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Proofs of ergodicity of piecewise Möbius interval maps using planar extensions

Kariane Calta^a, Cor Kraaikamp^b, Thomas A. Schmidt^{c,*}

^a *Vassar College, Department of Mathematics, Poughkeepsie, NY 12604, USA*

^b *Technische Universiteit Delft and Thomas Stieltjes Institute of Mathematics, EWI Mekelweg 4, 2628 CD, Delft, The Netherlands*

^c *Oregon State University, Department of Mathematics, Corvallis, OR 97331, USA*

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Abstract

We give two results for deducing dynamical properties of piecewise Möbius interval maps from their related planar extensions. First, eventual expansivity and the existence of an ergodic invariant probability measure equivalent to Lebesgue measure both follow from mild finiteness conditions on the planar extension along with a new property “bounded non-full range” used to relax traditional Markov conditions. Second, the “quilting” operation to appropriately nearby planar systems, introduced by Kraaikamp and co-authors, can be used to prove several key dynamical properties of a piecewise Möbius interval map. As a proof of concept, we apply these results to recover known results on the well-studied Nakada α -continued fractions; we obtain similar results for interval maps derived from an infinite family of non-commensurable Fuchsian groups.

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1. Introduction

For much of the technical vocabulary mentioned here, see Section 2 Background.

* Corresponding author.

E-mail addresses: kacalta@vassar.edu (K. Calta), c.kraaikamp@tudelft.nl (C. Kraaikamp), toms@math.orst.edu (T.A. Schmidt).

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1.1. Historical overview

Metric number theory can be said to have begun with Gauss's discovery of the invariant measure for the regular continued fraction map. The map, defined on $[0, 1]$, fixes $x = 0$ and for all nonzero x is given by $f : x \mapsto 1/x - \lfloor 1/x \rfloor$ where $\lfloor \cdot \rfloor$ denotes the floor or nearest integer function. In a letter to Laplace in 1812, Gauss stated what in modern terms is that the measure given by $dx/(\ln 2(1+x))$ is invariant under f . He did not, however, state how he found this measure.

Regular continued fractions appear in various settings in mathematics. For instance, in 1924 E. Artin used them to show that the existence of unit tangent vectors of the modular surface whose orbit under the geodesic flow is dense in the unit tangent bundle. In 1935, Hedlund used the connection to regular continued fractions to prove that the geodesic flow on the modular surface is ergodic with respect to the natural measure is "metrically transitive", a property which implies ergodicity. The following year, E. Hopf showed that the geodesic flow on the unit tangent bundle of any hyperbolic surface of finite volume is ergodic.

In fact, this last result can be used to show the ergodicity of the invariant measure for the regular continued fraction map and even determine the measure beforehand. In works of Adler–Flatto in the 1980s and 1990s and of C. Series in 1991, it is shown that one can find a cross section for the geodesic flow on the unit tangent bundle of the modular surface — thus a subset of the unit tangent bundle which every flow line meets transversely — and then show that the dynamical system of the first return map to this cross section by the flow is an extension of the regular continued fraction system.

In 1977, Nakada, Ito and Tanaka [34] gave a more elementary presentation of an extension of the regular continued fraction system. They considered the map \mathcal{T} on the unit square defined by $\mathcal{T}(x, y) = (1/x - \lfloor 1/x \rfloor, 1/(y + \lfloor 1/x \rfloor))$. Since the values $\lfloor 1/x \rfloor$ are locally constant, using the Jacobian determinant of \mathcal{T} one easily shows that the measure μ given by $(1+xy)^{-2} dx dy$ is \mathcal{T} -invariant. The *marginal measure* ν of this is the measure on $[0, 1]$ given by assigning to any Borel set E the value of μ on the subset of the square fibering over E . Here one easily finds Gauss's measure, up to the normalizing constant. Nakada, Ito and Tanaka showed that the system of \mathcal{T} gives the natural extension of the regular continued fraction map. (Keane [26] has suggested that Gauss may have found his invariant measure by use of a closely related system.)

In 1981, Nakada [33] introduced his α -continued fractions, which form a one dimensional family of interval maps, T_α with $\alpha \in [0, 1]$. (In fact, T_1 is the Gauss continued fraction map, and $T_{1/2}$ is the "nearest-integer continued fraction" map.) Using planar natural extensions, he gave the Kolmogorov–Sinai measure theoretic entropy — hereafter simply *entropy*, for those maps corresponding to $\alpha \in [1/2, 1]$. In 1991, Kraaikamp gave a more direct calculation of these entropy values by using his S -expansions, based upon inducing past subsets of the planar natural extension of the regular continued fraction map given in [34].

Let $h(T_\alpha)$ denote the entropy of T_α . In 2008 Nakada and Natsui [35] gave explicit intervals on which $\alpha \mapsto h(T_\alpha)$ is respectively constant, increasing, decreasing. Indeed, they showed this by exhibiting intervals of α such that $T_\alpha^{k'}(\alpha) = T_\alpha^k(\alpha - 1)$ for pairs of positive integers (k, k') — such intervals are now known as *matching intervals* — and

showed that the function $\alpha \mapsto h(T_\alpha)$ is constant (resp. increasing, decreasing) on such an interval if $k = k'$ (resp. $k > k'$, $k < k'$). That same year, Luzzi and Marmi [32] strongly suggested that $\alpha \mapsto h(T_\alpha)$ is a continuous function of α . They also asked if every T_α is the factor of some cross section to the geodesic flow on the unit tangent bundle of the modular surface. The continuity was proven in 2012 by both C. Carminati and G. Tiozzo [14,29]. This latter paper used explicit constructions of planar natural extensions. A presumably necessary commonality of the two papers was the description of the complement of the set of all matching intervals, called the *exceptional set*. The question of Luzzi–Marmi was answered affirmatively in [5], using what they called “Arnoux’s transversal” to find the cross sections.

Many generalizations of regular continued fractions have been studied. Katok–Ugarcovici [24] introduced the family of (a, b) -continued fraction maps and determined the full subset of its two-dimensional parameter set for which matching occurs; see also [13,25]. These continued fractions are also associated to the modular group. To each of the triangle Fuchsian groups known as the Hecke groups, [16] associated a one-parameter family of continued fraction maps and began the study of their entropy functions; see also [28].

1.2. Specific motivation, two main results

We call on the two main results of this paper in [11]. In [10], we studied a one parameter family of piecewise Möbius interval maps for each of a countably infinite number of triangle Fuchsian groups. Although planar extensions are barely mentioned there, our paper was informed by numerous calculations of them. As opposed to say the Nakada α -continued fractions, infinitely many of the maps considered in [10] are *not expansive* maps. A direct proof that each is *eventually expansive* seems tedious at best; this motivated us to seek a general result that can be easily applied to deduce eventual expansivity. We give such a result here as part of [Main Theorem 1](#).

One expects sufficiently nice continued fraction maps to be ergodic with respect to some measure which is absolutely continuous with respect to Lebesgue measure; the easiest setting to prove such results is when a Markov condition is fulfilled. In the setting of [10], and in many cases of continued fraction-like maps, Markov properties do not hold. In [Definition 1](#), we introduce a property that is often fulfilled in these settings. That this property and basic finiteness conditions satisfied by a planar extension for a map then imply ergodicity and more is also given in [Main Theorem 1](#). As an application, in [Section 3.6](#) we show that each of an infinite collection of maps is ergodic.

We also study a technique used to date for solving for the planar extension of a piecewise Möbius interval map beginning with such a planar extension for a sufficiently “nearby” map. This technique, called *quilting*, was introduced in [28], and has its roots in the discussion of the two-dimensional interpretation of “insertion” and “deletion” in the Ph.D. dissertation [27]. [Main Theorem 2](#) shows that one can use quilting to prove that fundamental dynamical properties are shared between appropriately nearby systems. We give applications of this in the setting of “matching intervals” in [Section 4.4](#).

One can thus pass from a system, say proven to have properties by use of [Main Theorem 1](#), to nearby systems and deduce that they also enjoy these properties. In

Section 5 we show that this approach gives an alternate path to proving properties of the well-studied Nakada α -continued fractions. While pursuing this path, we discovered what seems to be an unnoticed symmetry within the planar natural extensions of these maps, see the introductory paragraph of Section 5.2.2 for more on this symmetry.

Convention Throughout, we will allow ourselves the minor abuse of using adjectives such as injective, surjective and bijective to mean in each case *up to measure zero*, and thus similarly where we speak of disjointness and the like we again will assume the meaning being taken to include the proviso “up to measure zero” whenever reasonable.

1.3. Outline

In Section 2 we introduce basic terminology, notation and results from dynamical systems and ergodic theory; review the settings for our examples and illustrative applications; and summarize further background material for Propositions 18 and 19, which extend the second main result. Section 3 states and proves our first main result, Main Theorem 1, and gives an application. Section 4 states and proves our second main result, Main Theorem 2, and related results. Section 5 gives an application of our results in the setting of the Nakada α -continued fractions. Whereas the use of our main results are straightforward, here the setting is admittedly technical. We hope the reader will enjoy the rich details.

1.4. Thanks

It is a real pleasure to thank the referee for strongly recommending a clearer presentation, for mathematically helpful comments, and for additions to the bibliography.

2. Background

We collect standard background material in the Sections 2.1, 2.2 and 2.4; most of this material can be found in various textbooks, such as [15,17,22,23,40]. For a different perspective on matters of Section 2.2, see various works of P. Kůrka, such as the text [30] and the joint work [31]. In Section 2.3 we recall various results about Nakada’s α -continued fractions, both to illustrate the prior material and to use for motivation and application of the results of this paper. With the same ends, we give brief summaries of further results from the literature in the remaining portion of this section.

2.1. Basics of dynamical systems

A *dynamical system* is any (X, T, \mathcal{B}, μ) where X is a topological space X , $T : X \rightarrow X$ is a function, \mathcal{B} a sigma algebra, and μ a T -invariant measure on \mathcal{B} . (In all that follows, we consider only Borel sigma algebras, unless stated otherwise.) A dynamical system (X, T, \mathcal{B}, μ) is an *extension* of $(Y, S, \mathcal{B}', \nu)$ if there is a measurable map $\pi : X \rightarrow Y$ such that there are sets of full measure $Y' \subset Y$ and $X' \subset X$ such that $S(Y') \subset Y'$ and $T(X') \subset X'$ and a measurable surjective map $\pi : X' \rightarrow Y'$ such

that $\pi \circ T = S \circ \pi$ and $\mu \circ \pi^{-1} = \nu$. We also say that the second system is a *factor* of the first. The *natural extension* of a dynamical system was introduced by Rohlin, and defined by means of an inverse limit. It is a minimal invertible extension in the sense that any invertible system which is an extension of $(Y, S, \mathcal{B}', \nu)$ is also an extension of it. Naturally enough, the natural extension of a dynamical system is only well defined up to isomorphism; we will be most interested in planar extensions which give natural extensions, see Section 2.2.

The Kolmogorov–Sinai measure theoretic entropy, which as stated above we refer to simply as entropy and usually denote in the form $h(T)$ — is an invariant of a dynamic system, which roughly speaking measures its complexity. In fact, Rohlin introduced the notion of the natural extension system to aid in the study of entropy and showed that the original system and its natural extension share entropy values.

For a dynamical system (X, T, \mathcal{B}, μ) of finite measure and a subset $E \subset X$ of positive measure, the *induced transformation* on E is $T_E : E \rightarrow E$ given by $T_E(x) = T^k(x)$ where $k \in \mathbb{N}$ is minimal such that $T^k(x) \in E$. (By the Poincaré Recurrence Theorem, the set of $x \in E$ such that there is some such k has full measure in E , and in fact one defines T_E to be the identity on the complement of this subset.) We set μ_E to be the restriction to E of μ scaled by $1/\mu(E)$. This allows one to define a dynamical system for T_E . The following is a key tool in the study of planar extensions. *Abramov’s Formula* states that the entropy of the induced system on E is the quotient of the entropy of the original system divided by the measure of E , in short

$$h(T_E) = h(T)/\mu(E). \tag{1}$$

One also has *Rohlin’s Entropy Formula* for an interval map $T : \mathbb{I} \rightarrow \mathbb{I}$ which is ergodic with respect to an invariant probability measure ν and such that the derivative T' exists ν -almost everywhere,

$$h(T) = \int_{\mathbb{I}} \ln |T'(x)| \, d\nu. \tag{2}$$

2.2. Piecewise Möbius maps, cylinders, planar extensions

The group $GL(2, \mathbb{R})$ of invertible integral 2×2 matrices acts by way of Möbius transformations on the Riemann sphere, that is on the set of complex numbers union a point denoted ∞ . To wit, for

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3}$$

in $GL(2, \mathbb{R})$ and $z \in \mathbb{C}$, one has $M \cdot z = \frac{az+b}{cz+d}$. We use mainly the restriction of this action to the real numbers. Note that the action is projective in the sense that we can and do restrict to the case of $\det M = \pm 1$. Those M of determinant equal to 1 form $SL(2, \mathbb{R})$. Taking the quotient by its center $\{\pm I\}$, we obtain the group $PSL(2, \mathbb{R})$.

The following notation is perhaps slightly inelegant, but we will find it useful. Let $SL_2^{\pm 1}(\mathbb{R})$ denote the subgroup of $GL_2(\mathbb{R})$ comprised of those elements whose determinant is 1 or -1 , and let $PSL_2^{\pm 1}(\mathbb{R})$ be its quotient by $\{\pm I\}$. Then, $PSL_2^{\pm 1}(\mathbb{R})$ contains $PSL_2(\mathbb{R})$ as a subgroup of index two. It is a standard abuse to represent an element $[M] = \pm M \in PSL_2^{\pm 1}(\mathbb{R})$ simply by $M \in SL_2^{\pm 1}(\mathbb{R})$.

A *piecewise Möbius interval map* is a function T on a subinterval $\mathbb{I} \subset \mathbb{R}$ with values in \mathbb{I} such that there is a partition $\mathbb{I} = \cup_{\beta \in \mathcal{B}} K_\beta$ with $T(x) = M_\beta \cdot x$ for all $x \in K_\beta$. We will assume that each K_β is an interval and is taken as large as possible. We call these K_β the (rank one) *cylinders* for T . Similarly, a cylinder of rank $m > 1$ is the largest interval on which T^m is given by the action of some $M_{\beta_m} \cdots M_{\beta_1}$. We say that the cylinder K_β is *full* if $T(K_\beta) = \mathbb{I}$. Naturally enough, any cylinder which is not full is called *non-full*.

The standard number theoretic planar map associated to a Möbius transformation M is

$$\mathcal{T}_M(x, y) := \left(M \cdot x, RMR^{-1} \cdot y \right) \quad \text{for } x \in \mathbb{R} \setminus \{M^{-1} \cdot \infty\},$$

$$y \in \mathbb{R} \setminus \{(RMR^{-1})^{-1} \cdot \infty\},$$

where

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

Thus, $\mathcal{T}_M(x, y) = (M \cdot x, -1/(M \cdot (-1/y)))$. As mentioned in the introduction, an elementary Jacobian matrix calculation verifies that the measure μ on \mathbb{R}^2 given by

$$d\mu = \frac{dx \, dy}{(1 + xy)^2} \tag{5}$$

is (locally) \mathcal{T}_M -invariant.

For a piecewise Möbius interval map T we then set

$$\mathcal{T}(x, y) = \mathcal{T}_{M_\beta}(x, y) \quad \text{whenever } x \in K_\beta, \, y \in \mathbb{R} \setminus \{N^{-1} \cdot \infty\}. \tag{6}$$

Suppose that $\Omega \subset \mathbb{R}^2$ projects onto the interval \mathbb{I} and is a domain of bijectivity of \mathcal{T} , that is \mathcal{T} is bijective on Ω up to μ -measure zero. Let \mathcal{B} be the Borel algebra of Ω , we then call the system $(\mathcal{T}, \Omega, \mathcal{B}, \mu)$ a *planar extension* for T . We will occasionally abuse this terminology and say that Ω or \mathcal{T} is the planar extension. Similarly, we will make occasional use of the words T is of *positive planar extension* to mean that T has a planar extension with $0 < \mu(\Omega) < \infty$.

We will have occasional need to use a planar extension of T for which the invariant measure is Lebesgue measure. Let

$$\mathcal{Z}(x, y) = (x, y/(1 + xy)) \tag{7}$$

and for each $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as above, let $\widehat{\mathcal{T}}_M = \mathcal{Z} \circ \mathcal{T}_M \circ \mathcal{Z}^{-1}$. Then

$$\widehat{\mathcal{T}}_M(x, y) = (M \cdot x, \det M((cx + d)^2y - c(cx + d))). \tag{8}$$

We claim that Lebesgue measure is $\widehat{\mathcal{T}}_M$ -invariant. The derivative with respect to x of $x \mapsto M \cdot x$ equals $\det M/(cx + d)^2$. This also equals the multiplicative inverse of the partial derivative with respect to y of $\det M((cx + d)^2y - c(cx + d))$. An elementary Jacobian matrix calculation hence verifies that Lebesgue measure on \mathbb{R}^2 is (locally) $\widehat{\mathcal{T}}_M$ -invariant. See Fig. 2 for an indication of the effect of \mathcal{Z} .

Suppose that some T with its planar natural extension $(\mathcal{T}, \Omega, \mathcal{B}, \mu)$ is given. We let $\widehat{\mathcal{T}}(x, y) = \mathcal{Z} \circ \mathcal{T} \circ \mathcal{Z}^{-1}$, thus $\widehat{\mathcal{T}}$ is given piecewise by various $\widehat{\mathcal{T}}_M$ on the domain of

bijection $\Sigma = \mathcal{Z}(\Omega)$. We call the corresponding dynamical system the *Lebesgue planar extension* of T . See Fig. 2 for a view of both types of planar extensions for a particular map.

2.3. Nakada's α -continued fractions

The Nakada α -continued fractions form a one-parameter family of piecewise Möbius interval maps.

For $\alpha \in [0, 1]$, we let $\mathbb{I}_\alpha := [\alpha - 1, \alpha]$. Then Nakada's α -continued fraction map is defined as $T_\alpha : \mathbb{I}_\alpha \rightarrow [\alpha - 1, \alpha]$ by

$$T_\alpha(x) := \left\lfloor \frac{1}{x} \right\rfloor - \left\lfloor \left\lfloor \frac{1}{x} \right\rfloor + 1 - \alpha \right\rfloor \quad (x \neq 0),$$

$T_\alpha(0) := 0$. For $x \in \mathbb{I}_\alpha$, put

$$\varepsilon(x) := \begin{cases} +1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0, \end{cases} \quad \text{and} \quad d_\alpha(x) := \left\lfloor \left\lfloor \frac{1}{x} \right\rfloor + 1 - \alpha \right\rfloor,$$

with $d_\alpha(0) = \infty$.

The cylinder $\Delta_\alpha(\varepsilon, d)$ is the set of x such that $(\varepsilon(x), d_\alpha(x)) = (\varepsilon, d)$. Let

$$M_{(\varepsilon:d)} = \begin{pmatrix} -d & \varepsilon \\ 1 & 0 \end{pmatrix},$$

so that $T_\alpha(x) = M_{(\varepsilon:d)} \cdot x$ on $\Delta_\alpha(\varepsilon, d)$. (We will usually ignore the exceptional cylinder $\Delta_\alpha(+1, \infty)$ which contains only $x = 0$.) Note that the only endpoints of any cylinder which T_α could possibly send to an interior point of \mathbb{I}_α are $\alpha - 1$ or α . Thus, of the infinitely many cylinders at most two cylinders, those of $\alpha - 1$ or α , can be non-full. Furthermore, the fullness or non-fullness of each of these two depends only on the image of $\alpha - 1$ or α , respectively. See Fig. 1.

Furthermore, let

$$\varepsilon_n = \varepsilon_{\alpha,n}(x) := \varepsilon(T_\alpha^{n-1}(x)) \quad \text{and} \quad d_n = d_{\alpha,n}(x) := d_\alpha(T_\alpha^{n-1}(x)) \quad (n \geq 1).$$

This yields the α -continued fraction expansion of $x \in \mathbb{R}$:

$$x = d_0 + \frac{\varepsilon_1}{d_1 + \frac{\varepsilon_2}{d_2 + \dots}},$$

where $d_0 \in \mathbb{Z}$ is such that $x - d_0 \in [\alpha - 1, \alpha]$. Note that when $\alpha = 0$ this recovers the regular continued fractions, as mentioned in the introduction. The collection of all finite words in the $(\varepsilon : d)$ which arise in the expansions of any $x \in \mathbb{I}_\alpha$ form the language \mathcal{L}_α for T_α ; any word in \mathcal{L}_α is called α -admissible.

Also as indicated in the introduction, a matching interval of parameter α values is an interval such that $T_\alpha^k(\alpha) = T_\alpha^{k'}(\alpha - 1)$ for pairs of positive integers (k, k') for all α in the interval. Both [14,29] established that the complement in $[0, 1]$ of the union of the matching intervals is a set of measure zero. This complement is the *exceptional set*, denoted \mathcal{E} .

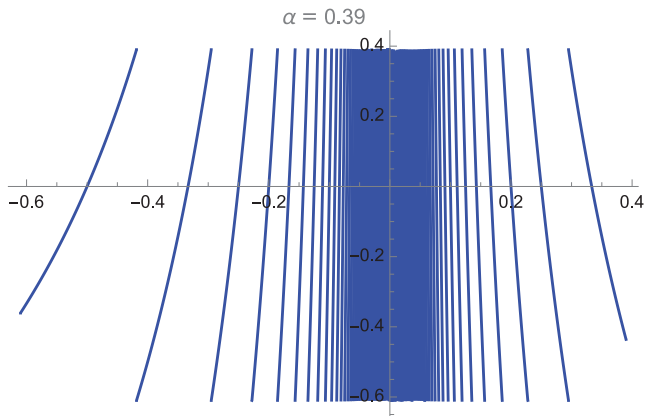


Fig. 1. Approximate graph of Nakada's $T_{0.39}$. (Aspect ratio of the plot unequal to 1 for aesthetic reasons only.) Each cylinder corresponds to a branch of the graph. Notice that the two extreme cylinders are not full; for each, this is due to exactly one endpoint of the cylinder having image in the interior of the interval of definition. See Fig. 8 for the planar extension of this function.

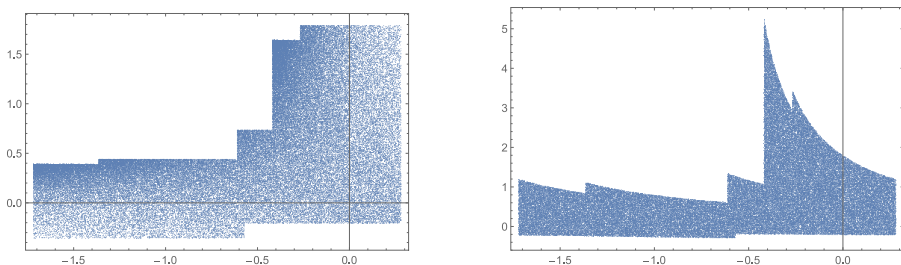


Fig. 2. Approximate plots of 100,000 points of a $\mathcal{T}_{3,0.14}$ -orbit (left), and its image under $Z(x, y)$ of (7).

For ε and d as above, let $N_{(\varepsilon:d)} = \begin{pmatrix} 0 & 1 \\ \varepsilon & d \end{pmatrix}$. Note that projectively, $N_{(\varepsilon:d)} = (M_{(\varepsilon:d)}^{-1})^t = RM_{(\varepsilon:d)}R^{-1}$. Define $\mathcal{T}_{(\varepsilon:d)}$ to be the map $(x, y) \mapsto (M_{(\varepsilon:d)} \cdot x, N_{(\varepsilon:d)} \cdot y)$. Thus for $x \in \Delta_\alpha(\varepsilon : d)$ and any y we have $\mathcal{T}_\alpha(x, y) = \mathcal{T}_{(\varepsilon:d)}(x, y)$ in accordance with (6).

2.4. Modular group and surface, geodesic flow, fuchsian groups

The upper half-plane is $\mathbb{H} = \{z = x + iy \mid x, y \in \mathbb{R}, y > 0\}$ as a subset of the complex numbers. The Möbius action of the group $SL(2, \mathbb{R})$ preserves \mathbb{H} . Indeed its elements act as isometries when we place the hyperbolic metric on \mathbb{H} , whose element of arclength squares to be $ds^2 = (dx^2 + dy^2)/y^2$. The geodesics of \mathbb{H} are either vertical lines and semi-circles, whose naive extensions meet the boundary real line perpendicularly.

For simplicity, let us say that a *unit tangent vector* on \mathbb{H} is $u = (z, \theta)$ where $z \in \mathbb{H}$ and θ denotes a direction. There is a unique hyperbolic geodesic passing through z with tangent line of the given direction. The collection of all of the unit tangent vectors is

called the *unit tangent bundle*, $T^1\mathbb{H}$. One can extend the action of $\mathrm{SL}_2(\mathbb{R})$ on \mathbb{H} to an action on $T^1\mathbb{H}$. The action is transitive and the stabilizer of the vertical unit tangent vector of basepoint $z = i$ is $\pm I$; this allows one to identify $T^1\mathbb{H}$ with $\mathrm{PSL}_2(\mathbb{R})$.

The *geodesic flow* on the unit tangent bundle is an action of the real numbers: given a nonnegative real number t and a unit tangent vector u , the unit tangent vector $t \cdot u$ is obtained by following the unique geodesic passing through u in the positive direction for arclength t and taking the unit tangent vector to this oriented geodesic at the new basepoint. If $t < 0$, we follow the geodesic in the opposite direction. In terms of $\mathrm{PSL}_2(\mathbb{R})$, the geodesic flow for time t is given by sending $A \in \mathrm{PSL}_2(\mathbb{R})$ to Ag_t , where $g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$.

A *Fuchsian group* Γ is a discrete (with respect to the natural topology) subgroup of $\mathrm{PSL}(2, \mathbb{R})$ and in particular, it is a subgroup acting properly discontinuously on \mathbb{H} ; the quotient $\Gamma \backslash \mathbb{H}$ is a surface (or orbifold) and inherits a hyperbolic metric.

The quotient $\Gamma \backslash \mathbb{H}$ has a unit tangent bundle, given by equivalence classes (thus Γ -orbits) of unit tangent vectors of \mathbb{H} . The geodesic flow descends so that for $t \in \mathbb{R}$, $[A] \in \Gamma \backslash \mathrm{PSL}_2(\mathbb{R})$ is sent to $[Ag_t]$, where the square brackets here denote Γ -cosets represented by the given elements.

The *modular group* is $\mathrm{PSL}(2, \mathbb{Z})$. The *modular surface* is $\Gamma \backslash \mathbb{H}$ with $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$. The modular group is in particular a *triangle Fuchsian group*, of finite covolume. The *signature* of this Fuchsian group is $(0; 2, 3, \infty)$, indicating that the modular surface is of genus zero and has quotient singularities of orders 2 and 3, and has a cusp: the modular surface is a punctured sphere with the puncture being at infinite hyperbolic distance from the other points.

2.5. Arnoux's method for cross sections to a geodesic flow

The following material is called upon in Section 4.3.

2.5.1. Cross sections to a measurable flow, Arnoux's transversal

Let (X, \mathcal{B}, μ) be a measure space and Φ_t a measure preserving flow on X , that is $\Phi : X \times \mathbb{R} \rightarrow X$ is a measurable function such that for $\Phi_t(x) = \Phi(x, t)$, $\Phi_{s+t} = \Phi_s \circ \Phi_t$. Then $\Sigma \subset X$ is a *measurable cross section* for the flow Φ_t if: (1) the flow orbit of almost every point meets Σ ; (2) for almost every $x \in X$ the set of times t such that $\Phi_t(x) \in \Sigma$ is a discrete subset of \mathbb{R} ; (3) for every Borel subset (in the subspace topology) $A \subset \Sigma$ and for every $\tau > 0$, *flow box* $A_{[0, \tau]} := \{\Phi_t(A) \mid A \in \Sigma, t \in [0, \tau]\}$ is μ -measurable.

The *return-time function* $r = r_\Sigma$ is $r(x) = \inf\{t > 0 : \Phi_t(x) \in \Sigma\}$ and the *return map* $R : \Sigma \rightarrow \Sigma$ is defined by $R(x) = \Phi_{r(x)}(x)$. The *induced measure* μ_Σ on Σ is defined from flow boxes: one sets $\mu_\Sigma(A) = \frac{1}{\tau} \mu(A_{[0, \tau]})$ for any $0 < \tau < \inf_{x \in A} \{r(x)\}$.

Convention: In all that follows, we write *cross section* to denote measurable cross-section.

A flow Φ_t is *ergodic* if for any invariant set either it or its complement is of measure zero. The result of Hopf mentioned in the introduction states that if a Fuchsian group Γ is of finite covolume, then the geodesic flow is ergodic with respect to the natural measure on the unit tangent bundle of $\Gamma \backslash \mathbb{H}$; see his reprisal in [20]. A flow is *recurrent* if the

Φ -orbit of almost every point meets any positive measure set infinitely often. By the Poincaré Recurrence Theorem, an ergodic flow on a finite measure space is recurrent. Given a cross-section, a first return-time transformation $(\Sigma, \mathcal{B}_\Sigma, \mu_\Sigma, R_\Sigma)$ is ergodic whenever the flow is.

There is a natural measure on the unit tangent bundle $T^1\mathbb{H}$: the *Liouville measure* is given as the product of the hyperbolic area measure on \mathbb{H} with the length measure on the circle of unit vectors at any point. Liouville measure is (left- and right-) $\mathrm{SL}_2(\mathbb{R})$ -invariant, and thus gives Haar measure on G . In particular, this measure is invariant for the geodesic flow. With standard normalizations, Liouville measure agrees with the Riemannian volume form. These both descend modulo any Fuchsian group Γ . For example, the volume of the unit tangent bundle of the modular surface is $\pi^2/3$.

Arnoux, see say [5], found an elementary manner to map Lebesgue planar extensions into the unit tangent bundle of appropriate surfaces so as to find cross sections to the geodesic flow. This often allows one to express an original interval map's system as a factor. For $(x, y) \in \mathbb{R}^2$, let

$$A(x, y) = \begin{pmatrix} x & xy - 1 \\ 1 & y \end{pmatrix}.$$

Arnoux's transversal, \mathcal{A} , is the projection to $\mathrm{PSL}_2(\mathbb{R})$ of the set of all $A(x, y)$. The complement to $\{[Ag_t] \mid A \in \mathcal{A}, t \in \mathbb{R}\}$ is a null set for Liouville measure. Furthermore Liouville measure restricts to \mathcal{A} to be a constant multiple of $dx dy dt$.

Given $M \in \mathrm{SL}_2^{\pm 1}(\mathbb{R})$ of the form (3), let $\tau(M, x) = 2 \ln |cx + d|$ for any $x \neq -d/c$ (if $c \neq 0$). The function $\tau(M, x)$ descends to be independent of representative M for an element of $\mathrm{PSL}_2^{\pm 1}(\mathbb{R})$. In what follows, we choose the representative M such that $|cx + d| = cx + d$. For ease of legibility, set $t_0 = \tau(M, x)$, $A = A(x, y)$ and $A' = A(\widehat{\mathcal{T}}_M(x, y))$. Then

$$MAg_{t_0} = \begin{cases} A' & \text{if } \det M = 1; \\ A' U & \text{if } \det M = -1, \end{cases} \tag{9}$$

where $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Given a Fuchsian group Γ , we have the function

$$\begin{aligned} \mathcal{P} = \mathcal{P}_\Gamma : \mathbb{R}^2 &\rightarrow \Gamma \backslash \mathrm{PSL}(2, \mathbb{R}) \\ (x, y) &\mapsto [A(x, y)], \end{aligned}$$

where again square brackets denote cosets. Note that this is measure preserving when we use Lebesgue measure on \mathbb{R}^2 and the measure given by $dx dy$ inherited by the projection \mathcal{A}_Γ , of \mathcal{A} to $\Gamma \backslash \mathrm{PSL}(2, \mathbb{R})$. When we use \mathcal{P} we will often commit the abuse of writing A to represent the corresponding coset.

2.5.2. Arnoux's method in the determinant one setting

Now suppose that T is a piecewise Möbius interval map. We say that the *group associated to T* is the group $\widehat{\Gamma}_T$ generated by the Möbius transformations of T ; thus, $\widehat{\Gamma}_T = \langle M_\beta, \beta \in \mathcal{B} \rangle \subset \mathrm{PSL}_2^{\pm 1}(\mathbb{R})$. We let $\Gamma_T = \widehat{\Gamma}_T \cap \mathrm{PSL}_2(\mathbb{R})$, this is a subgroup of

index $w_T \in \{1, 2\}$. In particular, if $\det M_\beta = 1$ for all β , then $w_T = 1$; we call this the *determinant one setting*.

If T has a positive planar extension, we conjugate via \mathcal{Z} to the Lebesgue planar extension of T . We then apply the measure preserving \mathcal{P} , with $\Gamma = \Gamma_T$, to $\mathcal{Z}(\Omega)$. When Γ_T is a Fuchsian group, this is a subset of $T^1(\Gamma_T \backslash \mathbb{H})$; using the fact that any Fuchsian group has countably many elements, in [6] it is shown that \mathcal{P} is injective up to measure zero on $\mathcal{Z}(\Omega)$.

Arnoux’s method in the determinant one setting is illustrated by the following result. When T is a piecewise Möbius interval map and $x \in \mathbb{I}$, we let $\tau(x) = \tau_T(x) = \tau(M, x)$ where $T(x) = M \cdot x$. The result here combines ([6], Theorem 5.4, Corollary 1, and Proposition 4).

Theorem 1 (Arnoux’s Method). *Let T be a piecewise Möbius interval map with positive planar extension, that each of Möbius transformations giving T is of determinant one, so $\widehat{\Gamma}_T = \Gamma_T$ and that this is Fuchsian group of finite covolume. Then*

$$\Sigma = \mathcal{P}_\Gamma(\mathcal{Z}(\Omega))$$

is a cross section to the geodesic flow on $T^1(\Gamma_T \backslash \mathbb{H})$. Furthermore, the system defined by

$$\begin{aligned} \phi : \Sigma &\rightarrow \Sigma \\ [A(x, y)] &\mapsto [MA(x, y)g_{\tau(x)}], \end{aligned}$$

with M such that $T(x) = M \cdot x$, is an extension of $T : \mathbb{I} \rightarrow \mathbb{I}$. Moreover, ϕ agrees with the first return map of the geodesic flow to Σ if and only if T is ergodic, eventually expansive, and with entropy satisfying $h(T)\mu(\Omega_f) = \text{vol}(T^1(\Gamma_T \backslash \mathbb{H}))$. When this holds, the first return to Σ gives a natural extension to T .

The initial statement of the theorem is shown by considering (9) with the projection. The subset $\{[A(x, y)g_t] \mid A(x, y) \in \Sigma, 0 \leq t \leq \tau(x)\} \subset T^1(\Gamma_T \backslash \mathbb{H})$ is invariant under the geodesic flow; Hopf’s result implies that this is all of $T^1(\Gamma_T \backslash \mathbb{H})$ up to measure zero. If ϕ agrees with the first return map, then the map to \mathcal{A} is such that the flow is expansive in the x -direction and contracting in the y -direction — recall that the map of (8) preserves Lebesgue measure — and one can argue as in [6] that the first return system is indeed the natural extension. The ergodicity of the flow implies that ϕ is ergodic, it then follows that \widehat{T}, \mathcal{T} and hence T itself are ergodic. The veracity of the equation involving the entropy is also in [6]; in brief, $\tau(x)$ is simultaneously the arclength of the geodesic path following the flow line from $[A(x, y)]$ to its image under ϕ and the (piecewise form of the) integrand in Rohlin’s formula (2). Recall that ν is the marginal measure of μ on Ω , and that both \mathcal{Z} and \mathcal{P}_Γ are measure preserving.

In fact, even if ϕ is not given by the first return map, if T is ergodic we can use Rohlin’s formula and find that $h(T)\mu(\Omega_f) \geq \text{vol}(T^1(\Gamma_T \backslash \mathbb{H}))$. Strict inequality holds exactly when there is a positive measure set of points in Σ whose flow paths return to first agree with ϕ only for some n th return with $n > 1$.

2.5.3. Arnoux’s method in the mixed determinant setting

One can extend the above results to the setting where not all $M = M_\beta$ are of determinant one.

Let $\Sigma_+ = \mathcal{P}_\Gamma(\mathcal{Z}(\Omega))$, thus equaling Σ as above. Fix $D \in \text{SL}_2^{\pm 1}(\mathbb{R})$ of determinant -1 , then let $\Sigma_- = \{[DAU] \mid A = A(x, y) \in \mathcal{A} \text{ with } (x, y) \in \mathcal{Z}(\Omega)\}$. Since Haar measure is both left- and right-multiplication invariant, we may and do assume that Σ_- has the Lebesgue measure in terms of x, y .

As usual, assume that $T(x) = M \cdot x$. We define maps under the restriction that $\det M = 1$

$$\begin{aligned} \alpha_M : \Sigma_+ &\rightarrow \Sigma_+ & \text{and} & & \beta_M : \Sigma_- &\rightarrow \Sigma_- \\ [A(x, y)] &\mapsto [MA(x, y)g_{\tau(x)}] & & & [DA(x, y)U] &\mapsto [DMA(x, y)g_{\tau(x)}U]. \end{aligned}$$

When $\det M = -1$ we define maps

$$\begin{aligned} \gamma_M : \Sigma_+ &\rightarrow \Sigma_- & \text{and} & & \delta_M : \Sigma_- &\rightarrow \Sigma_+ \\ [A(x, y)] &\mapsto [DMA(x, y)g_{\tau(x)}] & & & [DA(x, y)U] &\mapsto [MA(x, y)Ug_{\tau(x)}]. \end{aligned}$$

That these maps then do take values in the set indicated follows from considering (9) with the projection \mathcal{P}_Γ , and the facts that the diagonal matrices U and g_t commute and finally that U^2 is the identity. Note that $DMA(x, y)g_{\tau(x)}U = DMD^{-1}DA(x, y)Ug_{\tau(x)}$, thus β_M is given by left multiplication by a determinant one matrix and a right multiplication by a determinant one diagonal matrix. Similarly, the left multiplying matrices for the maps γ_M and δ_M are DM and MD^{-1} , respectively.

Theorem 2 (Arnoux’s Method, 2). *Let T be a piecewise Möbius interval map with positive planar extension, with Γ_T a Fuchsian group of finite covolume and $\Gamma_T \neq \widehat{\Gamma}_T$. Suppose $D \in \text{PSL}^{\pm 1}(\mathbb{Z})$ is such that (1) $\Sigma_+ \cap \Sigma_-$ is a null set; (2) $\forall M \in \Gamma_T$ also $DMD^{-1} \in \Gamma_T$; and, (3) $\forall M \in \widehat{\Gamma}_T \setminus \Gamma_T$ one has $DM \in \Gamma_T$. Then $\Sigma = \Sigma_+ \cup \Sigma_-$ is a cross section to the geodesic flow on $T^1(\Gamma_T \backslash \mathbb{H})$. Furthermore, the system defined by*

$$\begin{aligned} \psi : \Sigma &\rightarrow \Sigma \\ \sigma = \sigma(x, y) &\mapsto \begin{cases} \alpha_M(\sigma) & \text{if } \sigma \in \Sigma_+ \text{ and } \det M = 1; \\ \beta_M(\sigma) & \text{if } \sigma \in \Sigma_- \text{ and } \det M = 1; \\ \delta_M(\sigma) & \text{if } \sigma \in \Sigma_+ \text{ and } \det M = -1; \\ \gamma_M(\sigma) & \text{if } \sigma \in \Sigma_- \text{ and } \det M = -1, \end{cases} \end{aligned}$$

with M such that $T(x) = M \cdot x$, is an extension of $T : \mathbb{I} \rightarrow \mathbb{I}$. Moreover, ψ agrees with the first return map of the geodesic flow to Σ if and only if T is ergodic, eventually expansive, and has entropy satisfying $2h(T)\mu(\Omega_T) = \text{vol}(T^1(\Gamma_T \backslash \mathbb{H}))$. When this holds, the planar system \mathcal{T} on Ω_T gives a natural extension to T .

The following is the main result of [5].

Theorem 3. *For any $\alpha \in (0, 1]$ the Nakada α -continued fraction map T_α is a factor of a first return system of the geodesic flow on the unit tangent bundle of the modular surface.*

Proof (Sketch). Recall that the general form of the Nakada α -continued fraction maps is $T_\alpha(x) = \begin{pmatrix} -d & \varepsilon \\ 1 & 0 \end{pmatrix} \cdot x$. As $\begin{pmatrix} -d & \varepsilon \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -d & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\varepsilon \end{pmatrix}$, one can show that for

all α , the group Γ_{T_α} is the modular group. As well, for $\alpha > 0$, the nontrivial coset of Γ_{T_α} in $\widehat{\Gamma}_{T_\alpha}$ is represented by U .

Let $D = RU = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. One has $DA(x, y)U = \begin{pmatrix} 1 & -y \\ x & 1 - xy \end{pmatrix}$. For each $\alpha \in (0, 1)$, [29] give a positive planar extension Ω_α for T_α . One now easily verifies that $\Sigma = \Sigma_\alpha$ satisfies the hypotheses of Theorem 2 to form a cross section to the geodesic flow on the unit tangent bundle of the modular surface. By Arnoux’s [4] the result holds in the case of $\alpha = 1$, the case of regular continued fractions. By [29] (or by [14]), the product $h(T_\alpha)\mu(\Omega_\alpha)$ is constant for the set of $0 < \alpha \leq 1$. Thus, for each of these values of α , Theorem 2 applies to show that the *first* return system to Σ_α is an extension for the system of T_α . \square

In fact, [5], influenced by the formulation in the $\alpha = 1$ case due to the geometric approach of [4], work with the cross section obtained by applying R to Σ .

2.5.4. Realizable first return type

The above results motivated [6] (in the determinant one setting) to define a piecewise Möbius interval map T to be of *first return type* if: (1) T has a planar extension, with $0 < \mu(\Omega) < \infty$; (2) Γ_T is Fuchsian, of finite covolume; (3) $\tau_T(x) \geq 0$ for ν -a.e. x ; and, (4) for almost every $(x, y) \in \Omega$ and every non-trivial $N \in \widehat{\Gamma}_T$ with $\mathcal{T}_N(x, y) \in \Omega$ and $\tau(N, x) \geq 0$, we have $\tau_T(x) \leq \tau(N, x)$.

Thus, in the determinant one setting if T is of first return type then one can verify that the map ϕ of Theorem 1 does accord with the first return map of the geodesic flow. A natural extension is thus found for T , along with information about its entropy.

We say that T is of *realizable first return type* whenever T is of first return type and if $\widetilde{\Gamma}_T \neq \Gamma_T$ then also: (5) there is a $D \in \text{PSL}^{\pm 1}(\mathbb{Z})$ such the hypotheses (1)–(3) of Theorem 2 hold.

Note that when T is of *realizable first return type* then one of Theorem 1 or Theorem 2 holds. In particular, the system of T is a factor of the first return map to a cross section for the geodesic flow on the unit tangent bundle of the surface uniformized by Γ_T .

2.6. Bernoulli systems

The following material is called upon in Section 4.3.

A symbolic Bernoulli system is given by taking the shift map σ on the bi-infinite sequences $X = \mathcal{A}^{\mathbb{Z}}$ with $\mathcal{A} = \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$. There is a standard manner to define the distance between two sequences, making X into a (compact) metric space. Cylinders of rank m are defined as the set of sequences which agree for a specified consecutive sequence of length m . For each choice of probability vector (p_1, \dots, p_n) , thus with $p_1 + \dots + p_n = 1$, one gives a cylinder of rank m the measure defined by taking the product of the p_i corresponding to the ‘letters’ $a_i \in \mathcal{A}$ which define the cylinder. A limiting process results in a dynamical system $(X, \sigma, \mathcal{B}, \mu)$. Any dynamical system isomorphic to such a system is called a *Bernoulli system* (alternatively, a Bernoulli scheme). Note that one can also consider one-sided Bernoulli shifts, but we ignore that here.

Ornstein, both singly and with co-authors, established several celebrated results about Bernoulli systems.

Theorem 4 (Ornstein 1970, [36]). *Entropy is a complete invariant for Bernoulli systems. That is, if $(X, \mathcal{T}, \mathcal{B}, \mu)$ and (Y, U, \mathcal{C}, ν) are two Bernoulli systems, then these systems are isomorphic if and only if they have the same entropy value.*

A measure preserving flow Φ_t (as in Section 2.5.1) is called a *Bernoulli flow* if for each t , the system defined by Φ_t is a Bernoulli system. Due to a result of Abramov [1], it is traditional to say that the entropy value of a measure preserving flow is the entropy of its time one map, Φ_1 .

Theorem 5 (Ornstein–Weiss 1973, [38]). *The geodesic flow on the unit tangent bundle of a finite volume hyperbolic surface (or orbifold) is a Bernoulli flow of finite entropy.*

A nice application of the following result can be found in Haas' study of interval maps given by a single Möbius transformation, [18].

Theorem 6 (Rychlik 1983, [41]). *Suppose that T is a piecewise monotonic interval map with a unique invariant probability measure that is equivalent to Lebesgue measure; is such that every nonempty open subset is mapped onto the interval by some power of T ; and, whose Jacobian T' is such that $|1/T'|$ has bounded variation. Then the natural extension of T is a Bernoulli system.*

Two dynamical systems are called *quasi-isomorphic* if their natural extensions are isomorphic. See [43] for this term and pointers to the literature for examples of non-isomorphic but quasi-isomorphic systems.

2.7. Terse review of the setting of [10]

We call on this material for applications in Section 3.6, and for examples in Sections 3.2 and 4.4 (including the motivating figures, Figs. 6 and 7). As well, Remark 5 calls on this material to suggest how some of our results can be used in [11].

As mentioned in the introduction, we came to the present work in search of tools to further the study of a one-parameter family of piecewise Möbius interval maps associated to each of a countably infinite family of triangle Fuchsian groups begun in [10]. Here we give a quick review of some of the notation and terminology from that article. Besides providing direct motivation for the work here, as with the Nakada α -continued fractions, this material affords a setting for our illustrative applications throughout this paper.

For integer $n \geq 3$ we let $\nu = \nu_n = 2 \cos \pi/n$ and

$$t := t_n = 1 + 2 \cos \pi/n.$$

We use the group $G_n \subset \text{PSL}(2, \mathbb{R})$ generated by

$$A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}. \quad (10)$$

(We will have no direct use of the matrix $B = A^{-1}C$ of [10]. Furthermore, that paper studies a larger collection of groups, indexed by a pair of integers, (m, n) ; here we have set the index m equal to 3.) The group G_n is a Fuchsian triangle group of signature $(0; 3, n, \infty)$.

Fix $n \geq 3$. For each $\alpha \in [0, 1]$, let $\ell_0(\alpha) = (\alpha - 1)t, r_0(\alpha) = \alpha t$ and $\mathbb{I}_\alpha = \mathbb{I}_{n,\alpha} = [\ell_0(\alpha), r_0(\alpha)]$. Our interval maps are piecewise Möbius, of the form

$$T_\alpha = T_{n,\alpha} : [\ell_0(\alpha), r_0(\alpha)] \rightarrow \mathbb{I}_\alpha, \quad x \mapsto A^k C^\ell \cdot x \tag{11}$$

where $\ell \in \{1, 2\}$ is minimal such that $C^\ell \cdot x \notin \mathbb{I}_\alpha$ and k is the unique integer such that then $A^k C^\ell \cdot x \in \mathbb{I}_\alpha$. We call $b_\alpha(x) = (k, l)$ the α -digit of such an x , and say that x lies in the cylinder $\Delta_\alpha(k, l)$. That this does give continued fraction-like expansions of real numbers is shown in [10].

The parameter interval is naturally partitioned, $(0, 1] = (0, \gamma_n) \cup [\gamma_n, \epsilon_n) \cup [\epsilon_n, 1]$, where $\alpha < \gamma_n$ if $\forall x \in [\ell_0(\alpha), r_0(\alpha)]$ the α -digit (k, l) of x has $l = 1$ and $\alpha \geq \epsilon_n$ if and only if both $\alpha > \gamma_n$ and the α -digit of $\ell_0(\alpha)$ equals $(k, 1)$ with $k \geq 2$. See ([10], Figure 4.1) for plots indicating dynamical behavior related to this partition. We informally refer to the set of $\alpha < \gamma_n$ as the *small* α , and all others as *large* α ; see Figs. 2 and 7. For small α , we use *simplified digits*: since $l = 1$ we only report the exponent of A ; in this setting we use $\underline{d}_{[1,\infty)}^\alpha, \bar{d}_{[1,\infty)}^\alpha$ in place of $b_{[1,\infty)}^\alpha, \bar{b}_{[1,\infty)}^\alpha$, respectively. In general, the leftmost cylinder (whose left endpoint is $\ell_0(\alpha)$) and the rightmost (with right endpoint $r_0(\alpha)$) are possibly non-full; in the case of small α , these are the only possible non-full cylinders.

For large α , since the T_α -image of

$$b_\alpha = C^{-1} \cdot \ell_0(\alpha) \tag{12}$$

is exactly the image of $\ell_0(\alpha)$, the cylinder which has b_α as its left endpoint is also a candidate for non-fullness. Note that for $r_0(\alpha) \geq x \geq b_\alpha$ we have $T_\alpha(x) = A^k C^2 \cdot x$ for some $k \in \mathbb{Z}$.

From this last, if $\alpha < \alpha'$ then $b_\alpha < b_{\alpha'}$ and if α, α' are sufficiently close so that the α -digit of the left endpoint of \mathbb{I}_α equals $(k, 1)$ as does also the α' -digit of the left endpoint of $\mathbb{I}_{\alpha'}$, then there are $x \in \Delta_\alpha(k, 2) \cap \Delta_{\alpha'}(1, 1)$, see Fig. 7 for an indication of such a situation. With this condition, $T_\alpha(x) = (A^k C)A^{-1} \cdot T_{\alpha'}(x)$, but the T_α - and $T_{\alpha'}$ -images of any other $x \in \mathbb{I}_\alpha \cap \mathbb{I}_{\alpha'}$ either agree, or differ by an application of A or A^{-1} . We consider this matter in Proposition 21.

A main result of [10] is that there is a notation of *matching intervals* (there called synchronization intervals), and for each n the complement of the union of the matching intervals is a Lebesgue null subset of the parameter interval $[0, 1]$. We again call this complement the *exceptional set*, $\mathcal{E} = \mathcal{E}_n$.

We can define a map $\mathcal{T}_{n,\alpha}$ on that portion of the plane fibering over the interval of definition of $T_{n,\alpha}$, and hope for finding planar extensions. Again, see Figs. 2 and 7.

3. Bounded non-full range and finiteness of Ω implies ergodicity

In this section, we state and prove Main Theorem 1 and also give an application of it.

We use the term *eventually expansive* to describe an interval map T having some compositional power r that is expansive, thus there is some $c > 1$ so that all x in the domain of T^r satisfy $|(T^r)'(x)| \geq c$.

As stated in the Introduction, we studied an infinite countable collection of one parameter family of piecewise Möbius interval maps in [10]. As opposed to say the Nakada α -continued fractions, almost all of the maps we considered there are *not expansive* maps. A direct proof that each is eventually expansive seems tedious at best; this motivated us to seek a general result that can be easily applied to deduce eventual expansivity. We give such a result here.

One expects sufficiently nice continued fraction maps to be ergodic with respect to some measure which is absolutely continuous with respect to Lebesgue measure; the easiest setting to prove such results is when a Markov condition is fulfilled. In the setting of [10], and in many cases of continued fraction like maps, Markov properties do not hold. In Definition 1, we introduce a property that is often fulfilled in these settings. This property and basic finiteness conditions satisfied by a planar extension for a map then imply ergodicity and more.

3.1. Adler's 'Folklore Theorem'

Making an initial approach of Rényi much more practical, Adler [2,3] gave conditions implying that an interval map f has a unique ergodic measure that is equivalent to Lebesgue measure. In his afterword to [8], Adler sketched how to loosen one of his original conditions, with the following result (to which he referred there as a *folklore theorem*).

Theorem 7 (Adler, 1979; [2,3,8]). *Suppose that f is an interval map such that:*

- (i.) *All cylinders of f are full;*
- (ii.) *f is twice differentiable;*
- (iii.) *f is eventually expansive;*
- (iv.) *there is a finite bound on $|f''(x)|/f'(x)^2$ for x in the domain of f .*

Then f has a unique ergodic probability measure that is equivalent to Lebesgue measure.

Remark 1. Note that there can be at most one ergodic probability measure for f that is equivalent to Lebesgue measure. The reason for this is that any two ergodic measures are mutually singular; see say [17]. Thus, the existence result in the theorem implies the uniqueness result.

3.2. Statement of first main result

Our first main result shows that, in short, boundedness of fibers of Ω and a full cylinder for the given interval map implies ergodicity and more.

3.2.1. Cylinder covering property

We introduce a condition inspired by a condition introduced by Ito–Yuri [21]. Their *finite range property* holds for a map f if there is a finite set of measurable subsets, say $\mathcal{R} = \{V_0, \dots, V_N\}$ of the interval such that for every $n \in \mathbb{N}$ the image under f^n of any rank n cylinder is in \mathcal{R} . Of course, if every cylinder is full, then that f has the property as is shown by letting \mathcal{R} be the singleton consisting of the interval of definition itself. That is, the finite range property is a weakening of Adler’s condition of having full cylinders. We introduce a property implied by the finite range property whenever there are infinitely many full cylinders for f .

Definition 1. We say that an interval map has *bounded non-full range* if there is a full cylinder such that the orbits of the endpoints of all non-full cylinders (and hence of all cylinders) avoid the interior of this full cylinder.

Thus, under this property, the range under all positive compositional powers of the interval map of each endpoint of any non-full cylinder is bounded away from the interior of some full cylinder.

To illustrate the ease of verification of our property, we show that the bounded non-full range property holds for a large subset of one of the most studied families of continued fraction maps, the Nakada α -continued fractions [33]. Recall the review of these in Section 2.3.

Lemma 8. *Let \mathcal{E} denote the exceptional set for Nakada’s α -continued fractions. Both every rational $\alpha \in (0, 1]$ and every $\alpha \in \mathcal{E}$ is such that Nakada’s α -continued fraction map T_α has bounded non-full range.*

Proof. Whenever $\alpha \in \mathbb{Q} \cap (0, 1]$, the endpoints of $[\alpha - 1, \alpha)$ have finite α -expansion, both eventually reaching zero. Since for any α , of the infinitely many cylinders of T_α , the only possible non-full cylinders are those of these endpoints. Furthermore, whether each is non-full or not depends only on the image of $\alpha - 1$ or α , respectively. For these rational values of α , we thus have that the corresponding T_α have bounded non-full range.

Recall that T_1 is the regular continued fraction map. In the proof of ([29], Lemma 6.8; note that matching intervals are called synchronization intervals in that paper) a result of [14] is verified: $\alpha \in \mathcal{E}$ if and only if $T_1^n(\alpha) \geq \alpha$ for all $n \in \mathbb{N}$. In particular, if $\alpha \in \mathcal{E}$ then its regular continued fraction expansion is of the form $[0; a_1, a_2, \dots]$ with $a_1 \geq a_n$ for all $n > 1$ and in particular the a_i take on only finitely many values. Now, given this expansion of $\alpha \in \mathcal{E}$, from ([29], Proposition 4.1) the α -expansion of $\alpha - 1$ has digits contained in a finite set and hence from ([29], Lemma 6.7) also the α -digits of α itself are contained in a finite set. Since the α -digits correspond to cylinders, T_α is of bounded non-full range. \square

The above illustrates a setting where the finite range property fails, but our property holds. This is due to the fact that although for any $\alpha \in \mathcal{E}$ we have that both endpoints of \mathbb{I}_α have bounded digits, their orbits include infinitely many distinct points.

We further illustrate the verification of the property.

Example 1. As recalled in Section 2.7, in [10] we study families of maps $T_{n,\alpha}$, $n > 3$, $\alpha \in [0, 1]$ related to certain Fuchsian triangle groups, G_n . Fixing n , and α , the corresponding maps are of the basic form $x \mapsto A^k C^l \cdot x$, for various integers k, l , where $A, C \in G_n$ are explicitly given in Section 2.7. For each n , there is a real value γ_n such that for all $\alpha < \gamma_n$, the map indexed by α is of the simpler form $x \mapsto A^k C \cdot x$.

Fix n and a ‘small’ α , thus $\alpha \in (0, \gamma_n)$. In this setting, — similar to the case of the Nakada α -fractions — there are only at most two non-full cylinders, the leftmost cylinder whose left endpoint is $\ell_0(\alpha)$ and the rightmost with right endpoint $r_0(\alpha)$. Suppose further that α is in the exceptional set \mathcal{E}_n . Subsection 4.5 of [10] shows that the $T_{n,\alpha}$ -orbit of $\ell_0(\alpha)$ meets only the two leftmost cylinders, while the $T_{n,\alpha}$ -orbit of $r_0(\alpha)$ meets only the two rightmost cylinders. From this, we find that each of these maps has bounded non-full range. Indeed, here there are infinitely many full cylinders avoided by the orbits in question.

3.2.2. Statement of result

Main Theorem 1. *Suppose that T is a piecewise Möbius map on an interval I of finite Lebesgue measure and $\mathcal{T} : \Omega \rightarrow \Omega$ is a planar extension for T such that*

- (a) *the vertical fibers of Ω are of positive Lebesgue measure bounded away from both zero and infinity;*
- (b) *the vertical fibers are bounded away from the locus of $y = -1/x$;*
- (c) *T has at least one full cylinder for which the set of ratios of the Lebesgue measure of the T -image of each vertical fiber above this cylinder to the Lebesgue measure of its receiving fiber is bounded away from zero and one;*
- (d) *T has bounded non-full range.*

Let μ' be the normalization of μ to a probability measure on Ω and ν be the marginal measure of μ' . Also let $\mathcal{B}, \mathcal{B}'$ denote the Borel algebras of I, Ω respectively. Then

- (i.) $0 < \mu(\Omega) < \infty$;
- (ii.) T is eventually expansive;
- (iii.) T is ergodic with respect to ν ;
- (iv.) *the system $(\mathcal{T}, \Omega, \mathcal{B}', \mu')$ is the natural extension of (T, I, \mathcal{B}, ν) . In particular, the two dimensional system is also ergodic.*

That $\mu(\Omega)$ is finite is easily seen. We prove the remaining conclusions in three steps.

3.3. Eventual expansivity

Proposition 9. *Under the hypotheses (a)–(c) of Main Theorem 1, T is eventually expansive.*

Proof. Let $\widehat{\mathcal{T}} : \Sigma \rightarrow \Sigma$ denote the conjugate two-dimensional system where Σ is the image of Ω under the map $\mathcal{Z}(x, y)$ of (7). Lebesgue measure is invariant for this conjugate system. In particular, for each $x \in I$ the vertical fiber $F_x \subset \Sigma$ projecting to x is mapped by $\widehat{\mathcal{T}}$ into the vertical fiber F_{T_x} , with derivative along F_x constantly equal to

$(T'(x))^{-1}$. Hypotheses (a) and (b) imply that there are positive finite bounds $0 < b < B$ on the one-dimensional Lebesgue measure of the F_x .

The third hypothesis also carries over to the conjugate system. Let us temporarily use vertical bars to indicate the one-dimensional Lebesgue measure on vertical fibers of Σ . Denoting the chosen full cylinder by \mathcal{C} , we have that the set of ratios

$$\left\{ \frac{|\widehat{\mathcal{T}}(F_x)|}{|F_{Tx}|} : x \in \mathcal{C} \right\}$$

is also bounded away from zero and one. We can thus find a ρ with $0 < \rho < 1$ such that $1 - \rho$ is a lower bound and ρ an upper bound.

Now, for any $z \in I$, the fiber at z is the union of the images of the fibers over the preimages of z ; that is, $F_z = \cup_{Tx=z} \widehat{\mathcal{T}}(F_x)$. Given $x \in I \setminus \mathcal{C}$, set $z = Tx$. Since \mathcal{C} is a full cylinder, there is some $x' \in \mathcal{C}$ such that $T(x') = z$ and hence $|\widehat{\mathcal{T}}(F_{x'})|$ gives at least $1 - \rho$ of $|F_{Tx}|$. It follows that $|\widehat{\mathcal{T}}(F_x)| \leq \rho|F_{Tx}|$. Hence, for all $x \in I$, we have $|\widehat{\mathcal{T}}(F_x)| \leq \rho|F_{Tx}|$.

Recall that $\widehat{\mathcal{T}}$ has constant derivative along each vertical fiber. Thus, ratios of measures are preserved; in particular

$$\frac{|\widehat{\mathcal{T}}^2(F_x)|}{|\widehat{\mathcal{T}}(F_{Tx})|} = \frac{|\widehat{\mathcal{T}}(F_x)|}{|F_{Tx}|}.$$

Using a telescoping expansion and substituting the above, we deduce

$$\frac{|\widehat{\mathcal{T}}^2(F_x)|}{|F_{T^2x}|} = \frac{|\widehat{\mathcal{T}}(F_x)|}{|F_{Tx}|} \cdot \frac{|\widehat{\mathcal{T}}(F_{Tx})|}{|F_{T^2x}|} \leq \rho^2$$

and similarly for higher powers. Now let $r \in \mathbb{N}$ be such that $\rho^{r-1}B < b$. Then for any $x \in I$ we have $b \leq |F_x|$ but $|\widehat{\mathcal{T}}^r(F_x)| \leq \rho^r |F_{T^rx}| \leq \rho^r B < \rho b$. Thus, $|\widehat{\mathcal{T}}^r(F_x)| < \rho|F_x|$. Since $\widehat{\mathcal{T}}^r$ also preserves two-dimensional Lebesgue measure, we must have that $|(T^r)'(x)| > \rho^{-1}$, with now the vertical bars denoting the absolute value. Therefore, T^r is expansive. \square

3.4. Ergodicity

Proposition 10. *Under the hypotheses of Main Theorem 1, T is ergodic with respect to ν .*

Proof. Since T has bounded non-full range, there is a largest interval, say J , comprised of full cylinders avoided by the orbits of the endpoints of all non-full cylinders. Let \widetilde{T} be the first return map to J of orbits of T . We will show that the conditions for the Adler result, Theorem 7, hold for $f = \widetilde{T}$. Since ν is equivalent to Lebesgue, so is the probability measure it induces on J . As per Remark 1, the Adler result will then imply that this induced measure itself is ergodic. Ergodicity of a map induced from a general f implies that f itself is also ergodic under reasonable hypotheses (see Theorem 17.2.4 of [42]). Thus, the ergodicity of ν will follow. Again by the remark, this implies that ν is the unique ergodic invariant measure for T that is equivalent to Lebesgue measure.

The cylinders of \widetilde{T} are of the form Q_β where $\beta = (a_1, \dots, a_m)$ with a_1 an index of a T -cylinder meeting J , $[a_1, \dots, a_m]$ a rank m cylinder for T , and $Q_\beta = [a_1, \dots, a_m] \cap$

$T^{-m}(J)$. By Poincaré recurrence, up to measure zero J is the union of the Q_β . Since J consists of full cylinders for T which the T -orbits of the endpoints of T -cylinders never enter, each Q_β is a full \tilde{T} -cylinder.

The corresponding planar map \tilde{T} is bijective up to μ -measure zero on $\tilde{\Omega}$, the region defined by deleting the portion of Ω projecting to the complement of J . In particular, the arguments in the proof of [Main Theorem 1](#) apply, and thus \tilde{T} is eventually expansive.

We have ensured that \tilde{T} has full cylinders, and is eventually expansive. Our construction also preserves the property of being twice differentiable. The crux of the matter is thus to show that Adler's fourth condition holds. Since T is a piecewise Möbius map, certainly for ν -a.e. x , there is some matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\tilde{T}(x) = M \cdot x$. The first derivative here is $\det M (cx + d)^{-2}$, we must bound $|c(cx + d)|$ over all such x, M .

Since the vertical fibers of $\mathcal{Z}(\Omega)$ are bounded, so are those of $\mathcal{Z}(\tilde{\Omega})$. That is, every vertical fiber has Lebesgue measure in an interval $[b, B]$ bounded away from zero and infinity. To avoid notational unpleasanties, let us now use \hat{T} to denote the conjugate of \tilde{T} acting on $\mathcal{Z}(\tilde{\Omega})$; see (8). Recall that restricting \hat{T} to F_x (the vertical fiber at x) defines a map whose derivative equals $|\hat{T}'(x)|^{-1}$. It follows that $|\hat{T}'(x)|^{-1} \leq B/b$ for all $x \in I$. Hence, the set of values $(cx + d)^2 y$ is bounded. The boundedness of the fibers directly implies that the values $(cx + d)^2 y - c(cx + d)$ are bounded. We conclude that $c(cx + d)$ is bounded throughout I . This implies that Adler's fourth condition holds. Therefore, \tilde{T} is ergodic and, as argued in the first paragraph of this proof, the ergodicity of ν holds. \square

Remark 2. Given the above argument, one could ask whether it is always possible to induce past non-full cylinders and be sure that Adler's conditions hold. We strongly doubt this, as in general the return iteration number to the complement of those cylinders will be unbounded. As Zweimüller [44] states, this in general will cause Adler's condition (4) to fail. In our setting, of course, such "explosion" is impossible due to the boundedness of the vertical fibers of $\mathcal{Z}(\Omega)$.

3.5. Natural extension

Arnoux in particular has been a proponent of solving for planar presentations using properties of the interval map. In particular, the main results of [7] imply that (1) if a piecewise Möbius interval map $T : I \rightarrow I$ is (eventually) expansive then its associated naive two-dimensional map $\mathcal{T} : I \times \mathbb{R} \rightarrow I \times \mathbb{R}$ induces a contraction on the complete metric space of compact subsets of this product, where a modified Hausdorff metric is used. (The contraction sends a compact K to the closure of the union of the $\mathcal{T}_\beta(K_\beta)$, where K_β is the portion of K projecting to the β -cylinder.) And, (2) when the fixed point, say Ω , of this contraction has positive measure and \mathcal{T} is bijective on Ω up to measure zero, then this two-dimensional system is a planar natural extension of T .

We thus find the following.

Proposition 11. *Under the hypotheses of [Main Theorem 1](#), $(\mathcal{T}, \Omega, \mathcal{B}', \mu')$ is the natural extension of (T, I, \mathcal{B}, ν) .*

3.6. Application to each of an infinite collection of maps

The goal of this subsection is to apply [Main Theorem 1](#) to a family of maps whose possible ergodicity was unknown, and indeed, whose planar natural extensions had not been determined. We reach this goal in [Corollary 14](#).

Our maps are related to those studied in [10], again refer to Section 2.7. The dynamics of the maps $T_{n,1}$ (among others) are presented in Section 3 of [10]. Here we give a planar extension for each of these maps. Each is of infinite mass; by “accelerating” each of the interval maps past the domain of the parabolic element of the underlying group whose fixed point is responsible for the infinitude of the mass, we obtain an interval map of invariant probability measure, as verified by applying [Main Theorem 1](#).

Fix $n \geq 3$, let $T = T_{n,1}$ and $U = AC(AC^2)^{n-2}$, and note that $t = t_n = r_0(1)$. From [10], one has $T^i(t) = (AC^2)^{n-2} \cdot t$ for $1 \leq i \leq n - 2$ and $T^{n-1}(t) = U \cdot t = t$. Furthermore, all of the cylinders of T are full except for the right most cylinder, $\Delta(1, 2)$, of endpoints $\mu + 1/t = 1 + 1/t$ and t . It is easily verified that $(AC^2)^{n-2} \cdot t = 1$.

Compare the following with [Fig. 3](#).

Proposition 12. *Fix $n \geq 3$ and let $T = T_{n,1}$ and \mathcal{T} be the usual associated two-dimensional map. Let r_0, r_1, \dots, r_{n-2} be the T -orbit of $r_0 = t$. Then \mathcal{T} is bijective up to μ -measure zero on*

$$\Omega = [0, 1] \times [-1, 0] \cup \bigcup_{i=1}^{n-2} [r_i, r_{i-1}] \times [-1/r_{i-1}, 0].$$

Proof. We have that $(0, 1]$ is the union of the cylinders $\Delta(i, 1)$ with $i \in \mathbb{N}$. Similarly, $(1, 1+1/t]$ is the union of the $\Delta(j, 2)$ with $j \geq 2$. Since $1+1/t$ lies between $1 = r_{n-2}$ and r_{n-3} , the y -fiber of Ω above each of these $\Delta(j, 2)$ is $[-1/r_{n-3}, 0]$, whereas every $\Delta(i, 1)$ has y -fibers given by $[-1, 0]$. Recall that R is given in (4); since $RCR^{-1} \cdot 0 = -1$, it follows that $RA^kCR^{-1} \cdot -1 = RA^kC^2R^{-1} \cdot 0$ for any k . Hence, each rectangle $\Delta(k, 1) \times [-1, 0]$ is mapped above the image of $\Delta(k, 2) \times [-1/r_{n-3}, 0]$ so as to share exactly a common horizontal line.

Now, $AC^2 \cdot r_{n-3} = 1$ can be used to show that $RC^2AC^2R^{-1} \cdot -1/r_{n-3} = 0$ and a similar observation implies that each rectangle $\Delta(k, 1) \times [-1, 0]$ is mapped below the image of $\Delta(k+1, 2) \times [-1/r_{n-3}, 0]$ so as to share exactly a common horizontal line.

Therefore, \mathcal{T} sends $\Omega \cap \{x \leq 1 + 1/t\}$ bijectively up to measure zero to $(0, t] \times [-1/t, 0)$. Furthermore, since every \mathcal{T}_M preserves the locus $y = -1/x$, we easily find that $i = 1, \dots, n - 4$,

$$\mathcal{T}_{AC^2}([r_i, r_{i-1}] \times [-1/r_{i-1}, 0]) = [r_{i+1}, r_i] \times [-1/r_i, -1/t].$$

(Of course, when $n \leq 4$ we must make appropriate adjustments.) Furthermore, \mathcal{T}_{AC^2} sends $(1 + 1/t, r_{n-3}) \times [-1/r_{n-3}, 0]$ to $(0, 1] \times [-1, -1/t]$. The result thus holds. \square

Since U is a conjugate (up to sign) of A^{-1} we see that it is a parabolic matrix and thus t is parabolic fixed point under T . This in a sense is the cause of the invariant measure μ being infinite on Ω . Just as in the treatment of $T_{n,0}$ in [12], we “accelerate” our map by inducing past the cylinder (here of rank $n - 1$) related to the parabolic element. There is

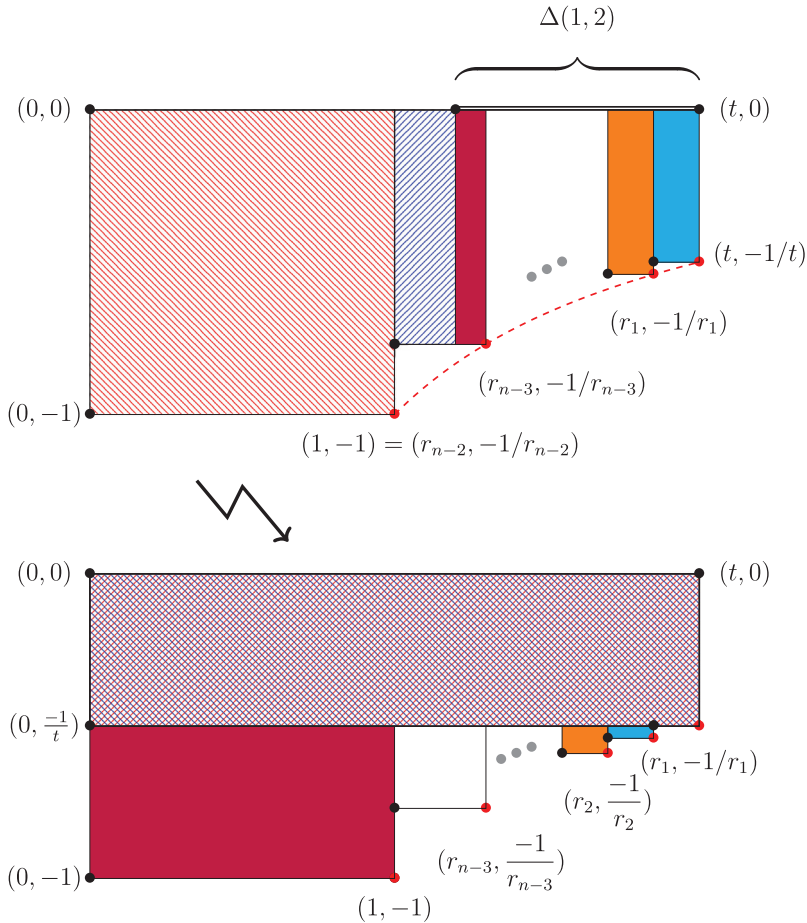


Fig. 3. Schematic representation of the domain $\Omega_{n,1}$, and its image under $\mathcal{T}_{n,1}$, $n \geq 3$ as discussed in Proposition 12. Compare with ([10], left side of Fig. 6). Here, on the left side, the red dotted curve plots $y = -1/x$ the only points of Ω on this are the (red) vertices coming from the orbit of $(t, -1/t)$. (Recall that our measure is given by $d\mu = (1 + xy)^{-2} dx dy$!) Blocks fibering over intervals whose endpoints are consecutive members of the orbit of $r_0 = t$ under the interval map are filled with solid colors. Red hatching indicates blocks fibering over cylinders indexed by $(i, 1)$, $i \in \mathbb{N}$. Blue hatching indicates blocks fibering over cylinders indexed by $(j, 2)$, $j \geq 2$. Images on the right hand side correspondingly colored, except that the cross-hatching indicates lamination from the hatched portions.

an easily determined domain of bijectivity for the corresponding two-dimensional map. (Indeed the following result is a specific case of a general phenomenon.) Compare the following with Fig. 4.

Lemma 13. Fix $n \geq 3$. Let $\epsilon_0 = U^{-1} \cdot 0$ and let $g(x)$ be the first return map of $T_{n,1}$ to $(0, \epsilon_0)$ and \mathcal{G} the corresponding two-dimensional map. Then \mathcal{G} is bijective up to

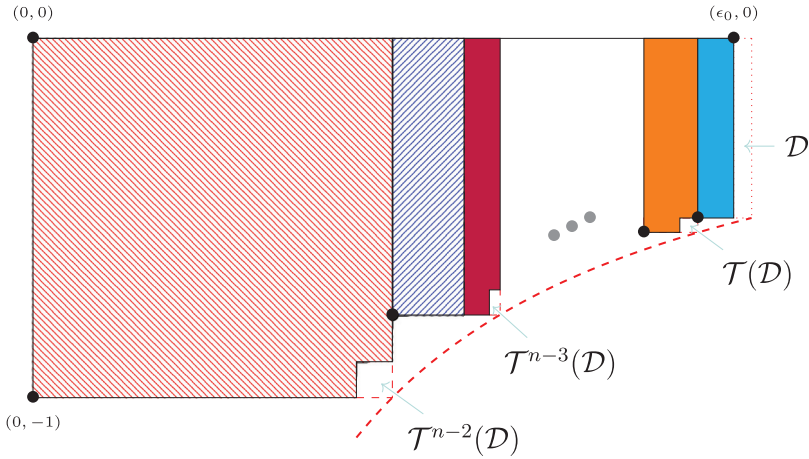


Fig. 4. Schematic representation of the domain Γ for the accelerated two-dimensional map. The domain is given by deleting from Ω the rectangle $\mathcal{D} = (\epsilon_0, t] \times [-t, 0]$ and its images under $\mathcal{T}, \dots, \mathcal{T}^{n-2}$. See Lemma 13.

μ -measure zero on

$$\Gamma = \Omega \setminus \bigcup_{i=0}^{n-2} \mathcal{T}_{(AC^2)^i}(\mathcal{D}),$$

where $\mathcal{D} = (\epsilon_0, t] \times [-1/t, 0]$.

Proof. By definition, $g(x) = T^i(x)$ where $i = i(x) \in \mathbb{N}$ is minimal such that $T^i(x) < \epsilon_0$. Since $(\epsilon_0, t] = \Delta((1, 1)^{n-2}(1, 2))$, we have that $g(x) = U^j \circ T(x)$ where $j \geq 0$ is minimal such that the image is outside of the rank $n - 1$ cylinder $\Delta((1, 2)^{n-2}(1, 1))$.

Now, $(x, y) \in \mathcal{D}$ if and only if $T(x) \in \Delta((1, 2)^{n-2}(1, 1))$ and therefore the \mathcal{T} -orbit of (x, y) includes the initial sequence $\{\mathcal{T}_{(AC^2)^i}(x, y)\}_{i=0}^{n-2}$. Furthermore, due to the bijectivity of \mathcal{T} , any $(x, y) \in \bigcup_{i=0}^{n-2} \mathcal{T}_{(AC^2)^i}(\mathcal{D})$ must belong to a length $n - 1$ orbit sequence with an initial orbit element in \mathcal{D} .

Suppose now that $(x, y) \in \Gamma$ and $\mathcal{T}(x, y) \notin \mathcal{D}$. From the previous paragraph, we have that $g(x) = T(x) = M \cdot x$ for some matrix M and hence both $\mathcal{G}(x, y) = \mathcal{T}(x, y) = \mathcal{T}_M(x, y)$ and $\mathcal{G}(x, y)$ must indeed belong to Γ .

On the other hand, if $(x, y) \in \Gamma$ with $\mathcal{T}(x, y) \in \mathcal{D}$, then $T(x) \in (\epsilon_0, t)$ and there is a $j \in \mathbb{N}$ such that $\mathcal{G}(x, y) = \mathcal{T}_{U^j} \circ \mathcal{T}(x, y) = \mathcal{T}_{AC} \circ \mathcal{T}_{(AC^2)^{j-1}} \circ \mathcal{T}_{U^{j-1}} \circ \mathcal{T}(x, y)$. But, $\mathcal{T}_{U^{j-1}} \circ \mathcal{T}(x, y) \in \mathcal{D}$ and hence while $(\mathcal{T}_{AC})^{-1} \circ \mathcal{G}(x, y) \in \bigcup_{i=0}^{n-2} \mathcal{T}_{(AC^2)^i}(\mathcal{D})$, the application of \mathcal{T}_{AC} must send this value outside of that union. That is, here also $\mathcal{G}(x, y)$ must belong to Γ .

The bijectivity of \mathcal{G} on Γ now follows immediately from that of \mathcal{T} on Ω . \square

Corollary 14. Fix $n \geq 3$ and let $g(x)$ be induced from $T_{n,1}$ and let Γ be as above. Then $g(x)$ is expansive and is ergodic with respect to the probability measure that is the normalized marginal measure from $\mu = (1 + xy)^{-2} dx dy$ on Γ .

Proof. From the definition of Ω , it follows that Ω meets the curve $y = -1/x$ exactly in the \mathcal{T} -orbit of $(t, -1/t)$, which is of course a point in \mathcal{D} . As well, the remainder of Ω lies above this curve (with $x \geq 0$). Since $\Gamma = \Omega \setminus \cup_{i=0}^{n-2} \mathcal{T}_{(AC^2)^i}(\mathcal{D})$, it follows that Γ not only does not meet the curve, but in fact stays a bounded distance away. From this, hypothesis (b) of [Main Theorem 1](#) is satisfied for the piecewise Möbius map $g(x)$ on the interval $(0, \epsilon_0)$ with \mathcal{G} bijective on Γ ; the other hypotheses are easily verified, and hence the result holds. \square

4. Quilting as a proof tool

We now discuss a technique used to date for solving for the planar extension of a piecewise Möbius interval map beginning with such a planar extension for a sufficiently “nearby” map. This technique, called *quilting*, was introduced in [28], and has its roots in the discussion of the two-dimensional interpretation of “insertion” and “deletion” in the Ph.D. dissertation [27]. [Main Theorem 2](#), shows that one can use quilting to prove that certain properties are shared between appropriately nearby systems.

4.1. Quilting defined, main properties announced

We give a basic definition.

Definition 2. Suppose that f, g are piecewise Möbius interval maps on $\mathbb{I}_f, \mathbb{I}_g$ each with uncountably many cylinders and with finite nonzero μ -measure planar two-dimensional domains of bijectivity Ω_f, Ω_g for corresponding two-dimensional maps \mathcal{F}, \mathcal{G} , respectively. For $x \in \mathbb{I}_f$ let $b_f(x)$ denote the f -digit of x (informally, this thus denotes the corresponding Möbius transformation which applied to x gives the value $f(x)$), and similarly for $b_g(x)$. Let

$$\Delta = \Delta_{f,g} = \{x \in \mathbb{I}_f \cap \mathbb{I}_g \mid b_f(x) \neq b_g(x)\} \text{ and}$$

$$\mathcal{C} = \mathcal{C}_f = \{(x, y) \in \Omega_f \mid x \in \Delta\}.$$

We then construct a domain on which we will show that \mathcal{G} is bijective (up to sets of measure zero) by deleting the forward \mathcal{F} -orbit of \mathcal{C} and adding in the forward \mathcal{G} -orbit of \mathcal{C} (here we extend \mathcal{G} to be the piecewise map on $\mathbb{I}_g \times \mathbb{R}$ given by the \mathcal{T}_M where g is given piecewise by $x \mapsto M \cdot x$, recall (6)). In general, each of these orbits is infinite and might even sweep out the respective domains up to measure zero. For a practical version of this approach, we introduce some finiteness conditions.

Recall that we are interested in measure theoretic results, and thus use disjointness of sets to mean that they meet in at most a null set. See [Figs. 6](#) and [7](#) for representations of quilting in specific settings. See [Fig. 5](#) for a representation of the basic relation (i) in the following definition.

Definition 3. We say that Ω_g can be *countably quilted* from Ω_f if \mathcal{C} has positive measure and

- (i.) There is an at most countable partition of \mathcal{C} by \mathcal{C}_i and corresponding integers d_i, a_i such that $\mathcal{F}_{|\mathcal{C}_i}^{1+d_i} = \mathcal{G}_{|\mathcal{C}_i}^{1+a_i}$;

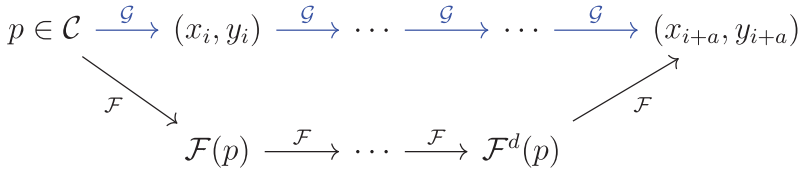


Fig. 5. When quilting, forward \mathcal{G} -orbit segments beginning at a point in \mathcal{C} rejoin forward \mathcal{F} -orbit segments.

- (ii.) The $\mathcal{F}^j(\mathcal{C}_i)$ indexed over all i and $1 \leq j \leq d_i$ is pairwise disjoint and their union has strictly less than full measure in Ω_f , and similarly the $\mathcal{G}^j(\mathcal{C}_i)$ are pairwise disjoint, with their union of finite measure; and,
- (iii.)

$$\Omega_g = \left(\Omega_f \setminus \prod_{i=1}^{\infty} \prod_{j=1}^{d_i} \mathcal{F}^j(\mathcal{C}_i) \right) \sqcup \prod_{i=1}^{\infty} \prod_{j=1}^{a_i} \mathcal{G}^j(\mathcal{C}_i). \tag{13}$$

Of course, when the partition is finite of cardinality n , then n replaces the infinite upper limits appearing in (13). We then speak of *finite quilting*. For simplicity’s sake, we will write *quilting* to denote countable quilting (see Fig. 5).

Main Theorem 2. Assume that Ω_f is of finite μ -measure. Quilting preserves the properties of: (a) ergodicity of two dimensional maps; (b) this planar extension giving the natural extension of the interval map’s system; and, when (b) holds allows for an explicit expression of the entropy of g in terms of that of f .

We give precise statements in the propositions of the subsequent subsection, which together prove the theorem.

Remark 3. Main Theorem 2 certainly holds when the systems of \mathcal{F} and of \mathcal{G} are isomorphic. This is the case in particular when $a_i = d_i$ holds for each \mathcal{C}_i of the quilting partition of \mathcal{C} . Indeed, we can then give an explicit isomorphism, $\varphi : \Omega_f \rightarrow \Omega_g$ by fixing $\Omega_f \setminus \prod_{i=1}^{\infty} \prod_{j=1}^{d_i} \mathcal{F}^j(\mathcal{C}_i)$ and applying $\mathcal{F}^j(x, y) \mapsto \mathcal{G}^j(x, y)$ for $1 \leq j \leq a_i$ for each $(x, y) \in \mathcal{C}_i$, for each \mathcal{C}_i .

4.2. Proofs of main properties

Proposition 15. Suppose that f, g are piecewise Möbius interval maps such that Ω_g can be quilted from Ω_f , and that $\mu(\Omega_f) < \infty$. If $\mathcal{F} : \Omega_f \rightarrow \Omega_f$ is ergodic with respect to the measure μ , then both \mathcal{G} and g are ergodic, with respect to μ on Ω_g and its marginal measure on \mathbb{I}_g , respectively.

Proof. Since the dynamical system of g is a factor of that of \mathcal{G} , its ergodicity will follow from that of this latter system. Now suppose $E \subset \Omega_g$ is μ -measurable, is not of full measure, and is invariant under \mathcal{G} . We aim to show that E is a null set. Recall that both \mathcal{F}, \mathcal{G} preserve the measure.

By definition of \mathcal{C} , we have that \mathcal{G} agrees with \mathcal{F} on $\Omega_g \cap (\Omega_f \setminus \mathcal{C})$. The ergodicity of \mathcal{F} shows that we may assume that each \mathcal{G} -orbit contained in E which meets $\Omega_g \cap (\Omega_f \setminus \mathcal{C})$ also exits this set. That is, we may assume that E is contained in the set of forward \mathcal{G} -orbits of points of \mathcal{C} . Each $p \in \mathcal{C}$ has an initial \mathcal{G} -orbit segment which meets its forward \mathcal{F} -orbit. By choosing the interpolating forward \mathcal{F} -orbit segments until their meeting and then through the next entrance to \mathcal{C} , we form an \mathcal{F} -invariant set. The ergodicity of \mathcal{F} shows that this is a nullset and therefore so must be E . \square

Proposition 16. *Suppose that f, g are piecewise Möbius interval maps such that Ω_g can be quilted from Ω_f and that the dynamical system of \mathcal{F} is the natural extension of that of f . Then the dynamical system of \mathcal{G} is the natural extension of that of g .*

Proof. The natural extension is the minimal invertible system of which our given system is a factor. Let us call any bi-infinite sequence $(x_i)_{i \in \mathbb{Z}}$ with each $x_i \in \mathbb{I}_g$ satisfying that for all i , $g(x_i) = x_{i+1}$ a *bi-infinite g -orbit* and similarly for our other maps. We then say that a bi-infinite \mathcal{G} -orbit $(x_i, y_i)_{i \in \mathbb{Z}}$ projects to the bi-infinite g -orbit $(x_i)_{i \in \mathbb{Z}}$.

To show that the system of \mathcal{G} is the natural extension of the system of g , it suffices to show that for each bi-infinite g -orbit there is a unique bi-infinite \mathcal{G} -orbit $(x_i, y_i)_{i \in \mathbb{Z}}$ in Ω_g projecting to it. By hypothesis, the analogous statement is true for the pair f, \mathcal{F} .

Step 1: From \mathcal{G} - to f -orbits. We first associate to each bi-infinite \mathcal{G} -orbit $(x_i, y_i)_{i \in \mathbb{Z}}$ in Ω_g a bi-infinite f -orbit. Let us discern two types of \mathcal{G} -orbit segments, those of type a , which begin at a point $p \in \mathcal{C}$ and end at the common point $\mathcal{F}^{d+1}(p) = \mathcal{G}^{a+1}(p)$; and those of type c , common to both by lying in $\Omega_g \cap (\Omega_f \setminus \mathcal{C})$. An orbit segment of type c but whose extension thereafter exits $\Omega_g \cap (\Omega_f \setminus \mathcal{C})$ must be such that the extension meets \mathcal{C} . That is, this orbit segment is then followed by an orbit segment of type a . Similarly, an orbit segment of each of type a is followed by a segment of one of two types. In each of these settings, there is a uniquely associated \mathcal{F} -orbit segment; this segment is given by equality when the \mathcal{G} -orbit segment is of type c , and otherwise by the \mathcal{F} -orbit segment demanded as part of the definition of the type a \mathcal{G} -orbit segment. Further note that an orbit segment of either of the two types begins at the ending of a segment of one of the two types. That is, we can uniquely extend the transcription from \mathcal{G} -orbit segments to \mathcal{F} -orbit segments infinitely in both directions. With this, we have uniquely associated to each bi-infinite \mathcal{G} -orbit $(x_i, y_i)_{i \in \mathbb{Z}}$ in Ω_g a bi-infinite \mathcal{F} -orbit in Ω_f . Finally, by projecting, we find a bi-infinite f -orbit.

Step 2: Uniqueness. Fix now a bi-infinite g -orbit $\gamma = (x_i)_{i \in \mathbb{Z}}$. Let us call any choice of \mathcal{G} -orbit in Ω_g projecting to γ a ‘lift’ of γ . From the above, to a bi-infinite g -orbit $\gamma = (x_i)_{i \in \mathbb{Z}}$ we associate one bi-infinite f -orbit per lift of γ . We now aim to show that these bi-infinite f -orbits are one and the same.

We first note that γ determines a unique bi-infinite sequence $(M_i)_{i \in \mathbb{Z}}$ such that $x_{i+1} = M_i \cdot x_i$; and hence any choice of bi-infinite \mathcal{G} -orbit in Ω_g projecting to γ satisfies $(x_{i+1}, y_{i+1}) = \mathcal{T}_{M_i}(x_i, y_i)$ for all i . The analogous statement holds for bi-infinite f - and \mathcal{F} -orbits.

Now, any lift of γ can be partitioned into segments of type a and c . Consider the set $\mathcal{J} = \{j \mid x_j \in \Delta\}$. If $\mathcal{J} = \emptyset$ then any \mathcal{G} -orbit in Ω_g projecting to γ lies in $\Omega_g \cap (\Omega_f \setminus \mathcal{C})$. This is then also an \mathcal{F} -orbit. The projections agree, and thus γ itself is our desired bi-infinite f -orbit. Now if \mathcal{J} is non-empty then any lift of γ is such that there is some j

with $p = (x_j, y_j) \in \mathcal{C}$ and the previous paragraph then shows that this lift's associated bi-infinite f -orbit contains the forward f -orbit of x_j . If another lift of γ is such that it contains a point $q = (x_j, y'_j)$ with $q \notin \mathcal{C}$ then q occurs in what we can call the 'middle' of some \mathcal{G} -orbit segment of type a . But then this segment is announced by a $p' = (x_{j'}, y_{j'}) \in \mathcal{C}$ with $j' < j$, and again by invoking the previous paragraph, we have that the forward f -orbit of $x_{j'}$ is contained in this lift's associated bi-infinite f -orbit. Furthermore, this forward f -orbit contains that of x_j . From this, if \mathcal{J} has a least element j , then every lift of γ has its associated bi-infinite f -orbit containing the forward f -orbit of the corresponding x_j . But, each such lift must then have backwards infinite orbit completely of type c . That is, the associated bi-infinite f -orbits must also all agree for indices less than j . By similar reasoning, in the case of \mathcal{J} having no least element, all of the lifts of γ share the same bi-infinite f -orbit.

Step 3: Conclusion. By hypothesis, each bi-infinite f -orbit has exactly one bi-infinite \mathcal{F} -orbit in Ω_f projecting to it. Since the bi-infinite \mathcal{F} -orbit in Ω_f associated to a bi-infinite \mathcal{G} -orbit in Ω_f certainly uniquely identifies this \mathcal{G} -orbit, we conclude that γ has exactly one bi-infinite \mathcal{G} -orbit in Ω_f projecting to it. \square

Proposition 17. *Suppose that f, g are piecewise Möbius interval maps such that Ω_g can be quilted from Ω_f . Then the entropy of \mathcal{F} and \mathcal{G} are related by*

$$h(\mathcal{G}) = h(\mathcal{F}) \frac{\mu(\Omega_f)}{\mu(\Omega_g)}.$$

If furthermore the dynamical system of \mathcal{F} is the natural extension of that of f , then the entropy of g is given by

$$h(g) = \left(1 + \sum_{i=1}^{\infty} (a_i - d_i) \nu(\Delta_i) \right)^{-1} h(f), \tag{14}$$

where for each i , C_i projects to $\Delta_i \subset \mathbb{I}_f$ and ν is the marginal probability measure induced from μ on Ω_f .

Proof. We can compute the entropy of \mathcal{G} in terms of that of \mathcal{F} by using Abramov's formula (as mentioned in 2.1) Let $\Omega_{f,g} := \Omega_f \setminus \coprod_{i=1}^{\infty} \coprod_{j=1}^{d_i} \mathcal{F}^j(C_i)$. The first return maps induced from each of \mathcal{F} and \mathcal{G} to $\Omega_{f,g}$ are equal. Therefore,

$$\frac{h(\mathcal{G}) \mu(\Omega_g)}{\mu(\Omega_{f,g})} = \frac{h(\mathcal{F}) \mu(\Omega_f)}{\mu(\Omega_{f,g})}.$$

Since our various maps are μ -measure preserving,

$$h(\mathcal{G}) = \left(1 + \sum_{i=1}^{\infty} (a_i - d_i) \frac{\mu(C_i)}{\mu(\Omega_a)} \right)^{-1} h(\mathcal{F}).$$

If \mathcal{F} gives the natural extension of f , then they have the same entropy. Furthermore, from the previous proposition, \mathcal{G} then gives the natural extension of g and thus these also have equal entropy. Since each Δ_i is the projection of C_i , we have that $\nu(\Delta_i) = \mu(C_i)/\mu(\Omega_a)$. Therefore, (14) holds. \square

4.3. Property of realizable first return type is also preserved

We use notation, terminology and results presented in Section 2.5.

Proposition 18. *Suppose that f, g are expansive piecewise Möbius interval maps such that Ω_g can be quilted from Ω_f . Suppose that $\Gamma_f = \Gamma_g$ is of finite covolume, $\widehat{\Gamma}_f = \widehat{\Gamma}_g$, and f is of realizable first return type. Then g is also of realizable first return type. Furthermore, both maps: are ergodic; have their planar extensions as natural extensions; and, are factors of the first return map to a section for the geodesic flow on $T^1(\Gamma_f \backslash \mathbb{H})$.*

Proof. Since f is of realizable first return type, from Theorem 1 or Theorem 2, we have that f is ergodic, expansive, that its planar extension gives a natural extension, and that $w_f h(f)\mu(\Omega_f) = \text{vol}(T^1(\Gamma_f \backslash \mathbb{H}))$, where $w_f \in \{1, 2\}$ is equal to 1 if only if $\widehat{\Gamma}_f = \Gamma_f$.

Now, Proposition 17 gives $h(\mathcal{G})\mu(\Omega_g) = h(\mathcal{F})\mu(\Omega_f)$. Proposition 16 and the fact that entropy is shared by a map and its natural extension, then gives that $h(g)\mu(\Omega_g) = h(f)\mu(\Omega_f)$. Hence, $h(g)\mu(\Omega_g)$ equals w_f times the volume of the unit tangent bundle of $\Gamma_g \backslash \mathbb{H}$. By hypothesis, g is expansive, Proposition 15 shows g is ergodic, and hence g is of realizable first return type. \square

We now also use terminology and results of Section 2.6. Recall that ‘quasi-isomorphic’ means having isomorphic natural extensions.

Proposition 19. *Assume the hypotheses of the previous proposition. If $|1/f'|$ is of bounded variation then the natural extension of f is a Bernoulli system, similarly for g ; if both have this property, then their systems are quasi-isomorphic if and only if they share the same entropy value.*

Proof. The fulfillment of the bounded variation condition for either map completes the hypotheses for the Rychlik result, Theorem 6, and thus guarantees that the natural extension system is Bernoulli. If this occurs for both maps, Ornstein’s fundamental result Theorem 4 shows that isomorphism of the natural extension systems is determined by entropy values. \square

Remark 4. In the above, if $|1/f'|$ is of bounded variation and f is of determinant one type (that is, if $\widehat{\Gamma}_f = \Gamma_f$) then the cross section to the geodesic flow on $T^1(\Gamma_f \backslash \mathbb{H})$ associated to f in Theorem 1, being a version of the natural extension, is Bernoulli. It is not always true that a cross section to a Bernoulli flow has Bernoulli first return system, as discussed in [37,39].

Remark 5. For each of the maps $T = T_{n,\alpha}$ discussed in Section 2.7 and for each x in its domain, one has $T'(x)$ being equal to either $(C \cdot x)'$ or $(C^2 \cdot x)'$ where C is given in (10). It easily follows that $|1/T'|$ is of bounded variation. Results of [10] and the determination in [11] of planar extension systems then allow one to invoke the results of this subsection for the $T_{n,\alpha}$.

4.4. Finite quilting for close neighbors that match

We quickly give basic definitions which capture the essence of the *matching* interval phenomenon — also known as: synchronization [10], or short cycles [24] — studied in various of families of continued fraction type maps. Thereafter we show that under reasonable assumptions upon matching intervals there are subintervals on which quilting applies. Our terminology and notation attempts to negotiate between that of [14] and of [10].

4.4.1. Matching: relations, intervals and their exponents

Suppose that we are given a one parameter family of piecewise Möbius interval functions, $\{T_\alpha\}_{\alpha \in \mathcal{I}}$, indexed by α ranging over some real interval \mathcal{I} , each of whose interval of definition $\mathbb{I}_\alpha = [\ell_0(\alpha), r_0(\alpha))$ is of fixed length λ . (We will always assume that also $|\mathcal{I}| \leq \lambda$.) In particular, letting

$$S : x \mapsto x + \lambda, \tag{15}$$

for all α we have $\ell_0(\alpha) = S^{-1} \cdot r_0(\alpha)$.

A *matching interval* $J \subset \mathcal{I}$ is a subinterval such (1) that there are some $m, n \in \mathbb{N}$ such that for all $\alpha \in J$ we have $T_\alpha^m(\ell_0(\alpha)) = T_\alpha^n(r_0(\alpha))$, where m, n are minimal except possibly at finitely many $\alpha \in J$ — we call the $\alpha \in J$ where this minimality holds *typical* —; (2) the digits of the expansions of the endpoints $\ell_0(\alpha)$ agree in that for all $1 \leq i < m$ there is a Möbius transformation M_i such that $T_\alpha^i(\ell_0(\alpha)) = M_i \cdot \ell_0(\alpha)$ for all $\alpha \in J$, and similarly for the $r_0(\alpha)$; and, (3) there are Möbius transformations L_J, R_J such that for all $\alpha \in J$ we have $T_\alpha^{n-1}(r_0(\alpha)) = R_J \cdot r_0(\alpha)$ and $T_\alpha^{m-1}(\ell_0(\alpha)) = L_J S^{-1} \cdot r_0(\alpha)$. We call m, n the *matching exponents* of J . In fact, we need a further property of the family: say that the family has a *matching relation* if (4) there is some Möbius transformation M such that for any matching interval, $ML_J S^{-1} = R_J$.

Remark 6. Note that the notion of matching is easily extended to orbits to the left and right of points of discontinuities, see Bruin et al. [9] where this is done for a related setting. We forgo doing this here, for simplicity's sake.

Many of the well-studied families of continued fractions have matching relations.

Example 2. We briefly indicate a few of these.

- The Nakada α -continued fractions has a matching relation: let

$$W = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \tag{16}$$

then combining ([29] Remark 6.9) with ([29] Lemmas 6.2, 6.4) shows that $M = W$ gives the relation for each matching interval J . (Note that [14] also showed that there are matching intervals in the Nakada family, but express the matching relations in a different manner than here.)

- In the setting of α -continued fraction expansions with odd partial quotients, [19] show a matching relation of the form $ML_J S^{-1} = R_J$, with $M = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$ see the final line on their p. 28.

- The countably many families of Section 2.7 coming from [10], are such that for each n the corresponding family has a matching relation for the small α , those with $\alpha \in (0, \gamma_n)$, see ([10], Prop. 5.2). There is a distinct matching relation on $(\gamma_n, 1]$ (more precisely for those parameter values, one splits each matching interval J into two pieces and finds that there is a matching relation for all of the left hand pieces and a nearly identical relation for all of the right hand pieces); see ([10], Lemma 6.2).

4.4.2. Close neighbors

We are interested in applying quilting when α, α' lie within the same synchronization interval; quilting succeeds in the most straightforward manner if we require that α, α' are particularly close.

To lighten notation, let us write ℓ_i and ℓ'_i for each of $T_\alpha^i(\ell_0(\alpha))$ and $T_{\alpha'}^i(\ell_0(\alpha'))$, respectively and similarly for the orbits of the other endpoints. We use the notation of Definition 2 in the following.

Definition 4. Suppose that J is a matching interval with corresponding matching exponents m, n . We say that $\alpha, \alpha' \in J$ are *close neighbors* if $\ell'_i, \ell_i, r'_j, r_j \in \mathbb{I}_{\alpha'} \cap \mathbb{I}_\alpha$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Note that the hypothesis on the orbit entries can equivalently be written as: The α -digit of ℓ'_i equals the α' -digit of ℓ_i and vice versa for each $1 \leq i < m$, and similarly for the various r_j, r'_j .

The majority of the aforementioned families have the following properties.

Definition 5. Fix a family of piecewise Möbius interval maps, $\mathcal{F} = \{T_\alpha \mid \alpha \in \mathcal{I}\}$.

1. We say that \mathcal{F} is of *purely shift digit changes* if for any $\alpha, \alpha' \in \mathcal{I}$ whenever $x \in \Delta_{T_\alpha, T_{\alpha'}}$ then $T_\alpha(x) = S^{\pm 1} T_{\alpha'}(x)$, where S is as in (15).
2. Suppose that $\alpha \in J$ with J a matching interval of matching exponents m, n . We say that Ω_α has *locally constant fibers* if its vertical fibers are constant between the points of the initial orbits of $\ell_0(\alpha)$ and $r_0(\alpha)$: that is, if the fibers are constant above the connected components of the complement in \mathbb{I}_α of $\{\ell_i(\alpha) \mid 0 \leq i \leq m - 1\} \cup \{r_j(\alpha) \mid 0 \leq j \leq n - 1\}$.

See Fig. 6 for a special case illustrating the following.

Proposition 20. Suppose that $\mathcal{F} = \{T_\alpha \mid \alpha \in \mathcal{I}\}$ is a family of piecewise Möbius interval maps of purely shift digit changes. Suppose further that α, α' are close neighbors with α typical for their common matching interval. Then $\Omega_{\alpha'}$ can be finitely quilted from Ω_α . Furthermore, Ω_α has locally constant fibers if and only if $\Omega_{\alpha'}$ does.

Proof. For ease, assume that $\alpha' < \alpha$, the other case follows by a symmetric argument. Thus, $\mathcal{T}_\alpha(\mathcal{C})$ fibers over $[r'_0, r_0)$ and hence we have $\mathcal{T}_{\alpha'}(\mathcal{C}) = \mathcal{T}_{S^{-1}} \circ \mathcal{T}_\alpha(\mathcal{C})$. Since α, α' are close neighbors, they share a common matching interval J , let m, n be its matching exponents. Let $U = U_\alpha$ be the Möbius transformation such that $\ell_m(\alpha) = UL_J \cdot \ell_0(\alpha)$ and

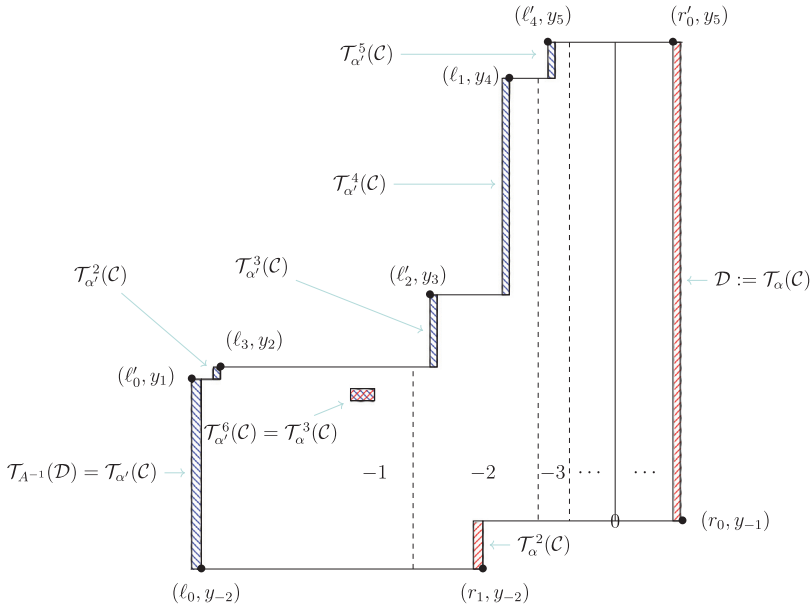


Fig. 6. Quilting from a close neighbor. Quilting in the setting of ‘small’ α of systems discussed in Section 2.7. Here $n = 3, \alpha = 0.14, \alpha' = 0.135$, and quilting given Ω_α results in $\Omega_{\alpha'}$. Domain Ω_α , details of which are in [11], not drawn fully to scale. Integers $-1, -2, -3$ indicate regions fibering over cylinders of corresponding ‘simplified digits’. The forward \mathcal{T}_α -orbit of \mathcal{C} is deleted, while the forward $\mathcal{T}_{\alpha'}$ -orbit of \mathcal{C} is added, until the ‘hole’ created by excising $\mathcal{T}_\alpha^3(\mathcal{C})$ is ‘patched’ in by $\mathcal{T}_{\alpha'}^0(\mathcal{C})$. See Proposition 20.

$V = V_\alpha$ be such that $r_n(\alpha) = VR_J \cdot r_0(\alpha)$. Since \mathcal{F} has only purely shift digit changes, the condition that $\ell'_i, \ell_i, r_j, r'_j \in \mathbb{I}_{\alpha'} \cap \mathbb{I}_\alpha$ for the various i, j implies that when $i < m$ and $j < n$ these $\ell'_i, \ell_i, r_j, r'_j$ lie outside of $\Delta_{\mathcal{T}_\alpha, \mathcal{T}_{\alpha'}}$. Hence, $\mathcal{T}_{\alpha'}^{m+1}(\mathcal{C}) = \mathcal{T}_{UL_J S^{-1}} \circ \mathcal{T}_\alpha(\mathcal{C}) = \mathcal{T}_{VR_J} \circ \mathcal{T}_\alpha(\mathcal{C}) = \mathcal{T}_\alpha^{n+1}(\mathcal{C})$.

We claim that $\bigcup_{i=1}^m \mathcal{T}_{\alpha'}^i(\mathcal{C})$ is disjoint from Ω_α . To this end, let k' be minimal such that $\mathcal{T}_{\alpha'}^{k'}(\mathcal{C}) \cap \Omega_\alpha$ has positive measure. (Since $\mathcal{T}_{S^{-1}} \circ \mathcal{T}_\alpha(\mathcal{C})$ projects to $[\ell'_0, \ell_0]$ clearly $k' > 1$.) Let $(x, y) \in \mathcal{C}$ such that $\mathcal{T}_{\alpha'}^{k'}(x, y) \in \Omega_\alpha$. Again the close neighbors property gives that thereafter the forward $\mathcal{T}_{\alpha'}$ -orbit of this point is given by α -admissible Möbius transformations, and thus agrees with its forward \mathcal{T}_α -orbit until we reach the $\mathcal{T}_{\alpha'}^{m+1}(x, y) = \mathcal{T}_\alpha^{n+1}(x, y)$. The bijectivity of \mathcal{T}_α then implies that there is some k such that $\mathcal{T}_\alpha^k \circ \mathcal{T}_{\alpha'}^{k'}(x, y) = \mathcal{T}_\alpha^{n+1}(x, y)$ and hence $\mathcal{T}_{\alpha'}^{k'}(x, y) = \mathcal{T}_\alpha^{n+1-k}(x, y)$. If $k' < m + 1$ then by the positivity of the measure of such points, we deduce that there are factorizations $L_J = L'_J U' L'_J$ and $R_J = R'_J V' R'_J$ with $L'_J = R'_J$ and $U' L'_J S^{-1} = V' R'_J$. Since $\alpha' < \alpha$ are close neighbors, there are other values $\alpha'' < \alpha' < \alpha$ that are also close neighbors of α and hence we find that there is an interval $J' \subseteq J$ with matching exponents m', n' . But, this contradicts the definition of J as the matching interval for its typical α, α' . Therefore, we must have that $k' = n + 1$ and the disjointness of $\bigcup_{i=1}^m \mathcal{T}_{\alpha'}^i(\mathcal{C})$ from Ω_α does hold.

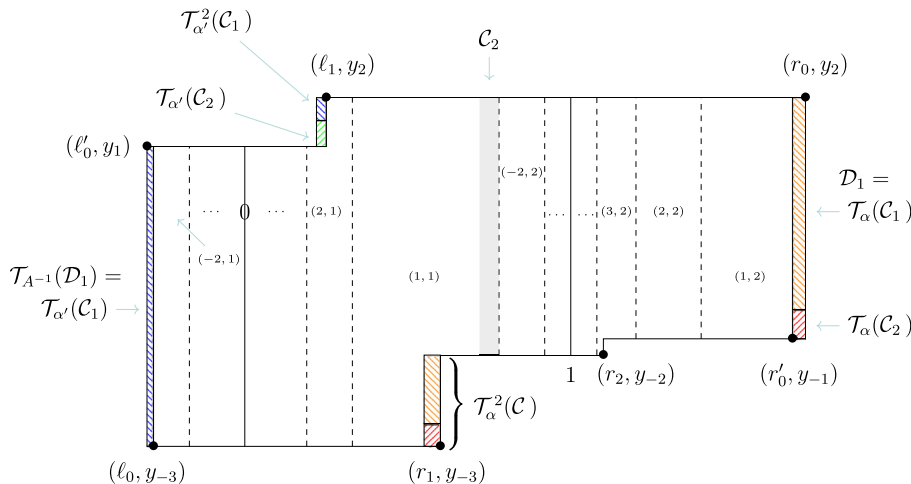


Fig. 7. Quilting for close neighbors, large α . Quilting from $\Omega_{3,0.86}$ to $\Omega_{3,0.855}$ (not fully to scale). Blocks $\mathcal{B}_{i,j}$, also denoted by (i, j) . The forward \mathcal{T}_α -orbit of $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ is deleted, while the forward $\mathcal{T}_{\alpha'}$ -orbit of \mathcal{C} is added, until synchronization causes a “hole” excised due to the first of these, but to be “patched” due to the second (not shown here, but compare with Fig. 6). See Proposition 21.

We next claim that the $\mathcal{T}_{\alpha'}^i(\mathcal{C})$ are pairwise disjoint. To this end, suppose that $\mathcal{T}_{\alpha'}^i(\mathcal{C})$ meets $\mathcal{T}_{\alpha'}^j(\mathcal{C})$ in positive μ -measure for some $1 \leq i \leq j \leq m + 1$. Then the same is true for $\mathcal{T}_{\alpha'}^{i+m+1-j}(\mathcal{C})$ and $\mathcal{T}_{\alpha'}^{m+1}(\mathcal{C}) = \mathcal{T}_{\alpha}^{m+1}(\mathcal{C})$ and arguing as above, we find that $i = j$. The analogous argument shows that the $\mathcal{T}_{\alpha}^j(\mathcal{C})$ are disjoint.

Finally, due to the disjointness properties which we have shown, it follows that Ω_α has locally constant fibers if and only if $\Omega_{\alpha'}$ does. \square

We desire to prove the analog of the above theorem holds also for the setting of ‘large’ α, α' as defined in Section 2.7. In that setting, digit changes other than shifts can occur. However, for close neighbors, the location of the second type of digit changes is constrained to a single interval and the digit change is completely explicit; for a hint of this, see Fig. 7. In brief, the following is a mild extension of the previous result, but one which we call on in [11].

Recall that the shift S is given in (15).

Proposition 21. *Suppose that $\mathcal{F} = \{T_\alpha \mid \alpha \in \mathcal{I}\}$ is a family of piecewise Möbius interval maps and $\alpha' < \alpha$ are close neighbors with α typical for their common matching interval. Fix M such that $T_\alpha(\ell_0(\alpha)) = M \cdot \ell_0(\alpha)$, and suppose further that $\Delta_{T_\alpha, T_{\alpha'}}$ is the union of its subsets*

$$\Delta_1 = \{x \mid T_{\alpha'}(x) = S^{-1} \cdot T_\alpha(x)\} \text{ and}$$

$$\Delta_2 = \{x \mid T_{\alpha'}(x) = MS^{-1} \cdot T_\alpha(x)\}.$$

Then $\Omega_{\alpha'}$ can be finitely quilted from Ω_α . Furthermore, Ω_α has locally constant fibers if and only if $\Omega_{\alpha'}$ does.

Proof. Let $\mathcal{C}_1, \mathcal{C}_2 \subset \Omega_\alpha$ be the sets projecting to Δ_1 and Δ_2 , respectively. The proof of Proposition 20 shows that the main interest here is understanding the \mathcal{T}_α - and $\mathcal{T}_{\alpha'}$ -orbits of \mathcal{C}_2 .

As in the previous proof, let the matching interval of α, α' be J , and let m, n be its matching exponents. We also again let $U = U_\alpha$ be the Möbius transformation such that $\ell_m(\alpha) = UL_J \cdot \ell_0(\alpha)$ and $V = V_\alpha$ be such that $r_n(\alpha) = VR_J \cdot r_0(\alpha)$.

We have that $\mathcal{T}_\alpha(\mathcal{C}_1 \cup \mathcal{C}_2)$ is that part of Ω_α fibering over $[r'_0, r_0]$ and hence $\mathcal{T}_{\alpha'}(\mathcal{C}_2)$ and $\mathcal{T}_\alpha^2(\mathcal{C}_1)$ are given by applying \mathcal{T}_M to non-intersecting subsets of the plane fibering over $[\ell'_0, \ell_0]$. The previous proof applies to show that $\mathcal{T}_\alpha^{m+1}(\mathcal{C}_1) = \mathcal{T}_{ULJS^{-1}} \circ \mathcal{T}_\alpha(\mathcal{C}_1) = \mathcal{T}_{VR_J} \circ \mathcal{T}_\alpha(\mathcal{C}_1) = \mathcal{T}_\alpha^{n+1}(\mathcal{C}_1)$. Since the $\mathcal{T}_\alpha(\mathcal{C}_1)$ and $\mathcal{T}_\alpha(\mathcal{C}_2)$ fiber over $[r'_0, r_0]$, their initial \mathcal{T}_α -orbits are given by the same sequence of Möbius transformations. Since α' and α are close neighbors, we have in fact that $\mathcal{T}_\alpha^{n+1}(\mathcal{C}_2) = \mathcal{T}_{VR_J} \circ \mathcal{T}_\alpha(\mathcal{C}_2)$. Since $\mathcal{T}_{\alpha'}(\mathcal{C}_2) = \mathcal{T}_{MS^{-1}} \mathcal{T}_\alpha(\mathcal{C}_2)$, we conclude that $\mathcal{T}_\alpha^{n+1}(\mathcal{C}_2) = \mathcal{T}_{\alpha'}^m(\mathcal{C}_2)$. The disjointness of the initial $\mathcal{T}_{\alpha'}$ -orbits of \mathcal{C}_1 and \mathcal{C}_2 follows from the disjointness of $\mathcal{T}_\alpha(\mathcal{C}_1)$ and $\mathcal{T}_\alpha(\mathcal{C}_2)$. The disjointness of each of these orbits up to the m th and $m - 1$ st step is argued as in the previous proof, as is the disjointness of these initial orbits with Ω_α . \square

5. Application: An alternate path to proving properties of Nakada’s α -continued fractions

We now give a rather technical application of our methods. We use an alternate description of the planar extension for each T_α given in [29] and certain results of [29] (not relying on the ergodicity of the T_α) about the planar extensions of Nakada’s α -continued fractions, to recover the following result.

Theorem 22 (Luzzi–Marmi 2008, [32]). *For every $0 < \alpha \leq 1$, the Nakada α -continued fraction is ergodic.*

In fact, we rely on Main Theorem 1 and thus find more than just ergodicity. In particular, we find the following.

Theorem 23. *For every $0 < \alpha \leq 1$, the dynamical systems of α, α' are quasi-isomorphic if and only if they have the same entropy value.*

5.1. Review of notation and results of [29]

Recall from the introduction and Section 2.3 that both [14,29] proved the continuity of the entropy function $\alpha \mapsto h(T_\alpha)$ for Nakada’s α -continued fractions, when $0 < \alpha \leq 1$. The second group of authors created planar extensions for the T_α in the form of $\Omega_\alpha = \{\mathcal{T}_\alpha^n(x, 0) \mid x \in [\alpha - 1, \alpha), n \geq 0\}$, showed the continuity of the μ -mass of these, and argued using Abramov’s formula (1) to reach the continuity result. All of this built upon the earlier result of Luzzi–Marmi [32] that each of the interval maps is ergodic with respect to the appropriate measure.

Here we wish to show that the techniques of this paper can be used to prove the ergodicity of the interval maps. In brief, whereas [29] prove by arguments based upon the

ergodicity of T_α that Λ_α given in (18) is also a valid expression for the planar extension of T_α , we turn this around and rather prove that Λ_α gives a planar extension and then apply Main Theorem 1 to deduce that T_α is ergodic.

We now briefly review some notation and arguments from [29].

Recall from Section 2.3 that for $\varepsilon \in \{-1, 1\}$ and $d \in \mathbb{N}$, we have $M_{(\varepsilon:d)} = \begin{pmatrix} -d & \varepsilon \\ 1 & 0 \end{pmatrix}$ and $N_{(\varepsilon:d)} = \begin{pmatrix} 0 & 1 \\ \varepsilon & d \end{pmatrix}$. It is easily verified both that exactly the digits $(+1 : d)$ and $(-1 : d + 1)$ are such that the image of the open interval $(0, 1)$ under $N_{(\varepsilon:d)}$ meets the open interval $(1/(d + 1), 1/d)$, and that for these two digits we have $N_{(\varepsilon:d)} \cdot [0, 1] = [1/(d + 1), 1/d]$. In particular, for each d and any y we find that

$$N_{(+1:d)} \cdot y = N_{(-1:d+1)} \cdot (1 - y). \tag{17}$$

Equivalently with W is as in (16), $N_{(+1:d)} \cdot y = N_{(-1:d+1)} W^t \cdot y$. Note that since W is of projective order two, this accords with the easily verified identity: $M_{(+1:d)} = M_{(-1:d+1)} W$.

For $\alpha \in (0, 1]$, we let $d_\alpha(\alpha)$ be the first α -digit of $r_0(\alpha) = \alpha$ and define

$$\mathcal{A}_\alpha = \{(-1 : d') \mid 2 \leq d' \leq d_\alpha(\alpha) + 1\} \cup \{(+1 : d_\alpha(\alpha))\}.$$

The approach of [29] is to list the matching intervals of parameter α by way of certain words v , the details of which are not necessary for the current application. For each v , [29] shows that the matching interval indexed by v contains a unique ‘atypical’ value, this is $\alpha = \chi_v$ which is identified by the T_{χ_v} -orbits of the endpoints of \mathbb{I}_{χ_v} both reaching $x = 0$ one step ‘earlier’ than for the matching for the typical values in this matching interval.

Recall that \mathcal{E} is the complement in $(0, 1]$ of the union of the matching intervals of α . For $\alpha \in \mathcal{E}$ or $\alpha = \chi_v$ for some v , let \mathcal{L}'_α be the words in \mathcal{A}_α which are admissible α -expansions (as well as the empty word). Then ([29], Lemma 7.11) shows that the region, which we rename for clarity’s sake,

$$\Lambda_\alpha = \overline{\bigcup_{w \in \mathcal{L}'_\alpha} T_\alpha^{|w|}(\Delta_\alpha(w)) \times N_w \cdot [0, \frac{1}{d_\alpha(\alpha)+1}]} \tag{18}$$

is a bijectivity domain for T_α . In fact, the lemma is stated for all α , upon making minor adjustments for the remaining α : Each such ‘remaining’ α is in the same matching interval as χ_v for some v , and one defines $\mathcal{L}'_\alpha = \mathcal{L}'_{\chi_v}$ and replaces $T_\alpha^{|w|}(\Delta_\alpha(w))$ by the J_w^α of ([29], (7.2)) — this last is only a change in the cases that w has a suffix which consists of a prefix of the digits of the α -expansion of either α or $\ell_0(\alpha) = \alpha - 1$ extending beyond where matching occurs (in a sense, the adjustment is to keep the digits up to one step before matching). To repeat, their proof (in all cases) relies in part on the ergodicity of the T_α and involves showing that the bijectivity domain Ω_α is equal to what we have denoted Λ_α . We now turn this around, and for $\alpha \in (0, 1)$ begin with Λ_α to show ergodicity of T_α and more.

5.2. Proving Λ_α is a bijectivity domain to conclude T_α is ergodic

We aim to show that T_α is bijective on Λ_α (as always, here and throughout, up to μ -measure zero sets).

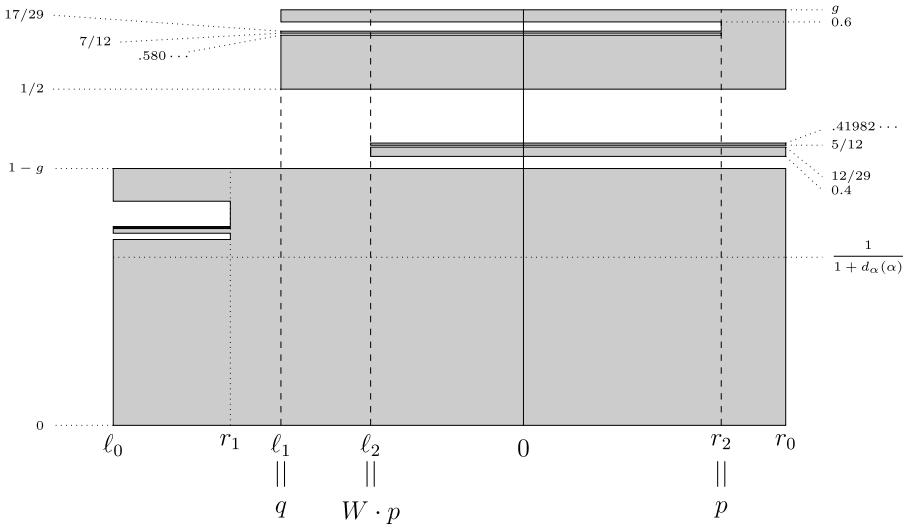


Fig. 8. The planar domain Ω_α for Nakada’s continued fraction of $\alpha = 0.39$, see Fig. 1 for the graph of T_α . Marked x -values include: $r_i = T_\alpha^i(\alpha)$, $\ell_i = T_\alpha^i(\alpha - 1)$, $0 \leq i \leq 2$. Here $d_\alpha(\alpha) = 3$. The y -fibers $\Phi(x)$ are constant for x between $W \cdot p$ and p , and $[0, 1]$ is the closure of $\Phi(x) \sqcup W^t \cdot \Phi(x)$ for each such x . The proof of Proposition 24 refers to $p, W \cdot p, q$ in general cases.

5.2.1. Surjectivity implies injectivity

Since a Möbius transformation is identified by its values on three points, and each $\mathcal{T}_{(\varepsilon;d)}$ is (locally) measure preserving, that surjectivity implies injectivity can be argued as in ([29], Lemma 5.2): In brief, Λ_α can be partitioned by blocks \mathcal{D}_a , each projecting to its cylinder indexed by a , upon each of which \mathcal{T}_α is injective and measure preserving; the sum over the various a of the $\mu(\mathcal{T}_\alpha(\mathcal{D}_a))$ hence equals $\mu(\Lambda_\alpha)$, which by the surjectivity equals $\mu(\mathcal{T}_\alpha(\Lambda_\alpha))$, but this in turn equals the measure of the union of the $\mathcal{T}_\alpha(\mathcal{D}_a)$. Since the sum of the $\mu(\mathcal{T}_\alpha(\mathcal{D}_a))$ equals the measure of the union of the $\mathcal{T}_\alpha(\mathcal{D}_a)$, injectivity holds up to measure zero.

Certainly the image of Λ_α under \mathcal{T}_α contains the union over the non-empty words $w \in \mathcal{L}'_\alpha$ of the $T_\alpha^{|w|}(\Delta_\alpha(w)) \times N_w \cdot [0, \frac{1}{d_\alpha(\alpha)+1}]$. The main challenge is to show that all of $\mathbb{I}_\alpha \times [0, \frac{1}{d_\alpha(\alpha)+1}]$ is in the image. For this, we introduce notation for the fiber in Λ_α over a point x : for each $x \in \mathbb{I}_\alpha$, let $\Phi_\alpha(x) = \{y \mid (x, y) \in \Lambda_\alpha\}$.

5.2.2. Surjectivity follows from fiber symmetry

We show surjectivity of \mathcal{T}_α by way of an interesting detail that seems not to have been observed in the literature. The fibers over the cylinders of the values not in \mathcal{S}_α satisfy a certain symmetry property. Note that the matrix W from (16) acts as $x \mapsto -1/(x + 1)$ while its transpose acts by $W^t \cdot y = 1 - y$. We will show that for any sufficiently large negative x , the sets $W^t \cdot \Phi_\alpha(x)$ and $\Phi_\alpha(W \cdot x)$ are disjoint and have union whose closure is $[0, 1]$. The reader is encouraged to view the various representations of planar domains Ω_α given in, say, [29] to see that this is reasonable. See also Fig. 8.

Proposition 24. For $\alpha \in (0, 1)$ the map \mathcal{T}_α is bijective from Λ_α to itself.

Proof. Fix α . From the definition of Λ_α , surjectivity onto the complement of $\mathbb{I}_\alpha \times [0, \frac{1}{d_\alpha(\alpha)+1}]$ is immediate.

The proof of ([29], Lemma 5.1) shows, based upon the fact that there is an explicit manner to rewrite T_α -orbits in terms of regular continued fraction T_1 -orbits, that the rectangle $[0, \frac{1}{d_\alpha(\alpha)+1}]^2$ is contained in the closure of the \mathcal{T}_α -orbits of the points contained in this rectangle. The admissible $(\varepsilon : d) \notin \mathcal{A}_\alpha$ are exactly those values such that $N_{(\varepsilon:d)} \cdot [0, 1] \subset [0, \frac{1}{d_\alpha(\alpha)+1}]$. Hence, the y -values here show that each \mathcal{T}_α -orbit returns to the rectangle only upon an application of some $\mathcal{T}_{(\varepsilon:d)}$ with $(\varepsilon : d) \notin \mathcal{A}_\alpha$.

It follows that for each $d \geq d_\alpha(\alpha) + 1$, we have that $[1/(d + 1), 1/d]$ equals the closure of the union of $N_{(+1:d)} \cdot \Phi_\alpha(x)$ with $N_{(-1:d+1)} \cdot \Phi_\alpha(x')$ whenever $x \in \Delta_\alpha(+1 : d)$ and $x' \in \Delta_\alpha(-1 : d + 1)$ are such that T_α sends them to the same value in $[0, \frac{1}{d_\alpha(\alpha)+1}]$. By (17) (in the equivalent form given in the line directly below it), this implies that $[0, 1] = \overline{W^t \cdot \Phi_\alpha(x) \cup \Phi_\alpha(x')}$ for each such pair.

Now, for $\alpha \in \mathcal{E}$ ([29], Lemma 7.9) shows that all of the digits of the expansions of both $\alpha - 1$ and α are contained in \mathcal{A}_α . Since the only non-full cylinders are associated with prefixes of these expansions, the fibers $\Phi_\alpha(x)$ are constant for all $x \in [\frac{-1}{d_\alpha(\alpha)+1}, \frac{1}{d_\alpha(\alpha)+1}]$. Therefore, $[0, 1] = \overline{W^t \cdot \Phi_\alpha(x) \cup \Phi_\alpha(x')}$ holds for every pair $x \in \Delta_\alpha(+1 : d)$ and $x' \in \Delta_\alpha(-1 : d + 1)$ that are sent by T_α to the same value. That is, the closure of the union of the images of Λ_α under the various $\mathcal{T}_{(+1:d)}$ and $\mathcal{T}_{(-1:d+1)}$ fills out all of $\mathbb{I}_\alpha \times [0, \frac{1}{d_\alpha(\alpha)+1}]$. Surjectivity holds in this case.

In the case of α of the form χ_v , ([29], Lemma 7.9) shows that the digits of the expansions of the two endpoints $\alpha - 1, \alpha$ remain in \mathcal{A}_α until they match at the value zero. One finds that the fibers $\Phi_\alpha(x)$ are constant for all $x > 0$ and also for all $x < 0$ whose α -digit is at least $(-1 : d_\alpha(\alpha) + 2)$. We conclude also in this case that $[0, 1] = \overline{W^t \cdot \Phi_\alpha(x) \cup \Phi_\alpha(x')}$ holds for every pair x, x' as above, and again the result holds.

In the remaining case, α is in the same matching interval as some χ_v , and ([29], Lemma 7.9) shows that up to their penultimate digits before matching, digits of the expansions of the two endpoints $\alpha - 1, \alpha$ remain in \mathcal{A}_α ; ([29], Lemma 6.2) implies that the values exactly before matching differ by an application of W . Let p denote the larger of these values, thus $W \cdot p$ is the other value; also let q be the maximum value of the remainder of T_α -orbits of the endpoints $\alpha - 1, \alpha$ up to these index values. The fibers $\Phi_\alpha(x)$ are constant over each of the intervals $[q, W \cdot p), [W \cdot p, p), [p, \alpha)$, with respective values $\Phi_\alpha(q), \Phi_\alpha(W \cdot p), \Phi_\alpha(p)$. Directly related to this is that for any $d > 0$, points $x \in \Delta_\alpha(+1 : d)$ and $x' \in \Delta_\alpha(-1 : d + 1)$ are sent by T_α to the same value if and only if $x' = W \cdot x$. (We could have used this in the previous cases, but preferred to minimize notation.)

Since $p > 0$, there exists some $d > d_\alpha(\alpha)$ such that p is strictly greater than all values in $\Delta_\alpha(+1 : d)$. Hence for all $x \in \Delta_\alpha(+1 : d)$ we have $\Phi_\alpha(x) = \Phi_\alpha(W \cdot p)$ and since also $W \cdot p < W \cdot x$ also $\Phi_\alpha(W \cdot x) = \Phi_\alpha(W \cdot p)$. Since there are $x \in \Delta_\alpha(+1 : d)$ such that $[0, 1] = \overline{\Phi_\alpha(x) \cup W^t \cdot \Phi_\alpha(W \cdot x)}$, we find that it is always the case that

$$[0, 1] = \overline{\Phi_\alpha(W \cdot p) \cup W^t \cdot \Phi_\alpha(W \cdot p)}. \tag{19}$$

From this for any $0 < x < p$ we find that $[0, 1] = \overline{\Phi_\alpha(x) \cup W^t \cdot \Phi_\alpha(W \cdot x)}$ and in particular for all $d > 0$ such that p is strictly greater than all values in $\Delta_\alpha(+1 : d)$ we have that $\mathbb{I}_\alpha \times [1/d, 1/(d + 1)]$ is contained in $\mathcal{T}_\alpha(\Lambda_\alpha)$.

We next claim that

$$\Phi_\alpha(p) \cup W^t \cdot \Phi_\alpha(q) = \Phi_\alpha(W \cdot p) \cup W^t \cdot \Phi_\alpha(W \cdot p).$$

To prove this, recall that [29] use k', k as matching exponents and use E to denote the matrix which acts as shift by -1 , and show that there is an M_v such that $T_\alpha^{k'-1}(\alpha - 1) = M_v \cdot (\alpha - 1) = M_v E \cdot \alpha$ and $T_\alpha^{k-1}(\alpha) = W M_v E \cdot \alpha$. Thus $\{p, W \cdot p\} = \{M_v E \cdot \alpha, W M_v E \cdot \alpha\}$ and $\Phi_\alpha(W \cdot p)$ is the union of $\Phi_\alpha(q)$ with one of either $N_v \cdot \Phi_\alpha(\alpha - 1)$ or $W^t N_v (E^{-1})^t \cdot \Phi_\alpha(\alpha)$. Similarly, $\Phi_\alpha(p)$ is the union of $\Phi_\alpha(q)$ with both $N_v \cdot \Phi_\alpha(\alpha - 1)$ and $W^t N_v (E^{-1})^t \cdot \Phi_\alpha(\alpha)$. Since W and hence W^t is of projective order two, the claim holds.

Using (19) and the claim, we find for all $d > d_\alpha(\alpha)$ that $\mathbb{I}_\alpha \times [1/d, 1/(d + 1)]$ is contained in $\mathcal{T}_\alpha(\Lambda_\alpha)$. Thus, our proof of surjectivity is complete.

For completeness, we recall that the previous subsection outlines a proof showing that injectivity now follows. \square

5.3. Proof of ergodicity

We now complete our proof of Theorem 22. It is immediate that each Λ_α has positive μ -measure. Since $N_\alpha \cdot [0, 1] \subset [0, 1]$ for every possible digit for any given α , it is clear that $\Lambda_\alpha \subset [\alpha - 1, \alpha] \times [0, 1]$. Therefore, the compact Λ_α is bounded away from $y = -1/x$ and has finite vertical fibers. Recall that Lemma 8 guarantees that every rational $\alpha \in (0, 1]$ and every $\alpha \in \mathcal{E}$ is of bounded non-full range. Furthermore, ([29] Theorem 5) shows that if α is an endpoint of a matching interval then both $\alpha - 1$ and α have periodic T_α -expansions and thus these maps also are of bounded non-full range. By Main Theorem 1 with $\Omega = \Lambda_\alpha$, we find ergodicity of T_α and the other properties listed in the statement of the theorem. Each of the remaining $\alpha' \in (0, 1)$ lies in the interior of some matching interval and the density of the rationals gives that α' has some typical α as a close neighbor. From Proposition 20, combined with Main Theorem 2 we have that each $\mathcal{T}_\alpha : \Lambda_\alpha \rightarrow \Lambda_\alpha$ gives the natural extension to the system of T_α , which is in particular ergodic.

5.4. Proof of quasi-isomorphism class determined by entropy

We now sketch the proof of Theorem 23. We desire to apply the Rychlik result, Theorem 6. By the ergodicity result, for each $\alpha \in (0, 1]$ the map T_α has a unique invariant probability measure equivalent to Lebesgue. Furthermore, each is such that every open subset of \mathbb{I}_α contains a full rank m cylinder for some $m \in \mathbb{N}$; hence all of \mathbb{I}_α is contained in the T_α^m -image of this open subset. Each $M_{(\varepsilon;d)} = \begin{pmatrix} -d & \varepsilon \\ 1 & 0 \end{pmatrix}$ defines a function of x whose derivative has absolute value x^{-2} . Thus, $|1/T'_\alpha|$ is certainly bounded on all of \mathbb{I}_α for any of the α . By Theorem 6, the natural extension of each T_α is Bernoulli. Therefore, the result holds due to Ornstein's fundamental result, Theorem 4.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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