Riemann Hypothesis

A complex function theory exploration

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Layman's Abstract

The Riemann hypothesis was first stated by Riemann in 1859. In its simplest form it asks the question "When is the Riemann zeta function equal to 0?" The answer to this question happens to be exceedingly difficult to find, and the theory surrounding it extremely rich. It also turns out that there exist deep mathematical links between this function and the study of prime numbers. We shall see that we first must build up certain mathematical tools to understand the zeta function on a particular domain. We will then follow in the footsteps of the eponymous B. Riemann and extend this domain greatly. This allows us to prove some theorems regarding the location of the zeroes and discuss the relevant literature. Finally, we shall see that the Riemann hypothesis has far-reaching consequences within mathematics, and we will end by discussing possible extensions of the hypothesis.

Abstract

The Riemann zeta function, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, was initially only studied for real $s > 1$. It is not difficult to see this can be extended to complex valued s provided the real part of s is greater than 1. We will study the zeta function on this domain first, and find the Euler product, $\prod_{p \text{ is prime}} 1/(1 - p^{-s})$, which links the zeta function to the primes. To do so, we will study the theory of infinite products.

Next we wish to find an analytic continuation of $\zeta(s)$ to the entire complex plane, and we will do so using the Euler-Maclaurin summation formula. To prove the formula holds we will study Bernoulli numbers and polynomials, which also allows us to find specific values of the zeta function.

Using the analytic continuation we will then obtain the functional equation, which allows us to study the function in more detail. The functional equation gives us the trivial zeroes, at which point we are able to study the critical strip $0 < \mathrm{Re}(s) < 1$, the critical line $s = \frac{1}{2} + it$, and attempt to find the non-trivial zeroes.

We investigate the Riemann hypothesis and prove certain classical results, such as Hardy's theorem, which states there are infinitely many zeroes on the critical line. Finally, we discuss certain related theorems from the literature and end by a brief discussion of the consequences of the Riemann hypothesis, as well as what the Generalised Riemann Hypothesis states.

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1 Introduction

In the year 1900 David Hilbert, one of the leading mathematicians of the time, listed 23 open problems to an audience of mathematicians. These questions were chosen in order to further the field of mathematics and its relation with physics. Hilbert told his audience [33]:

> "*Problems are a sign that the subject is alive. The benefit of problems is that, by solving them, the investigator gains a wider view of the subject.*"

In the year 2000 the Clay Mathematics Institute of Cambridge, Massachusetts (Clay Institute) also put forth a list of open problems. They listed 7 problems this time around, and these problems were chosen because they were some of the most difficult problems mathematicians had been working on at the turn of the millennium [28]. As the Clay Institute also offers a prize of $1,000,000 USD to anyone who manages to solve any of the problems, these problems have since become known as the Millennium Prize Problems.
Out of all the problems, only one managed to get listed by both Hilbert and the Clay Institute. This is the problem of the **Riemann hypothesis**, and is the topic of this text.

The hypothesis concerns the **Riemann zeta function**, named after B. Riemann who was the first person to study the function on the complex numbers. As it turns out, there exists a deep mathematical link between this function and the prime numbers and in fact, Riemann studied this function because of that property. The central point of interest regarding the function is the question of where precisely its zeroes are located. The Riemann hypothesis states that a certain subset of these zeroes lie on a perfectly straight line. While this has not been proven so far, the mathematics developed to try to find a proof of the hypothesis is incredibly deep and has been applied in many other branches of mathematics.
From that, one would not be wrong to contest Hilbert's statement that "*by solving [problems], the investigator gains a wider view of the subject*", by countering that in fact *attempting to solve* problems is plenty already.

2 Riemann Zeta Function

Back in the middle of the seventeenth century, around the time modern-day calculus was invented, mathematicians were playing with their newfound mathematical tools and posed each other questions left, right, and centre. While it had already been shown by N. Oresme [10], P. Mengoli gave a proof that the harmonic series, that is $\sum_{n=1}^{\infty} \frac{1}{n}$, diverged.
In the same text, his book *Novae Quadraturae Arithmeticae*, Mengoli then posed the following problem [7]:

Problem 2.1 (Basel Problem (1650))**.**
Does the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converge? And if so, what is its value?

In the nearly 100 years following Mengoli, many mathematicians including J. Wallis, John Bernoulli, James Bernoulli, Daniel Bernoulli, and G.W. Leibniz, worked on this problem. While convergence was established early on [8], only decimal approximations were found [7]. It took until 1735 for there to be a proper closed-form solution. Famously, L. Euler showed that the sum converges to $\frac{\pi^2}{6}$ [1]. While the appearance of the constant π might be strange, Euler was more excited than surprised. He then also showed that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} \tag{2.1}$$

among others.
In fact, he gave a method to solve

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

in a closed form for all even s [2]. (We will later find an expression for these values ourselves.) While calculating these sums for even s, he also attempted to find solutions for odd s, but he notes:

> *"however, in the cases where [s] is an odd number, all my effort to find their sum is a failure up to now."*

While Euler mainly worked on different, closely related, series, he did show in particular that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{p^s}{p^s - 1}.$$

Over 100 years later, Riemann gave this particular sum a name, and called it the *Zeta function.* Because of his work, and his introduction of the name, we have since denoted this infinite sum the Riemann zeta function.

Definition 2.1 (Riemann zeta function)**.**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

or equivalently

$$\zeta(s) = \prod_{p \text{ is prime}} \left(\frac{p^s}{p^s - 1} \right).$$

Euler himself only looked at s as an integer greater than 1, and never made any attempts to look further than that. It was P.G.L. Dirichlet in 1837 who first studied this function for the real numbers greater than 1 [11], and then in 1859 B. Riemann extended the domain even further, and considered complex values for s too. For the moment we will restrict s such that $\mathrm{Re}(s) = \sigma > 1$.

2.1 Infinite Products

We wish to show equivalence between our sum formulation and the product form that Euler derived. To do so, we need to use some results from the theory of infinite products.

Definition 2.2 (Convergence of products)**.** *Let $a_1, a_2, a_3, \dots$ be a sequence of numbers, let the partial product $a_1 \cdot a_2 \cdot \dots \cdot a_n = p_n$. The infinite product*

$$\prod_{n=1}^{\infty} a_n$$

*is said to **converge to** P if $\lim_{n\to\infty} p_n = P$ and $P \neq 0$.*

Remark. *If $P = 0$, the product is said to **diverge to** 0.*

If p_n converges to $P \neq 0$, we have the following two necessary and sufficient conditions:

a) There exists some $g > 0$ such that $|p_n| > g$ for all n.

b) For all $\varepsilon > 0$, there exists an $N > 0$ such that $|p_{n+m} - p_n| < \varepsilon$ for all $n \geq N$, $m \in \mathbb{N}$.

The first condition ensures that our sequence does not converge to 0, and the second condition ensures that our sequence is Cauchy, and thus convergent (in a complete space).
We can summarise both conditions by one equivalent condition, namely:

c) For all $\eta > 0$, there exists an N such that for all $n \geq N$, $m \in \mathbb{N}$, we have

$$\left| \frac{p_{n+m}}{p_n} - 1 \right| < \eta. \tag{2.2}$$

It is not difficult to see that a and b imply c.
Indeed, assume p_n converges to $P \neq 0$, and let $\eta > 0$, then by a we know $|p_n| > g$ for some $g > 0$, and by b, if we let $\varepsilon = g \cdot \eta$, there exists N such that for all $n \geq N$, $m \in \mathbb{N}$, we have

$$|p_{n+m} - p_n| < \varepsilon \implies \frac{1}{|p_n|} \cdot |p_{n+m} - p_n| < \frac{1}{|p_n|} g \cdot \eta.$$

Then since $|p_n| > g$, we have $1/|p_n| < 1/g$, so that we get

$$\left|\frac{p_{n+m}}{p_n} - 1\right| < \eta.$$

Let us show that c also implies a and b.

Proof. Let $\eta < 1$, then by c we have some N such that for all $n \geq N$, $m \in \mathbb{N}$.

$$\left|\left|\frac{p_{n+m}}{p_n}\right| - |1|\right| \leq \left|\frac{p_{n+m}}{p_n} - 1\right| < \eta,$$

where we apply the reverse triangle inequality on the left-hand side. This means that

$$-\eta < \left|\frac{p_{n+m}}{p_n}\right| - 1 < \eta \implies 1 - \eta < \left|\frac{p_{n+m}}{p_n}\right| < 1 + \eta.$$

Since $1 - \eta > 0$, we know that none of the partial products are ever zero. Furthermore, we can multply both sides of the inequality by $|p_n|$ to obtain

$$(1 - \eta)|p_n| < |p_{n+m}| < (1 + \eta)|p_n|,$$

which holds for all m. Clearly for any fixed n, $|p_n|$ must be bounded from above and below, so we have bounded p_k for $k \geq N$. Then we can take the maximum and minimum over the first $N - 1$ partial products as well, to bound p_k for all $k < N$ as well. Thus we conclude that there must exist both $g > 0$ such that $|p_k| > g$, and $G > 0$ such that $|p_k| < G$ for all $k \in \mathbb{N}$. So indeed c implies a.

Furthermore, multiplying (2.2) by $|p_n|$, we have that for every $\eta > 0$ we can find N such that for $n \geq N$, $m \in \mathbb{N}$, $|p_{n+m} - p_n| < \eta |p_n|$. If we let $\varepsilon > 0$ arbitrarily, and we choose $\eta = \frac{1}{2}\varepsilon/G$, we are left with $|p_{n+m} - p_n| < \frac{1}{2}\varepsilon/G \cdot G < \varepsilon$. Since $|p_n| < G$ for all n. We see that c also implies b. □

In general, we will use expression c as our criterion to check whether a product is convergent. We want to apply the following lemma:

Lemma 2.1. *The product*

$$\prod_{n=1}^{\infty}(1 + a_n)$$

converges whenever

$$\prod_{n=1}^{\infty}(1 + |a_n|)$$

converges, which converges whenever $\sum_{n=1}^{\infty}|a_n|$ *does. We call* $\prod_{n=1}^{\infty}(1 + a_n)$ ***absolutely convergent*** *in this case.*

Proof. Consider $p_n = \prod_{k=1}^{n}(1 + a_k)$. We first assume that $q_n = \prod_{k=1}^{n}(1 + |a_k|)$ converges. So let $\varepsilon > 0$ and choose N such that for all $n \geq N$, $m \in \mathbb{N}$, $|q_{n+m}/q_n - 1| < \varepsilon$, then:

$$\begin{aligned}\frac{p_{n+m}}{p_n} - 1 &= (1 + a_{n+1}) \cdot (1 + a_{n+2}) \cdot \ldots \cdot (1 + a_{n+m}) - 1 \\ &= a_{n+1} + a_{n+2} + \ldots + a_{n+1}a_{n+2} + \ldots,\end{aligned}$$

such that

$$\left|\frac{p_{n+m}}{p_n}-1\right| \leq |a_{n+1}|+|a_{n+2}|+\ldots+|a_{n+1}a_{n+2}|+\ldots$$
$$=(1+|a_{n+1}|)\cdot(1+|a_{n+2}|)\cdot\ldots\cdot(1+|a_{n+m}|)-1$$
$$=\frac{q_{n+m}}{q_n}-1<\varepsilon.$$

Since we know q_n converges and thus $-\varepsilon < q_{n+m}/q_n - 1 < \varepsilon$. We conclude that if $\prod_{n=1}^{\infty}(1+|a_n|)$ converges, then so does $\prod_{n=1}^{\infty}(1+a_n)$.
Secondly, we can see that

$$(1+|a_1|)\cdot(1+|a_2|)\cdot\ldots\cdot(1+|a_n|)=1+|a_1|+|a_2|+\ldots+|a_n|+|a_1a_2|+\ldots$$
$$>|a_1|+|a_2|+\ldots+|a_n|,$$

and so clearly convergence of $\sum_{n=1}^{\infty}|a_n|$ is necessary for the infinite product to converge.
Next we assume instead that $\sum_{n=1}^{\infty}|a_n|$ is convergent, we rewrite our product again and see

$$\left|\frac{p_{n+m}}{p_n}-1\right| \leq \frac{q_{n+m}}{q_n}-1$$
$$=|a_{n+1}|+|a_{n+2}|+\ldots+|a_{n+1}a_{n+2}|+\ldots$$
$$<\sum_{k=n+1}^{n+m}|a_k|+\left(\sum_{k=n+1}^{n+m}|a_k|\right)^2+\ldots+\left(\sum_{k=n+1}^{n+m}|a_k|\right)^m,$$

since all terms from the tail of our product appear in one of these sums.
Now if we let $\eta>0$, we want to show that $|p_{n+m}/p_n-1|<\eta$. Since our sum is absolutely convergent, we can choose $\varepsilon=\min\{\frac{1}{2},\frac{\eta}{2}\}$, then there exists an N such that for all $n\geq N$, $m\in\mathbb{N}$, we have $\sum_{k=n+1}^{n+m}|a_k|<\varepsilon$. Then:

$$\left|\frac{p_{n+m}}{p_n}-1\right|<\sum_{k=n+1}^{n+m}|a_k|+\left(\sum_{k=n+1}^{n+m}|a_k|\right)^2+\ldots+\left(\sum_{k=n+1}^{n+m}|a_k|\right)^m$$
$$<\varepsilon+\varepsilon^2+\ldots+\varepsilon^m<\frac{\varepsilon}{1-\varepsilon}<2\varepsilon<\eta.$$

We can use the geometric series since $0<\varepsilon<\frac{1}{2}$.
We conclude that convergence of $\sum_{n=1}^{\infty}|a_n|$ implies convergence of $\prod_{n=1}^{\infty}(1+|a_n|)$, which in turn implies convergence of $\prod_{n=1}^{\infty}(1+a_n)$. □

We will use the above lemma to show equivalence between our two definitions, namely our infinite sum and our infinite product.

Lemma 2.2.

$$\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p\ is\ prime}\left(\frac{p^s}{p^s-1}\right)$$

whenever $\mathrm{Re}(s)>1$.

Proof. Write $s = \sigma + it$, where $\sigma > 1$. Taking the absolute value of each term in the sum on the left we get $\sum_{n=1}^{\infty} 1/n^{\sigma}$, to which we can apply the p-series test (or the integral test) and conclude that this the sum must converge absolutely for $\sigma > 1$. Now consider the infinite product:

$$\prod_{p \text{ is prime}} \left(\frac{p^s}{p^s - 1}\right) = \prod_{p \text{ is prime}} \left(\frac{1}{1 - p^{-s}}\right) = \prod_{p \text{ is prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Since our infinite product converges whenever our partial products p_n converge for all n, we consider

$$\prod_{p \text{ is prime}} \left(1 - \frac{1}{p^s}\right)$$

instead. If we can show that this product converges to $P \neq 0$, then

$$\prod_{p \text{ is prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

must converge to P^{-1}.
We can use lemma 2.1, where we write $\prod(1 + (-b_p)) = \prod(1 + a_p)$, so that we only need to prove that

$$\sum_{p \text{ is prime}} \left|-\frac{1}{p^s}\right| = \sum_{p \text{ is prime}} \frac{1}{p^{\sigma}}$$

converges. Now we can compare this sum term by term to $\sum_{n=1}^{\infty} 1/n^{\sigma}$, and see that both sums only contain positive terms. Since the primes are a subset of the naturals, if the sum over all naturals converges, the sum over the primes must surely also converge.
Now we consider the factors on the right. Note that $p \geq 2$, and thus for $\sigma > 1$ we have $\left|\frac{1}{p^s}\right| < 1$, such that that we can use the geometric series and rewrite

$$\frac{1}{1 - \left(\frac{1}{p^s}\right)} = \sum_{n=0}^{\infty} \left(\frac{1}{p^s}\right)^n = 1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} + \frac{1}{(p^3)^s} + \dots.$$

Then multiplying out the first two terms we see that

$$\left(1 - \frac{1}{p_1^s}\right)^{-1} \left(1 - \frac{1}{p_2^s}\right)^{-1} =$$
$$(1 + \frac{1}{(p_1)^s} + \frac{1}{(p_1^2)^s} + \frac{1}{(p_1^3)^s} + \dots)(1 + \frac{1}{(p_2)^s} + \frac{1}{(p_2^2)^s} + \frac{1}{(p_2^3)^s} + \dots) =$$
$$1 + \frac{1}{(p_2)^s} + \frac{1}{(p_2^2)^s} + \dots + \frac{1}{(p_1)^s} + \frac{1}{(p_1 p_2)^s} + \frac{1}{(p_1 p_2^2)^s} + \dots + \frac{1}{(p_1^2)^s} + \dots.$$

Continuing this, and multiplying out the first P terms of the product, since every number $n \leq P$ is some unique product of prime numbers $p_1, p_2, \dots, p_k \leq P$, we see that the product

$$\prod_{\substack{p \leq P \\ p \text{ is prime}}} \left(1 - \frac{1}{p^s}\right)^{-1} = 1 + \frac{1}{n_1^s} + \frac{1}{n_2^s} + \dots,$$

where $n_1, n_2, \ldots$ are the natural numbers which have prime factors smaller or equal to P. So we have at least all natural numbers up to and including P. Now if we take the difference between $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and the product we see that

$$\left|\zeta(s) - \prod_{\substack{p \leq P \\ p \text{ is prime}}} \left(1 - \frac{1}{p^s}\right)^{-1}\right| = \left|\zeta(s) - 1 - \frac{1}{n_1^s} - \frac{1}{n_2^s} - \ldots\right|$$
$$\leq \frac{1}{(P+1)^\sigma} + \frac{1}{(P+2)^\sigma} + \ldots,$$

where we apply the triangle inequality.
This means that as $P \to \infty$, the difference between the sum and the product expressions goes to 0. We conclude they must be identical. □

2.2 Some Observations

While there is much more to be discussed regarding Riemann zeta function, and we will do so later, it is at least interesting to consider what we have so far. We have found a function, which we can express as an infinite sum over the natural numbers, that is equivalent to a product over the prime numbers.
And it is precisely this link between a function and the prime numbers, that has enticed mathematicians for years. In fact, Riemann was drawn to this function because he tried answering the question of how many prime numbers there exist up to some number T.
And this interest makes sense! The theory of functions, usually described by calculus and mathematical analysis, can be taught (in part) to high school students, while the subject of number theory is generally impenetrable until one is extremely familiar with university level algebra. Because of the relative ease with which mathematical analysis is taught, it has enabled mathematicians to try to apply results from analysis to number theory through the study of the Riemann zeta function. In doing so, they have managed to shed a light on the cornerstones of the integers; the prime numbers.
We will later see another one of the core results which links the two, namely the prime number theorem. But before we can do so, we wish to study the analytic properties of the Riemann zeta function in depth. We choose to do this through the theory of Bernoulli numbers and polynomials.

3 Bernoulli Numbers and Polynomials

Jacob Bernoulli is one of the many well known namesakes of the Bernoulli family name. He worked on many mathematical topics, ranging from probability theory to calculus. He discovered the law of large numbers [3] and he is credited with discovering the mathematical constant e [13] (which some say is the second most important mathematical constant after π). It was also most likely Jacob Bernoulli who first introduced Euler to the Basel problem [7].
In the posthumous text *Ars Conjectandi* (which famously contains the law of large numbers) Bernoulli introduced a new sequence of numbers, called the **Bernoulli numbers**, which we discuss in this chapter. He did so in order to find a closed-form formula for the sums of powers of consecutive integers, i.e. $1 + 2^k + 3^k + \cdots + n^k = \sum_{j=1}^{n} j^k$.

3.1 Bernoulli Numbers

We begin by giving the definition of the **Bernoulli numbers** which corresponds to the numbers that Bernoulli originally defined [14].

Definition 3.1 (Bernoulli numbers)**.** *Define B_n inductively by the following formula:*

$$\sum_{i=0}^{n} \binom{n+1}{i} B_i = n + 1 \quad n = 0, 1, 2, \ldots, \tag{3.1}$$

where B_0 is the 0-th Bernoulli number, B_1 is the first Bernoulli number, and so on.

Calculating the first few Bernoulli numbers gives us

$$B_0 = 1, \quad B_1 = \frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}.$$

More numbers can of course easily be found in online lists (such as the OEIS), or through mathematical software.

Remark. *This formulation gives rise to the same formulation as Bernoulli gave in his original paper, and it lends itself very well to some expressions about the zeta function. There is another definition of the Bernoulli numbers where $B_1 = -\frac{1}{2}$ instead, and all other values are the same. In practice these definitions can easily be interchanged, but there might be a slight difference in what the reader is familiar with.*

Nowadays the Bernoulli numbers are often defined as the coefficients of a generating function. We will show however that these definitions are identical.

Theorem 3.1 (Generating function of the Bernoulli numbers)**.** *Let B_n, $n = 0, 1, 2, \ldots,$ be the Bernoulli numbers, then we have:*

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{3.2}$$

Proof. Let us first prove that that $t = 0$ is a removable singularity:

$$\lim_{t \to 0} \frac{te^t}{e^t - 1} \overset{\text{l'Hôpital}}{=} \lim_{t \to 0} \frac{te^t + e^t}{e^t} = 1.$$

This means that around $t = 0$, our function is analytic, as both the numerator and denominator are analytic, and the denominator is not equal to 0. This means we can write a Taylor series for $te^t/(e^t - 1)$ around $t = 0$. This Taylor series will have a radius of convergence of 2π since $e^{2\pi i} = 1$ which is our closest singularity.
We wish to show that

$$\left(\sum_{n=0}^{\infty} B_n \frac{t^n}{n!}\right)(e^t - 1) = te^t,$$

writing e^t as a Taylor series we get

$$te^t = t\sum_{n=0}^{\infty} \frac{t^n}{n!} = \sum_{n=1}^{\infty} \frac{t^n}{(n-1)!}.$$

So it suffices to show that

$$\left(\sum_{n=0}^{\infty} B_n \frac{t^n}{n!}\right)(e^t - 1) = \sum_{n=1}^{\infty} \frac{t^n}{(n-1)!}. \tag{3.3}$$

First note that

$$e^t - 1 = -1 + \sum_{n=0}^{\infty} \frac{t^n}{n!} = \sum_{n=1}^{\infty} \frac{t^n}{n!},$$

so if we multiply these Taylor series together, we will not have a constant term, since the $e^t - 1$ does not have a constant term. We use the Cauchy product rule to find:

$$\left(\sum_{n=0}^{\infty} B_n \frac{t^n}{n!}\right)(e^t - 1) = \sum_{n=0}^{\infty} \overbrace{\frac{B_n}{n!}}^{a_n} t^n \sum_{n=1}^{\infty} \overbrace{\frac{1}{n!}}^{b_n} t^n = \sum_{n=1}^{\infty} c_n t^n,$$
$$\text{where } c_n = \sum_{j=0}^{n} a_j b_{n-j} = \sum_{j=0}^{n-1} a_j b_{n-j} \quad \text{as } b_0 = 0.$$

This means that

$$c_n = \sum_{j=0}^{n-1} \frac{B_j}{j!} \frac{1}{(n-j)!}$$
$$n! \cdot c_n = \sum_{j=0}^{n-1} \frac{n!}{j!(n-j)!} B_j = \sum_{j=0}^{n-1} \binom{n}{j} B_j.$$

But by the definition of our Bernoulli numbers we know that $\sum_{j=0}^{n-1} \binom{n}{j} B_j = (n-1)+1 = n$, which means that $n! \cdot c_n = n$, and thus

$$\left(\sum_{n=0}^{\infty} B_n \frac{t^n}{n!}\right)(e^t - 1) = \sum_{n=1}^{\infty} c_n t^n = \sum_{n=1}^{\infty} n! \cdot c_n \frac{t^n}{n!}$$
$$= \sum_{n=1}^{\infty} n \cdot \frac{t^n}{n!} = \sum_{n=1}^{\infty} \frac{t^n}{(n-1)!} = te^t.$$

Which proves (3.3). And thus we conclude:

$$\frac{te^t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \qquad \square$$

As a result of the generating function, we can prove the following:

Corollary 3.1. *All the odd numbered Bernoulli numbers except for B_1 are equal to* 0.

Proof. We remove the linear term (so the term corresponding to B_1) from our Taylor series:

$$\begin{aligned} h(t) =& \frac{te^t}{e^t-1} - B_1 t = \frac{te^t}{e^t-1} - \frac{t}{2} \\ =& \frac{2te^t - t(e^t-1)}{2(e^t-1)} = \frac{te^t+t}{2(e^t-1)}. \end{aligned}$$

Now we want to show that $h(t)$ is an even function, indeed:

$$\begin{aligned} h(-t) =& \frac{-te^{-t}-t}{2e^{-t}-2} \cdot \frac{e^t}{e^t} = \frac{-t-te^t}{2-2e^t} \\ =& \frac{te^t+t}{2e^t-2} = h(t). \end{aligned}$$

Clearly $h(t)$ is an even function, and thus we know that all odd-numbered coefficient in its Taylor series are 0, so this means $B_{2k+1} = 0 \; \forall k \in \mathbb{N}$ by theorem 3.1 $\square$

From this it immediately follows that $(-1)^k B_k = B_k$ as well. These are both important properties of the Bernoulli numbers, and we will gladly use these later.

3.2 Bernoulli Polynomials

While the Bernoulli numbers were described by Bernoulli himself, it was in fact Euler who properly defined the Bernoulli polynomials. And while there are again many different definitions, we opt for the following.

Definition 3.2 (Bernoulli polynomials)**.** *The Bernoulli polynomial $B_n(x)$ is defined as the unique polynomial that satisfies*

$$\int_x^{x+1} B_n(y)\mathrm{d}y = x^n \qquad (n = 0, 1, 2, \dots).$$

We will make use of the following shorthand:

$$I[f(x)] = \int_x^{x+1} f(y)\mathrm{d}y.$$

The first thing we see is that on the right hand side we have a polynomial of degree n, this means that $B_n(y)$ must be a polynomial of at most degree n, as otherwise we could not find a unique solution for our coefficients to solve the polynomial equation. From that observation we can find some Bernoulli polynomials ourselves.

In particular we know that as B_0 is a polynomial of degree 0, we see it must be constant. So if we write $B_0(x) = c$, we obtain

$$\int_x^{x+1} c\mathrm{d}y = 1 \implies c(x+1) - cx = 1 \implies c = 1,$$

and that means that $B_0(x) = 1$.
Writing $B_1(x)$ as $ax + b$ we get

$$\int_x^{x+1} ay + b\mathrm{d}y = x \implies a\frac{1}{2}(x+1)^2 + b(x+1) - a\frac{1}{2}x^2 - bx = x,$$
$$\implies ax + \frac{1}{2}a + b = x \implies a = 1,\ b = -\frac{1}{2},$$

and thus $B_1(x) = x - \frac{1}{2}$.
We immediately see some interesting things; $B_0(x)$ is a constant, and equal to $B_0 = 1$, and we see the value of $B_1 = \frac{1}{2}$ appear (with a sign change) in the constant term of $B_1(x)$. Let us now prove some facts about these polynomials.

Lemma 3.1.

1. *$B_n(1) = B_n$ for $n \geq 0$. Also if $n \neq 1$, $B_n(0) = B_n(1) = B_n$.*
2. *$B_n(x+1) - B_n(x) = nx^{n-1}$ for $n \geq 0$.*
3. *$B_n'(x) = nB_{n-1}(x)$ for $n \geq 0$.*
4. *$B_n(1-x) = (-1)^n B_n(x)$ for $n \geq 0$.*
5. *Generating function:*

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{te^{xt}}{e^t - 1}. \tag{3.4}$$

Remark. *Mind the notation, B_n is the n-th Bernoulli **number**, while $B_n(x)$ is the n-th Bernoulli **polynomial**.*

Proof.

1. Let us assume statement 5. holds. Then we can set $x = 1$ in the generating function, and we get
$$\sum_{n=0}^{\infty} B_n(1)\frac{t^n}{n!} = \frac{te^t}{e^t - 1}, \quad \text{which has Taylor series } \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}$$
by theorem 3.1. Then as Taylor series are equal whenever all terms are equal, we know that $B_n(1) = B_n$ for all n.
Assuming statement 2. if we set $x = 0$, we obtain
$$B_n(0+1) - B_n(0) = n \cdot 0^{n-1} = 0 \quad (\text{whenever } n \neq 1),$$
and so we know $B_n(1) = B_n(0)$ for $n \neq 1$.

2. Consider $I[B_n(x)] = x^n$. If we take the derivative on the left and right sides we are left with

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x+1} B_n(y)\mathrm{d}y &= \frac{\mathrm{d}}{\mathrm{d}x}x^n \\ B_n(x+1) - B_n(x) &= nx^{n-1}\end{aligned}$$

as desired.

3. Looking at $I[B_n'(x)]$ we see that by 2. we get

$$\begin{aligned}\int_x^{x+1} B_n'(y)\mathrm{d}y = B_n(x+1) - B_n(x) = nx^{n-1} \\ \int_x^{x+1} \frac{B_n'(y)}{n}\mathrm{d}y = x^{n-1}.\end{aligned}$$

So then we know from our definition that

$$\frac{B_n'(x)}{n} = B_{n-1}(x) \implies B_n'(x) = nB_{n-1}(x).$$

4. We consider $I[B_n(1-x)]$ and apply a change of variables $z = 1-y$ so that we get

$$\begin{aligned}\int_x^{x+1} B_n(1-y)\mathrm{d}y &= \int_{1-x}^{1-(x+1)} B_n(z)\cdot -\mathrm{d}z \\ &= -\int_{1-x}^{-x} B_n(y)\mathrm{d}y = \int_{-x}^{-x+1} B_n(y)\mathrm{d}y,\end{aligned}$$

which we can recognise as $I[B_n(-x)]$ and is of course equal to $(-x)^n$ such that

$$\int_x^{x+1} B_n(1-y)\mathrm{d}y = (-x)^n = (-1)^n x^n.$$

Multiplying the left and right hand side by $(-1)^n$ we obtain $I[(-1)^n B_n(1-x)] = x^n$ and so $B_n(1-x) = (-1)^n B_n(x)$.

5. First of all we need to note that $(te^{xt})\,/\,(e^t-1)$ is an analytic function of t, even around $t=0$ as this is merely a removable singularity:

$$\lim_{t\to 0}\frac{te^{xt}}{e^t-1} \overset{\text{l'Hôpital}}{=} \lim_{t\to 0}\frac{xte^{xt}+e^{xt}}{e^t} = 1.$$

This means that we can write down its Taylor series around 0. Since $e^{2\pi i} = 1$, the closest singularity sits at $t = 2\pi i$ and so the radius of convergence of the series is 2π.
So we can write

$$\sum_{n=0}^{\infty}\frac{a_n}{n!}t^n \text{ as a Taylor series for } \frac{te^{xt}}{e^t-1}. \tag{3.5}$$

Now the right hand side is a function of x and thus we know that our coefficients need to also be functions of x, so write $a_n(x)$ instead. We will show that $a_n(x) = B_n(x)$ which proves the statement. Then:

$$te^{xt} = (e^t-1)\frac{te^{xt}}{e^t-1} = (e^t-1)\sum_{n=0}^{\infty}\frac{a_n(x)}{n!}t^n.$$

Now using its Taylor series we can rewrite $e^t - 1 = \sum_{n=1}^{\infty} t^n/n!$, to get

$$te^{xt} = \sum_{n=1}^{\infty} \overbrace{\frac{1}{n!}}^{b_n} t^n \sum_{n=0}^{\infty} \frac{a_n(x)}{n!} t^n = \sum_{n=1}^{\infty} c_n t^n,$$

where

$$c_n = \sum_{j=0}^{n} \frac{a_j(x)}{j!} b_{n-j} = \sum_{j=0}^{n-1} \frac{a_j(x)}{j!} \frac{1}{(n-j)!} \quad (\text{as } b_0 = 0)$$

by the Cauchy product formula.
Secondly we can also write down the Taylor series for te^{xt},

$$\begin{aligned} te^{xt} &= \sum_{n=0}^{\infty} \frac{t(xt)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} t^{n+1} \\ &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} t^n = \sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{n!} t^n. \end{aligned}$$

Since we know that the Taylor series is unique, we can compare coefficients and see that

$$\frac{n \cdot x^{n-1}}{n!} = c_n = \sum_{j=0}^{n-1} \frac{a_j(x)}{j!} \frac{1}{(n-j)!} = \sum_{j=0}^{n-1} \frac{a_j(x)}{j!(n-j)!}.$$

Multiplying left and right by $n!$ we obtain

$$n \cdot x^{n-1} = \sum_{j=0}^{n-1} \frac{n!}{j!(n-j)!} a_j(x) = \sum_{j=0}^{n-1} \binom{n}{j} a_j(x). \tag{3.6}$$

<u>We want to show</u>: $a_j(x) = B_j(x)$, or equivalently,

$$\int_x^{x+1} a_j(y) \mathrm{d}y = x^j.$$

We will do this by induction on j. For $n = 1, 2$, we can see clearly from equation (3.6) that:

$$\begin{aligned} 1 \cdot x^{1-1} &= \sum_{j=0}^{1-1} \binom{1}{j} a_j(x) \\ 1 &= 1 a_0(x) \implies a_0(x) = 1 = B_0(x), \end{aligned}$$

and

$$\begin{aligned} 2 \cdot x^{2-1} &= \sum_{j=0}^{2-1} \binom{2}{j} a_j(x) \\ 2x &= 1 a_0(x) + 2 a_1(x) = 1 + 2 a_1(x) \\ &\implies a_1(x) = x - \frac{1}{2} = B_1(x). \end{aligned}$$

Now assume for the induction hypothesis that for $j < n$

$$a_j(x) = B_j(x) \iff I[a_j(x)] = \int_x^{x+1} a_j(y)\mathrm{d}y = x^j.$$

Then by equation (3.6) we have

$$\begin{aligned}(n+1)x^n &= \sum_{j=0}^{n} \binom{n+1}{j} a_j(x) \\ &= \binom{n+1}{n} a_n(x) + \sum_{j=0}^{n-1} \binom{n+1}{j} a_j(x) = (n+1)a_n(x) + \sum_{j=0}^{n-1} \binom{n+1}{j} a_j(x).\end{aligned}$$

Now we apply our operator I to both sides to get:

$$I[(n+1)x^n] = I[(n+1)a_n(x)] + I\left[\sum_{j=0}^{n-1} \binom{n+1}{j} a_j(x)\right],$$

$$\int_x^{x+1} (n+1)y^n \mathrm{d}y = \int_x^{x+1} (n+1)a_n(y)\mathrm{d}y + \int_x^{x+1} \sum_{j=0}^{n-1} \binom{n+1}{j} a_j(y)\mathrm{d}y.$$

We can evaluate the integral on the left-hand side, take out the $(n+1)$ term, and exchange summation and integration since we are working with a finite sum, so that we get

$$\left[y^{n+1}\right]_x^{x+1} = (n+1)\int_x^{x+1} a_n(y)\mathrm{d}y + \sum_{j=0}^{n-1} \binom{n+1}{j} \int_x^{x+1} a_j(y)\mathrm{d}y,$$

$$(x+1)^{n+1} - x^{n+1} = (n+1)\cdot I[a_n(x)] + \sum_{j=0}^{n-1} \binom{n+1}{j} x^j,$$

since by the induction hypothesis $I[a_j(y)] = x^j$ for $j = 0, 1, \ldots, n-1$.
Letting the sum run until $n+1$ instead (and taking away the extra terms), we get

$$(x+1)^{n+1} - x^{n+1} = (n+1)I[a_n(x)] + \sum_{j=0}^{n+1} \binom{n+1}{j} x^j - \binom{n+1}{n} x^n - \binom{n+1}{n+1} x^{n+1}.$$

Then by the binomial theorem we have $\sum_{j=0}^{n+1} \binom{n+1}{j} x^j = (x+1)^{n+1}$, such that

$$\begin{aligned}(x+1)^{n+1} - x^{n+1} &= (n+1)I[a_n(x)] + (x+1)^{n+1} - (n+1)x^n - x^{n+1} \\ 0 &= (n+1)I[a_n(x)] - (n+1)x^n.\end{aligned}$$

This means that $I[a_n(x)] = x^n$ and so $a_n(x) = B_n(x)$ for all $n \in \mathbb{N}$. So by equation (3.5),

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{te^{xt}}{e^t - 1}.$$

□

3.3 Related Series

Since we have now built up some knowledge about Bernoulli numbers and Bernoulli polynomials, we will look at series related to our Bernoulli numbers.
For the well-informed reader, some of these series are related to the sawtooth functions from the theory of Fourier series. In this text we will only study the series we need for later results however, and not delve deeper into the theory of Fourier series.

Definition 3.3 (Floor function)**.** *The floor function of a number, $\lfloor x \rfloor$ is the first integer smaller than or equal to x.*

As an example, $\lfloor \pi \rfloor = \lfloor 3 \rfloor = 3$, and $\left\lfloor \frac{299}{100} \right\rfloor = 2$. In older texts, one can find $[x]$ instead, which is sometimes called the Gauss symbol. Modern notation favours $\lfloor x \rfloor$ however [5].

Theorem 3.2. *Let k be a natural number. The formula*

$$B_k(x - \lfloor x \rfloor) = -\frac{k!}{(2\pi i)^k} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{e^{2\pi i n x}}{n^k} \tag{3.7}$$

holds for all real numbers x if $k \geq 2$; it holds for all real numbers $x \notin \mathbb{Z}$ if $k = 1$.
Here the sum is taken for all integers different from 0. If $k = 1$, the infinite sum on the right-hand side should be understood as

$$\lim_{N \to \infty} \sum_{\substack{n=-N \\ n \neq 0}}^{N} \frac{e^{2\pi i n x}}{n}.$$

If $k \geq 2$, then the right-hand side of (3.7) converges absolutely and uniformly with respect to x on $\mathbb{R}$.

Proof. First let $0 < x < 1$. Let us consider the function

$$f(z) = \frac{e^{xz}}{e^z - 1}.$$

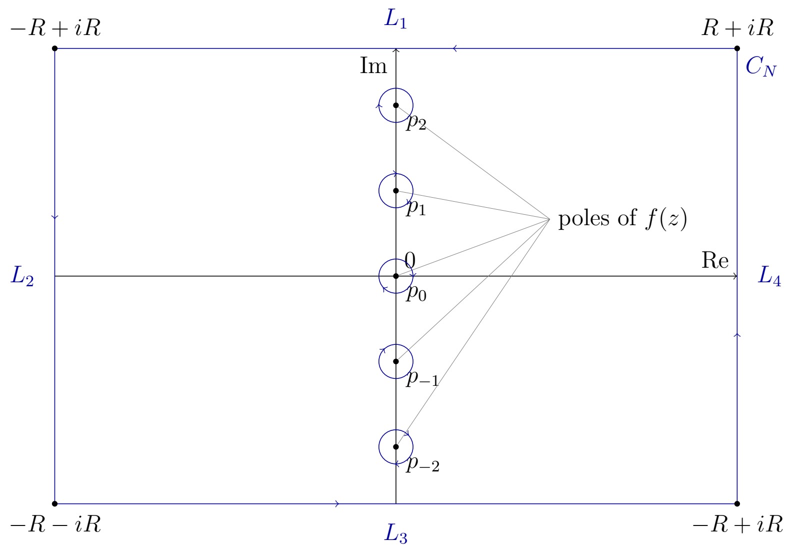
$f(z)$ has poles whenever $e^z - 1 = 0$, and thus whenever $z = 2\pi i n$, $n \in \mathbb{Z}$, and each is of order 1:

$$\begin{aligned} \lim_{z \to 2\pi i n} (z - 2\pi i n) \frac{e^{xz}}{e^z - 1} &\overset{\text{l'Hôpital}}{=} \lim_{z \to 2\pi i n} \frac{xze^{xz} + e^{xz} - 2\pi i n x e^{xz}}{e^z} \\ &= \frac{x(2\pi i n)e^{2\pi i n x} + e^{2\pi i n x} - 2\pi i n x e^{2\pi i n x}}{1} \\ &= e^{2\pi i n x}, \end{aligned}$$

and we know that $e^{2\pi i n x} \neq 0$ for any $n \in \mathbb{N}$, $x \in \mathbb{R}$.
Furthermore we also know that $e^{2\pi i n x}$ is the residue of $f(z)$ at $z = 2\pi i n$. Now let N be a sufficiently large natural number, and consider $R = 2\pi(N + \frac{1}{2})$.

Let C_N be a square passing through the four corner points $R+iR$, $-R+iR$, $-R-iR$, $R-iR$ in that order in the complex plane.
Let us denote L_1 as the line from $R+iR$ to $-R+iR$, let L_2 be the line from $-R+iR$ to $-R-iR$, and so on.

First we know that since $R = 2\pi N + \pi$, which is never an even multiple of π, we know that none of our singularities are ever on the square C_N.



Now if t is a point in C_N such that $t \neq 2\pi i n$, $n \in Z$, then by Cauchy's Residue Theorem we know that

$$\int_{C_N} \frac{f(z)}{z-t} \mathrm{d}z = 2\pi i \left(\sum_{z_j \text{ is a singularity}} Res\left(\frac{f(z)}{z-t}, z_j\right)\right).$$

Clearly $z_j = t$ is a singularity, and it is a simple pole since $z = t$ is not a simple pole of $f(z)$. In the same way we know that the simple poles of $f(z)$ are also simple poles of $f(z)/(z-t)$. Together those form all the singularities of $f(z)/(z-t)$ so the above becomes:

$$\int_{C_N} \frac{f(z)}{z-t} \mathrm{d}z = 2\pi i \left(\mathrm{Res}\left(\frac{f(z)}{z-t}, t\right) + \sum_{n=-N}^{N} \mathrm{Res}\left(\frac{f(z)}{z-t}, 2\pi i n\right)\right). \tag{3.8}$$

First we see that

$$\mathrm{Res}\left(\frac{f(z)}{z-t}, t\right) = \lim_{z\to t}(z-t)\frac{f(z)}{z-t} = f(t),$$

and also that

$$\begin{aligned}
\mathrm{Res}\left(\frac{f(z)}{z-t}, 2\pi i n\right) &= \lim_{z\to 2\pi i n}(z-2\pi i n)\frac{f(z)}{z-t} \\
&= -\frac{1}{t-2\pi i n}\lim_{z\to 2\pi i n}(z-2\pi i n)f(z) \\
&= -\frac{1}{t-2\pi i n}\mathrm{Res}((f(z), 2\pi i n) \\
&= -\frac{e^{2\pi i n}}{t-2\pi i n}.
\end{aligned}$$

Then equation (3.8) becomes:

$$\int_{C_N} \frac{f(z)}{z-t} \mathrm{d}z = 2\pi i \left(f(t) - \frac{1}{t} - \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{e^{2\pi i n x}}{t-2\pi i n}\right). \tag{3.9}$$

First let us look at the left hand side. We will consider

$$\lim_{N\to\infty}\int_{C_N}\frac{f(z)}{z-t}\mathrm{d}z$$

for some fixed $t\in\mathbb{C}$.
Note that as $N\to\infty$, we also have $R\to\infty$. We will now split our integral into 4 integrals along L_1, L_2, L_3, L_4. Let their corresponding paths be:

$$L_1:\gamma_1(s)=s+iR, R\geq s\geq -R, \qquad L_2:\gamma_2(s)=-R+is, R\geq s\geq -R,$$
$$L_3:\gamma_3(s)=s-iR, -R\leq s\leq R, \qquad L_4:\gamma_4(s)=R+is, -R\leq s\leq R.$$

Such that:

$$\begin{aligned}\int_{C_N}\frac{f(z)}{z-t}\mathrm{d}z&=\int_{L_1}\frac{f(z)}{z-t}\mathrm{d}z+\int_{L_2}\frac{f(z)}{z-t}\mathrm{d}z+\int_{L_3}\frac{f(z)}{z-t}\mathrm{d}z+\int_{L_4}\frac{f(z)}{z-t}\mathrm{d}z\\&=I_1+I_2+I_3+I_4.\end{aligned}$$

We apply the ML-inequality to I_2 and I_4.
As for I_2:

$$\left|\frac{f(-R+is)}{(-R+is-t)}\right|=\left|\frac{e^{x(-R+is)}}{(e^{-R+is}-1)(-R+is-t)}\right|\leq\frac{e^{-xR}}{|e^{-R}-1|\,|-R+is-t|}.$$

Then by the ML-inequality we have

$$\lim_{N\to\infty}|I_2|\leq\lim_{N\to\infty}2R\frac{e^{-xR}}{|e^{-R}-1|\,|-R+is-t|}=0.$$

Because $\left|e^{-R}-1\right|>0$, and $t\neq -R+is$; so the denominator is bound from below, and since $x>0$, we know $Re^{-xR}\to 0$ as $N\to\infty$, so we conclude the integral must vanish.
For I_4:

$$\left|\frac{f(R+is)}{(R+is-t)}\right|=\left|\frac{e^{x(R+is)}}{(e^{R+is}-1)(R+is-t)}\right|\leq\frac{e^{xR}}{(e^{R}-1)\,|R+is-t|}.$$

Again, by the ML-inequality we get

$$\lim_{N\to\infty}|I_4|\leq\lim_{N\to\infty}2R\frac{1}{(e^{(1-x)R}-e^{-xR})\,|R+is-t|}=0.$$

Because $x<1$ and thus $1-x>0$, and $t\neq R+is$, so that we have $R/(e^{(1-x)R})\to 0$ as $N\to\infty$. We conclude again that the integral must vanish.

Regarding I_1 and I_3, first recall that $R=\pi(2N+1)$ such that $e^{s\pm iR}=-e^{s}$. Now consider $|f(z)/(z-t)|$ on $z=s\pm iR$:

$$\left|\frac{f(z)}{z-t}\right|=\frac{|e^{xz}|}{|e^{z}-1|\,|z-t|}=\frac{e^{xs}}{(e^{s}+1)\,|z-t|}.$$

Now we can choose R big enough such that both $|s+iR-t| > \frac{R}{2}$ and $|s-iR-t| > \frac{R}{2}$, which means that

$$\left|\frac{f(z)}{z-t}\right| \leq \frac{2}{R}\frac{e^{xs}}{e^s+1}.$$

Then:

$$\begin{aligned}
\lim_{N\to\infty} |I_1| &= \lim_{N\to\infty} \left|\int_R^{-R} \frac{e^{x(s+iR)}}{(e^{s+iR}-1)(s+iR-t)}\mathrm{d}s\right| \\
&\leq \lim_{N\to\infty} \int_{-R}^{R} \left|\frac{e^{x(s+iR)}}{(e^{s+iR}-1)(s+iR-t)}\right| \mathrm{d}s \\
&\leq \lim_{N\to\infty} \int_{-R}^{R} \frac{2}{R}\frac{e^{xs}}{e^s+1}\mathrm{d}s = \lim_{N\to\infty} \frac{2}{R}\int_{-R}^{R} \frac{e^{xs}}{e^s+1}\mathrm{d}s.
\end{aligned}$$

But note for $0 < x < 1$, both $\lim_{s\to\infty} e^{xs}/(e^s+1) = 0$ and $\lim_{s\to-\infty} e^{xs}/(e^s+1) = 0$. We also know that $e^{xs}/(e^s+1)$ is bounded. And so the function is integrable and we can bound the integral from above by some value, say C_1, such that

$$\lim_{N\to\infty} |I_1| \leq \lim_{N\to\infty} \frac{2}{R} \cdot C_1 = 0.$$

By identical steps, except that the integral ranges from $-R$ to R, we find that

$$\lim_{N\to\infty} |I_3| \leq \lim_{N\to\infty} \frac{2}{R} \cdot C_2 = 0.$$

We conclude that:

$$\lim_{N\to\infty} \int_{C_N} \frac{f(z)}{z-t}\mathrm{d}z = 0.$$

So if we take the limit as $N \to \infty$ in equation (3.9) we get:

$$\begin{aligned}
\lim_{N\to\infty} \int_{C_N} \frac{f(z)}{z-t}\mathrm{d}z &= \lim_{N\to\infty} 2\pi i \left(f(t) - \frac{1}{t} - \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{e^{2\pi inx}}{t-2\pi in} \right) \\
0 &= 2\pi i \left(f(t) - \frac{1}{t} - \lim_{N\to\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{e^{2\pi inx}}{t-2\pi in} \right).
\end{aligned}$$

Now we can rearrange some terms to find an expression for $f(t)$ as follows:

$$\begin{aligned}
f(t) &= \frac{1}{t} + \lim_{N\to\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{e^{2\pi inx}}{t-2\pi in} \\
&= \frac{1}{t} - \lim_{N\to\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{e^{2\pi inx}}{2\pi in\left(1-\frac{t}{2\pi in}\right)} \\
&= \frac{1}{t} - \lim_{N\to\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{e^{2\pi inx}}{2\pi in} \cdot \frac{1}{1-\frac{t}{2\pi in}}. \qquad (3.10)
\end{aligned}$$

Whenever $0 < |t| < 2\pi$ we have $\left|\frac{t}{2\pi in}\right| = \frac{|t|}{|2\pi in|} < \frac{2\pi}{2\pi|n|} \leq 1$ and so we can write $1/\left(1 - \frac{t}{2\pi in}\right)$ as a geometric series:

$$\frac{1}{1-\frac{t}{2\pi in}} = \sum_{k=0}^{\infty}\left(\frac{t}{2\pi in}\right)^{k} = \sum_{k=1}^{\infty}\left(\frac{t}{2\pi in}\right)^{k-1}.$$

We use that in equation (3.10) to get:

$$\begin{aligned} f(t) &= \frac{1}{t} - \lim_{N\to\infty}\sum_{\substack{n=-N\\ n\neq 0}}^{N}\frac{e^{2\pi inx}}{2\pi in}\sum_{k=1}^{\infty}\left(\frac{t}{2\pi in}\right)^{k-1} \\ &= \frac{1}{t} - \lim_{N\to\infty}\sum_{\substack{n=-N\\ n\neq 0}}^{N}\left(\frac{e^{2\pi inx}}{2\pi in} + \sum_{k=2}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^k}t^{k-1}\right). \end{aligned}$$

Consider now that

$$\begin{aligned} \sum_{k=2}^{\infty}\left|\frac{e^{2\pi inx}}{(2\pi in)^k}t^{k-1}\right| &\leq \sum_{k=2}^{\infty}\frac{|t|^{k-1}}{(2\pi\,|n|)^k} \\ &= \sum_{k=2}^{\infty}\left(\frac{1}{2\pi\,|n|}\frac{|t|^{k-1}}{(2\pi\,|n|)^{k-1}}\right) \\ &= \frac{1}{2\pi\,|n|}\sum_{j=1}^{\infty}\left(\frac{|t|}{2\pi\,|n|}\right)^{j}. \end{aligned}$$

Now since $|t| < 2\pi$, we can rewrite the geometric series to:

$$\sum_{j=1}^{\infty}\left(\frac{|t|}{2\pi\,|n|}\right)^{j} = \frac{1}{1-|t|/(2\pi\,|n|)} - 1 = \frac{2\pi\,|n|}{2\pi\,|n|-|t|} - 1 \leq \frac{|n|}{|n|-1} - 1,$$

such that

$$\frac{1}{2\pi\,|n|}\sum_{j=1}^{\infty}\left(\frac{|t|}{2\pi\,|n|}\right)^{j} \leq \frac{1}{2\pi(|n|-1)} - \frac{1}{2\pi\,|n|}.$$

Which means that the double sum

$$\lim_{N\to\infty}\sum_{\substack{n=-N\\ n\neq 0}}^{N}\sum_{k=2}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^k}t^{k-1}$$

is absolutely convergent since we can bound the sum with a telescoping series for $|n| \neq 1$. For $|n| = 1$ we simply consider the limit of the geometric series, namely $\frac{1}{1-|t|/(2\pi|n|)}$, which is of course convergent as $|t| < 2\pi$ is strict. And so we can rewrite equation (3.10) once more to find:

$$f(t) = \frac{1}{t} - \lim_{N\to\infty}\sum_{\substack{n=-N\\ n\neq 0}}^{N}\frac{e^{2\pi inx}}{2\pi in} - \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\sum_{k=2}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^k}t^{k-1}. \tag{3.11}$$

By lemma 3.1 we know that for $|t| < 2\pi$

$$f(t) = \frac{1}{t}\frac{te^{xt}}{e^t - 1} = \frac{1}{t}\sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!} = \sum_{k=0}^{\infty} B_k(x)\frac{t^{k-1}}{k!}.$$

If we compare coefficients to 3.11 we see:

$$\begin{aligned} B_0(x) &= 1, \\ B_1(x) &= -\lim_{N\to\infty}\sum_{\substack{n=-N\\ n\neq 0}}^{N} \frac{e^{2\pi inx}}{2\pi in}, \\ B_k(x) &= -k!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^k} \quad \text{for } k \geq 2. \end{aligned}$$

For the sum for $B_k(x)$, we can apply the Weierstrass M-test with $M_n = 1/n^k$, $k \geq 2$, to see that the sum converges uniformly on $[0, 1]$. And since $B_k(x)$ is a polynomial and thus continuous, we know the above formula holds in $[0, 1]$. But note that the right hand side is 1-periodic. Now since $x - \lfloor x \rfloor \in [0, 1)$, we find

$$B_k(x - \lfloor x \rfloor) = -\frac{k!}{(2\pi i)^k}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{e^{2\pi inx}}{n^k} \qquad (k \geq 2).$$

We would like for this series to resemble a Fourier series, so we consider the endpoints, namely 0 and 1. Since we have uniform convergence we can put $x = 1$ (or 0) on the right. Now as $e^{2\pi in\cdot 1} = e^{2\pi in\cdot 0}$, we have the same endpoints on the left and right side of our interval.
Now on the left hand side, we also have a 1-periodic function, and by 3.1 we know that in fact $B_k(1) = B_k(0)$, and since $B_k(x)$ is a polynomial, we can just take the limit.
This means that our series converges uniformly on the interval $(0, 1)$, and the endpoints are also identical. We conclude that our series converges uniformly to the function $B_k(x - \lfloor x \rfloor)$ on the interval $[0, 1]$. □

For $B_1(x)$ we must be more careful, for if $x \in \mathbb{Z}$, we have $e^{2\pi inx} = 1$, but then we would get that

$$B_1(x) = -\lim_{N\to\infty}\sum_{\substack{n=-N\\ n\neq 0}}^{N}\frac{e^{2\pi inx}}{2\pi in} = \frac{i}{2\pi}\lim_{N\to\infty}\sum_{\substack{n=-N\\ n\neq 0}}^{N}\frac{1}{n}$$

which is not absolutely convergent.
So we need to assume $x \notin \mathbb{Z}$. Then we can recognise the sum as the Maclaurin series for $\mathrm{Log}(1+z) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}z^n$.[i] Where $z = -e^{\pm 2\pi ix} \neq 1$. This means the sum does converge as well. And so we see that

$$B_1(x) = -\lim_{N\to\infty}\sum_{\substack{n=-N\\ n\neq 0}}^{N}\frac{e^{2\pi inx}}{2\pi in} = \frac{i}{2\pi}\left(\mathrm{Log}(1 - e^{-2\pi ix}) - \mathrm{Log}(1 - e^{2\pi ix})\right),$$

where Log denotes the **principal branch** of the complex logarithm.

[i]See theorem A.1 in the appendix for why we can use this for $|z| = 1$, $z \neq 1$

Remark. *In this text we will choose the* ***principal branch*** *of the complex logarithm to mean the branch of the logarithm whose imaginary part is in the interval* $(-\pi, \pi]$*, which corresponds to having our branch cut lie on the negative real axis.*

Furthermore we can make the following observation about the series for odd and even k, which will be useful later on.

Observation. *If* k *is even, we know that* $n^k = (-n)^k$*. Recall that* $e^{2\pi inx} = \cos(2\pi nx) + i\sin(2\pi nx)$ *such that* $e^{2\pi inx} + e^{2\pi i(-n)x} = 2\cos(2\pi nx)$*. Also since* k *is even, we know* $-1/i^k = (-1)^{k/2-1}$*, which we can then multiply by* $(-1)^2$*.*
This means that for ***even*** k*, we can rewrite equation* (3.7) *into*

$$B_k(x - \lfloor x \rfloor) = 2(-1)^{k/2-1}k! \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{(2\pi n)^k}.$$

If k *is odd, then* $n^k = -(-n)^k$*, and* $e^{2\pi inx} - e^{2\pi i(-n)x} = 2i\sin(2\pi nx)$*. Also since* k *is odd we know* $-1/i^{k-1} = (-1)^{(k+1)/2}$*, which we can then multiply by* $(-1)^2$*.*
This means that for ***odd*** k *we can rewrite equation* (3.7) *into*

$$B_k(x - \lfloor x \rfloor) = 2(-1)^{(k+1)/2}k! \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{(2\pi n)^k}, \tag{3.12}$$

where we of course need to assume $x \notin \mathbb{Z}$ *whenever* $k = 1$*.*

As we have stated before, the series for $B_k(x - \lfloor x \rfloor)$ converge uniformly, except whenever $k = 1$. For this series specifically we shall introduce a weaker type of convergence which will suffice for our goals.

Definition 3.4 (Boundedly convergent series)**.**
A series $s_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$ *is called* ***boundedly convergent*** *on a set* U*, if it converges for all* $x \in U$*, and there exists some constant* M *such that*

$$|s_n(x)| \leq M,$$

for all $n \in \mathbb{N}$ *and* $x \in U$*.*

It is not hard to see that uniform convergence implies bounded convergence.

Lemma 3.2. *If a series* $s_n(x) = \sum_{j=1}^{n} u_j(x)$ *of bounded functions converges uniformly to a function* $s(x)$ *on a set* U*, then the series* $s_n(x)$ *is boundedly convergent on* U *as well.*

Proof. Assume $s_n(x)$ converges uniformly to $s(x)$.
Let $\varepsilon = 1$, then, by uniform convergence, there exists some N such that for all $on \geq N$:

$$|s_n(x) - s(x)| < 1 \text{ which means } |s_n(x)| \leq 1 + |s(x)|.$$

Denote $M_1 = 1 + \sup_{x \in U} |s(x)|$, which exists since $s_n(x)$ is bounded.
We know that $\sup_{x \in U} |u_j(x)| = \mu_j$ exists for $j = 1, 2, \ldots, N-1$. Then for $m < N$, $s_m(x)$ is the sum of a finite number of bounded terms.
Let $M_2 = \max_{j=1,\ldots,N-1}\{\mu_j\}$, then $|s_n(x)| \leq \max\{M_1, (N-1) \cdot M_2\} = M$ for all x, n. □

This means that bounded convergence is a weaker constraint than uniform convergence, but we will later see that it is still a strong enough result in our case. We will now show that the series for $B_k(x - \lfloor x \rfloor)$ are boundedly convergent.

Proposition 3.1. *The series*

$$B_1(x - \lfloor x \rfloor) = -\sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{\pi n} \quad (x \notin \mathbb{Z}),$$

which was obtained from equation (3.12)*, is boundedly convergent on* $\mathbb{R} \backslash \mathbb{Z}$*.*
And the series

$$B_k(x - \lfloor x \rfloor) = -k! \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^k} \quad (k \geq 2)$$

is boundedly convergent on $\mathbb{R}$*.*

Proof. Since we know that the series for $B_k(x - \lfloor x \rfloor)$ only contain bounded terms, and converge uniformly for $k \geq 2$, they must also be boundedly convergent by lemma 3.2. Let us consider the series for $k = 1$,

$$s_n(x) = -\sum_{k=1}^{n} \frac{\sin(2\pi k x)}{\pi k}.$$

First we observe that our function is 1-periodic, and thus we need to only show boundedness for x in some interval of length 1, and then we can extend the boundedness to all x by periodicity.
Consider the fact that

$$\int_0^x 2\cos(2\pi k u)\mathrm{d}u = \frac{\sin(2\pi k x)}{\pi k},$$

so that

$$\begin{aligned} s_n(x) &= 2\int_0^x \cos(2\pi u) + \cos(2\pi 2u) + \cdots + \cos(2\pi n u)\mathrm{d}u \\ &= \frac{1}{\pi}\int_0^{2\pi x} \cos(t) + \cos(2t) + \cdots + \cos(nt)\mathrm{d}t, \end{aligned}$$

where we use the change of variables $t = 2\pi u$.
Using the trigonometric identity $2\sin(A)\cos(B) = \sin(A + B) - \sin(B - A)$, we rewrite:

$$\begin{aligned} 2\sin(t/2)\sum_{k=1}^{n}\cos(kt) &= \sum_{k=1}^{n}\left[\sin(t/2 + kt) - \sin(kt - t/2)\right] \\ &= \sin((n + 1/2)t) - \sin(t/2), \\ \implies \sum_{k=1}^{n}\cos(kt) &= \frac{\sin\left(\left(n + \frac{1}{2}\right)t\right) - \sin\left(\frac{t}{2}\right)}{2\sin\left(\frac{t}{2}\right)}. \end{aligned}$$

This means we are left with

$$\begin{aligned}
\pi s_n(x) &= \int_0^{2\pi x} \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right) - \sin\left(\frac{t}{2}\right)}{2\sin\left(\frac{t}{2}\right)} \mathrm{d}t \\
&= \int_0^{2\pi x} \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{t} \mathrm{d}t + \int_0^{2\pi x} \left(\frac{1}{2\sin\left(\frac{t}{2}\right)} - \frac{1}{t}\right) \sin\left(\left(n+\frac{1}{2}\right)t\right) \mathrm{d}t - \pi x \\
&= \int_0^{2\pi(n+1/2)x} \frac{\sin(u)}{u} \mathrm{d}u + \int_0^{2\pi x} \left(\frac{1}{2\sin\left(\frac{t}{2}\right)} - \frac{1}{t}\right) \sin\left(\left(n+\frac{1}{2}\right)t\right) \mathrm{d}t - \pi x.
\end{aligned}$$

It is known that

$$\mathrm{Si}(h) = \int_0^h \frac{\sin(u)}{u} \mathrm{d}u$$

attains its global maximum whenever $h = \pi$[ii].
Now consider that for $x \geq 0$

$$\begin{aligned}
&\left| \int_0^{2\pi x} \left(\frac{1}{2\sin\left(\frac{t}{2}\right)} - \frac{1}{t}\right) \sin\left(\left(n+\frac{1}{2}\right)t\right) \mathrm{d}t \right| \\
&\leq \int_0^{2\pi x} \left| \left(\frac{1}{2\sin\left(\frac{t}{2}\right)} - \frac{1}{t}\right) \sin\left(\left(n+\frac{1}{2}\right)t\right) \right| \mathrm{d}t \\
&\leq \int_0^{2\pi x} \left| \left(\frac{1}{2\sin\left(\frac{t}{2}\right)} - \frac{1}{t}\right) \right| \mathrm{d}t = \int_0^{2\pi x} \frac{1}{2\sin\left(\frac{t}{2}\right)} - \frac{1}{t} \mathrm{d}t,
\end{aligned}$$

since we know that for $t \in (0, \pi]$, we have $0 \leq \sin(t) \leq t$, and thus $2\sin(t/2) \leq t \implies 1/t \leq 1/(2\sin(t/2))$.
For $x < 0$, we note that both terms in the integrand are odd functions in t, so integrating to the left is the same as integrating to the right with a sign change. Without loss of generality we look at $x > 0$.
Our integrand has a removable singularity at $t = 0$, indeed:

$$\lim_{t\to 0} \frac{1}{2\sin\left(\frac{t}{2}\right)} - \frac{1}{t} = \lim_{t\to 0} \frac{t - 2\sin\left(\frac{t}{2}\right)}{2t\sin\left(\frac{t}{2}\right)} \overset{\text{l'Hôpital twice}}{=} \lim_{t\to 0} \frac{\frac{1}{2}\sin(t/2)}{t\cos(t/2) + 2\cos(t/2)} = 0.$$

The next singularity appears at $t = 2\pi$. Thus, if we restrict x to the interval $[-\frac{1}{2}, \frac{1}{2}]$ instead of the interval $[0, 1]$, we know that

$$\int_0^{2\pi x} \frac{1}{2\sin\left(\frac{t}{2}\right)} - \frac{1}{t} \mathrm{d}t$$

is an integrable function on a bounded interval, and we conclude that the integral must be bounded. We also know that the integrand is a positive function for positive t, and so the integral must be increasing in x, and thus we have, for $x \in [-\frac{1}{2}, \frac{1}{2}]$

$$|s_n(x)| \leq \int_0^{\pi} \frac{\sin(u)}{u} \mathrm{d}u + \int_0^{\pi} \frac{1}{2\sin\left(\frac{t}{2}\right)} - \frac{1}{t} \mathrm{d}t + \frac{1}{2}\pi = M.$$

[ii] see lemma A.4 in the appendix

Where M is clearly independent of x and n.
Since each $\sin(2\pi kx)$ is 1-periodic, our series is 1-periodic. Then as we have shown the series is boundedly convergent for $x \in [-\frac{1}{2}, \frac{1}{2}]$, by periodicity, we have shown our series is boundedly convergent for all x. □

We will later use our weaker condition of bounded convergence when we wish to exchange summation and integration, since we do not have uniform convergence for $B_1(x - \lfloor x \rfloor)$. It is not hard to see that we can use Lebesgue's dominated convergence theorem if we multiply a boundedly convergent sequence and an integrable function, a result we will happily use later.

From theorem 3.2 we can finally obtain a result related to the zeta function, a result which took Euler great effort to prove.

Corollary 3.2. *Let $k \in \mathbb{N}$, then we have*

$$\zeta(2k) = -\frac{1}{2}\frac{B_{2k}}{(2k)!}(2\pi i)^{2k}.$$

Proof. Let $k \in \mathbb{N}$. If we set $x = 0$ in equation (3.7), we get

$$B_{2k}(0 - \lfloor 0 \rfloor) = -\frac{(2k)!}{(2\pi i)^{2k}} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{e^0}{n^{2k}}.$$

Now as $(-n)^{2k} = n^{2k}$, and $B_{2k}(0) = B_{2k}$ by lemma 3.1 we see that

$$B_{2k} = -\frac{(2k)!}{(2\pi i)^{2k}} \cdot 2\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$$
$$\zeta(2k) = -\frac{1}{2}\frac{B_{2k}}{(2k)!}(2\pi i)^{2k}.$$ □

Let us immediately use the above lemma to calculate some values of $\zeta(2k)$, we see that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2}\frac{B_2}{2!}(2\pi i)^2 = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = -\frac{1}{2}\frac{B_4}{4!}(2\pi i)^4 = \frac{\pi^4}{90},$$
$$\sum_{n=1}^{\infty} \frac{1}{n^6} = -\frac{1}{2}\frac{B_6}{6!}(2\pi i)^6 = \frac{\pi^6}{945}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^8} = -\frac{1}{2}\frac{B_8}{8!}(2\pi i)^8 = \frac{\pi^8}{9450}.$$

If we compare this to (2.1), which were the values that Euler found in 1735, we see that they indeed agree. We have thus found an explicit expression for some values of the zeta function.

3.4 Euler-Maclaurin Summation Formula

We will end this section with a result that Euler and Maclaurin found independently.

Theorem 3.3 (Euler-Maclaurin Summation formula)**.** *Let a and b be integers satisfying $a \leq b$ and let $M \in \mathbb{N}$. Suppose $f(x)$ is an M times continuously differentiable function on $[a, b]$. Then we have:*

$$\begin{aligned}\sum_{n=a}^{b} f(n) = \int_a^b f(x)\mathrm{d}x + \frac{1}{2}(f(a) + f(b)) + \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!}\left(f^{(k)}(b) - f^{(k)}(a)\right) \\ - \frac{(-1)^M}{M!}\int_a^b B_M(x - \lfloor x \rfloor) f^{(M)}(x)\mathrm{d}x.\end{aligned} \tag{3.13}$$

Whenever $M = 1$, the sum on the right is understood to be 0.

Remark. *We can regard $f(n)$ as the area of a rectangle whose base is in the interval $[n - \frac{1}{2}, n + \frac{1}{2}]$ and whose height is $f(n)$. Then we can approximate the difference between this sum and the integral by higher order derivatives at the end points. To make the approximation exact we then introduce an explicit error term.*

Proof. Let $g(x)$ be an M times continuously differentiable function on $[0, 1]$. Since $B_1(x) = x - \frac{1}{2}$, we know $B_1'(x) = 1$, so we can apply integration by parts to $g(x)$ as follows:

$$\begin{aligned}\int_0^1 g(x)\mathrm{d}x &= [B_1(x)g(x)]_0^1 - \int_0^1 B_1(x)g'(x)\mathrm{d}x \\ &= \frac{1}{2}\left(g(1) + g(0)\right) - \int_0^1 B_1(x)g'(x)\mathrm{d}x.\end{aligned}$$

By lemma 3.1 we can rewrite $B_k(x) = B_{k+1}'(x)/(k+1)$, if we apply this repeatedly we get:

$$\begin{aligned}\int_0^1 g(x)\mathrm{d}x =& \frac{1}{2}(g(1) + g(0)) - \frac{1}{2}[B_2(x)g'(x)]_0^1 + \frac{1}{2}\int_0^1 B_2(x)g''(x)\mathrm{d}x \\ =& \frac{1}{2}(g(1) + g(0)) - \frac{1}{2}[B_2(x)g'(x)]_0^1 + \frac{1}{2\cdot 3}[B_3(x)g''(x)]_0^1 - \frac{1}{2\cdot 3}\int_0^1 B_3(x)g'''(x)\mathrm{d}x \\ &\dots \\ =& \frac{1}{2}(g(1) + g(0)) + \sum_{k=1}^{M-1} \frac{(-1)^k}{(k+1)!}[B_{k+1}(x)g^{(k)}(x)]_0^1 + \frac{(-1)^M}{M!}\int_0^1 B_M(x)g^{(M)}(x)\mathrm{d}x.\end{aligned}$$

By lemma 3.1 we know that for $k \geq 2$, $B_k(0) = B_k(1) = B_k$, and by corollary 3.1 we know that for $n \in \mathbb{N}$, $(-1)^k B_k = B_k$, so we can rewrite the above to:

$$\int_0^1 g(x)\mathrm{d}x = \frac{1}{2}(g(1) + g(0)) - \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!}\left(g^{(k)}(1) - g^{(k)}(0)\right) + \frac{(-1)^M}{M!}\int_0^1 B_M(x)g^{(M)}(x)\mathrm{d}x,$$

and finally

$$\frac{1}{2}(g(1) + g(0)) = \int_0^1 g(x)\mathrm{d}x + \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!}\left(g^{(k)}(1) - g^{(k)}(0)\right) - \frac{(-1)^M}{M!}\int_0^1 B_M(x)g^{(M)}(x)\mathrm{d}x.$$

Now let $n \in \mathbb{N}$ be such that $a \leq n \leq b-1$, then if we write $f(x+n)$ for $g(x)$, and we note that whenever $n \leq x \leq n+1$ we have $B_M(x-n) = B_M(x - \lfloor x \rfloor)$, which was the periodic extension of $B_M(x)$ from $[0,1]$ to $\mathbb{R}$ by Theorem 3.2, we rewrite the statement to:

$$\begin{aligned}\frac{1}{2}(f(n+1)+f(n)) = \int_n^{n+1} f(x)\mathrm{d}x + \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!}\left(f^{(k)}(n+1) - f^{(k)}(n)\right) \\ - \frac{(-1)^M}{M!}\int_n^{n+1} B_M(x - \lfloor x \rfloor) f^{(M)}(x)\mathrm{d}x.\end{aligned}$$

We obtain the formula in the theorem by first adding up the above formula from $n = a$ to $n = b-1$:

$$\begin{aligned}\sum_{n=a}^{b-1}\left(\frac{1}{2}(f(n+1)+f(n))\right) = \sum_{n=a}^{b-1}\left(\int_n^{n+1} f(x)\mathrm{d}x\right) + \sum_{n=a}^{b-1}\left(\sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!}\left(f^{(k)}(n+1) - f^{(k)}(n)\right)\right) \\ - \sum_{n=a}^{b-1}\left(\frac{(-1)^M}{M!}\int_n^{n+1} B_M(x - \lfloor x \rfloor) f^{(M)}(x)\mathrm{d}x\right).\end{aligned}$$

Now since both sums are finite we can change their order, and we can also rewrite each sum integrals as one integral, so we obtain:

$$\begin{aligned}\frac{1}{2}f(a) + f(a+1) + \cdots + f(b-1) + \frac{1}{2}f(b) = \\ \int_a^b f(x)\mathrm{d}x + \sum_{k=1}^{M-1}\frac{B_{k+1}}{(k+1)!}\sum_{n=a}^{b-1}\left(f^{(k)}(n+1) - f^{(k)}(n)\right) - \frac{(-1)^M}{M!}\int_a^b B_M(x - \lfloor x \rfloor) f^{(M)}(x)\mathrm{d}x.\end{aligned}$$

Finally if we add $\frac{1}{2}(f(a)+f(b))$ to the left and right hand side, and recognise $\sum_{n=a}^{b-1} f^{(k)}(n+1) - f^{(k)}(n)$ as a telescoping series, and we are left with

$$\begin{aligned}\sum_{n=a}^{b} f(n) = \int_a^b f(x)\mathrm{d}x + \frac{1}{2}(f(a)+f(b)) + \sum_{k=1}^{M-1}\frac{B_{k+1}}{(k+1)!}(f^{(k)}(b) - f^{(k)}(a)) \\ - \frac{(-1)^M}{M!}\int_a^b B_M(x - \lfloor x \rfloor) f^{(M)}(x)\mathrm{d}x\end{aligned}$$

as desired. □

Euler used this summation formula to calculate the first few terms of $\sum_{n=1}^{\infty}\frac{1}{n^2}$ which most likely suggested to him the value of $\frac{\pi^2}{6}$ [7].

4 The analytic Character of $\zeta(s)$

While we have already considered $\zeta(s)$ for $\mathrm{Re}(s) > 1$, we will do so again using our newly found Euler-Maclaurin summation formula.
Consider for $s \in \mathbb{C}$, the sum:

$$\sum_{n=1}^{N} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \cdots + \frac{1}{N^s}.$$

Let $f : (0, \infty) \to \mathbb{C}$ be defined as

$$f(x) = x^{-s}.$$

We wish to use equation (3.13) for $f(x)$ with $a = 1$, $b = N$. Note that

$$\begin{aligned} f^{(k)}(x) &= (-s)(-s-1)(-s-2)\cdots(-s-k+1)x^{-s-k} \\ &= (-1)^k s(s+1)(s+2)\cdots(s+k-1)x^{-s-k} = (-1)^k (s)_k x^{-s-k}. \end{aligned}$$

Where $(s)_k = s(s+1)(s+2)\dots(s+k-1)$ is the **Pochhammer symbol**, it is sometimes also called the **rising factorial**.
It is easy to see that $(1)_k = k!$, which we will use later.
Additionally, whenever $s \neq 1$, we have

$$\int_1^N x^{-s}\mathrm{d}x = \left[\frac{1}{-s+1}x^{-s+1}\right]_1^N = \frac{-1}{s-1}\left(N^{-s+1} - 1\right) = \frac{1}{s-1}\left(1 - \frac{1}{N^{s-1}}\right).$$

We can substitute this into equation (3.13) to obtain:

$$\begin{aligned} \sum_{n=1}^{N} \frac{1}{n^s} =& \frac{1}{s-1}\left(1 - \frac{1}{N^{s-1}}\right) + \frac{1}{2}\left(1 - \frac{1}{N^s}\right) + \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!}(s)_k\left(1 - \frac{1}{N^{s+k}}\right) \\ & - \frac{(s)_M}{M!}\int_1^N B_M(x - \lfloor x \rfloor)x^{-s-M}\mathrm{d}x, \end{aligned} \tag{4.1}$$

where we use $(-1)^k B_{k+1} = -B_{k+1}$.
If $s = 1$, however, we know $\int_1^N x^{-1}\mathrm{d}x = \log(N)$ and so:

$$\sum_{n=1}^{N} \frac{1}{n} = \log(N) + \frac{1}{2}\left(1 + \frac{1}{N}\right) + \sum_{k=1}^{M-1} \frac{B_{k+1}}{k+1}\left(1 - \frac{1}{N^{1+k}}\right) - \int_1^N B_M(x - \lfloor x \rfloor)x^{-1-M}\mathrm{d}x. \tag{4.2}$$

Before we state the next result, for ease we formulate the following lemma:

Lemma 4.1. *For all $k \in \mathbb{R}$, the integral*

$$\int_1^N x^{-s-k}\mathrm{d}x$$

converges as $N \to \infty$, provided $\mathrm{Re}(s) > 1 - k$.

Proof.
$$\int_1^N x^{-s-k}\mathrm{d}x = \left[\frac{1}{-s-k+1}x^{-s-k+1}\right]_1^N = \frac{1}{s+k-1}\left(1-\frac{1}{N^{s+k-1}}\right).$$
If $\mathrm{Re}(s) > 1-k$, we have $\mathrm{Re}(s+k-1) = \mathrm{Re}(s)+k-1 > 0$, which is strict, and thus as $N \to \infty$ we have $1/(N^{s+k-1}) \to 0$. □

Which gives us the following corollary.

Corollary 4.1. *For all $M \in \mathbb{N}$, the integral*
$$\int_1^N B_M(x-\lfloor x\rfloor)x^{-s-M}\mathrm{d}x \tag{4.3}$$
converges as $N \to \infty$, provided $\mathrm{Re}(s) > 1-M$.

Proof. Note that $0 \le x - \lfloor x\rfloor \le 1$, and we know that B_M is a polynomial, which means that $B_M(x-\lfloor x\rfloor)$ is bounded for all M. This is because $0 \le \left|x^k\right| \le 1$ for $0 \le x \le 1$. Then let μ be the maximum of all coefficients in absolute value, then $|B_M(x-\lfloor x\rfloor)| \le M \cdot \mu$. This means we can bound the integral as follows:
$$\left|\int_1^N B_M(x-\lfloor x\rfloor)x^{-s-M}\mathrm{d}x\right| \le \int_1^N \left|B_M(x-\lfloor x\rfloor)x^{-s-M}\right|\mathrm{d}x \le M\cdot\mu\int_1^N x^{-\sigma-M}\mathrm{d}x.$$
Since by lemma 4.1 this integral converges as $N \to \infty$ for $\mathrm{Re}(s) > 1-M$, we know that the integral in equation (4.3) converges as well. □

As a direct consequence from these formulas, we obtain the following:

Proposition 4.1.

1. *The series* $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^s}$ *converges absolutely if* $\mathrm{Re}(s) > 1$, *and the series* $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ *diverges.*
2. *Let $\{a_N\}$ be the sequence defined by $a_N = \left(\sum_{n=1}^N \frac{1}{n}\right) - \log(N)$, then $\{a_N\}$ converges. The limit $\gamma = \lim_{N\to\infty} a_N$ is called the Euler-Mascheroni constant [6]. For any number $M \in \mathbb{N}$, we have*
$$\gamma = \sum_{k=0}^{M-1}\frac{B_{k+1}}{k+1} - \int_1^\infty \frac{B_M(x-\lfloor x\rfloor)}{x^{M+1}}\mathrm{d}x.$$
In particular, setting $M = 1$ we get
$$\gamma = 1 - \int_1^\infty \frac{x-\lfloor x\rfloor}{x^2}\mathrm{d}x.$$

Proof.

1. Let $\text{Re}(s) = \sigma$. Since $\sum_{n=1}^{N} \left| \frac{1}{n^s} \right| = \sum_{n=1}^{N} \frac{1}{n^\sigma}$, we will set $s = \sigma$ and $M = 1$ in equation (4.1), to get:
$$\sum_{n=1}^{N} \frac{1}{n^\sigma} = \int_1^N x^{-\sigma} \mathrm{d}x + \frac{1}{2}\left(1 + \frac{1}{N^\sigma}\right) - \sigma \int_1^N B_1(x - \lfloor x \rfloor) x^{-\sigma-1} \mathrm{d}x.$$
We can then solve the first integral and we are left with
$$\sum_{n=1}^{N} \frac{1}{n^\sigma} = \frac{1}{\sigma - 1}\left(1 - \frac{1}{N^{\sigma-1}}\right) + \frac{1}{2}\left(1 + \frac{1}{N^\sigma}\right) - \sigma \int_1^N B_1(x - \lfloor x \rfloor) x^{-\sigma-1} \mathrm{d}x. \tag{4.4}$$
If $\sigma > 1$, both $1/N^{\sigma-1}$ and $1/N^\sigma$ converge to 0 as $N \to \infty$. By corollary 4.1, the integral converges for $\sigma > 0$.
Since all terms on the left hand side of 4.4 converge for $\sigma > 1$, this means
$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s}$$
converges absolutely whenever $\sigma > 1$ (so whenever $\text{Re}(s) > 1$).
The fact that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges follows from taking the limit of $N \to \infty$ in equation (4.2). Every term is bounded, except for $\log(N)$, which we know grows to ∞ and thus we know the sum must diverge.

2. Rearranging equation (4.2) gives us
$$\sum_{n=1}^{N} \frac{1}{n} - \log(N) = \frac{1}{2}\left(1 + \frac{1}{N}\right) + \sum_{k=1}^{M-1} \frac{B_{k+1}}{k+1}\left(1 - \frac{1}{N^{1+k}}\right) - \int_1^N B_M(x - \lfloor x \rfloor) x^{-1-M} \mathrm{d}x, \tag{4.5}$$
where the left hand side is precisely a_N. Since $1 > 1 - M$ we can use corollary 4.1, with $s = 1$ to conclude that the integral in (4.5) also converges. Taking the limit on the left and right hand side gives us:
$$\gamma = \lim_{N\to\infty} a_N = \frac{1}{2} + \sum_{k=1}^{M-1} \frac{B_{k+1}}{k+1} - \int_1^\infty B_M(x - \lfloor x \rfloor) x^{1-M} \mathrm{d}x.$$
Since $B_{0+1}/(0+1) = B_1 = \frac{1}{2}$ we can put the loose term into the sum and are left with
$$\gamma = \sum_{k=0}^{M-1} \frac{B_{k+1}}{k+1} - \int_1^\infty B_M(x - \lfloor x \rfloor) x^{-1-M} \mathrm{d}x.$$
If $M = 1$, we have $B_1(x - \lfloor x \rfloor) = x - \lfloor x \rfloor - \frac{1}{2}$ and thus we get
$$\gamma = \frac{1}{2} - \int_1^\infty \frac{x - \lfloor x \rfloor - \frac{1}{2}}{x^2} \mathrm{d}x = \frac{1}{2} + \frac{1}{2}\int_1^\infty \frac{1}{x^2}\mathrm{d}x - \int_1^\infty \frac{x - \lfloor x \rfloor}{x^2}\mathrm{d}x$$
$$= \frac{1}{2} + \frac{1}{2}\left[-\frac{1}{x}\right]_1^\infty - \int_1^\infty \frac{x - \lfloor x \rfloor}{x^2}\mathrm{d}x = 1 - \int_1^\infty \frac{x - \lfloor x \rfloor}{x^2}\mathrm{d}x. \qquad \square$$

The constant γ has been the topic of study for many years. Over the years it has been linked to the gamma function, the digamma function, and Bessel functions, as well as Laplace and Mellin transforms.
It also appears in many number theoretic results, such as bounds to specific prime gaps and inequalities regarding Euler's totient function, among others. Despite its common occurrence and the continued interest into the number, it is currently still unknown whether γ is even irrational, and if so, if it is algebraic or transcendental [6].

If we now revisit definition 2.1 we can see clearly that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is well-defined for all complex numbers s such that $\mathrm{Re}(s) > 1$. While we already knew the sum converged, we have now shown some properties of the function $\zeta(s)$ as well.
Since $\zeta(s)$ converges absolutely for $\mathrm{Re}(s) > 1$, and it converges uniformly for $\mathrm{Re}(s) > 1 + \varepsilon$ where $\varepsilon > 0$, it must be analytic for $\mathrm{Re}(s) > 1$.

4.1 The Analytic Continuation

While the function $\zeta(s)$ for $\mathrm{Re}(s) > 1$ has been studied for a long time, we wish to extend our domain. If we can make sense of the behaviour of $\zeta(s)$ for $\mathrm{Re}(s) < 1$ as well, we can use this to deduce some interesting results later.
In order to do that, first of all suppose that $\mathrm{Re}(s) > 1$, if we use equation (4.1) and we let $N \to \infty$, we get

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{s-1} + \frac{1}{2} + \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!}(s)_k - \frac{(s)_M}{M!} \int_1^{\infty} B_M(x - \lfloor x \rfloor) x^{-s-M} \mathrm{d}x. \tag{4.6}$$

As we have described the zeta function in multiple ways, we can take the difference between equations (4.1) and (4.6), to get

$$\begin{aligned}
\zeta(s) - \sum_{n=1}^{N} \frac{1}{n^s} =& \frac{1}{s-1} \cdot \left(1 - \left(1 - \frac{1}{N^{s-1}}\right)\right) + \frac{1}{2} \cdot \left(1 - \left(1 - \frac{1}{N^s}\right)\right) \\
&+ \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!}(s)_k \cdot \left(1 - \left(1 - \frac{1}{N^{s+k}}\right)\right) \\
&- \frac{(s)_M}{M!} \cdot \left(\int_1^{\infty} B_M(x - \lfloor x \rfloor) x^{-s-M} \mathrm{d}x - \int_1^{N} B_M(x - \lfloor x \rfloor) x^{-s-M} \mathrm{d}x\right).
\end{aligned}$$

Simplifying some terms we are left with

$$\zeta(s) - \sum_{n=1}^{N} \frac{1}{n^s} = \frac{1}{s-1} \cdot \frac{1}{N^{s-1}} - \frac{1}{2N^s} + \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!}(s)_k \frac{1}{N^{s+k}} - \frac{(s)_M}{M!} \int_N^{\infty} B_M(x - \lfloor x \rfloor) x^{-s-M} \mathrm{d}x.$$

Then we can move the sum to the right hand side and we are left with another expression for $\zeta(s)$, namely

$$\begin{aligned}\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{1}{s-1}\frac{1}{N^{s-1}} - \frac{1}{2N^s} + \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!}\frac{(s)_k}{N^{s+k}} \\ - \frac{(s)_M}{M!}\int_N^\infty B_M(x-\lfloor x \rfloor)x^{-s-M}\mathrm{d}x.\end{aligned} \tag{4.7}$$

Now since N and M are simply natural numbers greater than 0, $(s)_j$ is a polynomial in terms of s, $B_M(x-\lfloor x\rfloor)$ is a polynomial in terms of x, and $s \neq 1$, all terms, except for possibly the integral, are clearly just analytic functions of s.
To show the integral is also analytic, we will use the following theorem, whose proof we will omit but can be found in the literature [4].

Theorem 4.1. *Let C be a path and let U be an open set. Let $\phi(s,x)$ be a function defined for $s \in U$ and $x \in C$. Suppose that $\phi(s,x)$ is continuous in $x \in C$, and analytic in $s \in U$, and that the complex derivative $\frac{\mathrm{d}\phi}{\mathrm{d}s}(s,x)$ is continuous in $x \in C$. Then the function*

$$g(s) = \int_C \phi(s,x)\mathrm{d}x$$

is analytic in U.

So then if U is the region where $\mathrm{Re}(s) > 1-M$, we know that U is open (since it does not contain its boundary). If we then let C be the path from N to the right along the real axis, we see that $\phi(s,x) = B_M(x-\lfloor x\rfloor)x^{-s-M}$ is continuous in $x \in C$, furthermore as $x > 0$ and we take the principal value of the logarithm, it is also entire in $s \in U$. Additionally we know that $\frac{\mathrm{d}\phi}{\mathrm{d}s} = -\ln(x)e^{(-s-M)\ln(x)}$, which is continuous in $x \in C$. We conclude by the theorem that the integral in equation (4.7) is an analytic function.
We need to be careful, however, that the integral does not diverge. But whenever $\mathrm{Re}(s) > 1-M$, we know by proposition 4.1 that the integral converges.

This means that we have constructed an analytic continuation of $\zeta(s)$ in the region $\mathrm{Re}(s) > 1-M$ through formula (4.7). Since we can put M to be any natural number, we have actually found an analytic continuation to the entire complex plane. But whenever $s = 1$ we run into problems because of the $1/(s-1)$ term, so we retain the divergence for $s = 1$.
Let us summarise this result in the following theorem, and also find some values of $\zeta(s)$:

Theorem 4.2.

1. *The function $\zeta(s)$ has an analytic continuation to the entire complex plane as an analytic function, except whenever $s = 1$, as it then has a pole of order 1, with residue 1.*

2. *Let $m \in \mathbb{N}$. The value of $\zeta(s)$ at $s = 1-m$ is given by*

$$\zeta(1-m) = -\frac{B_m}{m}.$$

Proof.

1. We have already seen this, except for the residue at $s = 1$. We are interested in finding $\lim_{s\to 1}(s-1)\zeta(s)$. Multiplying the right hand side with $(s-1)$ and taking the limit as $s \to 1$ in equation (4.6) we see that every term in the right hand side goes to 0 except for the first term, which becomes 1. So the residue is indeed 1.

2. Let $s = 1 - m$, setting $M \geq m$ in equation (4.7), then

$$(s)_M = (1-m)(1-m+1)\dots(1-m+M-1),$$

but since $M \geq m$, we know one of the terms has to be equal to $(1 - m + (m-1)) = 0$, and so the integral term vanishes. This means we can express $\zeta(1-m)$ as a finite sum.
Let now $s = 1 - m$ and $M = m$, then

$$\begin{aligned}\zeta(1-m) &= \sum_{n=1}^{N} n^{-(1-m)} + \frac{1}{1-m-1}N^{-(1-m-1)} - \frac{1}{2}N^{-(1-m)} \\ &+ \sum_{k=1}^{m-1} \frac{B_{k+1}}{(k+1)!}(1-m)_k N^{-(1-m+k)} \\ &= \sum_{n=1}^{N} n^{m-1} - \frac{N^m}{m} - \frac{N^{m-1}}{2} + \sum_{k=1}^{m-1} \frac{B_{k+1}}{(k+1)!}(1-m)_k N^{m-k-1}. \qquad (4.8)\end{aligned}$$

If $m = 1$, we have no summation, and we are left with

$$\zeta(0) = \sum_{n=1}^{N} 1 - N - \frac{1}{2} = -\frac{1}{2} = -\frac{B_1}{1}.$$

Suppose now that $m \geq 2$, such that

$$\begin{aligned}(1-m)_k &= (1-m)(2-m)\dots(k-m) = ((-1)(m-1))((-1)(m-2))\dots((-1)(m-k)) \\ &= (-1)^k (m-1)_k.\end{aligned}$$

Note that the sum runs from $k = 1$ to $k = m - 1$, and thus $m - k \geq 1$. So we can in fact use the normal factorial here as each term in $(m-1)_k$ is a positive integer. So $(1-m)_k = (-1)^k k!\binom{m-1}{k}$, which means that

$$\sum_{k=1}^{m-1} \frac{B_{k+1}}{(k+1)!}(1-m)_k N^{m-k-1} = \sum_{k=1}^{m-1} \frac{B_{k+1}}{k+1}(-1)^k \binom{m-1}{k} N^{m-k-1}.$$

Using the indentity

$$\frac{1}{k+1}\binom{m-1}{k} = \frac{(m-1)!}{(k+1)!(m-1-(k+1))!(m-1-k)} = \frac{1}{m-k-1}\binom{m-1}{k+1} \qquad (4.9)$$

as well as $(-1)^k B_k = B_k$ for $k \geq 2$, we see

$$\sum_{k=1}^{m-1} \frac{B_{k+1}}{k+1}(-1)^k \binom{m-1}{k} N^{m-k-1} = \sum_{k=1}^{m-2} (-1)^k B_{k+1} \binom{m-1}{k+1} \frac{N^{m-k-1}}{m-k-1} + (-1)^{m-1}\frac{B_m}{m},$$

where we take the $k = m-1$ term out of the sum. We can then change our index to start from $k = 2$ instead of $k = 1$, and reapply identity (4.9) so that we end up with

$$= \sum_{k=2}^{m-1} (-1)^{k-1} \binom{m-1}{k} B_k \frac{N^{m-k}}{m-k} + (-1)^{m-1} \frac{B_m}{m} = -\sum_{k=2}^{m-1} \binom{m-1}{k} B_k \frac{N^{m-k}}{m-k} - \frac{B_m}{m}.$$

Now since $B_0 = 1$ and $B_1 = \frac{1}{2}$, we get

$$\frac{N^m}{m} = \binom{m-1}{0} B_0 \frac{N^{m-0}}{m-0}, \qquad \text{and} \qquad \frac{N^{m-1}}{2} = \binom{m-1}{1} B_1 \frac{N^{m-1}}{m-1}.$$

So then equation (4.8) becomes

$$\begin{aligned}
\zeta(1-m) &= \sum_{n=1}^{N} n^{m-1} - \frac{N^m}{m} - \frac{N^{m-1}}{2} + \sum_{k=1}^{m-1} \frac{B_{k+1}}{(k+1)!} (1-m)_k N^{m-k-1} \\
&= \sum_{n=1}^{N} n^{m-1} - \frac{N^m}{m} - \frac{N^{m-1}}{2} - \sum_{k=2}^{m-1} \binom{m-1}{k} B_k \frac{N^{m-k}}{m-k} - \frac{B_m}{m} \\
&= \sum_{n=1}^{N} n^{m-1} - \sum_{k=0}^{m-1} \binom{m-1}{k} B_k \frac{N^{m-k}}{m-k} - \frac{B_m}{m}.
\end{aligned}$$

We can show[iii] that

$$\sum_{n=1}^{N} n^p = \sum_{k=0}^{p} \binom{p}{k} B_k \frac{N^{p+1-k}}{p+1-k}.$$

Setting $p = m-1$ we see that

$$\sum_{n=1}^{N} n^{m-1} = \sum_{k=0}^{m-1} \binom{m-1}{k} B_k \frac{N^{m-k}}{m-k}$$

So then (4.8) becomes

$$\zeta(1-m) = -\frac{B_m}{m}. \qquad \square$$

So we have found a beautifully simple expression for $\zeta(1-m)$. It is important to note here that the analytic continuation of $\zeta(s)$, which can be done in numerous ways, was one of the main contributions from Riemann (whom the function has been named after). This is also part of the reason why the function is even of interest to mathematicians. In general one could even argue that results about $\zeta(s)$ for $\mathrm{Re}(s) > 1$ are only noteworthy because of the importance of $\zeta(s)$ for $\mathrm{Re}(s) < 1$!

But that should not stop us from talking about $\zeta(s)$. In fact we can use the previous theorem to deduce the following infamous result:

$$\sum_{n=1}^{\infty} n = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \zeta(-1) = -\frac{B_2}{2} = -\frac{1}{12}.$$

[iii] See theorem A.2 in the appendix

If we take this at face value, we have now "shown" that the sum of all natural numbers, which is a sum of only positive terms, is equal to a negative number that is somewhat close to 0. This is of course not true.
What we need to realise at this point is that for $\text{Re}(s) > 1$ we have a (somewhat) nicely behaved function that is in essence an infinite sum. But beyond that region, our infinite sum does not converge. What we have done so far is that we have found an expression regarding $\zeta(s)$ only, which is a function that **coincides** with $\sum_{n=1}^{\infty} n^{-s}$ for $\text{Re}(s) > 1$. This means that outside that region we cannot express $\zeta(s)$ as an infinite sum (unless we are extremely cautious).

Recall that in corollary 3.2 we have shown that

$$\zeta(2k) = \frac{(-1)^{k-1}}{2} \frac{B_{2k}}{(2k)!} (2\pi)^{2k} \quad k \in \mathbb{N}$$

and we have just shown that

$$\zeta(1-m) = -\frac{B_m}{m} \quad m \in \mathbb{N}.$$

While it might be appealing to find out more closed-form solutions for $\zeta(k)$, where $k \in \mathbb{Z}$ or k is some other "nice" expression, this turns out to be extremely difficult. Famously, it took until 1978 for R. Apéry to prove that $\zeta(3)$ is an irrational number [17], and it is currently unknown whether it is a transcendental number or not. Because of his proof, $\zeta(3)$ is sometimes called **Apéry's constant**.
In 2000, T. Rivoal proved that there is an infinite number of odd integers k such that $\zeta(k)$ is irrational [18], and W. Zudilin showed in 2001 that certain sequences, such as $\{\zeta(5), \zeta(7), \zeta(9), \zeta(11)\}$, contain at least one irrational value [19]. It is currently unknown which specific ones are irrational, and research into values of $\zeta(s)$ is ongoing.
Interestingly Euler thought that for odd s $\zeta(s)$ would be some function of $\ln(2)$, but there does not seem to be any evidence for this claim [7].

4.2 The Functional Equation

Riemann's other very important result is the following, which we call the **functional equation**.

Theorem 4.3 (The functional equation of the Riemann zeta function)**.** *For all $s \in \mathbb{C}$, $s \neq 1$, we have*

$$\zeta(s) = -2^s s \pi^{s-1} \Gamma(-s) \sin\left(\frac{\pi s}{2}\right) \zeta(1-s). \tag{4.10}$$

Remark. *A detailed description of $\Gamma(z)$, which is called the **Gamma function** is given in the appendix, it suffices for now to recognise that*

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \mathrm{d}t.$$

Proof. If we put $N = M = 1$ in equation (4.7), we get

$$\zeta(s) = \sum_{n=1}^{1} n^{-s} + \frac{1}{s-1} - \frac{1}{2} - s \int_1^{\infty} B_1(x - \lfloor x \rfloor) x^{-s-1} \mathrm{d}x$$

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} - s \int_1^{\infty} \frac{x - \lfloor x \rfloor - \frac{1}{2}}{x^{s+1}} \mathrm{d}x. \tag{4.11}$$

We want to know when the integral converges. Write

$$f(x) = x - \lfloor x \rfloor - \frac{1}{2} \quad \text{and} \quad f_1(x) = \int_1^x f(y)\mathrm{d}y,$$

then for any integer k we know that

$$\int_k^{k+1} f(y)\mathrm{d}y = \int_0^1 y - \frac{1}{2}\mathrm{d}y = \left[\frac{1}{2}y^2\right]_0^1 - \frac{1}{2} = 0.$$

This means that $f_1(x)$ is bounded, so then we can integrate by parts to find

$$\int_{x_1}^{x_2} \frac{f(x)}{x^{s+1}}\mathrm{d}x = \left[\frac{f_1(x)}{x^{s+1}}\right]_{x_1}^{x_2} + (s+1)\int_{x_1}^{x_2} \frac{f_1(x)}{x^{s+2}}\mathrm{d}x, \tag{4.12}$$

since $\mathrm{d}f_1(x)/\mathrm{d}x = f(x)$.
Now if we first let $x_2 \to \infty$, then we see that everything remains bounded, since $f_1(x)$ is dominated by x^{s+2} for $\mathrm{Re}(s) > -1$ and so the integral still converges.
If we then let $x_1 \to \infty$, we see that for $\mathrm{Re}(s) > -1$ equation (4.12) tends to 0. So we conclude that the integral in equation (4.11) must be convergent.
Now as $\lfloor x \rfloor = 0$ for $0 < x < 1$, we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\left[-s\int_\varepsilon^1 \frac{x - \lfloor x \rfloor - \frac{1}{2}}{x^{s+1}}\mathrm{d}x\right] = \lim_{\varepsilon\downarrow 0}\left[-s\left(\int_\varepsilon^1 x^{-s}\mathrm{d}x - \frac{1}{2}\int_\varepsilon^1 x^{-s-1}\mathrm{d}x\right)\right] \\
=&\lim_{\varepsilon\downarrow 0}\left[-s\left(\left[\frac{1}{-s+1}x^{-s+1}\right]_\varepsilon^1 - \frac{1}{2}\left[\frac{1}{-s}x^{-s}\right]_\varepsilon^1\right)\right] = -s\left(\frac{1}{-s+1} - \frac{1}{2}\frac{1}{-s}\right) \\
=&\frac{s}{s-1} - \frac{1}{2} = \frac{s - \frac{1}{2}(s-1)}{s-1} = \frac{1}{2}\frac{s+1}{s-1} = \frac{1}{2}\left(\frac{s-1}{s-1} + \frac{2}{s-1}\right) \\
=&\frac{1}{s-1} + \frac{1}{2}.
\end{aligned}$$

We have to be extremely careful since 0^{-s+1} is not well defined, which is why we take the limit along the real axis.
For $\varepsilon \downarrow 0$, we know that $\lim_{\varepsilon\downarrow 0}\mathrm{Log}(\varepsilon) = \lim_{\varepsilon\downarrow 0}\ln|\varepsilon| + i\mathrm{Arg}(\varepsilon)$. While we cannot say anything interesting about $\mathrm{Arg}(\varepsilon)$, we know that $\lim_{r\downarrow 0}\ln(r) = -\infty$, and thus if $\mathrm{Re}(-s+1) > 0$, or equivalently $\mathrm{Re}(s) < 1$, we know that $\lim_{\varepsilon\downarrow 0}\varepsilon^{-s+1} = \lim_{\varepsilon\downarrow 0} e^{(-s+1)\mathrm{Log}(\varepsilon)} = 0$.
We can apply the same steps to $\lim_{\varepsilon\downarrow 0}\varepsilon^{-s}$, and conclude that whenever $\mathrm{Re}(s) < 0$ we can rewrite equation (4.11) to

$$\zeta(s) = -s\int_0^\infty \frac{x - \lfloor x \rfloor - \frac{1}{2}}{x^{s+1}}\mathrm{d}x \qquad (\text{for } -1 < \mathrm{Re}(s) < 0). \tag{4.13}$$

Now by equation (3.12) we know that

$$x - \lfloor x \rfloor - \frac{1}{2} = -\sum_{n=1}^{\infty}\frac{\sin(2\pi n x)}{n\pi}$$

almost everywhere, as we have convergence everywhere but on $\mathbb{Z}$, and thus the set where the series does not converge has measure $\mu(\mathbb{Z}) = 0$.
If we substitute this series into equation (4.13), and we integrate term by term we get

$$\begin{aligned}
\zeta(s) &= -s\int_0^\infty -\sum_{n=1}^\infty \frac{\sin(2\pi nx)}{n\pi x^{s+1}}\mathrm{d}x \\
&= \frac{s}{\pi}\sum_{n=1}^\infty \frac{1}{n}\int_0^\infty \frac{\sin(2\pi nx)}{x^{s+1}}\mathrm{d}x \quad \text{(substitute } y = 2\pi nx) \\
&= \frac{s}{\pi}\sum_{n=1}^\infty \frac{(2n\pi)^s}{n}\int_0^\infty \frac{\sin(y)}{y^{s+1}}\mathrm{d}y = \frac{s}{\pi}(2\pi)^s\sum_{n=1}^\infty \frac{1}{n^{1-s}}\int_0^\infty \frac{\sin y}{y^{s+1}}\mathrm{d}y.
\end{aligned}$$

Now we should recognise the sum as $\zeta(1-s)$, and we can use the fact[iv] that

$$\int_0^\infty \frac{\sin y}{y^\alpha}\mathrm{d}y = \Gamma(1-\alpha)\cos\left(\frac{\alpha\pi}{2}\right) \qquad (\text{for } 0 < \mathrm{Re}(\alpha) < 1),$$

which we can use since $-1 < \mathrm{Re}(s) < 0$. And so we are left with

$$\zeta(s) = \frac{s}{\pi}(2\pi)^s\zeta(-s+1)\Gamma(1-(s+1))\cos\left(\frac{(s+1)\pi}{2}\right).$$

Now since $\cos(\alpha + \pi/2) = -\sin(\alpha)$, after simplifying and rearranging we are left with

$$\zeta(s) = -2^s s\pi^{s-1}\Gamma(-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s),$$

as desired.
We do need to justify the interchanging of integration and summation, however.
Firstly, by proposition 3.1 we know that our series is boundedly convergent on $\mathbb{R}\backslash\mathbb{Z}$, which means that there exists an $M > 0$ such that

$$|s_n(x)| = \left|\sum_{k=1}^n \sin(2\pi kx)/\pi k\right| \leq M.$$

Then by the dominated convergence theorem, since $M/x^{\sigma+1}$ is integrable on any bounded interval, we can interchange integration and summation there. We want to show that

$$\lim_{\lambda\to\infty}\sum_{n=1}^\infty \frac{1}{n}\int_\lambda^\infty \frac{\sin(2\pi nx)}{x^{s+1}}\mathrm{d}x = 0 \quad \text{whenever } -1 \leq \mathrm{Re}(s) < 0,$$

because then we know that the tail of our integral goes to 0. And thus for every $\varepsilon > 0$, we can find λ such that $\left|\int_0^\lambda s_n(x)/x^{s+1}\mathrm{d}x - \int_0^\infty s_n(x)/x^{s+1}\mathrm{d}x\right| < \varepsilon$.
We write $\mathrm{Re}(s) = \sigma$, then we get

$$\begin{aligned}
\int_\lambda^\infty \frac{\sin(2\pi nx)}{x^{s+1}}\mathrm{d}x &= \left(\left[-\frac{\cos(2\pi nx)}{2n\pi x^{s+1}}\right]_\lambda^\infty - \frac{s+1}{2n\pi}\int_\lambda^\infty \frac{\cos(2\pi nx)}{x^{s+2}}\mathrm{d}x\right) \\
&= \mathcal{O}\left(\frac{1}{n\lambda^{\sigma+1}}\right) + \mathcal{O}\left(\frac{1}{n}\int_\lambda^\infty \frac{1}{x^{\sigma+2}}\mathrm{d}x\right) = \mathcal{O}\left(\frac{1}{n\lambda^{\sigma+1}}\right),
\end{aligned}$$

[iv] See lemma A.3 in the appendix

where we use the big O notation since we want to know the behaviour as λ grows. We know that all constants get dominated, and we can bound the trigonometric function so indeed our bound is true.
Finally we see that

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{\lambda}^{\infty} \frac{\sin(2\pi nx)}{x^{s+1}} = \mathcal{O}\left(\frac{1}{\lambda^{\sigma+1}} \sum_{n=1}^{\infty} \frac{1}{n^2}\right) = \mathcal{O}\left(\frac{1}{\lambda^{\sigma+1}}\right).$$

Taking the limit as $\lambda \to \infty$, we see that the tail of our integral indeed vanishes for $\sigma > -1$. We conclude that we can indeed interchange summation and integration on an infinite domain for $-1 < \mathrm{Re}(s) < 0$, and so all the steps we took were sound, and thus our functional equation holds. □

While we needed to ensure that $-1 < \mathrm{Re}(s) < 0$ in our proof, the right hand side of (4.10) is analytic whenever $\mathrm{Re}(s) < 0$, and so we can use the functional equation also as an analytic continuation of $\zeta(s)$ for $\mathrm{Re}(s) < 0$.
The functional equation has given us the possibility to study the zeta function in detail. While analytic continuation tells us the function must exist somewhere, we would still have a difficult time studying its properties. By making use of the functional equation however we can find those properties much easier.

4.3 Related Functions and Expressions

While this functional equation is essentially the "standard" functional equation, there are many equivalent statements which are all some type of functional equation. If we take our functional equation as in (4.10),

$$\zeta(s) = -2^s s \pi^{s-1} \Gamma(-s) \sin\left(\frac{\pi s}{2}\right) \zeta(1-s),$$

and we set $s = 1 - s$, we get

$$\begin{aligned} \zeta(1-s) &= -2^{1-s}(1-s)\pi^{1-s-1}\Gamma(-(1-s)) \sin\left(\frac{\pi(1-s)}{2}\right) \zeta(s) \\ &= 2^{1-s}\pi^{-s}(s-1)\Gamma(s-1) \sin\left(-\frac{\pi s}{2} + \frac{\pi}{2}\right) \zeta(s). \end{aligned}$$

Then since $z\Gamma(z) = \Gamma(z+1)$ and $\sin(\alpha - \pi/2) = -\cos(\alpha)$, we can rewrite the above to

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s), \tag{4.14}$$

which is another common functional equation for the Riemann zeta function.
There are also many closely related functions to the Riemann zeta function, such as the Riemann xi function:

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right) \zeta(s). \tag{4.15}$$

This function was defined in order to have a simple functional equation, indeed, we can see that

$$\xi(s) = \xi(1-s). \tag{4.16}$$

To show this we will make use of the following identities, whose proofs can be found in the appendix:

(A) $\Gamma(z+1) = z\Gamma(z)$,

(B) $\Gamma(z) = \dfrac{2^{z-1}}{\sqrt{\pi}}\Gamma\left(\dfrac{z}{2}\right)\Gamma\left(\dfrac{z+1}{2}\right)$ (Legendre's duplication formula A.2),

(C) $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$ (Euler's reflection formula A.1).

Now if we consider that

$$\begin{aligned}\xi(1-s) &= \frac{1}{2}(1-s)((1-s)-1)\pi^{-\frac{1}{2}(1-s)}\Gamma\left(\frac{1}{2}(1-s)\right)\zeta(1-s)\\ &= \frac{1}{2}(1-s)(-s)\pi^{\frac{1}{2}s-\frac{1}{2}}\Gamma\left(\frac{1}{2}(1-s)\right)\frac{1}{-2^s s\pi^{s-1}\Gamma(-s)\sin(\pi s/2)}\zeta(s),\end{aligned}$$

where we use the functional equation for $\zeta(1-s)$ as given in equation (4.10). Then rearranging terms we get

$$\xi(1-s) = 2^{-s-1}(1-s)\pi^{-\frac{1}{2}s+\frac{1}{2}}\frac{\Gamma\left(\frac{1}{2}(1-s)\right)}{\Gamma(-s)}\frac{\zeta(s)}{\sin(\pi s/2)}.$$

If we then use identity (C) with $z=-s$, we get $\Gamma(-s) = -\pi/(\Gamma(1+s)\sin(\pi s))$, such that

$$\begin{aligned}\xi(1-s) &= 2^{-s-1}(1-s)\pi^{-\frac{1}{2}s+\frac{1}{2}}\frac{\Gamma\left(\frac{1}{2}(1-s)\right)}{-\pi/(\sin(\pi s)\Gamma(1+s))}\frac{\zeta(s)}{\sin(\pi s/2)}\\ &= 2^{-s-1}(s-1)\pi^{-\frac{1}{2}s-\frac{1}{2}}\Gamma\left(\frac{1}{2}(1-s)\right)\Gamma(1+s)\frac{\sin(\pi s)\zeta(s)}{\sin(\pi s/2)}.\end{aligned} \tag{4.17}$$

Applying identity (B) with $z = 1+s$ to $\Gamma(1+s)$, then we obtain

$$\begin{aligned}\Gamma\left(\frac{1}{2}(1-s)\right)\Gamma(1+s) =& \Gamma\left(\frac{1}{2}(1-s)\right)2^s\pi^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}(1+s)\right)\Gamma\left(\frac{1}{2}s+1\right)\\ =& 2^s\pi^{-\frac{1}{2}}\Gamma\left(1-\frac{1}{2}(1+s)\right)\Gamma\left(\frac{1}{2}(1+s)\right)\left(\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}s\right),\end{aligned}$$

where we use identity (A) at the end. Then we can again apply identity (C) with $z = \frac{1}{2}(1+s)$ so that we are left with

$$\Gamma\left(\frac{1}{2}(1-s)\right)\Gamma(1+s) = 2^{s-1}\pi^{\frac{1}{2}}s\Gamma\left(\frac{1}{2}s\right)\frac{1}{\sin\left(\frac{\pi s}{2}+\frac{\pi}{2}\right)} = 2^{s-1}\pi^{\frac{1}{2}}s\Gamma\left(\frac{1}{2}s\right)\frac{1}{\cos\left(\frac{\pi s}{2}\right)},$$

since $\sin(\alpha+\frac{\pi}{2}) = \cos(\alpha)$. Now substituting everything into equation (4.17), and recognising that $\sin(2\alpha)/(\sin(\alpha)\cos(\alpha)) = 2$, we finally obtain

$$\xi(1-s) = 2^{-2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)2\zeta(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \xi(s),$$

which is what we set out to prove.
There is also the "uppercase" xi function:

$$\begin{aligned}\Xi(z) = \xi\left(\frac{1}{2}+iz\right) &= \frac{1}{2}\left(\frac{1}{2}+iz\right)\left(-\frac{1}{2}+iz\right)\pi^{-\frac{1}{4}-\frac{1}{2}iz}\Gamma\left(\frac{1}{4}+\frac{1}{2}iz\right)\zeta\left(\frac{1}{2}+iz\right)\\ &= -\frac{1}{2}\left(z^2+\frac{1}{4}\right)\pi^{-\frac{1}{4}-\frac{1}{2}iz}\Gamma\left(\frac{1}{4}+\frac{1}{2}iz\right)\zeta\left(\frac{1}{2}+iz\right).\end{aligned} \tag{4.18}$$

It was in fact this function that Riemann studied the most.
From the definition, and from the functional equation for $\xi(s)$, (4.16), it is not hard to see that

$$\Xi(-z) = \xi(\frac{1}{2} - iz) = \xi(1 - (\frac{1}{2} + iz)) = \xi(\frac{1}{2} + iz) = \Xi(z), \tag{4.19}$$

which is of course precisely the statement that $\Xi(z)$ is an even function.
Now we first make the observation that for $t > 1$, $\zeta(t)$ must be real since the partial sums form a Cauchy sequence in $\mathbb{R}$, and $\mathbb{R}$ is complete so $\zeta(t)$ must converge to a value in $\mathbb{R}$. Then if we reconsider our second functional equation (4.14),

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s),$$

we see that if we let t be some real number in the interval $(1, 2)$, such that $1 - t \in (0, 1)$, then $\zeta(1-t) = 2^{1-t}\pi^{-t}\Gamma(t)\cos(\pi t/2)\zeta(t)$ must be in $\mathbb{R}$ too.
Furthermore $\zeta(0) = -\frac{1}{2}$ by theorem 4.2. This means that we can use our functional equation to deduce that $\zeta(t)$ is in fact real for all $t \in \mathbb{R}$, except for the pole at $s = 1$.
From this it follows that $\xi(t) \in \mathbb{R}$ whenever $t \in \mathbb{R}$ also, as all terms in its definition are real for real t too. The astute reader may recognise that $\Gamma(s)$ has poles, but we will see later that these get cancelled by the zeroes of the other terms, and thus our conclusion is sound.
Now to prove more results about $\Xi(s)$, we need the following lemma.

Lemma 4.2. *Suppose that $f(z)$ is a function that is analytic on Ω, where Ω contains an interval that coincides with the real axis. If $f(x)$ is real whenever $x \in \Omega \cap \mathbb{R}$, then $f(\overline{z}) = \overline{f(z)}$ for $z \in \Omega$.*

Proof. Let $f(z)$ as in the theorem, and define $h(z) = \overline{f(\overline{z})}$. Now if we write $f(x+iy) = u(x,y) + iv(x,y)$, then we can write $h(x+iy) = u(x,-y) - iv(x,-y)$. Write $U(x,y) = u(x,-y)$ and $V(x,y) = -v(x,-y)$ such that $h(z) = U(x,y) + iV(x,y)$. Now we check the Cauchy-Riemann equations. Note that $f(z)$ is analytic on Ω and so u and v obey the Cauchy-Riemann equations. Then

$$\frac{\partial U}{\partial x} = u_x(x,-y) = v_y(x,-y) = \frac{\partial(-v(x,-y))}{\partial y} = \frac{\partial V}{\partial y},$$

and

$$\frac{\partial U}{\partial y} = \frac{\partial u(x,-y)}{\partial y} = -u_y(x,-y) = v_x(x,-y) = -\frac{\partial V}{\partial x}.$$

So we conclude that $h(z)$ is also analytic on Ω. Then let $x \in \mathbb{R} \cap \Omega$, then as we assumed that $f(x) \in \mathbb{R}$ as $x \in \mathbb{R}$, we know that $v(x,0) = 0$, which means that $h(x) = u(x,0) - iv(x,0) = f(x)$ for $x \in \mathbb{R}$. So by the identity principle we know that $h(z) = f(z)$ on Ω. This means that $\overline{f(\overline{z})} = f(z)$ and thus $f(\overline{z}) = \overline{f(z)}$. ☐

Since $\zeta(s)$ is analytic on $\mathbb{C}\backslash\{1\}$, and we know by theorem 4.2 $s = 1$ is a simple pole, which gets cancelled against the simple zero of the $s - 1$ term in the definition of $\xi(s)$, we conclude that $\xi(s)$ is entire. So lemma 4.2 applies to $\xi(s)$ such that we get

$$\overline{\xi\left(\frac{1}{2} + it\right)} = \xi\left(\overline{\frac{1}{2} + it}\right) = \xi\left(\frac{1}{2} - it\right),$$

and since $\xi(1-s) = \xi(s)$ we also know that

$$\xi\left(\frac{1}{2} - it\right) = \xi\left(1 - (\frac{1}{2} + it)\right) = \xi\left(\frac{1}{2} + it\right).$$

Finally we obtain that

$$\overline{\Xi(t)} = \overline{\xi\left(\frac{1}{2} + it\right)} = \xi\left(\frac{1}{2} - it\right) = \xi\left(\frac{1}{2} + it\right) = \Xi(t),$$

which means that the imaginary part of $\xi(\frac{1}{2} + it) = \Xi(t)$ must be 0 for real t. We summarise the result in the following corollary.

Corollary 4.2. *$\zeta(t)$, $\xi(t)$, and $\Xi(t)$ are real for real t.*

We will use this result to prove a result about the number of zeroes later.

5 Zeroes of the Zeta Function

Up until now we have analysed the Riemann zeta function and described its properties. We have also considered related functions and theory. The question remains, however: what do we do this for?
If we reconsider lemma 2.2 we see that there exists a link between the prime numbers and our zeta function. But this is no mere coincidence, in fact, this link is a core property of our zeta function. Mathematicians have used the zeta function as a tool to analyse the prime numbers for years. While we will discuss this later, we first shall make some observations and deduce some results.

5.1 The Critical Strip

If we reconsider lemma 2.2, namely that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ is prime}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\mathrm{Re}(s) > 1$, we should see that none of the terms on the right hand side are equal to 0 since each $1 - 1/p^s$ term is bounded from below by $1/2$, and from above by $3/2$. Then as p grows these terms tend to 1, and so we can make the following observation.

Observation. *$\zeta(s)$ has no zeroes on $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$.*

Furthermore, from theorem 4.2, we know that for $m \in \mathbb{N}$, we have $\zeta(1-m) = -\frac{B_m}{m}$. Since by corollary 3.1 $B_{2k+1} = 0$ for $k \in \mathbb{N}$, we conclude that $\zeta(1-(2k+1)) = \zeta(-2k)$ must be zero.

Observation. *$\zeta(-2k) = 0$ for all $k \in \mathbb{N}$.*

We call these zeroes the **trivial zeroes** of the Riemann zeta function.
Now we state an important consequence of these two observations, namely:

Theorem 5.1. *$\zeta(s)$ has no zeroes outside the strip $0 \le \mathrm{Re}(s) \le 1$, except for the trivial zeroes at $s = -2, -4, -6, \ldots$.*

Proof. Consider again the functional equation

$$\zeta(s) = -2^s s \pi^{s-1} \Gamma(-s) \sin\left(\frac{\pi s}{2}\right) \zeta(1-s).$$

Let us consider each term on the right hand side. It is not hard to see that both π^{s-1} and 2^s have no poles, nor any zeroes.
Next we consider $\Gamma(z)$, which has no zeroes, and simple poles for $z = 0, -1, -2, \ldots$.
Finally we consider $\sin\left(\frac{\pi s}{2}\right)$. Clearly $s = \ldots, -4, -2, 0, 2, 4, \ldots$ are all zeroes, and if we look at the Taylor series for the sine, they must be of order 1.
For $s = 2, 4, \ldots$ we know by the first observation above, that these cannot be zeroes of $\zeta(s)$, so what is happening? Well, while they might be (simple) zeroes of the sine term, they are (simple) poles of the gamma term, and thus they get cancelled. So indeed these do not correspond to zeroes of the zeta function.

Now consider in particular $s = 0$. Since $\zeta(1)$ is also a pole of order 1 by theorem 4.2, then on a small enough neighbourhood around $s = 0$, we can rewrite each term as follows:

$$\Gamma(-s) = \frac{\phi(s)}{s}, \qquad \sin(s) = s\lambda(s), \qquad \zeta(1-s) = \frac{\theta(s)}{s},$$

where $\phi(s)$, $\lambda(s)$, $\theta(s)$ are analytic and non-zero on that neighbourhood. Then we see that

$$\lim_{s\to 0} \zeta(s) = \lim_{s\to 0} -2^s s\pi^{s-1}\frac{\phi(s)}{s} s\lambda(s)\frac{\theta(s)}{s} = \lim_{s\to 0} -2^s\pi^{s-1}\phi(s)\lambda(s)\theta(s) \neq 0.$$

So $s = 0$ is not a zero of $\zeta(s)$.
Finally consider some $z_0 = \sigma + it$, such that $\sigma < 0$, but $z_0 \neq \ldots, -4, -2, 0, 2, 4, \ldots$. Assume that this is a zero of $\zeta(s)$. Then by the functional equation

$$\begin{aligned} \zeta(z_0) &= -2^{z_0} z_0 \pi^{z_0 - 1}\Gamma(-z_0)\sin\left(\frac{\pi z_0}{2}\right)\zeta(1 - z_0) \\ 0 &= -2^{z_0} z_0 \pi^{z_0 - 1}\Gamma(-z_0)\sin\left(\frac{\pi z_0}{2}\right)\zeta(1 - \sigma - it). \end{aligned}$$

Since we know that $\Gamma(-z_0) \neq 0$, and that $\sin(\frac{1}{2}\pi z_0) \neq 0$, we conclude that then $\zeta(1 - \sigma - it) = 0$, but since $\sigma < 0$, we know that $\text{Re}(1 - \sigma - it) = 1 - \sigma > 1$, so we have found a new zero with real part greater than 0. But this contradicts our first observation. We conclude no such z_0 can exist. Then we have, as desired, that all the zeroes outside the strip $0 \leq \text{Re}(s) \leq 1$ must be the trivial zeroes. □

Remark. *We call the strip $0 \leq \sigma \leq 1$ the **critical strip**, since any non-trivial zero must lie there.*

As a small intermezzo, the moniker "trivial zero" is quite interesting. One could argue that it fits since finding these zeroes was quite easy (i.e. trivial), or one might argue that since they are zeroes because of the sine term, they do not shed much light on the properties of $\zeta(s)$ at all and thus are uninteresting (i.e. trivial). There is, however, a third and more mathematical reason.
If we look back to our Riemann xi function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s),$$

it is not hard to see that our trivial zeroes of $\zeta(s)$, are **not** zeroes of $\xi(s)$. Indeed, as we know that our trivial zeroes are simple, since they are simple zeroes of the sine, and $\Gamma(\frac{1}{2}s)$ has a simple pole at these values, these must cancel. It is in fact not hard to see that the simple zero of the s term is also cancelled by the simple pole of $\Gamma(0)$, and the simple zero of $(1 - s)$ in turn gets cancelled by the simple pole of $\zeta(1)$.
We conclude that any zero of $\xi(s)$ must be a *non-trivial zero* of $\zeta(s)$.

Now that we have finally set our eyes on the critical strip, if we reconsider the functional equations, we see that we can always relate expressions of $\zeta(s)$ to $\zeta(1 - s)$. If for $s = \sigma + it$ we have $\sigma = \frac{1}{2}$, we see that the real part of s and $1 - s$ is the same. It almost seems like the line $s = \frac{1}{2} + it$ is some kind of line of symmetry for $\zeta(s)$.
One could hazard a guess, or hypothesise, that the zeroes of $\zeta(s)$ might only lie on this line, and that is precisely what Riemann thought in 1859:

Hypothesis 5.1 (The Riemann Hypothesis). *All non-trivial zeroes of the Riemann zeta function lie on the line* $s = \frac{1}{2} + it$.

Let us now prove some facts about these zeroes. One may ask themselves the question: "How many non-trivial zeroes does the zeta function have?" This turns out to be infinitely many. In order to prove this, we need to use the following two theorems whose proofs we omit but can be found in the literature [16].

Theorem 5.2 (Borel-Carathéodory). *Let* f *be analytic on a closed disc of radius* R*, centred at the origin. Let for* $r < R$, $\|f\|_r = \max_{|z|=r}\{|f(z)|\}$. *Then*

$$\|f\|_r \leq \frac{2r}{R-r} \sup_{|z|=R} \operatorname{Re}(f) + \frac{R+r}{R-r} |f(0)| .$$

The above theorem follows from applying the maximum modulus principle to the function $h(z) = f(z) - f(0)$.

Theorem 5.3 (Hadamard's three-circles theorem). *Let* f *be analytic on a closed annulus* $0 < r_1 < |z| < r_2$. *Write* $r = |z|$, *then let*

$$\alpha = \frac{\ln(r) - \ln(r_1)}{\ln(r_2) - \ln(r_1)}.$$

Write $M(r) = M_f(r) = \|f\|_r = \max |f(z)|$ *for* $|z| = r$, *then*

$$\ln(M(r)) \leq (1-\alpha)\ln(M(r_1)) + \alpha \ln(M(r_2)).$$

Which follows from the fact that $f(z)$ can only be zero on an entire circle if it is zero everywhere, and the fact that the logarithm of the maximum is a convex function. The theorem is essentially taking a convex combination of logarithms of maxima.
We will use these theorems to prove the following result.

Theorem 5.4. *The function* $\zeta(s)$ *has infinitely many non-real zeroes.*

Proof. First consider

$$\zeta(2) - 1 = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots < \frac{1}{2\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4\cdot 5} + \ldots,$$

where we can rewrite each term since $\frac{1}{n^2} < \frac{1}{(n^2-n)}$ for $n \geq 3$.
Next we use the fact that $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$, so we obtain

$$\zeta(2) - 1 \leq \frac{1}{4} + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots = \frac{3}{4},$$

since all terms other than $\frac{1}{4}$ and $\frac{1}{2}$ cancel.
We conclude that for $\operatorname{Re}(s) = \sigma \geq 2$ we get

$$|\zeta(s)| \leq 1 + \frac{1}{2^\sigma} + \frac{1}{3^\sigma} + \cdots \leq 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots < \frac{7}{4},$$

where we use the triangle inequality, and the fact that $\frac{1}{n^\sigma} \leq \frac{1}{n^2}$ for $n \geq 2$.
Furthermore, since each $-\frac{1}{n^\sigma} \geq -\frac{1}{n^2}$ (for $n \geq 2$), we find that

$$|\zeta(s)| \geq 1 - \frac{1}{2^\sigma} - \frac{1}{3^\sigma} - \cdots \geq 1 - \frac{1}{2^2} - \frac{1}{3^2} - \cdots > \frac{1}{4}. \tag{5.1}$$

If we now write $s = \sigma + it$, then we see that

$$\begin{aligned} \frac{1}{n^s} &= n^{-s} = e^{-s\ln(n)} = e^{-\sigma\ln(n) - it\ln(n)} = e^{-\sigma\ln(n)}e^{-it\ln(n)} \\ &= \frac{1}{n^\sigma}[\cos(-t\ln(n)) + i\sin(-t\ln(n))] = \frac{\cos(t\ln(n))}{n^\sigma} - i\frac{\sin(t\ln(n))}{n^\sigma}. \end{aligned}$$

Taking the real part we are left with $\mathrm{Re}(1/n^s) = \cos(t\ln(n))/n^\sigma$, such that

$$\mathrm{Re}(\zeta(s)) = 1 + \frac{\cos(t\ln(2))}{2^\sigma} + \frac{\cos(t\ln(3))}{3^\sigma} + \cdots \geq 1 - \frac{1}{2^2} - \frac{1}{3^2} - \cdots > \frac{1}{4},$$

just as before. So we conclude that $\frac{1}{4} < |\zeta(s)| < \frac{7}{4}$, and $\mathrm{Re}(\zeta(s)) > \frac{1}{4}$ for $\mathrm{Re}(s) > 2$.
In general, for $\sigma > 1$ we can write

$$\mathrm{Log}(\zeta(s)) = \ln|\zeta(s)| + i\mathrm{Arg}(\zeta(s)).$$

By the above, for $\sigma > 2$, since $\mathrm{Arg}(\zeta(s))$ and $|\zeta(s)|$ are bounded, we know that

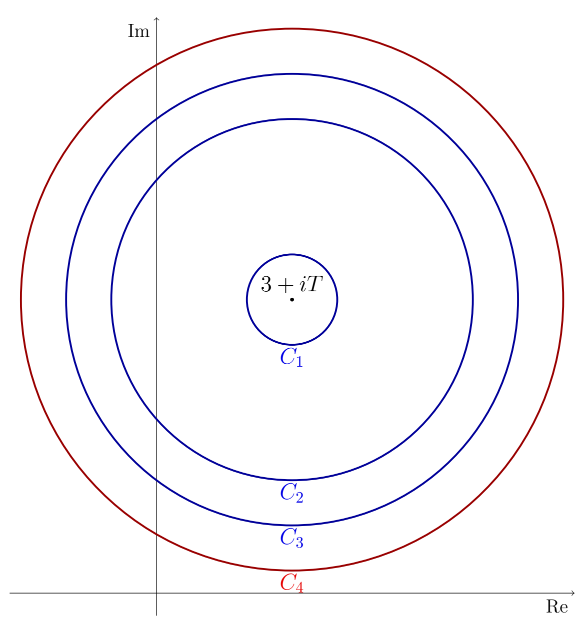
$$|\mathrm{Log}(\zeta(s))| < A \quad (\sigma \geq 2), \tag{5.2}$$

for some positive constant A.
For $\sigma < 2$, $t \neq 0$, we let $\mathrm{Log}(\zeta(s))$ be the analytic continuation of the above function along the straight line $(\sigma + it, 2 + it)$, provided that $\zeta(s) \neq 0$ on this line segment.

Consider now four circles, C_1, C_2, C_3, C_4, each with centre $3 + iT$, and radii 1, 4, 5, and 6 respectively. T must be big enough so that none of the circles cross the real axis.
Suppose now that $\zeta(s) \neq 0$ in or on C_4. Then the function $\log\zeta(s)$ (defined as before), is analytic inside C_4. Write

$$M_i = \max_{s \text{ on } C_i} |\log(\zeta(s))|,$$

for $i = 1, 2, 3$.
On C_4 there exists (see lemma 5.1 below) some positive constants B, B_0 such that $\zeta(s) = \mathcal{O}(t^{B_0})$, which means that $\mathrm{Re}\,|\log(\zeta(s))| < B\ln(T)$. We will apply the theorem by Borel-Carathéodory to the annulus between C_3 and C_4.



In that case we get, for T large enough,

$$M_3 \leq \frac{2\cdot 5}{6-5}B\ln(T) + \frac{6+5}{6-5}\ln|\zeta(3+iT)| < C\ln(T), \tag{5.3}$$

where C is some constant. Then we apply Hadamard's Three Circle Theorem to the circles C_1, C_2, C_3. In the theorem we write $r_1 = 1$, $r = 4$, $r_2 = 5$. So we get

$$\alpha = \frac{\ln(4) - \ln(1)}{\ln(5) - \ln(1)} = \frac{\ln(4)}{\ln(5)},$$

such that

$$\ln(M_2) \leq (1-\alpha)\ln(M_1) + \alpha \ln(M_3) = \ln(M_1^{1-\alpha}) + \ln(M_3^{\alpha}).$$

Since the logarithm is convex, we conclude

$$M_2 \leq M_1^{1-\alpha} M_3^{\alpha}.$$

Substituting $M_1 < A$, from equation (5.2), and $M_3 < C\ln(T)$ from equation (5.3), we are left with

$$M_2 \leq A^{1-\alpha} C^{\alpha} \ln(T)^{\alpha} = D\ln(T)^{\alpha},$$

for some constant D. Now since $M_2 = \max_{s \text{ on } C_2} |\log(\zeta(s))|$, and C_2 is a circle of radius 4 with centre $3 + iT$, we have found an upper bound as follows:

$$|\zeta(-1+iT)| < e^D e^{\ln(T)^{\alpha}} = \mathcal{O}(T). \tag{5.4}$$

Now instead look at functional equation (4.14), namely

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s).$$

Substituting $s = 2 + iT$, we get

$$\zeta(1-(2+iT)) = 2^{1-(2+iT)}\pi^{-(2+iT)}\cos\left(\frac{\pi(2+iT)}{2}\right)\Gamma(2+iT)\zeta(2+iT).$$

First we use Stirling's formula as before and see that, as $T \to \infty$,

$$\Gamma(2+iT) \sim \sqrt{2\pi}(2+iT)^{2+iT-\frac{1}{2}}e^{-2+iT}.$$

Now in particular

$$(2+iT)^{2+iT-\frac{1}{2}} = e^{(\frac{3}{2}+iT)\log(2+iT)} = e^{(\frac{3}{2}+iT)[\ln|2+iT| + i\mathrm{Arg}(2+iT)]}.$$

We know that $|2+iT| > T$, and thus $\ln|2+iT| > \ln(T)$, and also since T is positive, we have $0 < \mathrm{Arg}(2+iT) < \pi/2$, such that we get

$$\begin{aligned}\left|(2+iT)^{\frac{3}{2}+iT}\right| &= \left|e^{\frac{3}{2}\ln|2+iT| - T\mathrm{Arg}(2+iT) + i(T\ln|2+iT| + \frac{3}{2}\mathrm{Arg}(2+iT)}\right| \\ &= e^{\frac{3}{2}\ln|2+iT| - T\mathrm{Arg}2+iT} > e^{\frac{3}{2}\ln(T) - \pi T/2} = T^{\frac{3}{2}}e^{-\pi T/2}.\end{aligned}$$

We conclude that

$$|\Gamma(2+iT)| > \frac{\sqrt{2\pi}}{e^2}T^{\frac{3}{2}}e^{-\pi T/2}.$$

Furthermore,

$$\cos\left(\frac{(2+iT)\pi}{2}\right) = \frac{e^{-T\pi/2+\pi i} + e^{T\pi/2-\pi i}}{2} = -\frac{e^{T\pi/2} + e^{-T\pi/2}}{2},$$

which means that

$$\left|\cos\left(\frac{(2+iT)\pi}{2}\right)\right| > \frac{1}{2}e^{T\pi/2}.$$

Putting everything together, including inequality (5.1), we see that

$$\begin{aligned}|\zeta(-1+iT)| &= \left|2^{-1+iT}\pi^{-2-iT}\cos\left(\frac{\pi(2+iT)}{2}\right)\Gamma(2+iT)\zeta(2+iT)\right| \\ &> (2^{-1})(\pi^{-2})\left(\frac{1}{2}e^{T\pi/2}\right)\left(T^{\frac{3}{2}}e^{-\pi T/2}\right)\left(\frac{1}{4}\right) \\ &= \frac{1}{16\pi^2}T^{\frac{3}{2}}.\end{aligned}$$

Now comparing this to inequality (5.4), we see that for large enough T, both of these inequalities cannot be true at the same time! We have reached a contradiction. And since our only assumption was that $\zeta(s)$ had no zeroes on or in C_4, we conclude that there must be zeroes in or on C_4.
But since C_4 is a circle of radius 6 around $3+iT$, and the proof holds for all T that are large enough, we have shown that there are in fact infinitely many zeroes which are not the trivial zeroes. □

So we now know that $\zeta(s)$ has an infinitely many zeroes, and these zeroes must be somewhere within the region $0 \leq \text{Re}(s) \leq 1$. We call this region the **critical strip**, since that is where the non-trivial zeroes must be located.
Before we can continue, we need to justify a claim we have made.

Lemma 5.1. *For $s = \sigma + it$, $t \neq 0$, and σ is bounded within some interval $\sigma_0 < \sigma < \sigma_1$, there exists some positive constant A such that as $t \to \infty$,*

$$\zeta(\sigma + it) = \mathcal{O}(|t|^A).$$

Proof. For $\sigma > 1$ know that $|\zeta(\sigma + it)| \leq \zeta(\sigma)$ and so we know that $\zeta(s)$ is $\mathcal{O}(1)$ with respect to t. Then if we consider the functional equation (4.14),

$$\zeta(1-s) = -2^{1-s}\pi^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s),$$

we get $\zeta(1-\sigma-it)$ on the left hand side. Consider for $\sigma > 1$ the right hand side. It is clear that 2^{1-s} and π^{-s} are both $\mathcal{O}(1)$. Now for the $\cos(\pi s/2)$ term we write

$$\cos(\pi s/2) = \frac{1}{2}\left(e^{-t\pi/2+i\sigma\pi/2} + e^{t\pi/2-i\sigma\pi/2}\right),$$

which we see is $\mathcal{O}(e^{|t|\pi/2})$ as $t \to \infty$.
For the gamma term we use Stirling's formula,

$$\Gamma(z) \sim \sqrt{2\pi}z^{z-\frac{1}{2}}e^{-z},$$

which holds for $|z| \to \infty$, such that

$$\Gamma(\sigma + it) \sim \sqrt{2\pi}(\sigma + it)^{\sigma-\frac{1}{2}+it}e^{-\sigma-it}.$$

Only the $(\sigma+it)^{\sigma-\frac{1}{2}+it}$ term matters in the limit, which works out to

$$(\sigma+it)^{\sigma-\frac{1}{2}+it} = e^{(\sigma-\frac{1}{2})\ln|\sigma+it|-t\operatorname{Arg}(\sigma+it)+i(t\ln|\sigma+it|+(\sigma-\frac{1}{2})\operatorname{Arg}(\sigma+it))}.$$

We have $\ln|\sigma+it| \leq |t|$, and if $t<0$ we have $-\pi/2 < \operatorname{Arg}(\sigma+it) < 0$ and if $t>0$ we have $0 < \operatorname{Arg}(\sigma+it) < \pi/2$ so we see that the gamma term is $\mathcal{O}(|t|^{\sigma-\frac{1}{2}}\, e^{-|t|\pi/2})$. Putting everything together, we see that for $\sigma > 1$ (and thus $1-\sigma<0$), we have

$$\zeta(1-s) = \mathcal{O}(1\cdot |t|^{\sigma-\frac{1}{2}}\, e^{-|t|\pi/2}\cdot e^{|t|\pi/2}\cdot 1) = \mathcal{O}(|t|^{\sigma-\frac{1}{2}}).$$

This means we have proven the statement for $\sigma<0$ and $\sigma>1$.
On the critical strip ($0\leq\sigma\leq 1$), we can either use the Phragmen-Lindelöf principle [16], which states that the growth rate of an analytic function in the interior of some bounded region is determined by the growth rate of the function at the boundary of the region, which can be seen as a generalisation of the maximum modulus principle. Or if we wish to prove this directly, we consider equation (4.7) with $M=1$, in which case we get

$$\zeta(s) = \sum_{n=1}^{N}\frac{1}{n^s} + \frac{N^{1-s}}{s-1} + \frac{1}{2}N^{-s} - s\int_N^\infty \frac{x-\lfloor x\rfloor}{x^{s+1}}\mathrm{d}x. \tag{5.5}$$

Now for $0<\sigma<1$, we take[v] $N = \lceil |t| \rceil$, and we can take very rough estimates again. Namely for

$$\left|\sum_{n=1}^{N} n^{-s}\right| \leq \sum_{n=1}^{N} n^{-\sigma} \leq \sum_{n=1}^{N} 1 = N,$$

and so $\sum_{n=1}^{N} n^{-s} = \mathcal{O}(|t|)$. For $|t|$ large enough we see that $|s-1|\sim|t|$, and thus

$$\left|\frac{N^{1-s}}{s-1} + \frac{1}{2}N^{-s}\right| = \mathcal{O}\left(\frac{|t|^{1-\sigma}}{|t|} + |t|^{-\sigma}\right) = \mathcal{O}(|t|^{-\sigma}) = \mathcal{O}(1),$$

since $\sigma>0$.
Finally for the integral term we get

$$\left|s\int_N^\infty \frac{x-\lfloor x\rfloor}{x^{s+1}}\mathrm{d}x\right| \leq |s|\int_N^\infty x^{-\sigma-1}\mathrm{d}x = |s|\left[\frac{1}{-\sigma}x^{-\sigma}\right]_N^\infty = |s|\,N^{-\sigma}\frac{1}{\sigma},$$

which also means that $\left|s\int_N^\infty (x-\lfloor x\rfloor)x^{-s-1}\mathrm{d}x\right| = \mathcal{O}(|t|^{1-\sigma})$.
Since every term in equation (5.5) is of order at most $\mathcal{O}(|t|)$, we conclude that for $s=\sigma+it$, where $|t|$ is large enough, there must be some A such that for $\sigma<0$, $0<\sigma<1$, and $\sigma>1$ we have $\zeta(s)=\mathcal{O}(|t|^A)$. Now the lines $\sigma=0$ and $\sigma=1$ are much more difficult to study, but since $\zeta(s)$ is analytic everywhere except at $s=1$ (which is not in the domain we are considering), the function cannot grow at a faster rate there. We conclude the bound holds. □

So we indeed have infinitely many zeroes. If we look again at functional equation (4.14),

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s),$$

[v] $\lceil t\rceil = \lfloor t\rfloor + 1$, i.e. $\lceil t\rceil$ is the smallest integer **larger** than or equal to t.

then if we let $s = \sigma + it$, $0 \leq \sigma \leq 1$, then we know that if $\zeta(s) = 0$, $\zeta(1-s) = \zeta(1-\sigma+it) = 0$ too. This means that we have two options for a zero within the strip: either we find a pair $(s+it, s-1+it)$ of zeroes, **or** we find a zero on the line $s = \frac{1}{2} + it$, since then $\sigma = 1 - \sigma$. This gives us of course even more of a belief that Riemann hypothesis might be true.

But we can go a little further. By lemma 4.2, we know that $\overline{\zeta(\sigma+it)} = \zeta(\overline{\sigma+it}) = \zeta(\sigma - it)$. This means that if $s = \sigma + it$ is a zero, then also $s = \sigma - it$ is a zero. So in fact, if we wish to study the zeroes of the Riemann zeta function, we need only look at the strip $\frac{1}{2} \leq \sigma \leq 1$, $t > 0$. In fact, one could summarise a lot of research regarding the zeroes of the Riemann zeta function to finding $\epsilon > 0$ such that the zeroes of $\zeta(s)$ are restricted to $\frac{1}{2} \leq \sigma \leq 1 - \epsilon$.

Remark. *In the literature the term "zero-free region" is often mentioned. These are regions where the zeta function never vanishes. We can see $\sigma > 1$ as a "zero-free region", as well as $\sigma < 0$. A lot of core results regarding the zeta function, that we do not discuss here, are precisely about trying to find these zero-free regions. There are currently known zero-free regions that extend into the critical strip slightly. Proving the Riemann hypothesis is of course equivalent to showing that $\frac{1}{2} < \mathrm{Re}(s) \leq 1$ is also a zero-free region.*

While originally Riemann stated his hypothesis regarding $\Xi(t)$ having only real zeroes, looking at equation (4.18) we see that these hypotheses clearly coincide.
Over the years there have been numerous equivalent statements of this hypothesis. There are links to operator theory and eigenvalues of self adjoint operators, as well as links to random matrices, and many more, which we do not discuss here but can be found in the literature [20, 21].

5.2 The Line $\sigma = 1$

Because of its importance in discussing the Riemann zeta function, we would be amiss if we did not also discuss the **prime number theorem** (PNT) briefly. The PNT states the following:

Theorem 5.5 (Prime Number Theorem)**.** *Let $\pi(n)$ be the number of primes up to and including n, then as $n \to \infty$ we have*

$$\pi(n) \sim \frac{n}{\ln n}.$$

This means that asymptotically, the number of primes up to and including n is $n/\ln n$. The prime number theorem does not say anything about the error term; i.e. how good the approximation actually is for a given n. While the proof and detailed discussion of this theorem are beyond the scope of this text, they can be found in the literature [20, 25]. What we will mention is the fact that the prime number theorem is **equivalent** to the statement that $\zeta(1+it) \neq 0$ for all t. In fact, the first proof of the PNT, by J. Hadamard and C. de la Vallée-Poussin, proved that $\zeta(1+it) \neq 0$ and deduced the PNT from that [20]. It is possible that Euler actually studied the zeta function because he wanted to prove the prime number theorem [7].
Of course if we know that there are no zeroes on the line $\sigma = 1$, by the functional equations we know that the line $\sigma = 0$ also cannot contain any zeroes.

5.3 The Critical Line

As we stated before, the Riemann hypothesis states that all non-trivial zeroes of $\zeta(s)$ lie on the line $s = \frac{1}{2} + it$. We call this line the **critical line**. While we have infinitely many zeroes in the strip $0 \leq \sigma \leq 1$, we have no clue where exactly these are located.
From computer analysis we have known for years that the first zeroes do in fact lie on the critical line. In fact the Euler-Maclaurin summation formula, which we studied in section 3.4, was used to find the first few. As a quick historical aside:

- In 1903 J.P. Gram found the first few zeroes and showed the smallest one was found at $z = \frac{1}{2} \pm i \cdot 14.1347251\ldots$, and showed that the first 15 zeroes lie on the critical line. [22]
- By 1936 E.C. Titchmarsh had found the first 1041 zeroes, and showed that they all lie on the critical line [20].
- And in 2020 it was shown that all zeroes of the Riemann zeta function up to $3 \cdot 10^{12}$ are located on the critical line [29].

Our next point of interest is considering how many zeroes lie in the critical line. For that we need to define a new function, and prove two lemmas.

Lemma 5.2. *The function*

$$\psi(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x}$$

is convergent for x *on* $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ *and*

1. *obeys the functional equation*

$$\psi(x) = x^{-\frac{1}{2}} \psi\left(\frac{1}{x}\right) + \frac{1}{2} x^{-\frac{1}{2}} - \frac{1}{2},$$

2. *has the following Mellin transform [20]:*

$$\psi(x) = f(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \pi^{-s} \zeta(2s) \Gamma(s) x^{-s} \mathrm{d}s,$$

which gives in particular

$$\psi\left(\frac{1}{y^2}\right) = \frac{1}{4\pi i} \int_{\frac{1}{2} - iN}^{\frac{1}{2} + iN} \pi^{-\frac{1}{2}t} \zeta(t) \Gamma\left(\frac{1}{2}t\right) y^t \mathrm{d}t + \frac{1}{2} y.$$

3. *Furthermore* $\psi(x) + \frac{1}{2}$ *tends to* 0 *as* $x \to i$ *over an appropriately chosen path, and so do all derivatives of* $\psi(x) - \frac{1}{2}$.

Proof. To see that the infinite sum is convergent, we can apply the root test which gives $\lim_{n\to\infty} e^{-n\pi x}$, which tends to 0 for $\mathrm{Re}(x) > 0$ so we indeed have convergence for $\mathrm{Re}(x) > 0$. We shall prove the other claims in order.

1. It can be shown through Fourier analysis [20], that for $x > 0$,

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi x} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} e^{-n^2\pi/x}.$$

Since $n^2 = (-n)^2$, we see immediately that this implies

$$\begin{aligned} 2\left(\sum_{n=1}^{\infty} e^{-n^2\pi x}\right) + e^0 &= \frac{1}{\sqrt{x}} 2\left(\sum_{n=1}^{\infty} e^{-n^2\pi/x}\right) + \frac{1}{\sqrt{x}} e^0, \\ 2\psi(x) + 1 &= \frac{1}{\sqrt{x}}\left(2\psi\left(\frac{1}{x}\right) + 1\right). \end{aligned} \tag{5.6}$$

Multiplying the left hand side by $-x^{-\frac{1}{4}}$ and adding $x^{-\frac{1}{4}} + x^{\frac{1}{4}}$ to the left and right hand side we get

$$\begin{aligned} \left(-x^{\frac{1}{4}}\right)(2\psi(x) + 1) + x^{-\frac{1}{4}} + x^{\frac{1}{4}} &= \left(-x^{\frac{1}{4}}\right)\left(x^{-\frac{1}{2}} 2\psi\left(\frac{1}{x}\right) + x^{-\frac{1}{2}}\right) + x^{-\frac{1}{4}} + x^{\frac{1}{4}}, \\ x^{-\frac{1}{4}} - 2x^{\frac{1}{4}}\psi(x) &= x^{\frac{1}{4}} - 2x^{-\frac{1}{4}}\psi\left(\frac{1}{x}\right). \end{aligned}$$

Separating out $\psi(x)$ we get

$$\psi(x) = x^{-\frac{1}{2}}\psi\left(\frac{1}{x}\right) + \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}. \tag{5.7}$$

Since this clearly holds for all real $x > 0$, we can apply the identity principle and note that this must hold for all x on $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$.

2. The theory of Mellin transforms/inversions states that two functions $f(x)$ and $\mathcal{F}(s)$ are linked as follows [20]:

$$\mathcal{F}(s) = \int_0^{\infty} f(x) x^{s-1} \mathrm{d}x, \qquad f(x) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{F}(s) x^{-s} \mathrm{d}s. \tag{5.8}$$

Now if we put $f(x) = \psi(x)$, and we exchange integration and summation, we get

$$\int_0^{\infty} \psi(x) x^{s-1} \mathrm{d}x = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-n^2\pi x} x^{s-1} \mathrm{d}x.$$

Then we can substitute $u = n^2\pi x$, so that we get

$$\sum_{n=1}^{\infty} \int_0^{\infty} e^{-u} u^{s-1} (n^2\pi)^{-s+1} \cdot \frac{1}{n^2\pi} \mathrm{d}u = \sum_{n=1}^{\infty} (n^2\pi)^{-s} \int_0^{\infty} e^{-u} u^{s-1} \mathrm{d}u.$$

We can recognise the integral as the gamma function, take out the π^{-s} term, and recognise the sum as the zeta function, so we are left with

$$\mathcal{F}(s) = \pi^{-s}\zeta(2s)\Gamma(s),$$

which we know converges absolutely for all s such that $\text{Re}(s) > \frac{1}{2}$. Then by Fubini's theorem our interchanging of integration and summation was justified.
Furthermore, by equation (5.8) we get that

$$\psi(x) = f(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \pi^{-s}\zeta(2s)\Gamma(s)\, x^{-s}\mathrm{d}s.$$

Making the substitution $t = \frac{1}{2}s$, and writing $x = y^{-2}$, we get for $\sigma > 2$,

$$\psi\left(\frac{1}{y^2}\right) = \frac{1}{4\pi i}\int_{\frac{1}{2}\sigma-i\infty}^{\frac{1}{2}\sigma+i\infty} \pi^{-\frac{1}{2}t}\zeta(t)\Gamma\left(\frac{1}{2}t\right) y^t\mathrm{d}t. \tag{5.9}$$

Now if we integrate $\pi^{-\frac{1}{2}t}\zeta(t)\Gamma\left(\frac{1}{2}t\right) y^t$ over a positively oriented rectangle C_N with corners $2-iN$, $2+iN$, $\frac{1}{2}+iN$, $\frac{1}{2}-iN$, we see that we have one singularity in the interior, since $\zeta(t)$ has one at $t = 1$. Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (see the appendix), we see that $\text{Res}(\pi^{-\frac{1}{2}t}\zeta(t)\Gamma(\frac{1}{2}t)y^t, t = 1) = y$, because $\text{Res}(\zeta(t), t = 1) = 1$ by theorem 4.2.
So then

$$\int_{C_N} \pi^{-\frac{1}{2}t}\zeta(t)\Gamma\left(\frac{1}{2}t\right) y^t\mathrm{d}t = 2\pi i y.$$

We will first consider the following two integrals:

$$I_2 = \int_{2+iN}^{\frac{1}{2}+iN} \pi^{-\frac{1}{2}t}\zeta(t)\Gamma\left(\frac{1}{2}t\right) y^t\mathrm{d}t, \quad I_4 = \int_{2-iN}^{\frac{1}{2}-iN} \pi^{-\frac{1}{2}t}\zeta(t)\Gamma\left(\frac{1}{2}t\right) y^t\mathrm{d}t.$$

Since we know that there exists some positive constant A such that $\zeta(\sigma+iN) = \mathcal{O}(|N|^A)$, and that $\Gamma(\sigma+iN) = \mathcal{O}(|N|^{\sigma-\frac{1}{2}}\, e^{-|N|\pi/2})$ for $N \to \infty$ (or $N \to -\infty$), then by the ML-inequality we have $\lim_{N\to\infty} I_2 = 0$ and $\lim_{N\to\infty} I_4 = 0$, as the $e^{-|N|\pi/2}$ term dominates all others. And thus we get

$$\begin{aligned} 2\pi i y &= \lim_{N\to\infty}\int_{C_N} \pi^{-\frac{1}{2}t}\zeta(t)\Gamma\left(\frac{1}{2}t\right) y^t\mathrm{d}t \\ &= \lim_{N\to\infty}\int_{2-iN}^{2+iN} \pi^{-\frac{1}{2}t}\zeta(t)\Gamma\left(\frac{1}{2}t\right) y^t\mathrm{d}t + I_2 + \int_{\frac{1}{2}+iN}^{\frac{1}{2}-iN} \pi^{-\frac{1}{2}}\zeta(t)\Gamma\left(\frac{1}{2}t\right) y^t\mathrm{d}t + I_4, \end{aligned}$$

from which we obtain

$$\lim_{N\to\infty} 2\pi i y + \int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} \pi^{-\frac{1}{2}t}\zeta(t)\Gamma\left(\frac{1}{2}t\right) y^t\mathrm{d}t = \lim_{N\to\infty}\int_{2-iN}^{2+iN} \pi^{-\frac{1}{2}t}\zeta(t)\Gamma\left(\frac{1}{2}t\right) y^t\mathrm{d}t = 4\pi i\psi\left(\frac{1}{y^2}\right).$$

Dividing the left and right hand side by $4\pi i$ we have the desired result.

3. Consider

$$\psi(i+\delta) = \sum_{n=1}^{\infty} e^{-n^2\pi(i+\delta)} = \sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi\delta}.$$

This expression is the same as summing only over even n and then taking away the sum over only odd n. Now if we consider $\psi(4\delta)$, we see that $\sum_{n=1}^{\infty} e^{-n^2\pi 4\delta} = \sum_{n=1}^{\infty} e^{-(2n)^2\pi\delta}$, which sums only over the even natural numbers. So if we take this sum twice, and take away $\psi(\delta)$, we precisely have all the even terms minus all the negative terms. So we see

$$\begin{aligned}\psi(i+\delta) &= 2\psi(4\delta) - \psi(\delta)\\ &= \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{4\delta}\right) - \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{\delta}\right) - \frac{1}{2},\end{aligned}$$

by functional equation (5.7).

Now let us write $\delta = re^{i\theta}$. For every $\varepsilon > 0$, the function $\delta^{-\frac{1}{2}}\psi(1/\delta)$ converges uniformly on $\varepsilon < |\delta| < 1$ if we restrict δ so that $|\theta| < \frac{\pi}{2}$. We see that

$$\left|\delta^{-\frac{1}{2}}e^{-n^2\pi/\delta}\right| \leq r^{-\frac{1}{2}}e^{-\pi n^2/(r\cos(\theta))} \leq r^{-\frac{1}{2}}e^{-\pi n^2/r}.$$

Since the exponential term will dominate the $r^{-\frac{1}{2}}$ term, we know that for $0 < r < 1$ we have $\left|\delta^{-\frac{1}{2}}e^{-n^2\pi/\delta}\right| \leq e^{-\pi n^2}$. By the root test the sum $\sum_{n=1}^{\infty} e^{-n^2\pi/2}$ converges and so we conclude that by the Weierstrass-M test we have uniform convergence of $\sum_{n=1}^{\infty} \delta^{-\frac{1}{2}}e^{-n^2\pi/\delta}$ on the set $\{\delta = re^{i\theta} : \varepsilon \leq r \leq 1, |\theta| < \frac{\pi}{2}\}$, where ε is any positive number.

Now we can define

$$f_n(\delta) = \begin{cases}\delta^{-\frac{1}{2}}e^{-n^2\pi/\delta}, & \delta \neq 0\\ 0, & \delta = 0\end{cases}$$

which is continuous in δ, provided we do not cross the branch cut, and so we know that as $\delta \to 0$ (along the appropriate path), $f_n(\delta) \to 0$. This means that we can interchange the sum and the limit and conclude that

$$\frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{\delta}\right) \to 0, \quad \text{where we must choose } \delta \text{ along a path where } |\mathrm{Arg}(\delta)| < \frac{\pi}{2}.$$

And the same holds for $\delta^{-\frac{1}{2}}\psi\left(\frac{1}{4\delta}\right)$, which only tends to 0 faster, and so we indeed must have

$$\lim_{\delta\to 0}\psi(i+\delta) + \frac{1}{2} = \lim_{\delta\to 0}\frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{4\delta}\right) - \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{\delta}\right) = 0,$$

where we let $\delta \to 0$ along a path such that $|\mathrm{Arg}(\delta)| < \frac{\pi}{2}$.

For its derivatives we see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta}\left[\psi(i+\delta) - \frac{1}{2}\right] &= \frac{\mathrm{d}}{\mathrm{d}\delta}\left[\frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{4\delta}\right) - \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{\delta}\right)\right]\\ &= -\frac{1}{2}\delta^{-1\frac{1}{2}}\psi\left(\frac{1}{4\delta}\right) + \delta^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\delta}\psi\left(\frac{1}{4\delta}\right) + \frac{1}{2}\delta^{-1\frac{1}{2}}\psi\left(\frac{1}{\delta}\right) - \delta^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\delta}\psi\left(\frac{1}{\delta}\right).\end{aligned}$$

Now since we have uniform convergence on some region, we can simply differentiate the infinite sum to obtain

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\psi\left(\frac{1}{\delta}\right) = \sum_{n=1}^{\infty} n^2\pi\delta^{-2}e^{-n^2\pi/\delta}.$$

Now this means that the first derivative of $\psi(i+\delta) - \frac{1}{2}$, as well as the n-th derivative, only has terms of the form $\delta^{-m}\sum_{k=1}^{\infty} k^M e^{-n^2\pi/\delta}$, where m and M are some natural numbers, now since each term gets dominated by the $e^{-n^2\pi/\delta}$ term as $\delta \to 0$ by the exact same steps as before, we conclude that indeed

$$\left(\frac{\mathrm{d}}{\mathrm{d}\delta}\right)^n \left[\psi(i+\delta) - \frac{1}{2}\right] \to 0 \text{ as } \delta \to 0 \text{ over an appropriately chosen path.} \quad \square$$

A well informed reader may recognise the above $\psi(x)$ function in relation to the Jacobi theta functions. In fact $\psi(x) = \frac{1}{2}\vartheta_3(0, e^{-\pi x}) + \frac{1}{2}$, and so these results may be derived in a different manner [32].
Next we will consider a specific integral which we shall use to prove our desired result.

Lemma 5.3. *For all x we have*

$$\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} \cos(xt)\mathrm{d}t = -\pi e^{-\frac{1}{2}x}\psi\left(e^{-2x}\right) + \frac{1}{2}\pi e^{\frac{1}{2}x}.$$

Proof. In general we want to consider an integral of the form

$$\Phi(x) = \int_0^\infty f(t)\Xi(t)\cos(xt)\mathrm{d}t. \tag{5.10}$$

Let $f(t) = |\phi(it)|^2 = \phi(it)\phi(-it)$, where $\phi(t)$ is an analytic function which is real for real t (like in lemma 4.2). Then if we write $y = e^x$, we can rewrite the $\cos(xt) = \frac{1}{2}(e^{xit} + e^{-xit})$ term to $\frac{1}{2}(y^{it} + y^{-it})$. Now since $f(t)$ is an even function, and by equation (4.19) so is $\Xi(t)$, we can rewrite equation (5.10) to

$$\begin{aligned}
\Phi(x) &= \frac{1}{2}\left(\int_0^\infty \phi(it)\phi(-it)\Xi(t)y^{it}\mathrm{d}t + \int_0^\infty \phi(it)\phi(-it)\Xi(t)y^{-it}\mathrm{d}t\right), \\
&= \frac{1}{2}\int_{-\infty}^\infty \phi(it)\phi(-it)\Xi(t)y^{it}\mathrm{d}t \quad \text{(by a change of variables } t = -t\text{)}, \\
&= \frac{1}{2}\int_{-\infty}^\infty \phi(it)\phi(-it)\xi\left(\frac{1}{2}+it\right)y^{it}\mathrm{d}t,
\end{aligned}$$

since $\Xi(t) = \xi(\frac{1}{2}+it)$. Then we apply the change of variables $s = \frac{1}{2}+it$, so that

$$\begin{aligned}
\Phi(x) &= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \phi\left(s-\frac{1}{2}\right)\phi\left(\frac{1}{2}-s\right)\xi(s)y^{s-\frac{1}{2}} \cdot \frac{1}{i}\mathrm{d}s, \\
&= \frac{1}{i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \phi\left(s-\frac{1}{2}\right)\phi\left(\frac{1}{2}-s\right)\xi(s)y^{s}\mathrm{d}s.
\end{aligned}$$

If we then change $\xi(s)$ for its definition, as in equation (4.15), we get

$$\Phi(x) = \frac{1}{i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \phi\left(s-\frac{1}{2}\right)\phi\left(\frac{1}{2}-s\right)\frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s)y^s\mathrm{d}s.$$

Then if we set $\phi(s) = 1/(s+\frac{1}{2})$, which is analytic on $\mathbb{C}\backslash\{-\frac{1}{2}\}$ and satisfies the condition that it is real for real values of s, we obtain

$$\begin{aligned}\Phi(x) &= \frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{1}{s}\frac{1}{1-s}\frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s)y^s\mathrm{d}s \\ &= -\frac{1}{4i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s)y^s\mathrm{d}s, \qquad (5.11)\end{aligned}$$

where $y = e^x$ and so we do indeed have a function of x.
Now from lemma 5.2 we can recognise the last integral from the Mellin transform of

$$\psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x},$$

and so we get

$$\Phi(x) = -\frac{\pi}{\sqrt{y}}\left(\frac{1}{4\pi i}\int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN}\pi^{-\frac{1}{2}t}\zeta(t)\Gamma\left(\frac{1}{2}t\right)y^t\mathrm{d}t\right) = -\frac{\pi}{\sqrt{y}}\left(\psi\left(\frac{1}{y^2}\right)-\frac{1}{2}y\right).$$

Substituting $y = e^x$ we get

$$\Phi(x) = -\frac{\pi}{\sqrt{y}}\psi\left(\frac{1}{y^2}\right)+\frac{1}{2}\pi\sqrt{y} = -\pi e^{-\frac{1}{2}x}\psi\left(e^{-2x}\right)+\frac{1}{2}\pi e^{\frac{1}{2}x}.$$

Now since $f(t) = \phi(it)\phi(-it)$ and we chose $\phi(s) = 1/(s+\frac{1}{2})$ we can also rewrite equation (5.10) to

$$\Phi(x) = \int_0^{\infty}\frac{\Xi(t)}{t^2+\frac{1}{4}}\cos(xt)\mathrm{d}t,$$

which means finally we have

$$\int_0^{\infty}\frac{\Xi(t)}{t^2+\frac{1}{4}}\cos(xt)\mathrm{d}t = -\pi e^{-\frac{1}{2}x}\psi\left(e^{-2x}\right)+\frac{1}{2}\pi e^{\frac{1}{2}x}.$$ □

We will now be able to prove the following theorem, which was originally proven by G.H. Hardy in 1914.

Theorem 5.6. *There lie infinitely many zeroes of $\zeta(s)$ on the line $s = \frac{1}{2}+it$.*

Proof. Consider the function

$$\Xi(t) = -\frac{1}{2}\left(t^2+\frac{1}{4}\right)\pi^{-\frac{1}{4}-\frac{1}{2}it}\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)\zeta\left(\frac{1}{2}+it\right)$$

as defined in (4.18). A zero of $\zeta(s)$ for $\sigma = \mathrm{Re}(s) = \frac{1}{2}$ clearly corresponds to a real zero of $\Xi(t)$. We will show that $\Xi(t)$ has an infinite number of real zeroes.
First it is important to note that by corollary 4.2 we have that $\Xi(t)$ is real for real t. Now as $t\to\infty$, we know from before that $\zeta(\frac{1}{2}+it) = \mathcal{O}(t^A)$, and we can again apply Stirling's formula to

the gamma term for $\Xi(t)$ to see that $\Xi(t) = \mathcal{O}(t^B e^{-\frac{1}{4}\pi t})$. where A and B are some constants. If we set $x = -i\alpha$ in lemma 5.3, we get

$$\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} \cos(i\alpha t)\mathrm{d}t = -\pi e^{\frac{1}{2}\alpha i}\psi\left(e^{2\alpha i}\right) + \frac{1}{2}\pi e^{-\frac{1}{2}\alpha i}.$$

Since $e^{\frac{1}{2}\alpha i} = \cos(\frac{1}{2}\alpha) + i\sin(\frac{1}{2}\alpha)$, we can rewrite this to

$$\frac{2}{\pi}\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} \cosh(\alpha t)\mathrm{d}t = 2\cos\left(\frac{1}{2}\alpha\right) - 2e^{\frac{1}{2}i\alpha}\left(\frac{1}{2} + \psi\left(e^{2i\alpha}\right)\right).$$

Since both the left and right hand side are analytic functions in α, we can differentiate them any number of times, but since $\mathrm{Re}(e^{2i\alpha}) \leq 0$ for $\alpha \geq \frac{1}{4}\pi$, we require $\alpha < \frac{1}{4}\pi$. Differentiating $2n$-times with respect to α we get

$$\frac{2}{\pi}\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n}\cosh(\alpha t)\mathrm{d}t = \frac{(-1)^n\cos\left(\frac{1}{2}\alpha\right)}{2^{2n-1}} - 2\frac{\mathrm{d}^{2n}}{\mathrm{d}\alpha^{2n}}\left[e^{\frac{1}{2}i\alpha}\left(\frac{1}{2} + \psi\left(e^{2i\alpha}\right)\right)\right]. \tag{5.12}$$

To justify differentiating within the integral, note that $\cosh(\alpha t) = \mathcal{O}(e^{\alpha t})$, where α is **strictly** less than $\frac{1}{4}\pi$, but $\Xi(t) = \mathcal{O}(t^B e^{-\frac{1}{4}\pi t})$, which means that $\frac{\Xi(t)}{t^2+\frac{1}{4}}\cosh(\alpha t)$ is integrable and thus we can indeed exchange differentiation and integration by Lebesgue's dominated convergence theorem. We want to show that the last term goes to 0 as $\alpha \to \frac{1}{4}\pi$ for every fixed n. By lemma 5.2 we know this is indeed the case, since $e^{2i\cdot\frac{1}{4}\pi} = i$, and we indeed have the appropriate angle.
This means that for the integral in equation (5.12) we know that

$$\lim_{\alpha\to\frac{1}{4}\pi}\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n}\cosh(\alpha t)\mathrm{d}t = \frac{(-1)^n\pi\cos\left(\frac{1}{8}\pi\right)}{2^{2n}}. \tag{5.13}$$

Now we will assume that $\Xi(t)$ has no more sign changes for all $t > T$ where T is some constant. Say $\Xi(t)$ is positive for all $t > T$. Then since $\cosh(\alpha t) = \mathcal{O}(e^{\alpha t})$, where $\alpha < \frac{1}{4}\pi$, and $\Xi(t) = \mathcal{O}(t^B e^{-\frac{1}{4}\pi t})$, we know that $\frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh(\alpha t)$ is integrable by Lebesgue's dominated convergence theorem. Which means there exists some L such that

$$\lim_{\alpha\to\frac{1}{4}\pi}\int_T^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n}\cosh(\alpha t)\mathrm{d}t = L,$$

and then for all $\alpha < \frac{1}{4}\pi$ and $T' > T$ we have

$$\int_T^{T'} \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n}\cosh(\alpha t)\mathrm{d}t \leq L.$$

Then also, letting $\alpha \to \frac{1}{4}\pi$, we get

$$\int_T^{T'} \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n}\cosh\left(\frac{1}{4}\pi t\right)\mathrm{d}t \leq L.$$

We can conclude that in the integral

$$\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n}\cosh\left(\frac{1}{4}\pi t\right)\mathrm{d}t$$

must be convergent. So the integral on the left hand side of equation (5.13) converges with respect to α, for $0 \leq \alpha \leq \frac{1}{4}\pi$, and so for every n we know that

$$\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh\left(\frac{1}{4}\pi t\right) \mathrm{d}t = \frac{(-1)^n \pi \cos\left(\frac{1}{8}\pi\right)}{2^{2n}}.$$

But note that if we take n to be odd, the right hand side is negative, which would imply that

$$\int_T^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh\left(\frac{1}{4}\pi t\right) \mathrm{d}t < -\int_0^T \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh\left(\frac{1}{4}\pi t\right) \mathrm{d}t.$$

Now since $\Xi(t) = \mathcal{O}(t^B e^{-\frac{1}{4}\pi t})$, where $B > 0$, and $\cosh(\frac{1}{4}\pi t) = \mathcal{O}(e^{\frac{1}{4}\pi t})$, we know that the integrand is at least $\mathcal{O}(t^{2n-2})$, and so this would mean that

$$\int_T^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh\left(\frac{1}{4}\pi t\right) \mathrm{d}t < KT^{2n-1},$$

where K is independent of n. However since we assumed that $\Xi(t)$ was of one sign for $t > T$, we know there must exist some positive m (which may depend on T), such that $\Xi(t)/(t^2+\frac{1}{4}) \geq m$ for $2T \leq t \leq 2T+1$. And so that would also mean that

$$\int_T^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh\left(\frac{1}{4}\pi t\right) \mathrm{d}t \geq \int_{2T}^{2T+1} m t^{2n} \mathrm{d}t \geq m(2T)^{2n}.$$

But this would mean that the integral is both larger than $m(2T)^{2n}$ **and** smaller than KT^{2n-1}, which implies

$$m(2T)^{2n} < KT^{2n-1},$$
$$m2^{2n}T < K.$$

But we know that this cannot be true for large enough n. We conclude that there does not exist a $T > 0$ where for all $t > T$ $\Xi(t)$ is of the same sign, and thus $\Xi(t)$ has infinite sign changes and this we obtain infinite real zeroes of $\Xi(t)$ and thus infinite zeroes of $\zeta(\frac{1}{2}+it)$. ☐

Infinitely many zeroes on the critical line was a mathematical breakthrough at the time, and was widely celebrated. It is still one of the fundamental results in the study of the Riemann hypothesis. But this result *only* says that a lot of the zeroes lie in the critical line. We do not know anything more about zeroes being away from the line.

To strengthen our intuition, we mention the following, which was shown by Bohr and Landau in 1914 [26]. Consider the box with $0 < \mathrm{Re}(s) < 1$ and $0 \leq \mathrm{Im}(s) \leq T$. For any $\delta > 0$ the number of zeroes to the right of $\mathrm{Re}(s) = \frac{1}{2}+\delta$, divided by the total number of zeroes in the box, approaches 0 as $T \to \infty$.
J.E. Littlewood described this as "for any $\delta > 0$, all but an infinitesimal proportion of the roots p lie within δ of the line $\mathrm{Re}(s) = \frac{1}{2}$." (One might be tempted to say "almost all", but this is **not** the same).

Remark. *The aforementioned theorem is an example of a "zero-density" theorem. Theorems like these show, generally, something about where the majority of zeroes lie. These theorems do not, however, show that certain regions are absolutely zero-free, just that asymptotically, they contain at most a small proportion of total zeroes.*

It thus seems as if zeroes that do not lie on the critical line should be somewhat rare. In fact, it was shown in 1989 by B. Conrey [22] that at least 40% of all the zeroes in the critical strip are located on the line $s = \frac{1}{2} + it$.
As a quick aside, one might wonder what the order is of the zeroes of the zeta function. We know the trivial zeroes have order 1, and in fact from the aforementioned computer analysis we know all zeroes that have been found thus far are also of order 1, but this is still an open question for any general zero.

5.4 Some Consequences of the Riemann Hypothesis

We have worked hard to prove some results regarding the Riemann hypothesis, and we shall now discuss some of the results that would follow from the Riemann hypothesis being true. We do not give any proofs, but merely a discussion. In the following it is assumed the Riemann hypothesis is true.

Hypothesis 5.2 (Lindelöf Hypothesis)**.** *(As found in [20])*
For any $\varepsilon > 0$, we have

$$\zeta(\sigma + it) = \mathcal{O}(t^{\varepsilon}).$$

If the Lindelöf hypothesis were true that would bound the growth of $\zeta(s)$ significantly. We have shown that there is some constant A such that $\zeta(\sigma + it) = \mathcal{O}(t^A)$, but we had to work hard to show such A even existed. So the above statement is a great deal stronger!
Many results that would follow from the Riemann hypothesis are related to prime numbers and number theory. We list a few now.

Theorem 5.7. *(As found in [22])*
Write p_n for the n-th prime number, then we have

$$p_{n+1} - p_n << \sqrt{p_n} \ln(p_n).$$

Which is the strictest upper bound we can obtain from the Riemann hypothesis.
The current lowest bound [31] has been shown to be $p_{n+1} - p_n << p_n^{0.525}$, which is much weaker as n grows large.

We can also actually obtain an estimate on the error term in the prime number theorem. The following theorem has been proven without relying on the Riemann hypothesis:

Theorem 5.8. *(As found in [22])*
There exists some $C > 0$ such that

$$\pi(x) = \mathrm{li}(x) + \mathcal{O}\left(xe^{-C(\log(x)^{3/5}(\log\log(x))^{-1/5}}\right),$$

where $\mathrm{li}(x) = \int_0^x \frac{1}{\ln(t)} \mathrm{d}t$.

It can be shown that as $x \to \infty$ we have [25] $\mathrm{li}(x) = \mathcal{O}(\frac{x}{\ln(x)})$, which is where we recognise the prime number theorem again.
But if we assume the Riemann hypothesis, we can actually prove that [26]

$$|\mathrm{li}(s) - \pi(x)| = \mathcal{O}(x^{-1/2+\varepsilon}),$$

for any $\varepsilon > 0$, a great improvement!
And in the same vein we could then prove something regarding the Möbius function, which is defined as follows:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ has exactly } k \text{ distinct prime factors,} \\ 0 & \text{if } n \text{ is divisible by } p^2 \text{ where } p \text{ is some prime number.} \end{cases}$$

So as an example, $\mu(10) = (-1)^2 = 1$ since $10 = 2 \times 5$, and $\mu(8) = 0$ since $8 = 2^3$ and thus divisible by 2^2.
We can then define

$$M(x) = \sum_{n \leq x} \mu(n),$$

so that for example

$$\begin{aligned} M(6) &= \mu(1) + \mu(2) + \mu(3) + \mu(4) + \mu(5) + \mu(6) \\ &= 1 + (-1)^1 + (-1)^1 + 0 + (-1)^1 + (-1)^2 = -1. \end{aligned}$$

Now we state the following result

Theorem 5.9. *(As found in [25])*
The Riemann hypothesis is equivalent to proving

$$M(x) = \mathcal{O}(x^{\frac{1}{2}+\varepsilon}),$$

for any $\varepsilon > 0$.

Compared to the above, the prime number theorem is equivalent to the statement $M(x) = \mathcal{O}(x)$, which is substantially weaker.
Intuitively this means the prime number theorem implies that it is much more common for a number to have an even number of distinct prime factors, than to have an odd number. The Riemann hypothesis would then imply that it is actually only slightly more common to have an even number of distinct prime factors.

The direct consequences of the Riemann hypothesis are interesting on their own, but a proof of the Riemann hypothesis would most definitely pave the way for new mathematics. While the Riemann hypothesis is simple enough to understand (we want to know when a function is equal to 0), it has for over 150 years withstood any mathematical tool that was thrown at it. There are numerous books dedicated to the study of the Riemann zeta function, mathematicians from all over have worked on the function, and new tools had to be invented to solve problems related to the zeta function, but even then we are still not tangibly closer to a solution.
The express desire from mathematicians is that if someone found a proof (or a counter-proof) of

the Riemann hypothesis, it would give us new mathematical tools which we could use to solve different problems, and which would give us a deeper insight into analysis and number theory in general.
As an example, one of the other Millennium problems, the Poincaré conjecture, was solved in 2003 by G. Perelman [27]. He used a mathematical tool called Ricci flow together with surgery theory to prove the conjecture, and since his proof mathematicians have used his novel approaches to prove different results within numerous fields of mathematics.
On the other hand, one of the greatest mathematical upsets ever would be finding a non-trivial zero off the critical line in an elementary manner, say through computer analysis. As then we would gain no new knowledge beyond the fact that the Riemann hypothesis is false, when we would be mostly interested in *why* the Riemann hypothesis is false (or true for that matter). We could only hope this does not happen.

5.5 Generalised Riemann Hypothesis

We will give a very brief overview of the Generalised Riemann Hypothesis (GRH) to show that there exists a natural extension of what we have studied so far. A detailed description of the theory may be found in [25] and in [30].
The Hurwitz zeta function is defined as follows

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

We see that for $a = 1$, after shifting the index of our sum, this reduces to the normal Riemann zeta function. We call $\chi : \mathbb{Z} \to \mathbb{C}$ a Dirichlet character of modulus k (see [3, 25, 30]) if

1. χ is multiplicative, i.e. $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \mathbb{Z}$.
2. χ is k-periodic, i.e. $\chi(n + k) = \chi(n)$
3. $\chi(n) = 0$ if and only if $\gcd(n, k) > 1$, i.e. $\chi(n)$ is equal to zero if and only if n and k are **not** coprime.

Now if $\chi(n)$ is a character mod k, we define

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

It can be shown that we can express $L(s, \chi)$ as a linear combination of Hurwitz zeta functions, namely

$$L(s,\chi) = k^{-s} \sum_{r=1}^{k} \chi(r)\zeta\left(s, \frac{r}{k}\right).$$

This means that the properties of $L(s, \chi)$ depend on the properties of $\zeta(s, a)$, which we have now studied for only $a = 1$.

Such a series $L(s, \chi)$ is aptly named a **Dirichlet L-series**. Many concepts we have discussed such as the analytic continuation, functional equation, the Euler product, also (in general) apply to L-series. Because of the close link between these L-series and the Riemann/Hurwitz zeta function, it is believed that all L-series share the Riemann hypothesis, namely that their non-trivial zeroes all lie on the critical line $s = \frac{1}{2} + it$. This is the **Generalised Riemann Hypothesis**.

6 Discussion

The theory and study of the Riemann zeta function is incredibly rich and varied. We first studied the behaviour of $\zeta(s)$ for $\text{Re}(s) = \sigma > 1$, and saw that it coincided with the Euler product. We then built up the theory of Bernoulli numbers and polynomials, and used this to prove the Euler-Maclaurin summation formula. This formula helped us find an analytic continuation of $\zeta(s)$ for all $s \neq 1$. Then, following in Riemann's footsteps, we proved the functional equation, which helped us understand the behaviour of $\zeta(s)$ even better. This in turn allowed us to look into the Riemann hypothesis by studying the zeroes of $\zeta(s)$. We quickly found the trivial zeroes, and then showed that there must be infinitely many non-trivial zeroes which all had to lie in the critical strip. Finally, we showed that infinitely many of these zeroes had to lie on the critical line, which is where the Riemann hypothesis places all of them.

But we have also seen that much more is known regarding the Riemann zeta function and the Riemann hypothesis. In fact, most of the results that we proved were known as early as 1914; over a century ago! Since then, a lot more research has been done, but all research needs a jumping-off point, and that is precisely what this text is. We have used only mathematical tools that should be familiar to a mathematical undergraduate, which would allow someone to read this text in order to familiarise themselves with the topic before delving deeper into more difficult texts such as Titchmarsh's *The Theory of the Riemann Zeta-Function* or Edwards' *Riemann's Zeta Function*.

Most of the theorems that we have mentioned, such as Conrey's 1989 proof of 40% of zeroes being located on the critical line, or the Bohr and Landau's 1914 proof regarding proportions of zeroes, are asymptotic or density based. And sadly, even if we could prove that, for example, 100% of zeroes lie on the critical line, that still does not prove there are none that lie (slightly) off the critical line.
The difficulty in studying the Riemann zeta function and Riemann hypothesis is apparent from this text. Despite that, progress is still being made by sharpening bounds on previously known results. While this will most likely not lead to a proof of the Riemann hypothesis, it could still allow us to prove (weaker) results regarding related topics.

Beyond these improvements, the generalised Riemann hypothesis, which we briefly discussed, allows for a natural extension of the problems that we have investigated so far, and the possible consequences of it are still not exhausted. Additionally, the core concepts behind the Riemann hypothesis have been applied to domains other than $\mathbb{C}$, and this has also led to many interesting results and can in turn be extended even further.

All of which is to say that although we are not at all close to a proof of the Riemann hypothesis, the field of mathematics has only gotten much richer because of the work that was put towards trying to solve the problem. And while a proof is the ultimate desire for many mathematicians, the mathematical knowledge that has been gathered in its search is unquestionably invaluable.

Appendix

We will use the appendix to provide proofs to some statements that do not fit within the main text.

Theorem A.1. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n} z^n$$

converges for all $|z| \leq 1$*,* $z \neq 1$*, and it is equal to* $\operatorname{Log}(1-z)$*.*

Proof. First let $\{a_n\}$ and $\{b_n\}$ be sequences. Denote $A_n = \sum_{k=0}^{n} a_k$. Then

$$\sum_{n=k}^{L} a_n b_n = A_L b_L - A_{k-1} b_k - \sum_{n=k}^{L-1} A_n (b_{n+1} - b_n).$$

This is because

$$\begin{aligned} & A_L b_L - A_{k-1} b_k - \sum_{n=k}^{L-1} A_n (b_{n+1} - b_n) \\ =& A_L b_L - A_{k-1} b_k + A_k b_k - A_k b_{k+1} + A_{k+1} b_{k+1} \\ & - A_{k+1} b_{k+2} + A_{k+2} b_{k+2} - \cdots - A_{L-2} b_{L-1} + A_{L-1} b_{L-1} - A_{L-1} b_L \\ =& A_L b_L + a_k b_k + a_{k+1} b_{k+1} + a_{k+2} b_{k+2} + \cdots + a_{L-1} b_{L-1} - A_{L-1} b_L \\ =& a_k b_k + a_{k+1} b_{k+1} + a_{k+2} b_{k+2} + \cdots + a_{L-1} b_{L-1} + a_L b_L = \sum_{n=k}^{L} a_n b_n. \end{aligned}$$

If we let $a_n = z^n$ and $b_n = \frac{1}{n}$, then $A_n = (1 - z^{n+1})/(1-z)$, since

$$\begin{aligned} & (1-z)(1 + z + z^2 + \cdots + z^n) = 1 - z + z - z^2 + z^2 \cdots - z^n + z^n - z^{n+1} \\ & \implies 1 + z + z^2 + \cdots + z^n = (1 - z^{n+1})/(1-z). \end{aligned}$$

Now let us fix k and let $L > k$ be arbitrary, then for $|z| \leq 1$:

$$\begin{aligned} \left| \sum_{n=k}^{L} \frac{1}{n} z^n \right| &= \left| A_L \frac{1}{L} - A_{k-1} \frac{1}{k} - \sum_{n=k}^{L-1} A_n \left(\frac{1}{n+1} - \frac{1}{n} \right) \right| \\ &\leq \frac{2}{|1-z|} \left(\frac{1}{l} + \frac{1}{k} + \sum_{n=l}^{L-1} \left(\frac{1}{n} - \frac{1}{n+1} \right) \right) \\ &\leq \frac{2}{|1-z|} \frac{2}{k}. \end{aligned}$$

Where we have used the fact that $|1 - z^{n+1}| \leq 2$ as $-1 \leq z^{n-1} \leq 1$, and the telescoping series leaves us with only the terms $\frac{1}{k} - \frac{1}{L}$, where $\frac{1}{k} > \frac{1}{L}$.

Now note that for $z \neq 1$, the denominator is never equal to 0 and thus we have bounded, for a fixed z, the tail of the sum from above. We can conclude that the partial sums $\sum_{n=1}^{k} \frac{1}{n} z^n$ form a Cauchy sequence. We know that the complex plane, $\mathbb{C}$, is complete, and thus we know

$$\sum_{n=1}^{\infty} \frac{1}{n} z^n = \lim_{k \to \infty} \sum_{n=1}^{k} \frac{1}{n} z^n$$

must exist. We can denote this sum temporarily by $\sigma(z)$. We will show that the series converges for all z such that $|z| \leq 1$, and $z \neq 1$.
We have shown that

$$\left| \sum_{n=k}^{L} \frac{1}{n} z^n \right| \leq \frac{4}{k\,|1-z|}$$

holds for every z such that $|z| \leq 1$ and $z \neq 1$. This will give us uniform convergence on $E_r = \{z : |z| \leq 1, |1-z| \geq r\}$, since we can let $L \to \infty$ and thus the tail end of our sum is bounded.
Let $z \in E_r$, then we have:

$$\left| \sigma(z) - \sum_{n=1}^{k} \frac{1}{n} z^n \right| = \left| \sum_{n=k+1}^{\infty} \frac{1}{n} z^n \right| \leq \frac{4}{(k+1)\,|1-z|} \leq \frac{4}{(k+1)r}.$$

Let $\varepsilon > 0$, take N such that $4/(N+1)r < \varepsilon$. Then we know that

$$\left| \sigma(z) - \sum_{n=1}^{k} \frac{1}{n} z^n \right| < \varepsilon,$$

whenever $k \geq N$ and $z \in E_r$.
We know that for $|z| < 1$, strictly,

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

We also know that

$$\frac{1}{1-z} = \sum_{n=0}^{k} z^n + \frac{z^{k+1}}{1-z},$$

since $\sum_{n=0}^{k} z^n = (1 - z^{k+1})/(1-z)$. We can then integrate the left and right-hand side on the straight line L from 0 to z:

$$\begin{aligned}
\int_L \frac{1}{1-w} \mathrm{d}w &= \int_L \sum_{n=0}^{k} w^n \mathrm{d}w + \int_L \frac{w^{k+1}}{1-w} \mathrm{d}w, \\
[-\mathrm{Log}(1-w)]_0^z &= \sum_{n=0}^{k} \left[\frac{w^{n+1}}{n+1} \right]_0^z + \int_L \frac{w^{k+1}}{1-w} \mathrm{d}w, \\
-\mathrm{Log}(1-z) &= \sum_{n=0}^{k} \frac{1}{n+1} z^{n+1} + \int_L \frac{w^{k+1}}{1-w} \mathrm{d}w.
\end{aligned}$$

Where we can integrate the sum term by term because it is a finite sum and a definite integral. We will apply the ML-inequality to the integral. Since we assumed that $|z| < 1$, we have $\ell(L) \leq |z|$, and $|1-w| \geq ||1| - |w|| \geq |1-|z|| = 1 - |z|$. So then:

$$\left| \int_L \frac{w^{k+1}}{1-w} \mathrm{d}w \right| \leq |z| \frac{|z|^{k+1}}{1-|z|} = \frac{|z|^{k+2}}{1-|z|}.$$

And since $|z| < 1$, we know that

$$\lim_{k \to \infty} \int_L \frac{w^{k+1}}{1-w} \mathrm{d}w = 0.$$

And thus we know that for $|z| < 1$:

$$\sum_{n=1}^{\infty} \frac{1}{n} z^n = -\mathrm{Log}(1-z).$$

Since the function $\sigma(z)$ is continuous, and the logarithm is too, we know the formula must hold when $|z| = 1$ and $z \neq 1$ as well. □

Theorem A.2. *For any positive integer m, we have*

$$\sum_{n=1}^{N} n^m = \sum_{k=0}^{m} \binom{m}{k} B_k \frac{N^{m+1-k}}{m+1-k}.$$

Proof. Apply theorem 3.3 with $f(x) = x^m$, $a = 1$, $b = N$, then we get (for $M - 1 \leq m$):

$$\begin{aligned}
\sum_{n=1}^{N} n^m &= \int_1^N x^m \mathrm{d}x + \frac{1}{2}(1^m + N^m) + \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!} \left(k! \binom{m}{k} N^{m-k} - k! \binom{m}{k} 1^{m-k} \right) \\
&\quad - \frac{(-1)^M}{M!} \int_1^N B_M(x - \lfloor x \rfloor) \frac{\mathrm{d}^M (x^m)}{\mathrm{d}x^M} \mathrm{d}x, \\
&= \left[\frac{x^{m+1}}{m+1} \right]_1^N + \frac{1}{2} + \frac{1}{2} N^m + \sum_{k=1}^{M-1} \frac{B_{k+1}}{k+1} \binom{m}{k} \left(N^{m-k} - 1 \right) \\
&\quad - \frac{(-1)^M}{M!} \int_1^N B_M(x - \lfloor x \rfloor) \frac{\mathrm{d}^M (x^m)}{\mathrm{d}x^M} \mathrm{d}x.
\end{aligned}$$

Now if we set $M = m+1$, then $\mathrm{d}^{m+1}(x^m)/\mathrm{d}x^{m+1} = 0$, so we get rid of the integral, which means we are left with

$$\sum_{n=1}^{N} n^m = \frac{N^{m+1} - 1}{m+1} + \frac{N^m + 1}{2} + \sum_{k=1}^{m} \frac{B_{k+1}}{k+1} \binom{m}{k} (N^{m-k} - 1).$$

We now use the fact that

$$\begin{aligned}
\frac{1}{k+1} \binom{m}{k} &= \frac{1}{k+1} \frac{m!}{k!(m-k)!} = \frac{1}{m+1} \frac{m+1!}{(k+1)!(m+1-(k+1))!} \\
&= \frac{1}{m+1} \binom{m+1}{k+1},
\end{aligned}$$

so that we obtain

$$\begin{aligned}
&\frac{N^{m+1}-1}{m+1}+\frac{N^m+1}{2}+\sum_{k=1}^{m}B_{k+1}\frac{1}{m+1}\binom{m+1}{k+1}\left(N^{m-k}-1\right)\\
=&\frac{N^{m+1}-1}{m+1}+\frac{N^m+1}{2}+\frac{1}{m+1}\sum_{k=1}^{m}B_{k+1}\binom{m+1}{k+1}N^{m-k}\\
&-\frac{1}{m+1}\sum_{k=1}^{m}B_{k+1}\binom{m+1}{k+1}.
\end{aligned}$$

We want to rewrite these sums into sums representing Bernoulli numbers, so we will let them run from $k=-1$, so we need to remove those terms as well:

$$\begin{aligned}
=&\frac{N^{m+1}-1}{m+1}+\frac{N^m+1}{2}+\frac{1}{m+1}\sum_{k=-1}^{m}B_{k+1}\binom{m+1}{k+1}N^{m-k}\\
-&\frac{1}{m+1}\left(B_0\binom{m+1}{0}N^{m+1}+B_1\binom{m+1}{1}N^m\right)\\
-&\frac{1}{m+1}\sum_{k=-1}^{m}B_{k+1}\binom{m+1}{k+1}+\frac{1}{m+1}\left(B_0\binom{m+1}{0}+B_1\binom{m+1}{1}\right).
\end{aligned}$$

Working out the binomial coefficients, and splitting the fractions, we get:

$$\begin{aligned}
=&\frac{N^{m+1}}{m+1}-\frac{1}{m+1}+\frac{N^m}{2}+\frac{1}{2}+\frac{1}{m+1}\sum_{k=-1}^{m}B_{k+1}\binom{m+1}{k+1}N^{m-k}\\
-&\frac{N^{m+1}}{m+1}-\frac{N^m}{2}-\frac{1}{m+1}\sum_{k=-1}^{m}B_{k+1}\binom{m+1}{k+1}+\frac{1}{m+1}+\frac{1}{2}.
\end{aligned}$$

We see many terms cancel out, and we can rewrite the sums from $k=0$ instead, so we are left with:

$$=1+\frac{1}{m+1}\sum_{k=0}^{m+1}\binom{m+1}{k}B_kN^{m-(k-1)}-\frac{1}{m+1}\sum_{k=0}^{m+1}B_k\binom{m+1}{k}.$$

Now if we take the $1/(m+1)$ term all the way to the front, and we take out the $k=m+1$ term of the sums, we are left with:

$$\begin{aligned}
=&\frac{1}{m+1}\left((m+1)+\sum_{k=0}^{m}\binom{m+1}{k}B_kN^{m+1-k}-\sum_{k=0}^{m}B_k\binom{m+1}{k}\right)\\
+&\frac{1}{m+1}\binom{m+1}{m+1}B_{m+1}N^{m+1-(m+1)}-\frac{1}{m+1}\binom{m+1}{m+1}B_{m+1}+1.
\end{aligned}$$

Then note that

$$\begin{aligned}
\frac{1}{m+1}\binom{m+1}{k}&=\frac{1}{m+1}\frac{(m+1)!}{k!(m+1-k)!}=\frac{m!}{k!(m-k+1)!}\\
&=\frac{1}{m-k+1}\frac{m!}{k!(m-k)!}=\frac{1}{m+1-k}\binom{m}{k},
\end{aligned}$$

and by definition 3.1 we have:

$$\sum_{k=0}^{m} B_k \binom{m+1}{k} = m+1.$$

This means we can rewrite the binomial coefficient and our rightmost sum cancels against the $m+1$ term. Writing out the loose binomials, we are left with:

$$= \sum_{k=0}^{m} \frac{1}{m+1} \binom{m+1}{k} B_k N^{m+1-k} - \frac{B_{m+1}}{m+1} + \frac{B_{m+1}}{m+1} = \sum_{k=0}^{m} B_k \binom{m}{k} \frac{N^{m+1-k}}{m+1-k}.$$

We conclude that

$$\sum_{n=1}^{N} n^m = \sum_{k=0}^{m} \binom{m}{k} B_k \frac{N^{m+1-k}}{m+1-k},$$

as desired. □

Definition A.1. *The gamma function is defined as*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t,$$

for $\mathrm{Re}(z) > 0$*. Which then has an analytic continuation to the entire complex plane, except at the non-positive integers, through the relation*

$$\Gamma(z+1) = z\Gamma(z).$$

It is not hard to see that we require $\mathrm{Re}(z) > 0$ for the integral to converge, indeed

$$\left| \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t \right| \leq \int_0^\infty \left| t^{z-1} \right| \mathrm{d}t = \int_0^\infty t^{\mathrm{Re}(z)-1} \mathrm{d}t = \left[\frac{1}{z} t^{\mathrm{Re}(z)} \right]_0^\infty,$$

which we know converges only if $\mathrm{Re}(z) > 0$.
Furthermore, setting $z = 1$ we obtain

$$\Gamma(1) = \int_0^\infty e^{-t} \mathrm{d}t = \left[-e^t \right]_0^\infty = 1.$$

And integrating by parts once gives us, for $\mathrm{Re}(z) > 0$,

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} \mathrm{d}t = \left[-t^z e^{-t} \right]_0^\infty - \int_0^\infty z t^{z-1} \cdot -e^{-t} \mathrm{d}t = z \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t = z\Gamma(z).$$

So we see $\Gamma(1) = 0!$, and $\Gamma(z+1) = z\Gamma(z)$, and so we conclude that for $n \in \mathbb{N}$, $\Gamma(n) = (n-1)!$ which is one of the main reasons why the gamma function is so interesting; it is an extension of the factorial to non-integer values.
We can use the analytic continuation of $\Gamma(z+1) = z\Gamma(z)$ to extend the definition to the complex plane $\mathbb{C}$, except at the non-positive integers, since $\Gamma(0) = \frac{\Gamma(1)}{0}$, which clearly has a simple pole. Extending this by the same recurrence relation we see that this simple pole extends to all non-positive integers.

We shall now prove Euler's reflection formula and Legendre's duplication formula.

Lemma A.1 (Euler's reflection formula)**.** *For $z \notin \mathbb{Z}$ we have*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Proof. Let C_N be a positively oriented square with corners N, $N+2\pi i$, $-N+2\pi i$, $-N$. We will consider the function $e^{\alpha z}/(1+e^z)$, where $0<\alpha<1$. Clearly within C_N we have only one singularity, namely at $z=\pi i$, and by l'Hôptial's rule we have

$$\lim_{z\to\pi i}\frac{(z-\pi i)e^{\alpha z}}{1-e^z} = \lim_{z\to\pi i}\frac{e^{\alpha z}+\alpha e^{\alpha z}(z-\pi i)}{e^z} = -e^{\alpha\pi i},$$

and so $\text{Res}(e^{\alpha z}/(1+e^z), z=\pi i) = -e^{\alpha\pi i}$, such that by Cauchy's residue theorem

$$\int_{C_N}\frac{e^{\alpha z}}{1+e^z}\mathrm{d}z = -2\pi i e^{\alpha\pi i}.$$

Then write

$$\begin{aligned}\int_{C_N}\frac{e^{\alpha z}}{1+e^z}\mathrm{d}z &= \int_N^{N+2\pi i}\frac{e^{\alpha z}}{1+e^z}\mathrm{d}z + \int_{N+2\pi i}^{-N+2\pi i}\frac{e^{\alpha z}}{1+e^z}\mathrm{d}z + \int_{-N+2\pi i}^{-N}\frac{e^{\alpha z}}{1+e^z}\mathrm{d}z + \int_{-N}^{N}\frac{e^{\alpha z}}{1+e^z}\mathrm{d}z,\\ &= I_1+I_2+I_3+I_4.\end{aligned}$$

We first consider I_1,

$$|I_1| = \left|\int_0^{2\pi}\frac{e^{\alpha(N+it)}}{1+e^{N+it}}\mathrm{d}t\right| \le \int_0^{2\pi}\frac{e^{\alpha N}}{|1-e^N|}\mathrm{d}t \to 0 \text{ as } N\to\infty,$$

since $0<\alpha<1$ and so the denominator outgrows the numerator.
In the same vein for I_3, we get

$$|I_2| = \left|\int_{2\pi}^{0}\frac{e^{\alpha(-N+it)}}{1+e^{-N+it}}\mathrm{d}t\right| \le \int_0^{2\pi}\frac{e^{-\alpha N}}{|1+e^{-N}|}\mathrm{d}t \to 0 \text{ as } N\to\infty,$$

where the numerator goes to 0 and the denominator goes to 1 in the limit.
Now note that

$$I_2 = \int_{N+2\pi i}^{-N+2\pi i}\frac{e^{\alpha z}}{1+e^z}\mathrm{d}z = -\int_{-N}^{N}\frac{e^{\alpha(t+2\pi i)}}{1+e^{t+2\pi i}}\mathrm{d}t = -e^{2\pi i\alpha}\int_{-N}^{N}\frac{e^{\alpha t}}{1+e^t}\mathrm{d}t = -e^{2\pi i\alpha}I_4.$$

This means that

$$-2\pi i e^{\alpha\pi i} = \lim_{N\to\infty}\int_{C_N}\frac{e^{\alpha z}}{1+e^z}\mathrm{d}z = \lim_{N\to\infty} I_1+I_2+I_3+I_4 = (1-e^{2\pi i\alpha})\int_{-N}^{N}\frac{e^{\alpha t}}{1+e^t}\mathrm{d}t.$$

In other words

$$\int_{-\infty}^{\infty}\frac{e^{\alpha t}}{1+e^t}\mathrm{d}t = \frac{-2\pi i e^{\alpha\pi i}}{1-e^{2\alpha\pi i}} = \frac{-2\pi i}{e^{-\alpha\pi i}-e^{\alpha\pi i}} = \frac{-2\pi i}{-\sin(\alpha\pi)\cdot 2i} = \frac{\pi}{\sin(\alpha\pi)}.$$

Now consider $\Gamma(\alpha)\Gamma(1-\alpha)$:

$$\Gamma(\alpha)\Gamma(1-\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\mathrm{d}t\int_0^\infty u^{-\alpha}e^{-u}\mathrm{d}u = \int_0^\infty t^{\alpha-1}e^{-t}\int_0^\infty u^{-\alpha}e^{-u}\mathrm{d}u\mathrm{d}t,$$

by Fubini's theorem. Now we apply the change of variables $u = vt$, where $t > 0$, so we get

$$\begin{aligned}
\Gamma(\alpha)\Gamma(1-\alpha) &= \int_0^\infty t^{\alpha-1}e^{-t}\int_0^\infty (vt)^{-\alpha}e^{-vt}\cdot t\mathrm{d}v\mathrm{d}t \\
&= \int_0^\infty v^{-\alpha}\int_0^\infty e^{-t(1+v)}\mathrm{d}t\mathrm{d}v \\
&= \int_0^\infty v^{-\alpha}\left[\frac{-1}{v+1}e^{-t(1+v)}\right]_{t=0}^\infty \mathrm{d}v \\
&= \int_0^\infty \frac{v^{-\alpha}}{v+1}\mathrm{d}v.
\end{aligned}$$

We can apply the change of variables $v = e^{-x}$ to obtain

$$\Gamma(\alpha)\Gamma(1-\alpha) = \int_\infty^{-\infty} \frac{e^{\alpha x}}{e^{-x}+1}\cdot -e^{-x}\mathrm{d}x = \int_{-\infty}^\infty \frac{e^{\alpha x}}{1+e^x}\mathrm{d}x = \frac{\pi}{\sin(\alpha\pi)}.$$

Since the above holds for α on $(0,1)$, by the identity principle it must hold on any domain where both functions are analytic, which is $\mathbb{C}\backslash\mathbb{Z}$. □

We can immediately use Euler's reflection formula to find $\Gamma\left(\frac{1}{2}\right)$, since by the formula we have

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right) = \frac{\pi}{\sin(\frac{1}{2}\pi)} = \pi,$$

and so we conclude $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. We will use this in the following proof.

Lemma A.2 (Legendre's duplication formula)**.** *We have*

$$\Gamma(z) = \frac{2^{z-1}}{\sqrt{\pi}}\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right),$$

where z may be any complex number for which each Γ term is defined.

Proof. We first need the identity $B(z_1,z_2)\Gamma(z_1+z_2) = \Gamma(z_1)\Gamma(z_2)$, where $B(z_1,z_2)$ is the **beta function**, which is defined as

$$B(z_1,z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\mathrm{d}t.$$

We start as follows

$$\begin{aligned}
\Gamma(z_1)\Gamma(z_2) &= \int_0^\infty u^{z_1-1}e^{-u}\mathrm{d}u\cdot\int_0^\infty v^{z_2-1}e^{-v}\mathrm{d}v \\
&= \int_0^\infty e^{-u-v}u^{z_1-1}v^{z_2-1}\mathrm{d}u\mathrm{d}v.
\end{aligned}$$

Writing $u = st$ and $v = s(1-t)$, we obtain

$$\begin{aligned}
\Gamma(z_1)\Gamma(z_2) &= \int_0^\infty\int_0^1 e^{-s}(st)^{z_1-1}(s(1-t))^{z_2-1}\cdot s\mathrm{d}t\mathrm{d}s \\
&= \int_0^\infty e^{-s}s^{z_1-1}s^{z_2}\mathrm{d}s\cdot\int_0^1 t^{z_1-1}(1-t)^{z_2-1}\mathrm{d}t = \Gamma(z_1+z_2)B(z_1,z_2).
\end{aligned}$$

This means in particular that if we set $z_1 = z_2 = z$ we obtain

$$\Gamma(z)\Gamma(z) = \Gamma(2z)B(z,z),$$

such that

$$\frac{\Gamma^2(z)}{\Gamma(2z)} = \int_0^1 t^{z-1}(1-t)^{z-1}\mathrm{d}t.$$

Substituting $t = \frac{1}{2}(1+u)$ we get

$$\begin{aligned}\frac{\Gamma^2(z)}{\Gamma(2z)} &= \int_{-1}^1 \left(\frac{1}{2}\right)^{z-1} (1+u)^{z-1} \left(\frac{1}{2}\right)^{z-1} (1-u)^{z-1} \cdot \frac{1}{2}\mathrm{d}u \\ &= \frac{1}{2^{2z-1}} \int_{-1}^1 (1-u^2)^{z-1}\mathrm{d}u.\end{aligned}$$

Now the function $(1-u^2)$ is even and so we know that

$$2^{2z-1}\Gamma^2(z) = 2\Gamma(2z)\int_0^1 (1-u^2)^{z-1}\mathrm{d}u.$$

Note that

$$\begin{aligned}B\left(\frac{1}{2}, z\right) = \int_0^1 t^{\frac{1}{2}-1}(1-t)^{z-1}\mathrm{d}t &= \int_0^1 (u^2)^{\frac{1}{2}-1}(1-u^2)^{z-1} \cdot 2u\mathrm{d}u \\ &= 2\int_0^1 (1-u^2)^{z-1}\mathrm{d}u,\end{aligned}$$

by the change of variables $t = u^2$. Then using again the identity $B(z_1, z_2) = \Gamma(z_1)\Gamma(z_2)/\Gamma(z_1+z_2)$, we obtain

$$2^{2z-1}\Gamma^2(z) = \Gamma(2z)B\left(\frac{1}{2}, z\right) = \Gamma(2z)\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(z)}{\Gamma(z+\frac{1}{2})},$$

since we know $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, rearranging the terms gives us

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right).$$

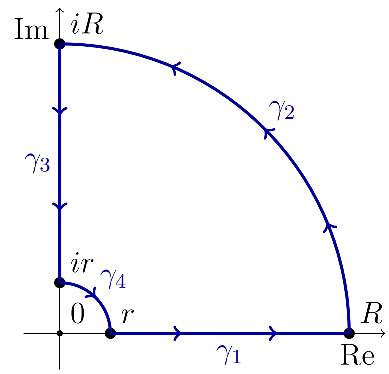
Setting $z = \frac{1}{2}z$ we obtain the result. □

Lemma A.3. *For* $0 < \mathrm{Re}(\alpha) < 1$

$$\int_0^\infty \frac{\sin y}{y^\alpha}\mathrm{d}y = \Gamma(1-\alpha)\cos\left(\frac{\alpha\pi}{2}\right).$$

Proof. We will integrate the function $f(z) = e^{iz}/(z^\alpha)$.

This means we must specify a branch since z^α is multi-valued. We will choose the principal branch, such that $z^\alpha = e^{z\mathrm{Log}(\alpha)}$.
We will integrate over the contour on the right. Now the function



$$\frac{e^{iz}}{z^\alpha} = \frac{e^{iz}}{e^{\alpha\mathrm{Log}(z)}}$$

is analytic inside and on γ, which means that

$$\begin{aligned} 0 &= \int_\gamma \frac{e^{iz}}{e^{\alpha\mathrm{Log}(z)}}\mathrm{d}z \\ &= \int_{\gamma_1} \frac{e^{iz}}{e^{\alpha\mathrm{Log}(z)}}\mathrm{d}z + \int_{\gamma_2} \frac{e^{iz}}{e^{\alpha\mathrm{Log}(z)}}\mathrm{d}z \int_{\gamma_3} \frac{e^{iz}}{e^{\alpha\mathrm{Log}(z)}}\mathrm{d}z \int_{\gamma_4} \frac{e^{iz}}{e^{\alpha\mathrm{Log}(z)}}\mathrm{d}z \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Now for γ_1, we have that $z = x$, $r < x < R$, and $\mathrm{d}z = \mathrm{d}x$, and we rewrite $e^{ix}/(e^{\alpha\mathrm{Log}(x)}) = e^{ix}/x^\alpha$, so then we get

$$\lim_{\substack{r\to 0\\ R\to\infty}} I_1 = \lim_{\substack{r\to 0\\ R\to\infty}} \int_r^R \frac{e^{ix}}{x^\alpha}\mathrm{d}x = \int_0^\infty \frac{e^{ix}}{x^\alpha}\mathrm{d}x = \int_0^\infty \frac{\cos(x)}{x^\alpha}\mathrm{d}x + i\int_0^\infty \frac{\sin(x)}{x^\alpha}\mathrm{d}x.$$

For γ_3, we have $z = iy$, $r < y < R$, $\mathrm{d}z = i\mathrm{d}y$, so then

$$\frac{e^{i(iy)}}{e^{\alpha\mathrm{Log}(iy)}}; = \frac{e^{-y}}{e^{\alpha(\ln(y)+i\mathrm{Arg}(iy))}} = \frac{e^{-y}}{y^\alpha e^{i\alpha\pi/2}} = e^{-i\alpha\pi/2}e^{-y}y^{-\alpha},$$

and we get

$$\begin{aligned} \lim_{\substack{r\to 0\\ R\to\infty}} I_3 &= \lim_{\substack{r\to 0\\ R\to\infty}} ie^{-i\alpha\pi/2}\int_R^r e^{-y}y^{-\alpha}\mathrm{d}y = -ie^{-i\alpha\pi/2}\int_0^\infty e^{-y}y^{-\alpha}\mathrm{d}y, \\ &= -ie^{i\alpha\pi/2}\Gamma(1-\alpha) = -i\left(\cos\left(\frac{\alpha\pi}{2}\right) - i\sin\left(\frac{\alpha\pi}{2}\right)\right)\Gamma(1-\alpha). \end{aligned}$$

Where the second to last equality follows from the definition of the gamma function, which converges whenever $\mathrm{Re}(\alpha) < 1$.
For γ_2, $z = Re^{i\theta}$, $0 \le \theta \le \pi/2$, then

$$|z^\alpha| = \left|e^{\alpha(\ln|z|+i\mathrm{Arg}(z))}\right| = e^{\mathrm{Re}(\alpha)\ln|z|}e^{-\mathrm{Im}(\alpha)\mathrm{Arg}(z)} = R^{\mathrm{Re}(\alpha)}e^{-\mathrm{Im}(\alpha)\theta},$$

and also

$$\left|e^{iz}\right| = \left|e^{R(-\sin(\theta)+i\cos(\theta))}\right| = e^{-R\sin(\theta)},$$

such that

$$\left|\frac{e^{iz}}{z^\alpha}\right| = \frac{e^{-R\sin(\theta)}}{R^{\mathrm{Re}(\alpha)}e^{-\mathrm{Im}(\alpha)\theta}}.$$

Note that $0 \leq \sin(\theta) \leq 1$ as $0 \leq \theta \leq \pi/2$, so that we know that $e^{-R\sin(\theta)} \leq 1$ (as $R > 0$). If $\mathrm{Im}(\alpha) > 0$, we know that $e^{-\mathrm{Im}(\alpha)\theta} \geq e^{-\mathrm{Im}(\alpha)\pi/2}$, and if $\mathrm{Im}(\alpha) < 0$, $e^{-\mathrm{Im}(\alpha)\theta} \geq 1$. Since we take the limit later, and α is fixed, we do not have to make a distinction between the two cases, and state in general that

$$\left|\frac{e^{iz}}{z^\alpha}\right| \leq \frac{1}{R^{\mathrm{Re}(\alpha)}e^{-\mathrm{Im}(\alpha)\theta}} = M(R).$$

Whenever $\mathrm{Re}(\alpha) > 0$, we know that $\lim_{R\to\infty} M(R) = 0$. This means that by Jordan's lemma we know that $\lim_{R\to\infty} I_2 = 0$.
For γ_4, $z = re^{i\theta}$, $0 \leq \theta \leq \pi/2$, and then, as before $|e^{iz}| \leq 1$, and $|z^\alpha| = r^{\mathrm{Re}(\alpha)}e^{-\mathrm{Im}(\alpha)\theta}$, where we can again bound $e^{-\mathrm{Im}(\alpha)\theta}$ from below. If we apply the ML-inequality to I_4 we get

$$|I_4| \leq \ell(\gamma_4)\frac{1}{r^{\mathrm{Re}(\alpha)}e^{-\mathrm{Im}(\alpha)\theta}} = \frac{\pi}{2}r^{1-\mathrm{Re}(\alpha)}e^{\mathrm{Im}(\alpha)\theta}.$$

Then for $1-\mathrm{Re}(\alpha) > 0$, we see that $\lim_{r\to 0}|I_4| = 0$, then taking the limits in the initial expression we get

$$\begin{aligned}0 = \lim_{\substack{r\to 0\\ R\to\infty}} \int_\gamma \frac{e^{iz}}{z^\alpha}\mathrm{d}z &= \lim_{\substack{r\to 0\\ R\to\infty}} I_1 + I_2 + I_3 + I_4 \\ &= \int_0^\infty \frac{\cos(x)}{x^\alpha}\mathrm{d}x + i\int_0^\infty \frac{\sin(x)}{x^\alpha}\mathrm{d}x + 0 - i\left(\cos\left(\frac{\alpha\pi}{2}\right) - i\sin\left(\frac{\alpha\pi}{2}\right)\right)\Gamma(1-\alpha) + 0.\end{aligned}$$

Now if we take α to be real, we can take the real and imaginary parts in the above, and see plainly that for $0 < \alpha < 1$

$$\int_0^\infty \frac{\cos(x)}{x^\alpha}\mathrm{d}x = \sin\left(\frac{\alpha\pi}{2}\right)\Gamma(1-\alpha),$$

and

$$\int_0^\infty \frac{\sin(x)}{x^\alpha}\mathrm{d}x = \cos\left(\frac{\alpha\pi}{2}\right)\Gamma(1-\alpha).$$

It is clear that each of these terms is an analytic function on the region $\Omega = \{z \mid 0 < \mathrm{Re}(z) < 1\}$. Then by the identity principle, since we have equality in the above for the real numbers, we conclude that

$$\int_0^\infty \frac{\cos(x)}{x^\alpha}\mathrm{d}x = \sin\left(\frac{\alpha\pi}{2}\right)\Gamma(1-\alpha),$$

and

$$\int_0^\infty \frac{\sin(x)}{x^\alpha}\mathrm{d}x = \cos\left(\frac{\alpha\pi}{2}\right)\Gamma(1-\alpha),$$

whenever $0 < \mathrm{Re}(\alpha) < 1$ as desired. □

Lemma A.4. *The function*

$$\mathrm{Si}(h) = \int_0^h \frac{\sin(x)}{x}\mathrm{d}x$$

is bounded from above and below, and attains its global maximum at $h = \pi$ (and its global minimum at $h = -\pi$).

Proof. Indeed, by l'Hôpital's rule $\lim_{t\to 0}\sin(t)/t = \cos(0) = 1$, so $t = 0$ is a removable singularity, and the integrand has no other singularities. We are left with a rational function of analytic terms, where the denominator is not 0, and so the function is integrable on any finite domain.
Taking the derivative with respect to h we get

$$\mathrm{Si}'(h) = \sin(h)/h$$

which is 0 whenever $h = k\pi$ for $k \in \mathbb{Z}$.
Taking the second derivative, we get

$$\mathrm{Si}''(h) = (h\cos(h) - \sin(h)/h^2.$$

As $\cos(2n\pi) = 1$ and $\cos((2n+1)\pi) = -1$ for $n \in \mathbb{N}_0$, we see that $\mathrm{Si}''(2n\pi) > 0$ and $\mathrm{Si}''((2n+1)\pi) < 0$, so we know that the odd multiples of π correspond to a maximum for $\mathrm{Si}(h)$.
Let $n \in \mathbb{N}_0$, consider:

$$\begin{aligned}\mathrm{Si}([2n+1]\pi) - \mathrm{Si}([2n+3]\pi) &= \int_0^{(2n+1)\pi} \frac{\sin(u)}{u}\mathrm{d}u - \int_0^{(2n+3)\pi} \frac{\sin(u)}{u}\mathrm{d}u \\ &= -\left[\int_{(2n+1)\pi}^{(2n+2)\pi} \frac{\sin(u)}{u}\mathrm{d}u + \int_{(2n+2)\pi}^{(2n+3)\pi} \frac{\sin(u)}{u}\mathrm{d}u\right].\end{aligned}$$

We apply the change of variables $t = u - (2n+1)\pi$ and $t = u - (2n+2)\pi$ to the first and second integral respectively to get

$$\begin{aligned}&-\left[\int_0^{\pi} \frac{\sin(t+(2n+1)\pi)}{t+(2n+1)\pi}\mathrm{d}t + \int_0^{\pi} \frac{\sin(t+(2n+2)\pi)}{t+(2n+2)\pi}\mathrm{d}t\right] \\ &= \int_0^{\pi} \frac{\sin(t)}{t+(2n+1)\pi}\mathrm{d}t - \int_0^{\pi} \frac{\sin(t)}{t+(2n+2)\pi}\mathrm{d}t.\end{aligned}$$

Since the sine is 2π-periodic, and $\sin(t+\pi) = -\sin(\pi)$. Now we can compare the left and right integrands.
It seems clear that

$$\frac{\sin(t)}{t+(2n+1)\pi} \geq \frac{\sin(t)}{t+(2n+2)\pi}$$

for $n \geq 0$ and $t \geq 0$.
This means that the left integral is bigger than the right integral, and we conclude that $\mathrm{Si}([2n+1]\pi) - \mathrm{Si}([2n+3]\pi) > 0$.
This means that the maxima of $\mathrm{Si}(h)$ decrease. Since the first maximum appears at $h = \pi$, we see that this must be the global maximum.
Applying a change of variables $t = -u$ to

$$\mathrm{Si}(-h) = \int_0^{-h} \frac{\sin(u)}{u}\mathrm{d}u = \int_0^{h} \frac{\sin(-t)}{-t} \cdot -\mathrm{d}t = -\mathrm{Si}(h),$$

we see that $\mathrm{Si}(-h) = -\mathrm{Si}(h)$, and so we can bound $\mathrm{Si}(h)$ for negative h as well, and immediately conclude that $\mathrm{Si}(h)$ attains its global minimum at $h = -\pi$.

□

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