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K-th order Hydrodynamic limits

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Abstract

In this thesis we study stochastic duality under hydrodynamic scaling in the context of interacting particles on a grid. The approach is inspired and motivated by the relation between duality and local equilibria. We identify duality relations in terms of the expectation of the density field for which the hydrodynamic limit is recovered. This is initially done both for symmetric inclusion and exclusion processes as well as for independent random walkers. We continue with the independent case and generalize to particles which also posses a, possibly scale dependent, internal energy state. The results in this context assume generator convergence under scaling and are illustrated using run-and-tumble systems. This work also includes examples concerning instances of run-and-tumble processes which do not have convergence on a generator level. Apart from run-and-tumble processes, we examine the effect of reservoirs on the relevant duality relations and macroscopic profiles. The reservoirs are found to correspond with boundary conditions for the macroscopic profile.

Contents

1 Introduction

Simple systems of particles on a grid have received considerable attention by mathematicians and physicists alike as they provide a convenient ground-zero for testing the properties of systems far from their equilibrium state, for which no general theory is available [6]. Regardless of this lack of theory, non-equilibrium states spark particular interest as they display peculiar behaviour like long-distance correlations. A first step in developing a rigorous and general theory is to understand systems which are in a so-called *local equilibrium*. In the macroscopic context, i.e. for systems which are scaled versions of particles on a grid, the concept of local equilibrium encompasses that, locally around a microscopic point, the measure describing the system is a product measure. However, this is parameterized by the macroscopic profile at the corresponding macroscopic point. Since product measures are believed to correspond to equilibrium states, one might interpret this as the equilibrium being established locally. Duality has proven to be a gateway into understanding this concept, as it offers exact methods of calculation to describe the propagation of the local equilibrium. That is, one can typically reformulate the canonical definition of local equilibrium in terms of a finite number of dual particles initialized around a macroscopic point. Through this reformulation one can understand propagation of the local equilibrium through the fact that at macroscopic times, the finite number of dual particles are typically at large distances. Ultimately this leads to macroscopic profiles given by the heat equation [1].

In this thesis we explore the scaling of the duality relations in the setting described above. That is, we study particles moving on a grid. Motivated by applications concerning non-equilibrium states, we mainly focus on the case where the dual system consists of a finite number of particles. We start from systems where the relevant dynamics purely depend on the spatial positions of the particles. We do this both in the case where the particles don't have interactions and in the case where they do. The interactions in question are either of the exclusion or inclusion type. We argue that, using a suited coupling, one can pass to independent particles. The next step is to generalize our findings to systems on a (possibly scale dependent) state space which is larger than just the integer grid. To illustrate this more general theory, we study the so-called run-and-tumble systems, where the particles possess an internal state which describes a drift dynamic. In the case where the internal state of these particles changes freely, our theory is sufficient to describe macroscopic behaviour. However, in the case where the internal state changes occur depending on position, we identify systems where our theory proves inadequate. Nonetheless we still managed to show results concerning the macroscopic profile, through the use of ergodic theory. Finally we also examine systems in a scenario where non-equilibrium steady states naturally occur, namely systems with the presence of reservoirs with different parameters.

Both scaling and duality techniques are common in the field of interactive particle systems. Before we get to the actual story, we informally introduce duality and hydrodynamic scaling to better contextualize our work.

Scaling limits

Behind the use of stochastics lies often the wish to gain a qualitative understanding of a system which is too complicated to describe using standard deterministic methods. In statistical physics, this strategy is particularly popular to understand the behaviour of physical systems consisting of a large number of small particles. The exact behaviour of each of these particles is a complex process highly dependent on presence of other particles in its direct environment.

On a macroscopic level this yields a system which has an unreasonable amount of degrees of freedom. However, in many cases it turns out that at a macroscopic level these systems behave in a predictable manner, described by a system of partial differential equations. Moreover, this predictable behaviour arises even from systems where the particles at microscale are hugely simplified. One might for example think of particles which live on a grid and jump and hop from one site to another, whereas the physical particles move along continuous paths in continuous space. In many cases, the resulting differential equations remain the same. Even the interactions between the particles can often be ignored or simplified. Through the study of simplified microscopic systems one may infer the type of micro scale dynamics responsible for the macroscopic behaviour of the system in question.

To meaningfully discuss these systems it is necessary to have a rigorous method to translate a microscopic system to the macroscopic world. This requires scaling of both space, time and often even the parameters describing the probability distributions at a microscopic level. usually, scaling results in the a particle density rather then the individual position of particles. One particularly popular method of scaling is so-called *hydrodynamic* scaling, where the space is scaled by ϵ and the time by ϵ^{-2} as ϵ tends to zero. This is inspired by the fact that a random walk on $\mathbb{Z}$ is typically distance of order $\sqrt{t}$ removed from its initial position at time t. Hence, one needs quadratic scaling of the time to allow the random walker to move away from its origin, without it escaping to infinity. In many physical systems the particles perform seemingly a Brownian motion when singled out, validating this choice of scaling in a physical context.

Stochastic duality

Duality is a recurrent theme throughout numerous branches of mathematics. Nonetheless, the precise meaning of the term is highly dependent on the area where it occurs. Generally it refers to some type of one-to-one correspondence, oftentimes between two problems which are in a sense equivalent. In such a scenario, one can translate an original problem to its dual problem, which is hopefully more approachable. The dual is then solved instead to obtain useful information about the original problem. In the context of stochastic processes, the duality between two Markov processes X and Y is understood in terms of a *duality function* D, mapping the Cartesian product of the state spaces to a real number. We consider the duality function acting on the processes evolving in time. More precise, we observe the mean of the duality function as we keep one process fixed and let the other evolve. If the mean is indifferent to which process, X or Y, is evolving, the processes are dual. Without getting formal about the notation, one can interpret duality as the following relation,

$$\mathbb{E}D(X_t, Y_0) = \mathbb{E}D(X_0, Y_t). \tag{1}$$

A more mathematically precise definition is given in subsection 2.6. Depending on the exact situation, duality provides the equivalence between the evolution of the two processes, potentially allowing to find quantities of interest about the original system via its dual. In practice this turns out to be an immensely powerful technique. In the literature there is an abundance of instances where stochastic processes are analysed via their dual. In most cases, the crux is that the dual is more comfortable to work with then the original system. We name a few ways the dual might be more desirable.

Nature of the state space:

The dual is in some cases a pure jump process on a countable state space, whereas the the original process has a continuous state space such as $\mathbb{R}^d$ [2]. In most cases a discrete state space is more convenient to work with, however there are also instances known where a discrete system is dual to a continuous process which is deterministic [1].

Number of particles:

In the context of interacting particle systems, one can often reduce the number of particles via a duality relation. This type is one of the most used simplifications one can derive from a duality result. This is also the major use of duality in this thesis. Technically, it works as follows. One considers a dual system which contains only a single particle. The duality function will be such that it returns the number of particles in the original system at the location of the dual particle. Hence one can recover the expected number of particles at any location by studying the dynamics of the single dual particle. Evidently this yields significant simplification as the single particle doesn't have interactions with other particles. Even when there are more dual particles, the interactions can often be ignored on a macroscopic scale.

Absorbing boundary sites

In statistical physics it is common to model transport using interacting particle systems. That is, one considers a finite grid where particles are brought in and removed with constant rates at the boundary sites. This models the contact with boundary reservoirs. Suppose the boundary parameters are not the same over the boundary, then the system will reach a steady state over the course of time, but it won't be in equilibrium. This is not contradictory as the terms steady state and equilibrium have different meaning in this context. We say the system is steady state because the distribution of the particles in the systems is stationary in time. It is non-equilibrium since there is a flow of particles moving from one reservoir to another. This corresponds to the microscopic model not being described by equilibrium measures. In the study of these systems duality has been a successful technique since it allows to replace the reservoirs by absorbing sites. That is, instead of placing and absorbing particles at a constant rate, the site will hold every particle which touches it. Hence, over time, the system will be drained, as all particles end up at the boundary sites. This gives the k-th order moments of the non-equilibrium steady state in terms of the absorption probabilities of k dual particles.

Duality under scaling

The hydrodynamic scaling is examined in the context of so-called *macroscopic fields*. More specifically, we focus on the *density field*. For a configuration η on $\mathbb{Z}^d$, the associated density field maps a test function $\varphi : \mathbb{Z}^d \to \mathbb{R}$ to a real number. The density field is then of the form

$$\chi_\epsilon(\eta, \varphi) = \epsilon^d \sum_{x \in \mathbb{Z}^d} \varphi(\epsilon x)\eta_x. \tag{2}$$

One could regard this expression as some sort of inner product between η and a scaled version of φ. In this expression, scaling φ is effectively the same as scaling the positions of the particles in configuration η to a finer grid $(\epsilon\mathbb{Z})^d$. Hence we could regard the system to be spatially scaled by a factor ϵ. Hydrodynamic scaling requires the time to be scaled by factor ϵ^{-2}, which yields

$$\chi_\epsilon(\eta(\epsilon^{-2}t), \varphi) = \epsilon^d \sum_{x \in \mathbb{Z}^d} \varphi(\epsilon x)\eta_x(\epsilon^{-2}t). \tag{3}$$

Typically η will scale hydrodynamically to a dynamic profile $\rho(t, x)$. That is,

$$\lim_{\epsilon\to 0}\chi_\epsilon(\eta(\epsilon^{-2}t),\varphi)=\int\varphi(x)\rho(t,x)dx. \tag{4}$$

We are interested in the remains of the duality relation after scaling. In the case where η consists of independent random walkers there is a duality function D_{micro} such that $D_{\text{micro}}(\delta_x,\eta)=\eta_x$. That is, for a single dual particle, the duality function yields the number of particles at the position of the dual particle. Macroscopically we have a similar result. Due to the connection between the heat equation and Brownian motion, which is the hydrodynamic limit of the random walk, we have

$$\mathbb{E}_{\rho(o,\cdot)}[\rho(t,x)]=\rho(t,x)=\mathbb{E}[\rho(0,B_x(t))]. \tag{5}$$

Here, $\{B_x(t):t\geq 0\}$ denotes the Brownian motion departing from x. What this statement says is that the evolution of the density at a fixed point x is the same as the evolution of (the function $\rho(0,\cdot)$ applied to) the dual particle. This is analogous to how $D_{\text{micro}}(\delta_x,\eta)=\eta_x$. Indeed, consider the duality function $D_{\text{macro}}(\delta_x,g)=g(x)$, which evaluates a density g in point x. We observe that equation 5 again has the form of a duality relation, but the expectation is redundant as η becomes deterministic on a macroscopic scale.

$$\mathbb{E}_{\rho(o,\cdot)}[D_{\text{macro}}(\rho(t,\cdot),x)]=\mathbb{E}_x[D_{\text{macro}}(\rho(0,\cdot),B_x(t))] \tag{6}$$

Notice that we can equivalently state that for each test function φ,

$$\int\rho(t,x)\varphi(x)dx=\int\rho(0,x)S_t^{\text{dual}}\varphi(x)dx \tag{7}$$

Here, $\{S_t^{\text{dual}},t\geq 0\}$ is the semigroup corresponding to the dual particle. A statement of this form is what we consider to be a duality relation on macroscopic scale. In this thesis we further generalize this concept of macroscopic duality to duality functions working on k dual particles. We do this for more general particle systems, rather then only for independent walkers. As is already apparent from the expression above, the macroscopic duality will appear directly from scaling the density field. Moreover, it will be equivalent with the k-th order hydrodynamic limit of the system in question. That is, the hydrodynamic limit of a density field involving k dual particles.

Structure of this thesis

In chapter 2 we review the basic theory relevant for interacting particle systems. This chapter does not contain any new work and is only included to avoid ambiguity in notation and terminology. In chapter 3 we explore the scaling of duality in an increasingly general setting. We start of with the case where the particles are independent in section 3.1. Once we have established macroscopic higher order duality in this setting we use coupling to derive similar results for interacting systems like SEP and SIP. This is done in section 3.2. Finally we formulate this result for general systems with state space containing $\mathbb{Z}^d$ and a sufficiently strong sense of convergence of the generator under the hydrodynamic limit. In chapter 4 we use our findings to analyse so-called run-and-tumble systems. These are introduced in subsection 4.1 and their microscopic self-duality is shown in subsection 4.2. We conclude this chapter by calculating the hydrodynamic limit in subsection 4.3. In this subsection we also treat run-and-tumble systems where the generator can not be scaled hydrodynamically. Chapter 5 deals with the higher-order hydrodynamics of systems with reservoirs. These are formally introduced in subsection 5.1, where also the relevant duality result is proved. The macroscopic profiles are recovered in 5.2.

2 Interacting Particle Systems

Before we address topics such as duality and scaling limits, we first sketch the larger picture we are working in, the world of so called *interacting particle systems*. Interacting particle systems were first introduced by Frank Spitzer in his 1970 seminal paper *Interaction of Markov processes* [16] and could be informally described as large systems of interacting Markov processes. These have far reaching applicability in statistical physics and also play an import role in modelling, for example, evolutionary biology [12], the spread of opinions [11] and the distribution of wealth [2]. We first provide the necessary background on Markov processes in subsections 2.1 and 2.2, after which we cover some relevant examples of interactive particle systems in 2.3. Next we introduce the Birkhoff ergodic theorem and the Dynkin martingale in subsections 2.4 and 2.5 respectively. These are some more advanced tools which are helpful in chapter 4. Finally we discuss the notion of stochastic duality in subsection 2.6.

2.1 Markov Processes

Since Markov theory lays at the core of interacting particle systems, it is important that we provide some basic elements of the theory first. The reader is referred to [11] for more background on Markov process theory. The majority of the theory presented here is adopted from [14, 17].

Intuitively one can think of a Markov processes on a state space Ω as a stochastic process $\{X_t, t \geq 0\}$ taking values in a measurable space $(\Omega, \mathscr{A})$ such that $\{X_t, t \geq 0\}$ is *memoryless*. This entails the following. Suppose we are provided with the state of the process up to some specific time $s \geq 0$, then the distribution of X_t at a future time $t \geq s$ only depends on the state X_s. In particular, the distribution of X_t does not depend on any of the states before time s. More precise we can say that $\forall t > 0$, $n \in \mathbb{N}$, $0 < t_1 < ... < t_2 < ... < t_n < t$ and for all $f : \Omega \to \mathbb{R}$ bounded and measurable:

$$\mathbb{E}(f(X_t)|X_{t_1, X_{t_2}}, ..., X_{t_n}) = \mathbb{E}(f(X_t)|X_{t_n}). \tag{8}$$

It is common to define the Markov property in terms of measure theoretic notations. This is done in the definition below.

Definition 2.1. *Let $\mathscr{F}_t = \sigma(X_r : r \leq t)$ denote the σ-algebra generated by the random variables X_r, $r \leq t$, then the Markov property means that for all $0 < s \leq t$*

$$\mathbb{E}(f(X_t)|\mathscr{F}_s) = \mathbb{E}(f(X_t)|X_s). \tag{9}$$

Furthermore, we say the Markov process is homogeneous if for each $s > 0$ the process $\{X_{t+s}, t \geq 0\}$ starting from $X_s = x$ has the same distribution as $\{X_t, t \geq 0\}$ starting from $X_0 = x$.

Remark 2.2. *So far we only defined Markov processes in the continuous time setting. However, one can easily give a similar definition for discrete processes. This definition would be of the same form as (8), but with the times $t_1, ..., t_n$ elements of $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, ...\}$.*

To illustrate the concept of a Markov process, we introduce the so called (finite state space) Markov chains, the simple symmetric random walk and Brownian motion. The latter two play a relevant role in the interactive particle systems we are ultimately interested in.

2.1.1 Finite state space Markov Chain

Let Ω be a finite state space. The idea of a Markov chain is that the system jumps from one state to a (random) other one. Let $x, y \in \Omega$. In the discrete time setting the system is described via a transition probability function $p : \Omega \times \Omega \to [0, 1]$ satisfying

$$\sum_{y \in \Omega} p(x, y) = 1 \text{ for all } x \in \Omega. \tag{10}$$

Suppose the system is in state $x \in \Omega$, then, after one unit of time elapses, the system jumps to state $y \in \Omega$ with probability $p(x, y)$. Since the transition probabilities are at any time only dependent on the current state of the system, the process is indeed Markov.

In this thesis we are in the first place interested in *continuous time* Markov chains. These Markov chains jump at random times and hence we define the system using rates rather than transition probabilities. This entails that we associate to each $x, y \in \Omega$ a rate $c(x, y)$ with the following properties

$$c(x, y) \geq 0 \qquad \& \qquad c_x := \sum_{z \in \Omega} c(x, z) > 0. \tag{11}$$

These rates can be intuitively understood as the probability that the system in state x jumps to state y per unit of time. Mathematically, a jump from x to y occurs at times which are exponentially distributed with parameter $c(x, y)$. One might wonder what would happen if we were to choose any other distribution with parameter $c(x, y)$ for the jumping time. In that case the process loses the Markov property. This stems from the memoryless property of the exponential distribution. Let T_x denote the time at which the state x is left for the first time. We assume the Markov property and verify that T_x has to be exponentially distributed.

$$\begin{aligned} \mathbb{P}(T_x > t + s | T_x > s) &= \mathbb{P}(X_r = x, \forall s \leq r \leq t + s | X_u = x, \forall 0 \leq u \leq s) \qquad (12) \\ &= \mathbb{P}(X_r = x, \forall s \leq r \leq t + s | X_s = x) \\ &= \mathbb{P}(T_x > t). \end{aligned} \tag{13}$$

This is in fact equivalent to

$$\mathbb{P}(T_x > t + s) = \mathbb{P}(T_x > t)\mathbb{P}(T_x > s). \tag{14}$$

A decreasing function $\phi : \mathbb{R}^+ \to [0, 1]$ has the property that $\phi(t + s) = \phi(t)\phi(s)$ has to be exponential, given that ϕ is measurable [5]. Hence we can conclude that $\mathbb{P}(T_x > t)$ has to be of the form $e^{-c_x t}$.

2.1.2 Simple symmetric random walk and Brownian motion

In the previous example we discussed Markov chains on a finite state space. Here we look at a continuous time system which is defined on a countable state space, namely the integers. We denote the continuous time *simple symmetric random walk* by $\{X_t : t \geq 0\}$. Suppose the random walk is at $x \in \mathbb{Z}$. It then jumps to another state $y \in \mathbb{Z}$ exponentially according to the following rate

$$c(x, y) = \mathbb{1}(|x - y| = 1). \tag{15}$$

One can imagine this process as the position of a particle jumping one unit left or right at exponentially distributed times. If we assume the system starts of at zero, it turns out that, by applying the appropriate (i.e. hydrodynamic) scaling, we can obtain the so called *Brownian motion* processes. That is, $\forall t \geq 0$,

$$\epsilon X_{\epsilon^{-2}t} \to W_t \text{ as } \epsilon \to 0 \tag{16}$$

The one dimensional Brownian motion $\{W_t : t \geq 0\}$ is a stochastic process taking values in the real numbers. It is defined by via the following properties,

- Starting from the origin: $W_0 = 0$.
- The increments independent and normally distributed: $\forall 0 < t_1 < t_2 < ... < t_n$,

$$W_{t_i} - W_{t_{i-1}} \tag{17}$$

  are independent normally distributed with mean zero and variance $t_i - t_{i-1}$.
- Continuity of paths: the map $t \mapsto W(t)$ is continuous.

One can extend this definition to a $d-$dimensional Brownian motion by joining d independent, one dimensional, Brownian motions: $W_t = (W_1(t), ..., W_d(t))$. For $x \in \mathbb{R}^d$ we refer to the stochastic process $W_t^x = \{x + W_t, t \geq 0\}$ as *the Brownian motion departing form* x. The Markov property follows from the independence of the increments.

2.2 Markov semigroups and generators

2.2.1 Markov semigroups

There is an intimate connection between Markov processes and strongly continuous contraction semigroups. One can use this connection to describe the macroscopic evolution of scaled interactive particle systems via (stochastic) partial differential equations. We first define the notion of a Markov semigroup on a complete, real-valued function space,

Definition 2.3. *Let $(F, ||\cdot||_\infty)$ be a Banach space of real-valued functions. A family $\{S_t, t \geq 0\}$ of bounded linear operators $S_t : F \to F$ is called a Markov semigroup if for all $f \in F$ and $s, t \geq 0$,*

(S_1) $S_0 f = f$

(S_2) Semigroup property: $S_{t+s} f = S_t(S_s f)$

(S_3) Strong continuity: $\lim_{t \downarrow 0} ||S_t f - f||_\infty = 0$

(S_4) $S_t \mathbb{1} = \mathbb{1}$

(S_5) Positivity: if $f \geq 0$ then $S_t f \geq 0$

(S_6) Contraction: $||S_t f||_\infty \leq ||f||_\infty$

The strong continuity essentially says that the semigroup acting on a function is right-continuous at zero with respect to the infinity norm. Combining this with the semigroup property yields the continuity in t of $\{S_t f, t \geq 0\}$ for each f. Given a fixed right-continuous homogeneous Markov process, it turns out that the following family of operators indeed defines a Markov semigroup.

Definition 2.4. *Let $X = \{X_t, t \geq 0\}$ be a homogeneous Markov process that is right-continuous. We define the following family of operators indexed by $t \geq 0$ acting on $f : \Omega \to \mathbb{R}$,*

$$S_t f(x) = \mathbb{E}_x[f(X_t)]. \tag{18}$$

Here $\mathbb{E}_x$ denotes the expectation with respect to the path space measure of $\{X_t, t \geq 0\}$ conditioned on $X_0 = x$.

As is already suggested by the notation, the operators $\{S_t, t \geq 0\}$ form a Markov semigroup. Notice that in this definition we are not specific about the domain of the operators. In fact we will choose the domain, denoted $F(\Omega)$, based on structure of the state space Ω. Some common choices are listed below.

Definition 2.5. *Let Ω be the state space of a right-continuous, homogeneous Markov process $X = \{X_t, t \geq 0\}$. We define the domain $F(\Omega)$ of the associated operators $\{S_t, t \geq 0\}$ as follows.*

- *Ω is a **compact metric space**: $F(\Omega) = C(\Omega)$, the continuous functions on Ω*
- *Ω is a **locally compact space**: either $F(\Omega) = C_0(\Omega)$ or $C_b(\Omega)$, depending on the situation. Here $C_0(\Omega)$ are the continuous functions which vanish at infinity and $C_b(\Omega)$ the continuous functions which are bounded.*
- *Ω is a general **measurable space**: $F(\Omega) = B(\Omega)$, the space of bounded measurable functions.*

We take a brief moment to discuss which type of systems correspond with these state spaces. A typical example for a compact metric space is $\Omega = E^S$, where E is a finite set and S is countable. S might for example correspond to a grid of particles and E to the spin state of each particle. This set is compact with respect to the with respect to the product topology and is metrizable via $d(\eta, \xi) = \sum_{n \in \mathbb{N}} 2^{-n} \mathbb{1}(\eta_{x_n} \neq \xi(x_n))$ with x_n is an enumeration of S. For processes like Brownian motion, with locally compact state space $\Omega = \mathbb{R}^d$, we take $F(\Omega) = C_0(\Omega)$. One can verify that for each choice, $F(\Omega)$ is a Banach space equipped with the infinity norm $||\cdot||_\infty$.

Theorem 2.6. *The operators $\{S_t, t \geq 0\}$ form a Markov semigroup on $(F(\Omega, ||\cdot||_\infty))$.*

Proof. Linearity of the operators, as well as properties S_1, S_4, S_5 and S_6 follow immediately from the definition of S_t. Hence we will focus on the semigroup property and the strong continuity. The latter is in fact rather difficult to prove, therefore we only show pointwise right continuity here and refer to the book "Functional Analysis" by Yoshida for the extension to strong continuity.

Semigroup property

We exploit the tower property of conditional expectation as well as the Markov property of X to derive the following equality,

$$S_{t+s} f(x) = \mathbb{E}_x[f(X_{t+s})] = \mathbb{E}_x[\mathbb{E}[f(X_{t+s})|\mathscr{F}_t]] = \mathbb{E}_x[\mathbb{E}[f(X_{t+s})|X_t]]. \tag{19}$$

The homogeneity of X yields

$$\mathbb{E}[f(X_{t+s})|X_t] = \mathbb{E}_{X_t}[f(X_s)] = S_s f(X_t). \tag{20}$$

Substituting this into the first expression gives that

$$S_{t+s}f(x) = \mathbb{E}_x[S_s f(X_t)] = S_t(S_s f(X_t)). \tag{21}$$

Strong continuity

We notice that $f \in F(\Omega)$ is continuous for each possible function space in definition 2.5. Since $X_0 = x$ and X is right-continuous, we have that $\lim_{t\downarrow 0} X_t = x$. This gives

$$\lim_{t\downarrow 0} S_t f(x) = \lim_{t\downarrow 0} \mathbb{E}_x[f(X_t)] = f(x). \tag{22}$$

Here we first use dominated convergence to bring the limit into the expectation and then use the continuity of f to obtain $f(x)$. This shows pointwise right-continuity. For the full argument showing uniform right-continuity we refer to section 1 of chapter IX [18]. □

Remark 2.7. *We remark that the theorem also works the other way around. That is, for a given a Markov semigroup there is a unique Markov process corresponding with that specific semigroup. Hence one may show existence of a Markov process by showing the existence of its semigroup generator. The latter will be introduced in the next subsection. More about this can be found in theorem 1.5 of the book of Liggett [10]. This semigroup is in turn constructed from a generator. We treat generators of Markov processes in the next subsection.*

2.2.2 Markov generators

In the case where S_t and L are matrices such that $S_t = \exp(tL)$, it is easy to check that S_t indeed satisfies the semigroup property. More over, for $f \in F(\Omega)$ we have

$$\frac{d}{dt} S_t f = \frac{d}{dt} e^{tL} f = L e^{tL} f = L S_t f. \tag{23}$$

Hence we have that, for each $f \in F(\Omega)$, the function $S_t f$ solves the differential equation

$$\begin{cases} \frac{d}{dt} g(t,\omega) = Lg(t,\omega) \\ g_0(\omega) = f(\omega). \end{cases} \tag{24}$$

This provides us with an intuition for how the semigroup is acting on the on the function f, hence we would like to generalize this relation "$S_t f = \exp(tL)$" to a more general setting where the semigroup operator is not necessarily a matrix. That is, we want to generalize the role of the matrix L, which we call the *generator* of the Markov process associated to $\{S_t, t \geq 0\}$. This is done in the following way.

Definition 2.8. *Let $\{S_t, t \geq 0\}$ be a Markov semigroup acting on $F(\Omega)$. We define the Markov generator as the operator L defined by*

$$Lf = \lim_{t\downarrow 0} \frac{S_t f - f}{t} \tag{25}$$

acting on the domain $D(L) := \{f \in F(\Omega) : \lim_{t\downarrow 0} \frac{S_t f - f}{t}$ exists$\}$.

Example 2.9 (Continuous time Markov chain)**.** *We calculate the generator for the continuous time Markov chain introduced in subsection 2.1.1. Let K_t denote the number of jumps made in the time interval $[0,t]$. We use the following two facts.*

$$\mathbb{P}(K_t > 1) = O(t^2) \qquad \& \qquad \mathbb{P}(X_0 = x, X_t = y | K_t = 1) = \frac{c(x,y)}{c_x}. \tag{26}$$

We use the law of total expectation to derive,

$$S_t f(x) - f(x) = \sum_{k=0}^{\infty} \mathbb{E}_x[f(X_t)|K_t = k]\mathbb{P}(K_t = k) - f(x) \tag{27}$$

$$= \mathbb{E}_x[f(X_t)|K_t = 1]\mathbb{P}(K_t = 1) + \mathbb{E}_x[f(X_t)|K_t = 0]\mathbb{P}(K_t = 0) - f(x) + O(t^2)$$

$$= (1 - e^{-c_x t}) \sum_{y \in \Omega} \Big[\frac{c(x,y)}{c_x} f(y)\Big] + e^{c_x t} f(x) - f(x) + O(t^2)$$

$$= \frac{1 - e^{-c_x t}}{c_x} \Big[\sum_{y \in \Omega} c(x,y)(f(y) - f(x))\Big] + O(t^2). \tag{28}$$

Since $\lim_{x \downarrow 0} \frac{1-e^{-x}}{x} = 1$,

$$Lf(x) = \lim_{t \downarrow 0} \frac{S_t f(x) - f(x)}{t} = \sum_{y \in \Omega} c(x,y)(f(y) - f(x)) \tag{29}$$

Example 2.10 (Brownian motion)**.** *We calculate the generator of Brownian motion. We use the notation introduced in subsection 2.1.2. Let $N(0,t)$ denote a normal random variable. Then*

$$\mathbb{E}_x f(W_t^x) = \mathbb{E} f(x + N(0,t)) = f(x) + f'(x)\mathbb{E}(N(0,t)) + \frac{1}{2} f''(x)\mathbb{E}(N(0,t)^2) + O(t) \tag{30}$$

with $O(t)/t \to 0$ as $t \to 0$. As a consequence

$$Lf(x) = \lim_{t \to 0} \frac{\mathbb{E}_x f(W_t^x) - f(x)}{t} = \frac{1}{2} f''(x). \tag{31}$$

For the d-dimensional Brownian motion this generalizes to

$$Lf = \frac{1}{2}\Delta f. \tag{32}$$

So far we have defined a semigroup operators corresponding to Markov processes and we have defined what their generators are. However, so far we don't know what generators correspond to a Markov semigroup. Using an application of the Hille-Yoshida theorem one can show that the following generators correspond uniquely to a Markov semigroup.

Definition 2.11. *Let $D(L) \subset F(\Omega)$, then the operator $L : D(L) \to F(\Omega)$ is called a Markov generator if the following properties hold:*

(G_1) $\mathbb{1} \in D(L)$ *and* $L\mathbb{1} = 0$

(G_2) $D(L)$ *is dense in* $F(\Omega)$

(G_3) L *is a closed operator, i.e.,* $\{(f, Lf) : f \in D(L)\}$ *is closed*

(G_4) The range of $(I - \lambda L)$ is $F(\Omega)$ for all $\lambda \geq 0$

(G_5) If $f \in D(L), \lambda \geq 0$ and $(I - \lambda L)f = g$, then

$$\min_{x \in \Omega} f(x) = \min_{x \in \Omega} g(x) \tag{33}$$

The following theorem states the connection to differential equations we derived in the case where the semigroup was the exponential of a matrix. It also provides a way to retrieve the semigroup from a given generator.

Theorem 2.12. *Let $\{S_t, t \geq 0\}$ and $(L, D(L))$ be a Markov semigroup and a Markov generator. We have the following two results.*

1. For all $t \geq 0$,

$$Lf = \lim_{t \downarrow 0} \frac{S_t f - f}{t} \iff S_t = \lim_{n \to \infty} \left(I - \frac{t}{n} L\right)^{-n}. \tag{34}$$

i.e. when L is the generator corresponding to $\{S_t, t \geq 0\}$, one can retrieve $\{S_t, t \geq 0\}$ via the formula on the right hand side.

2. Assume L is in correspondence with $\{S_t, t \geq 0\}$. For $f \in D(L)$, we have that $S_t f \in D(L)$ and

$$\frac{d}{dt} S_t f = S_t L f = L S_t f. \tag{35}$$

Moreover, $S_t f$ is the unique solution to this equation.

It is not always feasible to characterize the whole domain of the generator explicitly. However, oftentimes we can also study the generator on a smaller subspace $\mathscr{D} \subset D(L)$ which we will call a *core*. The precise definition is given below.

Definition 2.13. *A set $\mathscr{D} \subset D(L)$ is a core of the generator L if the closure of the restriction $L|_{\mathscr{D}}$ is again the generator L. That is, the graph $\{(f, Lf) : f \in \mathscr{D}\}$ is dense in the closed graph $\{(f, Lf) : f \in L(D)\}$*

Example 2.14. *Let K be a finite set. We consider the continuous-time Markov chain on $\Omega = K^{\mathbb{Z}}$. The Generator L is then given by*

$$Lf(x) = \sum_{y \in \Omega} c(x, y)\big(f(y) - f(x)\big), \tag{36}$$

with domain $D(L) = \{f \in B(\Omega) : ||Lf||_\infty < \infty\}$. In this case we could take local functions *as a core $\mathscr{D}$. That is, the functions which only depend on a finite number of coordinates in $\mathbb{Z}$. It is clear that $\mathscr{D} \subset D(L)$. Using the Stone-Weierstrass theorem one can show that the local functions are dense in $C(\Omega)$, hence they provide a valid core for L.*

The true power of working with a core lies in the fact that convergence of a Markov generator on a core implies the convergence of the corresponding semigroup and process. This is exactly the content of the so called *Trotter-Kurtz* theorem, which is stated below. The proof can be found in [8].

Theorem 2.15 (Trotter-Kurtz). *Let $(\{X_t^n, t \geq 0\})_{n \in \mathbb{N}}$, $\{X_t, t \geq 0\}$ be Markov processes on a compact space Ω, with corresponding semigroups $(\{S_t^n, t \geq 0\})_{n \in \mathbb{N}}$, $\{S_t, t \geq 0\}$ and generators $(L^n)_{n \in \mathbb{N}}$, L repsectively. Assume $\mathscr{D}$ is a core for L. The following are equivalent,*

- *for all $f \in \mathscr{D}$ there exists a sequence $(f^n)_{n\in\mathbb{N}}$ with $f^n \in D(L^n)$ such that $f^n \to f$ and $L^n f^n \to Lf$.*
- *$S_t^n f \to S_t f$ for every $f \in F(\Omega)$, uniformly for $t \in [0, T]$.*
- *if $X_0^n \to X_0$ in distribution, then $X^n \to X$ in distribution in the path space.*

2.3 Interacting Particle Systems: IRW, SIP & SEP

In the previous subsections we studied several Markov systems on their own. The next step is to consider systems which are defined as a collection of these Markov processes. Moreover, we are interested in case where there is an interactive dynamic between these processes, i.e. the evolution of one of the Markov processes depends on the state of the others. In this subsection we outline three interactive particle systems which are closely related to one another. To visualize these processes one can think of particles with position described by the simple symmetric random walk. That is, the particles jump left and right on the integer line with equal probabilities and exponential waiting times with parameter α. The process consisting of independent random walkers (IRW) without any interaction between them is the first of the three. The other two processes are known as the *symmetric inclusion process* (SIP) and the *symmetric exclusion process* (SEP) and have an attractive and a repulsive dynamic respectively. Here, "inclusion" refers to the fact that, in the SIP setting, the particles are inclusive towards each other in the sense that they prefer to heap together. Similarly,"exclusion" refers to the fact that the particles prefer to exclude each other from their site.

Consider a particle at an arbitrary position $x \in \mathbb{Z}$. In the SIP case, the particle jumps to a neighboring site $y \in \mathbb{Z}$ with rate $(\alpha + \eta_y)$, where η_y denotes the number of particles at site y. Clearly the probability to jump to neighboring site y increases with the number of particles at y. Hence we could intuitively understand this process as random walkers which attract each other. Similarly one could consider the system where the particle at x jumps to neighboring site y with rate $(\alpha - \eta_y)$. In this case the rate decreases with the number of particles at site y. We could interpret this as the particles repelling each other. This gives the other interactive system, SEP. Notice that for SEP the total number of particles can not rise above α.

We now formalize the processes introduced above on a grid of arbitrary dimension $d \in \mathbb{N}$. To each of the three systems we associate a value of parameter σ. That is, $\sigma = 0, 1, -1$ for IRW, SIP and SEP respectively. Consider the state space $\Omega = (\mathbb{Z}^d)^{\mathbb{N}}$ and let $\alpha > 0$. We associate to the pair (σ, α) a process η on Ω generated by the generator in definition 2.16.

Definition 2.16. *Let $D(L) = C_0(\Omega)$. We define IRW, SIP and SEP as the process on Ω generated by L. The Markov generator is L is given by,*

$$Lf(\eta) = \sum_{x,y \in F^d} \eta_x(\alpha + \sigma\eta_y)[f((\eta)^{x,y}) - f(\eta)]. \tag{37}$$

Here, $(\eta)^{x,y}$ denotes the the configuration where one particle moved from site x to site y, i.e. $(\eta)^{x,y} = \eta + \delta_y - \delta_x$.

2.4 Ergodic measures

In chapter 4.3.2 we encounter a system for which the generator does not converge under hydrodynamic scaling. Nonetheless, the particles in this system still have limiting distribution, resulting in a macroscopic field. To recover this limiting distribution, we mainly use two tools, namely the *Birkhoff egodic theorem* and the *Dynkin martingale*. We briefly introduce the necessary background here and in the next subsection. The material for this subsection is derived from [3].

In the upcoming definitions, $\{X_t, t \geq 0\}$ is a Markov process defined on the measure space $(\Omega, \mathscr{F})$. We write $\{S_t, t \geq 0\}$, L for the associated semigroup, generator and μ_t for the distribution of X_t at time $t \geq 0$.

Definition 2.17. *We say that $\mu := \mu_0$ is invariant with respect to $\{S_t, t \geq 0\}$ if*

$$\int f d\mu_t = \int f d\mu \tag{38}$$

for all $t \geq 0$ and $f \in F(\Omega)$. We denote the set of invariant measures by $\mathscr{I}$.

Definition 2.18. *Given an invariant measure μ, we call a measurable a set $A \in \mathscr{F}$ invariant with respect to $\{S_t, t \geq 0\}$ if for all $t \geq t$ we have that $S_t \mathbb{1}_A = \mathbb{1}_A$ almost surely. In the same fashion, we call a function $f \in L^p(\Omega, \mu)$ invariant if it remains unchanged under the semigroup operator: $\forall t \geq 0$, $S_t f = f$.*

Definition 2.19. *We say that a probability measure $\mu \in \mathscr{I}$ is ergodic with respect to $\{S_t, t \geq 0\}$ if for all invariant sets $A \in \mathscr{F}$ either $\mu(A) = 0$ or $\mu(A) = 1$.*

Given these definitions we can state the Birkhoff ergodic theorem. This theorem essentially says that the time average of a process is equal to the spatial average of the process, provided that the relevant measures are ergodic.

Theorem 2.20. *Let $\mu \in \mathscr{I}$ be ergodic, then, for every $f \in L^1(\Omega, \mu)$ we have that*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(X_t) dt = \int f d\mu, \tag{39}$$

where the convergence is μ-a.s.

Proof. The proof can be found in [3]. □

To characterize the invariant and ergodic measures, we use the following results.

Proposition 2.21. *$\mu \in \mathscr{I}$ if and only if for all $f \in \mathscr{D}$,*

$$\int L f d\mu = 0. \tag{40}$$

Here, $\mathscr{D}$ is a core for L.

Proposition 2.22. *$\mu \in \mathscr{I}$ is ergodic if and only if for any $p \geq 1$, all invariant functions $f \in L^p(\Omega, \mu)$ are μ-a.s. constant.*

2.5 Dynkin martingale

The following theorem and proof are adopted from [17].

Theorem 2.23. *Let $\{X_t, t \geq 0\}$ be an $\mathscr{F}_t$-adapted Markov process on $(\Omega, \mathscr{F}, \mathbb{P})$, generated by L. For any $f \in D(L)$, the process $\{M_t, t \geq 0\}$ defined by*

$$M_t := f(X_t) - f(X_0) - \int_0^t L f(X_s) ds \tag{41}$$

is a real-valued martingale on $(\Omega, \mathscr{F}, \mathbb{P})$ with respect to the filtration $\{\mathbb{F}, t \geq 0\}$.

Proof. The fact that this process is adapted follows from the adaptiveness of $\{X_t, t \geq 0\}$. To see that M_t is in $L^1(\Omega, \mathscr{F}, \mathbb{P})$, we use that

$$\mathbb{E}[|M_t|] \leq 2||f||_\infty + t||Lf||_\infty \leq \infty. \tag{42}$$

Now we only have to show the Markov property. Let $0 \leq s \leq t$, then

$$\begin{aligned}
\mathbb{E}[M_t - M_s|\mathscr{F}_s] &= \mathbb{E}\Big[f(X_t) - f(X_s) - \int_0^t Lf(X_r)dr|\mathscr{F}_s\Big] \qquad (43)\\
&= \mathbb{E}[f(X_t)|\mathscr{F}_s] - \mathbb{E}[f(X_s)|\mathscr{F}_s] - \int_0^{t-s} \mathbb{E}[Lf(X_{r+s})|\mathscr{F}_s]dr,
\end{aligned}$$

where the last inequality follows from Fubini. Since $\{X_t, t \geq 0\}$ is a Markov process with respect to the filtration $\{\mathbb{F}, t \geq 0\}$, the expectation given on the whole past $\mathscr{F}_s$ is equal to the expectation given X_s, i.e.,

$$\begin{aligned}
\mathbb{E}[M_t - M_s|\mathscr{F}_s] &= \mathbb{E}[f(X_t)|\mathscr{F}_s] - \mathbb{E}[f(X_s)|\mathscr{F}_s] - \int_0^{t-s} \mathbb{E}[Lf(X_{r+s})|\mathscr{F}_s]dr \qquad (44)\\
&= S_{t-s}f(X_s) - f(X_s) - \int_0^{t-s} S_rLf(X_s)dr. \qquad (45)
\end{aligned}$$

By the Hille-Yoshida theorem, we then that

$$\int_0^{t-s} S_rLf(X_s)dr = \int_0^{t-s} \frac{\partial}{\partial r}S_rf(X_s)dr = S_{t-s}f(X_s) - f(X_s), \tag{46}$$

Hence we indeed find that

$$\mathbb{E}[M_t - M_s|\mathscr{F}_s] = 0, \tag{47}$$

therefore $\{M_t, t \geq 0\}$ is a martingale with respect to $\{\mathscr{F}_t, t \geq 0\}$. □

The martingale introduced above is known as the *Dynkin martingale.* The following theorem provides an easy way to calculate its quadratic variation.

Theorem 2.24. *Suppose that both f, f^2 are in the domain of generator L. Then*

$$V_t = M_t^2 - \int_0^t [L(f^2)(X_s) - 2f(X_s)Lf(X_s)]ds \tag{48}$$

is a martingale with respect to the filtration $\{\mathscr{F}_t, t \geq 0\}$.

Proof. The proof can be found in [15]. □

Remark 2.25. *For a locally square integrable martingale M, the quadratic variation $\{[M, M]_t, t \geq 0\}$ is given by the unique right-continuous and increasing process starting at zero such that*

$$V_t = M_t^2 - [M, M]_t \tag{49}$$

is martingale[9]. Hence the theorem above indeed provides a way to calculate the quadratic variation of the Dynkin martingale.

2.6 Duality

In this chapter we state the same notion of duality we introduced in the introduction in a more formal setting. Throughout this thesis we will use this notion to translate systems with a large (or infinite) number of particles to a system containing only a few. The material presented here is adopted from [7].

We begin with the formal definition of duality.

Definition 2.26. *Suppose X and Y are Markov processes with state spaces E and F respectively. We write $\mathbb{E}_x$ and $\mathbb{E}^y$ for the expectations with respect to the path spaces of X and Y under the condition that the initial conditions of X and Y are given by $x \in E$ and $y \in F$. Let $D : E \times F \to \mathbb{R}$ be a measurable function. Then X and Y are dual with respect to D if and only if for all $x \in E$, $y \in F$ and $t > 0$*

$$\mathbb{E}_x D(X_t, y) = \mathbb{E}^y D(x, Y_t). \tag{50}$$

If X and Y are instances of the same Markov process, we say that they that X is self-dual with respect to D.

Note that one can write this definition in terms of the semigroup operators of X and Y as well. We denote these $\{S_t, t \geq 0\}$ and $\{\hat{S}_t, t \geq 0\}$ respectively, then we have the alternative relation

$$[S_t D(\cdot, y)](x) = [\hat{S}_t D(x, \cdot)](y). \tag{51}$$

Remark 2.27. *In the remainder of this thesis we will typically use η and ξ to denote the processes X and Y which exhibit a duality relation.*

Remark 2.28. *The stochastic duality which we introduced above can, in the majority of cases, be characterized with so-called* generator duality*. This entails that the action of the generators of two processes on the duality function is the same. Throughout this text, we will often prove generator duality instead of verifying the original definition 2.26. To be precise, we use the proposition below, which is adopted from [7]. However, it turns out that this proposition is false and one can find (pathological) examples for which it doesn't hold. Nonetheless, we still use the theorem as it is stated here, because in our setting it can be applied and it provides a powerful tool to show duality. In [1] one can find a complete theorem about the connection between generator and semigroup duality.*

Proposition 2.29. *Let X and Y be Markov processes with generators L^X and L^Y. Let $D : E \times F \to \mathbb{R}$ be continuous. If $D(x, \cdot), \hat{S}_t D(x, \cdot) \in D(L^Y)$ for all $x \in E, t \geq 0$ and $D(\cdot, y), S_t D(\cdot, y) \in D(L^X)$ for all $y \in F, t \geq 0$ and if*

$$[L^X D(\cdot, y)](x) = [L^Y D(x, \cdot)](y) \qquad \forall x \in E, y \in F, \tag{52}$$

then X and Y are dual with respect to D.

3 k-th order hydrodynamic equation

When interacting systems are properly rescaled, one can often obtain a macroscopic profile. In this chapter we focus on one such scaling, namely the so-called *hydrodynamic scaling*, which essential turns a simple symmetric random walk into a Brownian motion. Due to the relation between Brownian motion and the heat equation, this eventually leads to macroscopic profiles which evolve according to the heat equation. The theory which we present here is largely based on chapter XI in [1], a book in progress by Gioia Carinci, Cristian Giardinà and Frank Redig. However, the book is mainly concerned with first order fields. The major addition to the work done there is the generalization to *k-th order* fields. These fields can be used in order to prove a stronger version of the emergence of local equilibrium on a macroscopic scale.

3.1 Independent random walks

3.1.1 First order field

Before looking a the k-th order field it is insightful to first study the first order field. We consider a system of continuous-time independent random walkers on $\mathbb{Z}^d$ which we we denote η. We write $\eta_x(t)$ to denote the number of walkers at site $x \in \mathbb{Z}^d$ at time t. Given the necessary initial conditions, one can scale this system in space and time such that a deterministic, time-dependent, profile appears. This result is encapsulated in what we call the *hydrodynamic equation*. A weaker version, only claiming the first moment to converge, is stated below in terms of the density field associated to η. Apart from the density field, we also need the concept of consistent measures.

Definition 3.1. *The first order hydrodynamic field $\chi_\epsilon(\eta, .)$ associated with η is defined as the distribution which maps $\varphi \in C_c^\infty(\mathbb{R}^d)$ to the random variable*

$$\chi_\epsilon(\eta, \varphi) = \epsilon^d \sum_x \varphi(\epsilon x)\eta_x = \epsilon^d \sum_x \varphi(\epsilon x) D(\delta_x; \eta). \tag{53}$$

Here, D denotes the duality function

$$D(\eta, \xi) = \prod_x \frac{\eta_x!}{(\eta_x - \xi_x)!}. \tag{54}$$

for which a system of independent random walkers is self-dual.

Definition 3.2. *Let $\rho : \mathbb{R}^d \to [0, \infty)$ be a bounded smooth function. We say that the family of probability measures μ_ϵ, $\epsilon > 0$ has expected density consistent with ρ if for all $\epsilon > 0$, and $x \in \mathbb{Z}^d$,*

$$\int \eta_x d\mu_\epsilon(\eta) = \int D(x; \eta) d\mu_\epsilon(\eta) = \rho(\epsilon x). \tag{55}$$

Theorem 3.3. *Let $\{\mu_\epsilon, \epsilon > 0\}$ denote a family of probability measures on the configuration space consistent with a smooth and bounded profile $\rho_0 : \mathbb{R}^d \to \mathbb{R}$. Then, for all $t > 0$, the expectation of the time evolved first order field converges as*

$$\int \chi_\epsilon(\eta(\epsilon^{-2}t), \varphi) d\mu_\epsilon(\eta) \xrightarrow{\epsilon \to 0} \int \rho(t; x)\varphi(x)dx \tag{56}$$

$\rho(t; x)$ is the solution of the heat equation in $\mathbb{R}^d$,

$$\begin{cases} \frac{\partial \rho}{\partial t} = \frac{1}{2}\Delta\rho \\ \rho(0, x) = \rho_0(x). \end{cases} \tag{57}$$

Proof. We compute, using the duality and the fact that $D(\delta_x, \eta) = \eta_x$,

$$\begin{aligned}\int \eta_x(\epsilon^{-2}t)d\mu_\epsilon &= \int \mathbb{E}_\eta[\eta_x(\epsilon^{-2}t)]d\mu_\epsilon \\ &= \int \mathbb{E}_\eta[D(\delta_x, \eta(\epsilon^{-2}t))]d\mu_\epsilon \\ &= \int \mathbb{E}_x^{IRW}[D(\delta_{X(\epsilon^{-2}t)}, \eta)]d\mu_\epsilon \\ &= \mathbb{E}_0^{RW}[\rho(\epsilon x + \epsilon X(\epsilon^{-2}t))]d\mu_\epsilon.\end{aligned} \tag{58}$$

Here, $\mathbb{E}_x^{RW}$ denotes the expectation with respect to the random walk $\{X(t), t \geq 0\}$ starting from x at time $t = 0$. In the last step we used the translation invariance of the random walk.

Define now, for $x \in \mathbb{R}^d$, $t > 0$,

$$\rho(t, x) := \mathbb{E}^{BM}[\rho(W(t) + x)] = \int_{\mathbb{R}^d} \frac{e^{-\frac{(x-y)^2}{2t}}}{(2\pi t)^{d/2}} \rho(y)dy, \tag{59}$$

where $\mathbb{E}^{BM}$ refers to the expectation with respect to the standard Brownian motion $W(t)$, then, $\rho(t, x)$ is the solution to the heat equation with initial condition $\rho(0, x) = \rho(x)$.

We have, due to the fact that Brownian motion is the scaling limit of a random walk,

$$\lim_{\epsilon \to 0} \mathbb{E}_{\mu_\epsilon}[\chi_\epsilon(\varphi, \eta(\epsilon^{-2}t))] = \lim_{\epsilon \to 0} \epsilon^d \sum_x \varphi(\epsilon x)\mathbb{E}_0^{RW} \rho(\epsilon x + \epsilon X(\epsilon^{-2}t)) \tag{60}$$

$$= \lim_{\epsilon \to 0} \epsilon^d \sum_x \varphi(\epsilon x)\mathbb{E}_0^{BM} \rho(\epsilon x + W(t))$$

$$= \lim_{\epsilon \to 0} \epsilon^d \sum_x \varphi(\epsilon x)\rho(t, \epsilon x) = \int \varphi(x)\rho(t, x)dx \tag{61}$$

This concludes the proof. □

3.1.2 k-th order field

We now generalize what we did above, using a dual system with k particles instead of just one.

Definition 3.4. *The k-th order hydrodynamic field $\chi_\epsilon^k(\eta, .)$ associated with η is defined as the distribution which maps $\varphi \in C_c^\infty(\mathbb{R}^{kd})$ to the random variable*

$$\chi_\epsilon^k(\eta, \varphi) = \epsilon^{kd} \sum_{x_1,...,x_k} \varphi(\epsilon x_1, ..., \epsilon x_k)D(\delta_{x_1} + ... + \delta_{x_k}; \eta). \tag{62}$$

Here, D denotes the duality function

$$D(\eta, \xi) = \prod_x \frac{\eta_x!}{(\eta_x - \xi_x)!}. \tag{63}$$

To simplify notation we write $D(\delta_{x_1} + ... + \delta_{x_k}; \eta)$ as $D(x_1, ...x_k; \eta)$, which is then a symmetric function of $x_1, ..., x_k$.

Definition 3.5. *Let $\rho : \mathbb{R}^{kd} \to [0,\infty)$ be a bounded smooth function. We say that the family of probability measures μ_ϵ, $\epsilon > 0$ has expected density consistent with ρ if for all $\epsilon > 0$, and $x = (x_1, ..., x_k) \in \mathbb{Z}^{kd}$*

$$\int D(x_1, ..., x_k; \eta) d\mu_\epsilon(\eta) = \rho(\epsilon x). \tag{64}$$

Remark 3.6. *For our purposes this notion can be slightly weakened. In fact we could also have defined consistency as follows,*

$$\lim_{\epsilon \to 0} \int \chi^k_\epsilon(\eta, \varphi) d\mu_\epsilon = \int_{\mathbb{R}^{kd}} \rho(x_1, ..., x_k)\varphi(x_1, ..., x_k) dx_1...dx_k. \tag{65}$$

However, this definition involves a limit which will yield more complicated calculations later on. Hence, for the sake of clarity and convenience we will stick to definition using equality.

Theorem 3.7 (higher order hydrodynamic equation)**.** *Let $\{\mu_\epsilon, \epsilon > 0\}$ denote a family of probability measures on the configuration space consistent with a smooth and bounded profile $\rho_0 : \mathbb{R}^{kd} \to \mathbb{R}$. Then, for all $t > 0$, the time evolved k-th order field converges in the L^2-sense as*

$$\int \chi^k_\epsilon(\eta(\epsilon^{-2}t), \varphi) d\mu_\epsilon \xrightarrow{\epsilon \to 0} \int \rho(t; x_1, ..., x_k)\varphi(x_1, ..., x_k) dx_1...dx_k. \tag{66}$$

$\rho(t; x_1, ..., x_k)$ is the solution of the heat equation in $\mathbb{R}^{kd}$,

$$\begin{cases} \frac{\partial \rho}{\partial t} = \frac{1}{2}\Delta^k \rho \\ \rho(0, x) = \rho_0(x). \end{cases} \tag{67}$$

We go through the proof, which is similar to the proof for the first order field, attempting to interpret the macroscopic profile as the scaling limit of the duality relation. The solution to the heat equation is known to be $\rho(t, x) = \mathbb{E}_x[B(t)]$, where $B(t)$ is a kd-dimensional Brownian motion. One could in fact regard this expression for ρ as a duality result with respect to function $\bar{D} : C^\infty_b(\mathbb{R}^{kd}) \times \mathbb{R}^{kd} \to \mathbb{R}, (f, x) \mapsto f(x)$,

$$\mathbb{E}_{\rho_0}[\bar{D}(\rho(t, .), x)] = \rho(t, x) = \mathbb{E}_x[B(t)] = \mathbb{E}_x[\bar{D}(\rho_0, B(t))]. \tag{68}$$

Moreover, we can interpret this relation as the limit of the duality associated with D in the case where we consider a single dual particle. Indeed, we can regard ρ as being the scaling limit of η. Then $D(\eta, \delta_x) = \eta_x$ gives the evaluation of η at site x, similar to how $\bar{D}(\rho, x)$ gives the evaluation of ρ in x. Our aim is to retrieve the higher order hydrodynamic equation from a microscopic duality relation. The precise form of the scaled relation is the following,

$$\mathbb{E}_{\eta(0)}[Q(\eta(t), \varphi)] = \mathbb{E}_\varphi[Q(\eta(0), S^{k,\epsilon}_t \varphi)], \tag{69}$$

where $Q(\eta, \varphi) = \chi^k_\epsilon(\eta, \varphi)$. Let S^k_t be the Markov semi-group operator associated to k random walks in $\mathbb{Z}^d$ i.e. $(S^k_t \varphi)(x_1, .., x_k) = \mathbb{E}^{IRW}_{x_1,..,x_k}[\varphi(X_1(t), ..., X_k(t))]$ with $X_1, ..., X_k$ independent random walkers. We define scaling-operator Z^k_ϵ and its inverse $(Z^k_\epsilon)^{-1}$ as $[Z^k_\epsilon \varphi](x_1, ..., x_k) := \varphi(\epsilon x_1, ..., \epsilon x_k)$ and $[(Z^k_\epsilon)^{-1}\varphi](x_1, ..., x_k) := \varphi(x_1/\epsilon, ..., x_k/\epsilon)$ respectively. $S^{k,\epsilon}_t$ can now be constructed as $S^{k,\epsilon}_t := (Z^k_\epsilon)^{-1} S^k_t Z^k_\epsilon$.

$$\begin{aligned}
\mathbb{E}_{\eta(0)}[Q(\eta(t),\varphi)] &= \epsilon^{kd}\sum_{x_1,\dots,x_k}\varphi(\epsilon x_1,\dots,\epsilon x_k)\mathbb{E}_{\eta(0)}[D(x_1,\dots,x_k;\eta(t))] \\
&= \epsilon^{kd}\sum_{x_1,\dots,x_k}\varphi(\epsilon x_1,\dots,\epsilon x_k)\mathbb{E}^{IRW}_{x_1,\dots,x_k}[D(X_1(t),\dots,X_k(t);\eta(0))] \\
&= \epsilon^{kd}\sum_{x_1,\dots,x_k}[Z^k_\epsilon\varphi](x_1,\dots,x_k)[S^k_t D(.;\eta(0))](x_1,\dots,x_k) \\
&= \epsilon^{kd}\sum_{x_1,\dots,x_k}[S^k_t Z^k_\epsilon\varphi](x_1,\dots,x_k)D(x_1,\dots,x_k;\eta(0)) \\
&= \epsilon^{kd}\sum_{x_1,\dots,x_k}[(Z^k_\epsilon)^{-1}S^k_t Z^k_\epsilon\varphi](\epsilon x_1,\dots,\epsilon x_k)D(x_1,\dots,x_k;\eta(0)) \\
&= \mathbb{E}_\varphi[Q(\eta(0),S^{k,\epsilon}_t\varphi)]
\end{aligned} \tag{70}$$

For the second equality we use the duality with respect to D. Next we identify operators Z^k_ϵ and S^k_t and exploit the self-adjointness of the latter. Finally, we perform an extra scaling using $(Z^k_\epsilon)^{-1}$ to fit the definition of the hydrodynamic field. Notice that the expectation $\mathbb{E}_\varphi$ in the last line has no purpose other than achieving the typical form of a duality relation.

A quick calculation shows that $[S^{k,\epsilon}_t\varphi](\epsilon x_1,\dots,\epsilon x_k) = \mathbb{E}^{IRW}_{x_1,\dots,x_k}[\varphi(\epsilon X_1,\dots,\epsilon X_k)]$. Hence, taking the mean with respect to μ_ϵ and rescaling the time yields:

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0}\int \mathbb{E}_\varphi[Q(\eta(0),S^{k,\epsilon}_{\epsilon^{-2}t}\varphi)]d\mu_\epsilon(\eta) \\
&\quad= \lim_{\epsilon\downarrow 0}\epsilon^{kd}\sum_{x_1,\dots,x_k}\mathbb{E}^{IRW}_{x_1,\dots,x_k}[\varphi(\epsilon X_1(\epsilon^{-2}t),\dots,\epsilon X_k(\epsilon^{-2}t))]\rho(\epsilon x_1,\dots,\epsilon x_k) \\
&\quad= \int \mathbb{E}^{BM}_{x_1,\dots,x_k}[\varphi(X_1(t),\dots,X_k(t))]\rho(x_1,\dots,x_k)dx_1\dots dx_k \\
&\quad= \int [Q^k_t\varphi](x_1,\dots,x_k)\rho(x_1,\dots,x_k)dx_1\dots dx_k,
\end{aligned} \tag{71}$$

where $\mathbb{E}^{BM}_{x_1,\dots,x_k}$ and Q^k_t denote expectation and Markov semi-group associated with kd-dimensional Brownian motion. On the other hand,

$$\int \mathbb{E}_\eta[Q(\eta(\epsilon^{-2}t),\varphi)]d\mu_\epsilon(\eta) = \mathbb{E}_{\mu_\epsilon}[\chi^k_\epsilon(\eta(\epsilon^{-2}t),\varphi)]. \tag{72}$$

This shows the expectational part of the hydrodynamic equation. In fact one can also control the variance of the field, which shows that the profile is deterministic [1].

Remark 3.8. Notice that in the construction of (69) we put the semigroup operator on the test function, only to bring it back to the other leg of the inner product when we have taken the limit. At first sight this step might seem redundant in the derivation of the hydrodynamic equation. However, having the semigroup on φ tends to be convenient due to the properties of test functions. In the next subsection we consider systems with interaction. Again, we will be able to derive the expectational part of the hydrodynamic equation, assuming the existence of a coupling. To effectively use the coupling the properties of φ are essential.

3.2 Interacting systems

A similar result can be derived for the symmetric exclusion (SEP) and symmetric inclusion (SIP) process, provided that one has a suitable *coupling* between the dual particles and independent particles. This entails that, in the scaling limit, the dual particles behave as if they perform independent random walks. The existence of such a coupling is natural requirement because the interaction between the particles is local, and most of the time, in the scaling limit, particles will be at large distances from one another. Moreover, one can find explicit couplings for the SEP and SIP case are provided in [4] and [13] respectively. The precise definition of a coupling is given below.

Definition 3.9 (coupling)**.** *We say that* $\{(X_1(t), ..., X_n(t); Y_1(t), ..., Y_n(t)) : t \geq 0\}$ *is a suitable coupling of* n *particles with* n *independent particles if*

1. $\{(X_1(t), ..., X_n(t)) : t \geq 0\}$ *equals in distribution the* n *particle process started from* $(X_1(0), ..., X_n(0))$*, and* $\{(Y_1(t), ..., Y_n(t)) : t \geq 0\}$ *equals in distribution* n *independent particles starting from* $(Y_1(0), ..., Y_n(0))$*.*

2. $(\epsilon X_1(\epsilon^{-2}t), ..., \epsilon X_n(\epsilon^{-2}t)) - (\epsilon Y_1(\epsilon^{-2}t), ..., \epsilon Y_n(\epsilon^{-2}t)) \to 0$ *in probability as* $\epsilon \to 0$*.*

3. *At time* $t = 0$ *the coupled particles have the same position:* $X_1(0) = Y_1(0), ..., X_k(0) = Y_k(0)$

4. *The convergence in probability occurs uniformly in* $x_1, ..., x_k \in \mathbb{Z}^d$*. That is, for all* $\delta > 0$*,*

$$\sup_{x_1,...,x_k \in \mathbb{Z}^d} \mathbb{P}^C_{x_1,...,x_k}[(\epsilon X_1(\epsilon^{-2}t), ..., \epsilon X_n(\epsilon^{-2}t)) - (\epsilon Y_1(\epsilon^{-2}t), ..., \epsilon Y_n(\epsilon^{-2}t)) \geq \delta] \to 0, \tag{73}$$

as $\epsilon \to 0$*. Here,* $\mathbb{P}^C_{x_1,...,x_k}$ *is the coupling path space measure.*

Given a coupling, the relation in (69) can be generalized in the following sense. Define

$$\mathcal{D}_{\sigma,\alpha}(x_1, ..., x_k; \eta) = D_{\sigma,\alpha}(x_1, ..., x_k; \eta)\pi_{\sigma,\alpha}(x_1, ..., x_k). \tag{74}$$

The parameter σ determines the precise system. For SIP with parameter α, we have $\sigma = 1$. Similarly we have $\sigma = 0$ and $\sigma = -1$ for the independent random walkers and SEP(α) respectively. Here, $D_{\sigma,\alpha}$ and $\pi_{\sigma,\alpha}$ are the self-duality polynomial and the reversible finite measure of the process associated with σ. We replace D by $\mathcal{D}_{\sigma,\alpha}$ in the definition of the hydrodynamic field,

$$Q_{\sigma,\alpha}(\eta, \varphi) = \chi^k_\epsilon(\eta, \varphi) = \epsilon^{kd} \sum_{x_1,...,x_k} \varphi(\epsilon x_1, ..., \epsilon x_k)\mathcal{D}_{\sigma,\alpha}(x_1, ..., x_k; \eta). \tag{75}$$

Notice that for $\sigma = 0$ we have the same definition as in the independent case. The appearance of $D_{\sigma,\alpha}$ should not be a surprise, since self-duality is central in the derivation of the relation in (69). Apart from self-duality, also the self-adjointness of the semi-group operator plays an important role. This is exactly the purpose of $\pi_{\sigma,\alpha}$. Indeed, the fact that $\pi_{\sigma,\alpha}$ is reversible immediately implies the required self adjointness in $l_2((\mathbb{Z}^d)^k)$. This measure is unique up to a multiplicative constant, hence we can choose its weight to be one whenever the k dual particles are located at different sites i.e. $\pi_{\sigma,\alpha}(x_1, ..., x_k) = 1$ whenever $\forall i, j \in \{1, ..., k\}, i \neq j : x_i \neq x_j$. We have [1]

$$\pi_{1,\alpha}(x_1, ..., x_k) = \frac{1}{\alpha^k} \prod_{x \in \mathbb{Z}^d} \frac{\Gamma(\alpha + \xi_x)}{\Gamma(\alpha)} \qquad \text{and} \qquad \pi_{-1,\alpha}(x_1, ..., x_k) = \prod_{x \in \mathbb{Z}^d} \binom{\alpha}{\xi_x}. \tag{76}$$

Theorem 3.10. *Assume that there exists a suitable coupling of k dual particles with k independent particles in the sense of Definition 3.9. Let $Q_{\sigma,\alpha}$ be as defined above and let $\varphi \in \mathbb{C}_c^\infty(\mathbb{R}^{kd})$. Then we have*

$$\mathbb{E}_{\eta(0)}[Q_{\sigma,\alpha}(\eta(t),\varphi)] = \mathbb{E}_\varphi[Q_{\sigma,\alpha}(\eta(0), S_t^\epsilon \varphi)], \tag{77}$$

where $\eta = \{\eta(t), t \geq 0\}$ denotes the process describing k labeled particles with positions $X_1(t), ..., X_k(t)$ interacting according to σ. That is $\eta(t) = \sum_{i=1} \delta_{X_i(t)}$. We define S_t as the Markov semi-group associated to process η, i.e. $(S_t\varphi)(x_1, .., x_k) = \mathbb{E}_{x_1,..,x_k}[\varphi(X_1(t), ..., X_k(t))]$. S_t^ϵ is given by $S_t^\epsilon := (Z_\epsilon^k)^{-1} S_t^{\sigma,\alpha,k} Z_\epsilon^k$.

Proof.

$$\begin{aligned}
\mathbb{E}_{\eta(0)}[Q(\eta(t),\varphi)] &= \epsilon^{kd} \sum_{x_1,...,x_k} \varphi(\epsilon x_1, ..., \epsilon x_k) \mathbb{E}_{\eta(0)}[D_{\sigma,\alpha}(x_1, ..., x_k; \eta(t))] \pi_{\sigma,\alpha}(x_1, ..., x_k) \\
&= \epsilon^{kd} \sum_{x_1,...,x_k} \varphi(\epsilon x_1, ..., \epsilon x_k) \mathbb{E}^{\sigma,\alpha}_{x_1,...,x_k}[D_{\sigma,\alpha}(X_1(t), ..., X_k(t); \eta(0))] \pi_{\sigma,\alpha}(x_1, ..., x_k) \\
&= \epsilon^{kd} \sum_{x_1,...,x_k} [Z_\epsilon^k \varphi](x_1, ..., x_k)[S_t D_{\sigma,\alpha}(.; \eta(0))](x_1, ..., x_k) \pi_{\sigma,\alpha}(x_1, ..., x_k) \\
&= \epsilon^{kd} \sum_{x_1,...,x_k} [S_t Z_\epsilon^k \varphi](x_1, ..., x_k) D_{\sigma,\alpha}(x_1, ..., x_k; \eta(0)) \pi_{\sigma,\alpha}(x_1, ..., x_k) \\
&= \epsilon^{kd} \sum_{x_1,...,x_k} [(Z_\epsilon^k)^{-1} S_t Z_\epsilon^k \varphi](\epsilon x_1, ..., \epsilon x_k) D_{\sigma,\alpha}(x_1, ..., x_k; \eta(0)) \pi_{\sigma,\alpha}(x_1, ..., x_k) \\
&= \mathbb{E}_\varphi[Q(\eta(0), S_t^\epsilon \varphi)]
\end{aligned} \tag{78}$$

As in the independent case, we use duality for the second equation after which we identify operators Z_ϵ^k and S_t. Then we use the self-adjointness of the semi-group operator and rescale via $(Z_\epsilon^k)^{-1}$. □

Now that we have generalized (69) we will take the limit $\epsilon \to 0$ to recover the hydrodynamic equation, which also holds for SEP and SIP. Indeed, it is easy to check that $[S_t^\epsilon \varphi](\epsilon x_1, ..., \epsilon x_k) = \mathbb{E}_{x_1,...,x_k}[\varphi(\epsilon X_1, ..., \epsilon X_k)]$. Taking the mean with respect to μ_ϵ and rescaling the time yields

$$\begin{aligned}
\lim_{\epsilon \downarrow 0} \int \mathbb{E}_\varphi[Q(\eta(0), & S^{k,\epsilon}_{\epsilon^{-2}t} \varphi)] d\mu_\epsilon(\eta) \\
&= \lim_{\epsilon \downarrow 0} \epsilon^{kd} \sum_{x_1,...,x_k} \mathbb{E}_{x_1,...,x_k}[\varphi(\epsilon X_1(\epsilon^{-2}t), ..., \epsilon X_k(\epsilon^{-2}t))] \rho(\epsilon x_1, ..., \epsilon x_k) \\
&= \lim_{\epsilon \downarrow 0} \epsilon^{kd} \sum_{x_1,...,x_k} \mathbb{E}^{IRW}_{x_1,...,x_k}[\varphi(\epsilon X_1(\epsilon^{-2}t), ..., \epsilon X_k(\epsilon^{-2}t))] \rho(\epsilon x_1, ..., \epsilon x_k) + O(1) \\
&= \int \mathbb{E}^{BM}_{x_1,...,x_k}[\varphi(X_1(t), ..., X_k(t))] \rho(x_1, ..., x_k) dx_1 ... dx_k \\
&= \int [Q_t^k \varphi](x_1, ..., x_k) \rho(x_1, ..., x_k) dx_1 ... dx_k.
\end{aligned} \tag{79}$$

The coupling is used to obtain the second equality, where $O(1)$ indicates a term going to zero as ϵ goes to zero. This again yields the expectational part of the hydrodynamic equation,

$$\mathbb{E}_{\mu_\epsilon}[\chi_\epsilon^k(\eta(\epsilon^{-2}t), \varphi)] = \int \mathbb{E}_\eta[Q(\eta(\epsilon^{-2}t), \varphi)] d\mu_\epsilon(\eta) \to \int \varphi(x_1, ..., x_k) \rho(t; x_1, ..., x_k) dx_1 ... dx_k. \tag{80}$$

Now let us take a moment to discuss the use of the coupling in more technical detail. Let $\mathbb{P}^C_{x_1,...,x_k}$, $\mathbb{E}^C_{x_1,...,x_k}$ be the coupling path space measure and the corresponding expectation. We use $X^{\sigma_1,\alpha}(t),...,X_k^{\sigma_1,\alpha}(t)$ and $X^{IRW}(t),...,X_k^{IRW}(t)$ to denote the interacting and independent particles respectively. For notational convenience we write $v^{\sigma,\alpha} = (\epsilon X_1^{\sigma,\alpha}(\epsilon^{-2}t),...,\epsilon X_k^{\sigma,\alpha}(\epsilon^{-2}t))$ and $v^{IRW} = (\epsilon X_1^{IRW}(\epsilon^{-2}t),...,\epsilon X_k^{IRW}(\epsilon^{-2}t))$. Since φ is a test function, its support is a compact set $K \subset \mathbb{R}^{kd}$. We will write $\epsilon^{-1}K$ to denote the set $\epsilon^{-1}K := \{x \in \mathbb{Z}^{kd} : \epsilon x \in K\}$. For each $\delta > 0$,

$$\begin{aligned}&\epsilon^{kd} \sum_{x_1,...,x_k \in \epsilon^{-1}K} \mathbb{E}^C_{x_1,...,x_k}[\varphi(v^{\sigma,\alpha}) - \varphi(v^{IRW})] \leq \\ &\epsilon^{kd} \sum_{x_1,...,x_k \in \epsilon^{-1}K} \mathbb{P}^C_{x_1,...,x_k}(||v^{\sigma,\alpha} - v^{IRW}||_{l^2} \geq \delta) \cdot ||\varphi||_\infty + \mathbb{P}^C_{x_1,...,x_k}(||v^{\sigma,\alpha} - v^{IRW}||_{l^2} < \delta) \cdot \delta ||\varphi'||_\infty.\end{aligned} \tag{81}$$

Since the coupling is assumed to converge uniformly in $x_1,...,x_k$, we can bound the first probability by a constant depending on ϵ, which we call C_ϵ, which vanishes as ϵ tends to zero.

$$\begin{aligned}\epsilon^{kd} \sum_{x_1,...,x_k \in \epsilon^{-1}K} \mathbb{P}^C_{x_1,...,x_k}(||v^{\sigma,\alpha} - v^{IRW}||_{l^2} \geq \delta) \cdot ||\varphi||_\infty &\leq \epsilon^{kd} \sum_{x_1,...,x_k \in \epsilon^{-1}K} C_\epsilon ||\varphi||_\infty \\ &\approx \epsilon^{kd} \frac{\text{vol}(K)}{\epsilon^{kd}} C_\epsilon ||\varphi||_\infty \to 0\end{aligned} \tag{82}$$

We use that the number of elements in $\epsilon^{-1}K$ is approximately equal to the volume of the kernel divided by ϵ^{kd}. Here, the volume is defined as $\text{vol}(k) := \int_{\mathbb{R}^{kd}} \mathbb{1}_{\{K\}}(x)dx$. For the second term we can simply bound $\mathbb{P}^C_{x_1,...,x_k}(||v^{\sigma,\alpha} - v^{IRW}||_{l^2} < \delta) \leq 1$, which implies also the second term can be made arbitrarily small through the choice of δ,

$$\epsilon^{kd} \sum_{x_1,...,x_k \in \epsilon^{-1}K} \mathbb{P}^C_{x_1,...,x_k}(||v^{\sigma,\alpha} - v^{IRW}||_{l^2} < \delta) \cdot \delta ||\varphi'||_\infty \leq \epsilon^{kd} \frac{\text{vol}(K)}{\epsilon^{kd}} \delta ||\varphi'||_\infty \tag{83}$$

3.3 General independent particles

Our next step is to generalize the procedure we applied for independent random walkers. We assume a general particle system on $\mathbb{Z}^d$ where the particles don't have interaction. We will also assume the existence of a duality relation, where the dual system also lives on $\mathbb{Z}^d$. However, we won't explicitly assume the state space, only that it is the same for the original process and the dual. We only require the particles to have a well-defined position on $\mathbb{Z}^d$. The state spaces remain as general as possible. As it turns out, the independence of the particles will allow us to write the corresponding semigroup operator as a k-fold tensor product of semigroup operators describing a single particle. Using standard semigroup theory it will be easy to find the PDE describing the evolution of the initial density profile $\rho(0, x)$.

3.3.1 Notation

Before we state the duality result and the hydrodynamic equation, we introduce the necessary notation. Let $\epsilon > 0$ denote the scaling parameter. For some well known examples of particle systems, like the run and tumble systems in the next subsection, it is uncircumventable to scale the system parameters to obtain a hydrodynamic limit. Hence it is convenient to assume our system itself to be ϵ-dependent. We write η^ϵ for the particle system. That is, for $x \in \mathbb{Z}^d$, $\eta^\epsilon_x(t)$ denotes the number of particles on site x at time $t \geq 0$. Once again, we aim to keep the state space as general as possible. Let $(S, \mathscr{A})$ be a measurable space, we only assume the state space to be of the form

$$\Omega^\epsilon = \mathbb{Z}^{kd} \times S^\epsilon, \tag{84}$$

where $S^\epsilon \subseteq S$ is a set which may depend on ϵ. The measures λ^ϵ have support $\text{supp}(\lambda^\epsilon) \subseteq S^\epsilon$. Moreover, we assume there is a measure λ such that

$$\lambda^\epsilon \to \lambda \text{ in total variation as } \epsilon \to 0. \tag{85}$$

All notation for the dual process will be the same as for the original process, but with the addition of a hat. Hence we have that $\hat{\eta}^\epsilon_x(t)$ denotes the number of dual particles on site x at time $t \geq 0$. To define the field it is important that the dual state space is the same as the state space of the original process, $\hat{\Omega}^\epsilon = \mathbb{Z}^{kd} \times S^\epsilon$. The duality function is called $D^\epsilon : \Omega^\epsilon \times \Omega^\epsilon \to \mathbb{R}$. For completeness we explicitly state the duality relation.

$$\mathbb{E}_{\eta^\epsilon(0)}[D^\epsilon(\hat{\eta}^\epsilon(0), \eta^\epsilon(t))] = \hat{\mathbb{E}}_{\hat{\eta}^\epsilon(0)}[D^\epsilon(\hat{\eta}^\epsilon(t), \eta^\epsilon(0))]. \tag{86}$$

The duality function gives rise to a k-th order density field, which is defined analogously to definition 3.4,

$$\chi^k_\epsilon(\eta, \varphi) = \epsilon^{kd} \int_{(S)^k} \sum_{x_1,...,x_k} \varphi(\epsilon x_1, ..., \epsilon x_k) D^\epsilon(\delta_{(x_1,s_1)} + ... + \delta_{(x_k,s_k)}; \eta) d\lambda^\epsilon(s_1)...d\lambda^\epsilon(s_k).$$

We call the associated Markov semigroup operators for systems with k particles $S^{k,\epsilon}_t$ and $\hat{S}^{k,\epsilon}_t$ for η^ϵ and $\hat{\eta}^\epsilon$ respectively. That is, for $f, g \in F(\Omega^\epsilon)$,

$$[S^{k,\epsilon}_t f](x_1, ..., x_k; s_1, ..., s_k) = \mathbb{E}_{x_1,...,x_k;s_1,...,s_k}[f(X_1(t), ..., X_k(t); S_1(t), ..., S_k(t))] \tag{87}$$

and

$$[\hat{S}^{k,\epsilon}_t g](x_1, ..., x_k; s_1, ..., s_k) = \hat{\mathbb{E}}_{x_1,...,x_k;s_1,...,s_k}[g(\hat{X}_1(t), ..., \hat{X}_k(t); \hat{S}_1(t), ..., \hat{S}_k(t))]. \tag{88}$$

Here, $X_1(t), ..., X_k(t)$ and $\hat{X}_1(t), ..., \hat{X}_k(t)$ are k instances of particles in the original and the dual system respectively, which have initial state $(x_1, s_1), ..., (x_k, s_k)$ at time zero. For notional convenience we also introduce $\mathscr{S}^\epsilon_t$ and $\hat{\mathscr{S}}^\epsilon_t$, the scaled versions of $S^{k,\epsilon}_t$ and $\hat{S}^{k,\epsilon}_t$,

$$\mathscr{S}^{k,\epsilon}_t = (Z^k_\epsilon)^{-1} S^{k,\epsilon}_{\epsilon^{-2}t} Z^k_\epsilon \quad \& \quad \hat{\mathscr{S}}^{k,\epsilon}_t = (Z^k_\epsilon)^{-1} \hat{S}^{k,\epsilon}_{\epsilon^{-2}t} Z^k_\epsilon. \tag{89}$$

Due to the scaling, we can not define these scaled operators on $F(\Omega^\epsilon)$. Instead we define them on the functions in $F(\Omega)$ with

$$\Omega = \mathbb{R}^{k,d} \times S. \tag{90}$$

We use similar notation for the (scaled) generators as we did for the semigroup operators,

$$L^{k,\epsilon} f = \lim_{t \to 0} \frac{S^{k,\epsilon}_t f - f}{t} \quad \& \quad \hat{L}^{k,\epsilon} f = \lim_{t \to 0} \frac{\hat{S}^{k,\epsilon}_t f - f}{t} \tag{91}$$

and

$$\mathscr{L}^{k,\epsilon} = \lim_{t \to 0} \frac{\mathscr{S}^{k,\epsilon}_t f - f}{t} \quad \& \quad \hat{\mathscr{L}}^{k,\epsilon} = \lim_{t \to 0} \frac{\hat{\mathscr{S}}^{k,\epsilon}_t f - f}{t}. \tag{92}$$

on the functions $f \in F(\Omega)$ for which the above limits exist.

We work here under the assumption that the hydrodynamic limit exists for this general process. We denote the limiting semigroup operator $\mathscr{S}_t^k$ and the limiting generator $\mathscr{L}^k$, i.e. $\lim_{\epsilon\to 0}\mathscr{L}^{k,\epsilon} = \mathscr{L}^k$ and $\lim_{\epsilon\to 0}\mathscr{S}_t^{k,\epsilon} = \mathscr{S}_t^k$, where $\mathscr{L}^k$ generates $\mathscr{S}_t^k$. The limits here are in the following sense. For family of functions $\{f_\epsilon : f_\epsilon \in D(\mathscr{L}^{k,\epsilon})\}_{\epsilon>0}$ and $f \in D(\mathscr{L}^k)$ such that $\lim_{\epsilon\to 0} f_\epsilon = f$, we have that $\mathscr{L}^{k,\epsilon} f_\epsilon \to \mathscr{L}^k f$. In case of the semigroup operator we have for each $g \in F(\Omega)$, $\hat{S}_t^{k,\epsilon} g \to \hat{S}_t^k g$. In fact, the Trotter-Kurtz theorem provides the convergence of the semigroup operators whenever we have convergence of the generators, and the other way around.

A usual (unscaled) adjoint, with respect to the innerproduct on $L^2(\mathbb{Z}^{kd} \times (S^\epsilon)^k)$ or the innerproduct on $L^2(\mathbb{R}^{kd} \times S^k)$, is indicated using a $*$. We also need the scaled adjoint of $S_t^{k,\epsilon}$ and $L^{k,\epsilon}$, which we will equip with a $\star$.

$$(\mathscr{S}_t^{k,\epsilon})^\star = (Z_\epsilon^k)^{-1}(S_{\epsilon^{-2}t}^{k,\epsilon})^* Z_\epsilon \quad \& \quad (\mathscr{L}^{k,\epsilon})^\star = (Z_\epsilon^k)^{-1}(L^{k,\epsilon})^* Z_\epsilon \tag{93}$$

Finally we introduce a notation for a tensor product of operators where each operator is the identity, except operator number l, which is some operator A:

$$(A)_l = I^{\otimes l-1} \otimes A \otimes I^{\otimes k-l} \tag{94}$$

3.3.2 Statement

Theorem 3.11. *In the setting described above we have*

$$\mathbb{E}_{\eta^\epsilon(0)}[Q(\eta^\epsilon(t),\varphi)] = \mathbb{E}_\varphi[Q(\eta^\epsilon(0),(Z_\epsilon^k)^{-1}(\hat{S}_t^{k,\epsilon})^* Z_\epsilon^k\varphi)], \tag{95}$$

where $Q(\eta,\varphi) = \chi_\epsilon^k(\eta^\epsilon,\varphi)$ and $\varphi \in C_c^\infty(\Omega)$. It is understood that the scaling operator Z_ϵ only works on the spatial part of φ: $Z_\epsilon^k\varphi(x,s) = \varphi(\epsilon x, s)$. Moreover, for $\{\mu_\epsilon, \epsilon > 0\}$ a family of probability measures consistent with a profile $\rho_0 \in F(\Omega)$,

$$\int Q(\eta^\epsilon(t),\varphi)d\mu_\epsilon(\eta^\epsilon(0)) \xrightarrow{\epsilon\to 0} \int_{\mathbb{R}^{kd}}\int_{(S)^k} \rho(t,x_1,...,x_k,s_1,...,s_k)\varphi(x_1,...,x_k,s_1,...,s_k)d\lambda(s)dx. \tag{96}$$

$\rho(t,x_1,...,x_k)$ is the solution of the following differential equation,

$$\begin{cases} \frac{\partial\rho}{\partial t} = \sum_{l=1}^{k}(\hat{\mathscr{L}}^1)_l\rho \\ \rho(0,x) = \rho_0(x). \end{cases} \tag{97}$$

Proof. The theorem consists of three parts: the macroscopic duality, the hydrodynamic limit and the characterization in terms of a PDE. These correspond to equations 95, 96 and 158 respectively.

Macroscopic duality

The proof of the first part carries over verbatim from the proof of (69) in the case of independent random walkers,

$$\begin{aligned}
\mathbb{E}_{\eta^\epsilon(0)}[Q(\eta^\epsilon(t),\varphi)] &= \epsilon^{kd} \sum_{x\in\mathbb{Z}^{kd}} \int_{(S^\epsilon)^k} \varphi(\epsilon x,s)\mathbb{E}_{\eta^\epsilon(0)}[D^\epsilon(x,s;\eta^\epsilon(t))]d\lambda(s) \qquad (98)\\
&= \epsilon^{kd} \sum_{x\in\mathbb{Z}^{kd}} \int_{(S^\epsilon)^k} \varphi(\epsilon x,s)\hat{\mathbb{E}}_{x,s}[D^\epsilon(X_1(t),...,X_k(t),S_1(t),...,S_k(t);\eta^\epsilon(0))]d\lambda(s)\\
&= \epsilon^{kd} \sum_{x\in\mathbb{Z}^{kd}} \int_{(S^\epsilon)^k} [Z^k_\epsilon\varphi](x,s)[\hat{S}^{k,\epsilon}_t D^\epsilon(.;\eta^\epsilon(0))](x,s)d\lambda(s)\\
&= \epsilon^{kd} \sum_{x\in\mathbb{Z}^{kd}} \int_{(S^\epsilon)^k} [(\hat{S}^{k,\epsilon}_t)^* Z^k_\epsilon\varphi](x,s)D^\epsilon(x,s;\eta^\epsilon(0))d\lambda(s)\\
&= \epsilon^{kd} \sum_{x\in\mathbb{Z}^{kd}} \int_{(S^\epsilon)^k} [(Z^k_\epsilon)^{-1}(\hat{S}^{k,\epsilon}_t)^* Z^k_\epsilon\varphi](\epsilon x)D^\epsilon(x,s;\eta(0))d\lambda(s)\\
&= \mathbb{E}_\varphi[Q(\eta^\epsilon(0),(Z^k_\epsilon)^{-1}(\hat{S}^{k,\epsilon}_t)^* Z_\epsilon\varphi)].
\end{aligned}$$

For the second equality we use the duality. After that we identify the scaling and semigroup operator. Next, we interpret the sum over x and integral over $(S^\epsilon)^k$ as an inner product and put the semigroup operator on the scaled test function.

Hydrodynamic limit

To show the second part, we take the limit of the mean with respect to μ_ϵ.

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0}\int \mathbb{E}_\varphi[Q(\eta^\epsilon(0),(Z^k_\epsilon)^{-1}(\hat{S}^{k,\epsilon}_{\epsilon^{-2}t})^* Z_\epsilon\varphi)]d\mu_\epsilon(\eta) \qquad (99)\\
&= \lim_{\epsilon\downarrow 0}\epsilon^{kd} \sum_{x\in\mathbb{Z}^{kd}} \int_{(S^\epsilon)^k} [(\hat{\mathscr{S}}^{k,\epsilon}_t)^\star\varphi](\epsilon x,s)\rho(0;\epsilon x,s)d\lambda^\epsilon(s)\\
&= \lim_{\epsilon\downarrow 0}\epsilon^{kd} \sum_{x\in\mathbb{Z}^{kd}} \int_{(S)^k} [(\hat{\mathscr{S}}^{k}_t)^*\varphi](\epsilon x,s)\rho(0;\epsilon x)d\lambda(s) + o(1)\\
&= \int\int_{(S)^k} [(\hat{\mathscr{S}}^{k}_t)^*\varphi](x,s)\rho(0;x,s)d\lambda(s)dx\\
&= \int\int_{(S)^k} \varphi(x,s)[\hat{\mathscr{S}}^{k}_t\rho(0,\cdot)](x,s)d\lambda(s)d\lambda(s)dx
\end{aligned}$$

In second step we use both that $(\hat{\mathscr{S}}^{k,\epsilon}_t)^\star$ converges to $(\hat{\mathscr{S}}^{k}_t)^*$, which is easily deduced from the convergence of $\hat{\mathscr{S}}^{k,\epsilon}_t$, and that $\lambda^\epsilon \to \lambda$. We make this step rigorous. By the convergence of λ^ϵ in total variation we can replace $d\lambda^\epsilon(s)$ by $d\lambda(s)$,

$$\begin{aligned}
&\int_{S^k}\epsilon^{kd} \sum_{x\in\mathbb{Z}^{kd}} [(\hat{\mathscr{S}}^{k,\epsilon_1}_t)^*\varphi(\epsilon x,s)\rho(0;\epsilon x,s)d\lambda^\epsilon(s) = \qquad (100)\\
&\qquad \int_{(S)^k} \sum_{x\in\mathbb{Z}^{kd}} [(\hat{\mathscr{S}}^{k,\epsilon_1}_t)^*\varphi(\epsilon x,s)\rho(0;\epsilon x_1,...,\epsilon x_k)d\lambda(s)\Big| + O(1).
\end{aligned}$$

Due to the Trotter-Kurtz theorem, the convergence $\mathscr{L}^{k,\epsilon} \to \mathscr{L}^k$ implies that

$$||(\hat{\mathscr{S}}_t^k)^*\rho - (\hat{\mathscr{S}}_t^{k,\epsilon})^\star\rho||_\infty \to 0 \text{ as } \epsilon \to 0. \tag{101}$$

and hence

$$|\epsilon^{kd} \sum_{x_1,\dots,x_k} \int_{(S)^k} [(\hat{\mathscr{S}}_t^k)^*\varphi - (\hat{\mathscr{S}}_t^{k,\epsilon})^\star\varphi](\epsilon x_1,\dots,\epsilon x_k)\rho(0;\epsilon x_1,\dots,\epsilon x_k)| \tag{102}$$

$$\leq \epsilon^{kd} \sum_{x_1,\dots,x_k} \int_{(S)^k} |\varphi(\epsilon x_1,\dots,\epsilon x_k)| \cdot |[(\hat{\mathscr{S}}_t^k)^*\rho - (\hat{\mathscr{S}}_t^{k,\epsilon})^\star\rho](0;\epsilon x_1,\dots,\epsilon x_k)|d\lambda(s)$$

$$\leq \epsilon^{kd} \sum_{x_1,\dots,x_k} \int_{(S)^k} |\varphi(\epsilon x_1,\dots,\epsilon x_k)| \cdot ||(\hat{\mathscr{S}}_t^k)^*\rho - (\hat{\mathscr{S}}_t^{k,\epsilon})^\star\rho||_\infty d\lambda(s) = O(1)$$

The limiting PDE

We can now characterize $[\hat{\mathscr{S}}_t^k\rho](0;x_1,...,x_k)$ as the solution of $\frac{\partial\rho}{\partial t} = \mathscr{L}^k\rho$. moreover, due to the independence we can rewrite this in terms of one-particle generators. The argument is as follows. It is a well known fact that for measure spaces $(\Sigma_1,\mu_1),...,(\Sigma_N,\mu_N)$, $L^2(\Sigma_1,...,\Sigma_N,\mu_1,...,\mu_N) = L^2(\Sigma_1,\mu_1) \otimes ... \otimes L^2(\Sigma_N,\mu_N)$. Hence in our case $L^2(\Omega^k,\lambda^k) = L^2(\Omega,\lambda)^{\otimes k}$. Furthermore, the basis of this tensor space consists of functions of the form $f(x_1,...,x_k;s_1,...,s_k) = f_1(x_1,s_1) \otimes f_2(x_2,s_2) \otimes ... \otimes f_k(x_k,s_k) = f_1(x_1,s_1)f_2(x_2,s_2)...f_k(x_k,s_k)$ with $f_1,...,f_k$ basis functions for $L^2(\Omega,\lambda)$. We observe that

$$[\hat{\mathscr{S}}_t^k f_1...f_k](x_1,...,x_k;s_1,...,s_k) = \hat{\mathbb{E}}_{x_1,...,x_k}[f_1(\hat{X}_1(t),\hat{S}_1(t))...f_k(\hat{X}_k(t),\hat{S}_k(t))] \tag{103}$$

$$= \mathbb{E}_{x_1}[f(\hat{X}_1(t),\hat{S}_1(t))]...\mathbb{E}_{x_k}[f(\hat{X}_k(t),\hat{S}_k(t))]$$

$$= [(\hat{\mathscr{S}}_t^1)^{\otimes k} f_1...f_k](x_1,...,x_k). \tag{104}$$

Hence, due to the independence of the particles we can interpret $\hat{\mathscr{S}}_t^k$ as a tensor product when acting on a basis of $L^2(\mathbb{R}^{kd})$. Since φ is a test function it is in $L^2(\mathbb{R}^{kd})$. This yields

$$\lim_{\epsilon\downarrow 0} \int \mathbb{E}_\varphi[Q(\eta^\epsilon(0),(Z_\epsilon^k)^{-1}(\hat{S}_{\epsilon^{-2}t}^{k,\epsilon})^* Z_\epsilon\varphi)]d\mu_\epsilon(\eta) \tag{105}$$

$$= \int\int_{(S)^k} [(\hat{\mathscr{S}}_t^k)^*\varphi](x,s)\rho(0;x,s)d\lambda(s)dx$$

$$= \int\int_{(S)^k} [(\hat{\mathscr{S}}_t^{1*})^{\otimes k}\varphi](x,s)\rho(0;x,s)d\lambda(s)dx$$

$$= \int\int_{(S)^k} \varphi(x,s)[(\hat{\mathscr{S}}_t^1)^{\otimes k}\rho(0,\cdot)](x,s)d\lambda(s)dx.$$

Now we can use the chain rule to see that

$$\frac{\partial}{\partial t}[(\hat{\mathscr{S}}_t^1)^{\otimes k} f_1...f_k] = (\frac{\partial}{\partial t}\hat{\mathscr{S}}_t^1 f_1)(\hat{\mathscr{S}}_t^1 f_2)...(\hat{\mathscr{S}}_t^1 f_k) \tag{106}$$

$$+ (\hat{\mathscr{S}}_t^1 f_1)(\frac{\partial}{\partial t}\hat{\mathscr{S}}_t^1 f_2)...(\hat{\mathscr{S}}_t^1 f_k)$$

$$+ (\hat{\mathscr{S}}_t^1 f_1)(\hat{\mathscr{S}}_t^1 f_2)...(\frac{\partial}{\partial t}\hat{\mathscr{S}}_t^1 f_k)$$

$$= \sum_{l=1}^k (\mathscr{L}^1)_l(\hat{\mathscr{S}}_t^1 f_1)(\hat{\mathscr{S}}_t^1 f_2)...(\hat{\mathscr{S}}_t^1 f_k).$$

This concludes the proof. □

3.3.3 Alternative combinatorial approach

Due to a combinatorial property of the self-duality function associated with the independent random walkers,

$$D^{IRW}(\eta,\xi) = \prod_x \frac{\eta_x!}{(\eta_x-\xi_x)!}, \tag{107}$$

we can construct a similar kind of hydrodynamic limit as we did in the previous subsection. The property in question is the following.

Lemma 3.12. *Suppose that the number of particles, which we call l, in process η is conserved and finite. We label all these particles using numbers $1,...,l$ and for $i \in \{1,...,l\}$ we denote the position of particle i at time t as $X_i(t)$. Then*

$$\sum_{x_1,...,x_k} \varphi(x_1,...,x_k) D^{IRW}(x_1,...,x_k;\eta(t)) = \sum_{1\le i_1,...,i_k \le l}^{\neq} \varphi(X_{i_1}(t),...,X_{i_k}(t)), \tag{108}$$

where $\neq$ indicates that $\forall m,n \in \{1,...,k\}, m \neq n : i_m \neq i_n$.

Proof. To show this we can use induction.

Notice that for $x \in \mathbb{R}^d$ and $t > 0$,

$$\eta_x(t) = \sum_{1\le i\le l} \mathbb{1}\{X_i(t) = x\}. \tag{109}$$

The case $k = 1$ indeed holds,

$$\begin{aligned}\sum_x \varphi(x) D^{IRW}(x,\eta(t)) &= \sum_x \varphi(x)\eta_x(t) \\ &= \sum_x \varphi(x) \sum_{1\le i\le l} \mathbb{1}\{X_i(t) = x\} \\ &= \sum_{1\le i\le l} \sum_x \varphi(x)\mathbb{1}\{X_i(t) = x\} \\ &= \sum_{1\le i\le l} \varphi(X_i(t)).\end{aligned} \tag{110}$$

We proceed to the induction step. Assume the statement holds for some k, then

$$
\begin{aligned}
&\sum_{x_1,...,x_k,x_{k+1}} \varphi(x_1,...,x_k,x_{k+1}) D^{IRW}(x_1,...,x_k,x_{k+1};\eta(t)) \qquad (111)\\
&= \sum_{x_{k+1}} \sum_{x_1,...,x_k,x_{k+1}} \varphi(x_1,...,x_k,x_{k+1}) D^{IRW}(x_1,...,x_k;\eta(t))(\eta_{x_{k+1}}(t) - \sum_{j=1}^{k} \delta(x_{k+1},x_j))\\
&= \sum_{x_{k+1}} \sum_{1\le i_1,...,i_k\le l}^{\neq} \varphi(X_{i_1}(t),...,X_{i_k}(t),x_{k+1})(\eta_{x_{k+1}}(t) - \sum_{j=1}^{k} \delta(x_{k+1},X_{i_j}(t)))\\
&= \sum_{x_{k+1}} \sum_{1\le i_1,...,i_k\le l}^{\neq} \varphi(X_{i_1}(t),...,X_{i_k}(t),x_{k+1})\Big[\sum_{1\le i_{k+1}\le l} \mathbb{1}\{X_{i_{k+1}}(t) = x_{k+1}\} - \sum_{j=1}^{k} \delta(x_{k+1},X_{i_j}(t)\Big]\\
&= \sum_{1\le i_1,...,i_{k+1}\le l}^{\neq} \varphi(X_{i_1}(t),...,X_{i_{k+1}}(t))
\end{aligned}
$$

.

Hence we can conclude the statement holds for all k. □

Using this fact, it is not difficult to show the following duality result.

Theorem 3.13. *Let the notation be as described in the previous subsection. For*

$$Q(\eta,\varphi) = \epsilon^{kd} \sum_{x_1,...,x_k} \varphi(\epsilon x_1,...,\epsilon x_k) D^{IRW}(x_1,...,x_k;\eta), \qquad (112)$$

we have

$$\mathbb{E}_{\eta^\epsilon(0)}[Q(\eta^\epsilon(t),\varphi)] = \mathbb{E}_\varphi[Q(\eta^\epsilon(0),(Z^k_\epsilon)^{-1}(S^{k,\epsilon}_t)Z^k_\epsilon\varphi)], \qquad (113)$$

Proof.

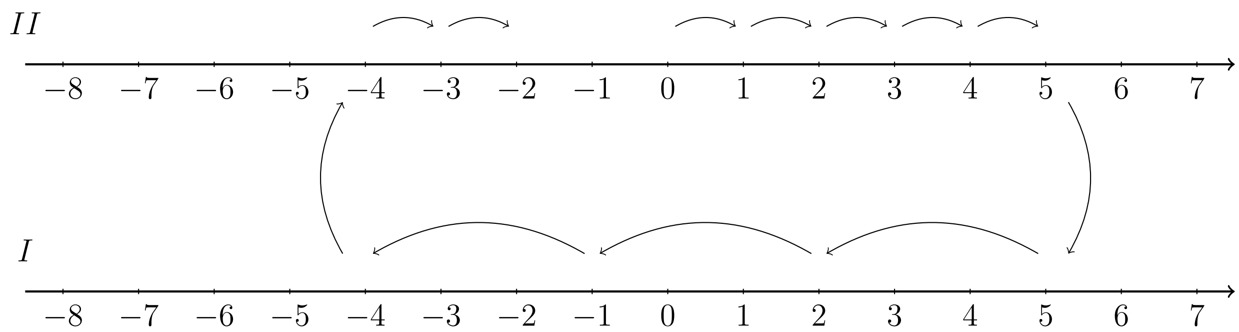
$$
\begin{aligned}
\mathbb{E}_{\eta^\epsilon(0)}[Q(\eta^\epsilon(t),\varphi)] &= \mathbb{E}_{\eta^\epsilon(0)}\Big[\epsilon^{kd} \sum_{x_1,...,x_k} \varphi(\epsilon x_1,...,\epsilon x_k) D^{IRW}(x_1,...,x_k;\eta)\Big] \qquad (114)\\
&= \mathbb{E}_{\eta^\epsilon(0)}\Big[\epsilon^{kd} \sum_{1\le i_1,...,i_{k+1}\le l}^{\neq} \varphi(\epsilon X_{i_1}(t),...,\epsilon X_{i_{k+1}}(t))\Big]\\
&= \epsilon^{kd} \sum_{1\le i_1,...,i_{k+1}\le l}^{\neq} \Big[(Z^k_\epsilon)^{-1} S^{k,\epsilon}_t Z^k_\epsilon \varphi\ \Big](\epsilon x_{i_1},...,\epsilon x_{i_k})\\
&= \epsilon^{kd} \sum_{x_1,...,x_k} \Big[(Z^k_\epsilon)^{-1} S^{k,\epsilon}_t Z^k_\epsilon \varphi\ \Big](\epsilon x_1,...,\epsilon x_k) D(x_1,...,x_k;\eta(0))
\end{aligned}
$$

□

4 Run-and-tumble processes

In this chapter we apply the generalized procedure from the previous chapter to so called *run-and-tumble* processes. Run-and-tumble particles are particles whose motion is a combination of a random walk and a directed motion according to an internal state. We illustrate this using an example.

Imagine a particle moving on the set of integers. That is, it jumps from one site to another at random times. The particle has two possible internal states. In state one the particle hops three units to the left and in state two it hops one unit to right. The particle also occasionally changes its internal state. This occurs again at randomly distributed times. One can visualize this process space using two copies of the integer line.



In this instance, the particle starts of at 0 with internal state two, and jumps five times before switching to state one, where it makes three jumps before switching back two state two. Due to this representation, the internal states are also commonly referred to as *layers*. Since physical particles are often subject to noise from the environment, we also add a random walk type of motion. This models random jumps due to environmental fluctuations. In this setting that would entail the particle also hopping symmetrically to the left and to the right. In order to have the Markov property, all jumping times are assumed to be exponentially distributed.

4.1 Definition

We adopt our terminology and notation from [17]. Let η denote a system of run-and-tumble particles which live on $\mathbb{Z}^d$, $d \in \mathbb{N}$. The associated state space is $\Omega = \mathbb{N}_0^{\mathbb{Z}^d \times S}$. Here, $S \subset \mathbb{N}_0^d$ contains the "velocities" of the particles associated to their internal state (in the introductory example the set S is $\{-3, 1\}$). Every $\sigma \in S$ corresponds to a layer where a particle at x will jump to $x + \sigma$ due to its internal state. One could in principle define the particles on any countable set G instead of $\mathbb{Z}^d$. However, we will for the most part focus on scaling limits, hence $\mathbb{Z}^d$ is the most natural choice. Unless stated otherwise, S is assumed to be a finite set. We write $\eta_{x,\sigma}(t)$ for the number of particles at site x, with internal state σ at time $t \geq 0$.

The generator of the process is given by L acting on the local functions [1] on Ω,

$$Lf = \lambda L_a f + \gamma L_i f + \kappa \mathcal{L} f \tag{115}$$

Here $\lambda, \gamma, \kappa \in \mathbb{R}_{\geq 0}$ are constants while L_a, L_i and $\mathcal{L}$ are the Markov generators associated to the three types of state changes the particles can experience.

[1] In this setting, the local functions are functions depending on a finite number of coordinates in $\mathbb{Z}^d \times S$.

L_a is the *active* part of the generator. It corresponds to the jumps from initial position (x, σ) to $(x + \sigma, \sigma)$ on the same layer, i.e. the jumps according to the internal state. These jumps make a particle move in a single direction, which one could interpret it as a "run". The generator is defined as

$$L_a f(\eta) = \sum_{x,\sigma} \eta_{x,\sigma}[f(\eta - \delta_{(x,\sigma)} + \delta_{(x+\sigma,\sigma)}) - f(\eta)]. \tag{116}$$

L_i is the *internal* part of the generator. It provides the change of the internal state. Due to L_i, a particle at (x, σ) may change its internal state from σ to σ' and ends up at (x, σ'). The change of internal state can be likened to the particle "tumbling" from one layer to another.

$$L_i f(\eta) = \sum_{x,\sigma\neq\sigma'} \eta_{x,\sigma} c_x(\sigma, \sigma')[f(\eta - \delta_{(x,\sigma)} + \delta_{(x,\sigma')}) - f(\eta)] \tag{117}$$

The rates $c_x(\sigma, \sigma')$ determine how often a particle on the σ layer jumps to the σ' layer. In the upcoming analysis we will use a duality result. It turns out that the result in question is only valid for rates which are symmetric, i.e. $c_x(\sigma, \sigma') = c_x(\sigma', \sigma)$. Therefore, symmetry will be assumed from now on. For generality we allow these rates to be x-dependent. However, this will significantly complicate the scaling procedure, as we will later discuss in more depth.

$\mathcal{L}$ is the random walk part of the generator.

$$\mathcal{L} f(\eta) = \sum_{x\neq y,\sigma} \eta_{x,\sigma} c(x, y)[f(\eta - \delta_{(x,\sigma)} + \delta_{(y,\sigma)}) - f(\eta)] \tag{118}$$

We have rates $c(x, y)$ to restrict or encourage a jump from site x to site y. Again we have to assume symmetry in order to have a duality result later on. Furthermore, we require that $\sup_x \sum_y c(x, y) < \infty$ and $c(x, y) = 0$ whenever $|x - y| > R$ for some real number R. Unless stated otherwise we will assume a d-dimensional simple random walk. More precise, for two different sites x and y, we assume $c(x, y) = (2d)^{-1}$ whenever $||x - y|| = 1$ and $c(x, y) = 0$ otherwise.

4.2 Duality

As mentioned before, there is a duality result [17] for the process defined above. We first introduce the dual operator and the duality function.

Let $\hat{L}$, acting on the local functions on Ω, be the following Markov generator,

$$\hat{L} f = \lambda \hat{L}_a f + \gamma L_i f + \kappa \mathcal{L} f. \tag{119}$$

Here, $\hat{L}_a$ denotes generator which sends particles in the opposite direction as L_a, that is,

$$\hat{L}_a f(\eta) = \sum_{x,\sigma} \eta_{x,\sigma}[f(\eta - \delta_{(x,\sigma)} + \delta_{(x-\sigma,\sigma)}) - f(\eta)]. \tag{120}$$

The duality function will, analogous to the self-duality function for the independent random walkers, be defined as a product of functions on $\mathbb{N} \times \mathbb{N}$ of the form

$$d(k, n) := \begin{cases} \frac{n!}{(n-k)!} & \text{for } n \geq k \\ 0 & \text{for } n < k. \end{cases} \tag{121}$$

More precise, we have $\mathscr{D} : \Omega \times \Omega \to \mathbb{R}$,

$$\mathscr{D}(\xi, \eta) = \prod_{(x,\sigma)\in\mathbb{Z}\times S} d(\xi_{x,\sigma}, \eta_{x,\sigma}), \tag{122}$$

as a duality function. The following theorem states the duality of L and $\hat{L}$ with respect to $\mathscr{D}$. This result, as well as the subsequent lemma are directly adopted from [17].

Theorem 4.1. *Let L, $\hat{L}$ and $\mathscr{D}$ be as described above. L and $\hat{L}$ are dual with respect to $\mathscr{D}$, i.e., for η and ξ processes generated by L and $\hat{L}$ respectively,*

$$[L\mathscr{D}(\xi(0), .)](\eta(t)) = [\hat{L}\mathscr{D}(., \eta(0))](\xi(t)). \tag{123}$$

Moreover, for the sub-generators we have the following duality results with respect to $\mathscr{D}$,

1. *L_a is dual to $\hat{L}_a$*
2. *L_i and $\mathcal{L}$ are self-dual.*

We show the duality results for each part of the generator separately. The proofs are mainly computational and rely heavily on the identity below.

Lemma 4.2. *Let $k, l, m, n \in \mathbb{N}$. The following identity holds,*

$$k\frac{d(k-1,m)d(l+1,n)}{d(k,m)d(l,n)} - n\frac{d(k,m+1)d(l,n-1)}{d(k,m)d(l,n)} = l - n, \tag{124}$$

where d denotes the function defined above in (121).

Proof.

$$\begin{aligned} k\frac{d(k-1,m)d(l+1,n)}{d(k,m)d(l,n)} - n\frac{d(k,m+1)d(l,n-1)}{d(k,m)d(l,n)} & \\ &= \frac{k\frac{m!\cdot n!}{(m-k+1)!(n-l-1)!} - n\frac{(m+1)!(n-1)!}{(m-k+1)!(n-l-1)!}}{\frac{m!\cdot n!}{(m-k)!(n-l)!}} \\ &= (k-m-1)\cdot\frac{n-l}{m-k+1} \\ &= l-n \end{aligned} \tag{125}$$

□

4.2.1 Duality for the active generator

We prove the duality for the active part of the generator first. That is, we show,

$$[L_a\mathscr{D}(\xi, \cdot)](\eta) = [\hat{L}_a\mathscr{D}(\cdot, \eta)](\eta). \tag{126}$$

Since L_a is defined on the local functions, we can restrict ourselves to finite configurations $\xi \in \Omega$. The active part only works on a single layer, hence for each $\sigma \in S$ we show

$$\sum_x \eta(x,\sigma)[\mathscr{D}(\xi, \eta^{(x,\sigma),(x+\sigma,\sigma)}) - \mathscr{D}(\xi, \eta)] = \sum_x \xi(x,\sigma)[\mathscr{D}(\xi^{(x,\sigma),(x-\sigma,\sigma)}, \eta) - \mathscr{D}(\xi, \eta)]. \tag{127}$$

We consider two cases, namely $\mathscr{D}(\xi, \eta) > 0$ and $\mathscr{D}(\xi, \eta) = 0$.

When $\mathscr{D}(\xi,\eta) > 0$, we can alternatively show,

$$\sum_x \eta(x,\sigma)\frac{\mathscr{D}(\xi,\eta^{(x,\sigma)(x+\sigma,\sigma)})}{\mathscr{D}(\xi,\eta)} - \xi(x+\sigma,\sigma)\frac{\mathscr{D}(\xi^{(x+\sigma,\sigma)(x,\sigma)}),\eta}{\mathscr{D}(\xi,\eta)} + \xi(x+\sigma,\sigma) - \eta(x,\sigma) = 0. \quad (128)$$

Here, we shift the terms in the sum on the right-hand side of (127) by σ. We also divide by $\mathscr{D}(\xi,\eta)$, which is strictly larger then zero, and bring everything to the left side. It is easy to see that for all $x \in \mathbb{Z}^d$ we have $\mathscr{D}(\xi,\eta^{(x,\sigma)(x+\sigma,\sigma)}) = 0$ if and only if $\mathscr{D}(\xi^{(x+\sigma,\sigma)(x,\sigma)},\eta) = 0$. In case we indeed have $\mathscr{D}(\xi,\eta^{(y,\sigma)(y+\sigma,\sigma)}) = \mathscr{D}(\xi^{(y+\sigma,\sigma)(y,\sigma)},\eta) = 0$ for some $y \in \mathbb{Z}^d$, we can derive $\xi(y,\sigma) = \eta(y,\sigma)$. Indeed, the assumption $\mathscr{D} > 0$ gives $\xi(x,\sigma) \le \eta(x,\sigma)$ for all $x \in \mathbb{Z}^d$ and $\sigma \in S$. on the contrary, $\mathscr{D}(\xi^{(y+\sigma,\sigma)(y,\sigma)},\eta) = 0$ yields $\xi(y,\sigma) + 1 > \eta(y,\sigma)$. Combining these gives the equality. As a consequence,

$$\begin{aligned}\eta(y,\sigma)\frac{\mathscr{D}(\xi,\eta^{(y,\sigma)(y+\sigma,\sigma)})}{\mathscr{D}(\xi,\eta)} - \xi(y+\sigma,\sigma)\frac{\mathscr{D}(\xi^{(y+\sigma,\sigma)(y,\sigma)},\eta)}{\mathscr{D}(\xi,\eta)} &+ \xi(y+\sigma,\sigma) - \eta(y,\sigma) \\ &= \xi(y+\sigma,\sigma) - \xi(y,\sigma).\end{aligned} \quad (129)$$

Similarly, it is also true that $\mathscr{D}(\xi,\eta^{(z,\sigma)(z+\sigma,\sigma)}) > 0$ implies $\mathscr{D}(\xi^{(z+\sigma,\sigma)(z,\sigma)},\eta) > 0$. For notational convenience we write

$$\begin{aligned}&\xi(z,\sigma) = k, && \eta(z,\sigma) = m \\ &\xi(z+\sigma,\sigma) = l, && \eta(z+\sigma,\sigma) = n\end{aligned} \quad (130)$$

Lemma 4.2 gives us a similar equation as in the case where $\mathscr{D}(\xi,\eta^{(z,\sigma)(z+\sigma,\sigma)}) = 0$,

$$\begin{aligned}\eta(y,\sigma)\frac{\mathscr{D}(\xi,\eta^{(y,\sigma)(y+\sigma,\sigma)})}{\mathscr{D}(\xi,\eta)} - \xi(y+\sigma,\sigma)\frac{\mathscr{D}(\xi^{(y+\sigma,\sigma)(y,\sigma)},\eta)}{\mathscr{D}(\xi,\eta)} & \quad (131) \\ = m\frac{d(k,m-1)d(l,n+1)}{d(k,m)d(l,n)} - l\frac{d(k+1,m)d(l-1,n)}{d(k,m)d(l,n)} & \\ = m - k & \\ = \eta(z,\sigma) - \xi(z,\sigma) & \quad (132)\end{aligned}$$

which implies

$$\begin{aligned}\eta(y,\sigma)\frac{\mathscr{D}(\xi,\eta^{(y,\sigma)(y+\sigma,\sigma)})}{\mathscr{D}(\xi,\eta)} - \xi(y+\sigma,\sigma)\frac{\mathscr{D}(\xi^{(y+\sigma,\sigma)(y,\sigma)},\eta)}{\mathscr{D}(\xi,\eta)} &+ \xi(z+\sigma,\sigma) - \eta(z,\sigma) \\ = \eta(z,\sigma) - \xi(z,\sigma) + \xi(z+\sigma,\sigma) - \eta(z,\sigma) & \\ = \xi(z+\sigma,\sigma) - \xi(z,\sigma) &\end{aligned}$$

This gives the duality we are after

$$\begin{aligned}\sum_x \eta(x,\sigma)\frac{\mathscr{D}(\xi,\eta^{(x,\sigma)(x+\sigma,\sigma)})}{\mathscr{D}(\xi,\eta)} - \xi(x+\sigma,\sigma)\frac{\mathscr{D}(\xi^{(x+\sigma,\sigma)(x,\sigma)},\eta)}{\mathscr{D}(\xi,\eta)} &+ \xi(x+\sigma,\sigma) - \eta(x,\sigma) \\ = \sum_x \xi(x+\sigma,\sigma) - \xi(x,\sigma) & \\ = 0. & \quad (133)\end{aligned}$$

Notice that for the last equality, we use that ξ is a finite configuration.

Next we look at the case where $\mathscr{D}(\xi,\eta)=0$. We show that, regardless of $x\in\mathbb{Z}^d$, the following two terms cancel.

$$\eta(x,\sigma)\mathscr{D}(\xi,\eta^{(x,\sigma),(x+\sigma,\sigma)})-\xi(x+\sigma,\sigma)\mathscr{D}(\xi^{(x+\sigma,\sigma),(x,\sigma)},\eta)=0 \tag{134}$$

This immediately implies (127). Again, $\mathscr{D}(\xi,\eta^{(x,\sigma)(x+\sigma,\sigma)})=0$ if and only if $\mathscr{D}(\xi^{(x+\sigma,\sigma)(x,\sigma)},\eta)=0$. In case both terms are zero the equation trivially holds. Assume that $\mathscr{D}(\xi^{(x+\sigma,\sigma),(x,\sigma)},\eta)>0$. Once more, we use k,l,m,n to facilitate notation,

$$\begin{aligned} \xi(x,\sigma)&=k, & \eta(x,\sigma)&=m \\ \xi(x+\sigma,\sigma)&=l, & \eta(x+\sigma,\sigma)&=n. \end{aligned} \tag{135}$$

In terms of k,l,m,n, we to show

$$m\cdot d(k,m-1)d(l,n+1)-l\cdot d(k+1,m)d(l-1,n)=0. \tag{136}$$

From $\mathscr{D}(\xi^{(x+\sigma,\sigma),(x,\sigma)},\eta)>0$ we have $\xi(x+\sigma,\sigma)-1\le\eta(x,\sigma)$. Furthermore, we also have $\xi(x+\sigma,\sigma)>\eta(x,\sigma)$. Indeed this follows from $\mathscr{D}(\xi,\eta)=0$. Combining these two gives $\xi(x+\sigma,\sigma)=\eta(x+\sigma,\sigma)+1$ or, using the notation introduced above, $l=n+1$. Therefore

$$\begin{aligned} m\cdot d(k,m-1)d(l,n+1)-l\cdot d(k+1,m)d(l-1,n)& \\ &=m\cdot d(k,m-1)d(l,l)-l\cdot d(K+1,m)d(l-1,l-1) \\ &=m\cdot\frac{(m-1)!}{(m-k-1)!}\cdot l!-l\frac{m!}{(m-k-1)!}\cdot(l-1)! \\ &=0. \end{aligned} \tag{137}$$

This concludes the proof of the duality relation between L_a and $\hat{L}_a$.

4.2.2 Self-duality for the internal generator

For the internal part of the generator there is self-duality, i.e.,

$$[L_i\mathscr{D}(\xi,\cdot)](\eta)=[L_i\mathscr{D}(\cdot,\eta)](\xi) \tag{138}$$

Our strategy is to show that for all $x\in\mathbb{Z}^d$ and $\sigma,\sigma'\in S$,

$$\begin{aligned} &\xi(x,\sigma)c_x(\sigma,\sigma')\Big[\mathscr{D}(\xi^{(x,\sigma),(x,\sigma')},\eta)-\mathscr{D}(\xi,\eta)\Big]+\xi(x,\sigma')c_x(\sigma',\sigma)\Big[\mathscr{D}(\xi^{(x,\sigma'),(x,\sigma)},\eta)-\mathscr{D}(\xi,\eta)\Big] \\ &=\eta(x,\sigma)c_x(\sigma,\sigma')\Big[\mathscr{D}(\xi,\eta^{(x,\sigma),(x,\sigma')})-\mathscr{D}(\xi,\eta)\Big]+\eta(x,\sigma')c_x(\sigma',\sigma)\Big[\mathscr{D}(\xi,\eta^{(x,\sigma'),(x,\sigma)})-\mathscr{D}(\xi,\eta)\Big]. \end{aligned} \tag{139}$$

The cases where either $\mathscr{D}(\xi,\eta)=0$, $\mathscr{D}(\xi^{(x,\sigma)(x,\sigma')},\eta)=0$ or $\mathscr{D}(\xi,\eta^{(x,\sigma)(x,\sigma')})=0$ can be approached similarly as we did for the active part of the generator. Hence we only show the case where $\mathscr{D}(\xi,\eta)>0$, $\mathscr{D}(\xi^{(x,\sigma)(x,\sigma')},\eta)>0$ and $\mathscr{D}(\xi,\eta^{(x,\sigma)(x,\sigma')})>0$. Once again we switch to a more convenient notation:

$$\begin{aligned} \xi(x,\sigma)&=k, & \eta(x,\sigma)&=m \\ \xi(x,\sigma')&=l, & \eta(x,\sigma')&=n. \end{aligned} \tag{140}$$

Due to our symmetry assumption concerning the rates, $c_x(\sigma,\sigma') = c_x(\sigma',\sigma)$, we can rewrite (139),

$$
\begin{aligned}
&k[d(k-1,m)d(l+1,n) - d(k,m)d(l,n)] + l[d(k+1,m)d(l-1,n) - d(k,m)d(l,n)] \\
&= m[d(k,m-1)d(l,n+1) - d(k,m)d(l,n)] + n[d(k,m+1)d(l,n-1) - d(k,m)d(l,n)]
\end{aligned} \tag{141}
$$

We divide both sides of the equation by $d(k,m)d(l,n)$. This yields, after reordering the terms,

$$
\begin{aligned}
&k\frac{d(k-1,m)d(l+1,n)}{d(k,m)d(l,n)} - n\frac{d(k,m+1)d(l,n-1)}{d(k,m)d(l,n)} + n - k \\
&= m\frac{d(k,m-1)d(l,n+1)}{d(k,m)d(l,n)} - l\frac{d(k+1,m)d(l-1,n)}{d(k,m)d(l,n)} + l - m.
\end{aligned} \tag{142}
$$

Now we are in a position to use Lemma 4.2. We obtain

$$
k\frac{d(k-1,m)d(l+1,n)}{d(k,m)d(l,n)} - n\frac{d(k,m+1)d(l,n-1)}{d(k,m)d(l,n)} = l - n \tag{143}
$$

and

$$
m\frac{d(k,m-1)d(l,n+1)}{d(k,m)d(l,n)} - l\frac{d(k+1,m)d(l-1,n)}{d(k,m)d(l,n)} = m - k \tag{144}
$$

for the terms on the left and right-hand side respectively. Substitution back in the original equation yields the required identity.

4.2.3 Self-duality for the random walk generator

Notice that the internal operator is essentially the same as the random walk operator. Both describe a random walk, the first on the set S and the latter on $\mathbb{Z}^d$. Hence the proof for the self-duality of $\mathcal{L}$ is essentially the same as the proof for L_i. It again relies heavily on the fact that the transition rates are symmetric.

4.3 Hydrodynamic Limit

In chapter 3.3 we generalized a procedure to obtain the hydrodynamic limit of a general process describing independent particles on a $d-$dimensional integer grid. In this subsection we determine the limiting dual generator such that we can directly apply theorem 3.11.

Before we proceed we notice that for generator $\hat{L}$, as it is defined above, one can not construct an operator according to the hydrodynamic limit such that one has convergence with respect to the strong operator topology. This is because the hydrodynamic limit is defined such that independent random walkers become independent Brownian motions. A simple symmetric random walker is typically a distance of order $\sqrt{t}$ away from its starting point at time $t \geq 0$. Meanwhile, a particle moving according to the active part of the generator moves only in one direction, which means it will be a distance of order t removed from its starting point. Therefore, the particle would move towards infinity too fast for a hydrodynamic scaling. We solve this by scaling the corresponding parameter λ by ϵ. The hydrodynamic scaling also doesn't work for the internal part. Indeed, L_i describes the jumps between layers, which are not spatially scaled in any way. Hence the time speedup would make the particles move infinitely fast, which again entails that the scaled operator doesn't converge. We can circumvent this by scaling the parameter γ by a factor ϵ^2. This gives

$$\hat{L} = \lambda \hat{L}_a + \gamma L_i + \kappa \mathcal{L} \qquad \longrightarrow \qquad \hat{L}^\epsilon = \epsilon\lambda \hat{L}_a + \epsilon^2 \gamma L_i + \kappa \mathcal{L} \tag{145}$$

To derive the k-th order equation, we switch to the generator and semigroup operator associated to the dual process with k particles. Taking the rescaling of the parameters into account, we end up with the following general expression of $\hat{\mathscr{L}}^{k,\epsilon}$,

$$\begin{aligned} \hat{\mathscr{L}}^{k,\epsilon} f = \lim_{t\to 0} \frac{\hat{\mathscr{S}}_t^{k,\epsilon} f - f}{t} &= \lim_{t\to 0} (Z_\epsilon^k)^{-1} \frac{\hat{S}^{k,\epsilon}_{\epsilon^{-2}t} - I}{t} Z_\epsilon^k f \\ &= (Z_\epsilon^k)^{-1} \epsilon^{-2} \hat{L}^{k,\epsilon} Z_\epsilon^k f \\ &= (Z_\epsilon^k)^{-1} (\epsilon^{-1} \lambda \hat{L}_a^k + \gamma L_i^k + \epsilon^{-2} \kappa \mathcal{L}^k) Z_\epsilon^k f. \end{aligned} \tag{146}$$

4.3.1 Location independent tumble rates

We first examine the scenario where the rates c_x in the internal part of the generator are x-independent, i.e. $\forall x, y \in \mathbb{Z}^d$ and $\forall \sigma, \sigma' \in S : c_x(\sigma, \sigma') = c_y(\sigma, \sigma') =: c(\sigma, \sigma')$. The following proposition states the limiting operator $\hat{\mathscr{L}}^k$.

Proposition 4.3. *Let $\hat{\mathscr{L}}^{k,\epsilon} = \epsilon^{-1}\lambda \hat{L}_a^k + \gamma L_i^k + \epsilon^{-2}\kappa \mathcal{L}^k$, with domain $\{f \in L^2(\mathbb{R}^{kd} \otimes S; dx \otimes \mu) : f(., s) \in C^2(\mathbb{R}^{kd}), \forall \sigma \in S\}$. Then $\lim_{\epsilon \to 0} \hat{\mathscr{L}}^{k,\epsilon}$ exists with respect to the strong operator topology. Moreover, the limit is given by:*

$$\hat{\mathscr{L}}^k f(\underline{x}, \underline{\sigma}) = \frac{\kappa}{2} \Delta f(\underline{x}, \underline{\sigma}) - \lambda \underline{\sigma} \cdot \nabla f(\underline{x}, \underline{\sigma}) + \sum_{i=1}^k \sum_{\sigma' \in S} c(\sigma_i, \sigma')[f(\underline{x}, \underline{\sigma} - \sigma_i (\vec{e}_i)^\intercal + \sigma' (\vec{e}_i)^\intercal) - f(\underline{x}, \underline{\sigma})], \tag{147}$$

where $\underline{x} = (x_1, ..., x_k) \in \mathbb{R}^{d\times k}$ and $\underline{\sigma} = (\sigma_1, ..., \sigma_k) \in S^k$ are matrices. The vector $\vec{e}_i \in \mathbb{R}^k$ is zero everywhere except in the i-th entry, where it is one.

Similarly, there is strong convergence for the operator $\mathscr{L}^{k,\epsilon} = \epsilon^{-1}\lambda L_a^k + \gamma L_i^k + \epsilon^{-2}\kappa \mathcal{L}^k$ and the limit $\mathscr{L}^k$ is given by

$$\mathscr{L}^k f(\underline{x}, \underline{\sigma}) = \frac{\kappa}{2} \Delta f(\underline{x}, \underline{\sigma}) + \lambda \underline{\sigma} \cdot \nabla f(\underline{x}, \underline{\sigma}) + \sum_{i=1}^k \sum_{\sigma' \in S} c(\sigma_i, \sigma')[f(\underline{x}, \underline{\sigma} - \sigma (\vec{e}_i)^\intercal + \sigma' (\vec{e}_i)^\intercal) - f(\underline{x}, \underline{\sigma})], \tag{148}$$

In fact we have that $\hat{\mathscr{L}}$ is the adjoint of $\mathscr{L}$ with respect to the inner product on $L^2(\mathbb{R}^{kd} \otimes S; dx \otimes \mu)$, with μ the counting measure.

Proof. We first show the convergence of $\hat{\mathscr{L}}^{k,\epsilon}$ by considering the active, internal and random walk part separately.

The active part gives:

$$
\begin{aligned}
(Z_\epsilon^k)^{-1}(\epsilon^{-1}\hat{L}_a^k)Z_\epsilon^k f(\underline{\mathrm{x}},\underline{\sigma}) &= \epsilon^{-1}\hat{L}_a^k Z_\epsilon^k f(\epsilon^{-1}\underline{\mathrm{x}},\underline{\sigma}) \\
&= \sum_{i=1}^k \epsilon^{-1}[Z_\epsilon^k f(\epsilon^{-1}\underline{\mathrm{x}} - \sigma_i(\vec{e}_i)^\intercal,\underline{\sigma}) - Z_\epsilon^k f(\epsilon^{-1}\underline{\mathrm{x}},\underline{\sigma})] \\
&= \sum_{i=1}^k \epsilon^{-1}[f(\underline{\mathrm{x}} - \epsilon\sigma_i(\vec{e}_i)^\intercal,\underline{\sigma}) - f(\underline{\mathrm{x}},\underline{\sigma})] \\
&= -\sum_{i=1}^k \mathrm{mat}_i(\sigma_i)\cdot\nabla f(\underline{\mathrm{x}},\underline{\sigma}) + O(\epsilon) \\
&= -\underline{\sigma}\cdot\nabla f(\underline{\mathrm{x}},\underline{\sigma}) + O(\epsilon)
\end{aligned} \tag{149}
$$

Here we write, for $v \in \mathbb{R}^d$, $\mathrm{mat}_i(v)$ to denote the $d \times k$ matrix which is empty, except for the i-th column, which is equal to v.

The random walk part gives:

$$
\begin{aligned}
&(Z_\epsilon^k)^{-1}(\epsilon^{-2}\hat{\mathcal{L}}^k)Z_\epsilon^k f(\underline{\mathrm{x}},\underline{\sigma}) \\
&\quad= \epsilon^{-2}\hat{\mathcal{L}}^k Z_\epsilon^k f(\epsilon^{-1}\underline{\mathrm{x}},\underline{\sigma}) \\
&\quad= \sum_{i=1}^k \sum_{y\in\mathbb{R}^d} \epsilon^{-2}c(\epsilon^{-1}x_i,y)[Z_\epsilon^k f(\epsilon^{-1}\underline{\mathrm{x}} - \mathrm{mat}_i(x_i) + \mathrm{mat}_i(y),\underline{\sigma}) - Z_\epsilon^k f(\epsilon^{-1}\underline{\mathrm{x}},\underline{\sigma})] \\
&\quad= \sum_{i=1}^k \sum_{j=1}^d \epsilon^{-2}\frac{1}{2d}[f(\underline{\mathrm{x}} - \epsilon^{-1}\mathrm{mat}_i(e_j),\underline{\sigma}) - 2f(\underline{\mathrm{x}},\underline{\sigma}) + f(\underline{\mathrm{x}} + \epsilon\cdot\mathrm{mat}_i(e_j),\underline{\sigma})] \\
&\quad= \sum_{i=1}^k \sum_{j=1}^d \frac{\partial^2 f}{\partial x_{ij}^2}(\underline{\mathrm{x}},\underline{\sigma}) + O(\epsilon)
\end{aligned} \tag{150}
$$

Since we assume the jump rates between the layers to be independent from x, the internal part is not scaled at all and thus remains the same.

The calculation for $\mathscr{L}^k$ is analogous, the only difference is the minus sign in front of the active part. This is due to the fact that $\hat{L}_a$ sends particles in the opposite direction as L_a and can easily be verified.

Next we argue that $\hat{\mathscr{L}}^k$ is the adjoint of $\mathscr{L}^k$. Let us first state the relevant inner product,

$$
\langle f,g\rangle_{L^2(\mathbb{R}^{kd}\otimes S)} = \sum_{\sigma\in S}\int f(x,\sigma)g(x,\sigma)dx. \tag{151}
$$

The active part in $\hat{\mathscr{L}}^k$, $-\lambda\underline{\sigma}\cdot\nabla$, is essentially a sum of differential operators in the directions given by $\underline{\sigma}$. Since differential operators are known to be anti-symmetric as operators on $L^2(\mathbb{R}^{kd})$, we can conclude that the active part in $\hat{\mathscr{L}}^k$ is indeed adjoint to the active part in $\mathscr{L}^k$, since they only differ by a minus sign. Now we consider the random walk part. It is well known that the Laplacian is self-adjoint on $L^2(\mathbb{R}^{kd})$, which immediately provides the self-adjointness of the random walk part. The internal part, $\sum_{i=1}^k\sum_{\sigma'\in S} c(\sigma_i,\sigma')(f(\underline{\mathrm{x}},\underline{\sigma} - \sigma(\vec{e}_i)^\intercal + \sigma'(\vec{e}_i)^\intercal) - f(\underline{\mathrm{x}},\underline{\sigma})$, is self-adjoint as well, but only under the assumption the symmetry assumption for the rates, i.e. $\forall\sigma,\sigma'\in S,\forall x,y\in\mathbb{R}^{kd} : c_x(\sigma,\sigma') = c_y(\sigma,\sigma')$. This is what we will show next. Due to the assumption that the hopping rates between the layers are x-independent, the internal part is an operator acting purely on $L^2(S,\mu)$, therefore we only have to check its self-adjointess with respect to the counting measure. It is straightforward to verify that for $g, f \in L^2(S,\mu)$,

$$\sum_{\sigma\in S} g(\sigma)Kf(\sigma) = \sum_{\sigma\in S} K^* g(\sigma) f(\sigma), \tag{152}$$

where

$$Kf(\sigma) = \sum_{\sigma'\in S} c(\sigma,\sigma')(f(\underline{\mathrm{x}}, \underline{\sigma} - \sigma(\vec{e}_i)^\intercal + \sigma'(\vec{e}_i)^\intercal) - f(\underline{\mathrm{x}},\underline{\sigma})) \tag{153}$$

and

$$K^* g(\sigma) = \sum_{\sigma'\in S} c(\sigma,\sigma') g(\underline{\mathrm{x}}, \underline{\sigma} - \sigma(\vec{e}_i)^\intercal + \sigma'(\vec{e}_i)^\intercal) - c(\sigma',\sigma) g(\underline{\mathrm{x}},\underline{\sigma}). \tag{154}$$

The symmetry of the rates immediately yields self-adjointness of the internal part.

□

Theorem 4.4. *Let $\rho_0 \in C_0\Big((\mathbb{R}\times S)^k\Big)$ be such that $\forall \sigma \in S$, $\rho_0(\cdot,\sigma)$ is twice differentiable and let $\{\mu_\epsilon\}$ a family of probability measures consistent with ρ_0. For $\eta \in \Omega$ and φ a test function we define*

$$Q(\eta,\varphi) = \epsilon^{kd} \sum_{\sigma_1,\dots,\sigma_k\in S} \sum_{x_1,\dots,x_k\in\mathbb{Z}^d} \varphi(\epsilon x_1,\dots,\epsilon x_k)\mathscr{D}(x_1,\dots,x_k,\sigma_1,..,\sigma_k;\eta). \tag{155}$$

We have that

$$\mathbb{E}_{\eta(0)}[Q(\eta(t),\phi)] = \mathbb{E}_{\varphi}[Q(\eta(0),(Z_\epsilon^k)^{-1}(\hat{S}_t^k))^* Z_\epsilon^k)] \tag{156}$$

Where $\hat{S}^k$ is defined as in 88. Moreover,

$$\int Q(\eta,\varphi)d\mu_\epsilon \to \sum_{\sigma_1,\dots,\sigma_k\in S} \int \dots \int \varphi(x_1,\dots,x_k)\rho(t;x_1,\dots,x_k,\sigma_1,\dots,\sigma_k)dx_1\dots dx_k. \tag{157}$$

as $\epsilon \to 0$. Here $\rho(t;x_1,\dots,x_k,\sigma_1,\dots,\sigma_k)$ is the solution to

$$\begin{cases} \frac{\partial\rho}{\partial t} = \frac{\kappa}{2}\Delta f(\underline{x},\underline{\sigma}) - \lambda\underline{\sigma}\cdot\nabla f(\underline{x},\underline{\sigma}) + \sum\limits_{i=1}^{k}\sum\limits_{\sigma'\in S} c(\sigma_i,\sigma')[f(\underline{x},\underline{\sigma} - \sigma_i(\vec{e}_i)^\intercal + \sigma'(\vec{e}_i)^\intercal) - f(\underline{x},\underline{\sigma})] \\ \rho(0;x_1,\dots,x_k,\sigma_1,\dots,\sigma_k) = \rho_0(x_1,\dots,x_k,\sigma_1,\dots,\sigma_k). \end{cases} \tag{158}$$

Proof. This is a direct consequence of theorem 3.11. Notice that, in fact, we did too much work in the proof of proposition 4.3. We could have calculated just $\hat{\mathscr{L}}^{1,\epsilon}$ since $\hat{\mathscr{L}}^k = \sum_{l=1}^{k}(\hat{\mathscr{L}}^1)_l$. □

4.3.2 Location dependent tumble rates

In this subsection we consider what happens if the tumble rates c_x are *x-dependent*. For mathematical convenience, we explore this scenario in the context of a system with two one dimensional layers, i.e. $d = 1$ and $S = \{\sigma^{(1)},\sigma^{(2)}\}$ where $\sigma^{(1)},\sigma^{(2)} \in \mathbb{Z}$. It turns out that in this case we don't have convergence of the generator $\hat{\mathscr{L}}^{k,\epsilon}$ in the sense of the strong operator topology. Indeed, the internal part of the generator does not converge,

$$(Z_\epsilon^k)^{-1}(L_a^k)Z_\epsilon^k f(\underline{\mathrm{x}},\underline{\sigma}) = L_a^k Z_\epsilon^k f(\epsilon^{-1}\underline{\mathrm{x}},\underline{\sigma}) \tag{159}$$

$$= \sum_{i=1}^{k}\sum_{\sigma'\in S} c_{\epsilon^{-1}x}[Z_\epsilon^k f(\epsilon^{-1}\underline{\mathrm{x}},\underline{\sigma}-\sigma(\vec{e}_i)^\intercal+\sigma'(\vec{e}_i)^\intercal) - Z_\epsilon^k f(\epsilon^{-1}\underline{\mathrm{x}},\underline{\sigma})]$$

$$= \sum_{i=1}^{k}\sum_{\sigma'\in S} c_{\epsilon^{-1}x}[f(\underline{\mathrm{x}},\underline{\sigma}-\sigma(\vec{e}_i)^\intercal+\sigma'(\vec{e}_i)^\intercal) - f(\underline{\mathrm{x}},\underline{\sigma})].$$

Indeed, the rates $c_{\epsilon^{-1}x}$ don't converge as $\epsilon \to \infty$. However, when we assume the tumble rates to be sampled from bounded i.i.d. random variables (see definition 4.5), the dual process itself still converges in distribution to a process with the a generator $\hat{\mathscr{L}}^{k,\epsilon}$ of the same form as for the location independent rates. The only difference being that the internal state changes with the *location-average* rate. That is,

$$\hat{\mathscr{L}}^k f(\underline{\mathrm{x}},\underline{\sigma}) = \frac{\kappa}{2}\Delta f(\underline{\mathrm{x}},\underline{\sigma}) - \lambda\underline{\sigma}\cdot\nabla f(\underline{\mathrm{x}},\underline{\sigma}) + \sum_{i=1}^{k}\sum_{\sigma'\in S} < c_x > [f(\underline{\mathrm{x}},\underline{\sigma}-\sigma_i(\vec{e}_i)^\intercal+\sigma'(\vec{e}_i)^\intercal) - f(\underline{\mathrm{x}},\underline{\sigma})], \tag{160}$$

where,

$$< c_x >= \lim_{N\to\infty}\sum_{x=-N}^{N}\frac{c_x}{2N+1}. \tag{161}$$

This can intuitively be understood through the fact that $c_{\epsilon^{-1}x_i}$ changes increasingly fast as $\epsilon \to \infty$, and hence "sees" the tumble rate at an increasing amount of locations. This eventually leads to the average appearing in the generator of the limiting process. We call this phenomenon *homogenization.*

Definition 4.5. *The numbers $\{c_x, x \in \mathbb{Z}\}$ are sampled from the family of independent identically distributed random variables $\{C_x, x \in \mathbb{Z}\}$ with joint distribution μ. We assume the marginals are bounded distributions: $\exists B > 0$ such that $\forall x \in \mathbb{Z}$,*

$$\mu\{C_x < |B|\} = 1. \tag{162}$$

First, we consider the case where there is only drift and no diffusion, i.e. $\lambda \neq 0$ and $\kappa = 0$. We treat this case separately to gain an intuition for how the homogenization appears. The next step is then to generalize to the case where $\kappa + \gamma > 0$. Here there is either diffusion, drift or both. Finally we also briefly discuss what happens in the case that both the diffusion and the drift are zero. It turns out that in this case the Markov property of the system is lost upon scaling. The reason is that the homogenization fails due to the fact that the particle only sees a single tumble rate.

Case 1: $\lambda \neq 0, \kappa = 0$

The argument presented here assumes the tumble rates to only take two values, a and b, with probabilities p_a and p_b respectively. One could generalize the method to tumble rates with a distribution as in definition 4.5, however, we opt to not do this since this argument is meant as a prelude for the case where there is also diffusion. There we will use the Birkhoff ergodic theorem to obtain the homogenization for more general μ.

Since we don't have interactions in our model we can simply examine the behaviour of a single particle and then, later on, generalize to the k-particle case. Consider the process $\{(X_t^\epsilon, \sigma_t^\epsilon)\}_{t\geq 0}$ on $\epsilon\mathbb{Z} \times \{\sigma^{(1)}, \sigma^{(2)}\}$ given by the generator

$$L^\epsilon f(x,\sigma) = \epsilon^{-1}\Big[f(x+\epsilon\sigma,\sigma) - f(x,\sigma)\Big] + c_{\epsilon^{-1}\cdot x}[f(x,\Theta\sigma) - f(x,\sigma)]. \tag{163}$$

Here, flip operator $\Theta : S \to S$ is an operator which maps each element of S to the other element, i.e. $\sigma^{(1)} \to \sigma^{(2)}$ and $\sigma^{(2)} \to \sigma^{(1)}$. We show that in the limit $\epsilon \to 0$ the process converges in distribution to $\{(X_t, \sigma_t)\}_{t\geq 0}$ generated by

$$Lf(x,\sigma) = \sigma\frac{df}{dx}(x,\sigma) + < c_{\epsilon^{-1}\cdot x} > [f(x,\Theta\sigma) - f(x,\sigma)]. \tag{164}$$

First we prove that $X_t = X_0 + \int_0^t \sigma_s ds$, i.e. X_t moves with velocity σ_t. This can be obtained by considering the Dynkin martingale associated to $g(x,\sigma) = x$ and the process $\{(X_t^\epsilon, \sigma_t^\epsilon)\}_{t\geq 0}$,

$$M_t^g = X_t^\epsilon - X_0^\epsilon - \int_0^t [L^\epsilon g](X_s^\epsilon, \sigma_s^\epsilon)ds. \tag{165}$$

A quick calculation shows that

$$[L^\epsilon g](x,\sigma) = \sigma \qquad \& \qquad [L^\epsilon g^2](x,\sigma) = 2x\sigma + \epsilon\sigma^2. \tag{166}$$

Hence,

$$M_t^g = X_t^\epsilon - X_0^\epsilon - \int_0^t \sigma_s^\epsilon ds \tag{167}$$

with quadratic variation

$$[M_t^g, M_t^g] = \int_0^t [L^\epsilon g^2](X_s^\epsilon, \sigma_s^\epsilon) - 2g(X_s^\epsilon, \sigma_s^\epsilon)[L^\epsilon g](X_s^\epsilon, \sigma_s^\epsilon)ds = \int_0^t \epsilon(\sigma_s^\epsilon)^2 ds \leq \epsilon \max(\sigma^{(1)}, \sigma^{(2)})t. \tag{168}$$

Taking the limit $\epsilon \to 0$, we find that

$$m_t^g = X_t - X_0 - \int_0^t \sigma_s ds \tag{169}$$

is a martingale with quadratic variation zero. Clearly, the paths of $\{X_t^\epsilon, t \geq 0\}$ become continuous as we take the limit $\epsilon \to 0$. This implies that m_t^g is a continuous martingale. Since it has quadratic variation equal to zero it must be constant. Moreover, since $m_0^g = 0$, it must be zero for all $t \geq 0$. We then have,

$$X_t = X_0 + \int_0^t \sigma_s ds. \tag{170}$$

The next step is to show that σ_t is an autonomous Markov process. To this end we calculate the distribution of the time T between two jumps of σ_t, occurring on t_0 and $t_1 > t_0$. It will turn out that this time is exponentially distributed with parameter $< c_x >$ and hence independent of the spatial process $\{X_t, t \geq 0\}$. By what we have shown above, X_t moves at constant velocity σ_t. Hence we can equivalently calculate the distribution of the distance covered by X_t between the jumps instead of the elapsed time T. Let N denote the number steps X_t takes before σ_t changes state. Without loss of generality we assume that $X_{t_0}^\epsilon = 0$. We have

$$\mathbb{P}_\epsilon(N \geq k) = \prod_{x=0}^{k-1} \frac{\epsilon^{-1}}{c_{\sigma \cdot x} + \epsilon^{-1}}. \tag{171}$$

Here $\mathbb{P}_\epsilon$ denotes the probability measure associated to the process $(X_t^\epsilon, \sigma_t^\epsilon)_{t\geq 0}$. Equation 171 stems from the fact that the particle jumps with rate ϵ^{-1} in the direction of σ and with rate $c_{\sigma \cdot x}$ it changes internal state. Recall that in a Markov chain the probability of a transition from state A to state B is given by the the rate between A and B divided by the sum of all rates. Due to the Markov property we can simply multiply the probability that the particle jumps in the direction of σ at each of the k jumps. Since X_t^ϵ makes jumps of size $\epsilon\sigma$, with σ the constant value of σ_t^ϵ for $t \in (t_0, t_1)$, we find that the distance covered by X_t^ϵ is given by $D = N \cdot \epsilon\sigma$. We obtain the following distribution for D.

$$\mathbb{P}_n(D \geq d) = \mathbb{P}\Big(N \geq \frac{d}{\epsilon \cdot \sigma}\Big) = \prod_{x=0}^{\frac{d}{\epsilon\sigma}-1} \frac{\epsilon^{-1}}{c_{\sigma \cdot x} + \epsilon^{-1}} \tag{172}$$

We can derive that the variance of this expression, with respect to μ, vanishes as $\epsilon \to 0$. To this end, we take the expectation with respect to μ, which we denote $\mathbb{E}_\mu$, and we write p_a and p_b for the probabilities associated with a and b respectively.

$$\mathbb{E}_\mu[\mathbb{P}_\epsilon(D \geq d)] = \prod_{x=0}^{\frac{d}{\epsilon\sigma}-1} \mathbb{E}_\mu\Big[\frac{1}{\epsilon \cdot c_{\sigma \cdot x} + 1}\Big] = \Big[p_a \frac{1}{\epsilon \cdot a + 1} + p_b \frac{1}{\epsilon \cdot b + 1}\Big]^{\frac{d}{\epsilon\sigma}-1} \xrightarrow{\epsilon \to 0} e^{-(p_a a + p_b b)\frac{d}{\sigma}} \tag{173}$$

In the last step we use the squeeze theorem and the fact that $\lim_{n\to\infty}(1 - \frac{y}{n})^n = e^{-y}$. That is, for $a \leq b$,

$$\lim_{\epsilon \to 0} \Big[p_a \frac{1}{\epsilon \cdot a + 1} + p_b \frac{1}{\epsilon \cdot b + 1}\Big]^{\epsilon^{-1}} = \lim_{n \to \infty} \Big[p_a \frac{n}{a+n} + p_b \frac{n}{b+n}\Big]^n = \lim_{n\to\infty} \Big[1 - \frac{p_a a}{a+n} - \frac{p_b b}{b+n}\Big]^n \tag{174}$$

and

$$\begin{aligned}\Big[1 - \frac{p_a a + p_b b}{a+n}\Big]^n &\leq \left[1 - \frac{p_a a + p_b b \frac{a+n}{b+n}}{a+n}\right]^n = \left[1 - \frac{p_a a}{a+n} - \frac{p_b b}{b+n}\right]^n = \left[1 - \frac{p_a a \frac{b+n}{a+n} + p_b b}{b+n}\right]^n \\ &\leq \Big[1 - \frac{p_a a + p_b b}{b+n}\Big]^n. \end{aligned} \tag{175}$$

We can interchanges the roles of a and b to show the same in the case that $b < a$. Next we calculate the second moment of $\mathbb{P}_\epsilon(D \geq d)$,

$$\begin{aligned}&\mathbb{E}_\mu[\mathbb{P}_\epsilon(D \geq d)^2] = \\ &\prod_{x=0}^{\frac{d}{\epsilon\sigma}-1} \mathbb{E}_\mu\Big[\frac{1}{(\epsilon \cdot c_{\sigma \cdot x} + 1)^2}\Big] = \Big[p_a \frac{1}{(\epsilon \cdot a + 1)^2} + p_b \frac{1}{(\epsilon \cdot b + 1)^2}\Big]^{\frac{d}{\epsilon\sigma}-1} \xrightarrow{\epsilon \to 0} e^{-2(p_a a + p_b b)\frac{d}{\sigma}}.\end{aligned} \tag{176}$$

The convergence follows from

$$\lim_{\epsilon\to 0}\left[p_a\frac{1}{(\epsilon\cdot a+1)^2}+p_b\frac{1}{(\epsilon\cdot b+1)^2}\right]^{\epsilon^{-1}} = \lim_{n\to\infty}\left[p_a\frac{n^2}{a^2+2an+n^2}+p_b\frac{n^2}{b^2+2bn+n^2}\right]^n \tag{177}$$
$$= \lim_{n\to\infty}\left[1-p_a\frac{a^2+2an}{a^2+2an+n^2}-p_b\frac{b^2+2bn}{b^2+2bn+n^2}\right]^n$$
$$= \lim_{n\to\infty}\left[1-p_a\frac{n^{-1}a^2+2a}{n^{-1}a^2+2a+n}-p_b\frac{n^{-1}b^2+2b}{n^{-1}b^2+2b+n}\right]^n$$
$$= e^{-2(p_a a+p_b b)}.$$

For the last equality one can again use the squeeze theorem in the same way as we used it for the expectation. Since the second moment has the same limit as the expectation squared, we have shown that the variance indeed becomes zero. We conclude that

$$\mathbb{P}_\epsilon(D\geq d)\to e^{-(p_a a+p_b b)\frac{d}{\sigma}} \tag{178}$$

in the L^2-sense as $\epsilon\to 0$. Now we can directly calculate the distribution of the hopping times of σ_t,

$$\mathbb{P}(T\geq t)=\lim_{\epsilon\to 0}\mathbb{P}_\epsilon(D\geq \sigma t)=e^{-t(p_a\cdot a+p_b\cdot b)}=e^{-t(<c_x>)}. \tag{179}$$

Case 2: $\kappa+\lambda>0$

Before we state and prove the homogenization for systems with drift and diffusion, we prove a useful lemma which is essentially an application of the Birkoff ergodic theorem. It say that we can treat the integral overtime-average of the tumble rates visited by a fast jumping particle as the space-average $< c_x >$ of the tumble rates.

Lemma 4.6. *Let μ and $\{c_x, x\in\mathbb{Z}\}$ be as in definition 4.5. Let $\sigma\in S$. Consider the Markov process $\{X_t, t\geq 0\}$ on the state space $\omega=\mathbb{Z}$. The process is generated by*

$$Lf(x)=\Big[f(x+\epsilon)-2f(x)+f(x-\epsilon)\Big]+\Big[f(x+\epsilon\sigma)-f(x)\Big] \tag{180}$$

acting on the core of local functions on Ω^ϵ. Then

$$\frac{1}{T}\int_0^T c_{X_s}ds \to < c_x > \quad as\ T\to\infty. \tag{181}$$

Proof. We consider the so-called environment process $\{c(t), t\geq 0\}$. Let $c:=\{c_x, x\in\mathbb{Z}\}$ denote the sequence of tumble rates, then $c(t)=\tau_{X_t}$, where τ_a shift sequences over $a\in\mathbb{Z}$. That is $(\tau_a c)_x=c_{x+a}$. One could interpret this process as follows. Instead of following the walker, we now observe the tumble rates as seen from the point of view of the walker. The environment process is generated by

$$\mathscr{L}f(c)=[f(\tau_1 c)-2f(c)+f(\tau_{-1}c)]+[f(\tau_{\epsilon\sigma}c)-f(c)]. \tag{182}$$

We show that the probability measure μ, is invariant and ergodic. These two properties are required for the Birkhoff ergodic theorem which we will apply in the end to translate the temporal averaging to spatial averaging.

To prove the invariance we compute

$$\int Lf(c)d\mu(c) = \int [(f(\tau_1 c) - 2f(c) + f(\tau_{-1}c)) + (f(\tau_{\epsilon\sigma}c) - f(c))]d\mu = 0 \tag{183}$$

Here we use that μ is translation invariant, which is a consequence of the assumption that the tumble rates are i.i.d. To prove the ergodicity, we show that every function f which is invariant with respect to $\mathscr{L}$, i.e. $\mathscr{L}f = 0$ is constant. This yields then the ergodicity. Let f be invariant, then we have

$$\begin{aligned}
0 &= \int f(c)(-\mathscr{L}f(c))d\mu(c) \qquad (184)\\
&= \int [f(c) - f(\tau_1 c)]d\mu + \int [f(c) - f(\tau_{-1}c)]d\mu + \int [f(c) - f(\tau_{\epsilon\sigma}c)]d\mu(c)\\
&= \frac{1}{2}\int [2f(c) - 2f(\tau_1 c)]d\mu - \frac{1}{2}\int [f^2(c) - f^2(\tau_1 c)]d\mu\\
&+ \frac{1}{2}\int [2f(c) - 2f(\tau_{-1} c)]d\mu - \frac{1}{2}\int [f^2(c) - f^2(\tau_{-1} c)]d\mu\\
&+ \frac{1}{2}\int [2f(c) - 2f(\tau_\sigma c)]d\mu - \frac{1}{2}\int [f^2(c) - f^2(\tau_\sigma c)]d\mu\\
&= \frac{1}{2}\int [f(c) - f(\tau_1 c)]^2 d\mu + \frac{1}{2}\int [f(c) - f(\tau_{-1} c)]^2 d\mu + \frac{1}{2}\int [f(c) - f(\tau_{\epsilon\sigma} c)]^2 d\mu(c)
\end{aligned}$$

Here we again used the translation invariance of μ. From the expression above we see immediately that f is indeed μ-a.s. constant.

We can now use the Birkhoff ergodic theorem, which yields the following. For all $f : \Omega \to \mathbb{R}$ which are μ integrable, we have, almost surely,

$$\frac{1}{T}\int_0^T f(c(s))ds \to \int f(c)d\mu(c) \tag{185}$$

In particular, when we choose $f(c) = c_0$ we find the almost sure convergence

$$\frac{1}{T}\int_0^T c(s)_0 = \frac{1}{T}\int_0^T c_{X_s} ds \to \int c_0 d\mu(c) =< c_x > \tag{186}$$

For the last step we the law of large numbers.

□

Theorem 4.7. *Let μ and $\{c_x, x \in \mathbb{Z}\}$ be as in definition 4.5. Consider the family of Markov processes $\{(X_t^\epsilon, \sigma_t^\epsilon), t \geq 0\}_{\epsilon > 0}$ on the state spaces $\Omega^\epsilon = \epsilon\mathbb{Z} \times S$. For each $\epsilon > 0$ the process is generated by the corresponding generator,*

$$\begin{aligned}
L^\epsilon f(x,\sigma) =& \epsilon^{-2}\Big[f(x+\epsilon,\sigma) - 2f(x,\sigma) + f(x-\epsilon,\sigma)\Big] \qquad (187)\\
&+ \epsilon^{-1}\Big[f(x+\epsilon\sigma,\sigma) - f(x,\sigma)\Big] + c_{\epsilon^{-1}\cdot x}[f(x,\Theta\sigma) - f(x,\sigma)]
\end{aligned}$$

acting on the core of local functions on Ω^ϵ. Then these processes converge in distribution to the process $\{(X_t, \Omega_t), t \geq 0\}$ on state space $\mathbb{R} \times S$, generated by

$$Lf(x,\sigma) = \frac{\partial^2}{\partial x^2} f(x,\sigma) + \frac{\partial}{\partial x} f(x,\sigma) + < c_x > [f(x,\Theta\sigma) - f(x,\sigma)]. \tag{188}$$

Proof. Like in the case where $\lambda \neq 0, \kappa = 0$, we show that the internal state $\{\sigma_t^\epsilon, t \geq 0\}$ becomes an autonomous Markov process as $\epsilon \to 0$ which switches between $\sigma^{(1)}$ and $\sigma^{(2)}$ with rate $< c_x >$. Then,using the Dynkin martingale, it is straightforward to show that $\{(X_t^\epsilon, \sigma_t^\epsilon), t \geq 0\}$ converges to $\{(X_t, \sigma_t), t \geq 0\}$. To find the limiting distribution of $\{\sigma_t^\epsilon, t \geq 0\}$ we consider the Markov processes $\{(\bar{X}_t^\epsilon, W_t^\epsilon), t \geq 0\}_{\epsilon > 0}$ on $\epsilon\mathbb{Z} \times \mathbb{R}$ with generators

$$\begin{aligned} \mathscr{L}^\epsilon f(x, w) =& \epsilon^{-2}\Big[f(x + \epsilon, w + w_{\epsilon^{-1}x}) - 2f(x, w) + f(x - \epsilon, w + w_{\epsilon^{-1}x})\Big] \\ &+ \epsilon^{-1}\Big[f(x + \epsilon\sigma, w + w_{\epsilon^{-1}x}) - f(x, w)\Big], \end{aligned} \tag{189}$$

acting on the core of local functions on $\epsilon\mathbb{Z} \times \mathbb{R}$. Here $\sigma \in S$ is fixed and w_x is given by

$$w_x := \log\Big(\frac{\epsilon^{-1} + 2\epsilon^{-2}}{c_x + \epsilon^{-1} + 2\epsilon^{-2}}\Big). \tag{190}$$

We assume the initial value of W_t^ϵ to be zero, $W_0 = 0$. Notice that $\{\bar{X}_t^\epsilon, t \geq 0\}$ has exactly the same distribution as $\{X_t^\epsilon, t \geq 0\}$ between two jumps of $\{\sigma_t^\epsilon, t \geq 0\}$, provided that $\sigma_t^\epsilon = \sigma$ during the time in between the jumps. The process $\{W_t^\epsilon, t \geq 0\}$ tracks the probability that the particle does not change internal state provided it takes the path described by $\bar{X}_t$ up to time t. More precise, for each $t > 0$,

$$e^{W_t} = \prod_{x \in V_t} \frac{\epsilon^{-1} + 2\epsilon^{-2}}{c_{\epsilon^{-1} \cdot x} + \epsilon^{-1} + 2\epsilon^{-2}}. \tag{191}$$

Here $V_t := \{X_s^\epsilon, s \leq t\}$ denotes the sites visited by $\bar{X}_t$. Indeed, at the point in time where the particle jumps, $\exp(W_t)$ is multiplied by the probability that this jump occurs before the internal state flip. We are interested in the limiting distribution of $\tau^\epsilon := \inf\{t \geq 0 : \sigma_t^\epsilon \neq \sigma_0^\epsilon\}$, the time when the internal state flips. The distribution of τ^ϵ can be written as

$$\begin{aligned} \mathbb{P}^\epsilon(\tau^\epsilon > t) &= \int \mathbb{P}^\epsilon(\tau^\epsilon > t | \{X_s^\epsilon, s \leq t\} = \pi_t) d\mathbb{P}^\epsilon(\{X_s^\epsilon, s \leq t\} = \pi_t) \\ &= \int e^{\nu_t} d\mathbb{P}^\epsilon(\{X_s^\epsilon, s \leq t\} = \pi_t). \end{aligned} \tag{192}$$

Here, ν_t is the value of W_t^ϵ corresponding to the fixed instance π_t of $\{X_s^\epsilon, s \leq t\}$. We argue that the process $\{W_t^\epsilon, t \geq 0\}$ converges in distribution to a constant process $\{W_t = -< c_x > t, t \geq 0\}$, which then immediately yields that the internal state flips are exponentially distributed with parameter $< c_x >$. To this end we calculate the Dynkin martingale associated to $\phi(x, w) = w$,

$$M_t^\epsilon = W_t^\epsilon - \int_0^t \mathscr{L}^\epsilon \phi(\bar{X}_s^\epsilon, W_s^\epsilon) ds. \tag{193}$$

A quick calculation shows that,

$$\mathscr{L}^\epsilon \phi(x, w) = (2\epsilon^{-2} + \epsilon^{-1}) w_{\epsilon^{-1}x} \tag{194}$$

and

$$\mathscr{L}^\epsilon \phi^2(x, w) = (2\epsilon^{-2} + \epsilon^{-1})(2w w_{\epsilon^{-1}x} + w_{\epsilon^{-1}x}^2). \tag{195}$$

Hence we can explicitly write the Dynkin martingale and its quadratic variation as

$$M_t^\epsilon = W_t^\epsilon - \int_0^t (2\epsilon^{-2} + \epsilon^{-1}) w_{\epsilon^{-1} X_s^\epsilon} ds \tag{196}$$

and

$$\begin{aligned}
[M_t^\epsilon, M_t^\epsilon] &= \int_0^t \Big[\mathscr{L}^\epsilon \phi^2(\bar{X}_s^\epsilon, W_s^\epsilon) - 2\phi(\bar{X}_s^\epsilon, W_s^\epsilon)\mathscr{L}^\epsilon \phi(\bar{X}_s^\epsilon, W_s^\epsilon)\Big] ds \\
&= \int_0^t (2\epsilon^{-2} - \epsilon^{-1}) w^2_{\epsilon^{-2}\bar{X}_s^\epsilon} ds.
\end{aligned} \tag{197}$$

We observe that the quadratic variance vanishes as $\epsilon \to 0$,

$$(2\epsilon^{-2} - \epsilon^{-1}) w^2_{\epsilon^{-1}\bar{X}_s^\epsilon} = (2\epsilon^{-2} - \epsilon^{-1}) \log^2\Big(\frac{\epsilon^{-1} + 2\epsilon^{-2}}{c_{\bar{X}_s^\epsilon} + \epsilon^{-1} + 2\epsilon^{-2}}\Big) \to 0 \text{ as } \epsilon \to 0. \tag{198}$$

Notice that one needs the boundedness of the tumble rates to calculate the limit. The fact the quadratic variance vanishes shows that the limiting martingal becomes constant. Moreover, since $M_0^\epsilon = 0$ for all ϵ, we can say that it vanishes. This gives

$$\begin{aligned}
W_t^\epsilon &= \int_0^t (2\epsilon^{-2} + \epsilon^{-1}) \log\Big(\frac{\epsilon^{-1} + 2\epsilon^{-2}}{c_{\bar{X}_s^\epsilon} + \epsilon^{-1} + 2\epsilon^{-2}}\Big) ds + o(\epsilon) \\
&= -\int_0^t c_{\epsilon^{-1}\bar{X}_s^\epsilon} ds + o(\epsilon) \\
&= -\frac{t}{\epsilon^2 t} \int_0^{\epsilon^{-2} t} c_{\epsilon^{-1}\bar{X}^\epsilon_{\epsilon^2 r}} dr + o(\epsilon) \\
&\sim -\frac{t}{\epsilon^2 t} \int_0^{\epsilon^{-2} t} c_{\bar{X}_r^1} dr + o(\epsilon) \to -t < c_x > \text{ as } \epsilon \to 0.
\end{aligned} \tag{199}$$

For the last step we use lemma 4.6. □

Case 3: $\lambda = 0, \kappa = 0$

The case where $\lambda = 0, \kappa = 0$ is a rather strange one, as there are no particles actually moving. In fact, the lack of movement makes that each particle only encounters the tumble rate at its own location, making homogenization inapplicable. It turns out that in the end we don't have a macroscopic profile which moves according to a PDE, but rater a profile evolving as a mixture of PDEs.

Before we calculate the field, we introduce a useful trick which allows us to homogenize while dealing with sums over $\mathbb{Z}$. This is done in proposition 4.8 We also need the powers of the internal part of the generator, which is in fact the only relevant part. These are calculated in proposition 4.9.

Proposition 4.8. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a test function, and let $c : \mathbb{Z} \to \mathbb{R}$ be bounded such that $< c_x >$ exists. Then*

$$\sum_{x\in\mathbb{Z}} \epsilon \cdot c_x \phi(\epsilon x) = \sum_{x\in\mathbb{Z}} \epsilon \cdot < c_x > \phi(x) \tag{200}$$

Proof.

$$\begin{aligned}
&\Big|\sum_{x\in\mathbb{Z}} \epsilon\cdot c_x\phi(\epsilon x) - \sum_{y:|x-y|\le(\epsilon N)^{-1}} \epsilon\frac{c_y}{2(\epsilon N)^{-1}+1}\phi(\epsilon x)\Big| \qquad (201)\\
&\qquad = \Big|\sum_{x\in\mathbb{Z}} \epsilon\cdot c_x\phi(\epsilon x) - \sum_{y:|x-y|\le(\epsilon N)^{-1}} \epsilon\frac{c_y}{2(\epsilon N)^{-1}+1}\phi(\epsilon x)\Big|\\
&\qquad = \Big|\sum_{x\in\mathbb{Z}} \epsilon\cdot c_x\Big(\phi(\epsilon x) - \sum_{y:|x-y|\le(\epsilon N)^{-1}} \frac{\phi(\epsilon x)}{2(\epsilon N)^{-1}+1}\Big)\Big|\\
&\qquad = \Big|\sum_{x\in\mathbb{Z}} \epsilon\cdot c_x \sum_{y:|x-y|\le(\epsilon N)^{-1}} \frac{\phi(\epsilon x)-\phi(\epsilon y)}{2(\epsilon N)^{-1}+1}\Big|\\
&\qquad \le \sum_{x\in\mathbb{Z}} \epsilon\cdot c_x \sum_{y:|x-y|\le(\epsilon N)^{-1}} \frac{|\phi(\epsilon x)-\phi(\epsilon y)|}{2(\epsilon N)^{-1}+1}\\
&\qquad \le \sum_{x\in\mathbb{Z}} \epsilon\cdot c_x \sum_{y:|x-y|\le(\epsilon N)^{-1}} N^{-1}\frac{|\max(\phi'(\epsilon x),\phi'(\epsilon y))|}{2(\epsilon N)^{-1}+1}\\
&\qquad = N^{-1}\sum_{x\in\mathbb{Z}} \epsilon\cdot c_x|\max(\phi'(\epsilon x),\phi'(\epsilon y))| \to 0 \text{ as } N\to\infty
\end{aligned}$$

□

Proposition 4.9. *Let $L : F(\Omega) \to F(\Omega)$ be the following operator.*

$$Lf = c_{\epsilon^{-1}x}[f(x,\Theta\sigma) - f(x,\sigma)]. \qquad (202)$$

Then, for all $n \in \mathbb{N}$,

$$L^n f = \frac{(-2c_{\epsilon^{-1}x})^n}{2}[f(x,\sigma) - f(x,\Theta\sigma)] \qquad (203)$$

Proof. We use induction. The case $n = 1$ is trivial. Assume the theorem holds for $n = k$, then we claim that the theorem also holds for $n = k + 1$.

$$\begin{aligned}
L^{k+1}f &= c_{\epsilon^{-1}x}[L^k f(x,\Theta\sigma) - L^k f(x,\sigma)] \qquad (204)\\
&= c_{\epsilon^{-1}x}\left[\frac{(-2c_{\epsilon^{-1}x})^k}{2}[f(x,\Theta\sigma)-f(x,\sigma)] - \frac{(-2c_{\epsilon^{-1}x})^k}{2}[f(x,\sigma)-f(x,\Theta\sigma)]\right]\\
&= \frac{(-2c_{\epsilon^{-1}x})^{k+1}}{2}[f(x,\sigma)-f(x,\Theta\sigma)]
\end{aligned}$$

This proves the claim and hence the theorem holds for general $n \in \mathbb{N}$. □

Theorem 4.10. *Let μ be as in definition 4.5, with support $\{a, b\}^{\mathbb{Z}}$ for some $a, b \in \mathbb{R}_{\ge 0}$ and define $p_a = \mu\{C_x = a\}$, $p_b = \mu\{C_x = b\}$. Let $\rho_0 \in C_0\Big((\mathbb{R} \times S)^k\Big)$. Then for each test function $\varphi : \mathbb{R} \to \mathbb{R}$,*

$$\epsilon^{-1}\sum_{x,\sigma}\varphi(\epsilon x)S_t\rho(\epsilon x,\sigma) \to \sum_{\sigma}\int \varphi(x)\rho(t;x,\sigma)dx. \qquad (205)$$

Here, $\{S_t, t \ge 0\}$ is the semigroup associated to the generator $L : F(\Omega) \to F(\Omega)$ defined by

$$Lf = c_{\epsilon^{-1}x}[f(x,\Theta\sigma) - f(x,\sigma)] \qquad (206)$$

and $\rho(t;x,\sigma) = p_a\rho^{(a)}(t;x,\sigma) + p_b\rho^{(b)}(t;x,\sigma)$ *, with for* $c \in \{a,b\}$*,* $\rho^{(c)}$ *the solution to*

$$\begin{cases} \frac{\partial \rho}{\partial t} = c[f(x,\Theta\sigma) - f(x,\sigma)] \\ \rho(0;x,\sigma) = \rho_0(x,\sigma). \end{cases} \tag{207}$$

Proof. Since L is a bounded operator, its semigroup has exponential representation $S_t = e^{Lt} = \sum_{k=0}^{\infty} \frac{(tL)^k}{k!}$. This gives, using proposition 4.9,

$$\begin{aligned} \epsilon^{-1}\sum_{x,\sigma}\varphi(\epsilon x)S_t\rho(\epsilon x,\sigma) &= \epsilon^{-1}\sum_{x,\sigma}\varphi(\epsilon x)\sum_{k=0}^{\infty}\frac{(tL)^k}{k!}\rho(\epsilon x,\sigma) \\ &= \sum_{k=0}^{\infty}\epsilon^{-1}\frac{t^k}{k!}\sum_{x,\sigma}\varphi(\epsilon x)L^k\rho(\epsilon x,\sigma) \\ &= \sum_{k=0}^{\infty}\epsilon^{-1}\frac{t^k}{2k!}\sum_{x,\sigma}\varphi(\epsilon x)(-2c_{\epsilon^{-1}x})^k[\rho(\epsilon x,\Theta\sigma) - \rho(\epsilon x,\sigma)] \\ &= \sum_{k=0}^{\infty}\epsilon^{-1}\frac{t^k}{2k!}\sum_{x,\sigma}\varphi(\epsilon x)(-2a\mathbb{1}(c_{\epsilon^{-1}x} = a))^k[\rho(\epsilon x,\Theta\sigma) - \rho(\epsilon x,\sigma)] \\ &+ \sum_{k=0}^{\infty}\epsilon^{-1}\frac{t^k}{2k!}\sum_{x,\sigma}\varphi(\epsilon x)(-2b\mathbb{1}(c_{\epsilon^{-1}x} = b))^k[\rho(\epsilon x,\Theta\sigma) - \rho(\epsilon x,\sigma)]. \end{aligned} \tag{208}$$

It is clear that for $c \in \{a,b\}$, $\left\langle(-2c)^k\mathbb{1}(c_{\epsilon^{-1}x} = c)\right\rangle = p_c(-2c)^k$. Hence can use proposition 4.8 to say

$$\begin{aligned} &\sum_{k=0}^{\infty}\epsilon^{-1}\frac{t^k}{2k!}\sum_{x,\sigma}\varphi(\epsilon x)(-2c\mathbb{1}(c_{\epsilon^{-1}x} = c))^k[\rho(\epsilon x,\Theta\sigma) - \rho(\epsilon x,\sigma)] \\ &= \sum_{k=0}^{\infty}\epsilon^{-1}\frac{t^k}{2k!}\sum_{x,\sigma}\varphi(\epsilon x)p_c(-2c)^k[\rho(\epsilon x,\Theta\sigma) - \rho(\epsilon x,\sigma)] \\ &= p_c\cdot\epsilon^{-1}\sum_{x,\sigma}\varphi(\epsilon x)\sum_{k=0}^{\infty}\frac{(-2ct)^k}{2k!}[\rho(\epsilon x,\Theta\sigma) - \rho(\epsilon x,\sigma)] \to p_c\rho^{(c)}(t;x,\sigma) \text{ as } \epsilon \to 0. \end{aligned} \tag{209}$$

this concludes the proof. □

5 Systems with reservoirs

In this chapter we examine systems which are coupled to so called reservoirs. We consider particles moving on a chain $\{1, ..., N\}$ where 1 and N are assumed to be coupled to larger systems which are not influenced by the process on the chain. This entails that at these sites particles appear and disappear at very specific rates. We say that 1 and N are coupled to a reservoir. It turns out that reservoirs can keep the system out of equilibrium in the sense that the measure describing the particles does not become reversible over time, even though, one a macroscopic level, steady state profiles appear. In other words, we reach a *non-equilibrium steady state*. Due to the measure not being reversible, long distance correlations will appear. This is where duality comes into play, the k-point correlation functions can be exactly mapped onto a diffusion equation for k particles [6]. This motivates our interest in k-th order hydrodynamic limits, as these quantify the k-th order correlation functions of the underlying process.

5.1 Model and duality

Model

We consider particles on a linear chain $V = \{1, ..., N\}$ where the first and the N-th site are coupled to a reservoir. Particles can this only leave the system through the left boundary site 1 and the right boundary site N. The particle densities on the boundary sites are $\rho_L \geq 0$ and $\rho_R \geq 0$ for 1 and N respectively. The interesting case, exhibiting non-equilibrium steady states, appears when $\rho_L \neq \rho_R$, hence we assume $\rho_L < \rho_R$. The generator, acting on the core of C^∞ functions with compact support, is defined as

$$L = L_{0,1} + L_{N,N+1} + \sum_{i=1}^{N-2} L_{i,i+1}. \tag{210}$$

Here

$$L_{i,i+1} f(\eta) = \eta_i(\alpha + \sigma\eta_{i+1})[f(\eta^{i,i+1}) - f(\eta)] + \eta_{i+1}(\alpha + \sigma\eta_i)[f(\eta^{i+1,i}) - f(\eta)] \tag{211}$$

and

$$L_{0,1} f(\eta) = \eta_1(1 + \sigma\rho_L)[f(\eta - \delta_1) - f(\eta)] + \rho_L(\alpha + \sigma\eta_1)[f(\eta + \delta_1) - f(\eta)], \tag{212}$$

$$L_{N,N+1} f(\eta) = \eta_N(1 + \sigma\rho_R)[f(\eta - \delta_N) - f(\eta)] + \rho_R(\alpha + \sigma\eta_N)[f(\eta + \delta_N) - f(\eta)].$$

We remind the reader that $\sigma \in \{-1, 0, 1\}$ fixes the type of interaction between the particles. More precise, $\sigma = -1$ corresponds to exclusion, $\sigma = 1$ corresponds to inclusion and $\sigma = 0$ means that the walkers have no interaction whatsoever. The notation $\eta^{x,y}$ has the usual meaning, $\eta^{x,y} = \eta + \delta_y - \delta_x$. As we already briefly mentioned before, this generator can be interpreted as putting particles on sites 1 and N as if the coupled reservoirs have constant particle densities ρ_L and ρ_R. On the sites which are not in contact with the reservoirs, which we will collectively call the *bulk*, the dynamics are akin to SIP, SEP or IRW, depending on σ.

Duality

As we have seen before, each process for $\sigma = -1, 0, 1$ has self-duality on $\mathbb{Z}^d$ without the reservoirs. The dual processes presented here still have the same dynamics as the original processes on the bulk sites. The reservoirs, however, will change into *absorbing* sites. Particles in the dual process will stop moving once they jump out of the chain $\{1, ..., N\}$ and hit either 0 or $N+1$. This makes it significantly easier to calculate the steady states and the corresponding correlation functions since we can rewrite both the profile and the correlation functions in terms of the probabilities that a particle ends up being absorbed at 0 or $N+1$. Notice that the dual process hence lives on $\{0, 1, ..., N, N+1\}$. The dual generator is given by,

$$\hat{L} = L_{\text{abs}} + \sum_{i=1}^{N-2} L_{i,i+1}. \tag{213}$$

Here,

$$L_{\text{abs}} f(\xi) = \xi_1[f(\xi^{1,0}) - f(\xi)] + \xi_N[f(\xi^{N,N+1}) - f(\xi)]. \tag{214}$$

The duality result is stated and proved below. This result is adopted from [1].

Theorem 5.1. *The process $\{\eta_t, t \geq 0\}$ with reservoirs, generated by L, is dual to the process $\{\xi_y, t \geq 0\}$ with the absorbing boundary sites, generated by $\hat{L}$. The duality function is given by*

$$D_{\sigma,\alpha}(\xi, \eta) = \rho_L^{\xi_1} \rho_R^{\xi_N} \cdot D_{\sigma,\alpha}^{bulk}(\xi, \eta), \tag{215}$$

where $D_{\sigma,\alpha}^{bulk}$ denotes the self-duality function for the process associated with σ without the reservoirs,

$$D_{\sigma,\alpha}^{bulk}(\xi, \eta) = \prod_{i=1}^{N} d_{\sigma,\alpha}(\xi_i, \eta_i). \tag{216}$$

The functions $d_{\sigma,\alpha}$ are given by

$$d_{\sigma,\alpha}(k, n) = \frac{n!}{(n-k)!}\pi_{\sigma,\alpha}(k) := \frac{n!}{(n-k)!}\mathbb{1}_{k \leq n} \cdot \begin{cases} \frac{1}{\alpha^k} & \text{for } \sigma = 0 \\ \frac{\Gamma(\alpha)}{\Gamma(\alpha+k)} & \text{for } \sigma = +1. \\ \frac{\Gamma(\alpha+1-k)}{\Gamma(\alpha+1)} & \text{for } \sigma = -1. \end{cases} \tag{217}$$

Proof. We know that the process generated by $L^{\text{bulk}} := \sum_{i=1}^{N-1} L_{i,i+1}$ is self-dual with duality function $D_{\sigma,\alpha}^{\text{bulk}}$. This means that the action of L^{bulk} on $D_{\sigma,\alpha}^{\text{bulk}}(\cdot, \eta)$ is the same as its action on $D_{\sigma,\alpha}^{\text{bulk}}(\xi, \cdot)$. Thus, since L^{bulk} does not act on ξ_1 and ξ_n, we have that

$$[L^{\text{bulk}} D_{\sigma,\alpha}^{\text{bulk}}(\cdot, \eta)](\xi) = [L^{\text{bulk}} D_{\sigma,\alpha}^{\text{bulk}}(\xi, \cdot)](\eta). \tag{218}$$

It remains to be checked that the actions on the duality function of the boundary components of L and $\hat{L}$ are the same. In other words, we have to verify that

$$(L_{0,1} + L_{N,N+1}) D_{\sigma,\alpha}(\xi, \cdot)(\eta) = L_{\text{abs}} D_{\sigma,\alpha}(\cdot, \eta)(\xi). \tag{219}$$

We have,

$$\begin{aligned}
&L_{0,1}D_{\sigma,\alpha}(\xi,\cdot)(\eta) \qquad (220)\\
&= \eta_1(\alpha+\sigma\rho_L)[D_{\sigma,\alpha}(\xi,\eta-\delta_1)-D_{\sigma,\alpha}(\xi,\eta)] + \rho_L(\alpha+\sigma\eta_1)[D_{\sigma,\alpha}(\xi,\eta+\delta_1)-D_{\xi,\sigma,\alpha}(\eta)]\\
&= D_{\sigma,\alpha}(\xi,\eta)\frac{(\eta_1-\xi_1)!}{\eta_1!}\cdot\Big\{\rho_L(\alpha+\sigma\eta_1)\Big[\frac{(\eta_1+1)!}{\eta_1-1+\xi_1}-\frac{\eta_1!}{(\eta_1-\xi_1)}\Big]\\
&\qquad\qquad + \eta_1(1+\sigma\rho_L)\Big[\frac{(\eta_1-1)!}{(\eta_1-1-\eta_1)}-\frac{\eta_1!}{(\eta_1-\xi_1)!}\Big]\Big\}\\
&= D_{\sigma,\alpha}(\xi,\eta)\frac{\xi_1}{\eta_1+1-\xi_1}\cdot\{\rho_L(\alpha+\sigma\eta_1)-(1+\sigma\rho_L)(\eta_1+1-\xi_1)\}\\
&= D_{\sigma,\alpha}(\xi,\eta)\frac{\xi_1}{\eta_1+1-\xi_1}\cdot\{\rho_L(\alpha+\sigma\eta_1-\sigma)-(\eta_1+1-\xi_1)\}\\
&= \xi_1\Big[\rho_L\frac{(\alpha+\sigma\xi_1-\sigma)}{\eta_1+1-\xi_1}D_{\sigma,\alpha}(\xi,\eta)-D_{\sigma,\alpha}(\xi,\eta)\cdot\pi_{\sigma,\alpha}\Big]\\
&= \xi_1[D_{\sigma,\alpha}(\xi^{1,0},\eta)-D_{\sigma,\alpha}(\xi,\eta)].
\end{aligned}$$

Similarly we can show

$$L_{N,N+1}D_{\sigma,\alpha}(\xi,\cdot)(\eta) = \xi_N[D_{\sigma,\alpha}(\xi^{N,N+1},\eta)-D_{\sigma,\alpha}(\xi,\eta)]. \qquad (221)$$

Hence we have,

$$\begin{aligned}
&(L_{0,1}+L_{N,N+1})D_{\sigma,\alpha}(\xi,\cdot)(\eta) = \qquad (222)\\
&\quad \xi_1[D_{\sigma,\alpha}(\xi^{1,0},\eta)-D_{\sigma,\alpha}(\xi,\eta)]+\xi_N[D_{\sigma,\alpha}(\xi^{N,N+1},\eta)-D_{\sigma,\alpha}(\xi,\eta)] = L_{\text{abs}}D_{\sigma,\alpha}(\cdot,\eta)(\xi).
\end{aligned}$$

This concludes the proof. □

5.2 Hydrodynamic field

5.2.1 Stationary field

As mentioned before, the system will eventually converge to a steady state in the sense that the macroscopic profile converges over time. Once again we study this profile as the spatial limit of the hydrodynamic field. That is, we assume the system to reach its steady state and then calculate the hydrodynamic limit of the k-th order field. The duality result allows us to exactly calculate the expectation of the k-th order duality function. Indeed, since the dual particles move on a finite chain, all of them must eventually end up being absorbed at one of the reservoir sites. This means that

$$\lim_{t\to\infty}\xi(t) =: \xi(\infty) \in \{l\delta_0+m\delta_{N+1} | l,m>0 \text{ and } l+m=|\xi(0)|\} \qquad (223)$$

with probability one. This yields

$$\hat{\mathbb{E}}_{x_1,\dots,x_k}D_{\sigma,\alpha}(X_1(\infty),\dots,X_k(\infty);\eta) = \sum_{\substack{l,m>0\\ l+m=k}}\rho_L^l\rho_R^m\cdot\hat{\mathbb{P}}_{x_1,\dots,x_k}(\xi(\infty)=l\delta_0+m\delta_{N+1}). \qquad (224)$$

Here $\hat{\mathbb{E}}_{x_1,\dots,x_k}$ and $\hat{\mathbb{P}}_{x_1,\dots,x_k}$ denote the expectation and probability with respect to the dual process conditioned on $\xi(0)=\delta_{x_1}+\dots+\delta_{x_k}$. In the case where $\sigma=0$ thing become a lot easier since we have can write the probability above explicitly in terms of $x_1, ..., x_k$ and N. This due to the fact that for all $i \in \{1, ..., k\}$

$$\hat{\mathbb{P}}_{x_i}(X_i(\infty) = 0) = 1 - \frac{x_i}{N+1} \qquad \& \qquad \hat{\mathbb{P}}_{x_i}(X_i(\infty) = N+1) = \frac{x_i}{N+1}. \tag{225}$$

In the cases where $\sigma = \pm 1$, we will use a coupling to ignore the interactions in the hydrodynamic limit.

Coupling

In chapter 3.2 we referred to the couplings provided in [13] and [4] for inclusion and exclusion particles respectively. We can again use these couplings since they both have discrepancies growing sufficiently slow in time. In other words, the distance between each interacting particle and its independent counterpart is of order smaller than $\sqrt{t}$.

Proposition 5.2. *Consider interacting particles $\{X_1(t), ..., X_k(t) : t \geq 0\}$ on $\mathbb{Z}$ subject to either inclusion or exclusion dynamics with parameter α. Then there is a coupling $\{(X_1(t), ..., X_k(t); \tilde{X}_1(t), ..., \tilde{X}_k(t)) : t \geq 0\}$ such that,*

1. the coupled particles are independent: $\{\tilde{X}_1(t), ..., \tilde{X}_k(t) : t \geq 0\}$ are, in distribution, independent particles.

2. The coupled particles have the same initial position: $(\tilde{X}_1(0), ..., \tilde{X}_k(0)) = (X_1(0), ..., X_k(0))$.

3. The discrepancies between the independent and the original particles are of order less than $\sqrt{t}$:

$$\lim_{t\to\infty} \frac{|X_i(t) - \tilde{X}_i(t)|}{\sqrt{t}} = 0 \tag{226}$$

with probability one.

Proof. See chapter 3 in [13] for the SIP case and chapter 3 in [4] for the SEP case. □

We can use the fact that the discrepancies grow slower then $\sqrt{t}$ for the case without reservoirs to ignore the interactions of the dual particles. More precise, we can show the following lemma.

Lemma 5.3. *In the context described above,*

$$\sup_{x_1,...,x_k} |\hat{\mathbb{P}}_{x_1,...,x_k}(\xi(\infty) = l\delta_0 + m\delta_{N+1}) - \hat{\mathbb{P}}^{IRW}_{x_1,...,x_k}(\xi^{IRW}(\infty) = l\delta_0 + m\delta_{N+1})| \to 0 \text{ as } N \to \infty. \tag{227}$$

Here, $\hat{\mathbb{P}}^{IRW}_{x_1,...,x_k}$ denotes the path space measure associated to the dual particles in the case that $\sigma = 0$, i.e. random walkers which get absorbed at 0 and $N+1$, with initial positions $x_1, ..., x_k$. This process is denoted $\{\xi^{IRW}(t), t \geq 0\}$.

Proof. Let $x_1, ..., x_n \in \{1, ..., N\}$ be arbitrary. We consider particles $\{(X_1(t), ..., X_k(t)), t \geq 0\}$ which move on the integer line without reservoirs. Due to 5.2, we can construct a coupling $\{(X_1(t), ..., X_k(t); \tilde{X}_1(t), ..., \tilde{X}_k(t)), t \geq 0\}$ where $\{\tilde{X}_1(t), ..., \tilde{X}_k(t), t \geq 0\}$ are independent particles which move freely, i.e. without reservoirs, on $\mathbb{Z}$.

To prove (227), it is enough to show that the dual particle gets absorbed at the same site as the coupled independent particle with probability close to one for N large. The are two cases we have to consider: the case where the dual particle gets absorbed first and the case where the interacting particle is absorbed first. Without loss of generality we may assume that the absorption takes place at zero.

We begin with the case where the independent particle is absorbed first. Consider the couple $\{(X_i(t), \tilde{X}_i(t)), t \geq 0\}$. We renew the process $\{(X_1(t), ..., X_k(t); \tilde{X}_1(t), ..., \tilde{X}_k(t)), t \geq 0\}$ at the time τ, which is the time when $\{\tilde{X}_i(t), t \geq 0\}$ hits zero. That is, we construct again a coupling starting from $y_1 = X_1(\tau), ..., y_k = X_k(\tau)$. Now $\{(Y_1(t), ..., Y_k(t); \tilde{Y}_1(t), ..., \tilde{Y}_k(t)), t \geq \tau\}$ denote the positions of coupled dual particles and independent random walkers with initial positions $y_1, ..., y_k$. We argue that $\{Y_i(t), t \geq \tau\}$ hits zero before it hits $N+1$. To this end we let $T > \tau$ be arbitrary and make two observations,

Observation 1

$$\begin{aligned}
\mathbb{P}_{y_1,...,y_k}\Big(\exists \tau \leq t < T : \frac{Y_i(N^2 t)}{N} = 0\Big) &\geq \mathbb{P}_{y_1,...,y_k}\Big(\exists \tau \leq t < T : \frac{Y_i(N^2 t)}{N} < 0\Big) \qquad (228) \\
&= \mathbb{P}_{y_1,...,y_k}\Big(\exists \tau \leq t < T : \frac{\tilde{Y}_i(N^2 t)}{N} < \frac{\tilde{Y}_i(N^2 t) - Y_i(N^2 t)}{N}\Big) \\
&= \mathbb{P}^{BM}_{\frac{y_i}{N}}\Big(\exists \tau \leq t < T : W(t) < 0\Big) + O_{N^{-1}}(1) \\
&= \mathbb{P}^{BM}_{0}\Big(\exists \tau \leq t < T : W(t) < 0\Big) + O_{N^{-1}}(1). \\
&= 1 + O_{N^{-1}}(1) \qquad (229)
\end{aligned}$$

Here $\mathbb{P}_{y_1,...,y_k}$ denotes the path space measure corresponding to the coupling with initial positions $y_1, ..., y_k$ and $\mathbb{P}^{BM}_y$ denotes the path space measure of Brownian motion $\{W(t), t \geq 0\}$ starting from y. We write $O_{N^{-1}}(1)$ for a term going to zero whenever N goes to infinity. In the first step we use the fact that $y_1 > 0$, this is true since we are looking at the case where the independent particle from the initial coupling hits zero first. In the third step we use that the discrepancies of the coupled particles vanish under hydrodynamic scaling:

$$\frac{\tilde{Y}_i(N^2 t) - Y_i(N^2 t)}{N} = \sqrt{t} \cdot \frac{\tilde{Y}_i(N^2 t) - Y_i(N^2 t)}{N\sqrt{t}} \to 0 \text{ as } N \to \infty, \qquad (230)$$

with probability one. The fourth step is the most tricky one. We use that $\frac{y_i}{N}$ goes to zero as $N \to \infty$. Recall that y_i was the position of the interacting particle with index i when its independent counterpart was absorbed. Since the discrepancies vanish with probability one under hydrodynamic scaling, we see that with probability one y_i/N goes to zero whenever N goes to infinity. Finally we use that the minimum of the Brownian motion before time $t > 0$ is distributed as minus the absolute value of the Brownian motion at time t. The latter is clearly smaller then zero with probability one.

Observation 2

$$\begin{aligned}
\mathbb{P}_{y_1,...,y_k}\Big(\exists \tau \leq t < T : \frac{Y_i(N^2 t)}{N+1} = 1\Big) &= \mathbb{P}^{BM}_{0}\Big(\exists \tau \leq t < T : W(t) = 1\Big) + O_{N^{-1}}(1) \qquad (231) \\
&= O_T(1) + O_{N^{-1}}(1).
\end{aligned}$$

For the first equality we use similar reasoning as for step two to four in observation one. For the second equality we observe that the probability that Brownian motion travels from zero to one in time T decays as T gets closer to zero. This can again be seen from the fact the maximum before time $t \geq 0$ of the Brownian motion starting from zero has the same distribution as the absolute value of Brownian motion starting from zero. We denote a term which vanishes as T get large by $O_T(1)$.

Since one can pick $T > 0$ arbitrarily close to zero, observations one and two yield that $\{X_i(t), t \geq 0\}$ hits 0 before it hits $N+1$ with probability one as $N \to \infty$.

The case where the interacting particle hits zero first can be shown using essentially the same method, only the first two steps of observation one won't be necessary. □

Remark 5.4. *In the proof above we actually show slightly more then is strictly required for lemma 5.3. The two observations not only show that each interacting dual particle gets absorbed at the same site as its non-interacting counterpart, they also show that the macroscopic time at which they get absorbed vanishes as well. That is, for the coupled particles with index* $i \in \{1, ..., k\}$, $\{(X_i(t), \tilde{X}_i(t)), t \geq 0\}$, *we have that the difference between the absorbing times*

$$\tau_{abs}^N = \inf\left\{t \geq 0, \frac{X_i(N^2 t)}{N} \in \{0, N/(N+1)\}\right\} \tag{232}$$

and

$$\tilde{\tau}_{abs}^N = \inf\left\{t \geq 0, \frac{\tilde{X}_i(N^2 t)}{N} \in \{0, N/(N+1)\}\right\} \tag{233}$$

vanishes under hydrodynamic scaling, $\forall \epsilon > 0$,

$$\mathbb{P}_{x_1,...,x_k}(|\tau_{abs}^N - \tilde{\tau}_{abs}^N| > \epsilon) \to 0 \text{ as } N \to \infty. \tag{234}$$

As a consequence, we can ignore interactions when we compute the hydrodynamic field for interacting particles with absorption. For $\rho : [0,1]^k \to \infty$,

$$\sup_{x_1,...,x_k} |\hat{\mathbb{E}}_{x_1,...,x_k}\rho(N^{-1}X_1(N^2t), ..., N^{-1}X_k(N^2t)) - \hat{\mathbb{E}}^{IRW}_{x_1,...,x_k}\rho(N^{-1}\tilde{X}_1(N^2t), ..., N^{-1}\tilde{X}_k(N^2t))| \to 0 \tag{235}$$

as $N \to \infty$. *Here,* $\hat{\mathbb{E}}^{IRW}_{x_1,...,x_k}$ *denotes the expectation with respect to the path space associated to k independent random walkers with initial positions* $x_1, ..., x_k$. *This fact will be used in the proof of theorem 5.8, which concerns the time dependent hydrodynamic field for systems with reservoirs.*

Steady state macroscopic field

At this point we are ready to compute the limit $t \to \infty$ of the k-th order macroscopic profile for the system with reservoirs, regardless of σ. In chapter 3 we established that the macroscopic profile for SIP, SEP and IRW, without reservoirs, evolves as described by the heat equation. It turns out the macroscopic steady state $\rho : [0,1]^k \to \mathbb{R}$ we find for the system with reservoirs is the k-fold product of the solution to the 1-dimensional Poisson equation $\bar{\rho} : [0,1] \to \mathbb{R}$ with *boundary conditions*. That is,

$$\rho(x_1, ..., x_k) = \bar{\rho}(x_1)...\bar{\rho}(x_k). \tag{236}$$

Since the steady state solution of the heat equation is given by the Poisson equation, one could interpret the addition of reservoirs on the microscale as equivalent to adding boundary conditions on the macroscale. Moreover, this analogy is also true for the steady state profile in almost the same way. As we will see in the next subsection, the macroscopic dynamic can be constructed as a product of the solution to the one dimensional heat equation evaluated in different points. However, we need that the initial condition is already in product form. Before we do so, we show our claim for the steady state. We first first explicitly state the form of the field.

$$\chi_N^k(\varphi,\eta) = N^{-k} \sum_{x_1,...,x_k=1}^{N} \varphi\Big(\frac{x_1}{N},...,\frac{x_k}{N}\Big) D_{\sigma,\alpha}(x_1,...,x_k;\eta). \tag{237}$$

Theorem 5.5. *Let $\sigma \in \{-1,0,1\}$ and let $\{\eta(t), t \geq 0\}$ be the associated process with reservoirs, as introduced in subsection 5.1. Then,*

$$\lim_{N\to\infty} \lim_{t\to\infty} \mathbb{E}_{\eta(0)}[\chi_N^k(\varphi,\eta(N^2 t)] = \int_0^1 ... \int_0^1 \varphi(x_1,...,x_k)\rho(x_1,...,x_k) dx_1...dx_k, \tag{238}$$

where $\rho : [0,1]^k \to \mathbb{R}$ is given by $\rho(x_1,...,x_k) = \bar{\rho}(x_1)...\bar{\rho}(x_k)$, with $\bar{\rho}(x) = \rho_L - (\rho_L - \rho_R))x$. Notice that $\bar{\rho}$ is in fact the solution to the one dimensional Poisson equation,

$$\begin{cases} 0 = \frac{\partial^2 \bar{\rho}}{dx^2} \\ \bar{\rho}(0) = \rho_L \\ \bar{\rho}(1) = \rho_R. \end{cases} \tag{239}$$

Proof. We begin by using duality to pass to the absorbed process and take the limit $t \to \infty$,

$$\begin{aligned} &\lim_{t\to\infty} \mathbb{E}_{\eta(0)}[\chi_N^k(\varphi,\eta(N^2t)] \\ &\qquad = \lim_{t\to\infty} N^{-k} \sum_{x_1,...,x_k=1}^{N} \varphi\Big(\frac{x_1}{N},...,\frac{x_k}{N}\Big) \mathbb{E}_{\eta(0)}[D_{\sigma,\alpha}(x_1,...,x_k;\eta(N^2t))] \\ &\qquad = \lim_{t\to\infty} N^{-k} \sum_{x_1,...,x_k=1}^{N} \varphi\Big(\frac{x_1}{N},...,\frac{x_k}{N}\Big) \hat{\mathbb{E}}_{x_1,...,x_k}[D_{\sigma,\alpha}(X_1(N^2t),...,X_k(N^2t);\eta)] \\ &\qquad = N^{-k} \sum_{x_1,...,x_k=1}^{N} \varphi\Big(\frac{x_1}{N},...,\frac{x_k}{N}\Big) \hat{\mathbb{E}}_{x_1,...,x_k}[D_{\sigma,\alpha}(X_1(\infty),...,X_k(\infty);\eta)] \end{aligned} \tag{240}$$

Next we make use what we stated in equation 224,

$$\hat{\mathbb{E}}_{x_1,...,x_k} D_{\sigma,\alpha}(X_1(\infty),...,X_k(\infty);\eta) = \sum_{\substack{l,m>0 \\ l+m=k}} \rho_L^l \rho_R^m \cdot \hat{\mathbb{P}}_{x_1,...,x_k}(\xi(\infty) = l\delta_0 + m\delta_{N+1}). \tag{241}$$

Recall that this holds due to the fact that the time when the last dual particle gets absorbed is finite with probability one. Due to lemma 5.3 we can ignore the interactions in the calculation of the probability above,

$$\begin{aligned} &\lim_{t\to\infty} \mathbb{E}_{\eta(0)}[\chi_N^k(\varphi,\eta(N^2t)] \\ &= N^{-k} \sum_{x_1,...,x_k=1}^{N} \varphi\Big(\frac{x_1}{N},...,\frac{x_k}{N}\Big) \sum_{\substack{l,m>0 \\ l+m=k}} \rho_L^l \rho_R^m \cdot \hat{\mathbb{P}}_{x_1,...,x_k}(\xi(\infty) = l\delta_0 + m\delta_{N+1}) \\ &= N^{-k} \sum_{x_1,...,x_k=1}^{N} \varphi\Big(\frac{x_1}{N},...,\frac{x_k}{N}\Big) \sum_{\substack{l,m>0 \\ l+m=k}} \rho_L^l \rho_R^m \cdot \hat{\mathbb{P}}^{IRW}_{x_1,...,x_k}(\xi^{IRW}(\infty) = l\delta_0 + m\delta_{N+1}) + O_{N^{-1}}(1). \end{aligned} \tag{242}$$

We write $(X_1^{IRW}(\infty), ..., X_k^{IRW}(\infty))$ for the absorbed independent particles in ξ^{IRW}. Then, since there are no interactions,

$$\begin{aligned}
&\hat{\mathbb{P}}^{IRW}_{x_1,...,x_k}(\xi^{IRW}(\infty) = l\delta_0 + m\delta_{N+1}) \qquad (243)\\
&= \sum_{(S_0,S_{N+1})\in S^{l,m}} \prod_{i\in S_0} \hat{\mathbb{P}}^{IRW}_{x_i}(X_i^{IRW}(\infty) = 0) \prod_{j\in S_{N+1}} \hat{\mathbb{P}}^{IRW}_{x_j}(X_j^{IRW}(\infty) = N+1)
\end{aligned}$$

Here, $S^{l,m} := \{(S_0, S_{N+1}) : \{S_0, S_{N+1}\} \text{ partition } \{1, ..., m\}), |S_0| = l, |S_{N+1}| = m\}$. We essentially sum over all possible ways the dual particles could form $\xi^{IRW}(\infty) = l\delta_0 + m\delta_{N+1}$ here. We notice the following combinatorial fact,

$$\begin{aligned}
&\sum_{\substack{l,m>0\\ l+m=k}} \rho_L^l \rho_R^m \Bigg[\sum \sum_{(S_0,S_{N+1})\in S^{l,m}} \prod_{i\in S_0} \hat{\mathbb{P}}^{IRW}_{x_i}(X_i^{IRW}(\infty) = 0) \prod_{j\in S_{N+1}} \hat{\mathbb{P}}^{IRW}_{x_j}(X_j^{IRW}(\infty) = N+1)\Bigg] \qquad (244)\\
&= \prod_{i=1}^{k} \Big[\rho_L \hat{\mathbb{P}}^{IRW}_{x_i}(X_i^{IRW}(\infty) = 0) + \rho_R \hat{\mathbb{P}}^{IRW}_{x_i}(X_i^{IRW}(\infty) = N+1)\Big].
\end{aligned}$$

It is easy to check that the, when expanded into a sum, both sides of the equation have exactly the same terms. The rest of the proof is just calculation, as we know explicit expressions for $\hat{\mathbb{P}}^{IRW}_{x_i}(X_i(\infty) = 0)$ and $\hat{\mathbb{P}}^{IRW}_{x_i}(X_i(\infty) = N+1)$. These are the following,

$$\hat{\mathbb{P}}^{IRW}_{x_i}(X_i^{IRW}(\infty) = 0) = 1 - \frac{x_i}{N+1} \qquad \& \qquad \hat{\mathbb{P}}^{IRW}_{x_i}(X_i^{IRW}(\infty) = N+1) = \frac{x_i}{N+1}. \qquad (245)$$

All together we have,

$$\begin{aligned}
&\lim_{N\to\infty} \lim_{t\to\infty} \mathbb{E}_{\eta(0)}[\chi_N^k(\varphi, \eta(N^2 t)] = \qquad (246)\\
&= \lim_{N\to\infty} N^{-k} \sum_{x_1,...,x_k=1}^{N} \varphi\Big(\frac{x_1}{N}, ..., \frac{x_k}{N}\Big) \sum_{\substack{l,m>0\\ l+m=k}} \rho_L^l \rho_R^m \cdot \hat{\mathbb{P}}^{IRW}_{x_1,...,x_k}(\xi^{IRW}(\infty) = l\delta_0 + m\delta_{N+1})\\
&= \lim_{N\to\infty} N^{-k} \sum_{x_1,...,x_k=1}^{N} \varphi\Big(\frac{x_1}{N}, ..., \frac{x_k}{N}\Big)\\
&\qquad \cdot \prod_{i=1}^{k} \Big[\rho_L \hat{\mathbb{P}}^{IRW}_{x_i}(X_i^{IRW}(\infty) = 0) + \rho_R \hat{\mathbb{P}}^{IRW}_{x_i}(X_i^{IRW}(\infty) = N+1)\Big]\\
&= \lim_{N\to\infty} N^{-k} \sum_{x_1,...,x_k=1}^{N} \varphi\Big(\frac{x_1}{N}, ..., \frac{x_k}{N}\Big) \prod_{i=1}^{k} \Big[\rho_L\Big(1 - \frac{x_i}{N+1}\Big) + \rho_R \frac{x_i}{N+1}\Big]\\
&= \int_0^1 ... \int_0^1 \varphi(x_1, ..., x_k) \prod_{i=1}^{k} \Big[\rho_L - (\rho_L - \rho_R)x_i\Big] dx_1 ... dx_k,
\end{aligned}$$

which is what we set out to prove. □

Remark 5.6. *Notice that the time scaling in (238) is in fact redundant. Indeed, we take the limit $t \to \infty$ first, which yields the final state of the dual process $\xi(\infty)$ in the proof above. Clearly $\xi(\infty)$ is independent from N.*

5.2.2 Dynamic field

We established that the k-th order stationary field for the systems with reservoirs is in fact a k-fold product of the first order stationary fields. Moreover, we saw that the first order stationary fields are steady states of the heat equation, where the boundary conditions are given by the reservoir parameters. One might suspect that, similarly to the stationary case, the field at each time $t > 0$ is a k-fold product of first order fields at time t, where the first order fields are simply the solution to the heat equation with boundary conditions given by the reservoir parameters. This intuition turns out to be true, but one should notice that such a statement only makes sense in the case where the initial macroscopic field $\rho_0 : [0, \infty) \times [0, 1]^k \to \mathbb{R}$ is already a k-fold product. This is made precise in the theorem below. Before we state the result, we reformulate the definition of consistency in this setting,

Definition 5.7. *(consistent measures) Let $\rho : [0, 1]^k \to \mathbb{R}$ be a smooth and bounded function. We say that the family of probability measures μ_N, $N \in \mathbb{N}$ has expected density consistent with ρ if for all $N \in \mathbb{N}$, and $x = (x_1, ..., x_k) \in [0, 1]^k$,*

$$\int D_{\sigma,\alpha}(x_1, ..., x_k; \eta) d\mu_N(\eta) = \rho\Big(\frac{x_1}{N}, ..., \frac{x_k}{N}\Big). \tag{247}$$

Theorem 5.8. *Let $\{\mu_N, N \in \mathbb{N}\}$ denote a family of probability measures on the state space compatible with a smooth and bounded profile $\rho_0 : [0, 1]^k \to \mathbb{R}$ of the form*

$$\rho_0(x_1, ..., x_k) = \rho_0^{(1)}(x_1) \cdot \rho_0^{(2)}(x_2) \cdot ... \cdot \rho_0^{(k)}(x_k) \tag{248}$$

with $\rho_0^{(i)} : [0, 1] \to \mathbb{R}$ smooth and bounded, $\forall i \in \{1, ..., k\}$. Then, for all $t \geq 0$, the time evolved k-th order field converges as

$$\int \chi_N^k(\eta(N^2 t), \varphi) d\mu_N(\eta) \xrightarrow{N\to\infty} \int_0^1 ... \int_0^1 \rho(t; x_1, ..., x_k)\varphi(x_1, ..., x_k) dx_1...dx_k. \tag{249}$$

Here χ_N^k is as in (237) and ρ is given by,

$$\rho(t; x_1, ..., x_k) = \rho^{(1)}(t; x_1) \cdot \rho^{(2)}(t; x_2) \cdot ... \cdot \rho^{(k)}(t; x_k). \tag{250}$$

The functions $\rho^{(i)} : [0, \infty) \times [0, 1] \to \mathbb{R}$ are the solution to

$$\begin{cases} \frac{\partial\rho}{\partial t} = \frac{\partial^2\rho}{dx^2} \\ \rho(0, t) = \rho_L \\ \rho(1, t) = \rho_R \\ \rho(0, x) = \rho_0^{(i)}(x). \end{cases} \tag{251}$$

Proof. As usually, we can exploit duality to obtain the field in terms of the dual process,

$$\begin{aligned} \mathbb{E}_{\eta(0)}[\chi_N^k(\varphi,\eta] &= N^{-k} \sum_{x_1,...,x_k=1}^{N} \varphi\Big(\frac{x_1}{N}, ..., \frac{x_k}{N}\Big) \mathbb{E}_{\eta(0)}[D_{\sigma,\alpha}(x_1, ..., x_k; \eta)] \\ &= N^{-k} \sum_{x_1,...,x_k=1}^{N} \varphi\Big(\frac{x_1}{N}, ..., \frac{x_k}{N}\Big) \hat{\mathbb{E}}_{x_1,...,x_k}[D_{\sigma,\alpha}(X_1, ..., X_k; \eta)] \end{aligned} \tag{252}$$

The scaling time and integration against μ_N give

$$\int \chi_N^k(\eta(N^2t),\varphi)d\mu_N(\eta) \tag{253}$$
$$= N^{-k}\sum_{x_1,...,x_k=1}^{N}\varphi\Big(\frac{x_1}{N},...,\frac{x_k}{N}\Big)\hat{\mathbb{E}}_{x_1,...,x_k}\rho(N^{-1}X_1(N^2t),...,N^{-1}X_k(N^2t))$$
$$= N^{-k}\sum_{x_1,...,x_k=1}^{N}\varphi\Big(\frac{x_1}{N},...,\frac{x_k}{N}\Big)\hat{\mathbb{E}}^{IRW}_{x_1,...,x_k}\rho(N^{-1}X_1(N^2t),...,N^{-1}X_k(N^2t))+O_{N^{-1}}(1)$$
$$= N^{-k}\sum_{x_1,...,x_k=1}^{N}\varphi\Big(\frac{x_1}{N},...,\frac{x_k}{N}\Big)\hat{\mathbb{E}}^{IRW}_{x_1}\rho_0^{(1)}(N^{-1}X_1(N^2t))\cdot...\cdot\hat{\mathbb{E}}^{IRW}_{x_k}\rho_0^{(k)}(N^{-1}X_k(N^2t))+O_{N^{-1}}(1).$$

Here, $\hat{\mathbb{E}}^{IRW}_{x_1,...,x_k}$ denotes the expectation with respect to the path space associated to k independent random walkers with initial positions $x_1,...,x_k$. In the second equality we use that we can couple the absorbed interactive particles to absorbed independent random walkers (see remark 5.4) and in the third equality we use the independence in combination with the assumption that ρ_0 is a k-fold product. From this point onwards we are concerned the time evolution of $\rho_0^{(i)}$ for a fixed value of i. To make the notation simpler we will write $\bar{\rho}_0$ instead of $\rho_0^{(i)}$.

It is clear that the absorbed random walk scales hydrodynamically to absorbed Brownian motion. Hence we have that

$$\hat{\mathbb{E}}^{IRW}_x[\bar{\rho}_0(N^{-1}X(N^2t))] = \hat{\mathbb{E}}^{BM}_{\frac{x}{N}}[\bar{\rho}_0(W^{\text{abs}}(t))]+o_{N^{-1}}(1), \tag{254}$$

where $\hat{\mathbb{E}}^{BM}$ denotes the expectation with respect to the path space measure of the absorbed Brownian motion, W^{abs}, with initial position x. All we have to do now is show that

$$\hat{\mathbb{E}}^{BM}_y\bar{\rho}_0(W^{\text{abs}}(t)) = \bar{\rho}_0(t,x). \tag{255}$$

This can easily be be seen from the following consideration. Let $\bar{\rho}_0^{\text{ext}}$ be an extension of $\bar{\rho}_0$ on the real numbers such that $\forall x\in\mathbb{R}$,

$$\bar{\rho}_0^{\text{ext}}(-x) = -\bar{\rho}_0^{\text{ext}}(x) \qquad \& \qquad \bar{\rho}_0^{\text{ext}}(1-x) = -\bar{\rho}_0^{\text{ext}}(1+x). \tag{256}$$

In other words, $\bar{\rho}_0^{\text{ext}}$ is an odd function around 0 and around 1. Notice that, under these two constraints, the extension is unique. We claim another property from this extension, namely

$$\hat{\mathbb{E}}^{BM}_y\bar{\rho}_0(W^{\text{abs}}(t)) = \mathbb{E}^{BM}_y\bar{\rho}_0^{\text{ext}}(W(t)). \tag{257}$$

Here $\mathbb{E}^{BM}_y$ denotes the expectation with respect to the path space measure associated to Brownian motion without absorption, $W(t)$, with initial position y. Indeed, by the symmetry of $\bar{\rho}^{\text{ext}}$ we have for all $t\geq 0$

$$\mathbb{E}^{BM}_0[\bar{\rho}_0^{\text{ext}}(W(t))] = \rho_L \qquad \& \qquad \mathbb{E}^{BM}_1[\bar{\rho}_0^{\text{ext}}(W_0(t))] = \rho_R, \tag{258}$$

so even though the free Brownian motion $W(t)$ keeps moving after hitting $\{0,1\}$, the relevant expectation remains as if it were absorbed. As we have seen in subsection 3.1, the function $\bar{\rho}^{\text{ext}}(t,x) := \mathbb{E}^{BM}_x\bar{\rho}^{\text{ext}}(W(t))$ is the solution to

$$\begin{cases}\frac{\partial\bar{\rho}^{\text{ext}}}{\partial t} = \frac{\partial^2\bar{\rho}^{\text{ext}}}{dx^2}\\ \bar{\rho}^{\text{ext}}(0,x) = \bar{\rho}_0^{\text{ext}}(x).\end{cases} \tag{259}$$

Because of the symmetry requirements and the fact that $\bar{\rho}_0$ is smooth on $[0,1]$, we must have that $\bar{\rho}_0^{\text{ext}}$ is smooth as well. Moreover,

$$\frac{\partial^2 \bar{\rho}_0^{\text{ext}}}{dx^2}(0) = \frac{\partial^2 \bar{\rho}_0^{\text{ext}}}{dx^2}(1) = 0. \tag{260}$$

Hence, the $\bar{\rho}^{\text{ext}}$ is stationary in 0 and 1, showing that $\forall x \in [0,1]$ and $\forall t \geq 0$

$$\bar{\rho}^{\text{ext}}(t,x) = \bar{\rho}(t,x). \tag{261}$$

Going back to our expression for the field, we obtain

$$\int \chi_N^k(\eta(N^2 t), \varphi) d\mu_N(\eta) \tag{262}$$

$$= N^{-k} \sum_{x_1,...,x_k=1}^{N} \varphi\Big(\frac{x_1}{N},...,\frac{x_k}{N}\Big) \hat{\mathbb{E}}_{x_1}^{IRW} \rho_0^{(1)}(N^{-1}X_1(N^2t)) \cdot ... \cdot \hat{\mathbb{E}}_{x_k}^{IRW} \rho_0^{(k)}(N^{-1}X_k(N^2t)) + o_{N^{-1}}(1).$$

$$= N^{-k} \sum_{x_1,...,x_k=1}^{N} \varphi\Big(\frac{x_1}{N},...,\frac{x_k}{N}\Big) \rho^{(1)}\Big(t,\frac{x_1}{N}\Big) \cdot ... \cdot \rho^{(k)}\Big(t,\frac{x_k}{N}\Big) + o_{N^{-1}}(1).$$

$$\to \int_0^1 \varphi(x_1,...,x_k)\rho(x_1,...,x_k;t) dx_1...dx_k \text{ as } N \to \infty. \tag{263}$$

□

Remark 5.9. *In case $\rho_0 : [0,\infty) \times [0,1]^k \to \mathbb{R}$ is not of the form*

$$\rho_0(x_1,...,x_k) = \rho_0^{(1)}(x_1) \cdot \rho_0^{(2)}(x_2) \cdot ... \cdot \rho_0^{(k)}(x_k), \tag{264}$$

one can use that for measure spaces $(\Sigma_1,\mu_1),...,(\Sigma_N,\mu_N)$, the L^2 space on the product of these measure spaces satisfies

$$L^2(\Sigma_1,...,\Sigma_N,\mu_1,...,\mu_N) = L^2(\Sigma_1,\mu_1) \otimes ... \otimes L^2(\Sigma_N,\mu_N). \tag{265}$$

Hence in our case $L^2([0,1]^k, dx^k) = L^2([0,1],dx)^{\otimes k}$. Furthermore, the basis of this tensor space consists of functions of the form $f(x_1,...,x_k) = f_1(x_1) \otimes f_2(x_2) \otimes ... \otimes f_k(x_k) = f_1(x_1)f_2(x_2)...f_k(x_k)$ with $f_1,...,f_k$ basis functions for $L^2([0,1],dx)$. Hence we can always express ρ_0 as a linear combination of functions which are of the form in 264.

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