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Abstract

While solving the Biot's poro-elasticity equations numerically often non-physical oscillations (also called 'wiggles') for the pressure function are obtained. For many applications the resulting functions with such numerical 'wiggles' are unacceptable. Therefore, the task of understanding the source of the wiggles and finding the numerical method, which allows to decrease or even avoid these oscillations in the solution, is of considerable significance. One of the available methods to increase the accuracy of the numerical solution is to choose sufficiently small elements in the regions where the gradient of pressure has a large value. The results for some test problems obtained using various approaches for solution quality improvement (including a 'stabilization' technique proposed by [Aguilar et al. 2008] along with its variations) have been analyzed. Through the series of numerical experiments the errors in the numerical solutions provided by different numerical approaches were compared for similar problems.

1 Introduction

Our aim here is to study a few cases when a numerical solution for the poro-elastic problem contains non-physical oscillations, and to investigate some numerical methods which allow to take the oscillations away.

In section 2 we start with the one dimensional problem. The source of the numerical wiggles in the solution for the pressure is studied and several ways to provide an oscillation free solution are considered. The test problem is solved with different numerical methods in order to determine whether or not they are applicable, and under what circumstances we can use them. The errors provided by such methods are estimated.

In section 3 we proceed with a two dimensional problem. The 'stabilization' techniques analogous to those from section 2 are introduced. After carrying out the series of the numerical experiments the accuracy provided by various 'stabilization' techniques is investigated.

2 One-dimensional problem for poro-elastic body

At the beginning of this section we introduce the 1D problem. While studying the 1D model we focus on the source of the numerical wiggles and on the numerical techniques allowing to decrease the influence of that 'source' on results being obtained.

In this section we are going to consider the 'worst' formulations of the problem to get familiar with what we could have to deal with when solving the practical problems. Though 1D problems have a little practical significance nowadays, since researches are equipped with tools and resources which allow to solve much more complicated problems, than 1D problems. But the information we get while investigating the 1D problems appears to be useful for the study of 2D and 3D poro-elastic models.

While formulating the 1D problem we consider the case, when the problem solution has very large gradients. Particularly, we are going to investigate the pressure solution for 1D poro-elastic problem, spatial derivative of which is infinitely large at the point $x = 0$ when $t = 0$, (actually $p(0, 0) = \delta(0)$, where $\delta(0)$ is Dirac delta function). The pressure itself is discontinuous initially. If the exact solution has such singularities the wiggles in the numerical solution are likely to appear. Hence the FEM mesh and solution approximations have to be chosen carefully in this case. Moreover, it does not seem to be possible to remove the wiggles in the numerical solution

at $t = 0$ through the mesh refinement for certain classes of elements widely used for solving a poro-elastic problems. Now we are going to study this problem thoroughly.

The problem is defined in the following way: find the functions $u(x,t)$ and $p(x,t)$ defined on the domain $[0,1] \times (0,T]$ which are differentiable with respect to t and twice differentiable with respect to x, and which for any $t \in (0,T]$ satisfy the following PDE's:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial p}{\partial x} = 0 \qquad \text{for} \quad (x,t) \in (0,1) \times (0,T] \tag{1}$$

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial x}) - \frac{\partial^2 p}{\partial x^2} = 0 \qquad \text{for} \quad (x,t) \in (0,1) \times (0,T] \tag{2}$$

Boundary conditions are given in the form:

$$\frac{\partial u}{\partial x}(0,t) = -1, \quad p(0,t) = 0, \quad t \in (0,T] \tag{3}$$

$$u(1,t) = 0, \quad \frac{\partial p}{\partial x}(1,t) = 0, \quad t \in (0,T] \tag{4}$$

One of the terms of equation (2) is a partial derivative of $\frac{\partial u}{\partial x}$ by time. Hence we have to specify the initial condition for $\frac{\partial u}{\partial x}$. It is not required for this problem to specify an initial conditions for the function p. But from the physical point of view it could be interesting to have the values of the pressure at the initial moment. Therefore, we consider functions $u_0 : [0,1] \to \Re$ and $p_0 : [0,1] \to \Re$, which represent the displacements and the pressure in one-dimensional poro-elastic body at $t = 0$, i.e.:

$$u(x,0) = u_0(x) \quad x \in [0,1] \qquad p(x,0) = p_0(x) \quad x \in [0,1]$$

Functions u_0 and p_0 are found from equilibrium equation (5) and incompressibility equation (6) :

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}x^2} - \frac{\mathrm{d}p_0}{\mathrm{d}x} = 0 \qquad \text{for} \quad x \in (0,1) \tag{5}$$

$$\frac{\mathrm{d}u_0}{\mathrm{d}x} = 0 \qquad \text{for} \quad x \in (0,1) \tag{6}$$

At the boundary points the following conditions are prescribed:

$$\frac{\mathrm{d}u_0}{\mathrm{d}x}(0) = -1, \quad p_0(0) = 0 \tag{7}$$

$$u_0(1) = 0, \quad \frac{\mathrm{d}p_0}{\mathrm{d}x}(0) = 0 \tag{8}$$

From the above equations we can derive the expression for $\frac{\mathrm{d}u_0}{\mathrm{d}x}$:

$$\frac{\mathrm{d}u_0}{\mathrm{d}x}(x) = \begin{cases} 0, & x \in (0,1] \\ -1, & x = 0 \end{cases} \tag{9}$$

or using a Heaviside step function H:

$$\frac{\mathrm{d}u_0}{\mathrm{d}x}(x) = 2H(x) - 2 \quad x \in [0,1] \tag{10}$$

$$H(x) = \begin{cases} 0, & x < 0 \\ 1/2, & x = 0 \\ 1, & x > 0 \end{cases}$$

Then the displacements at $t = 0$ are found as:

$$u_0(x) = 2(R(x) - x) \qquad x \in [0,1] \tag{11}$$

where R is the ramp function. Then from (5) and (10) we obtain:

$$\frac{\mathrm{d}p_0}{\mathrm{d}x}(x) = \frac{\mathrm{d}^2u_0}{\mathrm{d}^2x}(x) = 2H'(x) = 2\delta(x) \quad x \in [0,1] \tag{12}$$

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \in (-\infty, +\infty)\backslash\{0\} \end{cases}$$

where δ is a Dirac delta function. Therefore the first derivative of p_0 (it also corresponds to the gradient of pressure for $t = 0$ in one-dimensional case) and the second derivative of u_0 take infinitely large values at $x = 0$. From (12) and (7) we express:

$$p_0(x) = \begin{cases} 0, & x = 0 \\ 1, & x \in (0,1] \end{cases} \tag{13}$$

or

$$p_0(x) = 2H(x) - 1 \quad x \in [0,1] \tag{14}$$

Now we proceed with the solution of the problem for $0 < t \leq T$. It is given by the analytic expressions:

$$u(x,t) = 1 - x - \sum_{m=0}^{\infty} \frac{2}{\lambda_m^2} \cos \lambda_m x \; e^{-\lambda_m^2 t} \quad (x,t) \in [0,1] \times (0,T] \tag{15}$$

$$p(x,t) = \sum_{m=0}^{\infty} \frac{2}{\lambda_m} \sin \lambda_m x \; e^{-\lambda_m^2 t} \quad (x,t) \in [0,1] \times (0,T] \tag{16}$$

$$\lambda_m = \frac{2m+1}{2}\pi \quad m = 0, 1, 2, ... \tag{17}$$

It could be verified, that for functions u_0, p_0, u and p, defined in (11), (14), (15) and (16) correspondingly, the following relations hold:

$$\lim_{t \to 0} u(x,t) = u_0(x) \quad \text{for each } x \in [0,1] \tag{18}$$

$$\lim_{t \to 0} p(x,t) = p_0(x) \quad \text{for each } x \in [0,1] \tag{19}$$

$$\lim_{t \to 0} \frac{\partial u}{\partial x}(x,t) = \frac{\mathrm{d}u_0}{\mathrm{d}x}(x) = 0 \quad \text{for each } x \in [0,1] \tag{20}$$

Expressions (18), (19) and (18) denote a pointwise convergence.

For a small t the pressure gradient is characterized by the following equalities:

$$\lim_{t \to 0} \frac{\partial p}{\partial x}(x,t) = \frac{\mathrm{d}p_0}{\mathrm{d}x}(x) = 0 \quad \text{for each } x \in (0,1] \tag{21}$$

$$\lim_{t \to 0} \frac{\partial p}{\partial x}(0,t) = +\infty = \frac{\mathrm{d}p_0}{\mathrm{d}x}(0) \tag{22}$$

We find it is important to make some remarks about the behavior of analytical solution (15) and (16). At $t = 0$ the gradient of the pressure is infinite on boundary $x = 0$ (equation (12)). However, when considering the infinite sum for function $\frac{\mathrm{d}p}{\mathrm{d}x}$ derived from (16), it is verified, that for any fixed $t \in (0,T]$ (even though $t > 0$ is very small) this infinite sum converges. That means, that when the gradient of pressure has considerably large values in the small region close to $x = 0$ for a fixed small $t > 0$, it is still bounded in its domain (i.e. it is no longer infinite at $x = 0$). Gradually while t is increasing, the initial perturbation dissipates inside the poro-elastic body due to diffusion. The pressure gradient values in the neighborhood of $x = 0$ also becomes smaller and smaller in time, and the function p becomes 'more smooth'.

Now we have to introduce the weak formulation of the problem, and then we will derive the finite element model to get the discrete solution for the problem under consideration.

2.1 Weak formulation

Let us introduce first the function spaces in which the weak solutions should be found. $L^2[0,1]$ is a space of square-integrable functions. The inner product and the norm in this space are given by:

$$(p,q) = \int_0^1 pq dx \tag{23}$$

$$\|p\| = (p,p)^{1/2} \tag{24}$$

$H^1[0,1]$ is the subset of $L^2[0,1]$ containing square-integrable functions with square-integrable first derivatives ($H^1[0,1] = \{v \in L^2[0,1],\ \text{if}\, v' \in L^2[0,1]\}$). The inner product and the norm in $H^1[0,1]$ are given by:

$$(p,q)_1 = (p,q) + (\frac{\mathrm{d}p}{\mathrm{d}x}, \frac{\mathrm{d}q}{\mathrm{d}x}) \tag{25}$$

$$\|p\|_1 = (p,p)_1^{1/2} \tag{26}$$

To impose boundary conditions, we define the function spaces $H^1_{u0}[0,1] = \{v \in H^1[0,1],\ v(1) = 0\}$ and $H^1_{p0}[0,1] = \{q \in H^1[0,1],\ q(0) = 0\}$. Then if we multiply the equations (1) and (2) by functions $v \in H^1_{u0}[0,1]$ and $q \in H^1_{p0}[0,1]$ correspondingly, integrate them on the interval $[0,1]$, apply the divergence theorem and take into consideration boundary conditions (3) and (4) for functions $\frac{\partial u}{\partial x}$ and $\frac{\partial p}{\partial x}$ correspondingly, then we get the following equations:

$$(\frac{\partial u}{\partial x}, \frac{\mathrm{d}v}{\mathrm{d}x}) - (p, \frac{\mathrm{d}v}{\mathrm{d}x}) = v_b \tag{27}$$

$$(\frac{\partial \dot{u}}{\partial x}, q) + (\frac{\partial p}{\partial x}, \frac{\mathrm{d}q}{\mathrm{d}x}) = 0 \tag{28}$$

where $v_b = v(0)$.

Therefore the weak formulation of the problem is as follows. For each $t \in (0,T]$ find functions $u(t) \in H^1_{u0}[0,1]$ and $p(t) \in H^1_{p0}[0,1]$, which satisfy equations (27) and (28),

The necessary initial condition for $\frac{\partial u}{\partial x}$ is expressed as:

$$(\frac{\partial u}{\partial x}(0), q) = 0 \tag{29}$$

We note, that for the solution of the weak problem equations (27) and (28), the initial value of $(\frac{\partial u}{\partial x}, q)$ is only needed, due to the first term in left-hand part of equation (28). In other words, we need only equation (29) for the initial conditions. The pressure solution at $t = 0$ is important only from a physical point of view. In that case the pressure at $t = 0$ is found from:

$$(\frac{\mathrm{d}u}{\mathrm{d}x}(0), \frac{\mathrm{d}v}{\mathrm{d}x}) - (p(0), \frac{\partial v}{\partial x}) = v_b \tag{30}$$

2.2 Finite element model

When the weak formulation is introduced we are able to proceed to the finite element formulation.

Let us consider a discrete grid $\mathscr{G} = \{x_n \in [0,1]\}$ on the interval where variable x is defined. We define a function space $\mathscr{P}^k_h$ of piecewise polynomials on $\mathscr{G}$ of degree k; and then the function spaces $\mathscr{U}^k_h = \mathscr{P}^k_h \cap H^1_{u0}[0,1]$ and $\mathscr{Q}^{k'}_h = \mathscr{P}^{k'}_h \cap H^1_{p0}[0,1]$ with bases $\{\varphi_i \in \mathscr{U}^k_h, i = \overline{1,n_u}\}$ and $\{\psi_j \in \mathscr{Q}^{k'}_h, j = \overline{1,n_p}\}$. For each $t \in (0,T]$ we approximate functions $u(t)$ and $p(t)$ with functions $u_h(t) \in \mathscr{U}^k_h$ and $p_h(t) \in \mathscr{Q}^{k'}_h$, which can be given as:

$$u_h(t) = \sum_{i=1}^{n_u} U_i(t)\varphi_i \tag{31}$$

$$p_h(t) = \sum_{j=1}^{n_p} P_j(t)\psi_j \tag{32}$$

Then the Galerkin semi-discrete approximation of the problem (27), (28), (29) is defined as follows. For each $t \in (0, T]$ find functions $u_h(t) \in \mathscr{U}_h^k$ and $p_h(t) \in \mathscr{Q}_h^{k'}$, which satisfy the equations:

$$(\frac{\partial u_h}{\partial x}, \frac{\mathrm{d}\varphi_i}{\mathrm{d}x}) - (p_h, \frac{\mathrm{d}\varphi_i}{\mathrm{d}x}) = \varphi_i(0) \quad i = \overline{1, n_u} \tag{33}$$

$$(\frac{\partial u_h}{\partial x}, \psi_j) + (\frac{\partial p_h}{\partial x}, \frac{\mathrm{d}\psi_j}{\mathrm{d}x}) = 0 \quad j = \overline{1, n_p} \tag{34}$$

while for $t = 0$

$$(\frac{\partial u_h}{\partial x}(0), \psi_j) = 0 \quad j = \overline{1, n_p} \tag{35}$$

To proceed to the discrete problem formulation we choose the set $\{t_m\}$, where $m = \overline{0, M}$, $t_m = m\tau$, $\tau = \frac{T}{M}$ and denote $u_h^m = u_h(t_m)$, $p_h^m = p_h(t_m)$ $m = \overline{0, M}$. (Hence we can denote $u_h^m = \sum_{i=1}^{n_u} U_i^m \varphi_i$ and $p_h^m = \sum_{j=1}^{n_p} P_j^m \psi_j$, $m = \overline{0, M}$). Then Euler backward scheme is used to represent a time derivative. Thus the discrete Galerkin approximation is as follows. For $m = \overline{1, M}$ find functions $u_h^m \in \mathscr{U}_h^k$ and $p_h^m \in \mathscr{Q}_h^{k'}$, which satisfy equations:

$$(\frac{\mathrm{d}u_h^m}{\mathrm{d}x}, \frac{\mathrm{d}\varphi_i}{\mathrm{d}x}) - (p_h^m, \frac{\mathrm{d}\varphi_i}{\mathrm{d}x}) = \varphi_i(0) \quad i = \overline{1, n_u} \tag{36}$$

$$(\frac{\mathrm{d}u_h^m}{\mathrm{d}x}, \psi_j) + \tau(\frac{\mathrm{d}p_h^m}{\mathrm{d}x}, \frac{\mathrm{d}\psi_j}{\mathrm{d}x}) = (\frac{\mathrm{d}u_h^{m-1}}{\mathrm{d}x}, \psi_j) \quad j = \overline{1, n_p} \tag{37}$$

when for $m = 0$:

$$(\frac{\mathrm{d}u_h^0}{\mathrm{d}x}, \psi_j) = 0 \quad j = \overline{1, n_p} \tag{38}$$

Equation (38) and equation (36) for $m = 0$ written as:

$$(\frac{\mathrm{d}u_h^0}{\mathrm{d}x}, \frac{\mathrm{d}\varphi_i}{\mathrm{d}x}) - (p_h^0, \frac{\mathrm{d}\varphi_i}{\mathrm{d}x}) = \varphi_i(0) \quad i = \overline{1, n_u} \tag{39}$$

gives us approximate numerical solution for pressure and displacements at time $t = 0$.

2.2.1 Linear-linear elements

In this section we discuss the solution results for the Galerkin discrete approximation of the given 1D problem when linear-linear elements are used. In other words, displacements and pressure are approximated via piecewise linear continuous functions. As it will be shown later, linear-linear elements often do not provide satisfactory results. Though, when we have to mesh the problem region with a large number of elements, linear-linear elements have an advantage of a smaller calculation resources, comparing with elements of higher order.

The numerical solution of (38) and (39) with linear-linear elements is illustrated in Figure 1.

For linear-linear elements the wiggles in the initial solution for pressure do not disappear whatever the element size is. [Aguilar et al. 2008] proposed a 'stabilization' technique to remove these oscillations. This technique consist in introducing an additional term in equation (38). The additional term for the Galerkin discrete equations corresponds to the Laplacian of the pressure. This term is multiplied by 'stabilization' parameter β. Then equation (38) takes a form:

$$(\frac{\mathrm{d}u_h^0}{\mathrm{d}x}, \psi_j) + \beta(\frac{\mathrm{d}p_h^0}{\mathrm{d}x}, \frac{\mathrm{d}\psi_j}{\mathrm{d}x}) = 0 \quad j = \overline{1, n_p} \tag{40}$$

The other question is how to determine the value for β. It's desirable to take β as small as possible, in order that a new equation does not differ a lot from the original one. But if β is taken too small, then the oscillations remain in the solution. Therefore it is proposed to determine β through

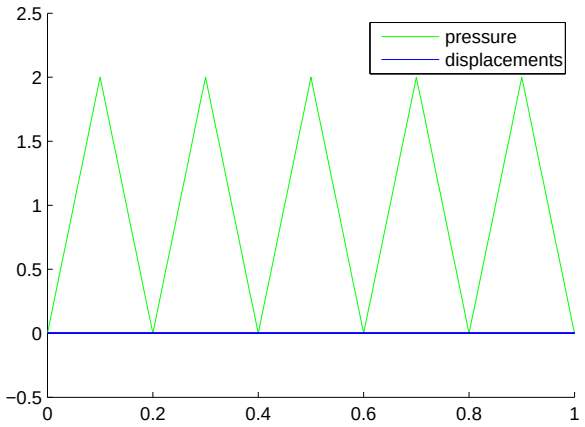

Figure 1: Plots of the numerical solution of the displacements and the pressure for a Galerkin FEM approximation of the 1D poro-elastic problem with linear-linear elements for $t = 0$.

analyzing the matrix of a 'stabilized' system of equations. The matrix form of the equations (39) and (40) is:

$$\begin{bmatrix} A & -G \\ G^T & \beta D \end{bmatrix} \begin{bmatrix} U^0 \\ P^0 \end{bmatrix} = \begin{bmatrix} f^0 \\ g^0 \end{bmatrix} \tag{41}$$

where A, $-G$, G^T and D correspond to inner products $(\frac{\mathrm{d}u_h^0}{\mathrm{d}x}, \frac{\mathrm{d}\varphi_i}{\mathrm{d}x})$, $-(p_h^0, \frac{\mathrm{d}\varphi_i}{\mathrm{d}x})$, $(\frac{\mathrm{d}u_h^0}{\mathrm{d}x}, \psi_j)$ and $(\frac{\mathrm{d}p_h^0}{\mathrm{d}x}, \frac{\mathrm{d}\psi_j}{\mathrm{d}x})$ respectively; $U^0 = [U_1^0 U_2^0 ... U_{n_u}^0]^T$, $P^0 = [P_1^0 P_2^0 ... P_{n_p}^0]^T$; and $[f^0 g^0]^T$ correspond to righthand part of the equations (39) and (40). We suppose that matrix A is invertible, and then introduce the Schur complement C of the block A:

$$C = G^T A^{-1} G + \beta D$$

Then

$$U^0 = A^{-1} G P^0 + A^{-1} f^0 \tag{42}$$

$$C P^0 = g^0 - G^T A^{-1} f^0 \tag{43}$$

It is proposed to choose the smallest β, for which matrix C is an L-matrix, i.e. for which the coefficients of C satisfy the following conditions:

$$C_{ii} > 0,\ C_{ij} \leq 0\ (i \neq j) \tag{44}$$

Therefore for a uniform mesh of $[0, 1]$ with elements of size h, conditions (44) lead to the inequality $\beta \geq h^2/4$. We can verify numerically, that for $\beta < h^2/4$ oscillations appear again. But for $\beta = h^2/4$ they disappear. (It is also confirmed by calculations that for $\beta = h^2/4$ matrix C is invertable and C^{-1} is a positive definite matrix, consequently matrix C is an M-matrix in this case).

The results obtained when a 'stabilization' term is added to the continuity equation are shown in Figure 2. The pressure approximation has no wiggles for this case. The value of displacements at $x = 0$ converges to zero when element size $h \to 0$. Therefore we can conclude, that for obtaining the initial solution of the 1D poro-elastic problem such a 'stabilization' technique works well and it removes the wiggles from the pressure solution, if the value of the 'stabilization' parameter β is chosen correctly. We also notice that the smaller elements are chosen in the neighborhood of $x = 0$, the numerical solution becomes more and more accurate (we can see that the numerical solution in this case differs from the analytical one only in the region of the element adjacent to $x = 0$).

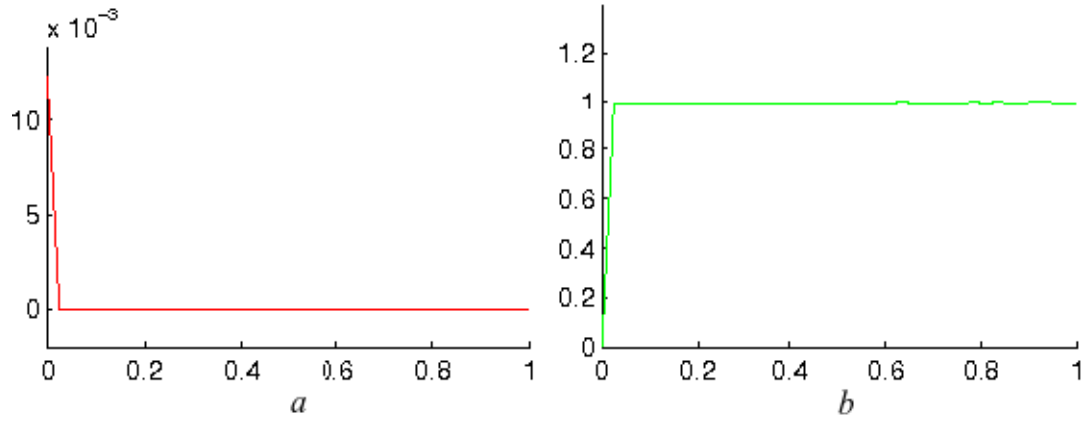

Figure 2: Plots of the numerical solution of the displacements (a) and the pressure (b) for a Galerkin FEM approximation of the 1D poro-elastic problem with linear-linear elements for $t = 0$, when the 'stabilization' term is used ($\beta = h^2/4$). The solution is obtained for a uniform mesh of 40 elements.

Let us consider the solution for $t = t_m$, $m = \overline{1, M}$. Numerical experiments show, that it does not matter, whether the solution at $t = 0$ contains oscillations or not, the oscillations still appear in the solution of (36) and (37) for a certain choice of the time step size and the element size. The oscillations appear for the small time steps and for the large element size, and they disappear when the time step size is increased, and also when a mesh refinement is introduced. Further, we are going to find the relations for the element size and the time step size, providing an oscillation free solution.

First we consider the matrix form of the (36) and (37).

$$\begin{bmatrix} A & -G \\ G^T & \tau D \end{bmatrix} \begin{bmatrix} U^m \\ P^m \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ G^T & 0 \end{bmatrix} \begin{bmatrix} U^{m-1} \\ P^{m-1} \end{bmatrix} + \begin{bmatrix} f^m \\ g^m \end{bmatrix} \quad m = \overline{1, M} \tag{45}$$

Then:

$$C_\tau = G^T A^{-1} G + \tau D \tag{46}$$

$$U^m = A^{-1} G P^m + A^{-1} f^m \tag{47}$$

$$C_\tau P^m = G^T A^{-1} G P^{m-1} + g^m + G^T A^{-1} (f^{m-1} - f^m) \tag{48}$$

If τ in (46) is replaced by β, then we get the same expressions for C from (43) and C_τ from (46). Hence we can deduce that, if the coefficient $\tau \geq h^2/4$, then coefficients of C_τ has the same properties as those of C. It is also confirmed by numerical experiments, that if $\tau \geq h^2/4$, then there are no wiggles in the solution regardless of whether we apply a 'stabilization' technique while obtaining initial solution, or not. However when $\tau < h^2/4$ the oscillations in the solution appear, and we have to refine the mesh again. In other words when it is possible to specify the smallest time step size τ used while the solution is being found and if time steps smaller than τ are not considered, then we could find the mesh size for which an oscillation free solution could be obtained for $t \geq \tau$ without any additional 'stabilization' techniques.

But it is not always possible to introduce necessary mesh refinement for small τ, because of excessively large dimensions of the global system of equations. Therefore, we again consider some 'stabilization' techniques proposed by [Aguilar et al. 2008] and aimed to remove the wiggles from the pressure solution without excessive mesh refinement. It is proposed now to add to equation (34) a term, corresponding to the Laplacian of the pressure or to a time derivative of the Laplacian of the pressure, multiplied by 'stabilization' parameters β_1 and β_2 respectively (i.e. $\beta_1 \Delta p$ and

$\beta_2 \Delta \dot{p}$). Then the semidiscrete equations takes the form respectively

$$(\frac{\partial \dot{u_h}}{\partial x}, \psi_j) + (\frac{\partial p_h}{\partial x}, \frac{\mathrm{d}\psi_j}{\mathrm{d}x}) + \beta_1 (\frac{\partial p_h}{\partial x}, \frac{\mathrm{d}\psi_j}{\mathrm{d}x}) = 0 \quad j = \overline{1, n_p} \tag{49}$$

or

$$(\frac{\partial \dot{u_h}}{\partial x}, \psi_j) + (\frac{\partial p_h}{\partial x}, \frac{\mathrm{d}\psi_j}{\mathrm{d}x}) + \beta_2 (\frac{\partial \dot{p_h}}{\partial x}, \frac{\mathrm{d}\psi_j}{\mathrm{d}x}) = 0 \quad j = \overline{1, n_p} \tag{50}$$

where dot denotes a partial derivative with respect to time, i.e. $\frac{\partial \dot{u_h}}{\partial x} = \frac{\partial^2 u_h}{\partial t \partial x}$.

Then the discrete equations are derived:

$$(\frac{\mathrm{d}u_h^m}{\mathrm{d}x}, \psi_j) + \tau(1+\beta_1)(\frac{\mathrm{d}p_h^m}{\mathrm{d}x}, \frac{\mathrm{d}\psi_j}{\mathrm{d}x}) = (\frac{\mathrm{d}u_h^{m-1}}{\mathrm{d}x}, \psi_j) \quad j = \overline{1, n_p} \tag{51}$$

or

$$(\frac{\mathrm{d}u_h^m}{\mathrm{d}x}, \psi_j) + (\tau+\beta_2)(\frac{\mathrm{d}p_h^m}{\mathrm{d}x}, \frac{\mathrm{d}\psi_j}{\mathrm{d}x}) = (\frac{\mathrm{d}u_h^{m-1}}{\mathrm{d}x}, \psi_j) + \beta_2 (\frac{\mathrm{d}p_h^{m-1}}{\mathrm{d}x}, \frac{\mathrm{d}\psi_j}{\mathrm{d}x}) \quad j = \overline{1, n_p} \tag{52}$$

If we analyze the matrices of the derived systems, then by analogy with the relations for β and τ obtained earlier we can get the conditions for β_1 and β_2. They are: $\tau(1+\beta_1) \geq h^2/4$ and $\tau + \beta_2 \geq h^2/4$. Hence $\beta_1 \geq \frac{h^2}{4\tau} - 1$ and $\beta_2 \geq \frac{h^2}{4} - \tau$. Since τ is supposed to be small, the smallest values of the 'stabilization' parameters are approximately $\beta_1 = \frac{h^2}{4\tau}$ and $\beta_2 = \frac{h^2}{4}$. Numerical experiments show, that if the initial solution is non-oscillatory, then the 'stabilization' techniques give solutions without wiggles even when $\tau < h^2/4$. However, if we start with initial pressure solution containing wiggles, then the equations of type (52) produce oscillations, while equations of type (51) provide solution without numerical wiggles

The main disadvantage of such a 'stabilization' techniques is that the modified system differs from the original one. And even when we get a 'good' solution for a modified system (i.e. solution without oscillations), we still have to remember, that we have solved another problem, not the original problem. The trick is to make the error due to changing the problem less than the error produced by imperfect discrete approximation of the original problem. When we say 'imperfect approximation', we imply that it could be improved, for example through mesh refinement or through using higher order elements. But often such methods of improvement lead to the systems of excessively large dimensions which are hard to solve.

To compare the various solution techniques the following numerical experiments were made. The uniform mesh of 50 linear-linear elements ($h = 2 \cdot 10^{-2}$) was considered. The solution for the first time step $t_1 = \tau = 10^{-6}$ was obtained with the use of different techniques introduced earlier. The time step τ is chosen small ($\tau \ll h^2/4 = 10^{-4}$) to motivate the use of the 'stabilization' techniques. The numerical solutions were compared with the analytic solution of the problem $(u(t), p(t))$. The errors for pressure $\varepsilon_p = \|p(t_1) - p_h^1\|_{L^2}$ given by various solution techniques are presented in Table 1.

We can conclude from Table 1, that for the given FEM mesh of the spatial domain of the problem a more accurate solution is obtained in the following cases:

- the 'stabilization' technique corresponding to adding a term $\nabla^2 p$ is used, (the one presented by the equations (36), (51)); the initial solution being not stabilized (equations (38), (39)). It appears that the numerical solution is oscillation free in this case (cf. Figure 3);

- the original equations (36), (37) for the first time step are solved, but the initial solution is stabilized (presented by equations (39), (40)). The numerical solution remains to be oscillatory for the given equation system, though the numerical oscillations is small enough and the resulting error is small too (cf. Figure 4);

- the 'stabilization' technique corresponding to adding a term $\nabla^2 \dot{p}$ is used, (the one presented by the equations (36), (52)); the initial solution is also stabilized (equations (39), (40)). The numerical solution is oscillation free in this case (cf. Figure 5).

	The initial solution obtained without 'stabilization' (equations (38), (39))	The initial solution obtained using 'stabilization' (equations (39), (40))
Solution for the first time step obtained without 'stabilization' (equations (36), (37))	0.1255	0.0723
Equations are stabilized by adding a term corresponding to $\nabla^2 p$ (equations (36), (51))	0.0720	0.0875
Equations are stabilized by adding a term corresponding to $\nabla^2 \dot{p}$ (equations (36), (52))	0.5705	0.0720

Table 1: Errors in the numerical solution of the pressure for the first time step $t_1 = 10^{-6}$ given by various solution techniques. Uniform mesh of 50 linear-linear elements is considered. Error=$(err_p, err_p)^{1/2}$, $err_p = p(t_1) - p_h^1$.

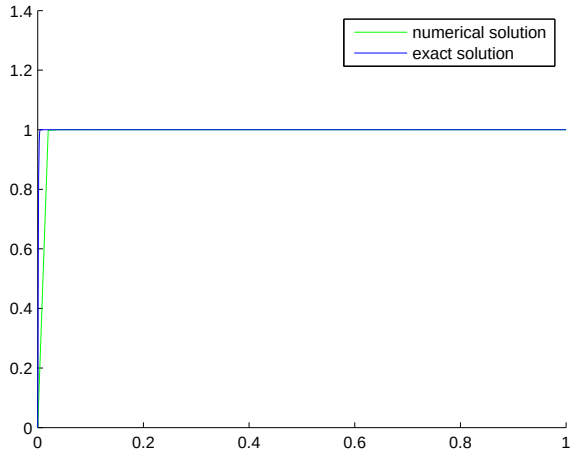

Figure 3: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of the 1D poro-elastic problem with linear-linear elements for $t_1 = 10^{-6}$, when the equations are stabilized by adding a term corresponding to $\nabla^2 p$ (system of equations (38), (39), (36), (51)). The solution is obtained for a uniform mesh of 50 elements. The initial solution is not 'stabilized'.

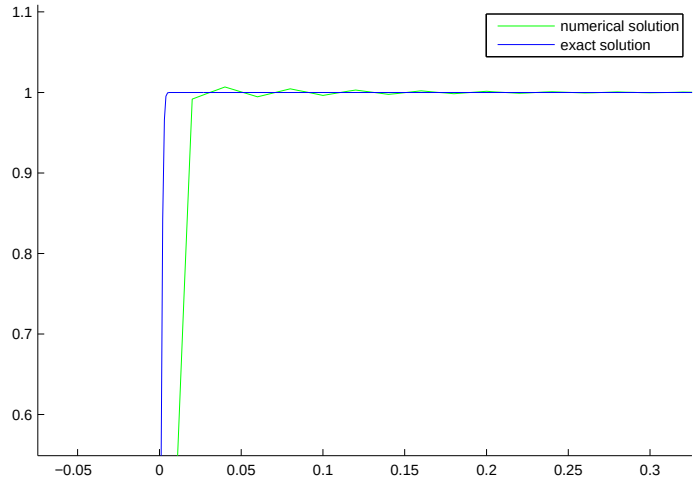

Figure 4: Plot of the numerical solution of the pressure near $x = 0$ for a Galerkin FEM approximation of the 1D poro-elastic problem with linear-linear elements for $t_1 = 10^{-6}$, when no 'stabilization' for the first time step term is used (system of equations (39), (40), (36), (37)). The solution is obtained for a uniform mesh of 50 elements. The initial solution is 'stabilized'.

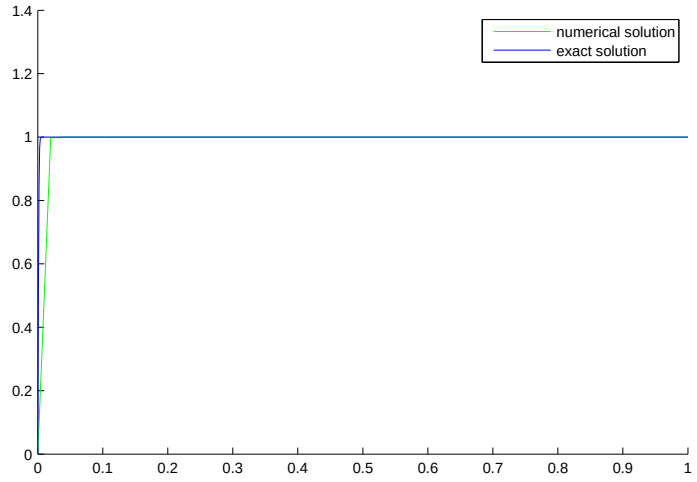

Figure 5: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of the 1D poro-elastic problem with linear-linear elements for $t_1 = 10^{-6}$, when the equations are stabilized by adding term corresponding to $\nabla^2 \dot{p}$ (system of equations (39), (40), (36), (52)). The solution is obtained for a uniform mesh of 50 elements. The initial solution is 'stabilized'.

In the numerical experiments introduced before we consider a uniform mesh. This was done just to compare the efficiency of the different 'stabilization' techniques for the case when the elements in the regions of a large gradient of the solution are not small enough. Nevertheless when solving practical problems, it is strongly recommended to adjust the mesh to the solution characteristics. Therefore if we choose a non-uniform grid with small elements near point $x = 0$ and large enough elements in the region where the solution is smooth, then we will be able to get a more accurate solution for a smaller number of elements. For example, to get an accurate enough solution for $t_1 = 10^{-6}$ without usage of any 'stabilization' technique, we consider a non-uniform grid with element of size $h_s = 2 \cdot 10^{-3}$ near $x = 0$; then the element size is increased gradually (with aspect ratio 1.5) when moving towards $x = 1$, so that the largest element is of size $h_l = 0.1$. That grid consists of 18 elements. The error for the pressure obtained after solving the system is 0.0053, which is much smaller than the error obtained in the previous numerical experiments (cf. Figure 6). Moreover the solution for the first time step is oscillation free, though we started from

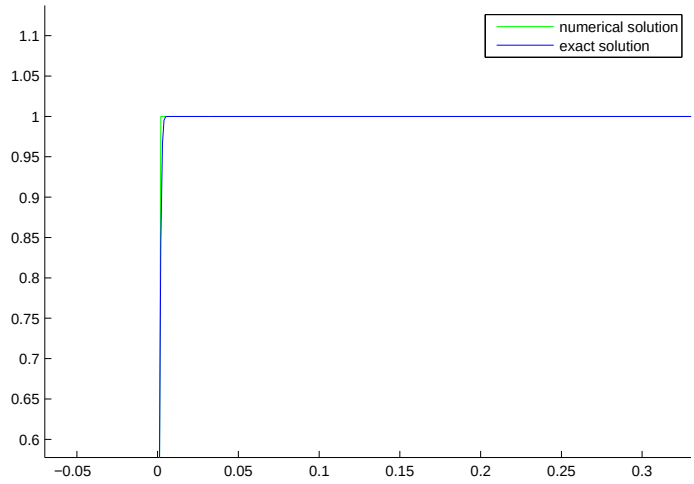

Figure 6: Plot of the numerical solution of the pressure near $x = 0$ for a Galerkin FEM approximation of the 1D poro-elastic problem with linear-linear elements for $t_1 = 10^{-6}$, when no 'stabilization' term is used (system of equations (38), (39), (36), (37)). The solution is obtained for a non-uniform mesh of 18 elements. The initial solution is not 'stabilized'.

the initial solution characterized by sufficient oscillatory behavior. No 'stabilization' was used, that means that the problem was not artificially modified.

Thus the non-uniform mesh of 18 elements without additional 'stabilization' of the governing system provided much more accurate numerical solution that the uniform mesh of 50 elements (even when the mentioned 'stabilization' techniques are applied). So we can conclude that if the order of elements is not changed, the best option to improve the numerical solution, is to refine the mesh in the region of large pressure gradient. This option gives better results than the 'stabilization' techniques introduced in this section do.

We notice again, that the element size needed to be obtained after refinement, depends on the chosen time step size. The smaller the time step, the smaller the element size we have to choose. However for practical problems it is often possible to define the smallest value of the time step for the current solution procedure. Then for a fixed minimal value of the time step a fixed size for the elements situated close to $x = 0$ could be chosen. However, if we are going to find the solution for $0 < t < \tau$, then we have to refine the mesh again.

2.2.2 Taylor-Hood elements

It is also possible to increase the order of elements in order to improve the solution quality. The elements with a second order approximation for displacements and a linear approximation for

pressure (also known as Taylor-Hood elements) are commonly used for solving an incompressible flow problem. In this section we consider results provided by these elements. For quadratic-linear elements numerical experiments show the same oscillatory behavior of the solution in the case when the element size is not small enough. Therefore it is also recommended to use mesh refinement to avoid the oscillations. 'Stabilization' techniques are the same as introduced in previous section, only that 'stabilization' parameters have a slightly different limits. Particularly, when solving 'stabilized' equations (40), (51) or (52) the following values have to be taken: $\beta = h^2/6$, $\beta_1 = h^2/6\tau$ and $\beta_2 = h^2/6$. If no 'stabilization' is used, then the element size near $x = 0$ should satisfy condition $h^2/6 \leq \tau$, in order to provide the non-oscillatory solution for first time step $t_1 = \tau$.

By analogy with the previous section the errors for pressure given by various solution techniques for uniform mesh of 50 elements are presented in Table 2.

	The initial solution obtained without 'stabilization' (equations (38), (39))	The initial solution obtained using 'stabilization' (equations (39), (40))
Solution for the first time step obtained without 'stabilization' (equations (36), (37))	0.0656	0.0721
Equations are stabilized by adding a term corresponding to $\nabla^2 p$ (equations (36), (51))	0.0720	0.0816
Equations are stabilized by adding a term corresponding to $\nabla^2 \dot{p}$ (equations (36), (52))	0.0656	0.0720

Table 2: Errors in the numerical solution of the pressure for the first time step $t_1 = 10^{-6}$ given by various solution techniques. Uniform mesh of 50 quadratic-linear elements is considered. Error=$(err_p, err_p)^{1/2}$, $err_p = p(t_1) - p_h^1$.

The series of numerical experiments showed that for the first time step and for the uniform mesh of 50 Taylor-Hood elements:

- the accuracy of the numerical solution is higher even when no 'stabilization' technique is used at all (i.e. when the discrete problem is solved in its initial form (equations (36), (37), (38), (39)). Since the element size is not small enough, the *numerical solution* for the first time step is still *oscillatory* (cf. Figure 7). At the same time the oscillations present only in the small region close to $x = 0$ and the resulting *error is the smallest*, comparing with the other 'stabilized' numerical solutions.

- the 'stabilization' technique corresponding to adding a term $\nabla^2 \dot{p}$ and non-'stabilized' initial conditions provided the *oscillatory numerical solution* with approximately the same accuracy of as the non-stabilized solution scheme (cf. Figure 8).

- the 'stabilization' technique corresponding to adding a term $\nabla^2 p$ and the 'stabilization' technique corresponding to adding a term $\nabla^2 \dot{p}$, when non-'stabilized' and 'stabilized' initial conditions were considered correspondingly (systems of the equations (38), (39), (36), (51) and (39), (40), (36), (52)), provided the *oscillation free* numerical solutions (cf. Figure 9 and Figure 10). Though the *accuracy* of these techniques *is lower* than in the case when no 'stabilization' technique is used.

- the original equations for the first time step (36), (37) and the 'stabilized' initial conditions (equations (39), (40)) provided *small numerical wiggles* in the solution and high enough accuracy (cf. Figure 11);

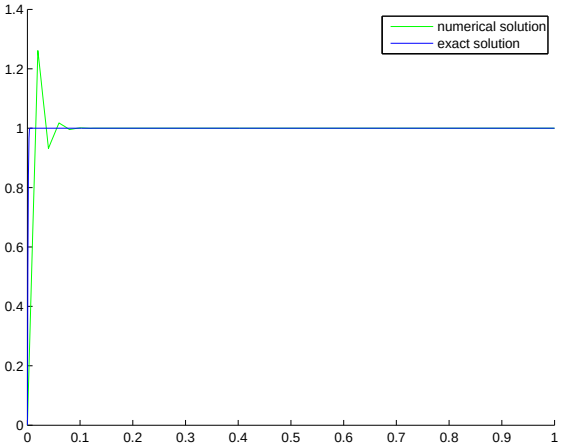

Figure 7: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of the 1D poro-elastic problem with quadratic-linear elements for $t_1 = 10^{-6}$, when no 'stabilization' for the first time step is used. (system of equations (38), (39), (36), (37)). The solution is obtained for a uniform mesh of 50 elements. The initial solution is not 'stabilized'.

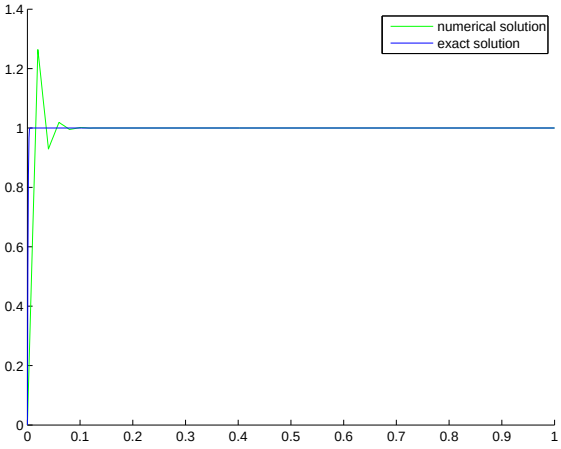

Figure 8: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of the 1D poro-elastic problem with quadratic-linear elements for $t_1 = 10^{-6}$, when the equations are stabilized by adding term corresponding to $\nabla^2 \dot{p}$ (system of equations (38), (39), (36), (52)). The solution is obtained for a uniform mesh of 50 elements. The initial solution is 'stabilized'.

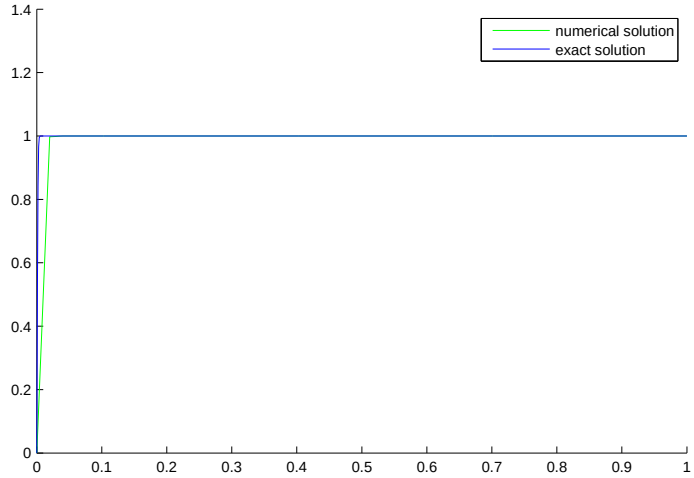

Figure 9: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of the 1D poro-elastic problem with quadratic-linear elements for $t_1 = 10^{-6}$, when the equations are stabilized by adding term corresponding to $\nabla^2 p$ (system of equations (38), (39), (36), (51)). The solution is obtained for a uniform mesh of 50 elements. The initial solution is not 'stabilized'.

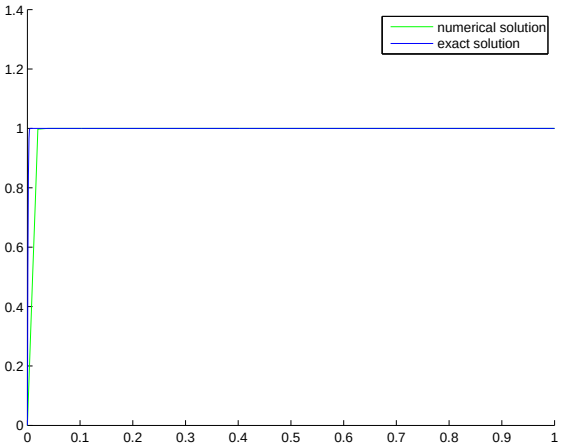

Figure 10: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of the 1D poro-elastic problem with quadratic-linear elements for $t_1 = 10^{-6}$, when the equations are stabilized by adding term corresponding to $\nabla^2 \dot{p}$ (system of equations (39), (40), (36), (52)). The solution is obtained for a uniform mesh of 50 elements. The initial solution is 'stabilized'.

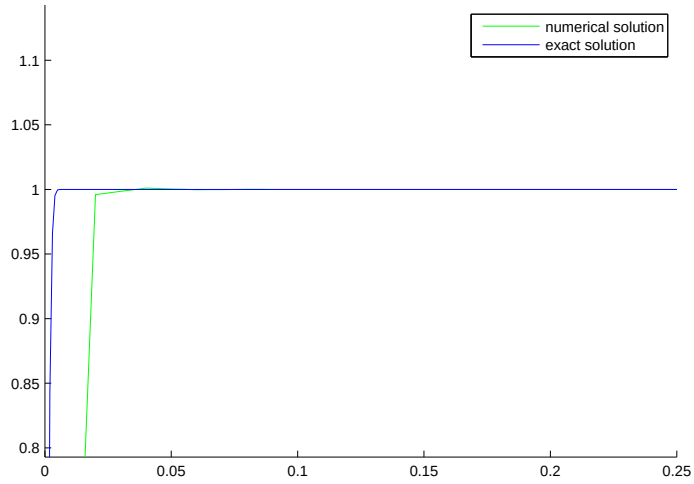

Figure 11: Plot of the numerical solution of the pressure near $x = 0$ for a Galerkin FEM approximation of the 1D poro-elastic problem with quadratic-linear elements for $t_1 = 10^{-6}$, when no 'stabilization' term for the first time step is used (system of equations (39), (40), (36), (37)). The solution is obtained for a uniform mesh of 50 elements. The initial solution is 'stabilized'.

- the 'stabilization' technique corresponding to adding a term $\nabla^2 p$ and the 'stabilized' initial conditions (equations (39), (40), (36), (51)), provided the oscillation free numerical solutions, though the *accuracy is the lowest* among the all solution schemes considered in this section (cf. Figure 12).

Then we consider a non-uniform mesh of 27 elements with element of size $h_s = 2 \cdot 10^{-3}$ near $x = 0$, aspect ratio of adjacent elements close to $x = 0$ being 1.2 and the largest element being of size $h_l = 0.1$. (We have decrease the aspect ratio and consequently to increase the number of elements comparing with the non-uniform mesh considered in the previous section in order to avoid the oscillations in the solution.) The error for pressure in this case is 0.0035 (cf. Figure 13). Again it is the smallest value of error for all the experiments considered in this section, and the solution for the first time step is oscillation free, while no 'stabilization' is used (initial solution is also not 'stabilized' and oscillatory).

As it was expected Taylor-Hood elements give more accurate results than linear-linear elements. But linear-linear elements provide a smaller number of equations to be solved. Thus again we have to choose between the solution quality and large dimensions of the global equation system.

To summarize we should say, that when solving the 1D poro-elastic problem characterized by large gradients of pressure for small $t > 0$ near point $x = 0$, the researcher should be careful with generating a mesh. To get a 'good' solution, element size near $x = 0$ should be taken small enough. Non-uniform mesh allows to increase the solution quality and to decrease the number of elements. Though, when it is not possible to refine the mesh to the necessary level, then several 'stabilization' techniques could be applied. If the 'stabilization' parameter is chosen correctly, then these techniques allow to obtain accurate solutions.

3 Two dimensional problem for poro-elastic body

We now proceed to the two dimensional problem. We consider $\mathbf{\Omega} \subset \mathbb{R}^2$ with smooth boundary Γ. The outward unit normal to the boundary is denoted as $\mathbf{n}$. The problem is formulated as follows.

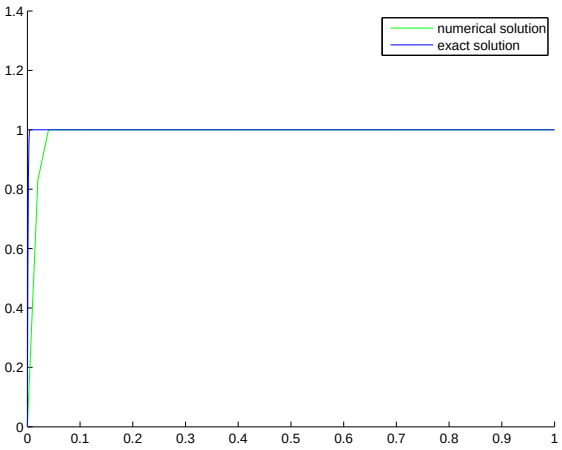

Figure 12: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of the 1D poro-elastic problem with quadratic-linear elements for $t_1 = 10^{-6}$, when the equations are stabilized by adding term corresponding to $\nabla^2 p$ (system of equations (38), (39), (36), (51)). The solution is obtained for a uniform mesh of 50 elements. The initial solution is 'stabilized'.

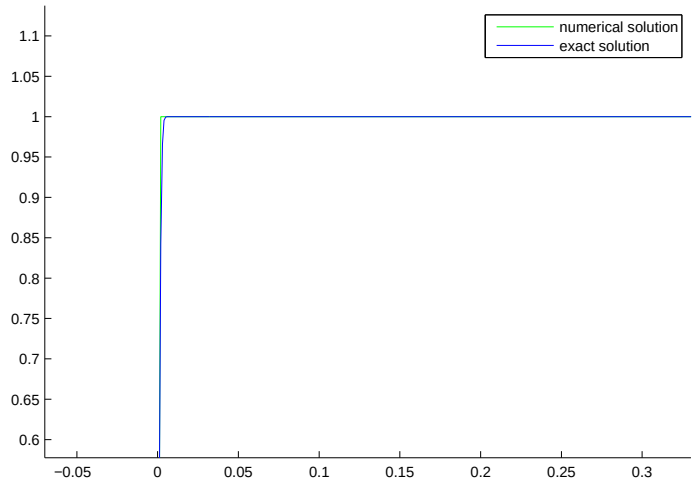

Figure 13: Plot of the numerical solution of the pressure near $x = 0$ for a Galerkin FEM approximation of the 1D poro-elastic problem with quadratic-linear elements for $t_1 = 10^{-6}$, when no 'stabilization' term is used (system of equations (38), (39), (36), (37)). The solution is obtained for a non-uniform mesh of 27 elements. The initial solution is not 'stabilized'.

Find functions $\mathbf{u} : \mathbf{\Omega} \times (0, \mathrm{T}] \to \Re^2$ and $p : \mathbf{\Omega} \times (0, \mathrm{T}] \to \Re$, which satisfy the equations:

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} - \nabla p = \mathbf{0} \tag{53}$$

$$\operatorname{div} \dot{\mathbf{u}} - \frac{\kappa}{\eta} \Delta p = 0 \tag{54}$$

Functions $\mathbf{u} = (u, v)$ and p represent the vector of the displacements of the pore skeleton and the hydraulic pressure in the poro-elastic body respectively. In the above equations the following parameters are used: λ, μ – Lamé constants, κ – permeability of the media, η – viscosity of the pore fluid.

Boundary conditions could be given, for example, in the form:

$$\mathbf{u} = 0, \quad \frac{\kappa}{\eta} \nabla p \cdot \mathbf{n} = Q \qquad \text{on} \quad \Gamma_1 \tag{55}$$

$$p = 0, \quad (\lambda \operatorname{div} \mathbf{u} \mathbf{I} + 2\mu\, \boldsymbol{\varepsilon}(\mathbf{u})) \cdot \mathbf{n} = \mathbf{h} \qquad \text{on} \quad \Gamma_2 \tag{56}$$

where $\Gamma_1 \subset \Gamma$, $\Gamma_2 \subset \Gamma$, $\Gamma_1 \cup \Gamma_2 = \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$; $\mathbf{I}$ – identity matrix; $\boldsymbol{\varepsilon}(\mathbf{u}) = (\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}})$ – strain tensor. Functions $\mathbf{h}$ and Q represent the traction and the discharge on Γ_2 and Γ_1 correspondingly.

An initial condition for the function $\operatorname{div} \mathbf{u}$ is needed, since its time derivative is contained in the equation (54). To specify the initial condition we suppose that the pore skeleton of the poro-elastic body is incompressible when $t = 0$, thus the condition takes the form:

$$\operatorname{div} \mathbf{u}(\mathbf{x}, 0) = 0 \quad \text{for} \quad \mathbf{x} \in \mathbf{\Omega} \tag{57}$$

Values of the function p at time $t = 0$ are not needed themselves for the solution of the problem. But they could be useful, for example, while choosing a numerical method for obtaining the solution at time $t \in (0, T]$. In this case the displacements and the pressure for $t = 0$ could be obtained by solving the equilibrium equation (53) for $(\mathbf{x}, t) \in \mathbf{\Omega} \times \{0\}$ and equation (57).

3.1 Weak formulation

We introduce the following function spaces: $H^1_{\mathbf{u}0}(\mathbf{\Omega}) = \{u \in H^1(\mathbf{\Omega}); u(\mathbf{x}_\Gamma) = 0, \mathbf{x}_\Gamma \in \Gamma_1\}$ and $H^1_{p0}(\mathbf{\Omega}) = \{q \in H^1(\mathbf{\Omega}); q(\mathbf{x}_\Gamma) = 0, \mathbf{x}_\Gamma \in \Gamma_2\}$. The inner product in the function space $L^2(\mathbf{\Omega})$ is denoted as:

$$(p, q) = \int_{\mathbf{\Omega}} pq d\mathbf{x} \tag{58}$$

Then if we multiply the equations (53) and (54) by functions $\mathbf{v} \in H^1_{\mathbf{u}0}(\mathbf{\Omega}) \times H^1_{\mathbf{u}0}(\mathbf{\Omega})$ and $q \in H^1_{p0}(\mathbf{\Omega})$ correspondingly, integrate them by the variable $\mathbf{x}$ over the region $\mathbf{\Omega}$, apply the divergence theorem and take into consideration boundary conditions (55) and (56), then we get the equations:

$$a(\mathbf{u}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) = \int_{\Gamma_2} \mathbf{h} \cdot \mathbf{v} d\mathbf{x} \tag{59}$$

$$(\operatorname{div} \dot{\mathbf{u}}, q) + b(p, q) = \int_{\Gamma_1} Q q d\mathbf{x} \tag{60}$$

where

$$a(\mathbf{u}, \mathbf{v}) = 2\mu \sum_{i,j=1}^{2} (\varepsilon_{ij}(\mathbf{u}), \varepsilon_{ij}(\mathbf{v})) + \lambda (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v}) \tag{61}$$

$$b(p, q) = \frac{\kappa}{\eta} \int_{\mathbf{\Omega}} \nabla p \cdot \nabla q \, d\mathbf{x} \tag{62}$$

Now we can define the weak formulation of the problem as follows. For each $t \in (0, T]$ find functions $\mathbf{u}(t) \in H^1_{\mathbf{u}0}(\mathbf{\Omega}) \times H^1_{\mathbf{u}0}(\mathbf{\Omega})$ and $p(t) \in H^1_{p0}(\mathbf{\Omega})$, which satisfy equations (59) and (60). The initial condition for the weak formulation is defined as:

$$(\operatorname{div} \mathbf{u}(0), q) = 0 \tag{63}$$

3.2 Finite element model

We proceed to the finite element formulation. Let us consider a discrete grid $\mathscr{G} = \{(x_n, y_m) \in \mathbf{\Omega}\}$. A function space $\mathscr{P}_h^k$ of piecewise polynomials on $\mathscr{G}$ of degree k is defined. Then we define function spaces $\mathscr{U}_h^k = (\mathscr{P}_h^k \times \mathscr{P}_h^k) \cap (H_{u0}^1(\mathbf{\Omega}) \times H_{u0}^1(\mathbf{\Omega}))$ and $\mathscr{Q}_h^{k'} = \mathscr{P}_h^{k'} \cap H_{p0}^1(\mathbf{\Omega})$ with bases $\{(\varphi_i, 0) \in \mathscr{U}_h^k, (0, \varphi_j) \in \mathscr{U}_h^k,\ i, j = \overline{1, n_u}\}$ and $\{\psi_j \in \mathscr{Q}_h^{k'},\ j = \overline{1, n_p}\}$. For each $t \in (0, T]$ we approximate functions $\mathbf{u}(t)$ and $p(t)$ with functions $\mathbf{u}_h(t) \in \mathscr{U}_h^k$ and $p_h(t) \in \mathscr{Q}_h^{k'}$, which can be given as:

$$\mathbf{u}_h(t) = \sum_{i=1}^{n_u} \mathbf{U}_i(t)\varphi_i \tag{64}$$

$$p_h(t) = \sum_{j=1}^{n_p} P_j(t)\psi_j \tag{65}$$

Then the Galerkin semi-discrete approximation of the problem (59), (60), (63) is defined as follows. For each $t \in (0, T]$ find functions $\mathbf{u}_h(t) \in \mathscr{U}_h^k$ and $p_h(t) \in \mathscr{Q}_h^{k'}$, which satisfy equations:

$$a(\mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) = \int_{\Gamma_2} \mathbf{h} \cdot \mathbf{v}_h \, d\mathbf{x} \qquad \forall \mathbf{v}_h \in \mathscr{U}_h^k \tag{66}$$

$$(\operatorname{div} \dot{\mathbf{u}}_h, q_h) + b(p_h, q_h) = \int_{\Gamma_1} Q\, q_h \, d\mathbf{x} \qquad \forall q_h \in \mathscr{Q}_h^{k'} \tag{67}$$

while for $t = 0$

$$(\operatorname{div} \mathbf{u}_h(0), q_h) = 0 \qquad \forall q_h \in \mathscr{Q}_h^{k'} \tag{68}$$

By analogy with the one dimensional case we choose time grid $\{t_m\}$, where $m = \overline{0, M}$, $t_m = m\tau$, $\tau = \frac{T}{M}$ and denote $\mathbf{u}_h^m = \mathbf{u}_h(t_m)$, $\mathbf{u}_h^m = \sum_{i=1}^{n_u} \mathbf{U}_i^m \varphi_i$ and $p_h^m = p_h(t_m)$, $p_h^m = \sum_{j=1}^{n_p} P_j^m \psi_j$, $m = \overline{0, M}$. Time derivative is represented through Euler Backward scheme. Then the discrete Galerkin formulation is given as follows. For $m = \overline{1, M}$ find functions $\mathbf{u}_h^m \in \mathscr{U}_h^k$ and $p_h^m \in \mathscr{Q}_h^{k'}$, which satisfy equations:

$$a(\mathbf{u}_h^m, \mathbf{v}_h) - (p_h^m, \operatorname{div} \mathbf{v}_h) = \int_{\Gamma_2} \mathbf{h} \cdot \mathbf{v}_h \, d\mathbf{x} \qquad \forall \mathbf{v}_h \in \mathscr{U}_h^k \tag{69}$$

$$(\operatorname{div} \mathbf{u}_h^m, q_h) + \tau b(p_h^m, q_h) = (\operatorname{div} \mathbf{u}_h^{m-1}, q_h) + \tau \int_{\Gamma_1} Q\, q_h \, d\mathbf{x} \qquad \forall q_h \in \mathscr{Q}_h^{k'} \tag{70}$$

when for $m = 0$:

$$(\operatorname{div} \mathbf{u}_h^0, q_h) = 0 \qquad \forall q_h \in \mathscr{Q}_h^{k'} \tag{71}$$

Approximate numerical solution for pressure and displacements at time $t = 0$ could be obtained from equation (71) and equation (69) considered for $m = 0$:

$$a(\mathbf{u}_h^0, \mathbf{v}_h) - (p_h^0, \operatorname{div} \mathbf{v}_h) = \int_{\Gamma_2} \mathbf{h} \cdot \mathbf{v}_h \, d\mathbf{x} \qquad \forall \mathbf{v}_h \in \mathscr{U}_h^k \tag{72}$$

3.2.1 Test problem

In this section we introduce a test problem and solution techniques, which can be used to solve it. The results provided by these techniques are studied and compared.

Therefore, for the test problem we consider $\mathbf{\Omega} = [0, 8] \times [0, 8]$. Boundary conditions are specified as follows:

$$\begin{aligned} p &= 0 && \text{on} \quad \Gamma_2 \\ \nabla p \cdot \mathbf{n} &= 0 && \text{on} \quad \Gamma \setminus \Gamma_2 \\ \sigma \mathbf{n} &= -\sigma_0 && \text{on} \quad \Gamma_1 \\ \sigma \mathbf{n} &= 0 && \text{on} \quad \Gamma_2 \\ \mathbf{u} &= \mathbf{0} && \text{on} \quad \Gamma \setminus (\Gamma_1 \cup \Gamma_2) \end{aligned} \tag{73}$$

where

$$\Gamma_1 = \{(x, y) \in \Gamma;\ 3.2 \le x \le 4.8,\ y = 8\}$$
$$\Gamma_2 = \{(x, y) \in \Gamma;\ x \in [0, 3.2) \cup (4.8, 8],\ y = 8\}$$
$$\sigma_0 = 10^4\, N/m^2$$

Material properties have the values:

$$E = 3 \cdot 10^4\, \mathrm{N/m^2}$$
$$\nu = 0.2$$
$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$
$$\mu = \frac{E}{2(1+\nu)}$$
$$\kappa = 10^{-8}\, \mathrm{m^2}$$
$$\eta = 10^{-4}\, \mathrm{Pa \cdot sec}$$

Now we are going to obtain the solution for the Galerkin discrete approximation of the test problem, using Q_1Q_1 quadrilateral elements and Q_2Q_1 quadrilateral elements. Inside Q_1Q_1 quadrilateral elements the bilinear interpolation functions for the displacements and the pressure are used. And Q_2Q_1 quadrilateral elements correspond to the biquadratic approximation for the displacements and the bilinear approximation for the pressure.

3.2.2 Q_1Q_1 quadrilateral elements

First we choose Q_1Q_1 quadrilateral elements (i.e. bilinear interpolation functions for the displacements and the pressure within each element are used). We are finding the solution of the test problem for time $t = t_1 = \tau$. It is obtained from the equations (69), (70) and (71). We also can suppose that when τ is small the solution for $t = \tau$ is close to the solution for $t = 0$. Therefore it could be helpful to look at the solution at time $t = 0$. This solution is given by equations (71) and (72). The calculations show, that the pressure approximation at $t = 0$ obtained from the numerical solution of the equations (71) and (72) contains oscillations (cf. Figure 14). This situation is analogous to the one dimensional case.

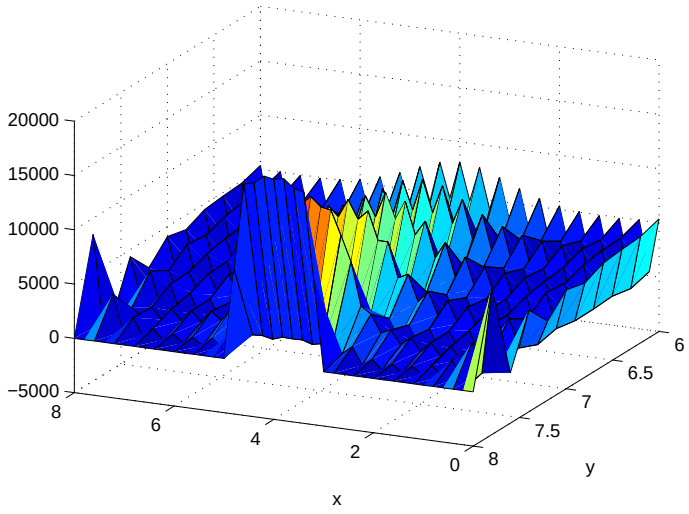

Figure 14: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of 2D poro-elastic problem with Q_1Q_1 quadrilateral elements for $t = 0$.

By analogy with the one dimensional case a stabilization technique could be applied to remove the oscillations from the initial solution for the pressure. This technique consists in adding to the equation (57) a term $-\beta\Delta p$. The 'stabilized' initial condition for the continuous formulation is:

$$\operatorname{div} \mathbf{u}(0) - \beta\Delta p(0) = 0 \quad \text{for} \quad \mathbf{x} \in \mathbf{\Omega} \tag{74}$$

Then the 'stabilized' initial condition for the Galerkin discrete approximation has the form:

$$(\operatorname{div} \mathbf{u}_h^0, q_h) + \beta' b(p_h^0, q_h) = \beta' \int_{\Gamma_1} Q\, q_h \, d\mathbf{x} \qquad \forall q_h \in \mathscr{Q}_h^{k'} \tag{75}$$

where $\beta' = \frac{\eta}{\kappa}\beta$. The value of the 'stabilization' parameter is taken as $\beta = \frac{h^2}{4(\lambda+2\mu)}$, where h is a characteristic size of the elements (while solving the problem we use a uniform mesh with the elements of a square shape, hence h is equal to the length of the element edge in this case). Usage of 'stabilization' while solving the problem for $t = 0$ provides the oscillation free numerical solution of the pressure (cf. Figure 15).

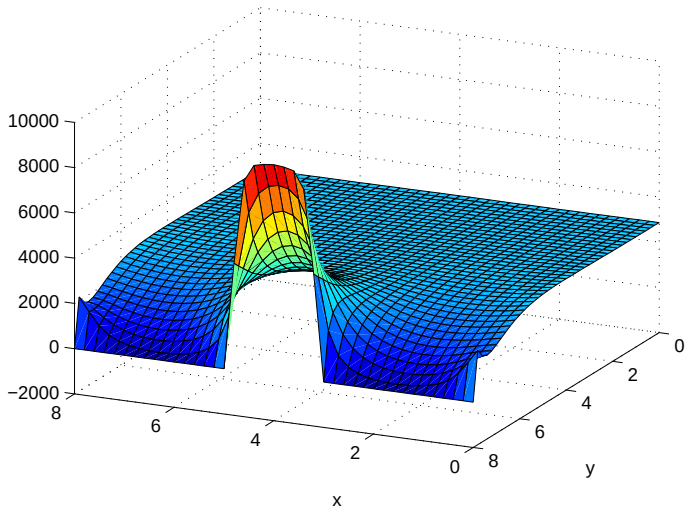

Figure 15: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of 2D poro-elastic problem with Q_1Q_1 quadrilateral elements for $t = 0$, when 'stabilization' technique for the initial solution is used.

Then we consider the solution for the first time step $t_1 = \tau = 10^{-6}$. If we take equation (71) as the initial condition, we obtain the numerical solution for the pressure at time $t = 10^{-6}$ represented in Figure 16. In this case the numerical solution for the pressure at the first time step again contains wiggles. By analogy with the one dimensional case there are several ways to take the oscillations away from the solution for the pressure. One of them is to refine the mesh at the boundary $\{(x, y) \in \Gamma;\, x \in [0, 8],\, y = 8\}$. Though usage of very small elements in the two dimensional case (to say nothing of 3D) could lead to extensively large dimensions of the global stiffness matrix and consequently, to the significant difficulties arising during the integration of the resulting system. Therefore, 'stabilization' techniques are worth of considering here.

We consider two 'stabilization' techniques for the 2D problem. They are analogous to those for the 1D problem described earlier. The first technique consists in adding to the equation (54) a term $-\beta_1 \Delta p$:

$$\operatorname{div} \dot{\mathbf{u}} - (\frac{\kappa}{\eta} + \beta_1)\Delta p = 0 \tag{76}$$

The Galerkin discrete formulation corresponding to the 'stabilized' continuous equation (76) is:

$$(\operatorname{div} \mathbf{u}_h^m, q_h) + \tau(1 + \beta_1')b(p_h^m, q_h) = (\operatorname{div} \mathbf{u}_h^{m-1}, q_h) + \tau(1 + \beta_1') \int_{\Gamma_1} Q\, q_h \, d\mathbf{x} \qquad \forall q_h \in \mathscr{Q}_h^{k'} \tag{77}$$

where $\beta_1' = \frac{\eta}{\kappa}\beta_1$. According to [Aguilar et al. 2008] the value of the 'stabilization' parameter $\beta_1 = \frac{h^2}{4\tau(\lambda+2\mu)}$ is chosen. Numerical experiments show that even when the initial condition is

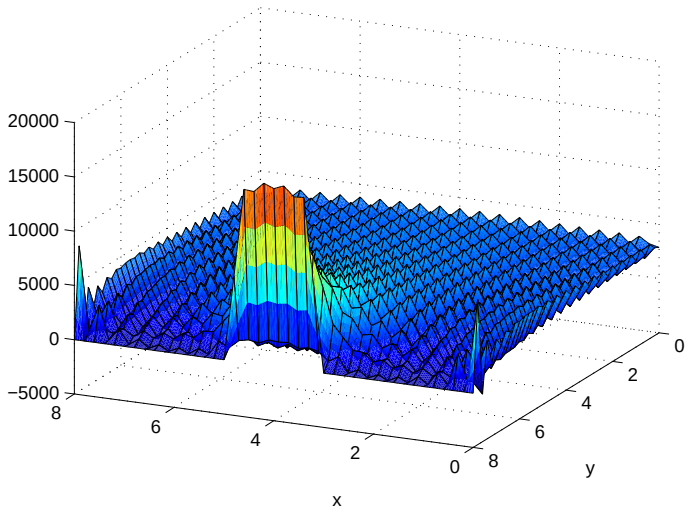

Figure 16: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of 2D poro-elastic problem with Q_1Q_1 quadrilateral elements for $t = 10^{-6}$, when 'stabilization' technique for the initial solution is not used.

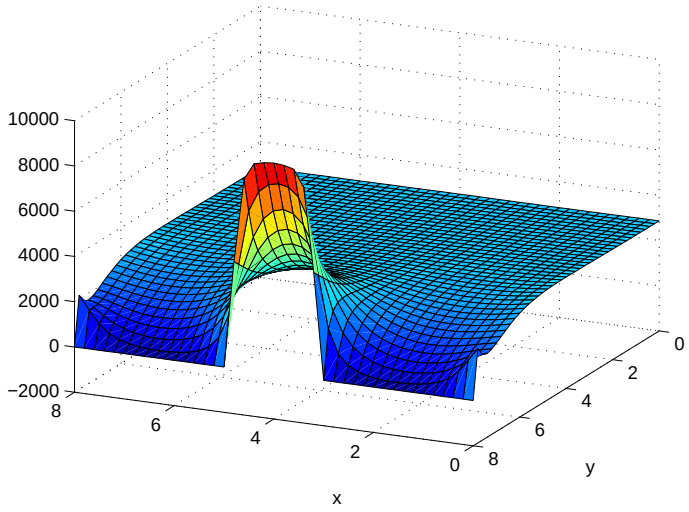

Figure 17: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of 2D poro-elastic problem with Q_1Q_1 quadrilateral elements for $t = 10^{-6}$, when 'stabilization' technique, corresponding to adding a term $-\beta_1\Delta p$, is used. The initial solution is not 'stabilized'.

taken in the 'non-stabilized' form (equation (71)), the numerical solution of the pressure provided by this 'stabilization' technique is oscillation free (cf. Figure 17).

The second 'stabilization' technique consists in adding to the equation (54) a term $-\beta_2 \Delta \dot{p}$:

$$\operatorname{div} \dot{\mathbf{u}} - \frac{\kappa}{\eta} \Delta p - \beta_2 \Delta \dot{p} = 0 \tag{78}$$

Then the Galerkin discrete approximation of (78) is:

$$(\operatorname{div} \mathbf{u}_h^m, q_h) + (\tau + \beta_2')b(p_h^m, q_h) = (\operatorname{div} \mathbf{u}_h^{m-1}, q_h) + \beta_2' b(p_h^{m-1}, q_h) + \tau \int_{\Gamma_1} Q\, q_h \, d\mathbf{x} \qquad \forall q_h \in \mathcal{Q}_h^{k'} \tag{79}$$

where $\beta_2' = \frac{\eta}{\kappa}\beta_2$. The value of the 'stabilization' parameter is chosen as $\beta_2 = \frac{h^2}{4(\lambda+2\mu)}$ [Aguilar et al. 2008].

As in the one dimensional case, this 'stabilization' technique do not provide oscillations free solution of the pressure at the first time step when the initial condition is not 'stabilized' (cf. Figure 18). But if we apply the second stabilization technique simultaneously with the 'stabilized'

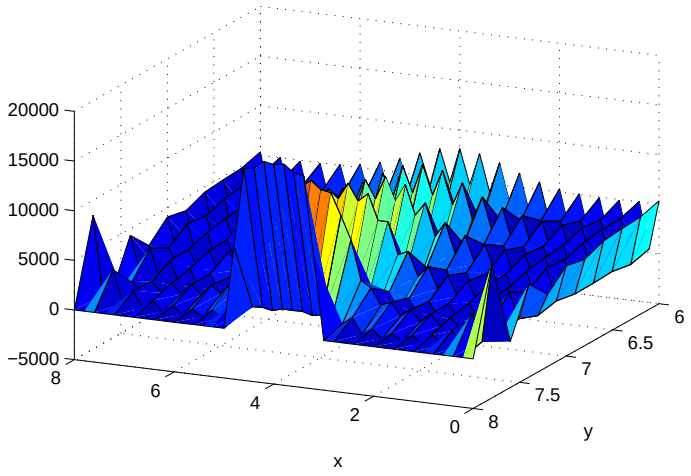

Figure 18: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of 2D poro-elastic problem with Q_1Q_1 quadrilateral elements for $t = 10^{-6}$, when 'stabilization' technique, corresponding to adding a term $-\beta_2 \Delta \dot{p}$, is used. The initial solution is not 'stabilized'.

initial condition (75), then we obtain a solution of the pressure without numerical wiggles, as it is shown in Figure 19.

However, if the 'stabilized' equation (75) is taken as the initial condition, then original (i.e. 'non-stabilized') equations (69) and (70) also provide oscillation free solution of the pressure (cf. Figure 20).

Finally, the first 'stabilization' technique gives a solution for pressure without wiggles also for the 'stabilized' form of the initial condition (75) (cf. Figure 21).

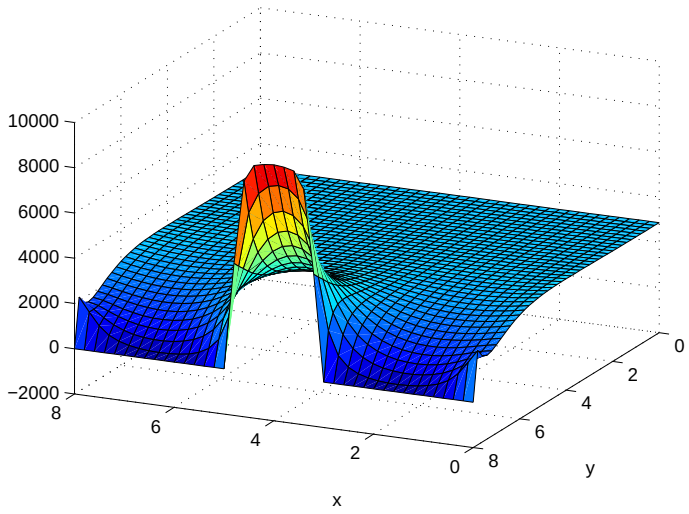

Figure 19: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of 2D poro-elastic problem with Q_1Q_1 quadrilateral elements for $t = 10^{-6}$, when 'stabilization' technique, corresponding to adding a term $-\beta_2\Delta\dot{p}$, is used. The initial solution is also 'stabilized'.

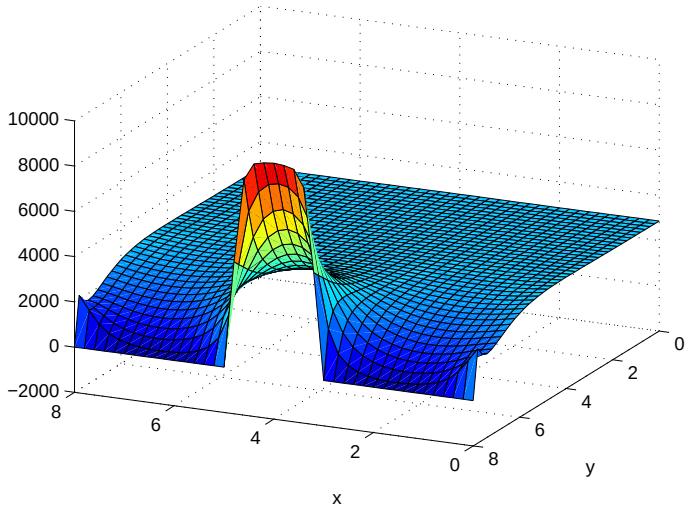

Figure 20: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of 2D poro-elastic problem with Q_1Q_1 quadrilateral elements for $t = 10^{-6}$, when 'stabilization' technique for the initial solution is used.

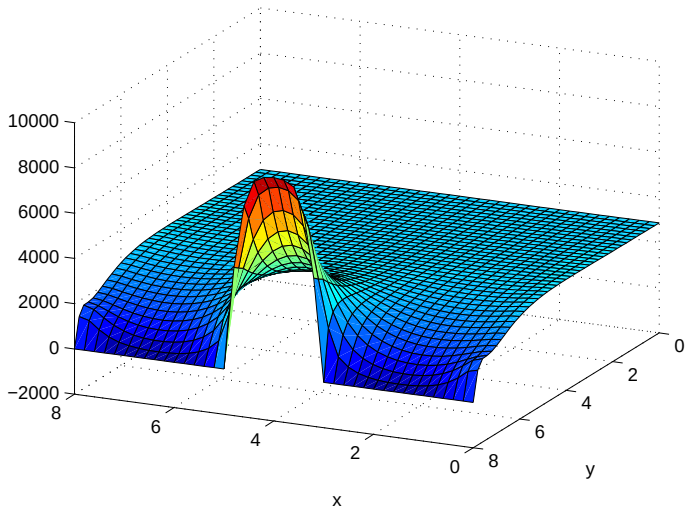

Figure 21: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of 2D poro-elastic problem with Q_1Q_1 quadrilateral elements for $t = 10^{-6}$, when 'stabilization' technique, corresponding to adding a term $-\beta_1 \Delta p$, is used. The initial solution is 'stabilized'.

3.2.3 Q_2Q_1 quadrilateral elements

Now we are going to look at the numerical solutions obtained with usage of Q_2Q_1 quadrilateral elements (biquadratic approximation for the displacements and bilinear approximation for the pressure). Though, usage of these elements leads to a much larger number of unknowns in the resulting system, if compared with the Q_1Q_1 elements. Nevertheless, Q_2Q_1 elements provide more accurate results for the given problem, than Q_1Q_1 quadrilateral elements do. However, these elements do not provide the oscillation free solution for the pressure in general case. But the oscillations, if appear, are local, i.e. the numerical solution is oscillatory only in the small regions in which the gradient of the pressure is large. The techniques for dealing with oscillations are analogous to those applied earlier for the solution of the problem with Q_1Q_1 elements. The 'stabilized' equations are the same, but the values of the 'stabilization' parameters are different. For Q_2Q_1 elements and for uniform mesh with square elements with edges of length h the following values for the parameters are proposed [Aguilar et al. 2008]: $\beta = \frac{h^2}{6(\lambda+2\mu)}$, $\beta_1 = \frac{h^2}{6\tau(\lambda+2\mu)}$ and $\beta_2 = \frac{h^2}{6(\lambda+2\mu)}$.

The plots of the numerical solutions of the pressure obtained with help of various techniques are shown below.

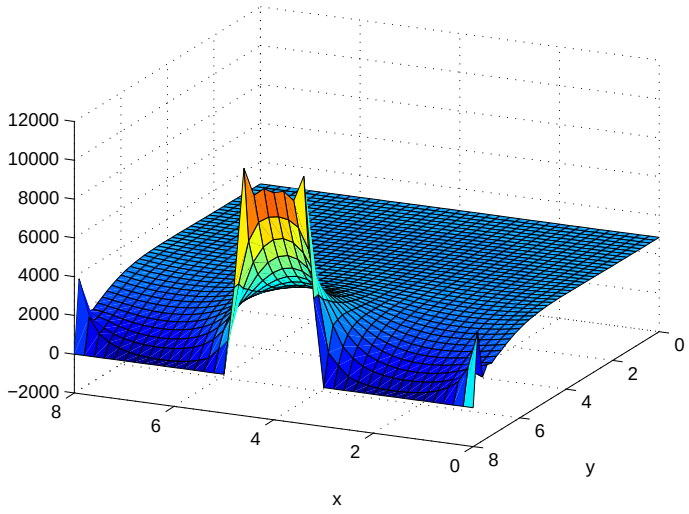

Figure 22: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of 2D poro-elastic problem with Q_2Q_1 quadrilateral elements for $t = 0$.

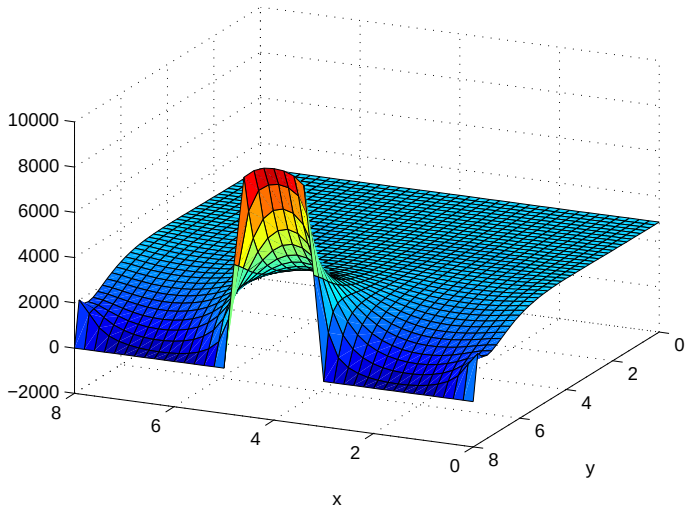

Figure 23: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of 2D poro-elastic problem with Q_2Q_1 quadrilateral elements for $t = 0$, when 'stabilization' technique for the initial solution is used.

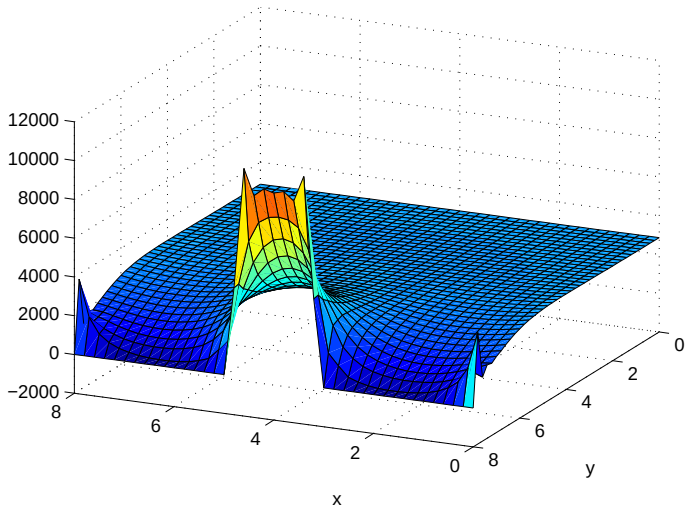

Figure 24: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of 2D poro-elastic problem with Q_2Q_1 quadrilateral elements for $t = 10^{-6}$, when 'stabilization' technique for the initial solution is not used.

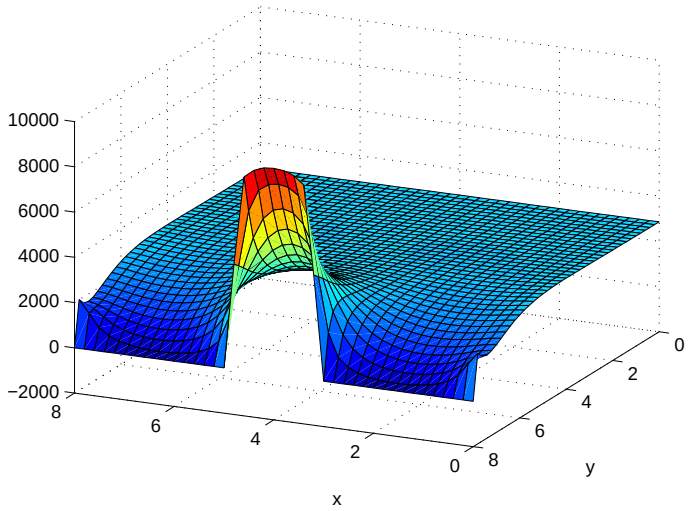

Figure 25: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of 2D poro-elastic problem with Q_2Q_1 quadrilateral elements for $t = 10^{-6}$, when 'stabilization' technique for the initial solution is used.

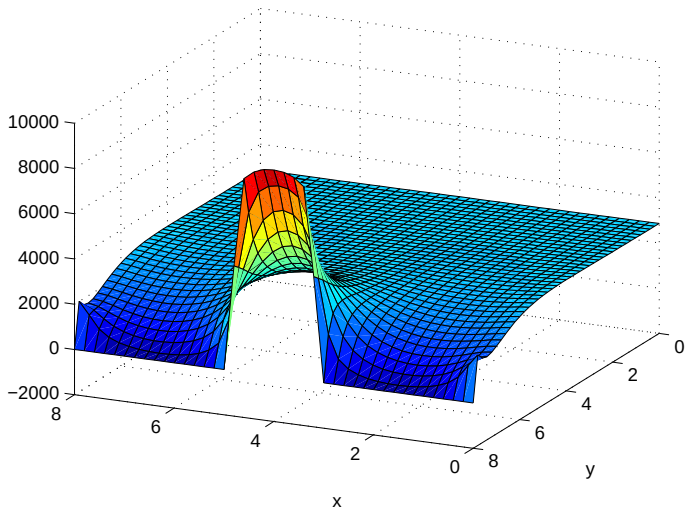

Figure 26: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of 2D poro-elastic problem with Q_2Q_1 quadrilateral elements for $t = 10^{-6}$, when 'stabilization' technique, corresponding to adding a term $-\beta_1\Delta p$, is used. The initial solution is not 'stabilized'.

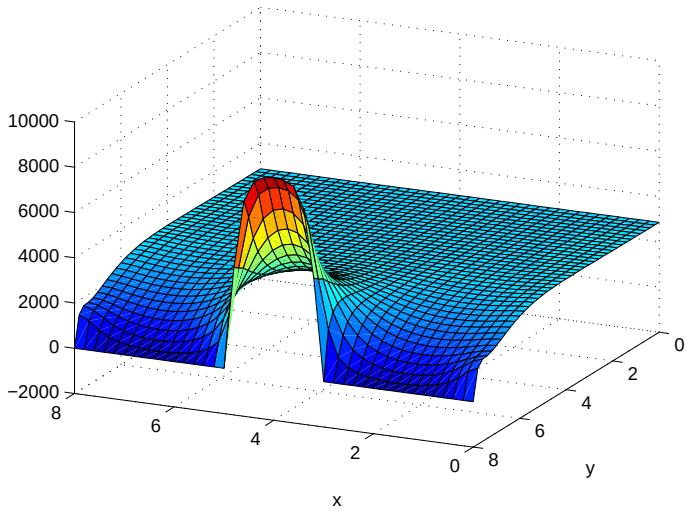

Figure 27: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of 2D poro-elastic problem with Q_2Q_1 quadrilateral elements for $t = 10^{-6}$, when 'stabilization' technique, corresponding to adding a term $-\beta_1\Delta p$, is used. The initial solution is 'stabilized'.

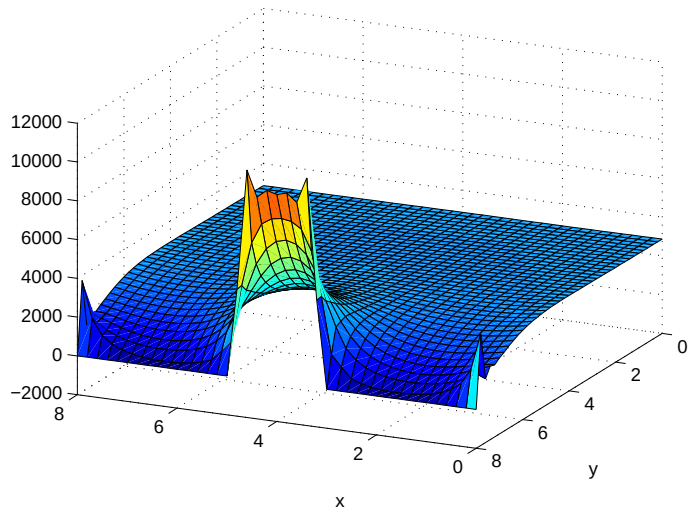

Figure 28: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of 2D poro-elastic problem with Q_2Q_1 quadrilateral elements for $t = 10^{-6}$, when 'stabilization' technique, corresponding to adding a term $-\beta_2\Delta\dot{p}$, is used. The initial solution is not 'stabilized'.

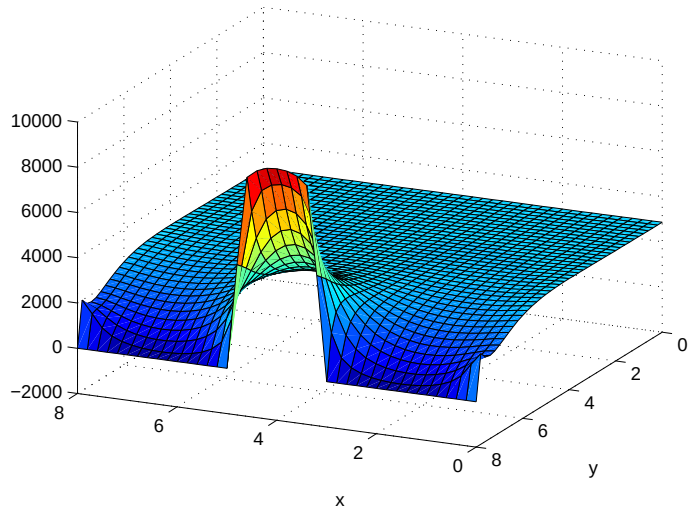

Figure 29: Plot of the numerical solution of the pressure for a Galerkin FEM approximation of 2D poro-elastic problem with Q_2Q_1 quadrilateral elements for $t = 10^{-6}$, when 'stabilization' technique, corresponding to adding a term $-\beta_2\Delta\dot{p}$, is used. The initial solution is also 'stabilized'.

4 Conclusions

In this report we have considered the numerical methods for the solution of the one dimensional and the two dimensional poro-elastic problems. The Galerkin discrete approximations of the original continuous poro-elasticity equations were derived.

Two types of approximations for the displacements and the pressure were considered: linear for the displacements and linear for the pressure; and quadratic for the displacements and linear for the pressure. The results of the numerical experiments, described in this report, allow to make the following conclusions:

- linear-linear approximation considerably reduce CPU time, comparing with quadratic-linear approximation for the same element size;
- both approximations lead to the numerical oscillations in the solution for small time steps when the element size is not small enough;
- mesh refinement in the regions with high pressure gradient allows to remove the numerical oscillations;
- for coarse grid and small time step linear-linear approximation leads to the numerical wiggles throughout the whole domain of the problem for small $t > 0$;
- quadratic-linear approximation leads to the numerical wiggles in the solution for small $t > 0$ only in small regions of the problem domain, where the presssure gradient is large;
- some 'stabilization' techniques allow to improve solution quality without mesh refinement:
  - for the linear-linear approximation the following techniques allow to remove the numerical wiggles:
    - the 'stabilization' technique corresponding to adding a term $\beta_1 \nabla p$ in a diffusion equation (equations (36), (51) for the 1D problem, and equations (69), (77) for the 2D problem), while the initial conditions are not 'stabilized' (equations (38), (39) for the 1D problem, and equations (71), (72) for the 2D problem);
    - the 'stabilization' technique corresponding to adding a term $\beta_1 \nabla \dot{p}$ in a diffusion equation (equation (36), (52) for the 1D problem, and equations (69), (79) for the 2D problem), along with the 'stabilization' of the initial conditions (equations (39), (40) for the 1D problem, and equations (72), (75) for the 2D problem);
  - for the quadratic-linear approximation for the small $t > 0$:
    - even the original discretization of the problem (equations (36), (37), (38), (39) for the 1D problem, and equations (69), (70), (71), (72) for the 2D problem) provide high enough accuracy of the numerical solution with oscillations in small regions, where the pressure gradient is large;
    - the 'stabilization' techniques analogous to those mentioned for the linear-linear approximation, provide oscillation free solution, though the numerical experiments for the 1D problem showed, that the accuracy of such 'stabilized' methods is lower than that of the 'non-stabilized' one.

Therefore such factors as the characteristic element size, the time step size, the order of approximation for the displacements and the pressure, the chosen 'stabilization' technique are interrelated and should be considered carefully before starting the simulation. The accuracy of the numerical solution and its oscillatory behavior strongly depends on all of that factors.

References

[Aguilar et al. 2008] G. Aguilar, F. Gaspar, F. Lisbona and C. Rodrigo: *Numerical stabilization of Biot's consolidation model by a perturbation on the flow equation*, Int. J. Num. Meth. Engng. 2008; **75** 1282–1300