio. We will show that not just the norm of the cross-ratio, but also the complex value itself, is canonical. To do this, we will study the *augmented height pairing* of $Y_P - Y_Q$ and $Y_R - Y_S$ in Chapter 5, which is now not just a real, but a complex number. We will recall some standard facts, showing that the singular homology group

$$H^{\mathrm{sing}}_{2(n-k)-1}(\mathbb{CP}^n \setminus |Y_P - Y_Q|, |Y_R - Y_S|),$$

can be equipped with the structure of an integral *mixed Hodge structure*, denoted by H. After twisting $H(-(n-k-1)) = H \otimes \mathbb{Z}(-(n-k-1))$ fits in a short exact sequence of mixed Hodge structures

$$0 \to \mathbb{Z}(1) \to H(-(n-k-1)) \to \mathbb{Z}(0) \to 0,$$

in other words, can be interpreted as an element of $\mathrm{Ext}^1_{\mathrm{MHS}_\mathbb{Z}}(\mathbb{Z}(0), \mathbb{Z}(1))$. In Proposition 5.12 and Corollary 5.13 we recall that this extension group is canonically isomorphic to $\mathbb{C}^\times$, the non-zero complex numbers. The height of this extension is called the *augmented height* of $Y_P - Y_Q$ and $Y_R - Y_S$, denoted by $\langle Y_P - Y_Q, Y_R - Y_S\rangle_{\mathrm{aug}}$, which is a non-zero *complex* number. Similarly to the local heights, we show in Section 5.4 that

$$\langle P - Q, R - S\rangle_{\mathrm{aug}} = \mathrm{CR}(P, Q; R, S),$$

for distinct points P, Q, R and S in $\mathbb{CP}^1$. We then generalise this result in Section 5.5.

Theorem E (Corollary 5.21)**.** Let Y_P and Y_Q be k-dimensional linear subspaces of $\mathbb{CP}^n$ and let Y_R and Y_S be $(n-k-1)$-dimensional linear subspaces of $\mathbb{CP}^n$, which are non-degenerate. Then the augmented height is given by

$$\langle Y_P - Y_Q, Y_R - Y_S\rangle_{\mathrm{aug}} = \mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S).$$

We would like to remark that (the proof of) Theorem E actually implies both Theorems A and C, see [Hai90, Proposition 3.3.12]. Nevertheless, we still present the direct proof in Chapter 2, because, although challenging, it is a far more intuitive direction to follow. This proof also requires far less technical machinery, which allows a broader audience to appreciate its result.

Conclusions. Theorems A and C show that the norms of the cross-ratio satisfy the relations

$$\begin{aligned} \langle Y_P - Y_Q, Y_R - Y_S\rangle_\infty &= \log |\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S)|, \\ \langle Y_P - Y_Q, Y_R - Y_S\rangle_p &= \log |\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S)|_p, \qquad \text{(for all primes } p). \end{aligned}$$

Theorem E, which relates the augmented height pairing with the generalised cross-ratio,

$$\langle Y_P - Y_Q, Y_R - Y_S\rangle_{\mathrm{aug}} = \mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S),$$

further confirms that the generalised cross-ratio is canonical not only through its norms, but also as a complex number. It is remarkable that such a simple constant completely governs the arithmetic complexity of linear subspaces in $\mathbb{CP}^n$. The simplicity of the cross-ratio made it possible to study asymptotics as complex linear subspaces degenerate, Theorem B, proving an instance of [Che25, Conjecture 1.5]. It also made it possible to to interpret the regularised limit geometrically, Theorem D, as described in [BdJS23, Bei25].

Chapter 1: The Archimedean Height Pairing

The main goal of this chapter is to introduce the Archimedean height pairing and prove a new formula that helps compute it. This chapter is mostly expository and serves to introduce the relevant notions for the rest of this thesis. In Sections 1.1 and 1.2 we will recall what differential forms and currents are and introduce a special kind, namely Green's forms and currents. Using these Green's forms we will recall the definition of the Archimedean height pairing in Section 1.3. Computing the Archimedean height explicitly is in general very hard. However, in Example 1.5, we observe that for a specific kind of cycles, it is easy and explicit. Lastly, in Sections 1.4, 1.5 and 1.6, we will work towards Lemma 1.23, which can reduce a priori difficult height computations to a case where we can compute them easily and explicitly. This is the main result of this Chapter, and will be used extensively in Chapter 2.

Section 1.1: Push-forward and pull-back of differential forms

In this section, we will recall what the push-forward of differential forms along a suitable map p is. We expect familiarity with the concepts of (complex) manifolds, their (co)tangent bundles and differential forms. For additional information on these topics, or an excellent introduction, we refer to "Introduction to Smooth Manifolds" by J. M. Lee [Lee13], in particular Chapter 14 on differential forms.

Let M be a smooth manifold. Recall that a *differential k-form* is a smooth section of $\wedge^k T^*M$, the bundle of alternating k-tensors on M. We denote the vector space of differential k-forms by

$$\Omega^k(M) := \Gamma(\Lambda^k T^*M),$$

and

$$\Omega(M) := \bigoplus_{k\geq 0} \Omega^k(M).$$

For a k-form ω and an l-form η, we define the wedge product $\omega \wedge \eta$, an $(k+l)$-form pointwise, i.e. $(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$. Since we are on a manifold, locally we can write a differential k-form ω as

$$\omega|_U = \sum_{|I|=k} \omega_I dx_I,$$

where $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, and $\omega_I \in \mathcal{C}^\infty(U)$. Now let $f : M \to N$ be a smooth map. The differential $df_p : T_pM \to T_{f(p)}N$ induces a map

$$f^* : \Omega^k(N) \to \Omega^k(M)$$

given by

$$(f^*\omega)_p(v_1, ..., v_k) := \omega_{f(p)}(df_p(v_1), ..., df_p(v_k)).$$

This map is $\mathbb{R}$-linear, and $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$, see [Lee13, Lemma 14.16]. When working with a complex manifold, the induced pull-back map is also $\mathbb{C}$-linear.

Remark 1.1. The pull-back map is also $\mathcal{C}^\infty(N, \mathbb{R})$- or $\mathcal{C}^\infty(N, \mathbb{C})$-linear, via

$$f^*(g\omega) = (g \circ f) f^*(\omega).$$

There is also a natural differential operator on differential k-forms, called the exterior derivative d,

$$d : \Omega^k(M) \to \Omega^{k+1}(M).$$

It is locally given by

$$d\left(\sum_{|I|=k} \omega_I dx_I\right) = \sum_{|I|=k} d\omega_I \wedge dx_I.$$

The exterior derivative commutes with respect to pull-back, i.e. $f^*(d\omega) = d(f^*\omega)$, see [Lee13, Proposition 14.32]. The push-forward of differential forms, also called fiber integration, is only defined for suitable smooth maps, such as *fiber bundles* with compact fibers, or for compactly supported differential forms. For a comprehensive introduction, we refer to "Lecture notes algebraic topology *II* 2024-2025" by T. Rot and R. Hoekzema [RH24].

We follow Chapter 7 of "Connections, Curvature, and Cohomology" by W. Greub et al. [GHV72]. Let $F \to E \xrightarrow{\pi} B$ be a fiber bundle. We say a differential form $\omega \in \Omega(E)$ has fiber-compact support if for all compact $K \subset B$ the intersection $\pi^{-1}(K) \cap \operatorname{Supp}(\omega)$ is compact, where

$$\operatorname{Supp}(\omega) = \overline{\{p \in E \mid \omega_p \neq 0\}}.$$

Denote those forms by $\Omega_F(E)$, and let compactly supported forms be denoted by $\Omega_c(E)$. We thus have a chain of inclusions

$$\Omega_c(E) \subset \Omega_F(E) \subset \Omega(E).$$

Now consider a vector bundle over the same base $H \to M \xrightarrow{\pi_M} B$. Write $\mathcal{B} = (F \to E \xrightarrow{\pi} B)$ and $\xi = (H \to M \xrightarrow{\pi_M} B)$. Let $V_E \subset TE$ be the vertical subbundle, i.e. $V_{E,x} = V_x(E) = \operatorname{Ker}((d\pi)_x : T_xE \to T_xB)$. Let $r = \dim F$ and let $\vartheta : \wedge^r V_E \to \xi$ be a bundle map inducing π at the base, i.e.

$$\begin{array}{ccc} \wedge^r V_E & \xrightarrow{\vartheta} & M \\ \downarrow & & \downarrow{\scriptstyle \pi_M} \\ E & \xrightarrow[\pi]{} & B \end{array}$$

We define $\operatorname{Supp}(\vartheta) := \overline{\{z \in E \mid \vartheta_z \neq 0\}} \subset E$. We say that ϑ has fiber-compact support if $\pi^{-1}(K) \cap \operatorname{Supp}(\vartheta)$ is compact for every compact subset $K \subset B$. Now assume $\mathcal{B}$ is oriented and ϑ has fiber-compact support. We will construct a section $\sigma : B \to M$ which is called the *integral over the fiber of* ϑ.

For all $x \in B$, ϑ determines an M_x-valued r-form on E_x, namely $\vartheta_x \in \Omega^r(E_x, M_x)$ given by

$$\vartheta_x(z; \eta_1, ..., \eta_r) = \vartheta(\eta_1 \wedge ... \wedge \eta_r)$$

for $z \in E_x$ and $\eta_i \in T_zE_x = V_z(E)$. Then finally, we define

$$\sigma(x) = \int_{E_x} \vartheta_x \in M_x,$$

where we note that this integral is not in general zero because ϑ_x is a top-degree form on F. We may write $\sigma = \int_F \vartheta$, which is smooth, see [GHV72, Proposition VII, p. 299].

In a less general setting, let us observe what happens when we push forward differential forms. In the same setup, let $\dim(F) = r$ and $\dim(B) = n$. Our goal is to construct a map $\int_F : \Omega_F(E) \to \Omega(B)$, which is $\mathbb{R}$- (or $\mathbb{C}$-)linear and homogeneous of degree $-r$. Let $\omega \in \Omega_F^{r+p}(E)$, for all $x \in B$, ω determines a compactly supported r-form, ω_x on F_x with values in $\wedge^p T_x(B)^*$. As follows, fix $z \in E_x$ and $\eta_1, ..., \eta_r \in V_z(E)$ and $\xi_1, ..., \xi_p \in T_x(B)$. Let $\zeta_i \in T_z(E)$ satisfy $(d\pi)_z \zeta_i = \xi_i$. Then ω_x is defined by

$$\langle \omega_x(z; \eta_1, ..., \eta_r), \xi_1 \wedge ... \wedge \xi_p \rangle = \omega(z; \zeta_1, ..., \zeta_p, \eta_1, ..., \eta_r).$$

Then we define
$$\left(\int_F \omega\right)(x) = \int_{E_x} \omega_x.$$
Note that if one chooses another lift ζ', then the difference $\zeta - \zeta'$ is an element of $V_z(E)$ which has dimension r, so this would mean we plug in, at least $r+1$, tangent vectors from $V_z(E)$ into ω giving 0 by anti-symmetry.

Example 1.2. Let us apply this to an easy example. Consider the fiber bundle $[0,1] \to [0,1] \times [0,1] \xrightarrow{\pi_1} [0,1]$. As before we let E denote the total space $[0,1] \times [0,1]$, B the base $[0,1]$ and F the fiber $[0,1]$. Consider the 2-form $\omega = dx \wedge dy \in \Omega^2_F(E) = \Omega^2(E)$. We will try to compute the 1-form $\int_F \omega = F_*\omega$. In this case $r = p = 1$. Fix $0 \in [0,1]$ and $z \in F_0 = \{0\} \times [0,1]$ say $z = (0,0)$. Now we choose $\eta \in V_z(E)$ to be an arrow of length 1 pointing up, so $(0,1) \in \mathbb{R}^2 \cong T_z(E)$, and ξ an arrow of length 1 pointing to the right, so $1 \in \mathbb{R} \cong T_x(B)$. Now we need to find $\zeta \in T_z(E)$ such that $(d\pi_1)_z\zeta = \xi$, choose for example an arrow of length 1 pointing to the right, so $(1,0) \in \mathbb{R}^2 \cong T_z(E)$. Then
$$\langle \omega_x(z;\eta), \xi\rangle = \omega(z;\zeta,\eta) = dx\left(\frac{\partial}{\partial x}\right) dy\left(\frac{\partial}{\partial y}\right) - dx\left(\frac{\partial}{\partial y}\right) dy\left(\frac{\partial}{\partial x}\right) = 1$$
Finally, we define $\int_F \omega$, a 1-form on $[0,1]$. We will show that this 1-form equals dx by testing it on the tangent vector ξ.
$$\left(\int_F \omega\right)_x(\xi) = \int_{F_x} \omega(z;\zeta,-) = \int_{\{x\}\times[0,1]} dy = \int_{y=0}^1 dy = 1.$$
Here we note that $\omega(z;\zeta,-) : T_zE \to \mathbb{R}$ behaves exactly like dy, when we consider the fact that $\omega(z;\zeta,\frac{\partial}{\partial y}) = 1$. Here we use notation slightly abusively, since we do not distinguish between the variable y in the fiber F_x and in the total space E, but this is a harmless identification.

Lastly, following Chapter 6 of “Differential Forms in Algebraic Topology” by R. Bott and L.W. Tu [BT82] we will make this concept even more explicit. First, consider a trivial vector bundle $M \times \mathbb{R}^n \xrightarrow{\pi} M$. We can decompose any $\omega \in \Omega^k(M \times \mathbb{R}^n)$ as parts which contain $dt_1 \wedge ... \wedge dt_n$ and those which do not. Write $\omega = (\pi^*\alpha)f(x,t_1,...,t_n)dt_1 \wedge ... \wedge dt_n + \beta$. Then for $x \in M$
$$(\pi_*\omega)_x := \alpha_x \cdot \int_{\mathbb{R}^n} f(x,t_1,...,t_n)dt_1...dt_n,$$
where (each) f has compact support for every x. Then, for an arbitrary orientable vector bundle $E \xrightarrow{\pi} M$ we do the above locally, it only remains to check that on the intersections $U_\alpha \cap U_\beta$ the different forms agree. Let ω be a differential form on U_α. Locally we can express $\omega|_{U_\alpha} = (\pi^*\alpha)f(x,t)dt_1 \wedge ... \wedge dt_n + \alpha'$ and $\omega|_{U_\beta} = (\pi^*\beta)g(x,s)ds_1 \wedge ... \wedge ds_n + \beta'$. Then we have that $s = \varphi(x,t)$ and $ds = \frac{\partial\varphi}{\partial t}(x,t)dt + \frac{\partial\varphi}{\partial x}(x,t)dx$, hence
$$ds_1 \wedge ... \wedge ds_n = \det\left(\frac{\partial\varphi}{\partial t}(x,t)\right) dt_1 \wedge ... \wedge dt_n + \varepsilon,$$
where ε is no longer a top-degree form with respect to the coordinates on the fiber. Hence
$$\alpha(x)f(x,t) = \beta(x)g(x,\varphi(x,t))\det\left(\frac{\partial\varphi}{\partial t}(x,t)\right).$$
Then we finally find:
$$\begin{aligned}\beta(x)\int_{\pi^{-1}(x)} g(x,s)ds &= \beta(x)\int_{\pi^{-1}(x)}\left(g(x,\varphi(x,t))\det\left(\frac{\partial\varphi}{\partial t}(x,t)\right)dt + \varepsilon\right)\\ &= \alpha(x)\int_{\pi^{-1}(x)} f(x,t)dt,\end{aligned}$$
and hence the push-forward, *integration along the fiber*, defines a well-defined global form on the base. This construction clearly shows that $(\pi_1)_*(dx \wedge dy) = dx$ as in the previous example.

Section 1.2: Green's Currents and Green's Forms

Following the paper "Arithmetic intersection theory" by H. Gillet and C. Soulé [GS90a], we will define Green's currents and forms associated to an algebraic cycle Z. These play a particularly important role in this thesis as they are used to define the Archimedean height pairing. It is in general very hard to compute these explicitly. However, in the special case where the cycle is a principal divisor, we can compute it explicitly, as in Example 1.5.

Let X denote a complex manifold of complex dimension n. Let $\mathcal{D}_k(X)$ denote the bornological dual of $\Omega_c^k(X)$. This dual space is called the space of currents. We can decompose the differential k-forms as

$$\Omega_c^k(X) = \bigoplus_{p+q=k} \Omega_c^{p,q}(X)$$

where a differential form ω of bidegree (p, q) can locally, in a holomorphic chart with local coordinates $z_1, \ldots, z_n$, be written as

$$\omega|_U = \sum_{|I|=p,|J|=q} \omega_{I,J} dz_I \wedge d\bar{z}_J,$$

meaning there are p holomorphic factors dz_j's and q anti-holomorphic factors $d\bar{z}_j$'s. This induces a similar decomposition on the space of currents, denoted by $\mathcal{D}_{p,q}(X)$. We can also decompose the exterior derivative as $d = \partial + \bar{\partial}$.

Example 1.3. Observe that $dz_j = dx_j + idy_j$ and $d\bar{z}_j = dx_j - idy_j$. Any smooth f can be differentiated

$$df = \partial f + \bar{\partial} f = \sum_{j=1}^{d} \frac{\partial f}{\partial z_j} dz_j + \sum_{j=1}^{d} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j,$$

where $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$ and $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$. Thus in particular, $\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X)$ and $\bar{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$. More concretely, if $\omega = f(z_1, z_2, \bar{z}_1, \bar{z}_2) dz_1 \wedge d\bar{z}_2 \in \Omega^{1,1}(X)$, then

$$\partial\omega = \frac{\partial f}{\partial z_2} dz_1 \wedge d\bar{z}_2 \wedge dz_2 = -\frac{\partial f}{\partial z_2} dz_1 \wedge dz_2 \wedge d\bar{z}_2$$
$$\bar{\partial}\omega = \frac{\partial f}{\partial \bar{z}_1} dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_1 = -\frac{\partial f}{\partial \bar{z}_1} dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2.$$

In a similar fashion, the dual of d, also denoted by d, decomposes as well.

Now any smooth oriented k-simplex $\sigma : \Delta^k \to X$ or k-chain $c = \sum_i^n a_i \sigma_i$ defines a current $\delta_c \in \mathcal{D}_k(X)$ defined by

$$\delta_c(\omega) = \int_c \omega = \sum_i^n a_i \int_{\Delta^k} \sigma_i^* \omega.$$

Because of Stokes' theorem we have $d(\delta_c) = \delta_{\partial c}$ as

$$(d\delta_c)(\omega) = \delta_c(d\omega) = \int_c d\omega = \sum_i^n a_i \int_{\Delta^k} \sigma_i^*(d\omega) = \sum_i^n a_i \int_{\Delta^k} d(\sigma_i^*\omega) = \sum_i^n a_i \int_{\partial\Delta^k} \sigma_i^*\omega = \delta_{\partial c}.$$

Next up, we will move to analytic subspaces. Endow $\mathbb{C}^k$ with coordinates $z_1, ..., z_k$, and the orientation form

$$\left(\frac{i}{2}\right)^k dz_1 \wedge d\bar{z}_1 \wedge ... \wedge dz_k \wedge d\bar{z}_k = dx_1 \wedge dy_1 \wedge ... \wedge dx_k \wedge dy_k.$$

Then any closed submanifold $Z \subset X$ of (complex) dimension k has a canonical induced orientation. Thus, we can associate to any such Z a current $\delta_Z \in \mathcal{D}_{2k}(X)$ defined by

$$\delta_Z(\omega) = \int_Z i^*\omega,$$

where $i : Z \to X$ is the inclusion and $\omega \in \Omega_c^{2k}(X)$. Observe that $\delta_Z \in \mathcal{D}_{k,k}(X)$, as $i^*\omega = 0$ unless ω is a form of bi-degree (k,k). We extend this linearly, so when Z is a *smooth analytic cycle* given by $Z = \sum_i n_i[Z_i]$ then $\delta_Z = \sum_i n_i \delta_{Z_i}$. By a result of Lelong [GS90a, Le57], we can extend this to analytic subspaces Z that are not necessarily smooth by setting

$$\delta_Z(\omega) = \int_{Z^{ns}} i^*\omega = \int_{\tilde{Z}} \pi^* i^*\omega,$$

where Z^{ns} is the dense and open subset of smooth points of Z, and $\pi : \tilde{Z} \to Z$ is a resolution of singularities of Z.

Let $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$, and consequently $dd^c = \frac{i}{\pi}\partial\bar{\partial}$ (we use the convention as in [Bos90]). For notational simplicity we define

$$\mathcal{D}^{p,q}(X) = \mathcal{D}_{n-p,n-q}(X)$$

and

$$\tilde{\Omega}^{p,q}(X) = \Omega^{p,q}(X)/(\partial\Omega^{p-1,q}(X) + \bar{\partial}\Omega^{p,q-1}(X)),$$
$$\tilde{\mathcal{D}}^{p,q}(X) = \mathcal{D}^{p,q}(X)/(\partial\mathcal{D}^{p-1,q}(X) + \bar{\partial}\mathcal{D}^{p,q-1}(X)).$$

We call a current $T \in \mathcal{D}^{k,k}$ *real* if

$$T(\omega) = \overline{T(\bar{\omega})}$$

for any $\omega \in \Omega_c^{k,k}(X)$. Note that we have a map

$$\Omega^{p,q}(X) \to \mathcal{D}_{n-p,n-q}(X),$$
$$\omega \mapsto [[\omega]],$$

where

$$[[\omega]](\eta) = \int_X \omega \wedge \eta$$

for any $\eta \in \Omega_c^{n-p,n-q}$.

Definition 1.4. Let X be a complex manifold, and $Z = \sum_{i=1}^n a_i[Z_i]$ be an analytic cycle on X of codimension k. We call an element $G_Z \in \tilde{\mathcal{D}}^{k-1,k-1}(X)$ a *Green's current* for Z if it is the class of a real current, such that

$$dd^c G_Z + \delta_Z = [\![\omega]\!],$$

where ω is a $\mathcal{C}^\infty$-form of bidegree (k,k).

As mentioned in the Introduction, for a specific type of cycle we can compute the Green's current explicitly, namely a principal divisor. Let X be a connected complex manifold and let $\mathcal{M}_X$ denote the sheaf of meromorphic functions on X. For a non-zero meromorphic function $f \in \mathcal{M}_X(X)^\times$, and for each irreducible analytic hypersurface $Z \subset X$ (i.e. an irreducible analytic subset of codimension 1), one defines the *order* $\mathrm{ord}_Z(f) \in \mathbb{Z}$ as follows. Choose a smooth point $z \in Z$. Then there exists a neighborhood U of z and a holomorphic function $g \in \mathcal{O}_X(U)$ with $Z \cap U = \{g = 0\}$ and $dg \neq 0$ along $Z \cap U$. On U one can write

$$f = g^m u,$$

where $u \in \mathcal{O}_X(U)^\times$ is a nowhere-vanishing holomorphic function (a unit) and $m \in \mathbb{Z}$. The integer m is independent of all choices and is called $\mathrm{ord}_Z(f)$, with positive values corresponding to zeros and negative values to poles. The *principal divisor* associated to f is

$$\mathrm{Div}(f) = \sum_Z \mathrm{ord}_Z(f)\,[Z],$$

where the sum runs over all irreducible analytic hypersurfaces $Z \subset X$ and is locally finite.

Example 1.5. Let X be a connected complex manifold and $Z = \mathrm{Div}(f)$ a principal divisor. We will show that $[\![-\log|f|]\!]$ is a Green's current for Z, i.e.,

$$dd^c[\![-\log|f|]\!] + \delta_{\mathrm{Div}(f)} = [\![\omega]\!],$$

for some $\mathcal{C}^\infty$-form ω.

Write $Z = \sum_{Z_i} \mathrm{ord}_{Z_i}(f)[Z_i]$ and locally near each component we can choose coordinates such that $Z_i = \{z_1 = 0\}$ and write

$$f = z_1^{\mathrm{ord}_{Z_i}(f)} \cdot u_i,$$

where u_i is some non-vanishing unit. Then locally $\log|f| = \mathrm{ord}_{Z_i}(f)\cdot\log|z_1| + \log|u_i|$. We now want to compute $dd^c[\![-\log|f|]\!]$, but first let us consider an even easier case. In just one variable z we will compute $dd^c[\![-\log|z|]\!]$ and show that it equals $-\delta_0$.

Approximate $-\log|z|$ by the functions $f_\varepsilon(z) := -\frac{1}{2}\log(|z|^2+\varepsilon)$. Then by explicit calculations and because f_ε is smooth for all $\varepsilon \neq 0$,

$$dd^c[\![f_\varepsilon(z)]\!] = [\![dd^c f_\varepsilon(z)]\!] = \left[\!\!\left[-\frac{i}{2\pi}\frac{\varepsilon}{(|z|^2+\varepsilon)^2}dz\wedge d\bar{z}\right]\!\!\right] = \left[\!\!\left[-\frac{1}{2}\frac{\varepsilon}{(|z|^2+\varepsilon)^2}dx\wedge dy\right]\!\!\right].$$

We say that $dd^c[\![f_\varepsilon]\!]$ converges to $-\delta_0$ as currents if for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ we have that

$$\int_{\mathbb{R}^2} dd^c f_\varepsilon(x,y)\varphi(x,y) \xrightarrow{\varepsilon\to 0} \int_{\mathbb{R}^2} -\delta_0(x,y)\varphi(x,y) = -\varphi(0,0).$$

Now let $x = r\cos\theta$ and $y = r\sin\theta$, then the integral above evaluates to

$$-\frac{\varepsilon}{\pi}\int_0^\infty\int_0^{2\pi}\frac{\varphi(r\cos\theta, r\sin\theta)r}{(r^2+\varepsilon)^2}d\theta dr,$$

now let $t = \frac{r^2}{\varepsilon}$. Then this becomes

$$-\frac{\varepsilon}{\pi}\int_0^\infty\int_0^{2\pi}\frac{\varphi(\sqrt{\varepsilon t}\cos\theta,\sqrt{\varepsilon t}\sin\theta)r}{(\varepsilon t+\varepsilon)^2}\frac{\varepsilon}{2}d\theta dt = -\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\frac{\varphi(\sqrt{\varepsilon t}\cos\theta,\sqrt{\varepsilon t}\sin\theta)r}{(t+1)^2}d\theta dt$$

Then using the dominated convergence theorem we get that this converges to

$$-\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\frac{\varphi(0,0)}{t^2+1}d\theta dt = -\varphi(0,0).$$

So indeed, we have now shown that $dd^c[\![-\log|z|]\!] = -\delta_0$ and in similar fashion $dd^c[\![-\log|z_1|]\!] = -\delta_{\{z_1=0\}} = -\delta_{Z_i}$. So, to answer our original question

$$\begin{aligned} dd^c[\![-\log|f|]\!] &= \sum_{Z_i}\mathrm{ord}_{Z_i}(f)\cdot dd^c[\![-\log|z_1|]\!] + dd^c[\![-\log|u_i|]\!] \\ &= \sum_{Z_i}\mathrm{ord}_{Z_i}(f)\cdot -\delta_{Z_i} + [\![dd^c(-\log|u_i|)]\!] \\ &= -\delta_Z + [\![\omega]\!]. \end{aligned}$$

Here $dd^c[\![-\log|u_i|]\!] = [\![dd^c(-\log|u_i|)]\!]$ because the $\log|u_i|$ are smooth.

We would like to stress that in the case of divisors (i.e., codimension 1 cycles) a Green's current is the class of a real current in $\tilde{\mathcal{D}}^{0,0}(X)$. Thus it is often locally given by a distribution.

Example 1.6. Let us make the previous example even more clear. Consider the ambient space $X = \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = \frac{(z-P)(z-Q)}{(z-R)^3},$$

so that $\mathrm{Div}(f) = [P]+[Q]-3[R]$. Locally near the point P, we can use the change of coordinates $w = z - P$ and we have that $[P] = \{w = 0\}$. So then f can be written as

$$f(w) = w^1 \cdot \frac{w+P-Q}{(w+P-R)^3},$$

so w times a unit. We conclude that locally the Green's current corresponding to $\mathrm{Div}(f)$ is given by $[\![-\log|w|]\!]$. Globally we can write

$$G_{\mathrm{Div}(f)} = \left[\!\!\left[-\log\frac{|z-P||z-Q|}{|z-R|^3}\right]\!\!\right] = [\![3\log|z-R| - \log|z-P| - \log|z-Q|]\!] .$$

Finally, we get to the definition of a Green's form. Let Z denote a closed analytic subset of X. If a form ω is only defined on $X \setminus Z$, i.e., $\omega \in \Omega^{p,q}(X \setminus Z)$, but is locally L^1 near Z, we can define the current on *all* of X, denoted by $[\![\omega]\!]$ given by

$$[\![\omega]\!](\eta) = \int_{X\setminus Z} \omega \wedge \eta, \qquad \eta \in \Omega_c^{n-p,n-q}(X).$$

Definition 1.7. Let X be a complex manifold, and $Z = \sum_{i=1}^n a_i[Z_i]$ be an analytic cycle on X of codimension k. A *Green's form* for Z is a smooth form $g_Z \in \tilde{\Omega}^{k-1,k-1}(X \setminus |Z|)$, which is locally L^1 integrable near $|Z|$, such that $[[g_Z]]$ is a Green's current for Z.

In the situation of Example 1.5, a Green's *form* for $\mathrm{Div}(f)$ is thus given by

$$g_Z = -\log|f|,$$

which is smooth on $X \setminus |Z|$.

Section 1.3: The Archimedean Height Pairing

In this section we will introduce the *Archimedean* height pairing, which assigns a real number to a pair of disjoint analytic cycles Z and Y on a complex manifold X. In this thesis we always work with a compact ambient space X, e.g. complex projective n-space $\mathbb{CP}^n$ or more generally complex Grassmannians

$$\mathrm{Gr}_k(\mathbb{C}^n) = \{k\text{-dimensional complex linear subspaces of } \mathbb{C}^n\}.$$

Definition 1.8. An analytic cycle $Z \in Z_k(X)$, of complex dimension k, determines a homology class $[Z] \in H_{2k}(X, \mathbb{Z})$ and via Poincaré duality a cohomology class $[Z] \in H^{2(\dim_{\mathbb{C}}(X)-k)}(X, \mathbb{Z})$. We call a cycle Z *homologically trivial* if $[Z] = 0$.

Under the assumption that both cycles Z and Y are homologically trivial, we will define the Archimedean height pairing. There is one more condition, we require Z and Y to be of *sub-complementary* dimension, i.e.

$$\dim_{\mathbb{C}}(Z) + \dim_{\mathbb{C}}(Y) = \dim_{\mathbb{C}}(X) - 1.$$

Definition 1.9. Let X be a compact complex manifold and let Z, Y be disjoint homologically trivial cycles of sub-complementary dimension. Then the Archimedean height pairing is defined as

$$\langle Z, Y\rangle_\infty = -\int_Y g_Z \tag{1.1}$$

where g_Z is a Green's form for Z on X.

Under the above assumptions, the Archimedean height pairing is symmetric, i.e. $\langle Z, Y\rangle_\infty = \langle Y, Z\rangle_\infty$, see [Bos90, p. 5]. Furthermore, we might explicitly write $\langle Z, Y\rangle_\infty^X$ to stress that the cycles Z and Y are in the ambient space X.

In general, it is very hard to compute Green's currents, Green's forms and Archimedean heights explicitly. We will work towards providing a non-trivial, yet explicit, calculation of the Archimedean height of disjoint differences of linear subspaces of sub-complementary dimension in complex projective space. The main idea is the fact that we *can* compute the Green's form for principal divisors very easily. From there computing the Archimedean height provides no additional issues, since integrating over points simply boils down to evaluation.

$$\langle \mathrm{Div}(f), P - Q\rangle_\infty = -\int_X \delta_{P-Q} \wedge g_Z = \log|f(P)| - \log|f(Q)|.$$

In the next two sections we will show that Green's currents and Green's forms are very well behaved under push-forward and pull-back along appropriate maps. It is via this construction that we will then relate the Archimedean height of linear subspaces in projective space to principal divisors and points in another ambient space, the *Grassmannians*. In sections 2.1 and 2.4 we will introduce these spaces briefly.

Section 1.4: Push-Forward and Pull-Back of Green's Currents

The following two subsections can be viewed as a recollection of standard results, written down carefully, to be used in the proof of Lemma 1.23. Throughout these sections let X and Y denote complex manifolds.

Lemma 1.10. *Let $p : X \to Y$ be a holomorphic map. Let g be a current on X, and let μ denote ∂ or $\overline{\partial}$, then*

$$\mu(p_* g) = p_*(\mu g).$$

Proof. Let $w_1, ..., w_m$ be local coordinates on Y and $z_1, ..., z_n$ on X. We can write a general (p, q)-form ω on Y as

$$\omega = \sum_{I,J} g_{I,J} dw_I \wedge d\bar{w}_J.$$

Now, by definition

$$p^*\omega = \sum_{I,J} (g_{I,J} \circ p)\ p^*(dw_I) \wedge p^*(d\bar{w}_J)$$

Then, by the Leibniz rule, we have

$$\begin{aligned}\partial(p^*\omega) &= \sum_{I,J} \partial(g_{I,J}\circ p)\ p^*(dw_I)\wedge p^*(d\bar{w}_J) + \sum_{I,J}(g_{I,J}\circ p)\ \partial(p^*(dw_I))\wedge p^*(d\bar{w}_J)\\ &\pm \sum_{I,J}(g_{I,J}\circ p)\ p^*(dw_I)\wedge \partial(p^*(d\bar{w}_J))\\ &= \sum_{I,J} \partial(g_{I,J}\circ p)\ p^*(dw_I)\wedge p^*(d\bar{w}_J) + \sum_{I,J}(g_{I,J}\circ p)\ \partial(p^*(dw_I))\wedge p^*(d\bar{w}_J)\\ &= \sum_{I,J}(\partial g_{I,J}\wedge dw_I\wedge d\bar{w}_J)\circ p + \sum_{I,J} g_{I,J}\circ p\ (\partial(dw_I)\wedge d\bar{w}_J)\circ p\\ &= p^*(\partial\omega)\end{aligned}$$

Here $\partial(p^*(d\bar{w}_j)) = 0$ for all j exactly because p is holomorphic. The last equality is justified because the pull-back along a holomorphic function of a (complex) function commutes with the ∂ operator. This follows simply by the chain rule

$$\partial(g\circ p) = \sum_{i=1}^{n}\frac{\partial(g\circ p)}{\partial z_j}dz_j = \sum_{j=1}^{m}\frac{\partial g}{\partial w_j}(p(z))\frac{\partial(w_j\circ p)}{\partial z_i} + \sum_{j=1}^{m}\frac{\partial g}{\partial \bar{w}_j}(p(z))\frac{\partial(\overline{w_j\circ p})}{\partial z_i}$$

where the last sum vanishes because p is holomorphic, and hence equals $p\circ\partial g$. Now similarly this works for $\mu = \bar{\partial}$, where now all the $\bar{\partial}(p^*(dw_j))$ terms will vanish. Now we will show equality of the currents $\mu(p_*g)$ and $p_*(\mu g)$ by showing they are equal when paired with any (suitable) form ω.

$$\langle\mu(p_*g),\omega\rangle = \langle p_*g,\mu\omega\rangle = \langle g, p^*(\mu\omega)\rangle = \langle g,\mu(p^*\omega)\rangle = \langle \mu g, p^*\omega\rangle = \langle p_*(\mu g),\omega\rangle.$$

This shows the desired equality of currents. □

Corollary 1.11. *Let $p : X \to Y$ be a holomorphic map. Let g be a current on X, then*

$$dd^c(p_*g) = p_*(dd^cg).$$

Proof. Observe that $dd^c = \frac{i}{\pi}\partial\bar{\partial}$, hence

$$dd^c(p_*g) = \frac{i}{\pi}\partial\bar{\partial}(p_*g) = \frac{i}{\pi}\partial p_*(\bar{\partial}g) = \frac{i}{\pi}p_*(\partial\bar{\partial}g) = p_*(\frac{i}{\pi}\partial\bar{\partial}g) = p_*(dd^cg),$$

as desired. □

This shows that the operator dd^c commutes with holomorphic push-forward. We will now show that it also commutes with pull-back along proper surjective submersions.

Lemma 1.12. *Let $p : X \to Y$ be a proper surjective submersion. Let g be a current on Y, and let μ denote ∂ or $\overline{\partial}$, then*

$$\mu(p^*g) = p^*(\mu g).$$

Proof. Because $p : X \to Y$ is a proper surjective submersion we can apply *Ehresmann's fibration theorem*, meaning p is a locally trivial smooth fiber bundle. Thus locally, above some neighbourhood $U \subset Y$ we have a commuting square

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi} & U\times F\\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_1}\\ U & = & U\end{array}$$

where $\varphi : p^{-1}(U) \to U \times F$ is a diffeomorphism. Thus we may reduce to the case where $p : Y \times F \to Y$ is the projection, with compact fiber F (without boundary). In this case we may decompose the exterior derivative

$$\mu = \mu_Y + \mu_F$$

where μ_Y takes the derivative in the Y-component and μ_F only considers the vertical derivative. Then

$$\begin{aligned} p_*(\mu\omega) &= p_*(\mu_Y\omega) + p_*(\mu_F\omega) \\ &= \int_F \mu_Y\omega + \int_F \mu_F\omega \\ &= \mu_Y \int_F \omega + \int_{\partial F} \omega \\ &= \mu(p_*\omega) \end{aligned}$$

Here we used Stokes' theorem and the fact that the boundary of F is empty, together with the dominated convergence theorem to bring the μ_Y operator outside of the integral over the fiber. Because push-forward and μ commute when applied to differential forms, it follows by duality that pull-back and μ commute when applied to currents. A partition of unity argument can glue these local pieces together to show this holds for general p, under the assumed conditions. □

Corollary 1.13. *Let $p : X \to Y$ be a proper surjective submersion. Let g be a current on Y, then*

$$dd^c(p^*g) = p^*(dd^cg).$$

Proof. Again, $dd^c = \frac{i}{\pi}\partial\bar{\partial}$, hence

$$dd^c(p^*g) = \frac{i}{\pi}\partial\bar{\partial}(p^*g) = \frac{i}{\pi}\partial p^*(\bar{\partial}g) = \frac{i}{\pi}p^*(\partial\bar{\partial}g) = p^*(\frac{i}{\pi}\partial\bar{\partial}g) = p^*(dd^cg),$$

as desired. □

Now we will define the pull-back and push-forward of analytic cycles.

Definition 1.14. Let $f : X \to Y$ be a holomorphic map. If f is a submersion of relative dimension r, then f induces a pull-back on the cycle groups

$$\begin{aligned} f^* : Z_k(Y) &\to Z_{k+r}(X) \\ \sum_i n_i[Z_i] &\mapsto \sum_i n_i[f^{-1}(Z_i)] \end{aligned}$$

If f is proper, then f induces a push-forward on the cycle groups

$$\begin{aligned} f_* : Z_k(X) &\to Z_k(Y) \\ \sum_i n_i[W_i] &\mapsto \sum_i n_i \deg(W_i/f(W_i))[f(W_i)], \end{aligned}$$

if $\dim(f(W_i)) = \dim(W_i) = k$, and otherwise we set $f_*[W_i] = 0$.

Proposition 1.15. *Let $p : X \to Y$ be a proper holomorphic map. Let $Z = \sum_i n_i[Z_i]$ be an analytic cycle on X, such that $p|_{Z_i} : Z_i \to p(Z_i)$ is a biholomorphism of analytic spaces for each i, then*

$$p_*\delta_Z = \delta_{p_*(Z)}.$$

Proof. Let ω be a form on Y, then

$$\begin{aligned}\langle p_*\delta_Z, \omega\rangle &= \langle \delta_Z, p^*\omega\rangle \\ &= \sum_i n_i \int_{Z_i} p^*\omega \\ &= \sum_i n_i \int_{p(Z_i)} \omega \\ &= \langle \delta_{p_*(Z)}, \omega\rangle.\end{aligned}$$

The only non-trivial step, $\int_{Z_i} p^*\omega = \int_{p(Z_i)} \omega$, follows from [Lee13, Proposition 16.1], together with the fact that $p|_{Z_i} : Z_i \to p(Z_i)$ is a diffeomorphism on the underlying smooth manifolds, for all i. □

Proposition 1.16. *Let $p : X \to Y$ be a holomorphic proper surjective submersion. Let $Z = \sum_i n_i[Z_i]$ be an analytic cycle on Y, then*

$$p^*\delta_Z = \delta_{p^*Z}.$$

Proof. We first prove the statement under the additional assumption that each component Z_i is smooth. The general case follows from the same argument applied to the regular loci Z_i^{reg}. Indeed, the integration current associated to an analytic cycle is defined by integration over its regular locus, and the singular locus has strictly smaller dimension, hence does not contribute to the current. Because $p : X \to Y$ is a proper surjective submersion we can apply Ehresmann's fibration theorem, meaning p is a locally trivial smooth fiber bundle. Thus locally, above some neighbourhood $U \subset Y$ we have a commuting square

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ U & = & U \end{array}$$

where $\varphi : p^{-1}(U) \to U \times F$ is a diffeomorphism. Thus we may reduce to the case where $p : Y \times F \to Y$ is the projection, with compact fiber F. Let ψ be a form on $Y \times F$, then we may (locally) write

$$\psi = \sum_{I,J} g_{I,J}(x, y) dx_I \wedge dy_J,$$

where $\{x_1, ..., x_m\}$ denotes the local coordinates on F and $\{y_1, ..., y_n\}$ denotes those on Y. Let $\tilde{g}_J$ be equal to $g_{I,J}$ in the special case where $I = \{1, ..., m\}$. Then (locally)

$$\begin{aligned}\langle p^*\delta_Z, \psi\rangle &= \langle \delta_Z, p_*\psi\rangle \\ &= \sum_i n_i \int_{Z_i} p_*\psi \\ &= \sum_i n_i \int_{Z_i} \sum_J \left(\int_{p^{-1}(y)} \tilde{g}_J(x, y) dx \right) dy_J \\ &= \sum_i n_i \int_{Z_i} \sum_J \left(\int_F \tilde{g}_J(x, y) dx \right) dy_J \\ &= \sum_i n_i \int_{Z_i \times F} \psi \\ &= \sum_i n_i \int_{p^{-1}(Z_i)} \psi \\ &= \langle \delta_{p^*(Z)}, \psi\rangle.\end{aligned}$$

Because the integral is defined locally, these local pieces glue together and hence the desired (global) conclusion follows. This is typically done with a *partition of unity* type argument. □

Proposition 1.17. *Let $p : X \to Y$ be a holomorphic proper surjective submersion map. Let $Z = \sum_i n_i[Z_i]$ be an analytic cycle, such that $p|_{Z_i} : Z_i \to p(Z_i)$ is a biholomorphism of analytic spaces for each i. Let G_Z be a Green's current for Z on X, then $p_*(G_Z)$ is a Green's current for $p_*(Z)$ on Y.*

Proof. This is a direct consequence of Proposition 1.15 and Corollary 1.11.

$$\begin{aligned} dd^c(p_*(G_Z)) &= p_*(dd^c(G_Z)) \\ &= p_*(-\delta_Z + [\![\omega_Z]\!]) \\ &= -\delta_{p_*(Z)} + p_*([\![\omega_Z]\!]). \end{aligned}$$

Because ω_Z is a smooth form on X, now $p_*([\![\omega_Z]\!]) = [\![p_*\omega_Z]\!]$ comes from a smooth form on Y. Hence we find that $p_*(G_Z)$ satisfies the defining equation for a Green's current for $p_*(Z)$ on Y, i.e.,

$$dd^c(p_*(G_Z)) + \delta_{p_*(Z)} = [\![\eta]\!],$$

for some smooth form η on Y. □

Proposition 1.18. *Let $p : X \to Y$ be a proper surjective submersion. Let $Z = \sum_i n_i[Z_i]$ be an analytic cycle on Y, and let G_Z be a Green's current for Z on Y, then $p^*(G_Z)$ is a Green's current for $p^*(Z)$ on X.*

This proof is analogous to that of Proposition 1.17, where we combine Proposition 1.16 and Corollary 1.13.

Section 1.5: Push-Forward and Pull-Back of Green's Forms

Whenever we talk about the Archimedean height pairings we must consider Green's *forms* and not Green's currents. Recall that for a differential form ω we let $[[\omega]]$ denote the associated current given by

$$[[\omega]](\eta) = \int_X \omega \wedge \eta,$$

for suitable η. Again, let X and Y denote complex manifolds and every map is holomorphic.

Lemma 1.19. *Let $p : X \to Y$ be a proper surjective submersion. Let η be a differential form on X, then we have an equality of currents*

$$p_*[[\eta]] = [[p_*(\eta)]].$$

Proof. As before, we can restrict to the case where $p : Y \times F \to Y$ is the projection map. Let ω be a suitable test form, then

$$\begin{aligned} p_*[[\eta]](\omega) &= [[\eta]](p^*(\omega)) \\ &= \int_{F\times Y} \eta \wedge p^*(\omega) \\ &= \int_{p^{-1}(Y)} \eta \wedge p^*(\omega) \\ &= \int_Y p_*(\eta \wedge p^*(\omega)) \\ &= \int_Y p_*(\eta) \wedge \omega \\ &= [[p_*(\eta)]](\omega). \end{aligned}$$

Since the currents agree on any test form ω, they agree as currents. □

Lemma 1.20. *Let $p : X \to Y$ be a proper surjective submersion. Let η be a form on Y, then*

$$p^*[[\eta]] = [[p^*(\eta)]].$$

Proof. We can do this locally, so assume that $p : Y \times F \to Y$ is the projection map. Let ω be a suitable test form, then

$$\begin{aligned} p^*[[\eta]](\omega) &= [[\eta]](p_*(\omega)) \\ &= \int_Y \eta \wedge p_*(\omega) \\ &= \int_Y p_*(p^*\eta \wedge \omega) \\ &= \int_{Y \times F} p^*(\eta) \wedge \omega \\ &= [[p^*(\eta)]](\omega). \end{aligned}$$

Since the currents agree on any test form ω, they agree as currents. $\square$

Proposition 1.21. *Let $p : X \to Y$ be a proper surjective submersion of complex manifolds. Let $Z \subset Y$ be an analytic subspace, and let g_Z be a Green's form for Z on Y, then $p^*(g_Z)$ is a Green's form for $p^*(Z)$ on X.*

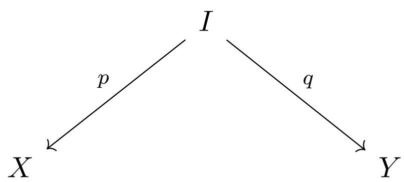
Proof. Because g_Z is a Green's form for Z on Y, we have by definition that $[[g_Z]]$ is a Green's current for Z on Y. Then by Proposition 1.18 we have that $p^*[[g_Z]]$ is a Green's current for $p^*(Z)$ on X. Moreover, by Lemma 1.20, we note that $p^*[[g_Z]] = [[p^*(g_Z)]]$ as currents on $X \setminus p^{-1}Z$. Since g_Z is L^1 near Z, so is p^*g_Z near $p^{-1}Z$ thus we may extend and conclude that $p^*[[g_Z]] = [[p^*(g_Z)]]$ as currents on X. Thus $p^*(g_Z)$ is a Green's form for $p^*(Z)$ on X. $\square$

Proposition 1.22. *Let $p : X \to Y$ be a holomorphic proper surjective submersion. Let $W \subset X$ be an analytic subspace, such that $p|_W : W \to p(W)$ is a biholomorphism of analytic spaces. Let g_W be a Green's form for W on X, then $p_*(g_W)$ is a Green's form for $p_*(W)$ on Y.*

Proof. Because g_W is a Green's form for W on X, we have by definition that $[[g_W]]$ is a Green's current for W on X. Then by Proposition 1.17 we have that $p_*[[g_W]]$ is a Green's current for $p_*(W)$ on Y. Moreover, by Lemma 1.19, we note that $p_*[[g_W]] = [[p_*(g_W)]]$ as currents on Y because g_W is L^1 near W, we have that p_*g_W is L^1 near $p(W)$ and hence the induced currents agree on Y. Thus, $p_*(g_W)$ is a Green's form for $p_*(W)$ on Y. $\square$

Section 1.6: Push-Pull of Archimedean Heights

Now we are ready to combine the push-forward and pull-back of Green's forms, Propositions 1.22 and 1.21 together with Propositions 1.15 and 1.16 to relate the Archimedean height pairings in different ambient spaces. Consider the following situation



Here X, Y and I are compact complex manifolds, p and q are holomorphic proper surjective submersions. Furthermore, let Z and W denote disjoint homologically trivial cycles of sub-complementary dimension in I. Write

$$Z = \sum_i n_i[Z_i]$$
$$W = \sum_j m_j[W_j]$$

Lemma 1.23. *In the situation as above assume that, for each j, $q|_{W_j} : W_j \to q(W_j)$ is a biholomorphism of analytic spaces. Lastly, suppose that, for all j, $p^{-1}(p(W_j)) = W_j$ and* $\deg(W_j/p(W_j)) = 1$*. Then*

$$\langle q_*(Z), q_*(W)\rangle_\infty^Y = \langle p_*(q^*(q_*(Z))), p_*(W)\rangle_\infty^X$$

Proof. Let g be a Green's form for $q_*(Z)$ on Y, then by Proposition 1.21 $q^*(g)$ is a Green's form for $q^*q_*(Z)$ on I. Then, by Proposition 1.22 $p_*q^*(g)$ is a Green's form for $p_*(q^*(q_*(Z)))$ on X. Then,

$$\begin{aligned}
\langle q_*(Z), q_*(W)\rangle_\infty^Y &= -\sum_j \int_{q(W_j)} g \\
&= -\sum_j \int_{W_j} q^*(g) \\
&= -\sum_j \int_{p^{-1}(p(W_j))} q^*(g) \\
&= -\sum_j \int_{p(W_j)} p_*(q^*(g)) \\
&= \langle p_*(q^*(q_*(Z))), p_*(W)\rangle_\infty^X
\end{aligned}$$

Note that all the induced maps must send homologically trivial cycles to homologically trivial cycles, since they are group homomorphisms, and hence the Archimedean heights are well-defined in the ambient spaces X and Y. □

Despite all the required conditions, we are now in an excellent position. We will show that a special type of correspondence, namely the *incidence correspondence* between $\mathbb{CP}^n$ and $\mathrm{Gr}_k(\mathbb{C}^{n+1})$ satisfies all conditions of Lemma 1.23, and this will become especially useful as it can transform one of the cycles Z into a principal divisor $p_*(q^*(q_*(Z)))$ and the other cycle W into a zero-cycle, i.e., a formal sum of points, $p_*(W)$.

Chapter 2: The Archimedean Height Pairing in Projective Space and the Generalised Cross-Ratio

In Section 2.1, we will introduce the projective space $\mathbb{P}(V)$, the space of lines in a vector space V. We will focus on *complex* projective n-space, i.e., the space of complex lines in $\mathbb{C}^{n+1}$, denoted by $\mathbb{CP}^n$. We call $\mathbb{CP}^1$ the (complex) projective line, which is a one-dimensional complex manifold. In $\mathbb{CP}^1$ analytic cycles of sub-complementary dimension simply consist of formal sums of points.

A homologically trivial cycle can be decomposed into differences of points, so when computing heights in $\mathbb{CP}^1$ we have to compute $\langle P-Q, R-S\rangle_\infty$ for distinct points $P, Q, R, S \in \mathbb{CP}^1$. We will carry out this computation in Section 2.2 and show that it is related to the *cross-ratio*, a number associated to an ordered quadruple of points P, Q, R and S in $\mathbb{CP}^1$, denoted by $\mathrm{CR}(P, Q; R, S)$.

In Section 2.3, we will generalise the cross-ratio, i.e., associate a complex number to an ordered pair of two k-dimensional and two $(n-k-1)$-dimensional subspaces which are non-degenerate in $\mathbb{CP}^n$. Our goal is to show that the Archimedean height of these linear subspaces is closely related to the generalised cross-ratio. However, this requires us to compute the Archimedean height in arbitrary dimension. As hinted at in the previous section, we will do this using Lemma 1.23 together with the incidence correspondence. For this we need some knowledge of the space of k-dimensional linear subspaces of $\mathbb{C}^{n+1}$ denoted by $\mathrm{Gr}_k(\mathbb{C}^{n+1})$, called a Grassmannian. These will be introduced in Section 2.4. Lastly we will combine all the previously introduced machinery and carry out the computation in Section 2.5 and prove the following theorem.

Theorem A (Theorem 2.20)**.** Let Y_P, Y_Q, Y_R and Y_S be as above. Then the Archimedean height is given by

$$\langle Y_P - Y_Q, Y_R - Y_S\rangle_\infty = \log|\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S)|.$$

Using this result, we have an explicit formula for the Archimedean height pairing. The cross-ratio is easy to compute and manipulate, and we will exploit this heavily in the next two chapters, first to study the asymptotic behaviour of the Archimedean height as the cycles degenerate, i.e., intersect, and later to compute the non-Archimedean contributions of the global height pairing on $\mathbb{CP}^n$.

Section 2.1: Complex Projective Space and the Classical Cross-Ratio

There are many equivalent ways to define projective space. We will give a hands-on definition and describe some properties. First, we will describe projective space as a topological space. Let V be an $(n+1)$-dimensional vector space, then define

$$\mathbb{P}(V) = (V \setminus \{0\})/\sim$$

where we declare $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$ if there exists some scalar $\lambda \in \mathbb{C}^\times$ such that $y_i = \lambda x_i$ for all i. We usually denote an element of $\mathbb{P}(V)$ by $[x_0 : \cdots : x_n]$ with respect to some basis. Clearly, a vector space has a topology induced by any norm. Then $V \setminus \{0\}$ has the subspace topology and finally $\mathbb{P}(V)$ has the quotient topology. In the special case where $V = \mathbb{C}^{n+1}$, we write $\mathbb{CP}^n$ for $\mathbb{P}(V)$. Now let V be a complex vector space (of complex dimension $n+1$), then $\mathbb{P}(V)$ is an n-dimensional complex manifold. Fix a basis for V, and define

$$U_i = \{[x_0 : \cdots : x_n] \mid x_i \neq 0\}$$

then note that $U_i \cong \mathbb{C}^n$ via the map that sends

$$[x_0 : \cdots : x_n] \mapsto (x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i).$$

The transition functions between such opens are given by x_i/x_j with inverse x_j/x_i, and this gives a holomorphic atlas for $\mathbb{P}(V)$ and in particular for $\mathbb{CP}^n$. For $\mathbb{CP}^1$ in particular, there is an easy description of all points and we have an equality of sets

$$\mathbb{CP}^1 = \{[1 : z] \mid z \in \mathbb{C}\} \cup \{[0 : 1]\}.$$

Via the inclusion $\mathbb{C} \to \mathbb{CP}^1$, which maps $z \mapsto [1 : z]$, we can describe the points of $\mathbb{CP}^1$ simply by the complex numbers, together with the *point at infinity* $[0 : 1]$ denoted by ∞.

Definition 2.1. Let $P, Q, R, S \in \mathbb{CP}^1$ be distinct points. There exists a unique projective transformation which maps

$$P \longmapsto 0, \qquad Q \longmapsto 1, \qquad R \longmapsto \infty.$$

Applying this transformation to the point S will give a complex number, which is called the cross-ratio and is denoted by $\mathrm{CR}(P, Q; R, S)$.

A projective transformation is an invertible 2×2 matrix, defined up to scaling, which acts in the usual way on an element $[x : y] \in \mathbb{CP}^1$. When all points P, Q, R and S are not equal to ∞, i.e., are given by complex numbers, it is well-known that we can compute the cross-ratio via the formula

$$\mathrm{CR}(P, Q; R, S) = \frac{(P - R)(Q - S)}{(P - S)(Q - R)}.$$

The cross-ratio corresponding to four distinct points depends on the order. A priori, there are $4! = 24$ ways to order four points. However, some symmetry does exist.

Proposition 2.2. *Let $P, Q, R, S \in \mathbb{CP}^1$ be distinct, then*

$$\begin{aligned}
\mathrm{CR}(P, Q; R, S) = \mathrm{CR}(Q, P; S, R) = \mathrm{CR}(R, S; P, Q) = \mathrm{CR}(S, R; Q, P) &= \lambda,\\
\mathrm{CR}(P, Q; S, R) = \mathrm{CR}(Q, P; R, S) = \mathrm{CR}(R, S; Q, P) = \mathrm{CR}(S, R; P, Q) &= \frac{1}{\lambda},\\
\mathrm{CR}(P, R; Q, S) = \mathrm{CR}(Q, S; P, R) = \mathrm{CR}(R, P; S, Q) = \mathrm{CR}(S, Q; R, P) &= 1 - \lambda,\\
\mathrm{CR}(P, R; S, Q) = \mathrm{CR}(Q, S; R, P) = \mathrm{CR}(R, P; Q, S) = \mathrm{CR}(S, Q; P, R) &= \frac{1}{1 - \lambda},\\
\mathrm{CR}(P, S; Q, R) = \mathrm{CR}(Q, R; P, S) = \mathrm{CR}(R, Q; S, P) = \mathrm{CR}(S, P; R, Q) &= \frac{\lambda - 1}{\lambda},\\
\mathrm{CR}(P, S; R, Q) = \mathrm{CR}(Q, R; S, P) = \mathrm{CR}(R, Q; P, S) = \mathrm{CR}(S, P; Q, R) &= \frac{\lambda}{\lambda - 1}.
\end{aligned}$$

In a similar fashion, the cross-ratio may be defined over any field. Namely, if k is a field and V is a two-dimensional k-vector space, then the cross-ratio exists for four distinct points in $\mathbb{P}(V)$.

In Section 2.3 we are going to define a cross-ratio associated to higher-dimensional linear subspaces of $\mathbb{CP}^n$. It is not immediately clear if this is possible using a similar universal property, as there are many more degrees of freedom in higher-dimensional vector spaces. However, there is another way to define or interpret the classical cross-ratio, one which does hint at a canonical generalisation.

As before, let $P, Q, R, S \in \mathbb{CP}^1$ be distinct points. By definition, these points correspond to lines $l_P, l_Q, l_R, l_S \subset \mathbb{C}^2$ passing through the origin. For each of these lines, choose a generator, i.e., a non-zero vector $v_i \in l_i$. Then pick any volume form, i.e., any $\omega \in \wedge^2(\mathbb{C}^2)^* \cong (\wedge^2\mathbb{C}^2)^* \cong \mathbb{C}$ which is non-zero.

Proposition 2.3. *With notation as above*

$$\mathrm{CR}(P,Q;R,S) = \frac{\omega(v_P \wedge v_R)\omega(v_Q \wedge v_S)}{\omega(v_P \wedge v_S)\omega(v_Q \wedge v_R)},$$

and consequently, the right-hand side is independent of the choice of representatives v_P, v_Q, v_R, v_S and volume form ω.

Proof. First, we show that the right-hand side expression is independent of the choice of representatives v_P, v_Q, v_R and v_S. If we replace v_i with $\lambda_i v_i$ then, for any scalars $\lambda_i \neq 0$,

$$\omega((\lambda_i v_i) \wedge (\lambda_j v_j)) = \lambda_i \lambda_j \omega(v_i \wedge v_j),$$

and so

$$\begin{aligned} \frac{\omega((\lambda_P v_P) \wedge (\lambda_R v_R))\omega((\lambda_Q v_Q) \wedge (\lambda_S v_S))}{\omega((\lambda_P v_P) \wedge (\lambda_S v_S))\omega((\lambda_Q v_Q) \wedge (\lambda_R v_R))} &= \frac{\lambda_P\lambda_R\lambda_Q\lambda_S}{\lambda_P\lambda_S\lambda_Q\lambda_R}\frac{\omega(v_P \wedge v_R)\omega(v_Q \wedge v_S)}{\omega(v_P \wedge v_S)\omega(v_Q \wedge v_R)} \\ &= \frac{\omega(v_P \wedge v_R)\omega(v_Q \wedge v_S)}{\omega(v_P \wedge v_S)\omega(v_Q \wedge v_R)}. \end{aligned}$$

Next, we show that the right-hand side is independent of the choice of volume form ω. Since $(\wedge^2\mathbb{C}^2)^* \cong \mathbb{C}$, any other volume form ω' can be written as $\omega' = \lambda\omega$. Then

$$\frac{\lambda\omega(v_P \wedge v_R)\lambda\omega(v_Q \wedge v_S)}{\lambda\omega(v_P \wedge v_S)\lambda\omega(v_Q \wedge v_R)} = \frac{\lambda^2}{\lambda^2}\frac{\omega(v_P \wedge v_R)\omega(v_Q \wedge v_S)}{\omega(v_P \wedge v_S)\omega(v_Q \wedge v_R)} = \frac{\omega(v_P \wedge v_R)\omega(v_Q \wedge v_S)}{\omega(v_P \wedge v_S)\omega(v_Q \wedge v_R)},$$

and so the right-hand side is well-defined. Finally, we will show that the right-hand side equals the cross-ratio. Without loss of generality, assume that none of the points is equal to ∞. If this is not the case, simply apply a projective transformation that moves the point at infinity away. By abuse of notation, write $P = [1 : P]$, $Q = [1 : Q]$, $R = [1 : R]$ and $S = [1 : S]$. Now choose representatives $v_P = (1, P)$, $v_Q = (1, Q)$, $v_R = (1, R)$ and $v_S = (1, S)$ in $\mathbb{C}^2$. Using the standard volume form, which is simply taking the determinant, we find

$$\frac{\omega(v_P \wedge v_R)\omega(v_Q \wedge v_S)}{\omega(v_P \wedge v_S)\omega(v_Q \wedge v_R)} = \frac{\begin{vmatrix} 1 & 1 \\ P & R \end{vmatrix}\begin{vmatrix} 1 & 1 \\ Q & S \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ P & S \end{vmatrix}\begin{vmatrix} 1 & 1 \\ Q & R \end{vmatrix}} = \frac{(P-R)(Q-S)}{(P-S)(Q-R)},$$

which is exactly the cross-ratio. □

Because the choice of volume form is irrelevant, we will denote this expression by

$$\frac{(v_P \wedge v_R)(v_Q \wedge v_S)}{(v_Q \wedge v_R)(v_P \wedge v_S)}.$$

We would like to give the reader the opportunity to come up with the *generalised* cross-ratio themselves, but we will give the formula in Section 2.3.

Section 2.2: The Archimedean Height in $\mathbb{CP}^1$

Before moving on to higher dimensions, we would like to compute the Archimedean height of four distinct points in $\mathbb{CP}^1$. We will show that the Archimedean height $\langle P - Q, R - S\rangle_\infty$ is closely related to the cross-ratio $\mathrm{CR}(P, Q; R, S)$.

A Green's current for $P = [a : b] \in \mathbb{CP}^1$ is given by

$$g_P(z_0, z_1) = -\frac{1}{2}\log\left(\frac{|bz_0 - az_1|^2}{|z_0|^2 + |z_1|^2}\right),$$

defined on $\mathbb{CP}^1 \setminus \{P\}$. Let us assume that $P \neq \infty$. Then, by letting $p = \frac{b}{a}$, we can write $P = [1 : p]$. The Green's current above reduces to

$$g_P(z) = -\frac{1}{2}\log\left(\frac{|z-p|^2}{1+|z|^2}\right),$$

up to the irrelevant constant $-\frac{1}{2}\log|a|^2$. Now observe that

$$dd^c g_P(z) = dd^c \frac{1}{2}\log(1+|z|^2) - dd^c \log|z-p| = \frac{i}{2\pi}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2} - \delta_P.$$

The first term, which we denote by ω_{FS}, is called the Fubini-Study form on $\mathbb{CP}^1$, which is smooth on $\mathbb{CP}^1$. Thus, we can indeed see that $g_P(z)$ is a Green's current corresponding to P.

Remark 2.4. One may ask why the extra $1 + |z|^2$ term is needed, as in $\mathbb{C}^n$ we only needed $\log|z-p|$. On the affine chart where $P = [1 : p]$ lives this would work, as locally $dd^c \log|z-p| = \delta_P$. However, because we are in projective space there is some point at infinity. In the other chart, introduce the variable $w = \frac{1}{z}$ on the overlap. Then we get, in terms of w:

$$-\log|z-p| = -\log\left|\frac{1}{w} - p\right| = -\log\left|\frac{1-pw}{w}\right| = -\log|1-pw| + \log|w|.$$

Thus we see that at infinity, i.e. where $w = 0$, our function blows up. The term $\log(1 + |z|^2)$ removes this singularity. In the w-coordinate:

$$\frac{1}{2}\log(1+|z|^2) = \frac{1}{2}\log\left(1+\left|\frac{1}{w}\right|^2\right) = \frac{1}{2}\log\left(\frac{1+|w|^2}{|w^2|}\right) = \frac{1}{2}\log(1+|w|^2) - \log|w|,$$

so they cancel out at the point $\infty = [0 : 1]$.

To compute the Archimedean height we need to consider a difference of two points $P - Q$, and not just the point P itself. By linearity of the operator dd^c we conclude that a Green's current corresponding to the cycle $P - Q$ is given by

$$g_{P-Q}(z) = -\frac{1}{2}\log\frac{|z-p|^2}{1+|z|^2} + \frac{1}{2}\log\frac{|z-q|^2}{1+|z|^2} = -\log\frac{|z-p|}{|z-q|}.$$

Here we make sure to choose a projective transformation such that both points P and Q lie in the same affine chart and are given by $[1 : p]$ and $[1 : q]$ respectively.

Computing the Archimedean height is now simple, since the other cycle $R-S$ is 0-dimensional.

Proposition 2.5. *Let P, Q, R and S be distinct points in $\mathbb{CP}^1$. Then the Archimedean height of the cycles $P - Q$ and $R - S$ is given by*

$$\langle P-Q, R-S\rangle_\infty = \log|\mathrm{CR}(P,Q;R,S)|.$$

Proof. The result follows immediately from the definition of the Archimedean height combined with the explicit formula for the Green's current. Since $P - Q$ is a divisor this current is the same as the Green's form and we are left to integrate over points.

$$\begin{aligned}
\langle P-Q, R-S\rangle_\infty &= -\int_{\mathbb{CP}^1} \delta_{R-S}\wedge g_{P-Q} \\
&= -\int_R g_{P-Q} + \int_S g_{P-Q} \\
&= \log\frac{|r-p|}{|r-q|} - \log\frac{|s-p|}{|s-q|} \\
&= \log\frac{|r-p|\cdot|s-q|}{|r-q|\cdot|s-p|} \\
&= \log(|\mathrm{CR}(P,Q;R,S)|).
\end{aligned}$$

□

We will do the same computation again, this time without choosing coordinates. So rather than working with points in $\mathbb{CP}^1$ we will work with lines in a complex vector space of dimension 2.

Let V be a complex vector space of dimension 2, so that $\mathbb{P}(V) \cong \mathbb{CP}^1$. Let $P, Q \in \mathbb{P}(V)$ be two distinct points. Choose non-zero vectors $v_P, v_Q \in V$ representing them. Choose a non-zero volume form

$$\omega \in (\Lambda^2 V)^\vee.$$

Using ω, we define linear forms $\varphi_P, \varphi_Q \in V^\vee$ by

$$\varphi_P(v) := \omega(v_P \wedge v), \qquad \varphi_Q(v) := \omega(v_Q \wedge v).$$

Since ω is alternating, we have $\varphi_P(v_P) = 0$ and $\varphi_Q(v_Q) = 0$, so $\ker(\varphi_P) = P$ and $\ker(\varphi_Q) = Q$. The ratio of these two linear functionals defines a rational map

$$\frac{\varphi_P}{\varphi_Q} : \mathbb{P}(V) \dashrightarrow \mathbb{C}, \qquad [v] \longmapsto \frac{\varphi_P(v)}{\varphi_Q(v)},$$

which is well-defined on $\mathbb{P}(V) \setminus Q$. Explicitly,

$$\frac{\varphi_P}{\varphi_Q}([v]) = \frac{\omega(v_P \wedge v)}{\omega(v_Q \wedge v)}.$$

Because $(\Lambda^2 V)^\vee \cong \mathbb{C}$ is one-dimensional, replacing ω by a non-zero scalar multiple does not change this ratio. Thus, the construction is independent of the choice of volume form.

Moreover, φ_P vanishes precisely at P and φ_Q vanishes precisely at Q. Hence the divisor of the rational function φ_P/φ_Q is

$$\operatorname{Div}\left(\frac{\varphi_P}{\varphi_Q}\right) = P - Q.$$

As seen previously, we know exactly what the corresponding Green's form is, namely $g_{P-Q} = -\log\left|\frac{\varphi_P}{\varphi_Q}\right|$.

Proposition 2.6. *Let V be a complex vector space of dimension 2, and let $[v_P], [v_Q], [v_R], [v_S] \in \mathbb{P}(V)$ be distinct elements. Then*

$$\langle [v_P] - [v_Q], [v_R] - [v_S] \rangle_\infty = \log\left|\frac{(v_P \wedge v_R)(v_Q \wedge v_S)}{(v_Q \wedge v_R)(v_P \wedge v_S)}\right| = \log\left|\operatorname{CR}([v_P], [v_Q]; [v_R], [v_S])\right|.$$

Proof. By the observations above, the divisor $P - Q$ is a principal divisor, given by $\operatorname{Div}(\varphi_P/\varphi_Q)$. By Example 1.5 we know a Green's current, and hence form, corresponding to this cycle.

$$\begin{aligned}\left\langle \operatorname{Div}\left(\frac{\varphi_P}{\varphi_Q}\right), [v_R] - [v_S] \right\rangle_\infty &= -\int_{\mathbb{P}(V)} \delta_{[v_R]-[v_S]} \wedge g_{\operatorname{Div}\left(\frac{\varphi_P}{\varphi_Q}\right)} \\ &= \log\left|\frac{\varphi_P}{\varphi_Q}([v_R])\right| - \log\left|\frac{\varphi_P}{\varphi_Q}([v_S])\right| \\ &= \log\left(\frac{|v_P \wedge v_R|}{|v_Q \wedge v_R|}\right) - \log\left(\frac{|v_P \wedge v_S|}{|v_Q \wedge v_S|}\right) \\ &= \log\left(\frac{|v_P \wedge v_R|}{|v_Q \wedge v_R|}\frac{|v_Q \wedge v_S|}{|v_P \wedge v_S|}\right).\end{aligned}$$

The last equality follows from Proposition 2.3 and the laws of logarithms. □

Section 2.3: The generalised Cross-Ratio

In this section we will extend the cross-ratio to higher dimensions. Throughout this section, let K denote a field. In [Olv01], Olver introduced the *volume cross-ratio* as a higher-dimensional analogue of the classical cross-ratio. Given non-zero vectors $v_0, \dots, v_{n-2}, v_j, v_k, v_l, v_n \in K^{n+1}$ he defines

$$C(v_0, \dots, v_{n-2}; v_j, v_k, v_l, v_n) = \frac{\det(v_0, \dots, v_{n-2}, v_j, v_k) \det(v_0, \dots, v_{n-2}, v_l, v_n)}{\det(v_0, \dots, v_{n-2}, v_j, v_l) \det(v_0, \dots, v_{n-2}, v_k, v_n)}.$$

In [KP81], Kaplenko and Ponomarev introduced a cross-ratio for suitable ordered quadruples $(V; E_1, E_2, E_3, E_4)$ where V is a finite-dimensional vector space over a field k, and E_1, E_2, E_3 and E_4 are subspaces of V. They call such a quadruple non-degenerate if for all $i \neq j$, $E_i + E_j = V$ and $E_i \cap E_j = 0$. Let e_{ij} denote the projection onto E_i along E_j, they define the cross-ratio denoted by

$$\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix} = (e_{14} - e_{32})^2$$

which is no longer a number but an operator on V.

We propose a different generalisation, one for which the connection with Archimedean heights remains clear. The classical cross-ratio takes as input two points $P, Q \in \mathbb{CP}^1$ together with two more points $R, S \in \mathbb{CP}^1$ just like the Archimedean height is defined for $P - Q$ and $R - S$. In higher dimensions, say in $\mathbb{CP}^n$, one still has the notion of Archimedean height for cycles of sub-complementary dimensions, i.e., if the analytic cycle $Y_P - Y_Q$ has dimension k, then we require the dimension of $Y_R - Y_S$ to be $n - k - 1$.

So, let V be a K-vector space of dimension $n + 1$, so that $\mathbb{P}(V)$ has dimension n. Let Y_P and Y_Q denote k-dimensional linear subspaces of $\mathbb{P}(V)$, and let Y_R and Y_S denote $(n - k - 1)$-dimensional linear subspaces of $\mathbb{P}(V)$, such that each of Y_P and Y_Q is disjoint from each of Y_R and Y_S. These are also called k-planes and $(n - k - 1)$-planes. Let V_P, V_Q, V_R and V_S be the vector subspaces of V corresponding to Y_P, Y_Q, Y_R and Y_S, so that $Y_P = \mathbb{P}(V_P)$, and similarly for Y_Q, Y_R and Y_S. Next, choose non-zero vectors in V such that

$$\begin{aligned} K\langle v_0^P, \dots, v_k^P \rangle &= V_P, \\ K\langle v_0^Q, \dots, v_k^Q \rangle &= V_Q, \\ K\langle v_0^R, \dots, v_{n-k-1}^R \rangle &= V_R, \\ K\langle v_0^S, \dots, v_{n-k-1}^S \rangle &= V_S. \end{aligned}$$

Let $v_P = v_0^P \wedge \dots \wedge v_k^P$ and similarly for v_Q, v_R and v_S. Lastly, choose a non-zero volume form $\omega \in (\Lambda^{n+1} V)^\vee$. We call a quadruple of two k-planes Y_P and Y_Q and two $(n - k - 1)$-planes Y_R and Y_S *non-degenerate* if

$$Y_P \cap Y_R = Y_P \cap Y_S = Y_Q \cap Y_R = Y_Q \cap Y_S = \varnothing.$$

The following formula was suggested by Emre Sertöz.

Definition 2.7. We define the *generalised* cross-ratio of two k-planes Y_P and Y_Q, and two $(n - k - 1)$-planes Y_R and Y_S, which are non-degenerate in $\mathbb{P}(V)$ to be

$$\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S) = \frac{\omega(v_P \wedge v_R)}{\omega(v_Q \wedge v_R)} \frac{\omega(v_Q \wedge v_S)}{\omega(v_P \wedge v_S)}.$$

First, we will show that this definition is well-defined.

Proposition 2.8. *The generalised cross-ratio* $\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S)$ *is independent of the choice of representatives and volume form. Therefore, we will write*

$$\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S) = \frac{(v_P \wedge v_R)}{(v_Q \wedge v_R)} \frac{(v_Q \wedge v_S)}{(v_P \wedge v_S)}.$$

Proof. For notational ease write $v_i^1 = v_i^P$, $v_i^2 = v_i^Q$, $v_i^3 = v_i^R$ and $v_i^4 = v_i^S$ for all suitable i. If we replace v_i^j with $\lambda_{ij} v_i^j$, then

$$\bigwedge_i \lambda_{ij} v_i^j = \prod_i \lambda_{ij} \bigwedge_i v_i^j$$

and so

$$\frac{\omega\left(\bigwedge_i \lambda_{i1} v_i^1 \wedge \bigwedge_i \lambda_{i3} v_i^3\right)}{\omega\left(\bigwedge_i \lambda_{i2} v_i^2 \wedge \bigwedge_i \lambda_{i3} v_i^3\right)} \frac{\omega\left(\bigwedge_i \lambda_{i2} v_i^2 \wedge \bigwedge_i \lambda_{i4} v_i^4\right)}{\omega\left(\bigwedge_i \lambda_{i1} v_i^1 \wedge \bigwedge_i \lambda_{i4} v_i^4\right)} = \frac{\prod_{i,j} \lambda_{i,j}}{\prod_{i,j} \lambda_{i,j}} \frac{\omega(v_P \wedge v_R)}{\omega(v_Q \wedge v_R)} \frac{\omega(v_Q \wedge v_S)}{\omega(v_P \wedge v_S)}.$$

The constants will cancel and hence the expression is independent of the chosen representatives. The fact that this expression is independent of the chosen volume form follows by the same argument as in the proof of Proposition 2.3. □

Thus, when computing the cross-ratio, note that $v_P \wedge v_R$ is simply given by

$$\det(v_0^P, \ldots, v_k^P, v_0^R, \ldots, v_{n-k-1}^R).$$

We will show that this generalised formula is invariant under projective transformations. This is well-known about the cross-ratio on $\mathbb{P}^1(\mathbb{C})$.

Proposition 2.9. *The generalised cross-ratio is invariant under projective transformations, i.e., for two* k*-planes* Y_P *and* Y_Q *and two* $(n-k-1)$*-planes* Y_R *and* Y_S*, non-degenerate in* $\mathbb{P}(V)$ *and* $T \in \mathrm{PGL}_n(K)$ *we have*

$$\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S) = \mathrm{CR}^n(T(Y_P), T(Y_Q); T(Y_R), T(Y_S)).$$

Proof. Choose a lift $\widetilde{T} \in \mathrm{GL}(V)$ of T. If $v_0^P, \ldots, v_k^P$ are representatives for Y_P, then $\widetilde{T}(v_0^P), \ldots, \widetilde{T}(v_k^P)$ represent $T(Y_P)$. This works similarly for Y_Q, Y_R and Y_S. Then

$$\begin{aligned}
(\widetilde{T}(v_P) \wedge \widetilde{T}(v_R)) &= \det\left(\widetilde{T}(v_0^P), \ldots, \widetilde{T}(v_k^P), \widetilde{T}(v_0^R), \ldots, \widetilde{T}(v_{n-k-1}^R)\right) \\
&= \det\left(\widetilde{T}\left(v_0^P, \ldots, v_k^P, v_0^R, \ldots, v_{n-k-1}^R\right)\right) \\
&= \det(\widetilde{T}) \det\left(v_0^P, \ldots, v_k^P, v_0^R, \ldots, v_{n-k-1}^R\right) \\
&= \det(\widetilde{T})(v_P \wedge v_R).
\end{aligned}$$

Combining this for all terms, we get

$$\begin{aligned}
\mathrm{CR}^n\big(T(Y_P), T(Y_Q); T(Y_R), T(Y_S)\big) &= \frac{(\widetilde{T}(v_P) \wedge \widetilde{T}(v_R))}{(\widetilde{T}(v_Q) \wedge \widetilde{T}(v_R))} \frac{(\widetilde{T}(v_Q) \wedge \widetilde{T}(v_S))}{(\widetilde{T}(v_P) \wedge \widetilde{T}(v_S))} \\
&= \frac{(\det \widetilde{T})\,(v_P \wedge v_R)}{(\det \widetilde{T})\,(v_Q \wedge v_R)} \frac{(\det \widetilde{T})\,(v_Q \wedge v_S)}{(\det \widetilde{T})\,(v_P \wedge v_S)} \\
&= \frac{(v_P \wedge v_R)}{(v_Q \wedge v_R)} \frac{(v_Q \wedge v_S)}{(v_P \wedge v_S)} \\
&= \mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S).
\end{aligned}$$

We conclude that the generalised cross-ratio is invariant under the action of projective transformations. □

In the same way as for the classical cross-ratio, the generalised cross-ratio also has some symmetric properties.

Proposition 2.10. *Let Y_P and Y_Q denote two k-planes, and Y_R and Y_S denote two $(n-k-1)$-planes, which are non-degenerate in $\mathbb{CP}^n$. Then the four expressions*

$$\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S) = \mathrm{CR}^n(Y_Q, Y_P; Y_S, Y_R) = \mathrm{CR}^n(Y_R, Y_S; Y_P, Y_Q) = \mathrm{CR}^n(Y_S, Y_R; Y_Q, Y_P)$$

are equal. Furthermore,

$$\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S) = \mathrm{CR}^n(Y_P, Y_Q; Y_S, Y_R)^{-1}.$$

Proof. This follows immediately from the symmetry in the definition of the generalised cross-ratio. □

Proposition 2.11. *Let Y_P and Y_Q denote two k-planes and let Y_R and Y_S denote two $(n-k-1)$-planes, non-degenerate in $\mathbb{CP}^n$. Then we can alternatively define the generalised cross-ratio to be*

$$\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S) = \frac{(\varphi_P \wedge \varphi_R)(\varphi_Q \wedge \varphi_S)}{(\varphi_P \wedge \varphi_S)(\varphi_Q \wedge \varphi_R)} = \frac{\varphi_R(v_P)\varphi_S(v_Q)}{\varphi_R(v_Q)\varphi_S(v_P)}.$$

Here $\varphi_P = \varphi_{P,1} \wedge \cdots \wedge \varphi_{P,n-k}$ where $\varphi_{P,i}$ are linear functionals on $\mathbb{C}^{n+1}$ such that $V_P = Z(\varphi_{P,1}, \ldots, \varphi_{P,n-k})$. Similarly for φ_Q, φ_R and φ_S, and $\varphi_P \wedge \varphi_R \in \Lambda^{n+1}(\mathbb{C}^{n+1})^\vee \cong \mathbb{C}$, again non-canonically, but the expression is well-defined and independent of the choices made.

Proof. First we will show that

$$\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S) = \frac{\varphi_R(v_P)\varphi_S(v_Q)}{\varphi_R(v_Q)\varphi_S(v_P)}.$$

Fixing Y_R, consider the element $F_R \in \Lambda^{k+1}(\mathbb{C}^{n+1})^\vee$ given by

$$F_R(v_1 \wedge \cdots \wedge v_{k+1}) = \omega(v_1 \wedge \cdots \wedge v_{k+1} \wedge v_{R,1} \wedge \cdots \wedge v_{R,n-k}) \in \mathbb{C},$$

for some volume form ω. Note that if any of the v_i lies in Y_R then $F_R(v_1 \wedge \cdots \wedge v_{k+1}) = 0$ so in fact $F_R \in \Lambda^{k+1}(Y_R)^\perp$ and $\Lambda^{k+1}(Y_R)^\perp \cong \mathbb{C}$ with basis φ_R and so $F_R = c_R\varphi_R$. Similarly define F_S and conclude $F_S = c_S\varphi_S$ and

$$\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S) = \frac{F_R(v_P)F_S(v_Q)}{F_R(v_Q)F_S(v_P)} = \frac{c_R\varphi_R(v_P)c_S\varphi_S(v_Q)}{c_R\varphi_R(v_Q)c_S\varphi_S(v_P)} = \frac{\varphi_R(v_P)\varphi_S(v_Q)}{\varphi_R(v_Q)\varphi_S(v_P)}.$$

From the dual standpoint we can analogously show

$$\frac{\varphi_R(v_P)\varphi_S(v_Q)}{\varphi_R(v_Q)\varphi_S(v_P)} = \frac{(\varphi_P \wedge \varphi_R)(\varphi_Q \wedge \varphi_S)}{(\varphi_P \wedge \varphi_S)(\varphi_Q \wedge \varphi_R)},$$

which completes the proof. □

Remark 2.12. This remark is due to an observation by Richard Kraaij. Suppose we are in the special case where $k = 0$, so Y_P and Y_Q are complex projective points, and Y_R and Y_S are two $(n-1)$-planes, non-degenerate in $\mathbb{CP}^n$. Then one can reduce the generalised cross-ratio to the classical one in the following way. Let L be a projective line passing through the points Y_P and Y_Q. Let $A = L \cap Y_R$ and $B = L \cap Y_S$, then the four points Y_P, Y_Q, A and B are all elements of $L \cong \mathbb{CP}^1$ and

$$\mathrm{CR}(Y_P, Y_Q; A, B) = \mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S).$$

It could be that Y_P, Y_Q, A and B are not all distinct. In that case the left-hand side still makes sense using Definition 2.7.

Proof. Let l_P and l_Q denote the complex lines in $\mathbb{C}^{n+1}$ corresponding to Y_P and Y_Q and let v_P and v_Q be any non-zero vector in l_P and l_Q respectively. Let V_R and V_S denote the n-dimensional linear subspaces corresponding to Y_R and Y_S, which are the zero loci of some linear functionals φ_R and φ_S. Give L coordinates

$$[x : y] \mapsto [xv_P + yv_Q] \in L.$$

In these coordinates $Y_P = [1 : 0]$ and $Y_Q = [0 : 1]$. Furthermore, A is precisely the point $[x : y]$ such that

$$\varphi_R(xv_P + yv_Q) = x\varphi_R(v_P) + y\varphi_R(v_Q) = 0$$

so $A = [\varphi_R(v_Q) : -\varphi_R(v_P)]$ and similarly $B = [\varphi_S(v_Q) : -\varphi_S(v_P)]$. So, we compute the determinants

$$\begin{aligned}
Y_P \wedge A &= \det \begin{bmatrix} 1 & \varphi_R(v_Q) \\ 0 & -\varphi_R(v_P) \end{bmatrix} = -\varphi_R(v_P), \\
Y_P \wedge B &= \det \begin{bmatrix} 1 & \varphi_S(v_Q) \\ 0 & -\varphi_S(v_P) \end{bmatrix} = -\varphi_S(v_P), \\
Y_Q \wedge A &= \det \begin{bmatrix} 0 & \varphi_R(v_Q) \\ 1 & -\varphi_R(v_P) \end{bmatrix} = -\varphi_R(v_Q), \\
Y_Q \wedge B &= \det \begin{bmatrix} 0 & \varphi_S(v_Q) \\ 1 & -\varphi_S(v_P) \end{bmatrix} = -\varphi_S(v_Q).
\end{aligned}$$

Thus

$$\mathrm{CR}(Y_P, Y_Q; A, B) = \frac{(-\varphi_R(v_P))(-\varphi_S(v_Q))}{(-\varphi_S(v_P))(-\varphi_R(v_Q))} = \frac{\varphi_R(v_P)\varphi_S(v_Q)}{\varphi_R(v_Q)\varphi_S(v_P)},$$

which equals exactly the generalised cross-ratio as in Proposition 2.11. ☐

This statement can even be upgraded slightly to the case where, now for general k, Y_P and Y_Q intersect in a $(k-1)$-dimensional subspace of $\mathbb{CP}^n$. As before, let V_P, V_Q, V_R and V_S denote the complex vector spaces corresponding to Y_P, Y_Q, Y_R and Y_S. By assumption $V_{P,Q} = V_P \cap V_Q \cong \mathbb{C}^k$. Let

$$\begin{aligned}
\overline{V}_P &= V_P/V_{P,Q} \cong \mathbb{C}^1, \\
\overline{V}_Q &= V_Q/V_{P,Q} \cong \mathbb{C}^1, \\
\overline{V}_R &= (V_R + V_{P,Q})/V_{P,Q} \cong \mathbb{C}^{n-k}, \\
\overline{V}_S &= (V_S + V_{P,Q})/V_{P,Q} \cong \mathbb{C}^{n-k}.
\end{aligned}$$

Then projectivising gives back the situation of two points $\overline{Y}_P = \mathbb{P}(\overline{V}_P)$ and $\overline{Y}_Q = \mathbb{P}(\overline{V}_Q)$ and two hyperplanes $\overline{Y}_R = \mathbb{P}(\overline{V}_R)$ and $\overline{Y}_S = \mathbb{P}(\overline{V}_S)$ in $\mathbb{CP}^{n-k}$. We leave it as an exercise to the reader to confirm that

$$\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S) = \mathrm{CR}^{n-k}(\overline{Y}_P, \overline{Y}_Q; \overline{Y}_R, \overline{Y}_S).$$

Then we are again in the situation described above, where we can draw a line between the complex projective points $\overline{Y}_P$ and $\overline{Y}_Q$ and further reduce to the classical cross-ratio.

Section 2.4: Grassmannians

The goal of the remaining part of this chapter is to compute the Archimedean height pairing of $Y_P - Y_Q$ and $Y_R - Y_S$ where Y_P and Y_Q are k-planes and Y_R and Y_S are $(n-k-1)$-planes, non-degenerate in an ambient $\mathbb{CP}^n$. By means of Lemma 1.23, we want to relate this height to that of divisors $W_P - W_Q$ and points $R - S$ which are related to the original linear subspaces Y_P, Y_Q, Y_R and Y_S, and are defined in Section 2.5. These new cycles are elements of a Grassmannian, another compact complex manifold which we will introduce in this section. We will give a definition, state and prove some properties and lastly introduce the *incidence correspondence*.

Definition 2.13. Let V be a complex vector space of dimension $n+1$. The *Grassmannian* $\mathrm{Gr}_k(V)$ is the set

$$\mathrm{Gr}_k(V) = \{W \subset V \mid W \text{ is a complex linear subspace and } \dim_{\mathbb{C}}(W) = k\}.$$

Our first goal is to show that $\mathrm{Gr}_k(V)$ is a complex manifold of dimension $k(n+1-k)$.

Proposition 2.14. *The Grassmannian* $\mathrm{Gr}_k(V)$ *admits the structure of a complex manifold of complex dimension* $k(n+1-k)$.

Proof. Fix a point $W_0 \in \mathrm{Gr}_k(V)$, and let $W_0^\perp$ denote its orthogonal complement with respect to any (fixed) Hermitian inner product on V. We define

$$\mathcal{U}_{W_0} = \{W \in \mathrm{Gr}_k(V) \mid W \cap W_0^\perp = \{0\}\}.$$

We claim that $\mathcal{U}_{W_0}$ is an open neighbourhood of W_0, and that it can be identified with the complex vector space $\mathrm{Hom}_{\mathbb{C}}(W_0, W_0^\perp)$. Indeed, let $W \in \mathcal{U}_{W_0}$. Since $W \cap W_0^\perp = \{0\}$ and $\dim W = \dim W_0 = k$, the projection

$$\pi_{W_0} \colon V = W_0 \oplus W_0^\perp \longrightarrow W_0$$

restricts to an isomorphism $W \xrightarrow{\sim} W_0$. Hence, there exists a unique linear map

$$A_W \colon W_0 \to W_0^\perp$$

such that

$$W = \{w + A_W(w) : w \in W_0\},$$

that is, W is the graph of A_W. Conversely, given any $A \in \mathrm{Hom}_{\mathbb{C}}(W_0, W_0^\perp)$, its graph

$$\Gamma(A) = \{w + A(w) : w \in W_0\}$$

is a k-dimensional complex subspace of V satisfying $\Gamma(A) \cap W_0^\perp = \{0\}$, hence $\Gamma(A) \in \mathcal{U}_{W_0}$. Thus, we obtain an isomorphism

$$\begin{aligned} \varphi_{W_0} \colon \mathcal{U}_{W_0} &\longrightarrow \mathrm{Hom}_{\mathbb{C}}(W_0, W_0^\perp), \\ W &\longmapsto A_W. \end{aligned}$$

This is a complex coordinate chart, because

$$\mathrm{Hom}_{\mathbb{C}}(W_0, W_0^\perp) \cong \mathbb{C}^{k(n+1-k)}.$$

It remains to check that the transition maps are holomorphic. Let

$$V = W_0 \oplus W_0^\perp = W_1 \oplus W_1^\perp$$

be two decompositions, with corresponding charts

$$\begin{aligned} \varphi_{W_0} \colon \mathcal{U}_{W_0} &\to \mathrm{Hom}_{\mathbb{C}}(W_0, W_0^\perp), \\ \varphi_{W_1} \colon \mathcal{U}_{W_1} &\to \mathrm{Hom}_{\mathbb{C}}(W_1, W_1^\perp). \end{aligned}$$

On the intersection $\mathcal{U}_{W_0} \cap \mathcal{U}_{W_1}$, the transition function

$$\varphi_{W_1} \circ \varphi_{W_0}^{-1}$$

sends a linear map $A \in \mathrm{Hom}_{\mathbb{C}}(W_0, W_0^\perp)$ to the unique map whose graph equals $\Gamma(A)$, now viewed relative to the decomposition $V = W_1 \oplus W_1^\perp$. More concretely, define

$$\begin{aligned} i_A &: W_0 \to V, \\ i_A(w) &= w + A(w), \end{aligned}$$

so that $\Gamma(A) = \mathrm{im}(i_A)$. Let

$$\pi_{W_1} : V \to W_1,$$
$$\pi_{W_1^\perp} : V \to W_1^\perp$$

denote the projections associated to the decomposition $V = W_1 \oplus W_1^\perp$. Since $\Gamma(A) \cap W_1^\perp = \{0\}$, the map

$$\pi_{W_1} \circ i_A : W_0 \to W_1$$

is an isomorphism, because it is an injective linear map between vector spaces of the same dimension. Hence $\Gamma(A)$ is the graph of the linear map

$$B = (\pi_{W_1^\perp} \circ i_A) \circ (\pi_{W_1} \circ i_A)^{-1} \in \mathrm{Hom}_{\mathbb{C}}(W_1, W_1^\perp).$$

Therefore the transition map $\varphi_{W_1} \circ \varphi_{W_0}^{-1}$ is given by $A \longmapsto (\pi_{W_1^\perp} \circ i_A) \circ (\pi_{W_1} \circ i_A)^{-1}$. This depends holomorphically on A, since it is obtained from A by linear operations and inversion on the open set where $\pi_{W_1} \circ i_A$ is invertible. Thus, the transition functions are holomorphic. Hence, these charts define a complex manifold structure on $\mathrm{Gr}_k(V)$, with complex dimension $k(n+1-k)$. □

We now turn to the Plücker embedding. Recall that $\Lambda^k V$ is a complex vector space, and hence one may form the projective space $\mathbb{P}(\Lambda^k V)$.

Definition 2.15. The *Plücker embedding* is the map

$$\iota \colon \mathrm{Gr}_k(V) \longrightarrow \mathbb{P}(\Lambda^k V)$$
$$W \longmapsto [w_1 \wedge \cdots \wedge w_k],$$

where $w_1, \ldots, w_k$ is any basis of W.

Firstly, we will show this map is well-defined. If $w_1', \ldots, w_k'$ is another basis of W, then there exists a matrix $M \in \mathrm{GL}_k(\mathbb{C})$ such that

$$(w_1', \ldots, w_k') = (w_1, \ldots, w_k)M.$$

Hence $w_1' \wedge \cdots \wedge w_k' = \det(M)\, w_1 \wedge \cdots \wedge w_k$, and because $\det(M) \neq 0$ these two elements define the same point in projective space.

Proposition 2.16. *The Plücker map $\iota \colon \mathrm{Gr}_k(V) \to \mathbb{P}(\Lambda^k V)$ is a holomorphic embedding.*

Proof. We first prove injectivity. Suppose that $\iota(W) = \iota(W')$. Choose bases $w_1, \ldots, w_k$ of W and $w_1', \ldots, w_k'$ of W'. Then $[w_1 \wedge \cdots \wedge w_k] = [w_1' \wedge \cdots \wedge w_k']$, so there exists $\lambda \in \mathbb{C}^\times$ such that $w_1 \wedge \cdots \wedge w_k = \lambda\, w_1' \wedge \cdots \wedge w_k'$, and hence the k-dimensional subspaces generated by $w_1, \ldots, w_k$ and $w_1', \ldots, w_k'$ are equal, i.e. $W = W'$. To see that ι is holomorphic, work in a chart $\mathcal{U}_{W_0}$. Choose bases $e_1, \ldots, e_k$ of W_0 and $f_1, \ldots, f_{n+1-k}$ of $W_0^\perp$. Let $A \in \mathrm{Hom}_{\mathbb{C}}(W_0, W_0^\perp)$, and write

$$A(e_i) = \sum_{j=1}^{n+1-k} a_{ji} f_j.$$

Then the graph $\Gamma(A)$ is spanned by

$$e_i + \sum_{j=1}^{n+1-k} a_{ji} f_j, \qquad i = 1, \ldots, k.$$

Hence

$$\iota(\Gamma(A)) = \Big[\big(e_1 + \sum_j a_{j1} f_j\big) \wedge \cdots \wedge \big(e_k + \sum_j a_{jk} f_j\big)\Big].$$

Expanding this wedge product in a basis of $\Lambda^k V$, we see that the homogeneous coordinates of $\iota(\Gamma(A))$ are polynomial functions in the entries a_{ji}. Thus, ι is holomorphic. Finally, in these local coordinates the image uniquely determines the graph, so ι is a holomorphic embedding. □

Choose a basis $e_0, \ldots, e_n$ of V. Then the induced basis of $\Lambda^k V$ is

$$e_{i_1} \wedge \cdots \wedge e_{i_k}, \qquad 0 \leq i_1 < \cdots < i_k \leq n.$$

Accordingly, points of $\mathbb{P}(\Lambda^k V)$ have homogeneous coordinates

$$[p_{i_1,\ldots,i_k}]_{0 \leq i_1 < \cdots < i_k \leq n},$$

called the *Plücker coordinates.* The image of the Plücker embedding is cut out by certain homogeneous quadratic equations in these coordinates, called the *Plücker relations.* We do not need their explicit form here. The important point is that the image of the Plücker embedding in $\mathbb{P}(\Lambda^k V)$ is closed.

Theorem 2.17. *The Grassmannian* $\mathrm{Gr}_k(V)$ *is compact.*

Proof. Since $\Lambda^k V$ is finite-dimensional, the projective space $\mathbb{P}(\Lambda^k V)$ is compact. The image of the Plücker embedding is defined by homogeneous polynomial equations, namely the Plücker relations, and is therefore Zariski closed in $\mathbb{P}(\Lambda^k V)$. In particular, it is closed in the usual complex topology. Hence $\iota(\mathrm{Gr}_k(V))$ is compact. Finally, since ι is an embedding, it is a homeomorphism from $\mathrm{Gr}_k(V)$ onto its image and therefore $\mathrm{Gr}_k(V)$ is compact. □

Section 2.5: The Archimedean Height of Linear Subspaces in $\mathbb{CP}^n$

At last, we will compute the Archimedean height pairing of $Y_P - Y_Q$ and $Y_R - Y_S$, where Y_P and Y_Q are k-planes and Y_R and Y_S are $(n-k-1)$-planes, non-degenerate in $\mathbb{CP}^n$. As before, choose generators for the corresponding vector subspaces such that

$$\begin{aligned} V_P &= \mathbb{C}\langle v_1^P, \ldots, v_{k+1}^P \rangle, \\ V_Q &= \mathbb{C}\langle v_1^Q, \ldots, v_{k+1}^Q \rangle, \\ V_R &= \mathbb{C}\langle v_1^R, \ldots, v_{n-k}^R \rangle, \\ V_S &= \mathbb{C}\langle v_1^S, \ldots, v_{n-k}^S \rangle. \end{aligned}$$

Let $v_P = v_1^P \wedge \cdots \wedge v_{k+1}^P$ and similarly for v_Q, v_R and v_S. Without loss of generality let us assume $k \geq n-k-1$, so all relevant planes can be parametrised by $\mathrm{Gr}_{n-k}(\mathbb{C}^{n+1})$. Let R and S denote the points in $\mathrm{Gr}_{n-k}(\mathbb{C}^{n+1})$ that correspond to the linear subspaces Y_R and Y_S. The compact complex manifolds $\mathbb{CP}^n$ and $\mathrm{Gr}_{n-k}(\mathbb{C}^{n+1})$ are related via the *incidence correspondence*

$$\begin{array}{ccc} & \mathcal{I} = \{(W,x) \in \mathrm{Gr}_{n-k}(\mathbb{C}^{n+1}) \times \mathbb{CP}^n \mid x \in W\} & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \mathrm{Gr}_{n-k}(\mathbb{C}^{n+1}) & & \mathbb{CP}^n \end{array}$$

Lastly, consider $W_P = \pi_1(\pi_2^{-1}(Y_P))$ and $W_Q = \pi_1(\pi_2^{-1}(Y_Q))$. Next, let $I_P = \pi_1^{-1}(W_P), I_Q = \pi_1^{-1}(W_Q), I_R = \pi_1^{-1}(R)$ and $I_S = \pi_1^{-1}(S)$. Observe that $(\pi_1)^*([W_P] - [W_Q]) = [I_P] - [I_Q]$ and similarly $(\pi_1)^*([R] - [S]) = [I_R] - [I_S]$. These will be the cycles we start with in Lemma 1.23. We will prove some preliminary results about this setup.

Lemma 2.18. *(i) The subspaces* $W_P, W_Q \subset \mathrm{Gr}_{n-k}(V)$ *are divisors, i.e., of codimension 1.*

(ii) As divisors we can write

$$[W_P] - [W_Q] = \mathrm{Div}(\varphi_P/\varphi_Q)$$

where

$$\begin{aligned} \varphi_P/\varphi_Q : \mathrm{Gr}_{n-k}(\mathbb{C}^{n+1}) &\dashrightarrow \mathbb{C}, \\ [x] &\longmapsto \frac{v_P \wedge \iota(x)}{v_Q \wedge \iota(x)}. \end{aligned}$$

(iii) *Alternatively we can define W_P and W_Q as follows. Consider the annihilators $Y_P^\perp \subset (\mathbb{C}^{n+1})^\vee$ and $Y_Q^\perp \subset (\mathbb{C}^{n+1})^\vee$ of Y_P and Y_Q, respectively. Because Y_P and Y_Q are of codimension $n-k$, as viewed in $\mathbb{C}^{n+1}$, we can write*

$$Y_P^\perp = \mathbb{C}\langle \varphi_1^P, \dots, \varphi_{n-k}^P \rangle,$$
$$Y_Q^\perp = \mathbb{C}\langle \varphi_1^Q, \dots, \varphi_{n-k}^Q \rangle.$$

Then let $\varphi_P = [\varphi_1^P \wedge \dots \wedge \varphi_{n-k}^P]$ and $\varphi_Q = [\varphi_1^Q \wedge \dots \wedge \varphi_{n-k}^Q]$ be points in $(\mathrm{Gr}_{n-k}(\mathbb{C}^{n+1}))^\vee$. Then

$$\varphi_P/\varphi_Q : \mathrm{Gr}_{n-k}(\mathbb{C}^{n+1}) \dashrightarrow \mathbb{C}$$
$$[x] \longmapsto \frac{\varphi_P([x])}{\varphi_Q([x])}$$

is a well-defined map on $\mathrm{Gr}_{n-k}(\mathbb{C}^{n+1}) \setminus W_Q$ and

$$[W_P] - [W_Q] = \mathrm{Div}(\varphi_P/\varphi_Q).$$

Proof. The first point follows immediately if we can realise $W_P = \ker(\varphi_P)$. To this end we need to fix a volume form $\omega \in \Lambda^{n+1}(\mathbb{C}^{n+1})^\vee \cong \mathbb{C}$, that is non-zero under the identification with $\mathbb{C}$. So we can define $\varphi_P : \mathrm{Gr}_{n-k}(\mathbb{C}^{n+1}) \to \mathbb{C}$ by sending $[x] \mapsto \omega(v_P \wedge \iota(x))$, which is well-defined up to a (non-zero) scalar. Hence, the ratio φ_P/φ_Q is well-defined on $\mathrm{Gr}_{n-k}(V) \setminus \ker(\varphi_Q)$. It remains to identify $\ker(\varphi_P)$. It consists of all $(n-k)$-planes Y in V such that the intersection $Y \cap Y_P \neq 0$. Thus, we see immediately that $\ker(\varphi_P) = W_P$ and similarly $\ker(\varphi_Q) = W_Q$. We conclude that $\mathrm{Div}(\varphi_P/\varphi_Q) = [W_P] - [W_Q]$ and that W_P and W_Q are of codimension 1.

To prove (iii) note that the ratio φ_P/φ_Q is exactly the same map in the descriptions of (ii) and (iii). Thus we may use either description. □

We will verify that we are in a position to apply Lemma 1.23 to the cycles $[I_P] - [I_Q]$ and $[I_R] - [I_S]$. Firstly, note that $\mathcal{I}$, $\mathbb{CP}^n$ and $\mathrm{Gr}_{n-k}(\mathbb{C}^{n+1})$ are all compact complex manifolds. Furthermore, π_1 is a surjective smooth submersion, with fibres that are copies of $\mathbb{CP}^n$ which have no boundary, and π_2 is holomorphic. Secondly, observe that $\pi_2|_{I_P} : I_P \to \pi_2(I_P) = Y_P$ and $\pi_2|_{I_Q} : I_Q \to \pi_2(I_Q) = Y_Q$ are orientation-preserving diffeomorphisms. Lastly, we need to check that both the cycles $[Y_P] - [Y_Q]$ and $[Y_R] - [Y_S]$ are homologous to zero.

Lemma 2.19. *The cycle $[Y_P] - [Y_Q]$ is homologically trivial.*

Proof. These are k-planes in $\mathbb{CP}^n$. Now note that $H_{2k}^{\mathrm{sing}}(\mathbb{CP}^n) \cong \mathbb{Z}$ and any non-zero k-plane is a generator. Thus, we find that the homology class associated with $[Y_P] - [Y_Q]$ is $1 - 1 = 0$. Note that, by the functoriality of the induced maps of π_1 and π_2 on homology, the divisor $[W_P] - [W_Q]$ is also homologous to zero. □

Let us now state the main result of Chapters 1 and 2.

Theorem 2.20. *The Archimedean height pairing of the cycles $Y_P - Y_Q$ and $Y_R - Y_S$, where Y_P and Y_Q are k-planes and Y_R and Y_S are $(n-k-1)$-planes, non-degenerate in $\mathbb{CP}^n$, is given by*

$$\langle [Y_P] - [Y_Q], [Y_R] - [Y_S] \rangle_\infty = \log \left| \mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S) \right|.$$

Proof. We apply Lemma 1.23 to the cycles $I_P - I_Q$ and $I_R - I_S$ to find

$$\begin{aligned}
\langle [Y_P] - [Y_Q], [Y_R] - [Y_S] \rangle_\infty^{\mathbb{CP}^n} &= \langle [W_P] - [W_Q], [R] - [S] \rangle_\infty^{\mathrm{Gr}_{n-k}(\mathbb{C}^{n+1})} \\
&= \log |\varphi_P/\varphi_Q([R])| - \log |\varphi_P/\varphi_Q([S])| \\
&= \log \left| \frac{v_P \wedge v_R}{v_Q \wedge v_R} \frac{v_Q \wedge v_S}{v_P \wedge v_S} \right| \\
&= \log |\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S)|.
\end{aligned}$$

□

Remark 2.21. Using the alternative characterisation of $[W_P] - [W_Q]$, as described in part (iii) of Lemma 2.18, we can also write the height pairing as

$$\langle [Y_P] - [Y_Q], [Y_R] - [Y_S] \rangle_\infty^{\mathbb{CP}^n} = \langle [W_P] - [W_Q], [R] - [S] \rangle_\infty^{\mathrm{Gr}_{n-k}(\mathbb{C}^{n+1})} = \log\left(\frac{|\varphi_P(v_R)|}{|\varphi_Q(v_R)|} \frac{|\varphi_Q(v_S)|}{|\varphi_P(v_S)|} \right).$$

Chapter 3: Asymptotic Behaviour of Archimedean Heights in Projective Space

In the previous section, we computed the Archimedean height pairing for *disjoint* k-planes $Y_P - Y_Q$ and $(n-k-1)$-planes $Y_R - Y_S$ in $\mathbb{CP}^n$. Suppose now that these planes depend on some parameter t, denoted by $Y_P(t), Y_Q(t), Y_R(t)$ and $Y_S(t)$ and that when $t = 0$, the cycles are no longer disjoint. For example, suppose that $Y_P(0) \cap Y_R(0) \neq \varnothing$ in $\mathbb{CP}^n$. For fixed t, let $v_P(t) = v_0^P(t) \wedge \cdots \wedge v_k^P(t)$ and $v_R(t) = v_0^R(t) \wedge \cdots \wedge v_{n-k}^R(t)$ be some generators. As t approaches zero, note that $v_P(t) \wedge v_R(t)$ also approaches zero, because of the anti-commutativity of the wedge product. More specifically, when $t = 0$, we can choose representatives for $Y_P(0)$ and $Y_R(0)$ that share at least one basis vector, say v_0. Then $v_P(0) \wedge v_R(0)$ contains $v_0 \wedge v_0 = 0$. So, if the configuration is otherwise non-degenerate, the Archimedean height diverges as t approaches zero. In this chapter, we will study this asymptotic behaviour. We will show that, for small t,

$$\langle [Y_P(t)] - [Y_Q(t)], [Y_R(t)] - [Y_S(t)] \rangle_\infty = A \log |t| + \log |u(t)|, \tag{3.1}$$

where $u(t)$ is some nowhere-vanishing holomorphic function and express A as an *intersection degree*.

In this chapter, we will first rephrase some of the previous results and definitions in the language of algebraic geometry. We assume familiarity with commutative algebra and the theory of schemes. For background on commutative algebra, we refer to [AM69], and for algebraic geometry to [EHKT22, JT23b].

Rephrasing the first part of this introduction we will let T denote a discrete valuation ring (DVR), such as $\mathbb{C}[\![t]\!]$, and consider families of k-planes $\overline{Y_P}, \overline{Y_Q}$ and $(n-k-1)$-planes $\overline{Y_R}, \overline{Y_S}$ over $\operatorname{Spec}(T)$. This means that, over both the *generic point* $\eta \in \operatorname{Spec}(T)$ and the special point $s \in \operatorname{Spec}(T)$, all fibres are k- or $(n-k-1)$-planes in $\mathbb{P}^n_K$ or $\mathbb{P}^n_\kappa$. In Section 3.1 we treat the case of proper intersections between the families. In Section 3.2 we show an example of all possible degenerations of families of lines in $\mathbb{CP}^3$. To be able to handle the case of excess intersection we introduce the necessary machinery in Sections 3.3 and 3.4. Lastly, in Section 3.5 we extend the result of Section 3.1 to general degenerations.

Theorem B (Theorem 3.5 & Theorem 3.21)**.** Let $Y_P(t)$ and $Y_Q(t)$ be moving k-dimensional linear subspaces of $\mathbb{CP}^n$, and let $Y_R(t)$ and $Y_S(t)$ be moving $(n-k-1)$-dimensional linear subspaces of $\mathbb{CP}^n$, all holomorphically dependent on t. Assume that, for all sufficiently small $t \neq 0$, the linear subspaces are non-degenerate in $\mathbb{CP}^n$. Then

$$\langle Y_P(t) - Y_Q(t), Y_R(t) - Y_S(t) \rangle_\infty = \deg\left((\overline{Y_P} - \overline{Y_Q}) \cdot (\overline{Y_R} - \overline{Y_S})\right) \log |t| + \log |u(t)|, \tag{3.2}$$

for some nowhere-vanishing holomorphic function $u(t)$.

In Chapter 4, we will also find a geometric interpretation of the remaining constant $u(0)$, as in Theorem 4.8.

Section 3.1: Proper Degenerations

Instead of working with complex linear subspaces of $\mathbb{C}^{n+1}$, we will work with linear subspaces of free T-modules, where T is a discrete valuation ring. Two important examples of discrete valuation rings are $\mathbb{C}[\![t]\!]$ and $\mathbb{Z}_p$, for any prime p. We will briefly recall why these rings are discrete valuation rings and what their fraction field and residue fields look like. First, the ring of *formal power series* $\mathbb{C}[\![t]\!]$. Observe that every non-zero element $f \in \mathbb{C}[\![t]\!]$ can be written uniquely in the form

$$f = t^m u$$

with $m \geq 0$ and $u \in \mathbb{C}[\![t]\!]^\times$ a unit. Thus, the unique maximal ideal is generated by t, so t is a uniformizer. Its fraction field is the field of *Laurent series*

$$\mathbb{C}((t)) = \left\{ \sum_{n \geq N} a_n t^n \mid N \in \mathbb{Z},\ a_n \in \mathbb{C} \right\},$$

and its residue field is given by

$$\mathbb{C}[\![t]\!]/(t) \cong \mathbb{C}.$$

The associated valuation

$$v_t \colon \mathbb{C}((t))^\times \to \mathbb{Z}$$

is given by

$$v_t \left(\sum_{n \geq N} a_n t^n \right) = \min\{\, n \mid a_n \neq 0 \,\}.$$

Second, the ring of p-adic integers $\mathbb{Z}_p$, for any prime $p \in \mathbb{Z}$, is a discrete valuation ring. Every non-zero element $x \in \mathbb{Z}_p$ can be written uniquely in the form

$$x = p^m u$$

with $m \geq 0$ and $u \in \mathbb{Z}_p^\times$ a unit. Hence, the unique maximal ideal is generated by p, so p is a uniformizer. Its fraction field is the field of p-adic numbers

$$\mathbb{Q}_p,$$

and its residue field is given by

$$\mathbb{Z}_p/(p) \cong \mathbb{F}_p.$$

The associated valuation

$$v_p \colon \mathbb{Q}_p^\times \to \mathbb{Z}$$

is characterized by the property that if

$$x = p^m \frac{a}{b},$$

with $a, b \in \mathbb{Z}$ not divisible by p, then

$$v_p(x) = m.$$

Equivalently, $v_p(x)$ is the exponent of p in the prime factorisation of x.

In both cases, the valuation measures the order of vanishing with respect to a chosen uniformizer: t in the case of $\mathbb{C}[[t]]$, and p in the case of $\mathbb{Z}_p$.

We will now properly introduce the objects of study. Let T be a discrete valuation ring with fraction field K, and residue field κ. Let $\mathcal{T} = \operatorname{Spec}(T)$.

Definition 3.1. Let $U \subset T^{n+1}$ be a direct summand of rank $k+1$, and let

$$\mathcal{U} = \widetilde{U} \subset \mathcal{O}_{\mathcal{T}}^{\oplus n+1}$$

be the corresponding locally free sheaf. We define the associated *family of k-planes* by

$$\overline{Y} = \operatorname{Proj}_{\mathcal{T}}(\operatorname{Sym}_{\mathcal{O}_{\mathcal{T}}}(\widetilde{U})^\vee) \subset \operatorname{Proj}_{\mathcal{T}}(\operatorname{Sym}_{\mathcal{O}_{\mathcal{T}}}(\mathcal{O}_{\mathcal{T}}^{\oplus n+1})^\vee) = \mathbb{P}^n_{\mathcal{T}}.$$

Because U is assumed to be a direct summand this family is flat over T and the fibers

$$\overline{Y}_\eta = \mathbb{P}_K(U \otimes_T K) \subset \mathbb{P}^n_K,$$
$$\overline{Y}_s = \mathbb{P}_\kappa(U \otimes_T \kappa) \subset \mathbb{P}^n_\kappa,$$

are true k-planes, isomorphic to $\mathbb{P}^k_K$ and $\mathbb{P}^k_\kappa$, respectively.

Equivalently, a family of k-planes in $\mathbb{P}^n_T$ is given by a free submodule $U \subset T^{n+1}$ of rank $k+1$, such that the quotient T^{n+1}/U is free of rank $n-k$. For such a submodule U, we write

$$\mathbb{P}_{\mathcal{T}}(U) = \mathrm{Proj}_{\mathcal{T}}(\mathrm{Sym}_{\mathcal{O}_{\mathcal{T}}}(\widetilde{U})^{\vee}).$$

We will study two families of k-planes, denoted $\overline{Y_P}$ and $\overline{Y_Q}$, and two families of $(n-k-1)$-planes, denoted $\overline{Y_R}$ and $\overline{Y_S}$. Let $U_P, U_Q \subset T^{n+1}$ be direct summands such that $\overline{Y_P} = \mathbb{P}_{\mathcal{T}}(U_P)$ and $\overline{Y_Q} = \mathbb{P}_{\mathcal{T}}(U_Q)$. To define $\overline{Y_R}$ and $\overline{Y_S}$, it is convenient to work dually. Let

$$(T^{n+1})^{\vee} = \mathrm{Hom}_T(T^{n+1}, T),$$

and choose direct summands

$$V_R, V_S \subset (T^{n+1})^{\vee}$$

of rank $k+1$. Each such submodule determines a natural morphism

$$\begin{aligned} T^{n+1} &\longrightarrow V_R^{\vee}, \\ v &\longmapsto (\varphi \mapsto \varphi(v)), \end{aligned}$$

and similarly for V_S. Since V_R and V_S are direct summands of $(T^{n+1})^{\vee}$, these maps are surjective, and their kernels are direct summands of T^{n+1} of rank $n-k$. We define

$$\overline{Y_R} = \mathbb{P}_{\mathcal{T}}(\ker(T^{n+1} \to V_R^{\vee})), \qquad \overline{Y_S} = \mathbb{P}_{\mathcal{T}}(\ker(T^{n+1} \to V_S^{\vee})).$$

Now choose bases for U_P, U_Q, V_R and V_S

$$\begin{aligned} U_P &= T\langle v_0^P, \dots, v_k^P \rangle, \\ U_Q &= T\langle v_0^Q, \dots, v_k^Q \rangle, \\ V_R &= T\langle \varphi_0^R, \dots, \varphi_k^R \rangle, \\ V_S &= T\langle \varphi_0^S, \dots, \varphi_k^S \rangle. \end{aligned}$$

Define

$$\begin{aligned} v_P &= v_0^P \wedge \dots \wedge v_k^P \in \bigwedge^{k+1} U_P \subset \bigwedge^{k+1} T^{n+1}, \\ v_Q &= v_0^Q \wedge \dots \wedge v_k^Q \in \bigwedge^{k+1} U_Q \subset \bigwedge^{k+1} T^{n+1}, \\ \varphi_R &= \varphi_0^R \wedge \dots \wedge \varphi_k^R \in \bigwedge^{k+1} V_R \subset \bigwedge^{k+1} (T^{n+1})^{\vee}, \\ \varphi_S &= \varphi_0^S \wedge \dots \wedge \varphi_k^S \in \bigwedge^{k+1} V_S \subset \bigwedge^{k+1} (T^{n+1})^{\vee}. \end{aligned}$$

The elements φ_R and φ_S define alternating $(k+1)$-linear functionals on T^{n+1}, and hence induce T-linear maps

$$\bigwedge^{k+1} T^{n+1} \longrightarrow T.$$

In particular, we may evaluate them on v_P and v_Q. Explicitly,

$$\varphi_R(v_P) = \det(\varphi_i^R(v_j^P))_{0 \le i,j \le k}, \qquad \varphi_R(v_Q) = \det(\varphi_i^R(v_j^Q))_{0 \le i,j \le k},$$

and similarly for $\varphi_S(v_P)$ and $\varphi_S(v_Q)$. Assume now that the generic fibers,

$$\overline{Y_P}_{\eta},\ \overline{Y_Q}_{\eta},\ \overline{Y_R}_{\eta},\ \overline{Y_S}_{\eta}$$

are non-degenerate. Then the above determinants are non-zero in the fraction field K. Then we may define

$$\mathrm{CR}^n(\overline{Y_{P\eta}}, \overline{Y_{Q_\eta}}; \overline{Y_{R\eta}}, \overline{Y_{S\eta}}) = \frac{\varphi_R(v_P)}{\varphi_R(v_Q)} \frac{\varphi_S(v_Q)}{\varphi_S(v_P)} \in K^\times.$$

This quantity is independent of the chosen bases of U_P, U_Q, V_R and V_S, since a change of basis multiplies the individual determinants by units, which cancel in the above expression. As in Proposition 2.11, we can give different equivalent descriptions. Choose generators

$$v_R \in \bigwedge^{n-k} \ker(T^{n+1} \to V_R^\vee), \qquad v_S \in \bigwedge^{n-k} \ker(T^{n+1} \to V_S^\vee).$$

After fixing an isomorphism

$$\bigwedge^{n+1} T^{n+1} \cong T,$$

the elements

$$v_P \wedge v_R, \quad v_Q \wedge v_R, \quad v_P \wedge v_S, \quad v_Q \wedge v_S$$

may be viewed as elements of T, well-defined up to a *fixed* unit. Then we define

$$\mathrm{CR}^n(\overline{Y_{P\eta}}, \overline{Y_{Q_\eta}}; \overline{Y_{R\eta}}, \overline{Y_{S\eta}}) = \frac{v_P \wedge v_R}{v_Q \wedge v_R} \frac{v_Q \wedge v_S}{v_P \wedge v_S} \in K^\times.$$

Similarly to the generalised cross-ratio for fields, the ratio is now a well-defined element of the fraction field $K^\times$. Equivalently, we can also do this on the dual side, and find

$$\mathrm{CR}^n(\overline{Y_{P\eta}}, \overline{Y_{Q_\eta}}; \overline{Y_{R\eta}}, \overline{Y_{S\eta}}) = \frac{\varphi_P \wedge \varphi_R}{\varphi_Q \wedge \varphi_R} \frac{\varphi_Q \wedge \varphi_S}{\varphi_P \wedge \varphi_S} \in K^\times.$$

This gives us three different ways to compute the same cross-ratio, associated to these k- and $(n-k-1)$-planes. Lastly, because T is a discrete valuation ring, we can now assign an integer to the cross-ratio

$$v_T\left(\mathrm{CR}^n(\overline{Y_{P\eta}}, \overline{Y_{Q_\eta}}; \overline{Y_{R\eta}}, \overline{Y_{S\eta}})\right),$$

We will show that this equals

$$\deg\left((\overline{Y_P} - \overline{Y_Q}) \cdot (\overline{Y_R} - \overline{Y_S})\right).$$

To establish this result we will make use of the following fact from linear algebra.

Lemma 3.2 (Smith Normal Form)**.** *Let M be a non-zero $m \times n$ matrix, with elements in a principal ideal domain R. Then, there exist invertible $m \times m$ and $n \times n$ matrices U and V such that the product UMV is given by*

$$\begin{bmatrix} \alpha_1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \alpha_2 & 0 & & & & \\ 0 & 0 & \ddots & & \vdots & & \vdots \\ \vdots & & & \alpha_r & & & \\ 0 & & \cdots & & 0 & \cdots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & & \cdots & & 0 & \cdots & 0 \end{bmatrix}.$$

The diagonal elements satisfy $\alpha_{i+1} | \alpha_i$ for all $1 \le i < r$.

Before stating and proving the main theorem of this section we need to recall what a *proper* intersection is.

Definition 3.3. Let X be a nonsingular variety. Let $W, V \subset X$ be closed subvarieties with $\dim(W) = s$ and $\dim(V) = r$. We say that W and V intersect *properly* if

$$\dim(V \cap W) \leq r + s - \dim(X).$$

In our setting the condition that $\overline{Y_P}$ and $\overline{Y_R}$ intersect properly means that $\overline{Y_P}_\eta \cap \overline{Y_R}_\eta = \varnothing$ and

$$\dim(\overline{Y_P}_s \cap \overline{Y_R}_s) \leq (k+1) + (n-k) - (n+1) = 0.$$

The reason proper intersection forces disjointness over the generic fiber is because the generic point η is dense in $\operatorname{Spec}(T)$. More precisely, suppose $\overline{Y_P}_\eta \cap \overline{Y_R}_\eta \neq \varnothing$. Choose an irreducible component Z of this intersection. Then the induced structure morphism

$$Z \to \operatorname{Spec}(T)$$

is *dominant*, i.e. the image is dense. Thus

$$\dim(Z) \geq \dim(\operatorname{Spec}(T)) = 1,$$

which implies that $\dim(\overline{Y_P} \cap \overline{Y_R}) \geq 1$, so the intersection is not proper. We assume familiarity with the intersection product on algebraic cycles. For a brief review, see Section 3.3. For a more detailed treatment, see [EH16, Ful98].

Theorem 3.4. *Let T be a discrete valuation ring, with valuation v_T and uniformizer π. Let $\overline{Y_P}$ and $\overline{Y_Q}$ be k-planes and $\overline{Y_R}$ and $\overline{Y_S}$ be $(n-k-1)$-planes in $\mathbb{P}^n_T$. Assume that $\overline{Y_P} - \overline{Y_Q}$ and $\overline{Y_R} - \overline{Y_S}$ intersect properly. Then*

$$\deg\left((\overline{Y_P} - \overline{Y_Q}) \cdot (\overline{Y_R} - \overline{Y_S})\right) = v_T\left(\mathrm{CR}^n(\overline{Y_P}_\eta, \overline{Y_Q}_\eta; \overline{Y_R}_\eta, \overline{Y_S}_\eta)\right).$$

Proof. Because of the linearity of both the degree and the valuation it suffices to show that

$$\deg(\overline{Y_P} \cdot \overline{Y_R}) = v_T(\varphi_R(v_P)). \tag{3.3}$$

By assumption, $\overline{Y_P}_\eta$ and $\overline{Y_R}_\eta$ are disjoint, so intersection can only happen in $\overline{Y_P}_s \cap \overline{Y_R}_s$. Furthermore, because $\overline{Y_P}_s$ and $\overline{Y_R}_s$ are linear subspaces of $\mathbb{P}^n_\kappa$, together with the fact that they intersect properly, we can conclude that they intersect in at most one point, call this point ξ. Thus

$$\deg(\overline{Y_P} \cdot \overline{Y_R}) = \sum_{x \in \mathbb{P}^n_T} m_x(\overline{Y_P}, \overline{Y_R}; \mathbb{P}^n_T) = m_\xi(\overline{Y_P}, \overline{Y_R}; \mathbb{P}^n_T).$$

Now

$$\begin{aligned} m_\xi(\overline{Y_P}, \overline{Y_R}; \mathbb{P}^n_T) &= l\left(\frac{\mathcal{O}_{\mathbb{P}^n_T,\xi}}{I_P + I_R}\right) \\ &= l\left(\left(\frac{T[x_0, \dots, x_n]_\xi}{(\varphi^P_0, \dots, \varphi^P_{n-k-1}, \varphi^R_0, \dots, \varphi^R_k)}\right)^{\deg=0}\right) \end{aligned}$$

Here φ^P_i and φ^R_j are such that $\overline{Y_P} = Z(\varphi^P_0, \dots, \varphi^P_{n-k-1})$ and $\overline{Y_R} = Z(\varphi^R_0, \dots, \varphi^R_k)$. These are equations cutting out $\overline{Y_P}$ and $\overline{Y_R}$. This notation means that $\overline{Y_P}$ is exactly the common zero-locus of these linear functions $\varphi^P_0, \dots, \varphi^P_{n-k-1}$. These functions are all linear, and hence they can be rewritten as

$$\begin{pmatrix} \varphi^P_0 \\ \vdots \\ \varphi^P_{n-k-1} \\ \varphi^R_0 \\ \vdots \\ \varphi^R_k \end{pmatrix} = M \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix},$$

for some matrix M with elements in T. Now, because T is a discrete valuation ring it is also a principal ideal domain and hence we can apply Lemma 3.2. So, there exist invertible matrices U and V and a diagonal matrix D such that $M = UDV$. Now, because $\overline{Y_P}_\eta$ and $\overline{Y_R}_\eta$ are assumed to be disjoint, M becomes invertible over K. Furthermore, because the intersection of $\overline{Y_P}$ and $\overline{Y_R}$ is assumed to be proper, the rank of M base changed to κ, denoted by M_s, is at least n. Now, a priori $D = \mathrm{Diag}(u_0\pi^{\alpha_0}, \dots, u_n\pi^{\alpha_n})$, where π denotes the uniformizer of T. However, because the rank of D may drop at most by 1, when base changed to $\kappa = T/(\pi)$, we can conclude that $\alpha_1 = \cdots = \alpha_n = 0$.

Next, we change coordinates, and let $[y_0 \cdots y_n]^T = V[x_0 \cdots x_n]^T$. Also, let $[\varphi_0 \cdots \varphi_n] = U^{-1}[\varphi_0^P \cdots \varphi_{n-k-1}^P\ \varphi_0^R \cdots \varphi_k^R]^T$. Then, because U is invertible, the ideals $(\varphi_0, \dots, \varphi_n)$ and $(\varphi_0^P, \dots, \varphi_{n-k-1}^P, \varphi_0^R, \dots, \varphi_k^R)$ are equal. Furthermore, observe that the point of intersection ξ becomes

$$\tilde{\xi} = (\pi, y_1, \dots, y_n),$$

in the new coordinates. Hence

$$\begin{aligned} l\left(\left(\frac{T[x_0, \dots, x_n]_\xi}{(\varphi_0^P, \dots, \varphi_{n-k-1}^P, \varphi_0^R, \dots, \varphi_k^R)}\right)^{\deg=0}\right) &= l\left(\left(\frac{T[y_0, \dots, y_n]_{\tilde{\xi}}}{(\varphi_0, \dots, \varphi_n)}\right)^{\deg=0}\right) \\ &= l\left(\left(\frac{T[y_0, \dots, y_n]_{(\pi, y_1, \dots, y_n)}}{(y_0\pi^{\alpha_0}, y_1, \dots, y_n)}\right)^{\deg=0}\right) \\ &= l\left(\frac{T}{(\pi^{\alpha_0})}\right) \\ &= \alpha_0 \end{aligned}$$

On the other hand, observe that $\varphi_P \wedge \varphi_R = \det(M)$, up to a unit in K, thus

$$\begin{aligned} v_T(\varphi_R(v_P)) &= v_T(\varphi_P \wedge \varphi_R) \\ &= v_T(\det(M)) \\ &= v_T(\det(UDV)) \\ &= v_T(\det(U)\det(D)\det(V)) \\ &= v_T(\det(D)) \\ &= v_T\left(u_0\pi^{\alpha_0} \cdot \prod_{i=1}^n u_i\right) \\ &= \alpha_0. \end{aligned}$$

So, we have shown that both sides of Equation 3.3 equal α_0. This concludes the proof. □

Next up, we want to relate this to the asymptotic behaviour of the Archimedean height pairing. Consider the special case where we take $T = \mathbb{C}\{t\}$, the ring consisting of *convergent* power series, with uniformizer t. Equivalently, it is the ring of germs of holomorphic functions at $0 \in \mathbb{C}$. Again, let $\overline{Y_P}$ and $\overline{Y_Q}$ be k-planes and $\overline{Y_R}$ and $\overline{Y_S}$ be $(n-k-1)$-planes in $\mathbb{P}_T^n$. Assume that $\overline{Y_P}_\eta, \overline{Y_Q}_\eta, \overline{Y_R}_\eta$ and $\overline{Y_S}_\eta$ are non-degenerate in $\mathbb{P}_K^n$ and that the families intersect properly. We can find equations cutting out these families, i.e., $\overline{Y_P} = Z(\varphi_0^P, \dots, \varphi_{n-k-1}^P)$ where each φ_i^P is a $\mathbb{C}\{t\}$-linear combination of the homogeneous coordinates $x_0, \dots, x_n$. Because the φ_i^P's are convergent power series, on a small enough disc

$$\Delta_\epsilon = \{t \in \mathbb{C} \mid |t| < \epsilon\},$$

we can actually evaluate the φ_i^P's on t to get equations $\varphi_i^P(t) \in \mathbb{C}[x_0, \dots, x_n]$. Let $Y_P(t) = Z(\varphi_0^P(t), \dots, \varphi_{n-k-1}^P(t)) \subset \mathbb{CP}^n$, similarly for $Y_Q(t), Y_R(t)$ and $Y_S(t)$. Now, by Theorem 2.20, we have shown that

$$\langle [Y_P(t)] - [Y_Q(t)], [Y_R(t)] - [Y_S(t)] \rangle_\infty = \log |\mathrm{CR}^n(Y_P(t), Y_Q(t); Y_R(t), Y_S(t))|.$$

Suppose now that the families *do* intersect over the special fiber, i.e., when $t = 0$. Then the cross-ratio could vanish or diverge, exactly with rate $v_T(\mathrm{CR}^n(\overline{Y_P}_\eta, \overline{Y_Q}_\eta; \overline{Y_R}_\eta, \overline{Y_S}_\eta))$. So, we can write

$$\langle [Y_P(t)] - [Y_Q(t)], [Y_R(t)] - [Y_S(t)] \rangle_\infty = v_T(\mathrm{CR}^n(\overline{Y_P}_\eta, \overline{Y_Q}_\eta; \overline{Y_R}_\eta, \overline{Y_S}_\eta)) \log |t| + \log |u(t)|,$$

for some nowhere-vanishing holomorphic function $u(t)$. Lastly, we note that every operation in the construction of the cross-ratio commutes with base changes. Thus, base changing the cross-ratio from $\mathbb{C}\{t\}$ to $\mathbb{C}$, i.e., for sufficiently small non-zero t,

$$\mathrm{CR}^n(\overline{Y_P}, \overline{Y_Q}; \overline{Y_R}, \overline{Y_S})(t) = \mathrm{CR}^n(Y_P(t), Y_Q(t); Y_R(t), Y_S(t)).$$

Combining this with Theorem 3.4 yields the first result about the asymptotic behaviour of the Archimedean height pairing, namely in the case of proper degenerations.

Theorem 3.5. *With the notation and assumptions as above, the following holds, for t small enough,*

$$\langle [Y_P(t)] - [Y_Q(t)], [Y_R(t)] - [Y_S(t)] \rangle_\infty = \deg\left((\overline{Y_P} - \overline{Y_Q}) \cdot (\overline{Y_R} - \overline{Y_S})\right) \log |t| + \log |u(t)|,$$

where $u : \Delta_\epsilon \to \mathbb{C}$ is some nowhere-vanishing holomorphic function.

Remark 3.6. Theorem 3.5 is an instance of a conjecture by Z. Chen and R. de Jong [Che25, Conjecture 1.5]. This conjecture states that the rate of divergence of the Archimedean height of certain families of algebraic cycles is controlled by their intersection degree, in *proper* degenerations. In [HdJ15, Theorem 2.1] such a result has been proven for degenerations of curves. We have now proven this for properly degenerating *higher-dimensional* projective linear subspaces. We aim to generalise this result to arbitrary degenerations in the next sections.

Section 3.2: An Example of Degenerating Families of Lines in $\mathbb{CP}^3_{\mathbb{C}\{t\}}$

Let us consider the following example to illustrate the existence of non-proper degenerations. Again, $T = \mathbb{C}\{t\}$ and we are dealing with four families of 1-planes, i.e., families of lines in $\mathbb{P}^3_T$. *For simplicity* we assume that the lines $Y_P(t), Y_Q(t), Y_R(t)$ and $Y_S(t)$, interpreted as $\overline{Y_P}_\eta, \overline{Y_Q}_\eta, \overline{Y_R}_\eta$ and $\overline{Y_S}_\eta$ base changed to $\mathbb{C}$, depend *linearly* on t. Because of this, the valuation v_T counts the dimension of the intersection,

$$v_T(\mathrm{CR}^n(\overline{Y_P}_\eta, \overline{Y_Q}_\eta; \overline{Y_R}_\eta, \overline{Y_S}_\eta) = \dim(\overline{Y_P}_s \cap \overline{Y_R}_s) + \dim(\overline{Y_Q}_s \cap \overline{Y_S}_s) - \dim(\overline{Y_P}_s \cap \overline{Y_S}_s) - \dim(\overline{Y_Q}_s \cap \overline{Y_R}_s).$$

This dimension can exceed the expected dimension 0. This is a first example of non-proper intersections, called *excess intersection*, which will be treated in Section 3.4. Up to symmetry, Figure 4 depicts all possible degenerations, together with the value of

$$A = \deg\left((\overline{Y_P} - \overline{Y_Q}) \cdot (\overline{Y_R} - \overline{Y_S})\right).$$

The four lines come with a sign, 1 for $Y_P(0)$ and $Y_R(0)$ and -1 for $Y_Q(0)$ and $Y_S(0)$. In the figure the positive lines are coloured in red and the negative lines are coloured in blue. Intersections are either red or blue, dependent on the product of signs of the two intersecting lines. For example, when $Y_P(0)$ intersects $Y_S(0)$ the intersection is coloured blue, because $1 \cdot (-1) = -1$.

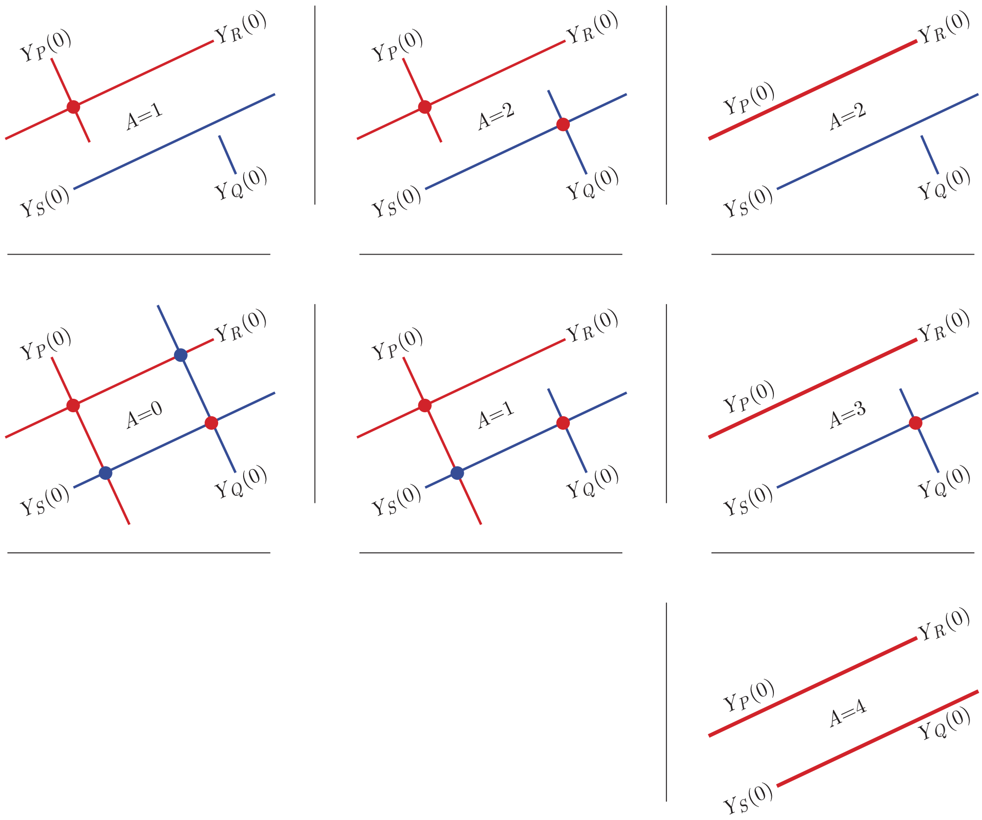


Figure 4: Linear degenerations of families of lines in $\mathbb{CP}^3_{\mathbb{C}\{t\}}$.

We will now start developing the necessary intersection theory to be able to compute the degree of excess and, more generally, *residual* intersections.

Section 3.3: An Introduction to Intersection Theory

In the remainder of this chapter we will recall the classical intersection product, between classes of algebraic cycles which intersect properly. Following *Intersection theory* by W. Fulton [Ful98], we will recall the generalised construction to handle non-proper intersections. In Section 3.4 we will assume that the intersection is defined using regular embeddings. Later, in Section 3.5, we will treat the most general case.

First, let us recall what the intersection product is. Let X be a smooth variety. Let $Z^k(X)$ denote the *cycle group*, which consists of formal sums of irreducible closed subvarieties of codimension k. Next up, we define the *Chow groups* $A^k(X) = Z^k(X)/\sim_{\text{rat}}$, where two irreducible subvarieties V_1 and V_2 are said to be *rationally equivalent* if there exists a flat family $W \subset X \times \mathbb{P}^1$ such that $W_0 = V_1$ and $W_\infty = V_2$. We leave it as an exercise to the reader to verify that this is an equivalence relation and extends linearly to the cycle groups $Z^k(X)$. We have an obvious additive group structure on these Chow groups, and we aim to upgrade this to a ring structure. This is where the intersection product comes in.

First, assume that two irreducible subvarieties V_1 and V_2 meet transversely, meaning that for all $p \in V_1 \cap V_2$ the tangent spaces of V_1 and V_2 together span all of the tangent space of X, i.e.,

$$T_pV_1 + T_pV_2 = T_pX.$$

There exists a unique intersection product, such that for all transverse intersections V_1 and V_2 this intersection is given by

$$\begin{aligned} A^i(X) \times A^j(X) &\to A^{i+j}(X) \\ [V_1] \cdot [V_2] &\mapsto [V_1 \cap V_2], \end{aligned}$$

see page 19 of [EH16]. We can upgrade this slightly to the case where V_1 and V_2 meet *generically transversely*, i.e. when there is a dense open subset $U \subset V_1 \cap V_2$ on which V_1 and V_2 meet transversely. Now we will state an important and well-known result.

Lemma 3.7 (Moving Lemma)**.** *Let X be a quasi-projective variety. Given $\alpha, \beta \in A^*(X)$ there exists an α' rationally equivalent to α such that α' meets β properly.*

There are, however, two important caveats concerning the moving lemma. First of all, it is unclear whether a complete proof exists, as all classical proofs turned out to contain errors. Secondly, the Lemma is an existence result and therefore not immediately useful to compute actual intersections.

Now in the case where X is also smooth, proper intersection implies generically transverse. Hence for any cycle we can find one that is rationally equivalent, which intersects the other cycle generically transversely. In that case we also declare $[V_1] \cdot [V_2] = [V_1 \cap V_2]$. When the intersection of V_1 and V_2 is transverse, in particular the subvariety $V_1 \cap V_2$ is reduced, and hence justifies calling it a sub*variety*. When we are studying families of varieties we do not want to move cycles using the moving Lemma and we may need to define an intersection product that allows non-transverse intersections. We omit the condition that V_1 and V_2 meet (generically) transversely, but only require them to intersect in the expected dimension, i.e.,

$$\operatorname{codim}(V_1 \cap V_2) = \operatorname{codim}(V_1) + \operatorname{codim}(V_2).$$

In this case we can compute, see [Sta, Section 43.1], the intersection product as follows,

$$[V_1] \cdot [V_2] = \sum_{Z \subset V_1 \cap V_2} m_Z(V_1, V_2; X)[Z], \tag{3.4}$$

where Z runs over the irreducible components (of expected dimension) of $V_1 \cap V_2$. Furthermore, m_Z is given by

$$m_Z = \sum_i (-1)^i \operatorname{length}_{\mathcal{O}_{X,Z}}(\operatorname{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{V_1,Z}, \mathcal{O}_{V_2,Z})),$$

this reduces to

$$m_Z = \operatorname{length}_{\mathcal{O}_{X,\eta_Z}}(\mathcal{O}_{V_1,\eta_Z} \otimes_{\mathcal{O}_{X,\eta_Z}} \mathcal{O}_{V_2,\eta_Z}) \tag{3.5}$$

when for example, W is cut out by a regular sequence in $\mathcal{O}_{X,Z}$ which also defines a regular sequence in $\mathcal{O}_{V,Z}$. Here $\mathcal{O}_X$ denotes the structure sheaf of X, and η_Z denotes the generic point of the irreducible subscheme Z.

Example 3.8. Consider the following example. Take $X = \mathbb{A}^2 = \operatorname{Spec} \mathbb{Z}[x, y]$ with coordinates x and y. Then we consider the two smooth irreducible subvarieties,

$$\begin{aligned} V_1 &= Z(y - x^2), \\ V_2 &= Z(y). \end{aligned}$$

Then set-theoretically $V_1 \cap V_2 = \xi$ where ξ is the single point given by $(0, 0)$. Unfortunately, V_1 and V_2 do not meet transversely, or generically transversely. In this example the intersections are cut out by regular sequences and we can compute $[V_1] \cdot [V_2]$ using Equations 3.4 and 3.5. Observe that the pull-back square of *affine* schemes,

$$\begin{array}{ccc} \xi & \longrightarrow & \operatorname{Spec} \mathbb{Z}[x,y]/(y-x^2) \\ \downarrow & \lrcorner & \downarrow j \\ \operatorname{Spec} \mathbb{Z}[x,y]/(y) & \xrightarrow{i} & \operatorname{Spec} \mathbb{Z}[x,y] \end{array}$$

corresponds exactly to the push-out square of rings

$$\begin{array}{ccc} \mathcal{O}_\xi(\xi) & \longleftarrow & \mathbb{Z}[x,y]/(y-x^2) \\ \uparrow & \lrcorner & \uparrow \\ \mathbb{Z}[x,y]/(y) & \longleftarrow & \mathbb{Z}[x,y] \end{array}$$

and we note that

$$\mathcal{O}_\xi(\xi) = \mathbb{Z}[x,y]/(y) \otimes_{\mathbb{Z}[x,y]} \mathbb{Z}[x,y]/(y-x^2) \cong \mathbb{Z}[x]/(x^2).$$

Thus finally we see

$$[V_1] \cdot [V_2] = \sum_{p \in V_1 \cap V_2} m_p(V_1, V_2; X)[p] = \operatorname{length}_{\mathbb{Z}[x]_{(x)}}(\mathbb{Z}[x]_{(x)}/(x^2))[\xi] = 2[\xi].$$

Remark 3.9. To show that these defining properties agree we have to show that the intersection multiplicity is always 1 in case of a transverse intersection. This is a standard result in commutative algebra and uses the famous *Nakayama lemma* (or rather, one of the many formulations of it), see [EH16, Theorem 1.26].

Section 3.4: Excess Intersection Theory

However, even with this refined definition, we still cannot handle the case where V_1 and V_2 intersect in a higher dimension than expected. This is called an excess intersection. To be able to handle this case we require some serious retooling and in this subsection we will go through this construction, as presented in [Ful98]. First, we will recall the construction in the case of proper intersections and show that it agrees with the definition as above. Then this will be upgraded to the case of excess intersections.

The setup is as follows. Let $i : X \to Y$ be a closed embedding of codimension d. Let V be a purely k-dimensional scheme and let $f : V \to Y$ be any morphism. Let $W = f^{-1}(X)$ denote the inverse image scheme of X so that we have a fiber square

$$\begin{array}{ccc} W & \xrightarrow{j} & V \\ \downarrow g & \lrcorner & \downarrow f \\ X & \xrightarrow{i} & Y \end{array} .$$

Consider the normal bundle of X in Y, denoted by $N_X Y$ which is defined via the short exact sequence of vector bundles over X

$$0 \to TX \to i^*TY \to N_X Y \to 0.$$

We let $N = g^* N_X Y$, viewed as a vector bundle over W. We can construct this because i is a regular embedding. However, j need not be a regular embedding so we have to consider the normal *cone* of W in V, denoted by $C_W V$ and defined by

$$C_W V = \operatorname{Spec} \bigoplus_{n \geq 0} \mathcal{G}^n / \mathcal{G}^{n+1},$$

where $\mathcal{G}$ denotes the ideal sheaf of W in V. Let $C = C_W V$, we want to realise C as a k-cycle on N. Let $s : W \to N$ denote the zero section of the vector bundle $N \to W$. Since s is a regular embedding, it has an associated refined Gysin homomorphism

$$s^! : A_k(N) \longrightarrow A_{k-\operatorname{rk} N}(W).$$

Geometrically, $s^!$ is intersection with the zero section. It takes a cycle on the total space of N and produces its refined intersection with W, viewed as the zero section inside N. Applying this to the cycle represented by the normal cone $C_W V \subset N$, we obtain a cycle

$$s^![C_W V] \in A_{k-\operatorname{rk} N}(W),$$

which is the desired excess/intersection cycle on W. First, we need the following lemma.

Lemma 3.10. *The ideal sheaf $\mathcal{F}$ of X in Y generates the ideal sheaf $\mathcal{G}$ of W in V.*

Proof. Consider the canonical short exact sequence of $\mathcal{O}_Y$-modules

$$0 \to \mathcal{F} \to \mathcal{O}_Y \to i_*\mathcal{O}_X \to 0.$$

Then apply f^*, the pull-back of $\mathcal{O}_Y$-modules as constructed in [JT23a, p. 65]. This is right exact so we get

$$f^*\mathcal{F} \to \mathcal{O}_V \to j_*\mathcal{O}_W \to 0,$$

where we used that $f^*\mathcal{O}_Y = \mathcal{O}_V$ and the fact that $f^*i_*\mathcal{O}_X = j_*g^*\mathcal{O}_X = j_*\mathcal{O}_W$. Thus we can construct a surjection $f^*\mathcal{F} \to \mathcal{G}$ because $\mathcal{G}$ fits in the short exact sequence of $\mathcal{O}_V$ modules

$$0 \to \mathcal{G} \to \mathcal{O}_V \to j_*\mathcal{O}_W \to 0.$$

This is exactly what it means for $\mathcal{F}$ to generate $\mathcal{G}$. □

Thus, as a direct consequence there is a surjection

$$\bigoplus_n f^*(\mathcal{F}^n/\mathcal{F}^{n+1}) \to \bigoplus_n \mathcal{G}^n/\mathcal{G}^{n+1},$$

and hence a closed immersion

$$\operatorname{Spec} \bigoplus_n \mathcal{G}^n/\mathcal{G}^{n+1} \to \operatorname{Spec} \bigoplus_n f^*(\mathcal{F}^n/\mathcal{F}^{n+1}).$$

Thus we obtain a closed immersion $C \hookrightarrow N$. Here we used the functoriality of pull-backs to write N as

$$\operatorname{Spec} \bigoplus_n j^*f^*(\mathcal{F}^n/\mathcal{F}^{n+1}) = \operatorname{Spec} \bigoplus_n g^*(\mathcal{F}^n/\mathcal{F}^{n+1}).$$

So in particular, we get a commutative diagram

$$\begin{array}{ccc} C & \hookrightarrow & N \\ & \searrow^{\pi_C} & \downarrow{\scriptstyle \pi_N} \\ & & W \end{array}$$

Now because V is a purely k-dimensional scheme $C = C_W V$ is also a purely k-dimensional scheme, see [Ful98, Appendix B.6.6]. Thus in particular, C determines a k-cycle on N given by $[C]$. Now we may define the intersection product $X \cdot V$ to be the unique class in $A_{k-d}(W)$ such that $\pi_N^*(X \cdot V) = [C]$ in $A_k(N)$, or equivalently $s^![C]$, where $s^!$ is the Gysin homomorphism.

Example 3.11. We will now compute the intersection of families of lines X and V in $\mathbb{P}^3_T$, where T is some discrete valuation ring. Let x, y, z, w denote the homogeneous coordinates on $\mathbb{P}^3$ and let π denote the uniformizer of T. Define the flat families

$$\begin{aligned} X &= Z(x, y - \pi^m), \\ V &= Z(y, z). \end{aligned}$$

Observe that the set-theoretic intersection $X \cap V = \{([0:0:0:1], s)\}$, again s denotes the closed point of $\operatorname{Spec}(T)$. Let us switch to the chart where $w \neq 0$. Then we get that

$$W = X \times_{\mathbb{P}^3_T} V = \operatorname{Spec} T[x, y, z]/(x, y, z, y - \pi^m) = \operatorname{Spec} T/(\pi^m),$$

and we get the following fiber square

$$\begin{array}{ccc} \operatorname{Spec} T/(\pi^m) & \xrightarrow{j} & \operatorname{Spec} T[x] \\ \downarrow g & \lrcorner & \downarrow f \\ \operatorname{Spec} T[y, z]/(y - \pi^m) & \xrightarrow{i} & \operatorname{Spec} T[x, y, z] \end{array}$$

Now observe that the ideal sheaf $\mathcal{F}$ of X in $\mathbb{P}^3_T$, and the ideal sheaf $\mathcal{G}$ of W in V are given by

$$\begin{aligned} \mathcal{F} &= (x, y - \pi^m), \\ \mathcal{G} &= (x, \pi^m), \\ f^*\mathcal{F} &= (x, \pi^m). \end{aligned}$$

Hence, we note that $C \cong N$, in particular C determines the class $[N] \in A_2(N)$, the fundamental class. Hence, we need a cycle $Z \in A_0(W)$ such that $\pi_N^* Z = [C]$, and the fundamental class of W satisfies this precisely, thus

$$X \cdot V = [W] = m \cdot [\xi] \in A_0(W).$$

So in particular,

$$\deg(X \cdot V) = m.$$

We can also compute the intersection using *Chern* and *Segre* classes.

Proposition 3.12 (Fulton's Proposition 6.1)**.** *With the notation as above*

$$X \cdot V = \{c(N) \cap s(W, V)\}_{k-d}$$

We will briefly introduce these concepts and quickly specialise to a simple case which will suffice for our purposes. Let E be a vector bundle of rank r on a scheme X. Then, the total Chern class is given by

$$c(E) = 1 + c_1(E) + \cdots + c_r(E) \in A^*(X),$$

where each *Chern class* $c_i(E) \in A^i(X)$ is determined by the following axioms. For any morphism of schemes $f : Y \to X$ we have $f^*c(E) = c(f^*E)$. Secondly, given an exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0,$$

we have $c(E) = c(E') \cdot c(E'')$. Thirdly, $c(\mathcal{O}(1)) = 1 + [H]$, for $\mathcal{O}(1)$ on $\mathbb{P}^n$ and where $[H]$ denotes the class of a codimension 1 hyperplane. Finally, the Chern classes satisfy the splitting principle. So if $E = L_1 \oplus \cdots \oplus L_r$ then

$$c(E) = \prod_{i=1}^{r} (1 + c_1(L_i)).$$

The Segre classes measure the deviation of a subscheme $Z \subset X$ from being a regular embedding. Let $C_Z X$ be the normal cone of Z in X and let $\pi : C_Z X \to X$ denote the projection. Then

$$s(Z, X) = \pi_* \left(\frac{[C_Z X]}{c(\mathcal{O}(1))} \right),$$

where $\mathcal{O}(1)$ is the tautological line bundle on $\mathbb{P}(C_Z(X) \oplus 1)$. However, in our case the subscheme $Z \subset X$ is locally defined by an ideal I. Then the formula reduces to

$$s(Z, X) = \pi_*([C_Z X]).$$

Example 3.13. We will compute the intersection $X \cdot V$ of example 3.11 again, this time using Proposition 3.12. For this we need to compute $c(N)$ and $s(W, V)$. Note that N is just the trivial vector bundle of rank 2 over W and hence $c(N) = 1$. Then

$$\begin{aligned} \{c(N) \cap s(W, V)\}_0 &= \{1 \cap (\pi_N)_*[C]\}_0 \\ &= \{(\pi_N)_*[N]\}_0 \\ &= \{[W]\}_0 \\ &= \{m \cdot [\xi]\}_0 \\ &= m \cdot [\xi]. \end{aligned}$$

Now finally we can introduce the Excess Intersection Formula [Ful98]. Consider the fiber square

$$\begin{array}{ccc} X' & \xrightarrow{j} & Y' \\ \downarrow{\scriptstyle g} & \lrcorner & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

where $i : X \to Y$ is a regular embedding of codimension d. We define homomorphisms

$$\begin{aligned} i^! : Z_k(Y') &\to A_{k-d}(X') \\ \sum_i n_i [V_i] &\mapsto \sum_i n_i X \cdot V_i \end{aligned}$$

where $X \cdot V_i$ is the intersection product as constructed in the previous subsection. We state that these descend to a homomorphism on the Chow groups

$$i^! : A_k(Y') \to A_{k-d}(X'),$$

see [Ful98, p. 98]. These maps are called the *refined Gysin homomorphisms.* Now consider the double fibre diagram

$$\begin{array}{ccc} X'' & \longrightarrow & Y'' \\ \downarrow{\scriptstyle q} & \lrcorner & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{j} & Y' \\ \downarrow{\scriptstyle g} & \lrcorner & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

Here i (resp. j) is a regular embedding of codimension d (resp. d') with normal bundle N (resp. N'). Then, as seen before, there is a closed immersion of N' in g^*N, because now both of these are truly vector bundles we can form the quotient bundle

$$E = g^*N/N',$$

called the *excess normal bundle* of the lower fibre square of dimension $e = d - d'$.

Theorem 3.14 (Excess intersection formula)**.** *For any* $\alpha \in A_k(Y'')$*,*

$$i^!(\alpha) = c_e(q^*E) \cap j^!(\alpha)$$

in $A_{k-d}(X'')$*.*

This formula can help when computing the intersection product in the case of an excess intersection.

Example 3.15. Consider a slight modification to the example 3.11, where we now let

$$X = Z(x + tz, y),$$
$$V = Z(x, y + tw).$$

Observe that over the special fiber, i.e. when $t = 0$, $X_s = Z(x, y) = V_s$. Thus, the set-theoretic intersection

$$X \cap V = L \times \{0\} \subset \mathbb{P}^3_T$$

where L is the line given by $\{[0 : 0 : z : w] : [z : w] \in \mathbb{P}^1\}$, which is of dimension 1 *and not* 0. This is an excess intersection. Note in this case that $d = 2$ and $d' = 1$ and so the excess bundle E is of rank $e = 2 - 1 = 1$, i.e. a line bundle. First let us compute $N_X Y$, the normal bundle of X inside Y. For this note that the normal bundle

$$N_X Y = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{O}_X),$$

where the ideal sheaf of X in Y is given by

$$\mathcal{I}_X = \widetilde{(x + tz, y)} \subset \mathcal{O}_Y.$$

On the affine chart $z \neq 0$, write $u = x/z$, $v = y/z$ and $s = w/z$. Then the defining equations become

$$v = 0, \quad u + t = 0.$$

Thus, in particular, locally $\mathcal{I}_X/\mathcal{I}_X^2$ is free of rank 2 generated by the classes $[v]$ and $[u + t]$. If we now switch to the other chart of X, namely $w \neq 0$, write $u' = x/w$, $v' = y/w$ and $r = z/w$. Then the defining equations become

$$v' = 0, \quad u' + tr = 0.$$

Furthermore, $v' = \frac{z}{w}v$ and $u' + tr = \frac{z}{w}(u + t)$, so the transition function from U_z to U_w is given by $\frac{z}{w}$ and hence these local pieces glue together to form $\mathcal{O}_X(-1) \oplus \mathcal{O}_X(-1)$ so in particular the dual

$$N_X Y \cong \mathcal{O}_X(1) \oplus \mathcal{O}_X(1).$$

Now consider the normal bundle of X' in Y', given by

$$\mathcal{I}_{X',Y'} = \widetilde{(tz, tw)} \subset \mathcal{O}_{Y'}.$$

On the chart U_z we have $(tz, tw) = (t)$ and on the chart U_w we also have $(tz, tw) = (t)$ so the pieces glue together trivially and we note that

$$N_{X'}Y' \cong \mathcal{O}_{X'}(0).$$

Now we have

$$E = \frac{g^* N_X Y}{N_{X'}Y'} \cong \frac{\mathcal{O}_{X'}(1) \oplus \mathcal{O}_{X'}(1)}{\mathcal{O}_{X'}(0)} \cong \mathcal{O}_{\mathbb{P}^1}(2),$$

and so we observe that $c_1(E) = 2[\mathrm{pt}]$. Thus, now finally the intersection formula gives us (where we take $Y'' = Y'$ and $X'' = X'$)

$$i^!([Y']) = c_1(E) \cap j^!([Y']) = 2[\mathrm{pt}] \cap [X'] = 2[\mathrm{pt}] \in A_0(X'').$$

Here we used that $j^!([Y']) = [X'] \cdot [Y'] = [X']$ because $[X']$ is a principal Cartier divisor. Hence we note that $\deg(X \cdot V) = 2$ and this agrees with

$$v_T \det \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ t & 0 & 0 & 0 \\ 0 & 0 & 0 & t \end{bmatrix} = v_T(-t^2) = 2.$$

Remark 3.16. The excess intersection formula gives a very nice and explicit formula for computing the intersection $X \cdot V$. However, there is a strong condition which is not always satisfied in our case, namely we require $W = V \cap X$ to be regularly immersed in V (or X). Consider the following example, in $\mathbb{P}^3_T$ we have two families of lines X and V such that the intersection of W in V is given by the ideal (xt^a, yt^b). This closed immersion is a regular immersion, of codimension 1, if and only if $a = b$ and $a, b > 0$. Of course, when $a = b = 0$ then $W = \varnothing$ and whenever only a or b is 0, then the intersection is in the expected dimension and the codimension of W in V is 2, so we will exclude these cases. Now suppose $a = b$, then locally, the ideal defining W is given by (t^a), we can check this on the two charts of $W \cong \mathbb{P}^1_{\mathbb{C}}$. On the chart where $x \neq 0$ we have $(xt^a, yt^a) = (t^a, yt^a) = (t^a)$ and similarly on the other chart. Now on the other hand, suppose that $a < b$, then we cannot find a regular sequence of non-zero-divisors of length 1 (or any length) defining W. Consider the chart where $y \neq 0$ then the ideal is given by (xt^a, t^b) which is not principal because $a < b$. Furthermore, the dimension of W is 1, and the dimension of V is 2 so if W is regularly immersed in V it would *have* to be of codimension 1, which is not possible. Because of this we will resort to the residual intersection formula, which generalises the excess intersection formula.

Section 3.5: General Degenerations

To handle the case where the ideal defining W in V and X does not define a regular immersion we must turn to residual intersections. First, we will state the relevant results from [Ful98, Chapter 9]. Consider the following situation.

$$\begin{array}{ccccc} & & R & & \\ & & \downarrow{\scriptstyle b} & & \\ D & \xrightarrow{a} & W & \xrightarrow{j} & V \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ & & X & \xrightarrow{i} & Y \end{array}$$

Here a, b, i, j are closed immersions, and V is a purely k-dimensional variety. Furthermore, assume i is a regular immersion of codimension d, $j \circ a$ embeds D as a Cartier divisor in V and R is the residual scheme to D in W. This means that

$$W = D \cup R,$$

and furthermore, the ideal sheaves on V are related by

$$\mathcal{I}(W) = \mathcal{I}(D) \cdot \mathcal{I}(R).$$

Then

Theorem 3.17 (Residual Intersection Theorem, Fulton 9.2)**.** *Let $N = g^* N_X Y$ and $\mathcal{O}(-D) = j^* \mathcal{O}_V(-D)$. Define the residual intersection class $\mathbb{R}$ in $A_{k-d}(R)$ by the formula*

$$\mathbb{R} = \{c(N \otimes \mathcal{O}(-D)) \cap s(R, V)\}_{k-d}.$$

Then

$$X \cdot V = \{c(N) \cap s(D, V)\}_{k-d} + \mathbb{R}.$$

First of all, following Corollary 9.2.1 of [Ful98], note that

$$\{c(N) \cap s(D, V)\}_{k-d} = \left\{ \frac{c(N)}{c(\mathcal{O}(D))} \right\}_{d-1} \cap [D].$$

With all the machinery as above, we will generalise the result of Theorem 3.4, to the case where the pairwise intersections need not be proper. As a last preliminary result we will compute the normal bundle of a regular immersion $i : X \to \mathbb{P}^n_T$ of codimension d, together with its total Chern class. We quickly note that in [Ful98] all schemes are assumed to be finite type over a field. In this case, these are not schemes over a field, but rather over the spectrum of a discrete valuation ring. Fulton notes that "the residual intersection theorem generalises without change", see [Ful98, p. 395], to the case where our schemes are of finite type over the spectrum of a discrete valuation ring.

Proposition 3.18. *Let $X = \mathbb{P}^{n-d}_T$ and $Y = \mathbb{P}^n_T$. Let $i : X \to Y$ be a regular immersion of codimension d, then*

$$N_X Y \cong \mathcal{O}_X(1)^{\oplus d}$$

and furthermore

$$c(N_X Y) = (1 + [H])^d,$$

where

$$[H] = i^* c_1(\mathcal{O}_Y(1)) = c_1(\mathcal{O}_X(1))$$

the class of a hyperplane on X.

Proof. Now, as in Example 3.2.11 in [Ful98], we have an exact sequence on $\mathbb{P}^n$

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0,$$

and similarly on $\mathbb{P}^{n-d}$

$$0 \to \mathcal{O}_{\mathbb{P}^{n-d}} \to \mathcal{O}_{\mathbb{P}^{n-d}}(1)^{\oplus(n-d+1)} \to T_{\mathbb{P}^{n-d}} \to 0.$$

Furthermore, as in Example 3.2.12 in [Ful98], we have an exact sequence on $\mathbb{P}^{n-d}$

$$0 \to T_{\mathbb{P}^{n-d}} \to i^* T_{\mathbb{P}^n} \to N_{\mathbb{P}^{n-d}} \mathbb{P}^n \to 0.$$

Then we have

$$\begin{aligned} N_{\mathbb{P}^{n-d}} \mathbb{P}^n &\cong i^* T_{\mathbb{P}^n} / T_{\mathbb{P}^{n-d}} \\ &\cong \left(i^* \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} / i^* \mathcal{O}_{\mathbb{P}^n} \right) / \left(\mathcal{O}_{\mathbb{P}^{n-d}}(1)^{\oplus(n-d+1)} / \mathcal{O}_{\mathbb{P}^{n-d}} \right) \\ &\cong \left(\mathcal{O}_{\mathbb{P}^{n-d}}(1)^{\oplus(n+1)} / \mathcal{O}_{\mathbb{P}^{n-d}} \right) / \left(\mathcal{O}_{\mathbb{P}^{n-d}}(1)^{\oplus(n-d+1)} / \mathcal{O}_{\mathbb{P}^{n-d}} \right) \\ &\cong \mathcal{O}_{\mathbb{P}^{n-d}}(1)^{\oplus(n+1)} / \mathcal{O}_{\mathbb{P}^{n-d}}(1)^{\oplus(n-d+1)} \\ &\cong \mathcal{O}_{\mathbb{P}^{n-d}}(1)^{\oplus d}. \end{aligned}$$

Then using the fact that $c(\mathcal{O}_{\mathbb{P}^{n-d}}(1)) = 1 + [H]$ for $[H]$ as above, together with the standard properties of Chern classes we have

$$c(\mathcal{O}_{\mathbb{P}^{n-d}}(1)^{\oplus d}) = \prod_{i=1}^{d} c(\mathcal{O}_{\mathbb{P}^{n-d}}(1)) = (1 + [H])^d,$$

as desired. □

Now let us state the main theorem of this section and in fact, of this thesis.

Theorem 3.19. *Let T be a discrete valuation ring, with valuation v_T and uniformizer π. Let $\overline{Y_P}$ and $\overline{Y_Q}$ be k-planes and $\overline{Y_R}$ and $\overline{Y_S}$ be $(n-k-1)$-planes in $\mathbb{P}^n_T$. Assume that $\overline{Y_P}_\eta, \overline{Y_Q}_\eta, \overline{Y_R}_\eta$ and $\overline{Y_S}_\eta$ are non-degenerate in $\mathbb{P}^n_K$. Then*

$$\deg\left((\overline{Y_P} - \overline{Y_Q}) \cdot (\overline{Y_R} - \overline{Y_S})\right) = v_T\left(\mathrm{CR}^n(\overline{Y_P}_\eta, \overline{Y_Q}_\eta; \overline{Y_R}_\eta, \overline{Y_S}_\eta)\right).$$

First, we will prove a convenient lemma, which we will use in the proof of Theorem 3.19.

Lemma 3.20. *Let T be a discrete valuation ring, with uniformizer π. Let V be a k-plane, and let X be an $(n-k-1)$-plane in $\mathbb{P}^n_T$. Then, after a change of coordinates, we may assume*

$$V = Z(x_{k+1}, \dots, x_n) \subset \mathbb{P}^n_T.$$

Furthermore, the scheme-theoretic intersection W of X and V inside V is given by

$$I_{W,V} = (x_0\pi^{\alpha_0}, \dots, x_k\pi^{\alpha_k}),$$

after possibly changing the first $k+1$ coordinates again.

Proof. Write the defining equations for V in an $(n-k) \times (n+1)$-matrix M, where each row is given by the coefficients of a T-linear equation in the coordinates $x_0, \dots, x_n$. Because V is a k-plane it is of constant full rank, so when we decompose $M = VDU$ in Smith-Normal form, see Lemma 3.2, with V, U invertible and D a diagonal matrix, D will be of the form

$$D = \begin{bmatrix} u_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & u_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & u_{n-k-1} & \cdots & 0 \end{bmatrix}.$$

Here u_j is a unit for all $0 \le j \le n-k-1$. After the change of coordinates given by U, we find

$$V = Z(u_0x_0, \dots, u_{n-k-1}x_{n-k-1}) = Z(x_0, \dots, x_{n-k-1}).$$

We then relabel the coordinates, e.g. $x_j := x_{n-j}$, and conclude

$$V = Z(x_{k+1}, \dots, x_n).$$

Then, the ideal defining W in V is given by

$$I_{W,V} = (\varphi_0, \dots, \varphi_k),$$

with $\varphi_j \in T[x_0, \dots, x_k]^{\deg=1}$. We can then apply the Smith normal form again, and after a change of coordinates we may assume

$$I_{W,V} = (x_0\pi^{\alpha_0}, \dots, x_k\pi^{\alpha_k}).$$

Note that here we could not assume that the diagonal elements were given by units, as W need not be flat over $\mathrm{Spec}(T)$. □

Now let us prove Theorem 3.19.

Proof. Similarly to the proof of Theorem 3.4, it suffices to show

$$\deg\left(\overline{Y_P}\cdot\overline{Y_R}\right) = v_T(\varphi_P \wedge \varphi_R).$$

First of all, let $Y = \mathbb{P}^n_T$, $X = \overline{Y_R}$ and $V = \overline{Y_P}$. Write W for the scheme-theoretic intersection of X and V. Let $i : X \to Y$ and $j : W \to V$ denote the closed immersions of X and V. Note that i is a regular immersion of codimension $k+1$ by Corollary 3.14 of [Liu02], as the discrete valuation ring T is in particular Noetherian. Using Lemma 3.20, find a suitable set of coordinates, such that V is stationary and given by $Z(x_{k+1},\dots,x_n) \subset T[x_0,\dots,x_n]$ and the ideal defining W inside V is given by

$$I_{W,V} = (x_0\pi^{\alpha_0},\dots,x_k\pi^{\alpha_k}).$$

We will prove

$$\deg(X\cdot V) = \sum_{i=0}^{k}\alpha_i = v_T(\varphi_P \wedge \varphi_R)$$

by induction on $l = \#\{\alpha_i \neq 0\} - 1$, the excess dimension. First, consider the case where $l = 0$, so the ideal defining W in V is, without loss of generality, given by $(x_0\pi^{\alpha_0}, x_1,\dots,x_k)$ and so X and V intersect properly. Then Theorem 3.4 gives that

$$\deg(X\cdot V) = v_T(\varphi_P \wedge \varphi_R) = \alpha_0 = \sum_{i=0}^{k}\alpha_i.$$

Now, suppose that the induction hypothesis holds for all $l \leq l_0$, then we will show that it also holds for l_0+1. So suppose the ideal defining W in V is given by

$$I_{W,V} = (x_0\pi^{\alpha_0},\dots,x_{l_0+1}\pi^{\alpha_{l_0+1}}, x_{l_0+2},\dots,x_k),$$

where, again without loss of generality, we assume $0 \leq \alpha_0 \leq \dots \leq \alpha_{l_0+1}$. Observe that $x_j = 0$ is a condition for X to intersect V. However, x_j need not be zero itself in X. We can find linear functionals ψ_j such that $\psi_j(x_0,\dots,x_k,0,\dots,0) = x_j$, for all $l_0+2 \leq j \leq k$, and $X \subset Z(\psi_{l_0+2},\dots,\psi_k)$. Now change V, call it V', to include the conditions $\psi_{l_0+2} = \psi_{l_0+3} = \dots = \psi_k = 0$, and remove them from X, call that X'. Note that we cannot do this for the other coordinates, as then V' would no longer be flat over $\operatorname{Spec}(T)$, whereas in this case V remains flat, i.e., a k-plane. Now we will show that the intersection

$$[X]\cdot[V] = [X']\cdot[V'].$$

To see this, let $H_j = \{\psi_j = 0\} \subset Y$ for $l_0+2 \leq j \leq k$. These are Cartier divisors on Y. Let

$$H = [H_{l_0+2}\cap H_{l_0+3}\cap\dots\cap H_k] = [H_{l_0+2}]\cdot[H_{l_0+3}]\cdots[H_k],$$

Furthermore,

$$[X]\cdot[V] = ([X']\cdot H)\cdot[V] = [X']\cdot(H\cdot[V]) = [X']\cdot[V'],$$

because neither X nor V is contained in any H_j, and the associativity and commutativity of the intersection product. Now X' is of codimension l_0+2 inside Y, and the ideal defining W in V' is given by

$$I_{W,V'} = (x_0\pi^{\alpha_0},\dots,x_{l_0+1}\pi^{\alpha_{l_0+1}}) = (\pi^{\alpha_0})\cdot(x_0, x_1\pi^{\alpha_1-\alpha_0},\dots,x_{l_0+1}\pi^{\alpha_{l_0+1}-\alpha_0}).$$

Now consider the principal Cartier divisor D globally given by $\operatorname{Div}(\pi^{\alpha_0})$. By the above equality, the residual scheme to D in W is given by

$$R = Z(x_0, x_1\pi^{\alpha_1-\alpha_0},\dots,x_{l_0+1}\pi^{\alpha_{l_0+1}-\alpha_0}).$$

Using Theorem 3.17, we find

$$X' \cdot V' = \left\{ \frac{c(N)}{c(\mathcal{O}(D))} \right\}_{d-1} \cap [D] + \mathbb{R}.$$

Now, first of all observe that the residual scheme $\mathbb{R}$ is given by an ideal with excess dimension less than or equal to l_0. To see this, consider X^{new} which is constructed as follows. X' is given by the zero locus of $l_0 + 2$ equations

$$X' = Z(\varphi_0, \dots, \varphi_{l_0+1}),$$

such that π^{α_0} divides $\varphi_j(x_0, \dots, x_k, 0, \dots, 0)$. Thus, in particular

$$\varphi_j = \sum_{i=0}^{k} \pi^{\alpha_0} a_{ij} x_i + \sum_{i=k+1}^{n} a_{ij} x_i,$$

where $a_{ij} \in T$. Then define

$$\varphi_j^{\text{new}} = \sum_{i=0}^{n} a_{ij} x_i,$$

and

$$X^{\text{new}} = Z(\varphi_0^{\text{new}}, \dots, \varphi_k^{\text{new}}).$$

Then

$$\begin{aligned}
\mathbb{R} &= \{c(N \otimes \mathcal{O}(-D)) \cap s(R, V')\}_0 && \text{(by definition)}\\
&= \{c(N) \cap s(R, V')\}_0 && \text{(as } \mathcal{O}(-D) \cong \mathcal{O}_W)\\
&= \{(1 + [H])^{l_0+2} \cap s(R, V')\}_0 && \text{(by Prop3.18)}\\
&= \{c(g^* N_{X^{\text{new}}} Y) \cap s(R^{\text{new}}, V')\}_0 && \text{(by Prop3.18 and } R = R^{\text{new}})\\
&= [X^{\text{new}}] \cdot [V'] && \text{(by definition).}
\end{aligned}$$

Here R^{new}, the ideal defining the intersection $X^{\text{new}} \cap V'$ in V', is precisely given by R. Furthermore, because $-D$ is a principal Cartier divisor $\mathcal{O}(-D)$ is a trivial vector bundle and hence

$$N \otimes \mathcal{O}(-D) \cong N.$$

Finally, we used that if g is the inclusion of the intersection W^{new} of X^{new} and V' in X^{new} then $g^* N_{X^{\text{new}}} Y \cong \mathcal{O}_{W^{\text{new}}}(1)^{\oplus l_0+2}$ by Proposition 3.18. Hence we may invoke the induction hypothesis to conclude

$$\deg(\mathbb{R}) = \sum_{i=1}^{l_0+1} (\alpha_i - \alpha_0) = \sum_{i=1}^{l_0+1} \alpha_i - (l_0 + 1)\alpha_0.$$

Now, observe that $[D] = \alpha_0 \cdot [\mathbb{P}_k^{l_0+1}]$, where k is the residue field of T, as this is

$$(\pi^{\alpha_0}) \cap V'.$$

Furthermore, because D is a principal Cartier divisor, $\mathcal{O}(D)$ is a trivial vector bundle and hence $c(\mathcal{O}(D)) = 1$. Lastly, using Proposition 3.18, we observe that $N \cong \mathcal{O}_W(1)^{\oplus l_0+2}$, because now $i : X \to Y$ is a regular immersion of codimension $l_0 + 2$ and so

$$c(N) = (1 + [H])^{l_0+2}$$

where $[H]$ is the class of any hyperplane in $\mathbb{P}_k^{l_0+1}$. Then

$$\begin{aligned}
X' \cdot V' &= \{(1 + [H])^{l_0+2}\}_{l_0+1} \cap \alpha_0[\mathbb{P}_k^{l_0+1}] + \mathbb{R}\\
&= (l_0 + 2)[H]^{l_0+1} \cap \alpha_0[\mathbb{P}_k^{l_0+1}] + \mathbb{R}\\
&= (l_0 + 2) \cdot \alpha_0[\text{pt}] + \mathbb{R}.
\end{aligned}$$

Therefore,

$$\deg(X \cdot V) = \deg(X' \cdot V') = \alpha_0 \cdot (l_0 + 2) + \sum_{i=1}^{l_0+1} \alpha_i - (l_0 + 1) \cdot \alpha_0 = \sum_{i=0}^{l_0+1} \alpha_i = v_T(\varphi_P \wedge \varphi_R).$$

This concludes the proof that $\deg(X \cdot V) = v_T(\varphi_P \wedge \varphi_R)$ by induction. □

Similarly to Theorem 3.5, when $T = \mathbb{C}\{t\}$ we find

Theorem 3.21. *Let $\overline{Y_P}$ and $\overline{Y_Q}$ be k-planes and $\overline{Y_R}$ and $\overline{Y_S}$ be $(n-k-1)$-planes in $\mathbb{P}^n_{\mathbb{C}\{t\}}$. Assume that $\overline{Y_P}_\eta, \overline{Y_Q}_\eta, \overline{Y_R}_\eta$ and $\overline{Y_S}_\eta$ are non-degenerate in $\mathbb{P}^n_{\mathrm{Frac}\mathbb{C}\{t\}}$. Then, for t small enough,*

$$\langle [Y_P(t)] - [Y_Q(t)], [Y_R(t)] - [Y_S(t)] \rangle_\infty = \deg\left((\overline{Y_P} - \overline{Y_Q}) \cdot (\overline{Y_R} - \overline{Y_S})\right) \log|t| + \log|u(t)|,$$

where $u : \Delta_\epsilon \to \mathbb{C}$ is some nowhere-vanishing holomorphic function.

Remark 3.22. Theorem 3.21 is a generalisation of Theorem 3.5, where non-proper degenerations of the projective linear spaces are now allowed. This result suggests that Chen's conjecture [Che25, Conjecture 1.5] could hold in a more general setting.

Remark 3.23. We highlight a simple aspect of the proof of Theorem 3.19. After modifying the setup in a convenient way, we are counting the exponents of the ideal I_W defining W in V. This ideal is of the form

$$I_W = (\pi^{\alpha_0} y_0, \ldots, \pi^{\alpha_l} y_l, y_{l+1}, \ldots, y_k),$$

where in this case l is the excess dimension. When proving the equality

$$\deg(X \cdot V) = \sum_{i=0}^{l_0+1} \alpha_i,$$

the right-hand side counts these exponents in the obvious way, $\alpha_0 + \alpha_1 + \cdots + \alpha_l$. Interestingly, the way the left-hand side is computed counts these exponents in a completely different way. I will highlight this difference by means of a simple example. Suppose the ideal I_W is given by $(\pi^2 x, \pi^4 y, \pi^4 z, \pi^5 w)$.

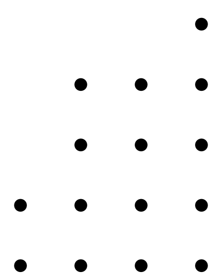


Figure 5: A figure representing the t-exponents of the ideal $(\pi^2 x, \pi^4 y, \pi^4 z, \pi^5 w)$.

As stated before, the right-hand side will simply add the number of dots in each column. The left-hand side takes a different approach. It will count the number of dots in rows, in the following way.

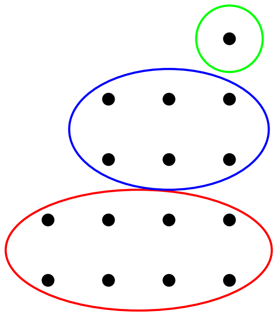


Figure 6: First the red dots $2 \cdot 4$, then the blue dots $2 \cdot 3$ and lastly the green dot.

Of course, counting the columns is the same as counting the rows (where we may count multiple rows at the same time). In some sense, that is why

$$\alpha_0 \cdot (l_0 + 2) + \left(\sum_{i=1}^{l_0+1} \alpha_i - \alpha_0(l_0 + 1) \right) = \sum_{i=0}^{l_0+1} \alpha_i.$$

Chapter 4: The Non-Archimedean Contributions

Number theory often studies geometric objects not only over the complex numbers, but also through their reductions modulo primes. Each prime p gives rise to a different local picture. Even when planes are non-degenerate over the complex numbers, it can happen that they intersect non-trivially when reduced modulo some prime p. The corresponding local height at p measures precisely this p-adic contribution to the arithmetic complexity of the cycles. In this chapter we apply the machinery from Chapter 3 to the ring $\mathbb{Z}_p$, for any prime p, to find the local heights whenever the planes Y_P, Y_Q, Y_R and Y_S are defined over $\mathbb{Z}$ or $\mathbb{Z}_p$. We will prove the following theorem.

Theorem C (Corollary 4.2)**.** Let p be a prime number. The p-adic local height of sub-complementary linear subspaces defined over $\mathbb{Z}_p$, whose generic fibres are non-degenerate, is given by

$$\langle Y_P - Y_Q, Y_R - Y_S\rangle_p = \log|\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S)|_p,$$

where $|\cdot|_p$ denotes the p-adic norm.

In the second part of this chapter, we will find a geometric interpretation of the constant $u(0)$ as in Theorem 3.19, namely a measurement for the rate of collision (or separation) of the moving planes. We will extend the definition of the generalised cross-ratio to the case where all planes are allowed to intersect, depending on chosen small perturbations. We call this the *regularised* cross-ratio. We will show that a degeneration of planes induces rate-of-collision vectors and show the following.

Theorem D (Corollary 4.10)**.** The absolute value of the constant $u(0)$ from Theorem 3.19 is given by the regularised cross-ratio of Y_P, Y_Q, Y_R and Y_S together with the induced rate-of-collision vectors $r_{P,R}, r_{P,S}, r_{Q,R}$ and $r_{Q,S}$.

We call $u(0)$ the *regularised limit*. It is very difficult to study these numbers themselves, because they lack an obvious structure. An algebraic variety which gives rise to such numbers is far easier to understand. The *limit geometry* of curves [BdJS23] and later nodal degenerations of certain odd-dimensional varieties [Bei25] have been studied. This thesis provides an entirely new class of examples in arbitrary dimension and for general degenerations of projective linear subspaces.

Section 4.1: Local Height Pairing at a Prime p

In this section we will consider linear subspaces of $\mathbb{P}^n_{\mathbb{Z}}$, instead of $\mathbb{P}^n_{\mathbb{C}}$. Let Y_P and Y_Q be k-dimensional linear subspaces and let Y_R and Y_S be $(n-k-1)$-dimensional linear subspaces of $\mathbb{P}^n_{\mathbb{Z}}$ such that they are flat, when viewed as families over $\operatorname{Spec}\mathbb{Z}$, and non-degenerate over the generic fibre. This means that as $\mathbb{Q}$-vector spaces the planes intersect trivially, and the dimension remains constant when we reduce modulo any prime p. Let $Y_{P,p}, Y_{Q,p}, Y_{R,p}$ and $Y_{S,p}$ denote the base changes of Y_P, Y_Q, Y_R and Y_S respectively to $\mathbb{Z}_p$. Then we can define the *local* height pairings.

Definition 4.1. For any prime p, the *local height pairing* for the prime p is given by

$$\langle Y_P - Y_Q, Y_R - Y_S\rangle_p = -\deg\left((Y_{P,p} - Y_{Q,p})\cdot(Y_{R,p} - Y_{S,p})\right)\log p,$$

and the *global height* is given by

$$\langle Y_P - Y_Q, Y_R - Y_S\rangle = \sum_{p\,\text{prime}} \langle Y_P - Y_Q, Y_R - Y_S\rangle_p + \langle Y_P - Y_Q, Y_R - Y_S\rangle_\infty.$$

Note that, by assumption, the underlying $\mathbb{Q}$-vector spaces of the planes Y_P, Y_Q, Y_R and Y_S form a non-degenerate quadruple, but may all intersect over $\mathbb{F}_p$ for any prime p. We can apply Theorem 3.19 to the discrete valuation ring $\mathbb{Z}_p$ for any prime p, to compute $\deg\left((Y_{P,p} - Y_{Q,p}) \cdot (Y_{R,p} - Y_{S,p})\right)$.

Corollary 4.2. *In the situation as above, the intersection degree is given by*

$$\deg\left((Y_{P,p} - Y_{Q,p}) \cdot (Y_{R,p} - Y_{S,p})\right) = v_p\left(\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S)\right)$$

and hence

$$\langle Y_P - Y_Q, Y_R - Y_S \rangle_p = \log \left|\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S)\right|_p .$$

Proof. The first equality follows immediately from Theorem 3.19. Furthermore, note that for any $x \in \mathbb{Q}^\times$ and any prime p, we have by definition

$$|x|_p = p^{-v_p(x)}.$$

The second equality follows immediately. □

Corollary 4.3. *The global height pairing of differences of linear subspaces of sub-complementary dimension on $\mathbb{P}^n$ vanishes.*

Proof. This follows from the fact that, for any $x \in \mathbb{Q}^\times$, we have

$$\log|x| + \sum_{p \text{ prime}} \log|x|_p = 0.$$

Because Y_P, Y_Q, Y_R and Y_S are defined over $\mathbb{Q}$, so is their cross-ratio. Hence

$$\begin{aligned}\langle Y_P - Y_Q, Y_R - Y_S \rangle &= \sum_{p \text{ prime}} \langle Y_P - Y_Q, Y_R - Y_S \rangle_p + \langle Y_P - Y_Q, Y_R - Y_S \rangle_\infty \\ &= \sum_{p \text{ prime}} \log|\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S)|_p + \log|\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S)| \\ &= 0.\end{aligned}$$

Here we used Theorem 2.20 to express the Archimedean height pairing in terms of the cross-ratio. □

Example 4.4. We give an example computing both the Archimedean and non-Archimedean height pairings for lines in $\mathbb{P}^3_\mathbb{Z}$. Consider the following four lines in $\mathbb{P}^3_\mathbb{Z}$, given by the following rank-two submodules of $\mathbb{Z}^4$

$$\begin{aligned}\widehat{Y_P} &= \mathbb{Z}\langle(1,0,0,0),(0,1,0,0)\rangle \\ \widehat{Y_Q} &= \mathbb{Z}\langle(0,0,1,0),(0,0,0,1)\rangle \\ \widehat{Y_R} &= \mathbb{Z}\langle(2,3,0,1),(4,0,2,3)\rangle \\ \widehat{Y_S} &= \mathbb{Z}\langle(5,5,5,1),(2,3,5,7)\rangle\end{aligned}$$

We note that these four submodules intersect pairwise trivially, i.e., only in the zero vector, *even when we base change to* $\mathbb{Q}$. Furthermore, reduction modulo p for any prime does not change the dimension of Y_P, Y_Q, Y_R or Y_S. Therefore, we can compute their Archimedean and non-Archimedean heights using their cross-ratio.

$$\begin{aligned}\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S) &= \frac{(v_P \wedge v_R)}{(v_Q \wedge v_R)} \frac{(v_Q \wedge v_S)}{(v_P \wedge v_S)} \\ &= \frac{-2}{-12} \frac{5}{30} \\ &= \frac{1}{36} \\ &= \frac{1}{2^2} \frac{1}{3^2}.\end{aligned}$$

Hence we find that, after base changing to $\mathbb{C}$ for the Archimedean height,

$$\begin{aligned}
\langle Y_P - Y_Q, Y_R - Y_S\rangle_\infty &= -2\log\left|\frac{1}{36}\right|,\\
\langle Y_P - Y_Q, Y_R - Y_S\rangle_2 &= \log\left|\frac{1}{36}\right|_2 = -2\log 2,\\
\langle Y_P - Y_Q, Y_R - Y_S\rangle_3 &= \log\left|\frac{1}{36}\right|_3 = -2\log 3,\\
\langle Y_P - Y_Q, Y_R - Y_S\rangle_p &= \log|1| = 0 \quad , \quad \mathrm{p} \notin \{2,3\}.
\end{aligned}$$

Therefore, we conclude that $\deg\left((Y_{P,2} - Y_{Q,2})\cdot(Y_{R,2} - Y_{S,2})\right) = 2$ and $\deg\left((Y_{P,3} - Y_{Q,3})\cdot(Y_{R,3} - Y_{S,3})\right) = 2$. It is not yet clear from which intersection these degrees appear. As shown before in Section 3.2, there are two possible ways, up to symmetry, in which these four lines can degenerate, resulting in degree 2. Note that intersections may appear with higher degree here, so there are more possible degenerations. Let us first treat the case $p = 2$. To examine the intersection mod 2, we compute the order to which the individual determinants vanish at 2. So, up to a sign,

$$\begin{aligned}
v_P \wedge v_R &= -2,\\
v_Q \wedge v_R &= -12 = -1\cdot 2^2\cdot 3,\\
v_Q \wedge v_S &= 5,\\
v_P \wedge v_S &= 30 = 2\cdot 3\cdot 5.
\end{aligned}$$

Thus, we see that modulo 2, $Y_{P,2}$ and $Y_{R,2}$ intersect with multiplicity one, $Y_{Q,2}$ and $Y_{R,2}$ intersect with multiplicity two, and lastly $Y_{P,2}$ and $Y_{S,2}$ also intersect with multiplicity one. Furthermore, intersections also occur over $\mathbb{Z}_3$ and $\mathbb{Z}_5$. In the following figure we have depicted the intersections and coloured their intersections according to the sign $+1$ for Y_P and Y_R and the sign -1 for Y_Q and Y_R.

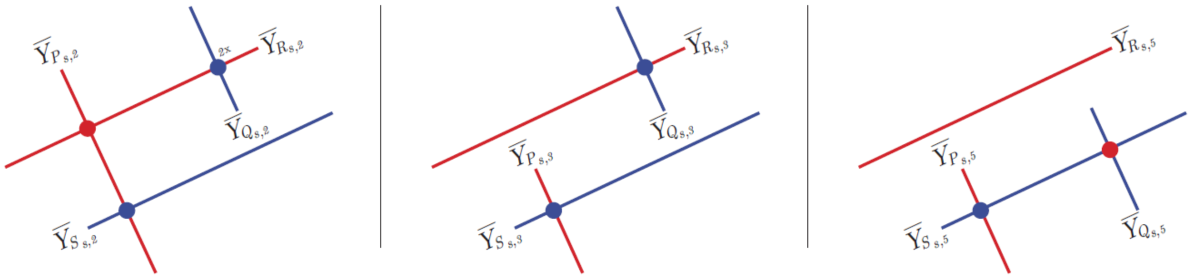


Figure 7: Left: intersections over $\mathbb{Z}_2$, Middle: intersections over $\mathbb{Z}_3$, Right: intersections over $\mathbb{Z}_5$.

An interesting phenomenon is that over $\mathbb{Z}_5$ there is an intersection. However, because the intersection is for both Y_P and Y_S and Y_Q and Y_S with the same multiplicity, their contributions cancel each other out.

Section 4.2: Geometric Interpretation of the Regularized Limit $u(0)$

In this section we will investigate the remaining nowhere-vanishing function $u(t)$ from Theorem 3.19, in particular $u(0)$. We can find a canonical interpretation well-defined up to sign when all planes are defined over $\mathbb{Z}$. This means that our moving planes now move over a two-dimensional base $\operatorname{Spec}\mathbb{Z}[\![t]\!]$. Since the constant $u(0)$ factors through the four different intersections, i.e.

$$u(0) = \frac{u_{P,R}(0)u_{Q,S}(0)}{u_{P,S}(0)u_{Q,R}(0)},$$

we will focus on a single intersection where $\overline{X}$ and $\overline{Y}$ are moving planes. In this setup we let $u(0)$ denote the following quantity. We can form the $(n+1) \times (n+1)$ matrix $v_X \wedge v_Y$ which has primitive generating vectors for $\hat{X}$ and $\hat{Y}$ in $\mathbb{Z}[\![t]\!]^{n+1}$ as its columns. Then we define $u(0)$ to be

$$\frac{1}{m!} \frac{\partial^m}{\partial t^m} \det(v_X \wedge v_Y)\Bigg|_{t=0}$$

where m is the smallest integer such that the expression is non-zero. Let $\overline{X}$ and $\overline{Y}$ denote the families in $\mathbb{P}^n_T$, where $T = \mathbb{Z}[\![t]\!]$ and $\operatorname{Spec}(T)$ is a two-dimensional base. By assumption, both $\overline{X}$ and $\overline{Y}$ are flat over the base $\operatorname{Spec} \mathbb{Z}[\![t]\!]$, i.e. when reduced modulo p we have induced families

$$\overline{X}_p \to \operatorname{Spec} \mathbb{F}_p[\![t]\!] \qquad \overline{Y}_p \to \operatorname{Spec} \mathbb{F}_p[\![t]\!],$$

which are families of k-planes, as before. More concretely,

$$\overline{X}_p = \overline{X} \times_{\mathbb{Z}[\![t]\!]} \mathbb{F}_p[\![t]\!], \qquad \overline{X}_{p,s} = \overline{X}_p \times_{\mathbb{F}_p[\![t]\!]} \mathbb{F}_p,$$

and similarly for $\overline{Y}_p$ and $\overline{Y}_{p,s}$. We also have induced families

$$\overline{X}_s \to \operatorname{Spec} \mathbb{Z}, \qquad \overline{Y}_s \to \operatorname{Spec} \mathbb{Z}, \quad \overline{X}_\eta \to \operatorname{Spec} \mathbb{Q}[\![t]\!], \quad \overline{Y}_\eta \to \operatorname{Spec} \mathbb{Q}[\![t]\!].$$

Lastly, we have a family

$$\overline{X}_s \cap \overline{Y}_s = \overline{X}_s \times_{\mathbb{P}^n_{\mathbb{Z}}} \overline{Y}_s \to \operatorname{Spec} \mathbb{Z}.$$

We denote by $\hat{X}$ and $\hat{Y}$ the underlying spaces in $\mathbb{Z}[\![t]\!]^{n+1}$, and $\hat{X}_s$ and $\hat{Y}_s$ the underlying spaces at $t = 0$ in $\mathbb{Z}^{n+1}$. We will first treat the case where $\overline{X}_s \cap \overline{Y}_s$ is flat over $\operatorname{Spec} \mathbb{Z}$, i.e. the intersection does not jump rank when further reduced modulo any prime p. We will write down some preliminary results, and then state and prove the most general case under the flatness assumption.

Lemma 4.5. *Suppose $\overline{X}$ and $\overline{Y}$ are moving lines in $\mathbb{P}^3_T$, where $T = \mathbb{Z}[\![t]\!]$. Suppose that $\overline{X}_\eta$ and $\overline{Y}_\eta$ intersect properly, that $\overline{X}_s \cap \overline{Y}_s$ is flat over* $\operatorname{Spec} \mathbb{Z}$*, and lastly that $\overline{X}$ and $\overline{Y}$ depend linearly on t. Then there is, up to sign, a canonical basis for the normal direction in which the planes $\hat{X}_s$ and $\hat{Y}_s$ collide, given by $\xi \in \mathbb{Z}^4$. Furthermore, the rate at which $\hat{X}_s$ and $\hat{Y}_s$ collide is given by $|u(0)|$, a length with respect to this basis.*

Proof. First, we will establish this normal basis vector ξ. Since $\overline{X}_\eta$ and $\overline{Y}_\eta$ intersect properly, they meet in a projective $\mathbb{Q}$-point, i.e. $\hat{X}_{\eta,s} \cap \hat{Y}_{\eta,s} = \mathbb{Q} \cdot \zeta \subset \mathbb{Q}^4$, for some generating vector ζ. Furthermore, since the intersection $\overline{X}_s \cap \overline{Y}_s$ is flat over $\operatorname{Spec} \mathbb{Z}$, the intersection also happens over $\mathbb{Z}$. So $\hat{X}_s \cap \hat{Y}_s = \mathbb{Z} \cdot \zeta \subset \mathbb{Z}^4$, for a primitive basis vector $\zeta \in \mathbb{Z}^4$. Since $\hat{X}_s \cap \hat{Y}_s \cong \mathbb{Z}$ the vector ζ is well-defined up to sign. So, write bases in $\mathbb{Z}[\![t]\!]^{n+1}$

$$\hat{X} = \mathbb{Z}\langle v_1^X(t), v_2^X(t) \rangle, \quad \hat{Y} = \mathbb{Z}\langle v_1^Y(t), v_2^Y(t) \rangle.$$

Now, since these bases share a common generator ζ, when evaluated at $t = 0$, we can find invertible matrices $U_X, U_Y \in \mathrm{GL}_2(\mathbb{Z})$ such that after changing the bases for $\hat{X}$ and $\hat{Y}$ to

$$\hat{X} = \mathbb{Z}\langle \zeta_X(t), a(t) \rangle, \quad \hat{Y} = \mathbb{Z}\langle \zeta_Y(t), b(t) \rangle,$$

we have that

$$\zeta_X(0) = \zeta_Y(0) = \zeta.$$

This in turn also gives us bases

$$\hat{X}_s = \mathbb{Z}\langle \zeta, a \rangle, \quad \hat{Y}_s = \mathbb{Z}\langle \zeta, b \rangle,$$

with $a = a(0)$ and $b = b(0)$. Because the intersection $X_s \cap Y_s$ is flat, this means that $\mathbb{Z}\langle \zeta, a, b\rangle$ is a direct sum decomposition, i.e.

$$\mathbb{Z}^4/\mathbb{Z}\langle \zeta, a, b\rangle \cong \mathbb{Z}.$$

Equivalently, the vectors ζ, a, b can be extended to a basis for $\mathbb{Z}^4$, call the additional basis vector the *normal* vector ξ, so

$$\mathbb{Z}^4 = \mathbb{Z}\langle \zeta, a, b, \xi\rangle.$$

Now, the rate of collision (or equivalently separation) is entirely captured by the ξ-direction. So, consider the class

$$[\zeta_X'(0) - \zeta_Y'(0)] \in \mathbb{Z}^4/\mathbb{Z}\langle \zeta, a, b\rangle \cong \mathbb{Z}\langle \xi\rangle.$$

We will show that the class of this difference is well-defined. Suppose we have other moving bases

$$\hat{X} = \mathbb{Z}\langle \tilde{\zeta}_X(t), \tilde{a}(t)\rangle, \quad \hat{Y} = \mathbb{Z}\langle \tilde{\zeta}_Y(t), \tilde{b}(t)\rangle,$$

with

$$\tilde{\zeta}_X(0) = \tilde{\zeta}_Y(0) = \zeta.$$

Then

$$\tilde{\zeta}_X(t) = \zeta_X(t) + f(t)a(t), \qquad \tilde{\zeta}_Y(t) = \zeta_Y(t) + g(t)b(t),$$

with

$$f(0) = g(0) = 0.$$

Now the difference of their derivatives is well-defined.

$$\begin{aligned}
[\tilde{\zeta}_X'(0) - \tilde{\zeta}_Y'(0)] &= [\zeta_X'(0) - \zeta_Y'(0) + f'(0)a(0) + f(0)a'(0) - g'(0)b(0) - g(0)b'(0)] \\
&= [\zeta_X'(0) - \zeta_Y'(0) + f'(0)a(0) - g'(0)b(0)] \\
&= [\zeta_X'(0) - \zeta_Y'(0)].
\end{aligned}$$

Next, let M denote the matrix whose columns are $v_1^X(t), v_2^X(t), v_1^Y(t)$ and $v_2^Y(t)$. Then by definition $u(0)$ is given by $\frac{\partial}{\partial t}\det M|_{t=0}$. Now let U be the 4×4 block matrix with U_X in the top left 2×2 minor and U_Y in the bottom right 2×2 minor. Now $\det(U) = \det(U_X)\det(U_Y) = \pm 1$ so

$$\det M = \pm\det(MU) = \pm\det\begin{bmatrix}\zeta_X(t) & a(t) & b(t) & \zeta_Y(t)\end{bmatrix}.$$

Now, by multi-linearity of the determinant, we have that

$$\frac{\partial}{\partial t}\det\begin{bmatrix}A_1(t)\cdots A_n(t)\end{bmatrix} = \det\begin{bmatrix}\frac{\partial}{\partial t}A_1(t) & A_2(t)\cdots A_n(t)\end{bmatrix} + \cdots + \det\begin{bmatrix}A_1(t)\cdots A_{n-1}(t) & \frac{\partial}{\partial t}A_n(t)\end{bmatrix}.$$

Since $\zeta_X(0) = \zeta_Y(0) = \zeta$ the second and third term in the right-hand sum vanish, after evaluating at $t = 0$. So we are left with

$$\begin{aligned}
\frac{\partial}{\partial t}\det\begin{bmatrix}\zeta_X(t) & a(t) & b(t) & \zeta_Y(t)\end{bmatrix}\Big|_{t=0} &= \det\begin{bmatrix}\zeta_X'(0) & a & b & \zeta\end{bmatrix} + \det\begin{bmatrix}\zeta & a & b & \zeta_Y'(0)\end{bmatrix} \\
&= \det\begin{bmatrix}\zeta_X'(0) & a & b & \zeta\end{bmatrix} - \det\begin{bmatrix}\zeta_Y'(0) & a & b & \zeta\end{bmatrix} \\
&= \det\begin{bmatrix}\zeta_X'(0) - \zeta_Y'(0) & a & b & \zeta\end{bmatrix}.
\end{aligned}$$

Now, $\zeta_X'(0) - \zeta_Y'(0) = c\cdot\xi$ under the isomorphism as above, modulo the basis $\mathbb{Z}\langle a, b, \zeta\rangle$. Hence,

$$\begin{aligned}
\det\begin{bmatrix}\zeta_X'(0) - \zeta_Y'(0) & a & b & \zeta\end{bmatrix} &= \det\begin{bmatrix}c\cdot\xi & a & b & \zeta\end{bmatrix} \\
&= c\det\begin{bmatrix}\xi & a & b & \zeta\end{bmatrix} \\
&= \pm c,
\end{aligned}$$

since ξ, a, b, ζ is a basis for $\mathbb{Z}^4$. Combining all the above, we find that

$$u(0) = \frac{\partial}{\partial t} \det M \Big|_{t=0} = \pm c,$$

and under the identification $\mathbb{Z}^4/\mathbb{Z}\langle \zeta, a, b\rangle \cong \mathbb{Z}$ also

$$[\zeta_X'(0) - \zeta_Y'(0)] \longmapsto c.$$

Thus, up to sign, the rate of collision $[\zeta_X'(0) - \zeta_Y'(0)]$ is given by $u(0)$. $\square$

Let us illustrate this in Figure 8. In blue we see the lattice $\hat{X}_s$, in red $\hat{Y}_s$ and the intersection $\hat{X}_s \cap \hat{Y}_s$ is coloured purple. We have indicated the generating vectors a, b, ζ as well as the normal direction ξ. Lastly, the *rate* of this degeneration is the length of the vector $|u(0)|$, in the ξ-direction.

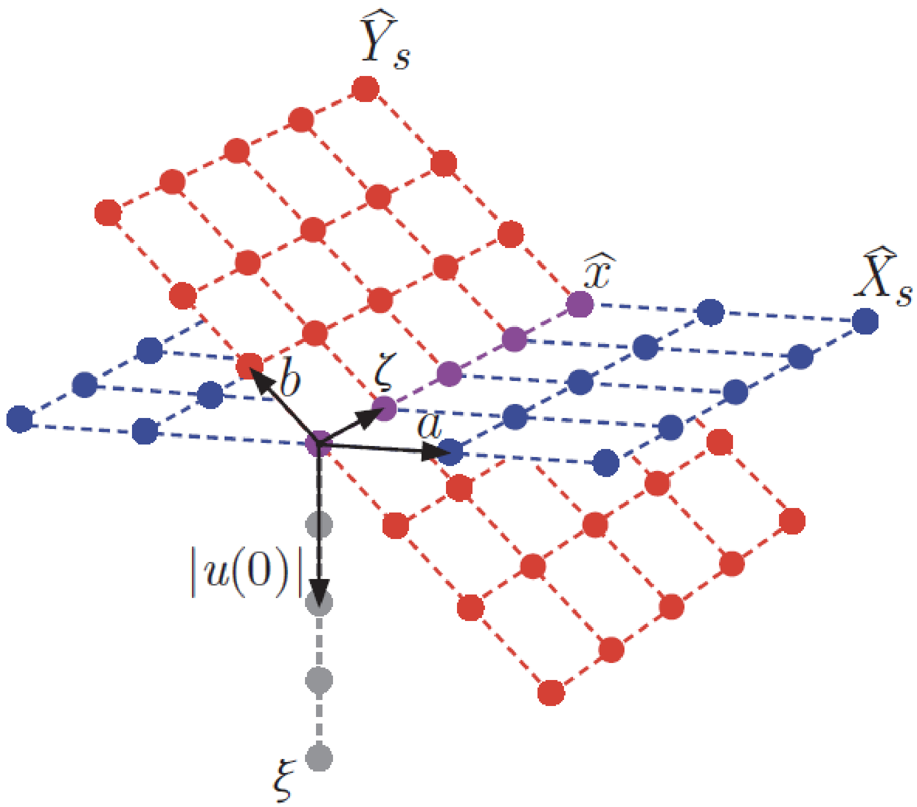


Figure 8: Interpreting $|u(0)|$ in a proper degeneration over $\operatorname{Spec} \mathbb{Z}[\![t]\!]$.

When $\overline{X}$ and $\overline{Y}$ are linearly moving k-planes and $(n-k-1)$-planes in $\mathbb{P}^n_T$ where $T = \mathbb{Z}[\![t]\!]$, the result of Lemma 4.5 can easily be generalised as follows. There is now a canonical normal direction $\xi \in \mathbb{Z}^{n+1}$ and up to sign the rate of collision is still given by $|u(0)|$ with respect to this basis. We will now extend the result to arbitrary t-dependence. Again, we will prove the case of moving lines in $\mathbb{P}^3_T$ and leave the more general proof to the reader.

Corollary 4.6. *Suppose $\overline{X}$ and $\overline{Y}$ are moving lines in $\mathbb{P}^3_T$, where $T = \mathbb{Z}[\![t]\!]$. Suppose that $\overline{X}_\eta$ and $\overline{Y}_\eta$ intersect properly and that $\overline{X}_s \cap \overline{Y}_s$ is flat over* $\operatorname{Spec} \mathbb{Z}$*. Then there is, up to sign, a canonical basis for the normal direction in which the planes $\hat{X}_s$ and $\hat{Y}_s$ collide, given by $\xi \in \mathbb{Z}^4$. Furthermore, the rate at which $\hat{X}_s$ and $\hat{Y}_s$ collide is given by $|u(0)|$, a length with respect to this basis.*

Proof. The only difference with respect to the statement and proof of Lemma 4.5 is the step where we define the element $[\zeta'_X(0) - \zeta'_Y(0)]$. The lines will intersect to order m, where m is the highest power of t which divides $\det M$, with M as before. For this reason, we let

$$\left[\frac{1}{m!}\left(\zeta_X^{(m)}(0) - \zeta_Y^{(m)}(0)\right)\right] \in \mathbb{Z}^4/\mathbb{Z}\langle \zeta, a, b\rangle.$$

By the same argument this class is well-defined. What remains to show is that

$$\frac{1}{m!}\frac{\partial^m}{\partial t^m}\det M\Big|_{t=0} = \pm c,$$

where c is the element corresponding to

$$\left[\frac{1}{m!}\left(\zeta_X^{(m)}(0) - \zeta_Y^{(m)}(0)\right)\right].$$

We will again use the multi-linearity of the determinant,

$$\frac{1}{m!}\frac{\partial^m}{\partial t^m}\det A\Big|_{t=0} = \sum_{m_1+\cdots+m_4=m} \det\left(\frac{A_1^{(m_1)}(0)}{m_1!}, \ldots, \frac{A_4^{(m_4)}(0)}{m_4!}\right).$$

So, when we compute

$$\begin{aligned}
&\frac{1}{m!}\frac{\partial^m}{\partial t^m}\det\left[\zeta_X(t) \quad a(t) \quad b(t) \quad \zeta_Y(t)\right]\Big|_{t=0} \\
&\quad= \frac{1}{m!}\frac{\partial^m}{\partial t^m}\det\left[\zeta_X(t) - \zeta_Y(t) \quad a(t) \quad b(t) \quad \zeta_Y(t)\right]\Big|_{t=0} \\
&\quad= \sum_{m_1+\cdots+m_4=m} \det\left(\frac{\zeta_X^{(m_1)}(0) - \zeta_Y^{(m_1)}(0)}{m_1!}, \frac{a^{(m_2)}(0)}{m_2!}, \frac{b^{(m_3)}(0)}{m_3!}, \frac{\zeta_Y^{(m_4)}(0)}{m_4!}\right) \\
&\quad= \det\left(\frac{\zeta_X^{(m)}(0) - \zeta_Y^{(m)}(0)}{m!}, a, b, \zeta\right),
\end{aligned}$$

because the m-th derivative, when evaluated at $t = 0$, is the first time $\zeta_X^{(k)}(0) - \zeta_Y^{(k)}(0)$ does not vanish. The final conclusion follows in the same way as in the proof of Lemma 4.5. ☐

As before, generalising this to the case of k and $(n-k-1)$-planes intersecting in a (projective) point is left to the reader. What remains is to consider excess intersections.

Proposition 4.7. *Let $\overline{X}$ be a moving k-plane and $\overline{Y}$ be a moving $(n-k-1)$-plane in $\mathbb{P}^n_T$, where $T = \mathbb{Z}[\![t]\!]$. Suppose that $\overline{X}_\eta$ and $\overline{Y}_\eta$ are disjoint over the generic fibre of* $\operatorname{Spec}\mathbb{Q}[\![t]\!]$ *and that $\overline{X}_s \cap \overline{Y}_s$ is flat over* $\operatorname{Spec}\mathbb{Z}$. *Let l be the (projective) dimension of the intersection $\overline{X}_{\eta,s} \cap \overline{Y}_{\eta,s} \subset \mathbb{P}^n_{\mathbb{Q}}$. Then, there is up to signs, a canonical basis for the normal directions in which the planes $\hat{X}_s$ and $\hat{Y}_s$ collide, given by $\xi_0, \ldots, \xi_l \in \mathbb{Z}^{n+1}$. There is a canonical $(l+1)$-dimensional volume, determined by the rate of collision in each direction with respect to this basis. We will show that this volume is given by $|u(0)|$.*

Proof. Choose adapted bases

$$\begin{aligned}
\hat{X} &= \mathbb{Z}\langle \zeta_0^X(t), \ldots, \zeta_l^X(t), a_{l+1}(t), \ldots, a_k(t)\rangle, \\
\hat{Y} &= \mathbb{Z}\langle \zeta_0^Y(t), \ldots, \zeta_l^Y(t), b_{l+1}(t), \ldots, b_{n-k-1}(t)\rangle,
\end{aligned}$$

such that $\zeta_i^X(0) = \zeta_i^Y(0) = \zeta_i$ where

$$\hat{X}_s \cap \hat{Y}_s = \mathbb{Z}\langle \zeta_0, \ldots, \zeta_l\rangle.$$

Let $r_i \geq 1$ be the smallest integer such that $(\zeta_i^X)^{(r_i)}(0) - (\zeta_i^Y)^{(r_i)}(0) \neq 0$. Let $\xi_0, \ldots, \xi_l$ be the unique basis, up to signs, of the normal directions

$$\mathbb{Z}\langle \xi_0, \ldots, \xi_l \rangle \cong \mathbb{Z}^{n+1}/\mathbb{Z}\langle \zeta_0, \ldots, \zeta_l, a_{l+1}(0), \ldots, a_k(0), b_{l+1}(0), \ldots, b_{n-k-1}(0)\rangle.$$

Because the elements

$$\frac{1}{r_i!}\left((\zeta_i^X)^{(r_i)}(0) - (\zeta_i^Y)^{(r_i)}(0)\right)$$

are well-defined classes in the quotient, they can be written in the basis $\xi_0, \ldots, \xi_l$. To this end, write

$$\frac{1}{r_i!}\left((\zeta_i^X)^{(r_i)}(0) - (\zeta_i^Y)^{(r_i)}(0)\right) = \sum_{j=0}^{l} c_{i,j}\xi_j.$$

Define $C = (c_{i,j})_{\{0 \leq i,j \leq l\}}$. Then the volume spanned by these $l+1$ vectors is exactly given by $|\det(C)|$. We will now show that this volume is equal to $|u(0)|$. So, let m be the smallest integer such that $\det M^{(m)}|_{t=0} \neq 0$, equal to $r_0 + \cdots + r_l$. To suppress all the indices we will denote by $a(t)$, all vectors $a_{l+1}(t), \ldots, a_k(t)$, and similar for the other vectors b, ζ^X and ζ^Y.

$$\begin{aligned}
u(0) &= \frac{1}{m!}\frac{\partial^m}{\partial t^m}\det\left(\zeta^X(t), a(t), \zeta^Y(t), b(t)\right)\Big|_{t=0} \\
&= \frac{1}{m!}\frac{\partial^m}{\partial t^m}\det\left(\zeta^X(t) - \zeta^Y(t), a(t), \zeta^Y(t), b(t)\right)\Big|_{t=0} \\
&= \det\left(\frac{(\zeta_i^X)^{(r_i)}(0) - (\zeta_i^Y)^{(r_i)}(0)}{r_i!}, a(0), \zeta^Y(t), b(0)\right) \\
&= \det\left(\frac{(\zeta_i^X)^{(r_i)}(0) - (\zeta_i^Y)^{(r_i)}(0)}{r_i!}, a(0), \zeta, b(0)\right).
\end{aligned}$$

The third equality is because the r_i are chosen in such a way that this is the only configuration without a zero column, so this is the only determinant that is non-zero. Now, modulo $\mathbb{Z}\langle a(0), \zeta, b(0)\rangle$, we can again express these first $l+1$ columns in the $\xi = (\xi_0, \ldots, \xi_l)$ basis. So the above equals

$$\det(C)\det\left(\xi, a(0), \zeta, b(0)\right) = \pm\det(C),$$

because $\xi, a(0), \zeta, b(0)$ form a basis for $\mathbb{Z}^{n+1}$, and hence must have determinant ± 1. Thus we conclude that $|u(0)| = |\det(C)|$. This finishes the proof. $\square$

Again, let us illustrate the situation in Figure 9. Let $\hat{X}_s$ and $\hat{Y}_s$ be two-dimensional lattices, which in this excess intersection coincide. We denote their common generators with ζ_1 and ζ_2. We let ξ_1 and ξ_2 span the normal directions. The vectors

$$\zeta_1^{X'}(0) - \zeta_1^{Y'}(0) \qquad \text{and} \qquad \zeta_2^{X'}(0) - \zeta_2^{Y'}(0)$$

depict the rate of collision (or separation) in both the ξ_1- and ξ_2-directions. Note that we have assumed the t-dependence to be linear, otherwise these vectors should have been replaced with

$$(\zeta_1^X)^{(r_1)}(0) - (\zeta_1^Y)^{(r_1)}(0) \qquad \text{and} \qquad (\zeta_2^X)^{(r_2)}(0) - (\zeta_2^Y)^{(r_2)}(0).$$

Lastly, we want to stress that these vectors need not land in $\mathbb{Z}\langle\xi_1\rangle$ or $\mathbb{Z}\langle\xi_2\rangle$, but rather somewhere in their joint span $\mathbb{Z}\langle\xi_1, \xi_2\rangle$.

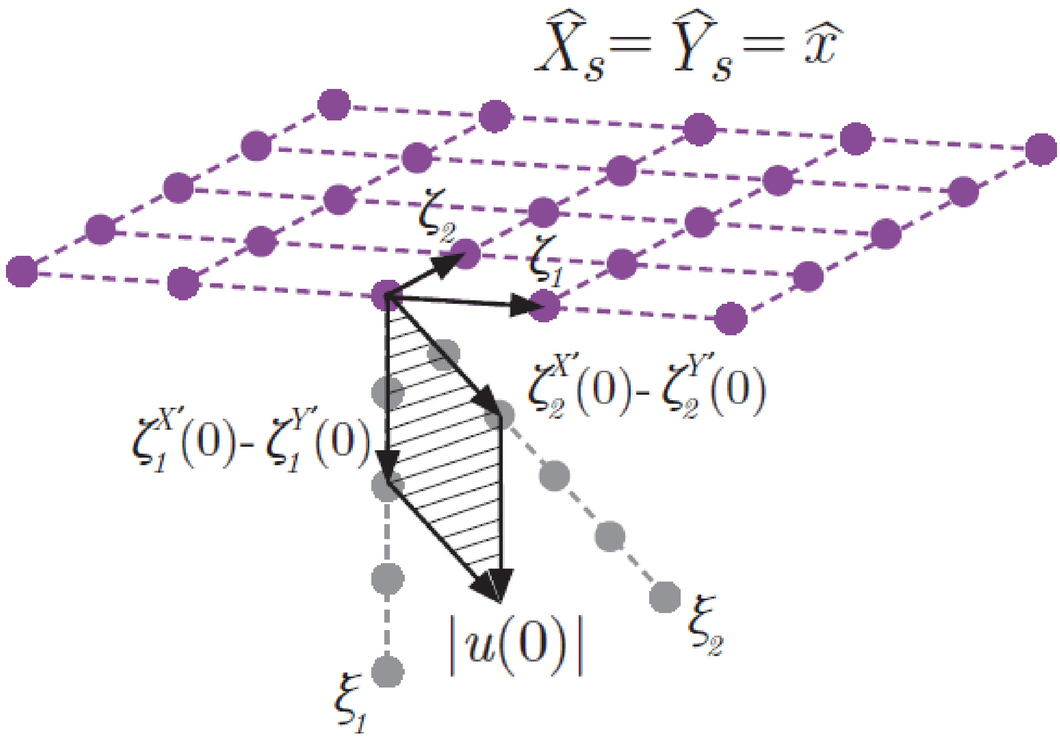


Figure 9: Interpreting $|u(0)|$ in a non-proper degeneration over $\operatorname{Spec} \mathbb{Z}[\![t]\!]$.

In the remaining part of this chapter, we want to generalise the result of Proposition 4.7 to the case where the intersection $\overline{X}_s \cap \overline{Y}_s$ is no longer flat over $\operatorname{Spec} \mathbb{Z}$. Suppose that $\hat{X}_s$ and $\hat{Y}_s$ are given by Y_P and Y_S as in Example 4.4. Then we see that

$$\dim_{\mathbb{Q}} \hat{X}_{s,\eta} \cap \hat{Y}_{s,\eta} = 0, \qquad \dim_{\mathbb{F}_2} \hat{X}_{s,2} \cap \hat{Y}_{s,2} = \dim_{\mathbb{F}_3} \hat{X}_{s,3} \cap \hat{Y}_{s,3} = \dim_{\mathbb{F}_5} \hat{X}_{s,5} \cap \hat{Y}_{s,5} = 1,$$

and for any prime $p > 5$

$$\dim_{\mathbb{F}_p} \hat{X}_{s,p} \cap \hat{Y}_{s,p} = 0.$$

This non-flatness is caused by torsion in the quotient $\mathbb{Z}^{n+1}/\left(\hat{X}_s + \hat{Y}_s\right)$. Let H denote the submodule $\hat{X}_s + \hat{Y}_s \subset \mathbb{Z}^4$. We can correct for this torsion by passing to the saturated submodule

$$H^{\text{sat}} = (H \otimes_{\mathbb{Z}} \mathbb{Q}) \cap \mathbb{Z}^4,$$

or equivalently

$$H^{\text{sat}} = \{x \in \mathbb{Z}^4 \mid \exists m \in \mathbb{N} : mx \in H\}.$$

In the example above,

$$\hat{X}_s = \mathbb{Z}\langle (1,0,0,0), (0,1,0,0)\rangle, \qquad \hat{Y}_s = \mathbb{Z}\langle (5,5,5,1), (2,3,5,7)\rangle$$

so

$$H = \mathbb{Z}\langle e_1, e_2, 5e_3 + e_4, 6e_4\rangle,$$

or after a change of basis (of $\mathbb{Z}^4$),

$$H = \mathbb{Z}\langle e_1, e_2, e_3, 2 \cdot 3 \cdot 5 e_4\rangle.$$

Clearly, $H^{\text{sat}} = \mathbb{Z}\langle e_1, e_2, e_3, e_4\rangle$ and

$$[H^{\text{sat}} : H] = 30.$$

Now in general H^{sat} is flat over $\operatorname{Spec}\mathbb{Z}$, so we can compute a canonical basis for the normal directions with respect to this saturated lattice $\mathbb{Z}^4/H^{\text{sat}}$. In this quotient, the degenerations $\overline{X}$ and $\overline{Y}$ induce the rate of collision vectors as in Proposition 4.7. Then $|u(0)|$ will be given by this volume, times the saturation index $[H^{\text{sat}} : H]$.

Theorem 4.8. *Let $\overline{X}$ be a moving k-plane and $\overline{Y}$ be a moving $(n-k-1)$-plane in $\mathbb{P}^n_T$, where $T = \mathbb{Z}[\![t]\!]$. Suppose that $\overline{X}_\eta$ and $\overline{Y}_\eta$ are non-degenerate over the generic fibre of* $\operatorname{Spec}\mathbb{Q}[\![t]\!]$. *Let l be the (projective) dimension of the intersection $\overline{X}_{\eta,s} \cap \overline{Y}_{\eta,s} \subset \mathbb{P}^n_{\mathbb{Q}}$. After saturating the submodule $\hat{X}_s + \hat{Y}_s \subset \mathbb{Z}^{n+1}$, there is up to signs, a canonical basis for the normal directions in which the planes $\hat{X}_s$ and $\hat{Y}_s$ collide, given by $\xi_0, \dots, \xi_l \in \mathbb{Z}^{n+1}$. There is a canonical $(l+1)$-dimensional volume, determined by the rate of collision in each direction with respect to this basis, denoted by V. Then*

$$|u(0)| = [(\hat{X}_s + \hat{Y}_s)^{\text{sat}} : (\hat{X}_s + \hat{Y}_s)] \cdot V.$$

Proof. We follow the same steps as in the proof of Proposition 4.7, up to the equality

$$u(0) = \det\left(\frac{(\zeta_i^X)^{(r_i)}(0) - (\zeta_i^Y)^{(r_i)}(0)}{r_i!}, a(0), \zeta, b(0)\right). \tag{4.1}$$

Unlike before we now cannot assume that

$$\mathbb{Z}^{n+1}/\mathbb{Z}\langle a_{l+1}(0), \dots, a_k(0), \zeta_0, \dots, \zeta_l, b_{l+1}(0), \dots, b_{n-k-1}(0)\rangle \cong \mathbb{Z}^{l+1},$$

since there can be torsion elements. Let $p_1, p_2, p_3, \dots$ denote the primes, and let $\epsilon_{1,i}$ be the largest integer for which the dimension of the intersection jumps rank after reduction modulo $p_i^{\epsilon_{1,i}}$, from $l+1$ to $l+1+\delta_{1,i}$. Then let $\epsilon_{2,i}$ denote the largest integer such that the dimension further increases after reduction modulo $p_i^{\epsilon_{2,i}}$, from rank $l+1+\delta_{1,i}$ to $l+1+\delta_{1,i}+\delta_{2,i}$. Keep repeating this process. If the intersection does not jump rank for any power $p_i, p_i^2, p_i^3, \dots$, then $\epsilon_{j,i} = \delta_{j,i} = 0$ for all j. Then we get

$$\mathbb{Z}^{n+1}/\mathbb{Z}\langle a_{l+1}(0), \dots, a_k(0), \zeta_0, \dots, \zeta_l, b_{l+1}(0), \dots, b_{n-k-1}(0)\rangle \cong \mathbb{Z}^{l+1} \oplus \bigoplus_{i,j\geq 1}^{r} (\mathbb{Z}/(p_i)^{\epsilon_{j,i}}\mathbb{Z})^{\oplus \delta_{j,i}}.$$

Now the saturation index equals

$$\left[\left(\hat{X}_s + \hat{Y}_s\right)^{\text{sat}} : \left(\hat{X}_s + \hat{Y}_s\right)\right] = \prod_{i,j\geq 1} p_i^{\delta_{j,i}\epsilon_{j,i}},$$

and in particular we can rewrite Equation 4.1 to

$$u(0) = \pm \prod_{i,j\geq 1} p_i^{\delta_{j,i}\epsilon_{j,i}} \det\left(\frac{(\zeta_i^X)^{(r_i)}(0) - (\zeta_i^Y)^{(r_i)}(0)}{r_i!}, v_1, \dots, v_{n-l}\right),$$

where

$$\left(\hat{X}_s + \hat{Y}_s\right)^{\text{sat}} = \mathbb{Z}\langle v_1, \dots, v_{n-l}\rangle.$$

Now we can proceed as before. Writing V for the $(l+1)$-dimensional volume spanned by the vectors $\frac{(\zeta_i^X)^{(r_i)}(0) - (\zeta_i^Y)^{(r_i)}(0)}{r_i!}$ in the normal directions $\xi_0, \dots, \xi_l$ we find that

$$|u(0)| = \left[\left(\hat{X}_s + \hat{Y}_s\right)^{\text{sat}} : \left(\hat{X}_s + \hat{Y}_s\right)\right] \cdot V.$$

$\square$

Using these techniques, we can define the *regularised* cross-ratio for possibly intersecting k-planes Y_P, Y_Q and $(n-k-1)$-planes Y_R, Y_S in $\mathbb{P}^n_{\mathbb{Z}}$.

Definition 4.9. Let Y_P, Y_Q, Y_R and Y_S be as above. Let V_P, V_Q, V_R and V_S denote their underlying lattices. Choose vectors $r^1_{P,R}, \ldots, r^m_{P,R} \in \mathbb{Z}^{n+1}$ whose images in $\mathbb{Z}^{n+1}/(V_P+V_R)^{\text{sat}}$ form a basis after tensoring with $\mathbb{Q}$. We write $r_{P,R} = (r^1_{P,R}, \ldots, r^m_{P,R})$. Similarly define $r_{P,S}, r_{Q,R}$ and $r_{Q,S}$. Note that, for example, when Y_P and Y_R are disjoint over $\mathbb{Q}$, then $(V_P+V_R)^{\text{sat}} = \mathbb{Z}^{n+1}$, so $r_{P,R}$ is the empty tuple. Whenever we write $V_P+V_R+r_{P,R}$, we mean the submodule generated by V_P+V_R together with all components of the tuple $r_{P,R}$. Let $r = (r_{P,R}, r_{P,S}, r_{Q,R}, r_{Q,S})$. Then we define the *regularised cross-ratio* to be

$$|\text{CR}^n_r(Y_P, Y_Q; Y_R, Y_S)| = \frac{[\mathbb{Z}^{n+1} : (V_P+V_R+r_{P,R})]}{[\mathbb{Z}^{n+1} : (V_Q+V_R+r_{Q,R})]} \frac{[\mathbb{Z}^{n+1} : (V_Q+V_S+r_{Q,S})]}{[\mathbb{Z}^{n+1} : (V_P+V_S+r_{P,S})]}.$$

When we let $l_{P,R}$ denote the dimension of the intersection of V_P and V_R over $\mathbb{Q}$, and similarly for $l_{P,S}, l_{Q,R}$ and $l_{Q,S}$, this equals

$$\frac{[(V_P+V_R)^{\text{sat}} : (V_P+V_R)]\,[\mathbb{Z}^{l_{P,R}} : \langle r_{P,R}\rangle]}{[(V_Q+V_R)^{\text{sat}} : (V_Q+V_R)]\,[\mathbb{Z}^{l_{Q,R}} : \langle r_{Q,R}\rangle]} \frac{[(V_Q+V_S)^{\text{sat}} : (V_Q+V_S)]\,[\mathbb{Z}^{l_{Q,S}} : \langle r_{Q,S}\rangle]}{[(V_P+V_S)^{\text{sat}} : (V_P+V_S)]\,[\mathbb{Z}^{l_{P,S}} : \langle r_{P,S}\rangle]}.$$

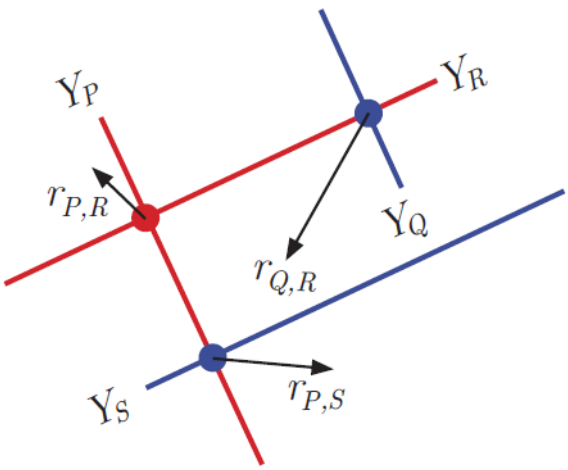


Figure 10: An example showing the vectors $r_{P,R}, r_{P,S}$ and $r_{Q,R}$ used to define the *regularised* cross-ratio for intersecting Y_P, Y_Q, Y_R and Y_S.

Corollary 4.10. *Let $\overline{Y_P}$ and $\overline{Y_Q}$ be two moving k-planes and let $\overline{Y_R}$ and $\overline{Y_S}$ be two moving $(n-k-1)$-planes in $\mathbb{P}^n_{\mathbb{Z}[\![t]\!]}$. Suppose that $\overline{Y_P}_\eta \cup \overline{Y_Q}_\eta$ and $\overline{Y_R}_\eta \cup \overline{Y_S}_\eta$ are disjoint over the generic fibre of* $\operatorname{Spec}\mathbb{Q}[\![t]\!]$. *Let $r_{P,R}$ denote the collection of rate of collision vectors for the (possible) collision of $\overline{Y_P}$ and $\overline{Y_R}$. Similarly define $r_{P,S}, r_{Q,R}$ and $r_{Q,S}$, and let $r = (r_{P,R}, r_{P,S}, r_{Q,R}, r_{Q,S})$. Then*

$$|u(0)| = \left|\text{CR}^n_r(\overline{Y_P}_s, \overline{Y_Q}_s; \overline{Y_R}_s, \overline{Y_S}_s)\right|,$$

so the constant $u(0)$ can, up to sign, be computed from the planes at $t=0$ together with induced vectors in the normal directions of all intersections.

Proof. This follows immediately from Definition 4.9 and Theorem 4.8. □

Remark 4.11. Note that via the generalised cross-ratio we can associate numbers to shapes, e.g. the planes in $\mathbb{P}^n$. From Theorem 4.8 it now follows that we can associate a shape to the regularised limit $|u(0)|$, namely the planes at $t = 0$, $Y_{P,s}, Y_{Q,s}, Y_{R,s}$ and $Y_{S,s}$, together with induced normal vectors $r = (r_{P,R}, r_{P,S}, r_{Q,R}, r_{Q,S})$. Finding such a *regularised limit of shapes* is in general very hard, and an interesting open problem. The case of curves [BdJS23] and nodal degenerations of certain odd-dimensional varieties [Bei25] have been handled. This thesis shows that the regularised limit is determined by the central geometry, also for degenerations of projective linear subspaces in arbitrary dimension.

Chapter 5: Mixed Hodge Structures and the Augmented Height Pairing

In this chapter, we aim to generalise the main result of Chapter 2, namely the Archimedean height computation of non-degenerate projective linear subspaces of $\mathbb{CP}^n$, see Theorem 2.20. For this, we will briefly introduce Hodge theory, which connects algebraic geometry, algebraic topology and complex geometry. First, in Section 5.1 we will define pure Hodge structures, together with an example. Then we will define the category of mixed Hodge structures in Section 5.2. In Section 5.3 we define the extension group $\mathrm{Ext}^1_{\mathrm{MHS}_\mathbb{Z}}(\mathbb{Z}(0), H)$ for a torsion-free integral pure Hodge structure H. In particular we will show

$$\mathrm{Ext}^1_{\mathrm{MHS}_\mathbb{Z}}(\mathbb{Z}(0), \mathbb{Z}(1)) \cong \mathbb{C}^\times,$$

and it is via this isomorphism that we will define the height of an extension $\mathcal{E}$, denoted by $\mathrm{ht}(\mathcal{E})$. Then, we will show that if $D = [Y_P] - [Y_Q]$ and $E = [Y_R] - [Y_S]$, where, as before, Y_P and Y_Q are k-dimensional linear subspaces and Y_R and Y_S are $(n-k-1)$-dimensional linear subspaces, forming a non-degenerate quadruple in $\mathbb{CP}^n$. Then there is a canonical extension associated to the pair (D, E) given by

$$H_{2(n-k)-1}(\mathbb{CP}^n \setminus |D|, |E|).$$

We define the *augmented height* of D and E to be the height of this extension, denoted by

$$\langle D, E\rangle_{\mathrm{aug}} := \mathrm{ht}(H_{2(n-k)-1}(\mathbb{CP}^n \setminus |D|, |E|)) \in \mathbb{C}^\times.$$

First, in Section 5.4 we assume that Y_P, Y_Q, Y_R and Y_S are distinct points in $\mathbb{CP}^1$ and prove

$$\langle Y_P - Y_Q, Y_R - Y_S\rangle_{\mathrm{aug}} = \mathrm{CR}(Y_P, Y_Q; Y_R, Y_S).$$

Finally, in Section 5.5, we remove this assumption and prove the last result of this thesis.

Theorem E (Corollary 5.21)**.** Let Y_P and Y_Q be k-dimensional linear subspaces of $\mathbb{CP}^n$ and let Y_R and Y_S be $(n-k-1)$-dimensional linear subspaces of $\mathbb{CP}^n$, non-degenerate. Then the augmented height is given by

$$\langle Y_P - Y_Q, Y_R - Y_S\rangle_{\mathrm{aug}} = \mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S).$$

This confirms that the proposed generalisation of the cross-ratio is a very *natural* choice.

Section 5.1: Pure Hodge Structures

In this section, we will define pure (K, L)-Hodge structures of weight $k \in \mathbb{Z}$ in two ways, via a Hodge filtration or equivalently via a Hodge decomposition. We will then explicitly compute the $(\mathbb{Q}, \mathbb{Q})$-Hodge structure on $\mathbb{P}^1 \setminus \{0, \infty\}$. Let us begin with the definition of an $\mathbb{R}$-Hodge structure.

Definition 5.1. An $\mathbb{R}$-Hodge structure of weight $k \in \mathbb{Z}$ on a finite-dimensional $\mathbb{C}$-vector space V is a decomposition

$$V = \bigoplus_{p+q=k} V^{p,q},$$

such that $\overline{V^{p,q}} = V^{q,p}$, for $p, q \in \mathbb{Z}$.

For example, when X is a complex Kähler manifold, then we have the decomposition

$$\mathcal{A}^k(X) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X),$$

where we decompose k-forms into forms of bidegree (p, q). It turns out, via the use of harmonic forms that this decomposition carries over to the de Rham cohomology groups, i.e. we get a decomposition

$$H^k_{dR}(X) = \bigoplus_{p+q=k} H^{p,q},$$

where $H^{p,q}$ consists of classes of closed forms of bidegree (p, q).
As mentioned before, we can equivalently define Hodge structures using filtrations. Let V be a finite-dimensional complex (or more generally L-) vector space.

Definition 5.2. A bounded decreasing filtration on V, denoted by $F^\bullet$, is a decreasing sequence of subspaces

$$0 \subset \cdots \subset F^{p+1}V \subset F^pV \subset F^{p-1}V \subset \cdots \subset V.$$

A morphism of filtered vector spaces $f : (V, F^\bullet_V) \to (W, F^\bullet_W)$ is a linear map $f : V \to W$ such that

$$f(F^p_V V) \subset F^p_W W.$$

Then we can equivalently define an $\mathbb{R}$-Hodge structure of weight $k \in \mathbb{Z}$, to be a finite dimensional $\mathbb{C}$-vector space, together with a bounded decreasing filtration $F^\bullet$, such that for every $p \in \mathbb{Z}$:

$$V = F^pV \oplus \overline{F^{k-p+1}V}.$$

To see the equivalence we can construct a filtration from the decomposition by defining

$$F^pV = \oplus_{r \geq p} V^{r,k-r}.$$

Conversely, given the filtration, we can recover

$$V^{p,k-p} = F^pV \cap \overline{F^{k-p}V}.$$

Let $K \subset \mathbb{R}$ and $L \subset \mathbb{C}$ be fields.

Definition 5.3. A (K, L)-Hodge structure of weight k is a tuple

$$H = (H_K, (H_L, F^\bullet), \rho : H_K \otimes_K \mathbb{C} \to H_L \otimes_L \mathbb{C}),$$

such that H_K is a finite dimensional K-vector space, H_L is a finite dimensional L-vector space, ρ is an isomorphism of complex vector spaces and $H_\mathbb{C} := H_K \otimes_K \mathbb{C}$ is a $\mathbb{R}$-Hodge structure of weight k. Morphisms are pairs (f, g) where $f : H_K \to H'_K$ and $g : H_L \to H'_L$ such that they are compatible with the isomorphism ρ.

It turns out that the category of (K, L)-Hodge structures of weight k is an Abelian category. In the next subsection we will give a reference for a stronger theorem, from which this observation follows. We will now compute two interesting examples.

Example 5.4. Let $X = \mathbb{P}^1_\mathbb{Q} \setminus \{0, \infty\}$, a smooth quasi-projective variety over $\mathbb{Q}$. We will compute the $(\mathbb{Q}, \mathbb{Q})$-Hodge structure of $H^1(X)$. Consider

$$H^1(X) = (H^1_{\text{sing}}(X^{an}; \mathbb{Q}), H^1_{\text{AdR}}(X/\mathbb{Q}), \rho : H^1_{\text{AdR}}(X/\mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C} \to H^1_{\text{sing}}(X; \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C}).$$

It remains to compute the first singular cohomology group of $X^{an} = X(\mathbb{C})$ with coefficients in $\mathbb{Q}$, the first algebraic de Rham cohomology group of X over $\mathbb{Q}$, and the comparison isomorphism between them. Now, as $(\mathbb{P}^1_\mathbb{C})^{an} \cong \mathbb{CP}^1 \simeq S^2$, we have that X^{an} is a sphere with two punctures, i.e. homotopic to the circumference S^1, so

$$H^1_{\text{sing}}(X; \mathbb{Q}) \cong H_1^{\text{sing}}(X; \mathbb{Q})^\vee \cong (\mathbb{Q}\langle[\gamma]\rangle)^\vee \cong \mathbb{Q}\langle[\gamma]^\vee\rangle.$$

Here γ is an oriented loop along the circumference of the punctured sphere X^{an}. Next, for algebraic de Rham cohomology, we note that X is an affine scheme, hence the complex of differential forms $\Omega^\bullet_{X/\mathbb{Q}}$ is acyclic, hence we can compute the hyper cohomology $\mathbb{H}^k(X, \Omega^\bullet_{X/\mathbb{Q}}) = H^k(\Omega_{X/\mathbb{Q}}(X)^\bullet)$, and so

$$\begin{aligned} H^1_{\mathrm{AdR}}(X/\mathbb{Q}) &\cong \frac{\Omega^1_{X/\mathbb{Q}}(X)}{\mathrm{Im}(d : \mathcal{O}_{X/\mathbb{Q}}(X) \to \Omega^1_{X/\mathbb{Q}}(X))} \\ &\cong \frac{\mathbb{Q}[x, \frac{1}{x}]dx}{\mathrm{Im}(d : \mathbb{Q}[x, \frac{1}{x}] \to \mathbb{Q}[x, \frac{1}{x}]dx)} \\ &\cong \mathbb{Q}\langle \left[\frac{dx}{x}\right] \rangle. \end{aligned}$$

It remains only to compute the comparison isomorphism. Since we have found bases for both $H^1_{\mathrm{AdR}}(X/\mathbb{Q}) \cong \mathbb{Q}\langle [\frac{dx}{x}] \rangle$ and $H^1_{\mathrm{sing}}(X; \mathbb{Q}) \cong \mathbb{Q}\langle [\gamma^\vee] \rangle$, and since we use the perfect pairing between algebraic cycles and algebraic forms, we conclude that the comparison isomorphism must send

$$\rho \left[\frac{dx}{x}\right] = \int_\gamma \frac{dx}{x} [\gamma^\vee] = 2\pi i [\gamma^\vee],$$

i.e., it is given by multiplication by $2\pi i$, which is a $\mathbb{C}$-linear isomorphism $\mathbb{C} \to \mathbb{C}$. It is not true that any complex variety X admits a $(\mathbb{Q}, \mathbb{Q})$-Hodge structure. A similar result is true if we allow *mixed* Hodge structures, as we will define in the next subsection.

Section 5.2: Mixed Hodge Structures

A mixed Hodge structure consists of, in particular, a bi-filtered vector space $H_\mathbb{C}$. First, let us define an increasing (bounded) filtration $W_\bullet = W^{-\bullet}$.

Definition 5.5. Let $(V, F^\bullet)$ be a filtered K-vector space. Let $W \subset V$, we get an induced filtration on W given by

$$F^p W = W \cap F^p V.$$

If $\pi : V \to Q$ is a surjection, then we get an induced filtration on Q given by

$$F^p Q = \mathrm{Im}(F^p V \to Q).$$

This allows us now to define a (K, L)-mixed Hodge structure, where, as before, $K \subset \mathbb{R}$ and $L \subset \mathbb{C}$ are subfields. Sometimes we can also take K to be $\mathbb{Z}$, and $L = \mathbb{C}$ for *integral* mixed Hodge structures.

Definition 5.6. A (K, L)-mixed Hodge structure is a triplet

$$H = ((H_K, W_\bullet), (H_L, W_\bullet, F^\bullet), \rho : (H_K \otimes_K \mathbb{C}, W_\bullet) \to (H_L \otimes_L \mathbb{C}, W_\bullet)),$$

where $(H_K, W_\bullet)$ is a finite dimensional filtered K-vector space and $(H_L, W_\bullet, F^\bullet)$ is a finite dimensional bi-filtered L-vector space, such that, for any $k \in \mathbb{Z}$, the graded piece

$$\mathrm{Gr}^W_k H = (\mathrm{Gr}^W_k H_K, (\mathrm{Gr}^W_k H_L, F^\bullet), \mathrm{Gr}^W_k(\rho))$$

is a pure (K, L)-Hodge structure of weight k.

Here the functor Gr^W_k is defined as follows, for a bifiltered vector space $(V, W_\bullet, F^\bullet)$, we define

$$\mathrm{Gr}^W_k V = \frac{W_k V}{W_{k-1} V},$$

and $\mathrm{Gr}^W_k(\rho)$ is simply ρ restricted to the k-th graded piece. We then let $F^\bullet$ be the induced subquotient filtration on $\mathrm{Gr}^W_k H_L$. We recall some celebrated results of Deligne from his papers [Del71a, Del71b, Del74].

Theorem 5.7 (Deligne [Del71b]). *The category of (K, L)-mixed Hodge structures is an abelian category.*

Theorem 5.8 (Deligne [Del71a]). *Let X be a smooth projective complex algebraic variety. Then for every $n \geq 0$, the singular cohomology group*

$$H^n_{\text{sing}}(X^{\text{an}}, \mathbb{Q})$$

carries a natural pure Hodge structure of weight n.

This result is later generalised to arbitrary complex algebraic varieties.

Theorem 5.9 (Deligne [Del74]). *Let X be a complex algebraic variety. Then for every $n \geq 0$, the singular cohomology group*

$$H^n_{\text{sing}}(X^{\text{an}}, \mathbb{Q})$$

admits a functorial mixed Hodge structure.

Section 5.3: Extensions of Mixed Hodge Structures

From now on we will work with *integral* mixed Hodge structures. This consists of a finitely generated Abelian group $H_{\mathbb{Z}}$ together with an increasing *weight* filtration $W_\bullet$ on $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, and a decreasing *Hodge* filtration $F^\bullet$ on $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, such that the triple

$$((H_{\mathbb{Q}}, W_\bullet), (H_{\mathbb{C}}, W_\bullet, F^\bullet), \mathrm{id}_{H_{\mathbb{C}}})$$

is a $\mathbb{Q}$-mixed Hodge structure. Before we define the extension classes of integral mixed Hodge structures we need to define the Tate Hodge structures.

Definition 5.10. For $n \in \mathbb{Z}$ we define the n-th integral Tate Hodge structure

$$\mathbb{Z}(n) = (\mathbb{Z}, \mathbb{C}, (2\pi i)^n : \mathbb{C} \to \mathbb{C}),$$

where the Hodge filtration is simply given by $F^p\mathbb{Z}(n)_{\mathbb{C}} = \mathbb{Z}(n)_{\mathbb{C}}$ if $p \leq -n$ and 0 otherwise. Equivalently, it is the unique rank one pure Hodge structure of weight $-2n$.

Definition 5.11. Let H be a torsion-free pure $\mathbb{Z}$-Hodge structure of weight $k \leq -1$. An *extension* of $\mathbb{Z}(0)$ by H is a short exact sequence of $\mathbb{Z}$-MHS

$$0 \to H \to E \to \mathbb{Z}(0) \to 0.$$

Two such short exact sequences are said to be isomorphic if there is an isomorphism of the central pieces extending the identity maps on H and $\mathbb{Z}(0)$. We denote the group of isomorphism classes of extensions by

$$\mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{Z}}}(\mathbb{Z}(0), H).$$

These are easily classified in this situation. Although classical and well-known, we will give a constructive proof.

Proposition 5.12. *There is a canonical identification*

$$\mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{Z}}}(\mathbb{Z}(0), H) \cong \frac{H_{\mathbb{C}}}{H_{\mathbb{Z}} + F^0 H_{\mathbb{C}}}.$$

Proof. We will construct a morphism in both directions. We leave it to the interested reader to confirm that these morphisms are each other's inverses. First, given an extension E, we get a short exact sequence of Abelian groups

$$0 \to H_{\mathbb{Z}} \to E_{\mathbb{Z}} \to \mathbb{Z} \to 0.$$

Let $e_{\mathbb{Z}} \in E_{\mathbb{Z}}$ be a lift of $1 \in \mathbb{Z}$. Similarly, we get a short exact sequence

$$0 \to F^0 H_{\mathbb{C}} \to F^0 E_{\mathbb{C}} \to \mathbb{C} \to 0,$$

and let $e_{\mathbb{C}} \in F^0 E_{\mathbb{C}}$ denote a lift of $1 \in \mathbb{C}$. By abuse of notation, we denote $\rho(e_{\mathbb{Z}} \otimes 1) \in E_{\mathbb{C}}$ also by $e_{\mathbb{Z}}$. Then observe that the element $e_{\mathbb{Z}} - e_{\mathbb{C}} \in E_{\mathbb{C}}$ maps to 0, so is in the image of $H_{\mathbb{C}}$. This element $[e_{\mathbb{Z}} - e_{\mathbb{C}}]$ is well-defined in $\frac{H_{\mathbb{C}}}{H_{\mathbb{Z}} + F^0 H_{\mathbb{C}}}$. Conversely, given an element $[h] \in \frac{H_{\mathbb{C}}}{H_{\mathbb{Z}} + F^0 H_{\mathbb{C}}}$, we will construct an extension E. First, we define $E_{\mathbb{Z}} = H_{\mathbb{Z}} \oplus \mathbb{Z}\langle e \rangle$ for some formal variable e, such that the projection to $\mathbb{Z}$ maps $e \mapsto 1$. Then $E_{\mathbb{C}} = E_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. Then, we specify the Hodge filtration on E.

$$\begin{aligned} F^p E_{\mathbb{C}} &= F^p H_{\mathbb{C}} \oplus \mathbb{C}\langle e \rangle && \text{for } p \leq -1 \\ F^0 E_{\mathbb{C}} &= F^0 H_{\mathbb{C}} \oplus \mathbb{C}\langle e - h \rangle \\ F^p E_{\mathbb{C}} &= F^p H_{\mathbb{C}} && \text{for } p \geq 1. \end{aligned}$$

Then E is an extension of $\mathbb{Z}(0)$ by H. □

In particular, when choosing $H = \mathbb{Z}(1)$ we obtain the following.

Corollary 5.13. *We have a canonical identification*

$$\operatorname{Ext}^1_{\mathrm{MHS}_{\mathbb{Z}}}(\mathbb{Z}(0), \mathbb{Z}(1)) \cong \mathbb{C}^{\times}.$$

Proof. Note that the isomorphism of $\mathbb{Z}(1)$ is given by multiplication with $2\pi i$, hence we get by Proposition 5.12

$$\operatorname{Ext}^1_{\mathrm{MHS}_{\mathbb{Z}}}(\mathbb{Z}(0), \mathbb{Z}(1)) \cong \frac{\mathbb{C}}{\mathbb{Z}\langle 2\pi i \rangle} \cong \mathbb{C}^{\times},$$

where the second isomorphism is given by the exponential map. □

Section 5.4: The Augmented Height of Points in $\mathbb{CP}^1$

In this subsection, we will finish with a computation. Consider four distinct points P, Q, R and S in $\mathbb{CP}^1$. We will show that $H = H_1^{\text{sing}}(\mathbb{CP}^1 \setminus |P - Q|, |R - S|)$ can be realised as an extension of $\mathbb{Z}(0)$ by $\mathbb{Z}(1)$, and thus corresponds to a unique non-zero complex number in $\mathbb{C}^{\times}$. This number is denoted by $\langle P - Q, R - S \rangle_{\text{Hodge}}$, and we will show that it is equal to $\mathrm{CR}(P, Q; R, S)$. Let us first recall some algebraic topology. Let X be a topological space and let A be a closed subspace of X. Then we have a short exact sequence of complexes

$$0 \to C_\bullet(A) \to C_\bullet(X) \to \frac{C_\bullet(X)}{C_\bullet(A)} \to 0.$$

Here $C_\bullet(X)$ denotes the chain complex of X and we define chains in X relative to A as $C_\bullet(X, A) = \frac{C_\bullet(X)}{C_\bullet(A)}$. We define the k-th singular homology group of X with coefficients in $\mathbb{Z}$ as

$$H_k^{\text{sing}}(X) = \frac{\operatorname{Ker}(\partial : C_k(X) \to C_{k-1}(X))}{\operatorname{Im}(\partial : C_{k+1}(X) \to C_k(X))}$$

and similarly we define relative homology

$$H_k^{\text{sing}}(X, A) = \frac{\operatorname{Ker}([\partial] : C_k(X, A) \to C_{k-1}(X, A))}{\operatorname{Im}([\partial] : C_{k+1}(X, A) \to C_k(X, A))}.$$

A short exact sequence of complexes gives rise to a long exact sequence on homology [Jon24]

$$\cdots \to H_k(A) \to H_k(X) \to H_k(X,A) \to H_{k-1}(A) \to \ldots$$

Now let us specialise to $(X \setminus Y, A)$ where $X = \mathbb{CP}^1$, $Y = P - Q$ and $A = R - S$, where P, Q, R and S are distinct points in $\mathbb{CP}^1$. Throughout this and the following subsection we will write $H_k^{\text{sing}}(X)$ to mean the Abelian group, and $H_k(X)$ the (mixed) integral Hodge structure. Furthermore, we will omit the $|-|$ notation to denote the support of the cycles Y and A. We will start with constructing an extension of $\mathbb{Z}(0)$ by $\mathbb{Z}(1)$ denoted by $\mathcal{E}_{Y,A}$, canonically associated to the pair (X, Y, A), which allows us to compute

$$\langle D, E\rangle_{\text{aug}} = \text{ht}(\mathcal{E}_{Y,A}).$$

Consider the tail of the long exact sequence on homology associated to the pair $(X \setminus Y, A)$,

$$H_1^{\text{sing}}(A) \to H_1^{\text{sing}}(X \setminus Y) \to H_1^{\text{sing}}(X \setminus Y, A) \to H_0^{\text{sing}}(A) \to H_0^{\text{sing}}(X \setminus Y) \to H_0^{\text{sing}}(X \setminus Y, A).$$

As H_0^{sing} describes the connected components, we observe that $H_0^{\text{sing}}(A) \cong \mathbb{Z}\langle R\rangle \oplus \mathbb{Z}\langle S\rangle$ and $H_0^{\text{sing}}(X \setminus Y) \cong \mathbb{Z}\langle X \setminus Y\rangle$ and $H_0^{\text{sing}}(X \setminus Y, A) = 0$. As the dimension of A is 0, $H_1^{\text{sing}}(A) = 0$, so we get

$$0 \to H_1^{\text{sing}}(X \setminus Y) \to H_1^{\text{sing}}(X \setminus Y, A) \to \mathbb{Z}\langle R\rangle \oplus \mathbb{Z}\langle S\rangle \xrightarrow{i} \mathbb{Z}\langle X \setminus Y\rangle \to 0.$$

As the map $i : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is simply induced from the inclusion of A in $X \setminus Y$, this sends $i(a, b) = a + b$. Thus, $\text{Ker}(i) = \mathbb{Z}\langle(1, -1)\rangle \cong \mathbb{Z}$ and so we get a short exact sequence

$$0 \to H_1^{\text{sing}}(X \setminus Y) \to H_1^{\text{sing}}(X \setminus Y, A) \xrightarrow{\partial} \mathbb{Z}\langle R - S\rangle \to 0,$$

where the map ∂ sends a chain γ with boundary $\partial\gamma = n \cdot (R - S)$ to $n \in \mathbb{Z}$. Because the construction of mixed Hodge structures on varieties is functorial, the induced morphisms on homology are morphisms of mixed Hodge structures. The group $\mathbb{Z}\langle R - S\rangle$ is isomorphic to $\mathbb{Z}(0)$ in the trivial way. We now identify $H_1(X \setminus Y)$ with $\mathbb{Z}(1)$, and then we have obtained a short exact sequence of integral mixed Hodge structures

$$0 \to \mathbb{Z}(1) \to H_1(X \setminus Y, A) \to \mathbb{Z}(0) \to 0,$$

and conclude that $H_1(X \setminus Y, A)$ can be interpreted as an extension of $\mathbb{Z}(0)$ by $\mathbb{Z}(1)$. First, as Abelian groups, note that $X \setminus Y$ is homotopic to a sphere punctured at P and Q. Thus, we choose γ_P, a positively oriented loop around P, as generator for $H_1^{\text{sing}}(X \setminus Y)$. Next, we use the perfect pairing

$$\langle -, -\rangle : H^1_{dR}(X \setminus Y) \times H_1^{\text{sing}}(X \setminus Y; \mathbb{C}) \to \mathbb{C}$$
$$([\omega], [\gamma]) \mapsto \int_\gamma \omega.$$

Thus we are looking for a differential form $\omega_{P,Q}$ such that

$$\int_{\gamma_P} \omega_{P,Q} = 2\pi i. \tag{5.1}$$

We present the following form

$$\omega_{P,Q} = \frac{dz}{z - P} - \frac{dz}{z - Q} = d\log\left(\frac{z - P}{z - Q}\right),$$

where z is a local coordinate on a chart such that P, Q, R and S all lie on that chart. Obviously $\omega_{P,Q}$ is closed, and if we switch to the other chart $w = \frac{1}{z}$ one can check that this form is globally well-defined. Furthermore, the form $\omega_{P,Q}$ satisfies equation 5.1.

Remark 5.14. This is the reason we cannot just choose $\omega_{P,Q} = d\log(z-P)$. Another beautiful observation is the following. Let γ_Q be a positively oriented loop around Q, then one can compute explicitly

$$\int_{\gamma_Q} \omega_{P,Q} = -2\pi i.$$

Now this is not a coincidence from the construction of $\omega_{P,Q}$, but rather forced by the equality $[\gamma_Q] = -[\gamma_P]$ in $H_1^{\text{sing}}(X \setminus Y)$, as

$$\int_{\gamma_Q} \omega_{P,Q} = -\int_{\gamma_P} \omega_{P,Q}.$$

Lastly, from Deligne's theorem we know that $H^1(X \setminus Y)$ has a pure Hodge structure of weight 2 because $X \setminus Y$ is smooth, and hence $H_1(X \setminus Y)$ is pure of weight -2. Since the isomorphism is given by $2\pi i$ we conclude $H_1(X \setminus Y) \cong \mathbb{Z}(1)$.

Proposition 5.15. *Let P, Q, R and S be distinct points in $\mathbb{CP}^1$. Then*

$$\langle P - Q, R - S\rangle_{\text{Hodge}} = \text{CR}(P, Q; R, S).$$

Proof. We will follow the construction of Proposition 5.12. First, choose an integral lift of 1 coming from the short exact sequence

$$0 \to H_1(X \setminus Y) \to H_1(X \setminus Y, A) \xrightarrow{\partial} \mathbb{Z}\langle R - S\rangle \to 0.$$

This is given by the class of a path $\gamma_{R,S}$ such that $\partial\gamma_{R,S} = R - S$. Then we construct a Hodge lift of 1 coming from the short exact sequence

$$0 \to F^0 H_1(X \setminus Y) \to F^0 H_1(X \setminus Y, A) \xrightarrow{\partial} \mathbb{C} \to 0.$$

Since $H_1(X \setminus Y) \cong \mathbb{Z}(1)$ we note that $F^0 H_1(X \setminus Y) = 0$, and so we have an isomorphism. We would like to remark that the unique inverse image of 1, denoted by h is a path with boundary $\partial h = R - S$ such that

$$\int_h \omega = 0,$$

for any holomorphic 1-form ω. This is because the filtration $F^1 H^1(X \setminus Y, A)$ on cohomology, consisting of only the holomorphic 1-forms, has $F^0 H_1(X \setminus Y, A)$ as its annihilator inside $H_1(X \setminus Y, A)$. So we get a class

$$[\gamma_{R,S} - h] \in \frac{H_1(X \setminus Y)_{\mathbb{C}}}{H_1(X \setminus Y)_{\mathbb{Z}}}.$$

Now we will use the constructed identification $H_1(X \setminus Y)_{\mathbb{C}} \cong \mathbb{C}$, to obtain

$$\begin{aligned}
\int_{\gamma_{R,S} - h} \omega_{P,Q} &= \int_{\gamma_{R,S}} \omega_{P,Q} - \int_h \omega_{P,Q} \\
&= \int_{\gamma_{R,S}} d\log\left(\frac{z-P}{z-Q}\right) \\
&= \int_{R-S} \log\left(\frac{z-P}{z-Q}\right) \\
&= \log(\text{CR}(P, Q; R, S)) \in \mathbb{C}/(2\pi i)\mathbb{Z}.
\end{aligned}$$

After exponentiating, we get a well-defined complex number

$$\langle P - Q, R - S\rangle_{\text{Hodge}} = \text{CR}(P, Q; R, S).$$

□

Hence, in this situation we see a clear connection between the Hodge height and the Archimedean height.

$$\mathrm{Re}\left(\log\langle P-Q, R-S\rangle_{\mathrm{Hodge}}\right) = \langle P-Q, R-S\rangle_\infty.$$

This relation between the height of extensions and the Archimedean height is in fact true in a much broader setting, see [Hai90].

Section 5.5: The Augmented Height of Linear Subspaces in $\mathbb{CP}^n$

We will generalise the previous result. As before, let Y_P and Y_Q denote k-dimensional linear subspaces of $\mathbb{CP}^n$, and Y_R and Y_S denote $(n-k-1)$-dimensional linear subspaces of $\mathbb{CP}^n$ such that they form a non-degenerate quadruple. We have previously computed

$$\langle Y_P - Y_Q, Y_R - Y_S\rangle_\infty = \log|\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S)|,$$

see Theorem 2.20. We now want to compute the Hodge theoretic height, i.e. a *complex* number. To do this, first we construct a short exact sequence of mixed integral Hodge structures

$$0 \to \mathbb{Z}(1) \to H_{2(n-k)-1}(\mathbb{CP}^n \setminus |Y_P - Y_Q|, |Y_R - Y_S|)(-(n-k-1)) \to \mathbb{Z}(0) \to 0,$$

i.e., this twisted relative homology group defines an element of $\mathrm{Ext}^1_{\mathrm{MHS}_\mathbb{Z}}(\mathbb{Z}(0), \mathbb{Z}(1)) \cong \mathbb{C}^\times$. Then we show that under this identification the extension corresponds to

$$\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S),$$

the generalised cross-ratio as defined in Definition 2.7. So, let us begin with the relevant part of the long exact sequence associated to the pair $(X \setminus Y, A)$, where now $X = \mathbb{CP}^n$, $Y = Y_P - Y_Q$ and $A = Y_R - Y_S$. We would like to point out that the *real* dimension of Y_R and Y_S is $2(n-k-1)$.

$$\begin{aligned} H^{\mathrm{sing}}_{2(n-k)-1}(A) &\longrightarrow H^{\mathrm{sing}}_{2(n-k)-1}(X\setminus Y) \longrightarrow H^{\mathrm{sing}}_{2(n-k)-1}(X\setminus Y, A) \\ &\longrightarrow H^{\mathrm{sing}}_{2(n-k-1)}(A) \xrightarrow{i} H^{\mathrm{sing}}_{2(n-k-1)}(X\setminus Y) \longrightarrow \cdots \end{aligned}$$

Some standard facts from algebraic topology now give us that $H^{\mathrm{sing}}_{2(n-k)-1}(A) = 0$ as the dimension of the connected components of A is $2(n-k-1)$. Furthermore, $H^{\mathrm{sing}}_{2(n-k-1)}(A) \cong \mathbb{Z}\langle Y_R\rangle \oplus \mathbb{Z}\langle Y_S\rangle$. Furthermore, the map denoted by i above has the property that $i(1,0) \neq 0$, $i(0,1) \neq 0$ and $i(1,-1) = 0$. The fact that $i(1,-1) = 0$ is because the cycle $Y_R - Y_S$ is homologically trivial. So the situation simplifies to

$$0 \to H^{\mathrm{sing}}_{2(n-k)-1}(X\setminus Y) \to H^{\mathrm{sing}}_{2(n-k)-1}(X\setminus Y, A) \to \mathbb{Z}\langle Y_R - Y_S\rangle \to 0$$

Next, we use Poincaré-Lefschetz duality for the first term. We have a functorial identification, compatible with mixed Hodge structures,

$$H_{2(n-k)-1}(X\setminus Y) \cong H^{2n-2(n-k)+1}(X, Y)(n) = H^{2k+1}(X, Y)(n). \tag{5.2}$$

This map is given by sending a cycle γ to the functional $\int_\gamma -$. Now we will use the long exact sequence of cohomology associated to the pair (X, Y)

$$H^{2k}_{\mathrm{sing}}(X) \xrightarrow{j} H^{2k}_{\mathrm{sing}}(Y) \to H^{2k+1}_{\mathrm{sing}}(X, Y) \to H^{2k+1}_{\mathrm{sing}}(X).$$

We know the cohomology of X, in particular $H^{2k}_{\mathrm{sing}}(X) \cong \mathbb{Z}\langle H^k\rangle$ and $H^{2k+1}_{\mathrm{sing}}(X) = 0$, where H is a generic hyperplane in X. Since the dimension of the connected components of Y is exactly $2k$

we have $H^{2k}_{\text{sing}}(Y) \cong \mathbb{Z}\langle \text{pt}_P\rangle \oplus \mathbb{Z}\langle \text{pt}_Q\rangle$. Generically, the codimension k-plane will intersect any codimension $(n-k)$-plane in a point, hence the map j is generated by $1 \mapsto (1,1)$. Hence,

$$H^{\text{sing}}_{2(n-k)-1}(X \setminus Y) \cong H^{2k+1}_{\text{sing}}(X, Y) \cong \text{Coker}(j) \cong \mathbb{Z}. \tag{5.3}$$

From this we conclude that the singular homology $H^{\text{sing}}_{2(n-k)-1}(X \setminus Y) \cong \mathbb{Z}$. Thus far, we have not *ordered* the pair Y_P and Y_Q in Y. We do this now by choosing $[S_P]$, the class of a linking sphere around Y_P as the generator of $H^{\text{sing}}_{2(n-k)-1}(X \setminus Y)$, as opposed to $[S_Q] = -[S_P]$. More explicitly defined as follows, consider the composition

$$H^{\text{sing}}_0(Y) \xrightarrow{\quad \cong\ \text{Th}_Y \quad} H^{\text{sing}}_{2(n-k)}(X, X \setminus Y) \xrightarrow{\quad \partial \quad} H^{\text{sing}}_{2(n-k)-1}(X \setminus Y),$$

where the first map is the Thom isomorphism. Since $H^{\text{sing}}_0(Y) = \mathbb{Z}\langle \text{pt}_{Y_P}\rangle \oplus \mathbb{Z}\langle \text{pt}_{Y_Q}\rangle$ we define

$$S_P = (\partial \circ \text{Th}_Y)(\text{pt}_{Y_P}), \qquad S_Q = (\partial \circ \text{Th}_Y)(\text{pt}_{Y_Q}).$$

From the isomorphism in equation 5.2 we see that $H_{2(n-k)-1}(X \setminus Y)$ is a pure Hodge structure of weight $2k - 2n = 2(k-n)$, since mixed Hodge structures of rank 1 are automatically pure Hodge structures. In summary, we have a short exact sequence of Abelian groups

$$0 \to \mathbb{Z}\langle [S_P]\rangle \to H_{2(n-k)-1}(X \setminus Y, A) \to \mathbb{Z}\langle [Y_R - Y_S]\rangle \to 0.$$

We have seen that $\mathbb{Z}\langle [S_P]\rangle$ has weight $-2(n-k)$ and similarly we have shown that $\mathbb{Z}\langle [Y_R - Y_S]\rangle$ has weight $-2(n-k-1)$. If we twist everything by $\mathbb{Z}(-(n-k-1))$ we have weights -2 and 0 in the outer rank 1 pure integral Hodge structures. It remains to identify $[S_P]$ with the complex number $2\pi i$ (and to choose the identity as comparison isomorphism for the weight 0 integral Hodge structure). In that case we get a short exact sequence of mixed Hodge structures $\mathcal{E}$,

$$0 \to \mathbb{Z}(1) \to H_{2(n-k)-1}(X \setminus Y, A)(-(n-k-1)) \to \mathbb{Z}(0) \to 0.$$

Remark 5.16. It turns out that this is hard to do explicitly. Functionals on homology are equivalently given by integration of differential forms, so we are searching for a differential form $\eta_{P,Q}$ such that

$$\int_{S_P} \eta_{P,Q} = 2\pi i.$$

Furthermore, we also require $\eta_{P,Q} \in F^{n-k}H^{2k+1}(X \setminus Y, A)$, which is equivalent to it annihilating $F^0 H_{2(n-k)-1}(X \setminus Y, A)(-(n-k-1)) = F^{n-k-1}H_{2(n-k)-1}(X \setminus Y, A)$. We want this because of the following reason, when we compute the height of this extension we find an integral lift, a path $\gamma_{R,S}$ with $\partial\gamma_{R,S} = Y_R - Y_S$, and a Hodge lift $h \in F^0 H_{2(n-k)-1}(X \setminus Y, A)(-(n-k-1)) = F^{n-k-1}H_{2(n-k)-1}(X \setminus Y, A)$. Then

$$\text{ht}(\mathcal{E}) = \int_{\gamma_{R,S} - h} \eta_{P,Q} = \int_{\gamma_{R,S}} \eta_{P,Q}.$$

What we will do instead, is construct an isomorphic extension $\mathcal{E}'$ associated to the pair $(\text{Gr}_{n-k-1}(\mathbb{C}^{n+1}), W_P - W_Q, R - S)$ where W_P, W_Q are the divisors corresponding to $p(q^{-1}(Y_P))$ and $p(q^{-1}(Y_Q))$ respectively, and R, S are the associated points to Y_R and Y_S. Because we are working with divisors and points here, computing the height $\text{ht}(\mathcal{E}')$ is easier.

We will now construct the extension $\mathcal{E}'$. As before, from the long exact sequence on homology, associated to the pair $(X \setminus Y, A)$, where $X = \text{Gr}_{n-k}(\mathbb{C}^{n+1})$, $Y = |W_P - W_Q|$ and $A = |R - S|$, we obtain

$$0 \to H^{\text{sing}}_1(X \setminus Y) \to H^{\text{sing}}_1(X \setminus Y, A) \to H^{\text{sing}}_0(A) \xrightarrow{i} H^{\text{sing}}_0(X \setminus Y) \to 0.$$

When we identify $\mathrm{Ker}(i) = \mathbb{Z}\langle R - S\rangle$, this further simplifies to

$$0 \to H_1^{\mathrm{sing}}(X \setminus Y) \to H_1^{\mathrm{sing}}(X \setminus Y, A) \to \mathbb{Z}\langle R - S\rangle \to 0.$$

As before, this is in fact a short exact sequence of integral mixed Hodge structures, we don't have to twist here since the end pieces are already of weight -2 and 0 respectively. To see that $H_1^{\mathrm{sing}}(X \setminus Y)$ is of weight -2 we use Poincaré-Lefschetz again.

$$H_1^{\mathrm{sing}}(X \setminus Y) \cong H^{2N-1}_{\mathrm{sing}}(X, Y)(N)$$

where $N = \dim_{\mathbb{C}}(\mathrm{Gr}_{n-k}(\mathbb{C}^{n+1}))$. Now when we use the long exact sequence on cohomology, associated to the pair (X, Y) we find that

$$H^{2N-2}_{\mathrm{sing}}(X) \xrightarrow{j} H^{2N-2}_{\mathrm{sing}}(Y) \to H^{2N-1}_{\mathrm{sing}}(X, Y) \to H^{2N-1}_{\mathrm{sing}}(X).$$

Now we know the cohomology groups of the Grassmannians (see [GH78, Chapter 1, Proposition on p. 196]), and conclude that $H^{2N-1}_{\mathrm{sing}}(X) = 0$ and $H^{2N-2}_{\mathrm{sing}}(X) \cong \mathbb{Z}$. Thus, the desired cohomology group $H^{2N-1}_{\mathrm{sing}}(X, Y)$ is of weight $2N - 2$ and so after shifting by N of weight -2. Now we will construct an explicit representative of the generator $[\gamma_P] \in H_1(X \setminus Y)$, which is again defined as the image of pt_{W_P} via

$$H_0^{\mathrm{sing}}(Y) \xrightarrow{\cong\ \mathrm{Th}_Y} H_2^{\mathrm{sing}}(X, X \setminus Y) \xrightarrow{\partial} H_1^{\mathrm{sing}}(X \setminus Y),$$

where we note that the Thom isomorphism now only shifts 2 degrees, because the codimension of Y in X is 1. Let $x \in W_P \setminus W_Q$, and choose in an open $U \ni x$ local holomorphic coordinates $(z_1, \dots, z_N)$ such that $x = (0, \dots, 0)$ and $U \cap |W_P - W_Q| = \{z_1 = 0\}$. Because $W_P - W_Q = \mathrm{Div}\left(\frac{\det \Theta_P(-)}{\det \Theta_Q(-)}\right) = \mathrm{Div}(f)$, where $\det \Theta_P(-)$ is exactly the map φ_P from Lemma 2.18, we can choose these holomorphic coordinates such that

$$f|_U(z) = u(z) z_1,$$

for some nowhere-vanishing holomorphic function u. Now consider

$$\Delta_P = \{(z_1, \dots, z_N) \in U \mid |z_1| \leq \varepsilon, z_2 = \dots = z_N = 0\},$$

and let $\gamma_P = \partial \Delta_P$, with the orientation coming from $\mathbb{C}$. Thus $\gamma_P(t) : [0, 1] \to X \setminus Y$ is given by $\gamma_P(t) = (\varepsilon e^{2\pi i t}, 0, \dots, 0)$. So $H_1^{\mathrm{sing}}(X \setminus Y) \cong \mathbb{Z}\langle \gamma_P \rangle$. In this situation we *can* find a differential form $\eta_{P,Q}$ which satisfies all the mentioned properties above.

Lemma 5.17. *The holomorphic 1-form on* $\mathrm{Gr}_{n-k} \setminus |W_P - W_Q|$ *given by*

$$\eta_{P,Q} = d \log \left(\frac{\det \Theta_P(-)}{\det \Theta_Q(-)} \right)$$

satisfies

$$\int_{\gamma_P} \eta_{P,Q} = 2\pi i,$$

and furthermore, for any $h \in F^0 H_1(X \setminus Y, A)$

$$\int_h \eta_{P,Q} = 0.$$

Proof. Again we write f for $\frac{\det \Theta_P}{\det \Theta_Q}$, and observe that f is non-zero on $X \setminus Y$, and hence holomorphic. Furthermore, locally we may write

$$(f \circ \gamma_P)(t) = (u \circ \gamma_P)(t)\varepsilon e^{2\pi i t}.$$

Then

$$\begin{aligned}
\int_{\gamma_P} d\log f &= \int_{\gamma_P} \frac{df}{f} \\
&= \int_{[0,1]} \gamma_P^* \frac{df}{f} \\
&= \int_{[0,1]} \frac{d(f \circ \gamma_P)}{f \circ \gamma_P} \\
&= \int_{[0,1]} d\log(f \circ \gamma_P) \\
&= \int_{[0,1]} d\log((u \circ \gamma_P)(t)\varepsilon e^{2\pi i t}) \\
&= \int_{[0,1]} \frac{d}{dt}\log((u \circ \gamma_P)(t)) + 2\pi i dt = 2\pi i.
\end{aligned}$$

For the second claim we observe that $h \in F^0 H_1(X \setminus Y, A)$ if and only if

$$\int_h \omega = 0,$$

for all *holomorphic* ω. Since $\eta_{P,Q}$ is a holomorphic 1-form the conclusion follows immediately. □

Now we can identify $\mathbb{Z}\langle \gamma_P \rangle$ with $\mathbb{Z}(1)$ by integrating $\eta_{P,Q}$ over a path. Thus we obtain the desired extension $\mathcal{E}'$ given by

$$0 \to \mathbb{Z}(1) \to H_1(\mathrm{Gr}_{n-k}(\mathbb{C}^{n+1}) \setminus |W_P - W_Q|, |R - S|) \to \mathbb{Z}(0) \to 0.$$

Proposition 5.18. *The height of the extension $\mathcal{E}'$ is given by*

$$\mathrm{ht}(\mathcal{E}') = \mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S).$$

Proof. Let $\gamma_{R,S}$ be an integral lift of 1, i.e., a path with $\partial\gamma_{R,S} = R - S$. We don't have to construct a Hodge lift, since integrating $\eta_{P,Q}$ over this path will vanish as shown in Lemma 5.17. Thus, after identifying $\mathbb{C}/(2\pi i \mathbb{Z}) \cong \mathbb{C}^\times$ via the exponential map, we get

$$\mathrm{ht}(\mathcal{E}') = \exp\left(\int_{\gamma_{R,S}} \eta_{P,Q}\right) = \exp\left(\log(f(R)) - \log(f(S))\right) = \frac{f(R)}{f(S)} = \mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S).$$

□

Theorem 5.19. *The extensions $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic as extensions of $\mathbb{Z}(0)$ by $\mathbb{Z}(1)$. As a result,* $\mathrm{ht}(\mathcal{E}) = \mathrm{ht}(\mathcal{E}')$.

Proof. First, let us recall what it means for two extensions $\mathcal{E}$ and $\mathcal{E}'$ to be isomorphic. This is given by an isomorphism $\varphi : \mathcal{E} \to \mathcal{E}'$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & \mathcal{E} & \longrightarrow & \mathbb{Z}(0) & \longrightarrow & 0 \\
 & & \Big\| \mathrm{id} & & \Big\downarrow \varphi & & \Big\| \mathrm{id} & & \\
0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathbb{Z}(0) & \longrightarrow & 0
\end{array}$$

For notational simplicity, define

$$\begin{aligned} X &= \mathbb{CP}^n, \\ G &= \mathrm{Gr}_{n-k}(\mathbb{C}^{n+1}), \\ W &= |W_P - W_Q|, \\ L &= |R - S|, \\ Y &= |Y_P - Y_Q|, \\ V &= |Y_R - Y_S|, \\ q^*Y &= |q^{-1}(Y_P) - q^{-1}(Y_Q)|, \\ p^*L &= |p^{-1}(R) - p^{-1}(S)|. \end{aligned}$$

First, observe that we have an incidence of pairs given by

$$\begin{array}{ccccc} & & (I \setminus q^*Y, p^*L) & & \\ & \swarrow^{p} & & \searrow^{q} & \\ (G \setminus W, L) & & & & (X \setminus Y, V) \end{array}$$

We will define a morphism of mixed Hodge structures $p^!$ as the composition of

$$\begin{array}{ccc} H_i(G \setminus W, L) & \overset{p^!}{\dashrightarrow} & H_{i+2(n-k-1)}(I \setminus q^*Y, p^*L)(-(n-k-1)) \\ \downarrow \cong & & \uparrow \cong \\ H^{2m-i}(G \setminus L, W)(m) & \xrightarrow{p^*} & H^{2m-i}(I \setminus p^*L, q^*Y)(m) \end{array}$$

Here the duality isomorphisms are isomorphisms of mixed Hodge structures ([Ste99] remark on page 8), and m denotes the complex dimension of $\mathrm{Gr}_{n-k}(\mathbb{C}^{n+1})$. Then, let $\varphi = q_* \circ p^!$. For compact K, $p^![K] = [p^{-1}(K)]$ (see page 69 of [BT82]). In particular, $\varphi(R - S) = Y_R - Y_S$. Furthermore, $\varphi([\gamma_P]) = [S_P]$ where γ_P is the generator for $H_1^{\mathrm{sing}}(\mathrm{Gr}_{n-k}(\mathbb{C}^{n+1}) \setminus |W_P - W_Q|)$ and S_P is the generator for $H^{\mathrm{sing}}_{2(n-k)-1}(\mathbb{CP}^n \setminus |Y_P - Y_Q|)$, see Lemma 5.20. Let $\eta^{\mathrm{Gr}}_{P,Q}$ denote the form which we used to identify loops in the Grassmannians with integer multiples of $2\pi i$. Let $\eta^{\mathbb{CP}}_{P,Q}$ be a representative of the unique class of forms, which we cannot write down explicitly, with the property that

$$\int_{S_P} \eta^{\mathbb{CP}}_{P,Q} = 2\pi i.$$

Then we have a commutative diagram, where we omit the Tate twists in the relevant homology groups,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & H_1(G \setminus W, L) & \longrightarrow & \mathbb{Z}(0) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \int_- \eta^{\mathrm{Gr}}_{P,Q}} & & \Vert{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle R-S\mapsto 1} & & \\
0 & \longrightarrow & \mathbb{Z}\langle \gamma_P \rangle & \longrightarrow & H_1(G \setminus W, L) & \longrightarrow & \mathbb{Z}\langle L \rangle & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & \mathbb{Z}\langle S_P \rangle & \longrightarrow & H_{2(n-k)-1}(X \setminus Y, V)(-(n-k-1)) & \longrightarrow & \mathbb{Z}\langle V \rangle & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \int_- \eta^{\mathbb{CP}^n}_{P,Q}} & & \Vert{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle Y_R-Y_S\mapsto 1} & & \\
0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & H_{2(n-k)-1}(X \setminus Y, V)(-(n-k-1)) & \longrightarrow & \mathbb{Z}(0) & \longrightarrow & 0
\end{array}$$

Note that the compositions

$$\left(\int_- \eta^{\mathbb{CP}^n}_{P,Q}\right) \circ \varphi \circ \left(\int_- \eta^{\mathrm{Gr}}_{P,Q}\right)^{-1} = \mathrm{id}_{\mathbb{Z}(1)},$$
$$(Y_R - Y_S \mapsto 1) \circ \varphi \circ (R - S \mapsto 1)^{-1} = \mathrm{id}_{\mathbb{Z}(0)}.$$

Thus, we can conclude that $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic extensions of $\mathbb{Z}(0)$ by $\mathbb{Z}(1)$. □

Lemma 5.20. *With notation as above,* $[\varphi(\gamma_P)] = [S_P]$.

Proof. Let W denote $W_P - W_Q$, let Y denote $Y_P - Y_Q$ and let q^*Y denote $q^{-1}(Y_P) - q^{-1}(Y_Q)$. Let G denote the Grassmannian $\mathrm{Gr}_{n-k}(\mathbb{C}^{n+1})$, let X denote $\mathbb{CP}^n$ and lastly let I denote the incidence correspondence. By functoriality of $p^!$ and q_*, we have a commutative diagram

$$\begin{array}{ccccc}
H_0^{\mathrm{sing}}(W) & \xrightarrow{\cong\ \mathrm{Th}_W} & H_2^{\mathrm{sing}}(G, G \setminus W) & \xrightarrow{\partial} & H_1^{\mathrm{sing}}(G \setminus W) \\
\downarrow{\scriptstyle p^!} & & \downarrow{\scriptstyle p^!} & & \downarrow{\scriptstyle p^!} \\
H_0^{\mathrm{sing}}(q^*Y) & \xrightarrow{\cong\ \mathrm{Th}_{q^*Y}} & H_{2(n-k)}^{\mathrm{sing}}(I, I \setminus q^*Y) & \xrightarrow{\partial} & H_{2(n-k)-1}^{\mathrm{sing}}(I \setminus q^*Y) \\
\downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} \\
H_0^{\mathrm{sing}}(Y) & \xrightarrow{\cong\ \mathrm{Th}_Y} & H_{2(n-k)}^{\mathrm{sing}}(X, X \setminus Y) & \xrightarrow{\partial} & H_{2(n-k)-1}^{\mathrm{sing}}(X \setminus Y).
\end{array}$$

Since $[\gamma_P] = (\partial \circ \mathrm{Th}_W)([\mathrm{pt}_{W_P}])$ and $[S_P] = (\partial \circ \mathrm{Th}_Y)([\mathrm{pt}_{Y_P}])$ it suffices to show that

$$(q_* \circ p^!)([\mathrm{pt}_{W_P}]) = [\mathrm{pt}_{Y_P}],$$

since by commutativity it then follows that

$$(q_* \circ p^!)([\gamma_P]) = \varphi([\gamma_P]) = [S_P].$$

Now consider the commutative square

$$\begin{array}{ccc}
H_0^{\mathrm{sing}}(W) & \xrightarrow{p^!} & H_0^{\mathrm{sing}}(q^*Y) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
H^N_{\mathrm{sing}}(W) & \xrightarrow{p^*} & H^N_{\mathrm{sing}}(q^*Y),
\end{array}$$

where $N = \dim_{\mathbb{R}}(W)$. Since $p : q^*Y \to W$ is generically isomorphic, i.e. on the dense open

$$\Sigma = \{L \in G \mid \dim_{\mathbb{C}}(L \cap Y_P) = \dim_{\mathbb{C}}(L \cap Y_Q) = 1\}$$

the degree of p is 1. Thus, since the pull-back map on top cohomology classes is given by multiplication with the degree [Pet, Chapter 3.4], p^* maps $\mathrm{PD}([\mathrm{pt}_{W_P}]) \longmapsto \mathrm{PD}([\mathrm{pt}_{q^{-1}(Y_P)}])$. Thus $p^!([\mathrm{pt}_{W_P}]) = [\mathrm{pt}_{q^{-1}(Y_P)}]$ and $q_*[\mathrm{pt}_{q^{-1}(Y_P)}] = [q(\mathrm{pt}_{q^{-1}(Y_P)})] = [\mathrm{pt}_{Y_P}]$ which finishes the proof. □

Finally, we can compute the augmented height pairing of $Y_P - Y_Q$ and $Y_R - Y_S$.

Corollary 5.21. *Let Y_P and Y_Q denote k-dimensional linear subspaces of $\mathbb{CP}^n$, and let Y_R and Y_S denote $(n-k-1)$-dimensional linear subspaces of $\mathbb{CP}^n$ such that they form a non-degenerate quadruple. Then*

$$\langle Y_P - Y_Q, Y_R - Y_S\rangle_{\mathrm{aug}} = \mathrm{ht}(\mathcal{E}) = \mathrm{ht}(\mathcal{E}') = \mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S).$$

Proof. The first equality is by definition. The second equality follows from Theorem 5.19. The last equality follows from Proposition 5.18. □

Remark 5.22. Theorems 2.20 and 4.2 show that the norms of the cross-ratio satisfy the relations

$$\langle Y_P - Y_Q, Y_R - Y_S\rangle_\infty = \log\left|\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S)\right|,$$
$$\langle Y_P - Y_Q, Y_R - Y_S\rangle_p = \log\left|\mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S)\right|_p, \qquad \text{(for all primes } p\text{)}.$$

Corollary 5.21, which relates the augmented height pairing with the generalised cross-ratio,

$$\langle Y_P - Y_Q, Y_R - Y_S\rangle_{\mathrm{aug}} = \mathrm{CR}^n(Y_P, Y_Q; Y_R, Y_S),$$

further confirms that the generalised cross-ratio is canonical not only through its norms, but also as a complex number. It is remarkable that such a simple constant completely governs the arithmetic complexity of linear subspaces in $\mathbb{CP}^n$.

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