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RIESZ TRANSFORMS ON COMPACT QUANTUM GROUPS AND STRONG SOLIDITY

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Abstract One of the main aims of this paper is to give a large class of strongly solid compact quantum groups. We do this by using quantum Markov semigroups and noncommutative Riesz transforms. We introduce a property for quantum Markov semigroups of central multipliers on a compact quantum group which we shall call 'approximate linearity with almost commuting intertwiners'. We show that this property is stable under free products, monoidal equivalence, free wreath products and dual quantum subgroups. Examples include in particular all the (higher-dimensional) free orthogonal easy quantum groups.

We then show that a compact quantum group with a quantum Markov semigroup that is approximately linear with almost commuting intertwiners satisfies the immediately gradient- $\mathcal{S}_2$ condition from [10] and derive strong solidity results (following [10]). Using the noncommutative Riesz transform we also show that these quantum groups have the Akemann–Ostrand property; in particular, the same strong solidity results follow again (now following [27]).

Keywords: quantum Markov semigroups, compact quantum groups, strong solidity, Riesz transforms

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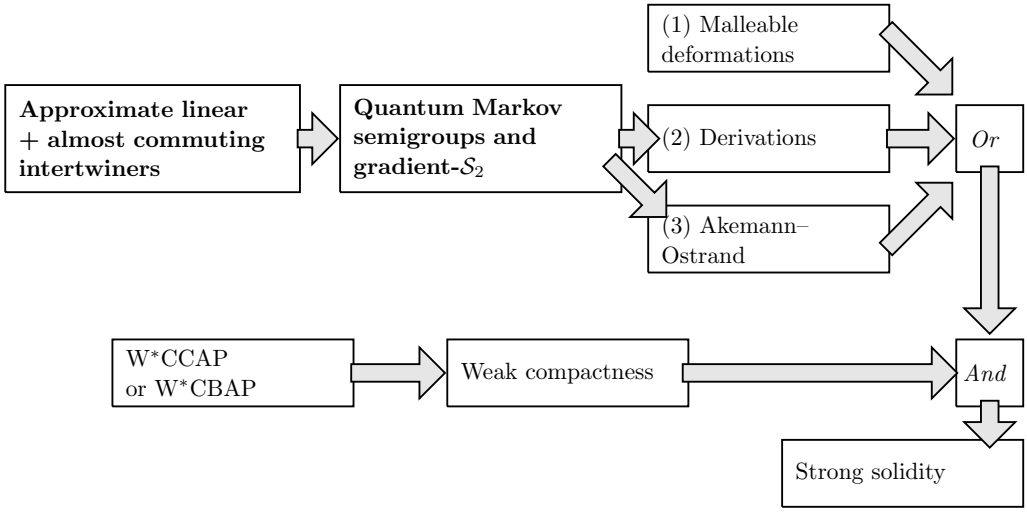
In their fundamental papers, Voiculescu [48] and Ozawa and Popa [35] prove that the free group factors $\mathcal{L}(\mathbb{F}_n), n \geq 2$, do not contain a Cartan subalgebra. This means that $\mathcal{L}(\mathbb{F}_n)$ does not contain a maximal abelian von Neumann subalgebra whose normaliser generates $\mathcal{L}(\mathbb{F}_n)$. Consequently, $\mathcal{L}(\mathbb{F}_n)$ does not admit a natural crossed product decomposition and is therefore distinguishable from the class of group measure space von Neumann algebras. The proof of Ozawa and Popa in fact shows a stronger property: that the normaliser of any diffuse amenable von Neumann subalgebra of $\mathcal{L}(\mathbb{F}_n)$ generates a von Neumann algebra that is amenable again. This property has become known as strong solidity. After [35], many von Neumann algebras were proven to be strongly solid.

These strong solidity results required several techniques that come from approximation properties and the geometry of groups. The proof of Ozawa and Popa [35] essentially splits into two parts. First, they show that weak amenablity of a group (or the weak* completely contractive approximation property [W*CCAP] of its von Neumann algebra) can be used to prove a so-called weak compactness property. Second, using weak compactness and

Popa's deformation and spectral gap techniques, they obtain the results. For the second part, a number of alternative approaches have been presented. Essentially they split into three methods, using (1) malleable deformations [35], (2) closable derivations in 1-cohomology and HH^+-type properties [35] or (3) the Akemann–Ostrand property [38] or quasicohomological methods [15]. The second and third methods are closely related (see also [12] and Section 6). Each of these approaches provides new classes of von Neumann algebras that are strongly solid.

We believe it is instructive to include the following diagram at this point, since these global methods shall not appear very explicitly in this paper (but rather in the references). Our focus here is to show that the input for methods (2) and (3) can be proved for a reasonably large class of quantum groups. We shall thus concentrate on the boldface part of the diagram, on which we expound later. The arrows should not always be understood as strict implications; sometimes additional conditions are needed.



In [27], Isono provided the first examples of von Neumann algebras coming from the theory of compact quantum groups that are strongly solid. The approach falls into category (3) already described. In particular, Isono proved that free orthogonal quantum groups are strongly solid. Later different proofs of this fact were given in [23] (see also the earlier paper [46] on solidity). In [8], strong solidity results for quantum automorphism groups were obtained.

We note that [26, Theorem C] also covers free products of free orthogonal/unitary quantum groups and quantum automorphism groups. In the present paper, we shall deal with a property that implies strong solidity and is stable under free products and monoidal equivalence. One advantage of this approach is that our methods apply to a free product of (certain) compact quantum groups followed by a monoidal equivalence. This is especially important for the treatment of free wreath products [4].

In [10] it was proved that the type III deformations of free orthogonal and unitary quantum groups are also strongly solid. The proof builds upon the weak compactness properties from [6] and follows the path of method (2) already described. The theory of quantum Markov semigroups (QMSs) is used to construct the closable derivations in method (2) from [17]. This is done for the specific examples of free orthogonal and unitary quantum groups.

This paper continues the line of [10] by involving two new ideas. First, we look at [10] from the viewpoint of a rigid C*-tensor category. Although this paper is not written in the abstract language of C*-tensor categories (as we found it less accessible), this is precisely the structure of $\mathrm{Irr}(\mathbb{G})$ that occurs in our proofs.

Second, we refine the method from [10]. We introduce a new property for a QMS of central multipliers on a compact quantum group which we call 'approximate linearity with almost commuting intertwiners' (see Definition 2.2). The definition is certainly technical in nature, but it has some clear advantages, namely, it is immediately clear that it is invariant under monoidal equivalence of quantum groups. A first consequence is that since the free orthogonal quantum groups O_N^+ are monoidally equivalent to $SU_q(2), q \in (0,1)$ with $q + q^{-1} = N$, the estimates from [10] can be carried out on $SU_q(2)$. We also prove a couple of other stability properties, including free wreath products.

Theorem 0.1. *Approximate linearity with almost commuting intertwiners of a QMS of central multipliers is stable under the following:*

(1) *Monoidal equivalence.*
(2) *Free products.*
(3) *Taking dual quantum subgroups.*
(4) *Free wreath products with S_N^+ (more precisely, Theorem 5.1).*

The proof for free wreath products is a combination of [30, Theorem 5.11] (see also [44]), the other stability properties and the fact that $SU_q(2)$ carries a QMS that is approximately linear with almost commuting intertwiners. To prove the latter statements we provide a conceptual way to construct QMSs from suitable families of unital completely positive maps. This makes use of generating functionals and differentiation at 0. The proof also simplifies [10, Section 6.1]. We are indebted to Adam Skalski for sharing this argument.

We then show that indeed the strong solidity and Akemann–Ostrand-type results as in the diagram are implied. We first show the following (in path (2)):

Theorem 0.2. *Let $\mathbb{G}$ be a compact quantum group of Kac type such that $L_\infty(\mathbb{G})$ has the weak* completely bounded approximation property (W*CBAP). Suppose that $\mathbb{G}$ carries a QMS of central multipliers that is approximately linear with almost commuting intertwiners and which is immediately L_2-compact. Then $L_\infty(\mathbb{G})$ is strongly solid.*

Then we show the following theorem using noncommutative Riesz transforms (see also [12]). Since the Akemann–Ostrand property could be of independent interest, we record it in this paper in a separate section.

Theorem 0.3. *Let $\mathbb{G}$ be a compact quantum group of Kac type such that $C_r(\mathbb{G})$ is locally reflexive. Suppose that $\mathbb{G}$ carries a QMS of central multipliers that is approximately linear with almost commuting intertwiners and which is immediately L_2-compact. Then $L_\infty(\mathbb{G})$ satisfies the Akemann–Ostrand property (more precisely, AO^+ from [27]).*

In [27] it was proved in the factorial case that together with the W*CBAP, Theorem 0.3 implies strong solidity. In that case, Theorem 0.3 implies Theorem 0.2.

We now turn to the examples. Most of the work is contained in the following theorem, from which a diversity of results follow by stability properties. Its proof heavily uses the estimates [46, Appendix]; it is interesting that these estimates are precisely sharp enough for our purposes.

Theorem 0.4. *$SU_q(2)$ carries a QMS of central multipliers that is approximately linear with almost commuting multipliers and immediately L_2-compact.*

We can now harvest our results using the stability properties and several monoidal equivalence and isomorphism results for compact quantum groups that have been proved by others, most notably [4].

Theorem 0.5. *The following (Kac-type) compact quantum groups are strongly solid and satisfy AO^+:*

(1) *All seven series of free orthogonal easy quantum groups classified in [50] under the names $O_{N_3}^+$, $S_{N_5}^+$, $H_{N_5}^+$, $B_{N_4}^+$, $S_{N_5}'^+$, $B_{N_4}'^+$ and $B_{N_4}^{\#+}$ for $N_3 \geq 3$, $N_4 \geq 4$, $N_5 \geq 5$ (see [8]).*

(2) *The quantum reflection groups $H_N^{s+} \simeq \widehat{\mathbb{Z}_s} \wr_* S_N^+$ for $N \geq 5$, $\infty \geq s \geq 2$, where $\mathbb{Z}_\infty = \mathbb{Z}$.*

(3) *The free unitary quantum groups U_N^+ for $N \geq 3$ (see [26]).*

The selection of examples presented in Theorem 0.5 is a bit random and not exhaustive. We have chosen to present examples that relate to attempts to classify easy quantum groups. The representation category of the families in Theorem 0.5 are precisely the ones whose representation categories can be described in terms of noncoloured, noncrossing partitions. One may wonder what happens when more colours are added to the partitions, like in [25]. Our theorem shows that some cases are already covered.

It should be mentioned that part of Theorem 0.5 was proved in the literature already using different methods (we have given references in the theorem). Our method gives a unified way to treat all examples at once. To our knowledge, strong solidity for H_N^+ and the more general quantum reflection groups has not been covered, nor has AO^+. Other new examples include all free wreath products of these examples with S_N^+.

Structure

Section 1 introduces preliminary notation. In Section 2 we introduce almost linearity with almost commuting intertwiners and show stability properties. We conclude most of Theorem 0.1 except for the wreath products. Section 3 contains the implications for strong solidity and proves Theorem 0.2. In Section 4 we show that $SU_q(2)$ carries a good QMS and prove Theorem 0.4. From this we can conclude the proof of the case of wreath

products in Theorem 0.1 as well as strong solidity of the examples in Theorem 0.5; this is done in Section 5. In Section 6 we prove the corresponding statements for the Akemann–Ostrand property, which concludes Theorem 0.3.

1. Preliminaries

By $\delta(x \in X)$ we denote the function that is 1 if $x \in X$ and 0 otherwise. Inner products are linear in the left leg. For ξ,η vectors in a Hilbert space H we write $\omega_{\xi,\eta}(x) = \langle x\xi,\eta\rangle$.

The standard theory of von Neumann algebras can be found in [42]. For operator spaces we refer to [22].

1.1. Finite-dimensional approximations and strong solidity

See [9] for the following notions.

Definition 1.1. We say that a von Neumann algebra M has the *W* CBAP* if there exists a net $(\Phi_i)_i$ of normal completely bounded finite-rank maps $M \to M$ such that:

(1) there exists $\Lambda \geq 1$ such that for all i we have $\|\Phi_i\|_{cb} \leq \Lambda$ and
(2) for every $x \in M$ we have $\Phi_i(x) \to x$ σ-weakly.

Λ is called the Cowling–Haagerup constant. If $\Lambda = 1$, then we say that M has the *W* CCAP*.

For quantum groups of Kac type, the W*CBAP (resp., W*CCAP) is equivalent to weak amenability of the quantum group (resp., weak amenability with Cowling–Haagerup constant 1). For the Haagerup property, see also [13].

Definition 1.2. We say that a finite von Neumann algebra with faithful normal state (M,τ) has the *Haagerup property* if there exists a net $(\Phi_i)_i$ of normal unital completely positive maps $M \to M$ such that $\tau \circ \Phi_i = \tau$, such that Φ_i is compact as a map $L_2(M,\tau) \to L_2(M,\tau)$ and such that for every $x \in M$ we have $\Phi_i(x) \to x$ strongly.

We further need the notions of solidity [9] and strong solidity as in the next definition.

Definition 1.3. A finite von Neumann algebra M is called *strongly solid* if for every diffuse amenable von Neumann subalgebra $P \subseteq M$, $\mathsf{Nor}_M(P)''$ is amenable, where the normaliser is defined as

$$\mathsf{Nor}_M(P) = \{u \in M \mid u \text{ unitary such that } uPu^* = P\}.$$

1.2. Compact quantum groups and represenations

The theory of compact quantum groups has been established by Woronowicz [51].

Definition 1.4. A compact quantum group $\mathbb{G}$ is a pair $(C(\mathbb{G}),\Delta_{\mathbb{G}})$ of a unital C*-algebra $C(\mathbb{G})$ and a unital $*$-homomorphism $\Delta_{\mathbb{G}} : C(\mathbb{G}) \to C(\mathbb{G}) \otimes_{\min} C(\mathbb{G})$ (comultiplication) satisfying $(\Delta_{\mathbb{G}} \otimes \mathrm{id}) \circ \Delta_{\mathbb{G}} = (\mathrm{id} \otimes \Delta_{\mathbb{G}}) \circ \Delta_{\mathbb{G}}$ (coassociativity) and such that both $\Delta_{\mathbb{G}}(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)$ and $\Delta_{\mathbb{G}}(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))$ are dense in $C(\mathbb{G}) \otimes_{\min} C(\mathbb{G})$.

A compact quantum group $\mathbb{G}$ admits a unique state φ on $C(\mathbb{G})$ called the *Haar state* which satisfies left and right invariance:

$$(\varphi\otimes\mathrm{id})\circ\Delta_{\mathbb{G}}(x)=\varphi(x)1=(\mathrm{id}\otimes\varphi)\circ\Delta_{\mathbb{G}}(x).$$

$\mathbb{G}$ is called *Kac* if τ is tracial. We let $C_r(\mathbb{G})=\pi_\varphi(C(\mathbb{G}))$ and $L_\infty(\mathbb{G})=\pi_\varphi(C(\mathbb{G}))''$ be the C*-algebra and von Neumann algebra generated by the GNS-representation π_φ of φ. A (finite dimensional unitary) representation of $\mathbb{G}$ is a unitary element $u\in C(\mathbb{G})\otimes M_n(\mathbb{C})$ such that $(\Delta_{\mathbb{G}}\otimes\mathrm{id})(u)=u_{13}u_{23}$, where $u_{23}=1\otimes u$ and u_{13} is u_{23} with the flip map applied to its first two tensor legs. We also set $u_{12}=u\otimes 1_n$. All representations are assumed to be unitary and finite dimensional, and we shall just call them *representations*. The elements $(\mathrm{id}\otimes\omega)(y)$ with $\omega\in M_n(\mathbb{C})^*$ are called the *matrix coefficients* of u. We shall use the Woronowicz quantum Peter–Weyl theorem [51], which states that for every $\alpha,\beta\in\mathrm{Irr}(\mathbb{G})$ there exists positive $Q_\alpha\in M_{n_\alpha}(\mathbb{C})$ with $\mathrm{qdim}(\alpha):=\mathrm{Tr}(Q_\alpha)=\mathrm{Tr}\left(Q_\alpha^{-1}\right)$ such that

$$\varphi\left(\left(u^\beta_{\mu,\nu}\right)^* u^\alpha_{\xi,\eta}\right)=\delta_{\alpha,\beta}\,\mathrm{qdim}(\alpha)^{-1}\left\langle Q_\alpha^{\frac12}\xi,Q_\alpha^{\frac12}\mu\right\rangle\langle\xi,\eta\rangle,\quad \xi,\eta,\mu,\nu\in\mathbb{C}^{n_\alpha}. \tag{1.1}$$

The quantity $\mathrm{qdim}(\alpha)$ is called the *quantum dimension*.

After these preliminaries the comultiplication $\Delta_{\mathbb{G}}$ shall never be used, and we stress that all occurrences of the greek letter Δ (without subscript $\mathbb{G}$) concern generators of quantum Markov semigroups.

Set $u^1\in C(\mathbb{G})\otimes M_{n_1}(\mathbb{C})$ and $u^2\in C(\mathbb{G})\otimes M_{n_2}(\mathbb{C})$. The tensor product $u^1\otimes u^2$ is defined as the representation $u^1_{12}u^2_{13}$. We call u irreducible if the matrix algebra generated by $(\omega\otimes\mathrm{id})(u),\omega\in C(\mathbb{G})^*$, is simple. A morphism between $u^1\in C(\mathbb{G})\otimes M_{n_1}(\mathbb{C})$ and $u^2\in C(\mathbb{G})\otimes M_{n_2}(\mathbb{C})$ is a map $T:\mathbb{C}^{n_1}\to\mathbb{C}^{n_2}$ such that $u^1(1\otimes T)=(1\otimes T)u^2$. Let $\mathrm{Mor}\left(u^1,u^2\right)$ be the (normed) vector space of morphisms. There is a quantum version of Schur's lemma which states that u is irreducible if and only if $\mathrm{Mor}(u,u)=\mathbb{C}1$. If $\mathrm{Mor}\left(u^1,u^2\right)$ contains a unitary element, then u^1 and u^2 are called equivalent. We write $\mathrm{Irr}(\mathbb{G})$ for the equivalence classes of irreducible representations and $\mathrm{Rep}(\mathbb{G})$ for the equivalence classes of all finite dimensional representations. Its elements shall typically be denoted by α,β and γ. The dimension of $\alpha\in\mathrm{Rep}(\mathbb{G})$ is denoted by n_α and satisfies $n_\alpha\leq\mathrm{qdim}(\alpha)$. Tensor products and Mor are well defined on equivalence classes. For $\alpha,\beta\in\mathrm{Irr}(\mathbb{G})$, the tensor product $\alpha\otimes\beta$ is equivalent to a direct sum of irreducibles $\oplus_{\gamma\in\mathrm{Irr}(\mathbb{G})}m_\gamma\cdot\gamma$, where $m_\gamma\cdot\gamma=\oplus_{i=1}^{m_\gamma}\gamma$ is an m_γ-fold copy. This decomposition is unique up to equivalence, and the set of all such decompositions is referred to as the fusion rules. We write $\alpha\subseteq\beta$ if $\mathrm{Mor}(\alpha,\beta)$ contains an isometry. For $\alpha\in\mathrm{Rep}(\mathbb{G})$ we denote by $\overline{\alpha}$ its *contragredient representation*.

Proposition 1.5 (Frobenius duality)**.** *For $\alpha,\beta,\gamma\in\mathrm{Rep}(\mathbb{G})$, we have $\mathrm{Mor}(\alpha,\beta\otimes\gamma)\simeq\mathrm{Mor}\left(\overline{\beta}\otimes\alpha,\gamma\right)$ linearly. Consequently, if α and γ are irreducible, then $\alpha\subseteq\beta\otimes\gamma$ if and only if $\gamma\subseteq\overline{\beta}\otimes\alpha$.*

Lemma 1.6. *Set $\alpha,\gamma\in\mathrm{Irr}(\mathbb{G})$. There are only finitely many $\beta\in\mathrm{Irr}(\mathbb{G})$ such that $1\subseteq\alpha\otimes\beta\otimes\gamma$.*

Proof. If $1\subseteq\alpha\otimes\beta\otimes\gamma$, then by Frobenious duality we have $\beta\subseteq\overline{\alpha}\otimes\overline{\gamma}$, and there are only finitely many such β. □

We let $\mathrm{Pol}(\mathbb{G})$ be the $*$-algebra of matrix coefficients of (finite-dimensional) representations of $\mathbb{G}$. It is given by the linear span of $(\mathrm{id}\otimes\omega)(u)$ for all representations $u\in C(\mathbb{G})\otimes M_n(\mathbb{C})$ and $\omega\in M_n(\mathbb{C})^*$. There is a distinguished faithful $*$-homomorphism $\epsilon:\mathrm{Pol}(\mathbb{G})\to\mathbb{C}$ called the *counit* that satisfies

$$(\epsilon\otimes\mathrm{id})\circ\Delta_{\mathbb{G}}=\mathrm{id}=(\mathrm{id}\otimes\epsilon)\circ\Delta_{\mathbb{G}}.$$

$\mathrm{Pol}(\mathbb{G})$ carries the inner product $\langle x,y\rangle=\varphi(y^*x)$ and norm $\|x\|_2^2=\langle x,x\rangle$. The completion of $\mathrm{Pol}(\mathbb{G})$ with respect to this norm is called $L_2(\mathbb{G})$ and may be identified with the GNS-space of φ. For $\alpha\in\mathrm{Irr}(\mathbb{G})$ we let $P_\alpha:\mathrm{Pol}(\mathbb{G})\to\mathrm{Pol}(\mathbb{G})$ be the orthogonal projection onto the matrix coefficients of α.

For compact quantum groups $\mathbb{H}$ and $\mathbb{G}$ we say that $\mathbb{H}$ is a dual quantum subgroup of $\mathbb{G}$, notation $\widehat{\mathbb{H}}<\widehat{\mathbb{G}}$, if $L_\infty(\mathbb{H})\subseteq L_\infty(\mathbb{G})$ and the von Neumann algebraic comultiplication of $L_\infty(\mathbb{G})$ restricts to $L_\infty(\mathbb{H})$ as the comultiplication of $\mathbb{H}$. In this case $\mathrm{Irr}(\mathbb{H})\subseteq\mathrm{Irr}(\mathbb{G})$ naturally, and the fusion rules and morphisms of $\mathrm{Irr}(\mathbb{G})$ restrict to $\mathrm{Irr}(\mathbb{H})$ (it is a full subcategory).

A *central multiplier* $\Phi:L_\infty(\mathbb{G})\to L_\infty(\mathbb{G})$ is a map such that for every $\alpha\in\mathrm{Irr}(\mathbb{G})$ there exist $\Delta_\alpha\in\mathbb{C}$ such that $\Phi((\mathrm{id}\otimes\omega)(\alpha))=\Delta_\alpha(\mathrm{id}\otimes\omega)(\alpha)$ for all $\alpha\in\mathrm{Irr}(\mathbb{G})$ and $\omega\in M_{n_\alpha}(\mathbb{C})^*$. We refer to [29] for more general background on multipliers.

Remark 1.7. We have that $(\mathrm{Irr}(\mathbb{G}),\mathrm{Mor})$ with the tensor products, fusion rules and contragredients forms a rigid C*-tensor category. A large part of this paper can be directly translated in terms of the abstract setting of rigid C*-tensor categories. However, since our many applications are in quantum group theory, our presentation follows the terminology of quantum group theory. Recall that by Tannaka–Krein duality, rigid C*-tensor categories with specified fibre functor are always of the form $(\mathrm{Irr}(\mathbb{G}),\mathrm{Mor})$ [52].

1.3. Quantum Markov semigroups

Let M be a von Neumann algebra with a faithful normal state φ. A *quantum Markov semigroup (QMS)* $\Phi=(\Phi_t)_{t\geq0}$ is a semigroup of normal unital completely positive maps $\Phi_t:M\to M$ such that for every $x\in M$, the map $t\mapsto\Phi_t(x)$ is strongly continuous. Moreover, we assume that a QMS is GNS-symmetric in the sense that $\varphi(\Phi_t(x)y)=\varphi(x\Phi_t(y))$ for all $x,y\in M$. The QMS Φ is called φ-modular (or modular) if $\Phi_t\circ\sigma_s^\varphi=\sigma_s^\varphi\circ\Phi_t$ for all $t\geq0,s\in\mathbb{R}$, where σ^φ is the modular automorphism group of φ [43]. The QMSs occuring in this paper are QMSs of central multipliers which are always modular and GNS-symmetric. Further, they are norm-continuous on $\mathrm{Pol}(\mathbb{G})$. It should also be stressed that the most important of our applications are for finite von Neumann algebras and φ tracial. However, in the analysis we shall also need the Haar state on $\mathbb{G}_q=SU_q(2),q\in(-1,1)$, which is nontracial even though $L_\infty(\mathbb{G}_q)$ is of type I.

If Φ is a QMS of central multipliers, then for every $\alpha\in\mathrm{Irr}(\mathbb{G})$ there exists $\Delta_\alpha\geq0$ such that $\Phi_t(x_\alpha)=\exp(-t\Delta_\alpha)x_\alpha$ for every matrix coefficient x_α of α. The values $(\Delta_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})}$ completely determine Φ. We set the generator $\Delta:\subseteq L_2(\mathbb{G})\to L_2(\mathbb{G})$ to be the closure of

$$\mathrm{Pol}(\mathbb{G})\to\mathrm{Pol}(\mathbb{G}):x_\alpha\mapsto\Delta_\alpha x_\alpha.$$

The QMS Φ is called *immediately L_2-compact* if Δ has compact resolvent. The generator Δ is closely related to the associated quantum Dirichlet form. In [14] it was proved that a (general) von Neumann algebra has the Haagerup property if and only if it admits an immediately L_2-compact QMS.

1.4. Free products

To two compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ one can associate a free product quantum group $\mathbb{G}_1 * \mathbb{G}_2$ [49]. It satisfies $L_\infty(\mathbb{G}) = L_\infty(\mathbb{G}_1) * L_\infty(\mathbb{G}_2)$, where free products are taken with respect to the von Neumann algebraic Haar states. The quantum groups's Haar state is the free product of the Haar states of the two compact quantum groups. Moreover, the quantum group can be equipped with a natural comultiplication, which shall not be used in this paper. What is relevant for us is the following proposition, which describes $\mathrm{Irr}(\mathbb{G})$ as a fusion category:

Proposition 1.8 ([49] or [11, Theorem 3.4])**.** *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be compact quantum groups. A tensor product $\gamma_1 \otimes \cdots \otimes \gamma_n$ with $\gamma_i \in \mathrm{Irr}(\mathbb{G}_{k_i})$ and $k_i \neq k_{i+1}$ is called reduced. All such reduced tensor products form a well-defined complete set of mutually inequivalent irreducible representations of $\mathbb{G}_1 * \mathbb{G}_2$. In other words, they constitute $\mathrm{Irr}(\mathbb{G}_1 * \mathbb{G}_2)$. The fusion rules are as follows for reduced tensors $\beta_1 \otimes \cdots \otimes \beta_l$ and $\gamma_1 \otimes \cdots \otimes \gamma_n$. If β_l and γ_1 are not representations of the same quantum group, then*

$$\beta_1 \otimes \cdots \otimes \beta_l \cdot \gamma_1 \otimes \cdots \otimes \gamma_n = \beta_1 \otimes \cdots \otimes \beta_l \otimes \gamma_1 \otimes \cdots \otimes \gamma_n.$$

If β_l and γ_1 are representations of the same quantum group, then

$$\begin{aligned}(\beta_1 \otimes \cdots \otimes \beta_l) \otimes (\gamma_1 \otimes \cdots \otimes \gamma_n) = &\left((\beta_1 \otimes \cdots \otimes \beta_{l-1}) \otimes \left(\bigoplus_{i, \alpha_i \neq 1} \alpha_i \right) \otimes (\gamma_2 \otimes \cdots \otimes \gamma_n) \right) \\ &\oplus \left(\bigoplus_{i, \alpha_i = 1} (\beta_1 \otimes \cdots \otimes \beta_{l-1}) \otimes (\gamma_2 \otimes \cdots \otimes \gamma_n) \right),\end{aligned} \tag{1.2}$$

where $\beta_l \otimes \gamma_1 = \oplus_i \alpha_i$ is the decomposition of $\beta_l \otimes \gamma_1$ into irreducibles (with possible multiplicity). Note that in equation (1.2) the latter summand is not necessarily reduced, but the fusion rules are hereby defined inductively.

We shall use the shorthand notation

$$\gamma_1 \cdots \gamma_n = \gamma_1 \otimes \cdots \otimes \gamma_n$$

for a reduced word.

1.5. Multiplicity freeness

A compact quantum group $\mathbb{G}$ is called *multiplicity free* if for $\alpha, \beta, \gamma \in \mathrm{Irr}(\mathbb{G})$, the space $\mathrm{Mor}(\gamma, \alpha \otimes \beta)$ is ≤ 1-dimensional. That is, γ occurs at most once in the decomposition of $\alpha \otimes \beta$ into irreducible representations. When $\mathbb{G}_1$ and $\mathbb{G}_2$ are multiplicity free, then in equation (1.2) the last summation is in fact a single summand if $\beta_k = \overline{\gamma_1}$, and it vanishes otherwise (it follows, for example, by Frobenius duality, Proposition 1.5). So, with the

summation over α going over irreducible representations, we record that

$$\begin{aligned}&(\beta_1\cdots\beta_k)\otimes(\gamma_1\cdots\gamma_n)\\&=\bigoplus_{i=1}^{L}\bigoplus_{1\neq\alpha\subseteq\beta_{k-i+1}\otimes\gamma_i}(\beta_1\cdots\beta_{k-i})\alpha(\gamma_{i+1}\cdots\gamma_n),\end{aligned}\tag{1.3}$$

where $L-1$ is the maximum index i for which $\gamma_i=\overline{\beta_{k-i+1}}$. We note that the summands in equation (1.3) are reduced. This decomposition shall be used without further reference in the rest of the paper.

Assumption. Throughout the entire paper we assume that all compact quantum groups (e.g., $\mathbb{H}$, $\mathbb{G}$, $\mathbb{G}_1$ and $\mathbb{G}_2$) are multiplicity free.

The following result should be well known and is easy to prove:

Proposition 1.9. *If $\mathbb{G}_1$ and $\mathbb{G}_2$ are compact quantum groups that are multiplicity free, then so is $\mathbb{G}_1*\mathbb{G}_2$. If $\widehat{\mathbb{H}}<\widehat{\mathbb{G}}$ and $\mathbb{G}$ is multiplicity free, then so is $\mathbb{H}$.*

Proof. Suppose that we have an irreducible representation $\alpha=\alpha_1\cdots\alpha_l$ contained in $(\beta_1\cdots\beta_k)\otimes(\gamma_1\cdots\gamma_n)$. Then by considering the length, α must be one of the ith summands in equation (1.3), with i satisfying $2i=k+n-l+1$. But all those summands are mutually inequivalent, by Proposition 1.8 and the fact that $\mathbb{G}_1$ and $\mathbb{G}_2$ are multiplicity free. □

That $\mathbb{G}$ is multiplicity free has the following consequence. For $\beta,\gamma\in\mathrm{Irr}(\mathbb{G})$ and $\alpha\subseteq\beta\otimes\gamma$, there exists an intertwiner

$$V_\alpha^{\beta,\gamma}\in\mathrm{Mor}(\alpha,\beta\otimes\gamma)$$

that is moreover unique up to a phase factor. All expressions and proofs occuring in this paper are independent of this phase factor unless mentioned otherwise.

1.6. Monoidal equivalence

Definition 1.10. Two compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ are called *monoidally equivalent* if there exists a bijection $\pi:(\mathrm{Irr}(\mathbb{G}_1),\mathrm{Mor}_{\mathbb{G}_1})\to(\mathrm{Irr}(\mathbb{G}_2),\mathrm{Mor}_{\mathbb{G}_2})$ that maps the trivial representation of $\mathbb{G}_1$ to the trivial representation of $\mathbb{G}_2$ and which for any morphisms S,T and unit $1_\alpha\in\mathrm{Mor}(\alpha,\alpha),\alpha\in\mathrm{Rep}(\mathbb{G})$, satisfies

$$\begin{aligned}&\pi(1_\alpha)=1_\alpha,\qquad\pi(S\otimes T)=\pi(S)\otimes\pi(T),\\&\pi(S^*)=\pi(S)^*,\qquad\pi(ST)=\pi(S)\pi(T),\end{aligned}$$

where in the last equality we assume that S and T are composable. The bijection π is then called a monoidal equivalence.

Proposition 1.11. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be monoidally equivalent compact quantum groups, so that we may identify $\mathrm{Irr}(\mathbb{G}_1)=\mathrm{Irr}(\mathbb{G}_2)$. Let $\left(\Phi_t^1\right)_{t\geq0}$ be a QMS of central multipliers on $L_\infty(\mathbb{G}_1)$ such that $\Phi_t^1(x_\alpha)=\exp(-t\Delta_\alpha)x_\alpha$ for every matrix coefficient x_α of $\alpha\in\mathrm{Irr}(\mathbb{G}_1)$. Then there exists a QMS of central multipliers $\left(\Phi_t^2\right)_{t\geq0}$ on $L_\infty(\mathbb{G}_2)$ such that $\Phi_t^2(x_\alpha)=\exp(-t\Delta_\alpha)x_\alpha$ for every matrix coefficient x_α of $\alpha\in\mathrm{Irr}(\mathbb{G}_2)$.*

Proof. The proof of this fact is the same as [24, Proposition 6.3], and is based on [5, Theorems 3.9 and 6.1] together with a transference method. □

In a sense, one could also say that a central QMS lives on the level of the rigid C*-tensor category [31].

2. A rigid C*-tensor category approach to gradient estimates

To a QMS on a tracial von Neumann algebra one can associate a canonical bimodule (in principle only defined over a dense subalgebra of M) which is called the gradient bimodule H_∇. In [10], sufficient conditions were given to assure that H_∇ is in fact a von Neumann bimodule that is moreover quasicontained in the coarse bimodule. In this section we provide a categorical viewpoint on the approach in [10]. What we show is that the methods and estimates that occur in the proofs of [10] actually live on the level of a monoidal category. In particular, all computations in [10] can be carried out on the level of $SU_q(2)$, after which they transfer to a much larger class of quantum groups. A particular feature of our current approach is that the properties we consider are stable under *repeated* applications of constructions like free products, wreath products, taking dual quantum subgroups and monoidal equivalence. This should be compared to, for instance, [26, Theorem C], where such results (and consequences for rigidity properties) were limited to free products of quantum groups in a specific class. We thus cover a richer class of quantum groups than has occurred in the literature so far. In particular, this approach allows us to use the main result of [30], and we cover in particular free wreath products and H_N^+. We prove, for instance, that H_N^+ is strongly solid. We will come back to these results in subsequent sections. In the current section we introduce the main technical definition of being 'approximately linear with almost commuting intertwiners' and prove that is stable under free products, monoidal equivalence and taking quantum subgroups.

2.1. Approximately linear with almost commuting intertwiners

Let $\mathbb{G}$ be a compact quantum group and recall that it is assumed to be multiplicity free. For $\alpha,\beta,\gamma \in \mathrm{Irr}(\mathbb{G}),\beta_2 \subseteq \alpha\otimes\beta\otimes\gamma$, we define

$$\begin{aligned}
L_\beta^{\alpha,\gamma} &= \{(\beta_1,\beta_2) \in \mathrm{Irr}(\mathbb{G})\times\mathrm{Irr}(\mathbb{G}) \mid \beta_1 \subseteq \alpha\otimes\beta, \beta_2 \subseteq \beta_1\otimes\gamma\},\\
R_\beta^{\alpha,\gamma} &= \{(\beta_1,\beta_2) \in \mathrm{Irr}(\mathbb{G})\times\mathrm{Irr}(\mathbb{G}) \mid \beta_1 \subseteq \beta\otimes\gamma, \beta_2 \subseteq \alpha\otimes\beta_1\},\\
L_{\beta,\beta_2}^{\alpha,\gamma} &= \left\{\beta_1 \in \mathrm{Irr}(\mathbb{G}) \mid (\beta_1,\beta_2) \in L_\beta^{\alpha,\gamma}\right\},\\
R_{\beta,\beta_2}^{\alpha,\gamma} &= \left\{\beta_1 \in \mathrm{Irr}(\mathbb{G}) \mid (\beta_1,\beta_2) \in R_\beta^{\alpha,\gamma}\right\}.
\end{aligned}$$

Lemma 2.1. *Given $\alpha,\gamma \in \mathrm{Irr}(\mathbb{G})$, the number of elements in the sets $L_\beta^{\alpha,\gamma}$, $R_\beta^{\alpha,\gamma}$, $L_{\beta,\beta_2}^{\alpha,\gamma}$, $R_{\beta,\beta_2}^{\alpha,\gamma}$ is bounded uniformly in β,β_2.*

Proof. Suppose that $\beta_1 \subseteq \alpha\otimes\beta,\beta_1 \in \mathrm{Irr}(\mathbb{G})$; then by Frobenius duality (Proposition 1.5) we have that $\beta \subseteq \overline{\alpha}\otimes\beta_1$. But this can only happen if $\dim(\beta) \leq \dim(\overline{\alpha})\dim(\beta_1)$, so that $\dim(\beta_1) \geq \dim(\beta)\dim(\overline{\alpha})^{-1}$. By counting dimensions we see that $\alpha\otimes\beta$ can therefore have

at most $\dim(\overline{\alpha})$ irreducible inequivalent subrepresentations. Applying the same argument in turn to $\beta_1 \otimes \beta \otimes \gamma$, we see that there are at most $\dim(\overline{\gamma})$ irreducible representations contained in this representation. □

Let $\Phi := (\Phi_t)_{t\geq 0}$ be a QMS of central multipliers on $\mathbb{G}$. The following definition is our main technical tool. Recall that we need $\mathbb{G}$ to be multiplicity free to define up to a phase factor uniquely determined intertwiners $V_\gamma^{\alpha,\beta}$, $\alpha,\beta,\gamma \in \mathrm{Irr}(\mathbb{G})$. So from this point the multiplicity freeness is being used.

Definition 2.2. We say that Φ is *approximately linear with almost commuting intertwiners* if the following holds. For every $\alpha,\gamma \in \mathrm{Irr}(\mathbb{G})$, there exists a finite set $A_{00} := A_{00}(\alpha,\gamma) \subseteq \mathrm{Irr}(\mathbb{G})$ such that for every $\beta \in \mathrm{Irr}(\mathbb{G})\backslash A_{00}$ and $\beta_2 \subseteq \alpha\otimes\beta\otimes\gamma$, there exist bijections (called the *v-maps*)

$$v^{\alpha,\gamma}(\,\cdot\,;\beta,\beta_2) := v(\,\cdot\,;\beta,\beta_2) : L^{\alpha,\gamma}_{\beta,\beta_2} \to R^{\alpha,\gamma}_{\beta,\beta_2},$$

such that the following holds. There exists a set $A \subseteq \mathrm{Irr}(\mathbb{G})\backslash A_{00}$ and a constant $C := C(\alpha,\gamma) > 0$ such that the following are true:

(1) For all $\beta \in A, (\beta_1,\beta_2) \in L^{\alpha,\gamma}_\beta$, we have

$$\left|\Delta_\beta - \Delta_{\beta_1} - \Delta_{v(\beta_1;\beta,\beta_2)} + \Delta_{\beta_2}\right| \leq C\,\mathrm{qdim}(\beta)^{-1} \tag{2.1}$$

and

$$|\Delta_\beta - \Delta_{\beta_1}| \leq C, \tag{2.2}$$

and for all $\beta \in \mathrm{Irr}(\mathbb{G})\backslash(A\cup A_{00}), (\beta_1,\beta_2) \in L^{\alpha,\gamma}_\beta$, we have

$$\Delta_\beta - \Delta_{\beta_1} - \Delta_{v(\beta_1;\beta,\beta_2)} + \Delta_{\beta_2} = 0. \tag{2.3}$$

(2) For all $\beta \in A, (\beta_1,\beta_2) \in L^{\alpha,\gamma}_\beta$, we have

$$\inf_{z\in\mathbb{T}} \left\| V^{\beta_1,\gamma}_{\beta_2}\left(V^{\alpha,\beta}_{\beta_1}\otimes \mathrm{id}_\gamma\right) - zV^{\alpha,v(\beta_1;\beta,\beta_2)}_{\beta_2}\left(\mathrm{id}_\alpha \otimes V^{\beta,\gamma}_{v(\beta_1;\beta,\beta_2)}\right)\right\| \leq C\,\mathrm{qdim}(\beta)^{-1} \tag{2.4}$$

and for all $\beta \in \mathrm{Irr}(\mathbb{G})\backslash(A\cup A_{00}), (\beta_1,\beta_2) \in L^{\alpha,\gamma}_\beta$, we have

$$\inf_{z\in\mathbb{T}} \left\| V^{\beta_1,\gamma}_{\beta_2}\left(V^{\alpha,\beta}_{\beta_1}\otimes \mathrm{id}_\gamma\right) - zV^{\alpha,v(\beta_1;\beta,\beta_2)}_{\beta_2}\left(\mathrm{id}_\alpha \otimes V^{\beta,\gamma}_{v(\beta_1;\beta,\beta_2)}\right)\right\| = 0. \tag{2.5}$$

(3) There exists a polynomial P such that for every $N \in \mathbb{N}$ we have

$$\#\{\beta \in A \mid \Delta_\beta < N\} \leq P(N) \tag{2.6}$$

and $\beta \mapsto \delta(\beta \in A)\,\mathrm{qdim}(\beta)^{-1}$ is square summable.

Remark 2.3. In summary, Definition 2.2 entails the following. We cut $\mathrm{Irr}(\mathbb{G})$ into three disjoint sets. Each of these sets has a size condition and a condition on estimates of eigenvalues of Δ, as well as certain almost commutations of intertwiners:

A_{00}	A	The rest: $\mathrm{Irr}(\mathbb{G})\backslash(A_{00}\cup A)$
-Finite set	-Grows polynomially compared to Δ	-No size restrictions
-No conditions	-Estimates (2.1),(2.2) and (2.4)	-Vanishing of equations (2.3),(2.5)

We shall usually refer to property (1) as being approximately linear and (2) as having almost commuting intertwiners. We note that they have to be satisfied for the same choice of A and A_{00}, which is why we did not define 'approximate linearity' and 'almost commuting intertwiners' as independent notions.

Theorem 2.4. *The property of Φ being approximately linear with almost commuting intertwiners is stable under monoidal equivalence of compact quantum groups.*

Proof. Monoidally equivalent compact quantum groups have the same representation category seen as a rigid C*-tensor category. In particular, the quantum dimension, norms of intertwiners and irreducible representations with their fusion rules are invariant under monoidal equivalence (see [5, Remarks 3.2, 3.4 and 3.5]). Since all properties in Definition 2.2 are expressed in these terms, the theorem follows directly. □

The following theorem is clear to specialists; for completeness, we give its proof:

Theorem 2.5. *Suppose that Φ is a QMS of central multipliers on a compact quantum group $\mathbb{G}$. Suppose that $\mathbb{H}$ is a compact quantum group with $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$. Then $\mathrm{Irr}(\mathbb{H}) \subseteq \mathrm{Irr}(\mathbb{G})$ and $L_\infty(\mathbb{H}) \subseteq L_\infty(\mathbb{G})$. In particular, the restriction of Φ to $L_\infty(\mathbb{H})$ is a QMS of central multipliers. Furthermore, if Φ is approximately linear with almost commuting intertwiners, then so is its restriction to $L_\infty(\mathbb{H})$.*

Proof. Indeed, if $\widehat{\mathbb{H}} < \widehat{\mathbb{G}}$, then there exists a surjective $*$-homomorphism $\widehat{\pi} : \ell_\infty\left(\widehat{\mathbb{G}}\right) \to \ell_\infty\left(\widehat{\mathbb{H}}\right)$. Since $\ell_\infty\left(\widehat{\mathbb{G}}\right)$ is an ℓ_∞-direct sum of finite-dimensional simple C*-algebras (i.e., matrix algebras), $\widehat{\pi}$ must be either 0 or faithful on each of the simple matrix blocks. Then $\ell_\infty\left(\widehat{\mathbb{H}}\right)$ is given by the ℓ_∞-direct sum of all matrix blocks for which $\widehat{\pi}$ is faithful. The matrix blocks of $\ell_\infty\left(\widehat{\mathbb{G}}\right)$ are labelled by $\mathrm{Irr}(\mathbb{G})$ and the matrix blocks of $\ell_\infty\left(\widehat{\mathbb{H}}\right)$ are labelled by $\mathrm{Irr}(\mathbb{H})$, which is thus a subset of $\mathrm{Irr}(\mathbb{G})$. Since $L_\infty(\mathbb{H})$ is generated by the matrix coefficients of $\mathrm{Irr}(\mathbb{H})$, it must thus be a subalgebra of $L_\infty(\mathbb{G})$. We see that Φ restricts to $L_\infty(\mathbb{H})$ and is again a QMS of central multipliers. It is clear that Φ restricted to $L_\infty(\mathbb{H})$ satisfies Definition 2.2, since one has to check fewer conditions than for the original Φ (in particular, the sets $L^{\alpha,\gamma}_{\beta,\beta_2}$ and $R^{\alpha,\gamma}_{\beta,\beta_2}$ and the bijection $v^{\alpha,\gamma}(\cdot;\beta,\beta_2)$ stay the same but need only be considered for $\alpha,\beta,\gamma \in \mathrm{Irr}(\mathbb{H}), \beta_2 \subseteq \alpha\otimes\beta\otimes\gamma$). □

2.2. Free products

Our next aim is to show that Definition 2.2 is stable under free products.

Theorem 2.6. *Let Φ^1 and Φ^2 be QMSs of central multipliers on respective compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$. Let $\Phi = \Phi^1 * \Phi^2$ be the free product QMS of central multipliers on $\mathbb{G}_1 * \mathbb{G}_2$. If Φ^1 and Φ^2 are both approximately linear with almost commuting intertwiners, then so is Φ.*

The proof of this theorem will take the rest of this section, for which we will fix the following notation. First we let Δ be the generator of Φ with eigenvalues $\Delta_\alpha, \alpha \in \mathrm{Irr}(\mathbb{G})$. In particular, this defines Δ_α for the subsets $\mathrm{Irr}(\mathbb{G}_1)$ and $\mathrm{Irr}(\mathbb{G}_2)$ of $\mathrm{Irr}(\mathbb{G})$. The straightforward proof of the following lemma can be found at [10, Beginning of Section 5]:

Lemma 2.7 (Leibniz rule)**.** *For $\beta = \beta_1 \cdots \beta_l \in \mathrm{Irr}(\mathbb{G})$ a reduced word, we have*

$$\Delta_\beta = \sum_{r=1}^{l} \Delta_{\beta_r}.$$

Now let

$$\alpha = \alpha_1 \cdots \alpha_k, \qquad \gamma = \gamma_1 \cdots \gamma_m,$$

in $\mathrm{Irr}(\mathbb{G})$ be reduced words of representations of lengths k and m, respectively. So $\alpha_i, i = 1, \ldots, k$, is alternatingly in $\mathrm{Irr}(\mathbb{G}_1)$ and $\mathrm{Irr}(\mathbb{G}_2)$, and similarly for γ_i. When $\alpha_i, \gamma_j \in \mathrm{Irr}(\mathbb{G}_1)$ (resp., $\alpha_i, \gamma_j \in \mathrm{Irr}(\mathbb{G}_2)$), we define $A_{00}^1(\alpha_i, \gamma_j)$ and $A^1(\alpha_i, \gamma_j)$ (resp., $A_{00}^2(\alpha_i, \gamma_j)$ and $A^2(\alpha_i, \gamma_j)$) to be the sets A_{00} and A of Definition 2.2 for $\mathbb{G}_1$ (resp., $\mathbb{G}_2$) with respect to α_i, γ_j and Φ^1 (resp., Φ^2). This makes sense because of the assumption that α_i and γ_j are representations of the same quantum group.

Definition of A_{00} and A associated to $\alpha, \gamma \in \mathbb{G}$. The set $A_{00} \subseteq \mathrm{Irr}(\mathbb{G})$ will consist of all representations $\beta \in \mathrm{Irr}(\mathbb{G})$ of the following form:

- β equals a reduced word $\beta = \overline{\alpha}_k \cdots \overline{\alpha}_{k-i+1} \overline{\gamma}_j \cdots \overline{\gamma}_1$ for some $0 \leq i \leq k$, $0 \leq j \leq m$.
- β equals a reduced word $\beta = \overline{\alpha}_k \cdots \overline{\alpha}_{k-i+1} \beta_{i+1} \overline{\gamma}_j \cdots \overline{\gamma}_1$ for some $0 \leq i < k$, $0 \leq j < m$, and at least one of the following holds:
  - $\beta_{i+1} \in A_{00}^s(\alpha_{k-i}, \gamma_{j+1})$ when there is $s \in 1,2$ such that $\alpha_{k-i}, \gamma_{j+1} \in \mathrm{Irr}(\mathbb{G}_s)$,
  - $1 \subseteq \alpha_{k-i} \otimes \beta_{i+1} \otimes \gamma_{j+1}$.

Since $A_{00}^s, s = 1,2$, is finite (for the first sub-bullet) and we have Lemma 1.6 (for the second sub-bullet), we see that A_{00} is a finite set. We set $A \subseteq \mathrm{Irr}(\mathbb{G})$ to be the set of representations $\beta \in \mathrm{Irr}(\mathbb{G})$ of the following form:

- β equals a reduced word $\beta = \overline{\alpha}_k \cdots \overline{\alpha}_{k-i+1} \beta_{i+1} \overline{\gamma}_j \cdots \overline{\gamma}_1$ for some $0 \leq i < k$, $0 \leq j < m$, and $\beta_{i+1} \in A^s(\alpha_{k-i}, \gamma_{j+1})$ when there is $s \in 1,2$ such that $\alpha_{k-i-1}, \gamma_{j+1} \in \mathrm{Irr}(\mathbb{G}_s)$.

As part of the proof of Theorem 2.6, we shall at this point establish that Definition 2.2, property (2.2) holds.

Lemma 2.8. *Property (2.2) holds for $\mathbb{G}$ and the choice of A.*

Proof. The QMSs on $\mathbb{G}_1$ and $\mathbb{G}_2$ are both approximately linear with almost commuting intertwiners. Therefore, let P be a polynomial such that for all possible choices $s = 1,2$ and $1 \leq i \leq k$, $1 \leq j \leq m$ such that $\alpha_i, \gamma_j \in \mathrm{Irr}(\mathbb{G}_s)$, we have for all $N \in \mathbb{N}$ that

$$\#\left\{\widetilde{\beta} \in A^s(\alpha_i, \gamma_j) \mid \Delta_{\widetilde{\beta}} \leq N\right\} \leq P(N).$$

Suppose that $\beta \in A$. Then from the definition of A we see that the length of the reduced expression $\beta = \beta_1 \cdots \beta_l$ cannot be longer than the sum of the lengths of α and γ minus 1 – that is, $l \leq k+m-1$. Moreover, we may write $\beta = \beta_1 \cdots \beta_l = \overline{\alpha}_k \cdots \overline{\alpha}_{k-i+1} \beta_{i+1} \overline{\gamma}_j \cdots \overline{\gamma}_1$ for some $0 \leq i < k$, $0 \leq j < m$, with $i+j+2 = l$, and there is an $s = 1,2$ such that $\beta_{i+1} \in A^s(\alpha_{k-i}, \gamma_{j+1})$. We have by the Leibniz rule $\Delta_\beta := \Delta_{\beta_1 \cdots \beta_l} = \sum_{r=1}^{l} \Delta_{\beta_r}$. If $\Delta_\beta \leq N$, then certainly $\Delta_{\beta_{i+1}} \leq N$. Therefore, we crudely estimate

$$\#\{\beta \in A \mid \Delta_\beta \leq N\} \leq (k+m-1)^2 P(N).$$

This concludes the proof of the growth bound on A as in Definition 2.2(2.2). From a similar reasoning, it also follows that $\beta \mapsto \delta(\beta)\,\mathrm{qdim}(\beta)^{-1}$ is square summable. □

Definition of the bijections $v^{\alpha,\beta}(\,\cdot\,;\beta,\beta_2)$ *for* $\mathbb{G}$. Take $\beta \in \mathrm{Irr}(\mathbb{G}) \backslash A_{00}$. There are three cases to be treated.

Case 1. Assume that there exists some $i < j$ such that we have a decomposition as a reduced word

$$\beta = (\beta_1 \cdots \beta_i)(\beta_{i+1} \cdots \beta_{j-1})(\beta_j \cdots \beta_l),$$

where $1 \leq i$ is the smallest index for which β_i is not the conjugate of α_{k-i+1} (and if this does not exist, then $i = 1$) and $j \leq l$ is the largest index such that β_j is not the conjugate of γ_{l-j+1} (and if this does not exist, then $j = l$). Heuristically, this means that in $\alpha \otimes \beta \otimes \gamma$, the letters of α can annihilate at most the first $i-1$ letters of β, and the letters of γ can annihilate at most the last $l-j$ letters of β. More precisely, we get the following. The irreducible representations contained in $\alpha \otimes \beta \otimes \gamma$ are precisely given by representations that have a reduced expression

$$\beta'(\beta_{i+1} \cdots \beta_{j-1})\beta'', \quad \text{with } \beta' \subseteq \alpha \otimes (\beta_1 \cdots \beta_i), \beta'' \subseteq (\beta_j \cdots \beta_l) \otimes \gamma \text{ irreducible.}$$

Furthermore, we have singleton sets

$$L^{\alpha,\gamma}_{\beta,\beta'(\beta_{i+1}\cdots\beta_{j-1})\beta''} = \{\beta'(\beta_{i+1}\cdots\beta_l)\} \quad \text{and} \quad R^{\alpha,\gamma}_{\beta,\beta'(\beta_{i+1}\cdots\beta_{j-1})\beta''} = \{(\beta_1\cdots\beta_{j-1})\beta''\}.$$

We therefore set the bijection from $L^{\alpha,\gamma}_{\beta,\beta'(\beta_{i+1}\cdots\beta_{j-1})\beta''}$ to $R^{\alpha,\gamma}_{\beta,\beta'(\beta_{i+1}\cdots\beta_{j-1})\beta''}$ by

$$v(\beta'(\beta_{i+1}\cdots\beta_l);\beta,\beta'(\beta_{i+1}\cdots\beta_{j-1})\beta'') = (\beta_1\ldots\beta_{j-1})\beta''.$$

Case 2. Assume that we have a reduced expression

$$\beta = \beta_1 \cdots \beta_l = \overline{\alpha}_k \cdots \overline{\alpha}_{k-i+1} \beta_{i+1} \overline{\gamma}_j \cdots \overline{\gamma}_1, \tag{2.7}$$

for some $0 \leq i < k$, $0 \leq j < m$, with $i+j+1 = l$. Moreover, since $\beta \notin A_{00}$, we assume that $\beta_{i+1} \notin A^s_{00}(\alpha_{k-i}, \gamma_{j+1}), s = 1,2$.

A representation contained in $\alpha \otimes \beta \otimes \gamma$ can have two different forms, which determine Case 2 and Case 3. In Case 2 we assume that α_{k-i}, γ_{j+1} and β_{i+1} are representations of the same quantum group. Moreover, we assume that we have a subrepresentation of $\alpha \otimes \beta \otimes \gamma$ of the form $\alpha_1 \cdots \alpha_{k-i-1} \beta''_{i+1} \gamma_{j+2} \cdots \gamma_m$, where $\beta''_{i+1} \subseteq \alpha_{k-i} \otimes \beta_{i+1} \otimes \gamma_{j+1}$ is irreducible. Further, β''_{i+1} is nontrivial, since $\beta \notin A_{00}$. So the expression $\alpha_1 \cdots \alpha_{k-i-1} \beta''_{i+1} \gamma_{j+2} \cdots \gamma_m$

is reduced. In this case, since we already observed that $\beta_{i+1} \notin A_{00}^{s}(\alpha_{k-i},\gamma_{j+1}), s=1,2$, so that the sets below are defined, we have

$$L^{\alpha,\gamma}_{\beta,\alpha_1\cdots\alpha_{k-i-1}\beta''_{i+1}\gamma_{j+2}\cdots\gamma_m} = \left\{\alpha_1\cdots\alpha_{k-i-1}\beta'_{i+1}\beta_{i+2}\cdots\beta_l \mid \beta'_{i+1}\in L^{\alpha_{k-i},\gamma_{j+1}}_{\beta_{i+1},\beta''_{i+1}}\right\},$$
$$R^{\alpha,\gamma}_{\beta,\alpha_1\cdots\alpha_{k-i-1}\beta''_{i+1}\gamma_{j+2}\cdots\gamma_m} = \left\{\beta_1\cdots\beta_i\beta'_{i+1}\gamma_{j+2}\cdots\gamma_m \mid \beta'_{i+1}\in R^{\alpha_{k-i},\gamma_{j+1}}_{\beta_{i+1},\beta''_{i+1}}\right\}.$$

Since there is by assumption a bijection $v\left(\cdot;\beta_{i+1},\beta''_{i+1}\right): L^{\alpha_{k-i},\gamma_{j+1}}_{\beta_{i+1},\beta''_{i+1}} \to R^{\alpha_{k-i},\gamma_{j+1}}_{\beta_{i+1},\beta''_{i+1}}$, we may set

$$\begin{aligned} &v(\alpha_1\cdots\alpha_{k-i-1}\beta'_{i+1}\beta_{i+2}\cdots\beta_l;\beta,\alpha_1\cdots\alpha_{k-i-1}\beta''_{i+1}\gamma_{j+2}\cdots\gamma_m)\\ &\quad=\beta_1\cdots\beta_i v(\beta'_{i+1};\beta_{i+1},\beta''_{i+1})\gamma_{j+2}\cdots\gamma_m, \end{aligned}$$

for $\beta'_{i+1}\in L^{\alpha_{k-i},\gamma_{j+1}}_{\beta_{i+1},\beta''_{i+1}}$. By the previous, then, this is a bijection

$$v(\cdot;\beta,\alpha_1\cdots\alpha_{k-i-1}\beta''_{i+1}\gamma_{j+2}\cdots\gamma_m): L^{\alpha,\gamma}_{\beta,\alpha_1\cdots\alpha_{k-i-1}\beta''_{i+1}\gamma_{j+2}\cdots\gamma_m} \to R^{\alpha,\gamma}_{\beta,\alpha_1\cdots\alpha_{k-i-1}\beta''_{i+1}\gamma_{j+2}\cdots\gamma_m}.$$

Case 3. We still assume that β is written as equation (2.7) and treat the remaining case. The other form that a representation contained in $\alpha\otimes\beta\otimes\gamma$ can have is a *reduced* expression $\beta'\beta''$ with either $\beta'\subseteq\alpha\otimes(\overline{\alpha}_k\cdots\overline{\alpha}_{k-i-1}\beta_{i+1})$ and $\beta''\subseteq\left(\overline{\gamma}_j\cdots\overline{\gamma}_1\right)\otimes\gamma$ or $\beta'\subseteq\alpha\otimes(\overline{\alpha}_k\cdots\overline{\alpha}_{k-i-1})$ and $\beta''\subseteq\left(\beta_{i+1}\overline{\gamma}_j\cdots\overline{\gamma}_1\right)\otimes\gamma$. We treat the first of these cases; the second one can be treated similarly. In fact, both are rather close to Case 1. We get

$$L^{\alpha,\gamma}_{\beta,\beta'\beta''}=\left\{\beta'\overline{\gamma}_j\cdots\overline{\gamma}_1\right\},\qquad R^{\alpha,\gamma}_{\beta,\beta'\beta''}=\left\{\overline{\alpha}_k\cdots\overline{\alpha}_{k-i-1}\beta_{i+1}\beta''\right\}.$$

Therefore we may set the bijection $L^{\alpha,\gamma}_{\beta,\beta'\beta''}\to R^{\alpha,\gamma}_{\beta,\beta'\beta''}$ by

$$v(\beta'\overline{\gamma}_j\ldots\overline{\gamma}_1;\beta,\beta'\beta'')=\overline{\alpha}_k\ldots\overline{\alpha}_{k-i-1}\beta_{i+1}\beta''.$$

Remark 2.9. Note that we have exhausted all the cases for $\beta\notin A_{00}$. Indeed, the only other possible form that a $\beta\in\mathrm{Irr}(\mathbb{G})$ can have is $\beta=\beta_1\cdots\beta_l=\overline{\alpha}_k\cdots\overline{\alpha}_{k-i}\overline{\gamma}_j\cdots\overline{\gamma}_1$ for suitable i,j, but those representations are in A_{00}. It should also be noted that if β falls into Case 1, then $\beta\notin A$.

In the following proof we need the following notation. Set $V: K_1\otimes K_2\to K_3$ and $W: H_1\otimes H_2\to H_3$, with K_i and H_i Hilbert spaces. Then

$$V\boxtimes W: K_1\otimes H_1\otimes H_2\otimes K_2\to K_3\otimes H_3$$

is the map that sends $\xi_1\otimes\eta_1\otimes\eta_2\otimes\xi_2$ to $V(\xi_1\otimes\xi_2)\otimes W(\eta_1\otimes\eta_2)$. Note that if $H_3=\mathbb{C}$, then the range space simplifies to $K_3\otimes H_3=K_3$.

Proposition 2.10. *Properties* (2.2) *and* (2.2) *of Definition 2.2 hold for the foregoing choices.*

Proof. We treat the three cases previously described separately. In Remark 2.9 we already noted that for $\beta\in\mathrm{Irr}(\mathbb{G})$ as in Case 1, we have $\beta\notin A\cup A_{00}$. So in Case 1 we must prove equations (2.3) and (2.5) only.

Proof of equation (2.3) **in Case 1.** Take $\beta \in \mathrm{Irr}(\mathbb{G})$ as in Case 1, so that $\beta \notin A \cup A_{00}$. We recall from the discussion of Case 1 that any irreducible representation contained in $\alpha \otimes \beta \otimes \gamma$ can be written as a reduced expression of the form $\beta'(\beta_{i+1}\cdots\beta_{j-1})\beta''$, $i<j$, with $\beta' \subseteq \alpha \otimes \beta_1 \cdots \beta_i$ and $\beta'' \subseteq \beta_j \cdots \beta_l \otimes \gamma$ irreducible. Further, we have one-point sets

$$L^{\alpha,\gamma}_{\beta,\beta'(\beta_{i+1}\cdots\beta_{j-1})\beta''} = \{\beta'\beta_{i+1}\cdots\beta_l\}, \qquad R^{\alpha,\gamma}_{\beta,\beta'(\beta_{i+1}\cdots\beta_{j-1})\beta''} = \{\beta_1\cdots\beta_{j-1}\beta''\},$$

and the v-bijection maps the one set to the other. We therefore conclude that equation (2.3) equals

$$\begin{aligned}&\Delta_\beta - \Delta_{\beta'\beta_{i+1}\cdots\beta_l} - \Delta_{\beta_1\cdots\beta_{j-1}\beta''} + \Delta_{\beta'(\beta_{i+1}\cdots\beta_{j-1})\beta''}\\ =&\sum_{r=1}^{l}\Delta_{\beta_r} - \left(\Delta_{\beta'} + \sum_{r=i+1}^{l}\Delta_{\beta_r}\right) - \left(\Delta_{\beta''} + \sum_{r=1}^{j-1}\Delta_{\beta_r}\right) + \left(\Delta_{\beta'} + \Delta_{\beta''} + \sum_{r=i+1}^{j-1}\Delta_{\beta_r}\right) = 0.\end{aligned}$$

Proof of equation (2.5) **in Case 1.** To prove equation (2.5), we note that for a suitable choice of phase factors,

$$\begin{aligned}V^{\alpha,\beta}_{\beta'\beta_{i+1}\cdots\beta_l} &= V^{\alpha,\beta_1\cdots\beta_i}_{\beta'} \otimes \mathrm{id}_{\beta_{i+1}\cdots\beta_l}, & V^{\alpha,\beta_1\cdots\beta_{j-1}\beta''}_{\beta'\beta_{i+1}\cdots\beta_{j-1}\beta''} &= V^{\alpha,\beta_1\cdots\beta_i}_{\beta'} \otimes \mathrm{id}_{\beta_{i+1}\cdots\beta_{j-1}\beta''},\\ V^{\beta,\gamma}_{\beta_1\cdots\beta_{j-1}\beta''} &= \mathrm{id}_{\beta_1\cdots\beta_{j-1}} \otimes V^{\beta_j\cdots\beta_l,\gamma}_{\beta''}, & V^{\beta'\beta_{i+1}\cdots\beta_l,\gamma}_{\beta'\beta_{i+1}\cdots\beta_{j-1}\beta''} &= \mathrm{id}_{\beta'\beta_{i+1}\cdots\beta_{j-1}} \otimes V^{\beta_j\cdots\beta_l,\gamma}_{\beta''}.\end{aligned}$$

By using these identities in the first and last equations we find the following. The second equation is elementary, since the intertwiners commute as they act on different tensor legs. So we get

$$\begin{aligned}V^{\beta'\beta_{i+1}\cdots\beta_l,\gamma}_{\beta'\beta_{i+1}\cdots\beta_{j-1}\beta''} \circ \left(V^{\alpha,\beta}_{\beta'\beta_{i+1}\cdots\beta_l} \otimes \mathrm{id}_\gamma\right) &= \left(\mathrm{id}_{\beta'\beta_{i+1}\cdots\beta_{j-1}} \otimes V^{\beta_j\cdots\beta_l,\gamma}_{\beta''}\right)\\ &\quad\circ \left(V^{\alpha,\beta_1\cdots\beta_i}_{\beta'} \otimes \mathrm{id}_{\beta_{i+1}\cdots\beta_l} \otimes \mathrm{id}_\gamma\right)\\ &= \left(V^{\alpha,\beta_1\cdots\beta_i}_{\beta'} \otimes \mathrm{id}_{\beta_{i+1}\cdots\beta_{j-1}\beta''}\right)\\ &\quad\circ \left(\mathrm{id}_\alpha \otimes \mathrm{id}_{\beta_1\cdots\beta_{j-1}} \otimes V^{\beta_j\cdots\beta_l,\gamma}_{\beta''}\right)\\ &= V^{\alpha,\beta_1\cdots\beta_{j-1}\beta''}_{\beta'\beta_{i+1}\cdots\beta_{j-1}\beta''} \circ \left(\mathrm{id}_\alpha \otimes V^{\beta,\gamma}_{\beta_1\cdots\beta_{j-1}\beta''}\right).\end{aligned}$$

This proves that equation (2.5) is true for β in Case 1.

Proof of formulas (2.1) **and** (2.3) **in Case 2.** Now set $\beta \in \mathrm{Irr}(\mathbb{G})$ and assume that we are in Case 2, so $\beta \notin A_{00}$. Take $\beta'' \subseteq \alpha \otimes \beta \otimes \gamma$, which in Case 2 is assumed to be of the form of a reduced expression $\alpha_1\cdots\alpha_{k-i-1}\beta''_{i+1}\gamma_{j+2}\cdots\gamma_m$, where α_{k-i}, β_{i+1} and γ_{j+1} are representations of the same quantum group and $\beta''_{i+1} \subseteq \alpha_{k-i} \otimes \beta_{i+1} \otimes \gamma_{j+1}$ is irreducible, nontrivial and not contained in $A^s_{00}(\alpha_{k-i},\gamma_{j+1})$. Take $\beta'_{i+1} \in L^{\alpha_{k-i},\gamma_{j+1}}_{\beta_{i+1},\beta''_{i+1}}$ so that

$$\beta' := \alpha_1\cdots\alpha_{k-i-1}\beta'_{i+1}\beta_{i+2}\cdots\beta_l \in L^{\alpha,\gamma}_{\beta,\alpha_1\cdots\alpha_{k-i-1}\beta''_{i+1}\gamma_{j+2}\cdots\gamma_m}$$

and the v-image of β' is $\beta_1\cdots\beta_i v\left(\beta'_{i+1}\right)\gamma_{j+2}\cdots\gamma_m$. We have

$$\begin{aligned}\Delta_\beta-\Delta_{\beta'}-\Delta_{v(\beta')}+\Delta_{\beta''}&=\left(\sum_{r=1}^{l}\Delta_{\beta_r}\right)-\left(\sum_{r=1}^{k-i-1}\Delta_{\alpha_r}+\Delta_{\beta'_{i+1}}+\sum_{r=i+2}^{l}\Delta_{\beta_r}\right)\\&\quad-\left(\sum_{r=1}^{i}\Delta_{\beta_r}+\Delta_{v\left(\beta'_{i+1}\right)}+\sum_{r=j+2}^{m}\Delta_{\gamma_r}\right)\\&\quad+\left(\sum_{r=1}^{k-i-1}\Delta_{\alpha_r}+\Delta_{\beta''_{i+1}}+\sum_{r=j+2}^{m}\Delta_{\gamma_r}\right)\\&=\Delta_{\beta_{i+1}}-\Delta_{\beta'_{i+1}}-\Delta_{v(\beta'_{i+1})}+\Delta_{\beta''_{i+1}}.\end{aligned}$$

So since the QMSs on $\mathbb{G}_1$ and $\mathbb{G}_2$ are approximately linear, we can conclude as follows. When $\beta_{i+1}\in A^s\left(\alpha_{k-i},\gamma_{j+1}\right),s=1,2$, we see that there is a constant $C>0$ depending only on α_{k-i} and γ_{j+1}, such that

$$\left|\Delta_{\beta_{i+1}}-\Delta_{\beta'_{i+1}}-\Delta_{v\left(\beta'_{i+1}\right)}+\Delta_{\beta''_{i+1}}\right|\leq C\,\mathrm{qdim}(\beta_{i+1})^{-1}.$$

So by equation (2.7) and the multiplicativity of the quantum dimension,

$$\left|\Delta_{\beta_{i+1}}-\Delta_{\beta'_{i+1}}-\Delta_{v\left(\beta'_{i+1}\right)}+\Delta_{\beta''_{i+1}}\right|\leq C\left(\prod_{r=1}^{i}\mathrm{qdim}(\alpha_r)\right)\left(\prod_{r=i+2}^{m}\mathrm{qdim}(\gamma_r)\right)\mathrm{qdim}(\beta)^{-1}.$$

This concludes formula (2.1). When $\beta_{i+1}\notin A^s\left(\alpha_{k-i},\gamma_{j+1}\right),s=1,2$, and as we have also assumed that $\beta_{i+1}\notin A^s_{00}\left(\alpha_{k-i},\gamma_{j+1}\right),s=1,2$, we find

$$\left|\Delta_{\beta_{i+1}}-\Delta_{\beta'_{i+1}}-\Delta_{v\left(\beta'_{i+1}\right)}+\Delta_{\beta''_{i+1}}\right|=0,$$

and we conclude equation (2.3).

Proof of formula (2.2) in Case 2. We stay in the setting of the previous subproof and assume that $\beta_{i+1}\in A^s\left(\alpha_{k-i},\gamma_{j+1}\right),s=1,2$. Recall that in Case 2 β' must be of the form $\alpha_1\cdots\alpha_{k-i-1}\beta'_{i+1}\beta_{i+2}\cdots\beta_l$, with $1\neq\beta'_{i+1}\in L^{\alpha_{k-i},\gamma_{j+1}}_{\beta_{i+1},\beta''_{i+1}}$. In that case, $\beta_1\cdots\beta_i=\overline{\alpha}_k\cdots\overline{\alpha}_{k-i+1}$. This gives

$$\begin{aligned}\Delta_\beta-\Delta_{\beta'}&=\left(\sum_{r=1}^{l}\Delta_{\beta_r}\right)-\left(\sum_{r=i+2}^{l}\Delta_{\beta_r}+\Delta_{\beta'_{i+1}}+\sum_{r=1}^{k-i+1}\Delta_{\alpha_r}\right)\\&=\sum_{r=0}^{i-1}\Delta_{\overline{\alpha}_{k-r}}+\Delta_{\beta_{i+1}}-\Delta_{\beta'_{i+1}}-\sum_{r=1}^{k-i+1}\Delta_{\alpha_r}.\end{aligned}$$

We therefore estimate

$$|\Delta_\beta-\Delta_{\beta'}|\leq\left|\sum_{r=0}^{i-1}\Delta_{\overline{\alpha}_{k-r}}-\sum_{r=1}^{k-i+1}\Delta_{\alpha_r}\right|+\left|\Delta_{\beta_{i+1}}-\Delta_{\beta'_{i+1}}\right|\leq\left|\sum_{r=0}^{i-1}\Delta_{\overline{\alpha}_{k-r}}-\sum_{r=1}^{k-i+1}\Delta_{\alpha_r}\right|+C,$$

for some constant C that depends only on α and γ, since both the QMSs on $\mathbb{G}_1$ and $\mathbb{G}_2$ are approximately linear. This proves that formula (2.2) holds for $\beta \in A$ in Case 2.

Proof of formulas (2.4) and (2.5) in Case 2. To prove formula (2.4) for Case 2, note that up to a phase factor,

$$\begin{aligned}
V^{\alpha,\beta}_{\alpha_1\cdots\alpha_{k-i-1}\beta'_{i+1}\beta_{i+2}\cdots\beta_l} &= V^{\alpha,\beta_1\cdots\beta_{i+1}}_{\alpha_1\cdots\alpha_{k-i-1}\beta'_{i+1}} \otimes \mathrm{id}_{\beta_{i+2}\cdots\beta_l},\\
V^{\alpha,\beta_1\cdots\beta_i v(\beta'_{i+1})\gamma_{j+2}\cdots\gamma_m}_{\alpha_1\cdots\alpha_{k-i-1}\beta''_{i+1}\gamma_{j+2}\cdots\gamma_m} &= V^{\alpha,\beta_1\cdots\beta_i v(\beta'_i)}_{\alpha_1\cdots\alpha_{k-i-1}\beta''_{i+1}} \otimes \mathrm{id}_{\gamma_{j+2}\ldots\gamma_m},\\
V^{\beta,\gamma}_{\beta_1\cdots\beta_i v(\beta'_{i+1})\gamma_{j+2}\cdots\gamma_m} &= \mathrm{id}_{\beta_1\cdots\beta_i} \otimes V^{\beta_{i+1}\cdots\beta_l,\gamma}_{v(\beta'_{i+1})\gamma_{j+2}\cdots\gamma_m},\\
V^{\alpha_1\cdots\alpha_{k-i-1}\beta'_{i+1}\beta_{i+2}\cdots\beta_l,\gamma}_{\alpha_1\cdots\alpha_{k-i-1}\beta''_{i+1}\gamma_{j+2}\cdots\gamma_m} &= \mathrm{id}_{\alpha_1\cdots\alpha_{k-i-1}} \otimes V^{\beta'_{i+1}\beta_{i+2}\cdots\beta_l,\gamma}_{\beta''_{i+1}\gamma_{j+2}\cdots\gamma_m}.
\end{aligned}$$

Write $x \approx_D y$ for $\|x-y\| \le D$. Let $D = C\,\mathrm{qdim}(\beta_{i+1})^{-1}$ if $\beta'_{i+1} \in A^s(\alpha_{k-i},\gamma_{j+1})$, and let $D=0$ otherwise. We find since $\mathbb{G}_1$ and $\mathbb{G}_2$ have almost commuting intertwiners that

$$\begin{aligned}
&V^{\alpha_1\cdots\alpha_{k-i-1}\beta'_{i+1}\beta_{i+2}\cdots\beta_l,\gamma}_{\alpha_1\cdots\alpha_{k-i-1}\beta''_{i+1}\gamma_{j+2}\cdots\gamma_m} \circ \left(V^{\alpha,\beta}_{\alpha_1\cdots\alpha_{k-i-1}\beta'_{i+1}\beta_{i+2}\cdots\beta_l} \otimes \mathrm{id}_\gamma\right)\\
&= \left(\mathrm{id}_{\alpha_1\cdots\alpha_{k-i-1}} \otimes V^{\beta'_{i+1}\beta_{i+2}\cdots\beta_l,\gamma}_{\beta''_{i+1}\gamma_{j+2}\cdots\gamma_m}\right) \circ \left(V^{\alpha,\beta_1\cdots\beta_{i+1}}_{\alpha_1\cdots\alpha_{k-i-1}\beta'_{i+1}} \otimes \mathrm{id}_{\beta_{i+2}\cdots\beta_l} \otimes \mathrm{id}_\gamma\right)\\
&= \left(\mathrm{id}_{\alpha_1\cdots\alpha_{k-i-1}} \otimes V^{\beta'_{i+1},\gamma_{j+1}}_{\beta''_{i+1}} \boxtimes V_1^{\beta_{i+2}\cdots\beta_l,\gamma_1\cdots\gamma_j} \otimes \mathrm{id}_{\gamma_{j+2}\cdots\gamma_m}\right)\\
&\quad \circ \left(\mathrm{id}_{\alpha_1\cdots\alpha_{k-i-1}} \otimes V^{\alpha_{k-i},\beta_{i+1}}_{\beta'_{i+1}} \boxtimes V_1^{\alpha_{k-i+1}\cdots\alpha_k,\beta_1\cdots\beta_i} \otimes \mathrm{id}_{\beta_{i+2}\cdots\beta_l} \otimes \mathrm{id}_\gamma\right)\\
&\approx_D \left(\mathrm{id}_{\alpha_1\cdots\alpha_{k-i-1}} \otimes V^{\alpha_{k-i},v(\beta'_{i+1})}_{\beta''_{i+1}} \boxtimes V_1^{\alpha_{k-i+1}\cdots\alpha_k,\beta_1\cdots\beta_i} \otimes \mathrm{id}_{\gamma_{j+2}\cdots\gamma_m}\right)\\
&\quad \circ \left(\mathrm{id}_\alpha \otimes \mathrm{id}_{\beta_1\cdots\beta_i} \otimes V^{\beta_{i+1},\gamma_{j+1}}_{v(\beta'_{i+1})} \boxtimes V_1^{\beta_{i+2}\cdots\beta_l,\gamma_1\cdots\gamma_j} \otimes \mathrm{id}_{\gamma_{j+2}\cdots\gamma_m}\right)\\
&= \left(V^{\alpha,\beta_1\cdots\beta_i v(\beta'_{i+1})}_{\alpha_1\cdots\alpha_{k-i-1}\beta''_{i+1}} \otimes \mathrm{id}_{\gamma_{j+2}\cdots\gamma_m}\right) \circ \left(\mathrm{id}_\alpha \otimes \mathrm{id}_{\beta_1\cdots\beta_i} \otimes V^{\beta_{i+1}\cdots\beta_l,\gamma}_{v(\beta'_{i+1})\gamma_{j+2}\cdots\gamma_m}\right)\\
&= V^{\alpha,\beta_1\cdots\beta_i v(\beta'_{i+1})\gamma_{j+2}\cdots\gamma_m}_{\alpha_1\cdots\alpha_{k-i-1}\beta''_{i+1}\gamma_{j+2}\cdots\gamma_m} \circ \left(\mathrm{id}_\alpha \otimes V^{\beta,\gamma}_{\beta_1\cdots\beta_i v(\beta'_{i+1})\gamma_{j+2}\cdots\gamma_m}\right).
\end{aligned}$$

So formulas (2.4) and (2.5) hold in Case 2.

Proof of formulas (2.1) and (2.3) in Case 3. We shall write

$$\beta = \beta_1\cdots\beta_l = \overline{\alpha}_k\cdots\overline{\alpha}_{k-i+1}\beta_{i+1}\overline{\gamma}_j\cdots\overline{\gamma}_1.$$

This case is essentially the same as Case 1 for $i+1=j$ (so that the terms $\beta_{i+1}\cdots\beta_{j-1}$ in the proof of Case 1 vanish). Nevertheless, we provide full details here.

Consider the subrepresentation of $\alpha\otimes\beta\otimes\gamma$ given by the reduced word $\beta'\beta''$, where $\beta' \subseteq \alpha\otimes(\overline{\alpha}_k\cdots\overline{\alpha}_{k-i+1}\beta_{i+1})$ and $\beta'' \subseteq (\overline{\gamma}_j\cdots\overline{\gamma}_1)\otimes\gamma$. (The case where $\beta' \subseteq \alpha\otimes(\overline{\alpha}_k\cdots\overline{\alpha}_{k-i+1})$ and $\beta'' \subseteq (\beta_{i+1}\overline{\gamma}_j\cdots\overline{\gamma}_1)\otimes\gamma$ can be treated in the same manner, or by taking adjoints.) We recall that

$$L^{\alpha,\gamma}_{\beta,\beta'\beta''} = \{\beta'\overline{\gamma}_j\cdots\overline{\gamma}_1\} \quad \text{and} \quad R^{\alpha,\gamma}_{\beta,\beta'\beta''} = \{\overline{\alpha}_k\cdots\overline{\alpha}_{k-i+1}\beta_{i+1}\beta''\}.$$

We write v as shorthand for the bijection between these singleton sets. We now find that

$$\begin{aligned}&\Delta_{\overline{\alpha}_k\cdots\overline{\alpha}_{k-i+1}\beta_{i+1}\overline{\gamma}_j\cdots\overline{\gamma}_1}-\Delta_{\beta'\overline{\gamma}_j\cdots\overline{\gamma}_1}-\Delta_{\overline{\alpha}_k\cdots\overline{\alpha}_{k-i+1}\beta_{i+1}\beta''}+\Delta_{\beta'\beta''}\\&=\left(\Delta_{\beta_{i+1}}+\sum_{r=k-i+1}^{k}\Delta_{\overline{\alpha}_r}+\sum_{r=1}^{j}\Delta_{\overline{\gamma}_r}\right)-\left(\Delta_{\beta'}+\sum_{r=1}^{j}\Delta_{\overline{\gamma}_r}\right)\\&\quad-\left(\Delta_{\beta_{i+1}}+\Delta_{\beta''}+\sum_{r=k-i+1}^{k}\Delta_{\overline{\alpha}_r}\right)+(\Delta_{\beta'}+\Delta_{\beta''})=0.\end{aligned}$$

This proves equation (2.3) and certainly formula (2.1); in fact the expression always is 0.

Proof of formula (2.2) **in Case 3.** Formula (2.2) can be proved as in Case 2; we omit the details here.

Proof of formula (2.4) **and** (2.5) **in Case 3.** For suitable phase factors for the intertwiners, we have

$$\begin{aligned}V^{\alpha,\beta}_{\beta'\beta_{i+2}\cdots\beta_l}&=V^{\alpha,\beta_1\cdots\beta_{i+1}}_{\beta'}\otimes\mathrm{id}_{\beta_{i+2}\cdots\beta_l},&\qquad V^{\alpha,\beta_1\cdots\beta_{i+1}\beta''}_{\beta'\beta''}&=V^{\alpha,\beta_1\cdots\beta_{i+1}}_{\beta'}\otimes\mathrm{id}_{\beta''},\\V^{\beta,\gamma}_{\beta_1\cdots\beta_{i+1}\beta''}&=\mathrm{id}_{\beta_1\cdots\beta_{i+1}}\otimes V^{\beta_{i+2}\cdots\beta_l,\gamma}_{\beta''},&\qquad V^{\beta'\beta_{i+2}\cdots\beta_l,\gamma}_{\beta'\beta''}&=\mathrm{id}_{\beta'}\otimes V^{\beta_{i+2}\cdots\beta_l,\gamma}_{\beta''}.\end{aligned}$$

By using these identities in the first and last equations we find the following. The second equation is elementary, since the intertwiners commute as they act on different tensor legs. So we get

$$\begin{aligned}V^{\beta'\beta_{i+2}\cdots\beta_l,\gamma}_{\beta'\beta''}\circ\left(V^{\alpha,\beta}_{\beta'\beta_{i+2}\cdots\beta_l}\otimes\mathrm{id}_\gamma\right)&=\left(\mathrm{id}_{\beta'}\otimes V^{\beta_{i+2}\cdots\beta_l,\gamma}_{\beta''}\right)\circ\left(V^{\alpha,\beta_1\cdots\beta_{i+1}}_{\beta'}\otimes\mathrm{id}_{\beta_{i+2}\cdots\beta_l}\otimes\mathrm{id}_\gamma\right)\\&=\left(V^{\alpha,\beta_1\cdots\beta_{i+1}}_{\beta'}\otimes\mathrm{id}_{\beta''}\right)\circ\left(\mathrm{id}_\alpha\otimes\mathrm{id}_{\beta_1\cdots\beta_{i+1}}\otimes V^{\beta_{i+2}\cdots\beta_l,\gamma}_{\beta''}\right)\\&=V^{\alpha,\beta_1\cdots\beta_{i+1}\beta''}_{\beta'\beta''}\circ\left(\mathrm{id}_\alpha\otimes V^{\beta,\gamma}_{\beta_1\cdots\beta_{i+1}\beta''}\right).\end{aligned}$$

This proves that equation (2.5) and certainly formula (2.4) are true in Case 3. □

3. Approximate linearity with almost commuting intertwiners implies immediately gradient-$\mathcal{S}_2$

One of the main tools introduced in [10] is the notion of a QMS being immediately gradient Hilbert–Schmidt or immediately gradient-$\mathcal{S}_2$, where $\mathcal{S}_2$ refers to the Schatten–von Neumann noncommutative L_2-space. The aim of this section is to show that if a QMS is approximately linear with almost commuting intertwiners, then it is immediately gradient-$\mathcal{S}_2$. The immediately gradient-$\mathcal{S}_2$ property, together with some additional assumptions, implies rigidity results for von Neumann algebras. The proofs of the latter facts were given in [10] and shall not be repeated here.

We note in the following definition that since Φ is a QMS of central multipliers, the $*$-algebra $\mathrm{Pol}(\mathbb{G})$ is in the domain of the generator Δ.

Definition 3.1. Let $\Phi=(\exp(-t\Delta))_{t\geq 0}$ be a QMS of central multipliers on a compact quantum group $\mathbb{G}$. The QMS Φ is called *immediately gradient-$\mathcal{S}_2$* if for every $a,c\in\mathrm{Pol}(\mathbb{G})$

the map

$$\Psi_t^{a,c} : x\Omega_\varphi \mapsto \exp(-t\Delta)(\Delta(axc) - \Delta(ax)c - a\Delta(xc) + a\Delta(x)c)\Omega_\varphi, \quad x \in \mathrm{Pol}(\mathbb{G}),$$

is bounded $L_2(\mathbb{G}) \to L_2(\mathbb{G})$ for $t \geq 0$ and moreover Hilbert–Schmidt for $t > 0$.

We first need the following estimate for the isotypical projections.

Proposition 3.2. *Suppose that Φ is approximately linear with almost commuting intertwiners. Let $a,c \in \mathrm{Pol}(\mathbb{G})$ be matrix coeffiecients of, respectively, $\alpha,\gamma \in \mathrm{Irr}(\mathbb{G})$. Let $A_{00} = A_{00}(\alpha,\gamma)$ and the v-map be as in Definition 2.2. There exists a constant $C = C(a,c) > 0$ such that for all $\beta \in \mathrm{Irr}(\mathbb{G})\backslash A_{00}, (\beta_1,\beta_2) \in L_\beta^{\alpha,\gamma}$, and every matrix coefficient x of β we have*

$$\left\| P_{\beta_2}(P_{\beta_1}(ax)c) - P_{\beta_2}\left(aP_{v(\beta_1;\beta,\beta_2)}(xc)\right)\right\|_2 \leq C\,\mathrm{qdim}(\beta)^{-1}\delta(\beta \in A)\|x\|_2,$$

where v and A are as in Definition 2.2.

Proof.

In this proof we identify $L_\infty(\mathbb{G}) \otimes M_s(\mathbb{C})$ with $M_s(L_\infty(\mathbb{G}))$. For an element $X \in M_s(L_\infty(\mathbb{G}))$ and vectors $\xi,\eta \in \mathbb{C}^s$, we thus have under this correspondence $\langle X\xi,\eta\rangle = (\mathrm{id} \otimes \omega_{\xi,\eta})(X) \in L_\infty(\mathbb{G})$. We shall also write $m := (0,\ldots,0,1,0,\ldots,0)^t, 1 \leq m \leq s$ (1 at the mth coordinate) for the orthonormal basis vectors in $\mathbb{C}^s$. Let u^α, u^β and u^γ be some concrete representatives for α, β and γ.

Set $a = \langle u^\alpha i,j\rangle$, $c = \langle u^\gamma m,n\rangle$ and $x = \langle u^\beta k,l\rangle$, with $\beta \in \mathrm{Irr}(\mathbb{G})\backslash A_{00}$. By the Woronowicz quantum Peter–Weyl theorem we find

$$\begin{aligned} \|a\|_2^2 &= \mathrm{qdim}(\alpha)^{-1}\left\langle Q_\alpha^{\frac{1}{2}} i, Q_\alpha^{\frac{1}{2}} i\right\rangle, \qquad \|x\|_2^2 = \mathrm{qdim}(\beta)^{-1}\left\langle Q_\beta^{\frac{1}{2}} k, Q_\beta^{\frac{1}{2}} k\right\rangle, \\ \|c\|_2^2 &= \mathrm{qdim}(\gamma)^{-1}\left\langle Q_\gamma^{\frac{1}{2}} m, Q_\gamma^{\frac{1}{2}} m\right\rangle. \end{aligned} \tag{3.1}$$

We have

$$\begin{aligned} P_{\beta_2}\left(aP_{v(\beta_1;\beta_2)}(xc)\right) = \Big\langle u^{\beta_2}\left(V_{\beta_2}^{\alpha,v(\beta_1;\beta_2)}\right)^*\left(1 \otimes V_{v(\beta_1;\beta_2)}^{\beta,\gamma}\right)^* i \otimes m \otimes k, \left(V_{\beta_2}^{\alpha,v(\beta_1;\beta_2)}\right)^* \\ \times \left(1 \otimes V_{v(\beta_1;\beta_2)}^{\beta,\gamma}\right)^* j \otimes n \otimes l\Big\rangle \end{aligned}$$

and

$$P_{\beta_2}(P_{\beta_1}(ax)c) = \left\langle u^{\beta_2}\left(V_{\beta_2}^{\beta_1\otimes\gamma}\right)^*\left(V_{\beta_1}^{\alpha,\beta} \otimes 1\right)^* i \otimes m \otimes k, \left(V_{\beta_2}^{\beta_1\otimes\gamma}\right)^*\left(V_{\beta_1}^{\alpha,\beta} \otimes 1\right)^* j \otimes n \otimes l\right\rangle.$$

For any $z \in \mathbb{T}$, we have

$$\begin{aligned}
&P_{\beta_2}\left(aP_{v(\beta_1;\beta_2)}(xc)\right) - P_{\beta_2}\left(P_{\beta_1}(ax)c\right)\\
&= \Big\langle u^{\beta_2}\left(\left(V_{\beta_2}^{\alpha,v(\beta_1;\beta_2)}\right)^*\left(1\otimes V_{v(\beta_1;\beta_2)}^{\beta,\gamma}\right)^*\right.\\
&\qquad \left. -z\left(V_{\beta_2}^{\beta_1\otimes\gamma}\right)^*\left(V_{\beta_1}^{\alpha,\beta}\otimes 1\right)^*\right) i\otimes m\otimes k, \left(V_{\beta_2}^{\alpha,v(\beta_1;\beta_2)}\right)^*\left(1\otimes V_{v(\beta_1;\beta_2)}^{\beta,\gamma}\right)^* j\otimes n\otimes l\Big\rangle\\
&\qquad -\Big\langle u^{\beta_2} z\left(V_{\beta_2}^{\beta_1\otimes\gamma}\right)^*\left(V_{\beta_1}^{\alpha,\beta}\otimes 1\right)^* i\otimes m\otimes k, \left(z\left(V_{\beta_2}^{\beta_1\otimes\gamma}\right)^*\left(V_{\beta_1}^{\alpha,\beta}\otimes 1\right)^*\right.\\
&\qquad \left. -\left(V_{\beta_2}^{\alpha,v(\beta_1;\beta_2)}\right)^*\left(1\otimes V_{v(\beta_1;\beta_2)}^{\beta,\gamma}\right)^*\right) j\otimes n\otimes l\Big\rangle.
\end{aligned} \tag{3.2}$$

We shall estimate the last two lines. The norm of the first of these lines can be expressed by the Peter–Weyl theorem (equation (3.1)) as

$$\begin{aligned}
&\Big\|\Big\langle u^{\beta_2}\left(\left(V_{\beta_2}^{\alpha,v(\beta_1;\beta_2)}\right)^*\left(1\otimes V_{v(\beta_1;\beta_2)}^{\beta,\gamma}\right)^* - z\left(V_{\beta_2}^{\beta_1\otimes\gamma}\right)^*\left(V_{\beta_1}^{\alpha,\beta}\otimes 1\right)^*\right) i\otimes m\otimes k,\\
&\qquad \times\left(V_{\beta_2}^{\alpha,v(\beta_1;\beta_2)}\right)^*\left(1\otimes V_{v(\beta_1;\beta_2)}^{\beta,\gamma}\right)^* j\otimes n\otimes l\Big\rangle\Big\|_2\\
&= \operatorname{qdim}(\beta_2)^{-1}\Big\|Q_{\beta_2}^{\frac12}\left(\left(V_{\beta_2}^{\alpha,v(\beta_1;\beta_2)}\right)^*\left(1\otimes V_{v(\beta_1;\beta_2)}^{\beta,\gamma}\right)^*\right.\\
&\qquad \left. -z\left(V_{\beta_2}^{\beta_1\otimes\gamma}\right)^*\left(V_{\beta_1}^{\alpha,\beta}\otimes 1\right)^*\right) i\otimes m\otimes k\Big\|_2\\
&\qquad \times\Big\|\left(V_{\beta_2}^{\alpha,v(\beta_1;\beta_2)}\right)^*\left(1\otimes V_{v(\beta_1;\beta_2)}^{\beta,\gamma}\right)^* j\otimes n\otimes l\Big\|_2\\
&\le \operatorname{qdim}(\beta_2)^{-1}\Big\|\left(\left(V_{\beta_2}^{\alpha,v(\beta_1;\beta_2)}\right)^*\left(1\otimes V_{v(\beta_1;\beta_2)}^{\beta,\gamma}\right)^*\right.\\
&\qquad \left. -z\left(V_{\beta_2}^{\beta_1\otimes\gamma}\right)^*\left(V_{\beta_1}^{\alpha,\beta}\otimes 1\right)^*\right) Q_\alpha^{\frac12} i\otimes Q_\beta^{\frac12} m\otimes Q_\gamma^{\frac12} k\Big\|_2.
\end{aligned}$$

By Definition 2.2 there exists a constant $C>0$ and $z_0\in\mathbb{T}$ such that

$$\begin{aligned}
&\Big\|\Big\langle u^{\beta_2}\left(\left(V_{\beta_2}^{\alpha,v(\beta_1;\beta_2)}\right)^*\left(1\otimes V_{v(\beta_1;\beta_2)}^{\beta,\gamma}\right)^* - z_0\left(V_{\beta_2}^{\beta_1\otimes\gamma}\right)^*\left(V_{\beta_1}^{\alpha,\beta}\otimes 1\right)^*\right) i\otimes m\otimes k,\\
&\qquad \times\left(V_{\beta_2}^{\alpha,v(\beta_1;\beta_2)}\right)^*\left(1\otimes V_{v(\beta_1;\beta_2)}^{\beta,\gamma}\right)^* j\otimes n\otimes l\Big\rangle\Big\|_2\\
&\le C\operatorname{qdim}(\beta)^{-1}\operatorname{qdim}(\beta_2)^{-1}\delta(\beta\in A)\Big\|Q_\alpha^{\frac12} i\otimes Q_\beta^{\frac12} m\otimes Q_\gamma^{\frac12} k\Big\|_2.
\end{aligned} \tag{3.3}$$

Similarly, the second line in equation (3.2) can be estimated with the same $z_0\in\mathbb{T}$ as

$$\begin{aligned}
&\Big\|\Big\langle u^{\beta_2} z_0\left(V_{\beta_2}^{\beta_1\otimes\gamma}\right)^*\left(V_{\beta_1}^{\alpha,\beta}\otimes 1\right)^* i\otimes m\otimes k, \left(z_0\left(V_{\beta_2}^{\beta_1\otimes\gamma}\right)^*\left(V_{\beta_1}^{\alpha,\beta}\otimes 1\right)^*\right.\\
&\qquad \left. -\left(V_{\beta_2}^{\alpha,v(\beta_1;\beta_2)}\right)^*\left(1\otimes V_{v(\beta_1;\beta_2)}^{\beta,\gamma}\right)^*\right) j\otimes n\otimes l\Big\rangle\Big\|_2\\
&\le C\operatorname{qdim}(\beta)^{-1}\operatorname{qdim}(\beta_2)^{-1}\delta(\beta\in A)\Big\|Q_\alpha^{\frac12} i\otimes Q_\beta^{\frac12} m\otimes Q_\gamma^{\frac12} k\Big\|_2.
\end{aligned} \tag{3.4}$$

Combining formulas (3.2), (3.3) and (3.4), we get

$$\begin{aligned}\left\|P_{\beta_2}\left(aP_{v(\beta_1;\beta_2)}(xc)\right)-P_{\beta_2}\left(P_{\beta_1}(ax)c\right)\right\|_2 &\leq 2C\,\mathrm{qdim}(\beta)^{-1}\\ &\quad\times \mathrm{qdim}(\beta_2)^{-1}\delta(\beta\in A)\left\|Q_\alpha^{\frac12}i\otimes Q_\beta^{\frac12}m\otimes Q_\gamma^{\frac12}k\right\|_2.\end{aligned}$$

Then using equation (3.1),

$$\begin{aligned}&\left\|P_{\beta_2}\left(aP_{v(\beta_1;\beta_2)}(xc)\right)-P_{\beta_2}\left(P_{\beta_1}(ax)c\right)\right\|_2\\ &\leq 2C\,\mathrm{qdim}(\beta)^{-1}\delta(\beta\in A)\frac{\mathrm{qdim}(\alpha)\,\mathrm{qdim}(\beta)\,\mathrm{qdim}(\gamma)}{\mathrm{qdim}(\beta_2)}\|a\|_2\|x\|_2\|c\|_2.\end{aligned}$$

This concludes the proof, since the fraction $\frac{\mathrm{qdim}(\beta)}{\mathrm{qdim}(\beta_2)}$ is bounded for all pairs β,β_2 with $\beta_2\subseteq\alpha\otimes\beta\otimes\gamma$. □

Theorem 3.3. *Suppose that Φ is approximately linear with almost commuting intertwiners. Then Φ is immediately gradient-$\mathcal{S}_2$.*

Proof. We use the same notation as in the proof of Proposition 3.2. Let $a,c\in\mathrm{Pol}(\mathbb{G})$ be matrix coefficients of, respectively, $\alpha,\gamma\in\mathrm{Irr}(\mathbb{G})$. Say that $a=\langle\alpha\xi,\eta\rangle$ and $c=\langle\gamma\zeta,\psi\rangle$. Let e_i^β, $1\leq i\leq n_\beta$, be orthogonal vectors in H_β such that $\left\langle\beta e_i^\beta,e_j^\beta\right\rangle$ is orthogonal in $L_2(\mathbb{G})$ [19, Proposition 2.1]. We must show that for any $t>0$,

$$\sum_{\beta\in\mathrm{Irr}(\mathbb{G})}\sum_{i,j=1}^{n_\beta}\frac{\left\|\Psi_t^{a,b}\left(\langle\beta e_i,e_j\rangle\right)\right\|_2}{\|\langle\beta e_i,e_j\rangle\|_2}<\infty. \tag{3.5}$$

Let $x=\langle\beta e_i,e_j\rangle$ for some fixed $\beta\in\mathrm{Irr}(\mathbb{G})$, $1\leq i\leq n_\beta$. We start by examining the term

$$\begin{aligned}\Psi_0^{a,b}(x)&=\Delta(axc)-\Delta(ax)c-a\Delta(xc)+a\Delta(x)c\\ &=\sum_{(\beta_1,\beta_2)\in L_\beta^{\alpha,\gamma}}\left(\Delta_{\beta_2}P_{\beta_2}\left(P_{\beta_1}(ax)c\right)-\Delta_{\beta_1}P_{\beta_2}\left(P_{\beta_1}(ax)c\right)\right)\\ &\quad+\sum_{(\beta_1',\beta_2)\in R_\beta^{\alpha,\gamma}}\left(-\Delta_{\beta_1'}P_{\beta_2}\left(aP_{\beta_1'}(xc)\right)+\Delta_\beta P_{\beta_2}\left(aP_{\beta_1'}(xc)\right)\right).\end{aligned}$$

Now if $\beta\notin A_{00}$, then we may write this expression as

$$\begin{aligned}\Psi_0^{a,b}(x) &= \Delta(axc) - \Delta(ax)c - a\Delta(xc) + a\Delta(x)c \\ &= \sum_{(\beta_1,\beta_2)\in L_\beta^{\alpha,\gamma}} \Big(\Delta_{\beta_2} P_{\beta_2}\left(P_{\beta_1}(ax)c\right) - \Delta_{\beta_1} P_{\beta_2}\left(P_{\beta_1}(ax)c\right) \\ &\qquad - \Delta_{v(\beta_1;\beta_2)} P_{\beta_2}\left(aP_{v(\beta_1;\beta_2)}(xc)\right) + \Delta_\beta P_{\beta_2}\left(aP_{v(\beta_1;\beta_2)}(xc)\right) \Big) \\ &= \sum_{(\beta_1,\beta_2)\in L_\beta^{\alpha,\gamma}} \left(\Delta_{\beta_2} - \Delta_{\beta_1} - \Delta_{v(\beta_1;\beta_2)} + \Delta_\beta\right) P_{\beta_2}(aP_{v(\beta_1;\beta_2)}(xc)) \\ &\qquad + (\Delta_{\beta_1} - \Delta_\beta)(P_{\beta_2}(aP_{v(\beta_1;\beta_2)}(xc)) - P_{\beta_2}(P_{\beta_1}(ax)c))\end{aligned}$$

Since $\left\|P_{\beta_2}\left(aP_{v(\beta_1;\beta_2)}(xc)\right)\right\|_2 \le \|a\|\|c\|\|x\|_2$, we estimate

$$\begin{aligned}\left\|\Psi_t^{a,b}(x)\right\|_2 \le &\sum_{(\beta_1,\beta_2)\in L_\beta^{\alpha,\gamma}} \exp\left(-t\Delta_{\beta_2}\right)\left|\Delta_{\beta_2} - \Delta_{\beta_1} - \Delta_{v(\beta_1;\beta_2)} + \Delta_\beta\right| \|a\|\|c\|\|x\|_2 \\ &+ \exp\left(-t\Delta_{\beta_2}\right)|\Delta_{\beta_1} - \Delta_\beta| \left\|P_{\beta_2}\left(P_{\beta_1}(ax)c\right) - P_{\beta_2}\left(aP_{v(\beta_1;\beta_2)}(xc)\right)\right\|_2.\end{aligned} \tag{3.6}$$

Since the semigroup is approximately linear with almost commuting intertwiners, we see by Proposition 3.2 that there exists a constant $C > 0$ depending only on a and c, such that

$$\left\|P_{\beta_2}\left(P_{\beta_1}(ax)c\right) - P_{\beta_2}\left(aP_{v(\beta_1;\beta_2)}(xc)\right)\right\|_2 \le C^{\frac{1}{2}} \operatorname{qdim}(\beta)^{-1}\delta(\beta\in A)\|x\|_2,$$

as well as

$$\left|\Delta_{\beta_2} - \Delta_{\beta_1} - \Delta_{v(\beta_1;\beta_2)} + \Delta_\beta\right| \le C \operatorname{qdim}(\beta)^{-1}\delta(\beta\in A)$$

and, when $\beta \in A$,

$$|\Delta_{\beta_1} - \Delta_\beta| \le C^{\frac{1}{2}}.$$

Combining this with formula (3.6), and estimating $\exp\left(-t\Delta_{\beta_2}\right) \le C'\exp(-t\beta)$ for some constant $C' > 0$ for all β,β_2 in the summations, we find

$$\begin{aligned}\left\|\Psi_t^{a,b}(x)\right\|_2 &\le C(1+\|a\|\|c\|)\operatorname{qdim}(\beta)^{-1}\delta(\beta\in A) \sum_{(\beta_1,\beta_2)\in L_\beta^{\alpha,\gamma}} \exp\left(-t\Delta_{\beta_2}\right)\|x\|_2 \\ &\le CC'(1+\|a\|\|c\|)\operatorname{qdim}(\beta)^{-1}\delta(\beta\in A)\left(\#L_\beta^{\alpha,\gamma}\right)\exp(-t\Delta_\beta)\|x\|_2.\end{aligned}$$

By Lemma 2.1 we have that $\#L_\beta^{\alpha,\gamma}$ is bounded in β, with the bound depending only on α and γ. We may therefore assemble terms and conclude that there exists a constant $C(a,c)$ depending only on a and c, such that

$$\left\|\Psi_t^{a,b}(x)\right\|_2 \le C(a,c)\operatorname{qdim}(\beta)^{-1}\delta(\beta\in A)\exp(-t\Delta_\beta)\|x\|_2. \tag{3.7}$$

We can now estimate term (3.5) as follows, where in the last line we use the fact that the classical dimension is smaller than or equal to the quantum dimension:

$$\begin{aligned}\sum_{\beta\in\mathrm{Irr}(\mathbb{G})\setminus A_{00}}\sum_{i,j=1}^{n_\beta}\frac{\left\|\Psi_t^{a,b}(\langle\beta e_i,e_j\rangle)\right\|_2^2}{\|\langle\beta e_i,e_j\rangle\|_2^2}&\leq C(a,c)^2\sum_{\beta\in A}\sum_{i,j=1}^{n_\beta}\mathrm{qdim}(\beta)^{-2}\exp(-2t\Delta_\beta)\\&\leq C(a,c)^2\sum_{\beta\in A}\mathrm{qdim}(\beta)^{-2}n_\beta^2\exp(-2t\Delta_\beta)\\&\leq C(a,c)^2\sum_{\beta\in A}\exp(-2t\Delta_\beta).\end{aligned}\tag{3.8}$$

In turn, we may estimate using Definition 2.2(2.2) and get

$$\sum_{\beta\in A}\exp(-2t\Delta_\beta)=\sum_{N\in\mathbb{N}}\sum_{\substack{\beta\in A,\\ N<\Delta_\beta\leq N+1}}\exp(-2tN)\leq\sum_{N\in\mathbb{N}}P(N)\exp(-2tN)<\infty.\tag{3.9}$$

Combining formula (3.8) and (3.9), we see that for $t>0$,

$$\begin{aligned}\sum_{\beta\in\mathrm{Irr}(\mathbb{G})\setminus A_{00}}\sum_{i,j=1}^{n_\beta}\frac{\left\|\Psi_t^{a,b}(\langle\beta e_i,e_j\rangle)\right\|_2^2}{\|\langle\beta e_i,e_j\rangle\|_2^2}\leq&\sum_{\beta\in\mathrm{Irr}(\mathbb{G})\setminus A_{00}}\sum_{i,j=1}^{n_\beta}\frac{\left\|\Psi_t^{a,b}(\langle\beta e_i,e_j\rangle)\right\|_2^2}{\|\langle\beta e_i,e_j\rangle\|_2^2}\\&+C(a,c)^2\sum_{N\in\mathbb{N}}P(N)\exp(-2tN)<\infty.\end{aligned}$$

So formula (3.5) is finite as A_{00} is finite.

Finally, set $x\in\mathrm{Pol}(\mathbb{G})$ and write $x_\beta=P_\beta(x)$ so that $x=\sum_{\beta\in\mathrm{Irr}(\mathbb{G})}x_\beta$. By the triangle inequality, formula (3.7) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\|\Psi_0^{a,c}(x)\|_2&\leq\left\|\sum_{\beta\in A_{00}}P_\beta(x)\right\|_2+\sum_{\beta\in A}C(a,c)\,\mathrm{qdim}(\beta)^{-1}\|P_\beta x\|_2\\&\leq\left\|\sum_{\beta\in A_{00}}P_\beta(x)\right\|_2+C(a,c)\left(\sum_{\beta\in A}\mathrm{qdim}(\beta)^{-2}\right)^{\frac12}\left(\sum_{\beta\in A}\|P_\beta x\|_2^2\right)^{\frac12}\\&\leq\left(1+C(a,c)\left(\sum_{\beta\in A}\mathrm{qdim}(\beta)^{-2}\right)^{\frac12}\right)\left(\left\|\sum_{\beta\in A_{00}}P_\beta(x)\right\|_2+\left\|\sum_{\beta\in A}P_\beta x\right\|_2\right)\\&\leq\sqrt{2}\left(1+C(a,c)\left(\sum_{\beta\in A}\mathrm{qdim}(\beta)^{-2}\right)^{\frac12}\right)\left\|\sum_{\beta\in\mathrm{Irr}(\mathbb{G})}P_\beta x\right\|_2.\end{aligned}$$

This gives the boundedness of $\Psi_0^{a,c}$ and concludes that Φ is immediately gradient-$\mathcal{S}_2$ by Definition 2.2(2.2). □

Recall that we say that a QMS is *immediately L_2-compact* if for every $t > 0$, the map $x\Omega_\varphi \mapsto \Phi_t(x)\Omega_\varphi$ is compact as a map on $L_2(\mathbb{G})$. Equivalently, the generator $\Delta \geq 0$ has compact resolvent.

Theorem 3.4. *Let $\mathbb{G}$ be a compact quantum group with the W*CBAP with constant Λ. Suppose that $\mathbb{G}$ admits a QMS that is immediately gradient-$\mathcal{S}_2$ and immediately L_2-compact. Then the following are true:*

(1) *If $\mathbb{G}$ is of Kac type, then $L_\infty(\mathbb{G})$ is strongly solid.*
(2) *If $L_\infty(\mathbb{G})$ is solid and $\Lambda = 1$, then $L_\infty(\mathbb{G})$ is strongly solid.*

Proof. Part (3.4) was proved in [10, Proposition 7.9] and is based on the results of [6]. For part (3.4) we see by [10, Section 3.2] (based on [17]) that there exists a closable real derivation $\partial : \mathrm{Pol}(\mathbb{G}) \to H_\partial$ into an $L_\infty(\mathbb{G})$-$L_\infty(\mathbb{G})$ H_∂ such that $\Delta = \partial^*\overline{\partial}$. Further, since Φ is immediately L_2-compact, Δ has compact resolvent. Moreover, by [10, Proposition 4.3] (see also [12, Theorem 3.9]), this bimodule H_∂ can be constructed in such a way that it is weakly contained in the coarse bimodule of $L_\infty(\mathbb{G})$. It follows then from the main results of [34, Corollary B] that $L_\infty(\mathbb{G})$ is strongly solid; we note that [34, Corollary B] is stated only for group von Neumann algebras, but it holds in this context as well (see, e.g., [10, Appendix]). □

Combining Theorems 3.3 and 3.4, we conclude the following main results of this paper:

Corollary 3.5. *Let $\mathbb{G}$ be a compact quantum group of Kac type with the W*CBAP. Suppose that $\mathbb{G}$ admits a QMS of central multipliers that is approximately linear with almost commuting intertwiners and immediately L_2-compact. Then $L_\infty(\mathbb{G})$ is strongly solid.*

We also get the following corollary, which shall not be used further in this paper:

Corollary 3.6. *Let $\mathbb{G}$ be a compact quantum group with the W*CCAP such that $L_\infty(\mathbb{G})$ is solid. Suppose that $\mathbb{G}$ admits a QMS of central multipliers that is approximately linear with almost commuting intertwiners and immediately L_2-compact. Then $L_\infty(\mathbb{G})$ is strongly solid.*

4. Quantum Markov semigroups and differentiable families of states

We prove that $SU_q(2)$ admits a QMS of central multipliers that is approximately linear with almost commuting intertwiners. Parts of the proof compare to our analysis from [10]. However, we present a much more conceptual and shorter approach by making use of generating functionals. We are indebted to Adam Skalski for showing us the argument contained in Section 4.2.

4.1. Preliminaries on quantum $SU(2)$

Definition 4.1. Let $\mathbb{G}_q, q \in (-1,1)\backslash\{0\}$, be the quantum $SU(2)$ group. It may be defined as follows. Consider the Hilbert space $\ell_2(\mathbb{N}_{\geq 0}) \otimes \ell_2(\mathbb{Z})$ with natural orthonormal basis

$e_i \otimes f_k, i \in \mathbb{N}_{\geq 0}, k \in \mathbb{Z}$. Define the operators

$$\alpha e_i \otimes f_k = \sqrt{1-q^{2i}} e_{i-1} \otimes f_k,$$
$$\gamma e_i \otimes f_k = q^i e_i \otimes f_{k+1},$$

and the comultiplication determined by

$$\Delta_{\mathbb{G}_q}(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \qquad \Delta_{\mathbb{G}_q}(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

It was proved in [1] that $\mathrm{Irr}(\mathbb{G}_q) = \mathbb{N}_{\geq 0}$, and the fusion rules of $\mathbb{G}_q$ are described by

$$\alpha \otimes \beta = |\alpha - \beta| \oplus |\alpha - \beta| + 2 \oplus \cdots \oplus |\alpha + \beta| - 2 \oplus |\alpha + \beta|, \quad \alpha, \beta \in \mathbb{N}_{\geq 0}.$$

4.2. QMSs on quantum $SU(2)$

We construct a natural QMS of central multipliers on $\mathbb{G}_q$ – that is, quantum $SU(2)$. The QMS is the same as the one from [10, Section 6.1], but the approach is more conceptual. See also [7] for related results.

Definition 4.2. A *generating functional* is a (linear) functional $L : \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ such that $L(1) = 0$ and $L(x^*) = \overline{L(x)}$ (i.e., L is self-adjoint) and such that if we have $\epsilon(x) = 0$ for $x \in \mathrm{Pol}(\mathbb{G})$, then $L(x^*x) \leq 0$ (i.e., L is conditionally negative definite).

A state on the unital $*$-algebra $\mathrm{Pol}(\mathbb{G})$ is a map $\mu : \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ such that $\mu(x^*x) \geq 0, x \in \mathrm{Pol}(\mathbb{G})$, and $\mu(1) = 1$. Recall that ϵ denotes the counit.

Proposition 4.3. *Let $\mathbb{G}$ be a compact quantum group and let $(\mu_t)_{t \geq 0}$ be a family of states on $\mathrm{Pol}(\mathbb{G})$ (not necessarily forming a convolution semigroup). Assume that for every $x \in \mathrm{Pol}(\mathbb{G})$, the limit*

$$L(x) := \lim_{t \searrow 0} \frac{1}{t}(\epsilon(x) - \mu_t(x))$$

exists. Then $L : \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ is a generating functional.

Proof. Let $x \in \mathrm{Pol}(\mathbb{G})$ be such that $\epsilon(x) = 0$. Then

$$\mu_t(x^*x) - \epsilon(x^*x) = \mu_t(x^*x) - \epsilon(x)^*\epsilon(x) = \mu_t(x^*x) \geq 0.$$

All other properties are clear. □

Let $U_\alpha, \alpha \in \mathbb{N}$, be the Chebyshev polynomials of the second kind with derivative U'_α. They are orthogonal polynomials satisfying $U_0 = 1, U_1(\lambda) = \lambda$ and the recursion relation

$$\lambda U_\alpha(\lambda) = U_{\alpha+1}(\lambda) + U_{\alpha-1}(\lambda), \quad \lambda \in \mathbb{R}, \alpha \in \mathbb{N}_{\geq 1}.$$

In [21, Theorem 17] (see also [7]), it was proved that for every $t \in [-1,1]$ there exists a state $\mu_t : \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ characterised by

$$\mu_t\left(u_{ij}^\alpha\right) = \left(\frac{U_\alpha\left(q^t + q^{-t}\right)}{U_\alpha\left(q + q^{-1}\right)}\right)^3 \delta_{i,j}, \quad \alpha \in \mathbb{N}_{\geq 0}, 1 \leq i,j \leq n_\alpha. \tag{4.1}$$

Proposition 4.4. *There exists a generating functional $L : Pol(\mathbb{G}_q) \to \mathbb{C}$ given by*

$$(L \otimes id)u^\alpha = \Delta_\alpha id_{n_\alpha}, \qquad \text{with} \qquad \Delta_\alpha = \frac{U'_\alpha\left(q^1+q^{-1}\right)}{U_\alpha\left(q+q^{-1}\right)}.$$

Proof. Consider the function

$$c_\alpha(t) := \left(\frac{U_\alpha\left(q^t+q^{-t}\right)}{U_\alpha\left(q+q^{-1}\right)}\right)^3, \quad [-1,1].$$

The derivative of this function is

$$c'_\alpha(t) = \frac{U'_\alpha\left(q^t+q^{-t}\right)}{U_\alpha\left(q+q^{-1}\right)}\left(q^t-q^{-t}\right)\log(q).$$

Proposition 4.3 and equation (4.1) show that there is a generating functional $L_0 : \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ determined by

$$(L_0 \otimes \mathrm{id})(u^\alpha) = c'_\alpha(1)\mathrm{id}_{n_\alpha}.$$

Then also $L = \log(q)^{-1}(q-q^{-1})^{-1}L$ is a generating functional and the proposition is proved. □

Theorem 4.5. *Let $\mathbb{G} = SU_q(2)$ with $q \in (-1,1)\backslash\{0\}$. There exists a QMS $\Phi = (\Phi_t)_{t\geq 0}$ on $L_\infty(\mathbb{G})$ determined by*

$$(\Phi_t \otimes id)u^\alpha = \exp(-t\Delta_\alpha)u^\alpha, \quad \alpha \in \mathbb{N}_{\geq 0}.$$

Here Δ_α is defined in Proposition 4.4. Moreover, Φ is approximately linear with almost commuting intertwiners.

Proof. Let $L : \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ be the generating functional from Proposition 4.4. By [20, Lemma 6.14] we see that

$$\exp(-tL) := \sum_{k=0}^{\infty} \frac{1}{k!}(-tL)^{*k}$$

is a convolution semigroup of states. We set

$$\Phi_t = (\exp(-tL) \otimes \mathrm{id}) \circ \Delta, \quad t \geq 0,$$

which then forms a QMS. We have, writing $u^\alpha_{i,j}$ for the matrix coefficients with respect to some orthonormal basis of $\mathbb{C}^{n_\alpha}$,

$$\Phi_t\left(u^\alpha_{ij}\right) = (\exp(-tL) \otimes \mathrm{id})\left(\sum_{k=1}^{n_\alpha} u^\alpha_{ik} \otimes u^\alpha_{kj}\right) = \exp(-t\Delta_\alpha)u^\alpha_{ij}.$$

It follows that $(\Phi_t)_{t\geq 0}$ is a QMS with the desired properties. □

4.3. Approximate linearity

In this section and the next, we prove that the QMS from Theorem 4.5 is approximately linear with almost commuting intertwiners. In order to do so we fix the following notation.

Recall that $\mathrm{Irr}(\mathbb{G}) = \mathbb{N}_{\geq 0}$. Take $\alpha,\gamma \in \mathbb{N}_{\geq 0}$. Set $A_{00} = \{0,1,\ldots,\max(\alpha,\gamma)\}$ and let $A = \mathbb{N}_{\geq 0}\backslash A_{00}$. We note that A and A_{00} partition $\mathbb{N}_{\geq 0}$ and therefore we do not need to check equations (2.3) and (2.5). Take $\beta \in A$. Then if $\beta_2 \subseteq \alpha \otimes \beta \otimes \gamma$, we must have $\beta_2 \in \{\beta - \alpha - \gamma, \beta - \alpha - \gamma + 2, \ldots, \beta + \alpha + \gamma\}$. We have

$$L^{\alpha,\gamma}_{\beta,\beta_2} = \{\beta - \alpha, \beta - \alpha + 2, \ldots, \beta + \alpha\},$$
$$R^{\alpha,\gamma}_{\beta,\beta_2} = \{\beta - \gamma, \beta - \gamma + 2, \ldots, \beta + \gamma\}.$$

We set $v(\beta_1;\beta,\beta_2) = \beta + \beta_2 - \beta_1$.

The proof of the next proposition is the same as [10, Section 6.1 and 6.2]:

Proposition 4.6. *The QMS defined in Theorem 4.5 is approximately linear.*

Proof. For any $m,n \in \mathbb{Z}\backslash\{0\}$, we have

$$\frac{1+q^{-2m}}{1-q^{-2m}} - \frac{1+q^{-2n}}{1-q^{-2n}} = \frac{2\left(q^{-2m} - q^{-2n}\right)}{(1-q^{-2m})(1-q^{-2n})} = \frac{2\left(q^{2n} - q^{2m}\right)}{(q^{2m}-1)(q^{2n}-1)}.$$

Let $N_q = q + q^{-1}$, which is the quantum dimension of the fundamental representation. By [23, Lemma 4.4], we have the explicit expression

$$\Delta_\alpha = \frac{\alpha}{\sqrt{N_q^2 - 4}}\left(\frac{1+q^{-2\alpha-2}}{1-q^{-2\alpha-2}}\right) + \frac{2}{(1-q^2)\sqrt{N_q^2-4}}. \tag{4.2}$$

Therefore it follows that for $\beta,\beta_1 \in \mathrm{Irr}(\mathbb{G})$ we have

$$\begin{aligned}|\Delta_\beta - \Delta_{\beta_1}| &\leq |\beta - \beta_1|\frac{1}{\sqrt{N_q^2-4}}\frac{1+q^{-2\beta-2}}{1-q^{-2\beta-2}} + \frac{\beta_1}{\sqrt{N_q^2-4}}\left|\frac{1+q^{-2\beta-2}}{1-q^{-2\beta-2}} - \frac{1+q^{-2\beta_1-2}}{1-q^{-2\beta_1-2}}\right| \\ &= |\beta - \beta_1|\frac{1}{\sqrt{N_q^2-4}}\frac{1+q^{-2\beta-2}}{1-q^{-2\beta-2}} + \frac{\beta_1}{\sqrt{N_q^2-4}}\left|\frac{2\left(q^{2\beta+2} - q^{2\beta_1+2}\right)}{(q^{2\beta+2}-1)(q^{2\beta_1+2}-1)}\right|.\end{aligned}$$

This expression can be estimated uniformly over all $\beta,\beta_1 \in \mathbb{N}_{\geq 0}$ with $|\beta - \beta_1| \leq \alpha + \gamma$. This yields formula (2.2). Further,

$$\begin{aligned}
&\sqrt{N_q^2-4}\,|\Delta_\beta-\Delta_{\beta_1}-\Delta_{\beta+\beta_2-\beta_1}+\Delta_{\beta_2}|\\
&=\left|\beta\frac{1+q^{-2\beta-2}}{1-q^{-2\beta-2}}-\beta_1\frac{1+q^{-2\beta_1-2}}{1-q^{-2\beta_1-2}}-(\beta+\beta_2-\beta_1)\frac{1+q^{-2(\beta+\beta_2-\beta_1)-2}}{1-q^{-2(\beta+\beta_2-\beta_1)-2}}+\beta_2\frac{1+q^{-2\beta_2-2}}{1-q^{-2\beta_2-2}}\right|\\
&\le\beta\left|\frac{1+q^{-2\beta-2}}{1-q^{-2\beta-2}}-\frac{1+q^{-2(\beta+\beta_2-\beta_1)-2}}{1-q^{-2(\beta+\beta_2-\beta_1)-2}}\right|+\beta_1\left|\frac{1+q^{-2\beta_1-2}}{1-q^{-2\beta_1-2}}-\frac{1+q^{-2(\beta+\beta_2-\beta_1)-2}}{1-q^{-2(\beta+\beta_2-\beta_1)-2}}\right|\\
&\quad+\beta_2\left|\frac{1+q^{-2\beta_2-2}}{1-q^{-2\beta_2-2}}-\frac{1+q^{-2(\beta+\beta_2-\beta_1)-2}}{1-q^{-2(\beta+\beta_2-\beta_1)-2}}\right|\\
&\le\beta\left|q^{2\beta}-q^{2(\beta+\beta_2-\beta_1)}\right|+\beta_1\left|q^{2\beta_1}-q^{2(\beta+\beta_2-\beta_1)}\right|+\beta_2\left|q^{2\beta_2}-q^{2(\beta+\beta_2-\beta_1)}\right|.
\end{aligned}$$

As $\operatorname{qdim}(\beta)\approx q^{-\beta}$ asymptotically, we see that there exists a constant $C>0$ such that for all $\beta,\beta_1,\beta_2\in\mathbb{N}_{\ge0}$ with $|\beta-\beta_1|\le\alpha+\gamma$ and $|\beta-\beta_2|\le\alpha+\gamma$, we have

$$\sqrt{N_q^2-4}\,|\Delta_\beta-\Delta_{\beta_1}-\Delta_{\beta+\beta_2-\beta_1}+\Delta_{\beta_2}|\le C\beta\operatorname{qdim}(\beta)^{-2}\le C\operatorname{qdim}(\beta)^{-1}.$$

This yields the desired estimate (2.1). □

4.4. Almost commuting intertwiners

In this section we extend the results from [46, Appendix] on almost commuting intertwiners. In fact these results are self-improving, in the sense that the main estimates are already proved in [46]. Here we show that they automatically imply the same results for a larger range of representations.

The following lemma and proposition pertain to $\mathbb{G}_q, q\in(-1,1)\backslash\{0\}$. Note however that the principle of proof of Lemma 4.7 actually works for any compact quantum group. In the following statements we require that $\alpha+k$ be even or odd (and $\gamma+l$ be even). In other words, α and k have either the same parity or different parity. This is because otherwise the intertwiner $V_{\beta+k}^{\alpha,\beta}$ or $V_{\beta+k}^{\alpha+1,\beta}$ would be 0 by the fusion rules, and the statements to come would thus be trivial.

Lemma 4.7. *Set $\alpha,\beta\in\mathbb{N}_{\ge0}$ with $\alpha\le\beta$. Set $k\in\mathbb{Z}$ with $|k|\le\alpha$ and $\alpha+k$ odd. Then we have, up to a phase factor,*

$$\sum_{k'=-\alpha,-\alpha+2,\dots,\alpha}V_{\beta+k}^{1,\beta+k'}\left(id_1\otimes V_{\beta+k'}^{\alpha,\beta}\right)\left(V_{\alpha+1}^{1,\alpha}\otimes id_\beta\right)^*=V_{\beta+k}^{\alpha+1,\beta}.\tag{4.3}$$

Proof. We may decompose $1\otimes\alpha\otimes\beta=\oplus_{\delta\in\mathbb{N}_{\ge0}}m_\delta\cdot\delta$, where m_δ denotes the multiplicity. Each of the intertwiners $V_{\beta+k}^{1,\beta+k'}\left(\mathrm{id}_1\otimes V_{\beta+k'}^{\alpha,\beta}\right)$ intertwines $1\otimes\alpha\otimes\beta$ with a copy of $\beta+k$, and the copies are orthogonal for different k'. Moreover,

$$\sum_{k'=-\alpha,-\alpha+2,\dots,\alpha}V_{\beta+k}^{1,\beta+k'}\left(\mathrm{id}_1\otimes V_{\beta+k'}^{\alpha,\beta}\right)$$

intertwines $1\otimes\alpha\otimes\beta$ with $m_{\beta+k}\cdot(\beta+k)$ – that is, it exhausts all the summands.

The the total expression on the left-hand side of equation (4.3) intertwines $(\alpha+1)\otimes\beta$ with $\beta+k$ and therefore by Schur's lemma must be a scalar multiple of the isometry

$V_{\beta+k+l}^{\alpha+1,\beta+l}$. By the first paragraph of this proof and the fact that $\left(V_{\alpha+1}^{1,\alpha}\right)^*$ is an isometry that maps $\alpha+1$ into its isotypical component in $1\otimes\alpha$, we find that this scalar multiple must be in $\mathbb{T}$. □

Proposition 4.8. *Set $\alpha,\gamma\in\mathbb{N}_{\geq 0}$. There exists a constant $C>0$ such that for all $\beta\in\mathbb{N}_{\geq 0}$, $\beta\geq\max(\alpha,\gamma)$, and $k,l\in\mathbb{Z}$ with $|k|\leq\alpha,|l|\leq\gamma$ and $\alpha+k$ and $\gamma+l$ even, we have*

$$\inf_{z\in\mathbb{Z}}\left\|V_{\beta+k+l}^{\beta+k,\gamma}\left(V_{\beta+k}^{\alpha,\beta}\otimes id_\gamma\right)-zV_{\beta+k+l}^{\alpha,\beta+l}\left(id_\alpha\otimes V_{\beta+l}^{\beta,\gamma}\right)\right\|\leq C\,\mathrm{qdim}(\beta)^{-1}. \tag{4.4}$$

Proof. This lemma was proved in [46, Lemmas A.1 and A.2] for the case when $\alpha,\gamma=1$ and (k,l) is equal to either $(1,1)$, $(1,-1)$ or $(-1,1)$. For $(k,l)=(-1,-1)$, the same conclusion can be derived, as follows. Both $V_{\beta-2}^{\beta-1,\gamma}\left(V_{\beta-1}^{\alpha,\beta}\otimes\mathrm{id}_\gamma\right)$ and $V_{\beta-2}^{\alpha,\beta-1}\left(\mathrm{id}_\alpha\otimes V_{\beta-1}^{\beta,\gamma}\right)$ are intertwiners from $\alpha\otimes\beta\otimes\gamma$ to $\beta-2$. But by the fusion rules, $\beta-2$ occurs at most once in the decomposition of $\alpha\otimes\beta\otimes\gamma$ in terms of irreducibles. Therefore such an intertwiner is unique up to a phase factor. So the left-hand side of formula (4.4) is 0. We note that actually also in the case when $(k,l)=(1,1)$, the left-hand side of formula (4.4) is 0 for the analogous reason.

We now prove the general case by an induction argument. Suppose that the statement is true for α and γ. Then we shall prove it for $\alpha+1$ and γ. Consider the composition of maps with α,β,γ,k,l as in the proposition and $|k'|\leq\alpha$ such that $\alpha+k'$ is even:

$$\begin{array}{lccccccc}
A_\beta: 1\otimes\alpha\otimes\beta\otimes\gamma & \xrightarrow{\mathrm{id}_1\otimes\mathrm{id}_\alpha\otimes V_{\beta+l}^{\beta,\gamma}} & 1\otimes\alpha\otimes(\beta+l) & \xrightarrow{\mathrm{id}_1\otimes V_{\beta+k'+l}^{\alpha,\beta+l}} & 1\otimes(\beta+k'+l) & \xrightarrow{V_{\beta+k+l}^{1,\beta+k'+l}} & \beta+k+l,\\
B_\beta: 1\otimes\alpha\otimes\beta\otimes\gamma & \xrightarrow{\mathrm{id}_1\otimes V_{\beta+k'}^{\alpha,\beta}\otimes\mathrm{id}_\gamma} & 1\otimes(\beta+k')\otimes\gamma & \xrightarrow{\mathrm{id}_1\otimes V_{\beta+k'+1}^{\beta+k',\gamma}} & 1\otimes(\beta+k'+l) & \xrightarrow{V_{\beta+k+l}^{1,\beta+k'+l}} & \beta+k+l,\\
C_\beta: 1\otimes\alpha\otimes\beta\otimes\gamma & \xrightarrow{\mathrm{id}_1\otimes V_{\beta+k'}^{\alpha,\beta}\otimes\mathrm{id}_\gamma} & 1\otimes(\beta+k')\otimes\gamma & \xrightarrow{V_{\beta+k}^{1,\beta+k'}\otimes\mathrm{id}_\gamma} & (\beta+k)\otimes\gamma & \xrightarrow{V_{\beta+k+l}^{\beta+k,\gamma}} & \beta+k+l.
\end{array}$$

By the induction hypothesis, we have

$$\inf_{z\in\mathbb{T}}\|A_\beta-zB_\beta\|\leq C\,\mathrm{qdim}(\beta)^{-1}\quad\text{and}\quad\inf_{z\in\mathbb{T}}\|B_\beta-zC_\beta\|\leq C\,\mathrm{qdim}(\beta)^{-1}$$

for some constant $C>0$ that depends only on α and γ. By the triangle inequality,

$$\inf_{z\in\mathbb{T}}\|A_\beta-zC_\beta\|\leq 2C\,\mathrm{qdim}(\beta)^{-1}.$$

For every β, let $z_\beta\in\mathbb{T}$ be the phase factor where this infimum is attained, so that

$$\|A_\beta-z_\beta C_\beta\|\leq 2C\,\mathrm{qdim}(\beta)^{-1}.$$

By multiplying one of the intertwiners in the expression of C_β with z_β, we may assume without loss of generality that $z_\beta=1$ for all β. Now consider the following expressions, where D_β is obtained from A_β by summing over all k' and multiplying with $\left(V_{\alpha+1}^{1,\alpha}\otimes\mathrm{id}_\beta\otimes\mathrm{id}_\gamma\right)^*$ on the right. Similarly, E_β is obtained from C_β by summing over all

k' and multiplying with $\left(V_{\alpha+1}^{1,\alpha}\otimes \mathrm{id}_\beta\otimes \mathrm{id}_\gamma\right)^*$. We set

$$\begin{aligned} D_\beta &= \sum_{k'} V_{\beta+k+l}^{1,\beta+k'+l}\left(\mathrm{id}_1\otimes V_{\beta+k'+l}^{\alpha,\beta+l}\right)\left(\mathrm{id}_1\otimes \mathrm{id}_\alpha\otimes V_{\beta+l}^{\beta,\gamma}\right)\left(V_{\alpha+1}^{1,\alpha}\otimes \mathrm{id}_\beta\otimes \mathrm{id}_\gamma\right)^* \\ &= \sum_{k'} V_{\beta+k+l}^{1,\beta+k'+l}\left(\mathrm{id}_1\otimes V_{\beta+k'+l}^{\alpha,\beta+l}\right)\left(V_{\alpha+1}^{1,\alpha}\otimes \mathrm{id}_{\beta+l}\right)^*\left(\mathrm{id}_1\otimes \mathrm{id}_\alpha\otimes V_{\beta+l}^{\beta,\gamma}\right), \\ E_\beta &= \sum_{k'} V_{\beta+k+l}^{\beta+k,\gamma}\left(V_{\beta+k}^{1,\beta+k'}\otimes \mathrm{id}_\gamma\right)\left(\mathrm{id}_1\otimes V_{\beta+k'}^{\alpha,\beta}\otimes \mathrm{id}_\gamma\right)\left(V_{\alpha+1}^{1,\alpha}\otimes \mathrm{id}_\beta\otimes \mathrm{id}_\gamma\right)^*. \end{aligned}$$

It follows from the triangle inequality that

$$\|D_\beta - E_\beta\| \le CK\,\mathrm{qdim}(\beta)^{-1}, \tag{4.5}$$

where K is the total number of summands in D_β and E_β, which depends only on α and γ. But by Lemma 4.7 we have, for suitable phase factors $z_1, z_2 \in \mathbb{T}$,

$$\begin{aligned} V_{\beta+k+l}^{\alpha+1,\beta+l}\left(\mathrm{id}_{\alpha+1}\otimes V_{\beta+l}^{\beta,\gamma}\right) &= z_1\sum_{k'} V_{\beta+k+l}^{1,\beta+k'+l}\left(\mathrm{id}_1\otimes V_{\beta+k'+l}^{\alpha,\beta+l}\right)\left(V_{\alpha+1}^{1,\alpha}\otimes \mathrm{id}_{\beta+l}\right)^*\left(\mathrm{id}_{\alpha+1}\otimes V_{\beta+l}^{\beta,\gamma}\right), \\ V_{\beta+k+l}^{\beta+k,\gamma}\left(V_{\beta+k}^{\alpha+1,\beta}\otimes \mathrm{id}_\gamma\right) &= z_2\sum_{k'} V_{\beta+k+l}^{\beta+k,\gamma}\left(V_{\beta+k}^{1,\beta+k'}\otimes \mathrm{id}_\gamma\right)\left(\mathrm{id}_1\otimes V_{\beta+k'}^{\alpha,\beta}\otimes \mathrm{id}_\gamma\right)\left(V_{\alpha+1}^{1,\alpha}\otimes \mathrm{id}_\beta\otimes \mathrm{id}_\gamma\right)^*. \end{aligned}$$

So formula (4.4) is just estimate (4.5). By induction, the lemma is proved for any $\alpha\in\mathbb{N}_{\ge 1}$ and $\gamma = 1$. Analogously, we can do induction on γ, and the proof follows. □

In conclusion we record the following result:

Theorem 4.9. *The QMS defined in Theorem 4.5 is approximately linear with almost commuting intertwiners.*

Proof. Formulas (2.1), (2.2) and (2.4) follow from Propositions 4.4 and 4.8. Finally, by equation (4.2) we see that formula (2.6) holds for P a linear polynomial. □

5. Applications to strong solidity: Free wreath products and easy quantum groups

In this section we gather the consequences of our main results. For the definition of the free wreath product we refer to [4] (and [30] for the main properties we need).

Theorem 5.1. *Let $\mathbb{G}$ be a compact quantum group. If $\mathbb{G}$ carries a QMS of central multipliers that is approximately linear with almost commuting intertwiners, then so does the free wreath product $\mathbb{G}\wr_* S_N^+, N\ge 5$. If the QMS on $\mathbb{G}$ is immediately L_2-compact, then so is the one on $\mathbb{G}\wr_* S_N^+$.*

Proof. By [30, Theorem 5.11], the free wreath product $\mathbb{G}\wr_* S_N^+$ is monoidally equivalent to a compact quantum group $\mathbb{H}$ whose dual $\widehat{\mathbb{H}}$ is a quantum subgroup of $\mathbb{G}*\widehat{SU_q}(2)$ for $q\in(0,1)$ such that $q+q^{-1}=\sqrt{N}$. By Theorem 4.9, $SU_q(2)$ has a QMS of central multipliers that is approximately linear with almost commuting intertwiners, which is moreover immediately L_2-compact. Now since approximate linearity with almost commuting

intertwiners (and immediate L_2-compactness) passes to free products (Theorem 2.6), monoidal equivalence (Theorem 2.4) and dual quantum subgroups (Theorem 2.5), we are done. □

For the definition of the almost completely positive approximation property, we refer to [21].

Theorem 5.2. *Let $\mathbb{G}$ be a compact quantum group of Kac type either with the almost completely positive approximation property or such that $L_\infty\left(\mathbb{G}\wr_* S_N^+\right)$ has the W*CBAP. If $\mathbb{G}$ carries a QMS of central multipliers that is approximately linear with almost commuting intertwiners and which is immediately L_2-compact, then the free wreath product $\mathbb{G}\wr_* S_N^+, N \geq 5$, is strongly solid.*

Proof. If $\mathbb{G}$ is of Kac type, then so is $\mathbb{G}\wr_* S_N^+$. It follows from [30, Theorem 6.4 and Remark 6.6] that $\mathbb{G}\wr_* S_N^+$ has the W*CBAP. Then we conclude by Theorem 5.1 and Corollary 3.5. □

Remark 5.3. Theorem 5.7 gives an answer to [30, Remark 6.6]. We note that in [30, Remark 6.6], the strong solidity statement as suggested can only hold under additional assumptions on $\mathbb{G}$ like the ones in Theorem 5.7. Indeed, if there would not but such assumptions, then we could consider for instance the case where $\mathbb{G}$ decomposes as a product of two nonamenable quantum groups whose von Neumann algebras are type II_1 factors with the W*CCAP (which exist by [24]). Then $L_\infty(\mathbb{G})$ is not strongly solid and neither is the ambient von Neumann algebra $L_\infty\left(\mathbb{G}\wr_* S_N^+\right)$.

To our knowledge the following result has not appeared explicitly in the literature so far. We refer to [41, Theorem 5.11] for strong solidity results for a related series of compact quantum groups, and to [2] for the hyperoctahedral series. In the proofs to follow, the symbol $\simeq$ stands for an isomorphism of compact quantum groups (not necessarily preserving the fundamental representation).

Corollary 5.4. *The hyperoctahedral compact quantum groups $H_N^+ \simeq \mathbb{Z}_2\wr_* S_N^+$ are strongly solid for $N \geq 5$.*

Proof. This follows directly from Theorem 5.2. □

Theorem 5.5. *The seven series of free orthogonal easy quantum groups that were classified in [50] under the names $O_{N_3}^+$, $S_{N_5}^+$, $H_{N_5}^+$, $B_{N_4}^+$, $S_{N_5}'^+$, $B_{N_4}'^+$ and $B_{N_4}^{\#+}$ are strongly solid for $N_3 \geq 3$, $N_4 \geq 4$, $N_5 \geq 5$.*

Proof. It is known that all these examples have the almost completely positive approximation property (and hence the von Neumann algebras have the W*CCAP) by [21], [30, Theorem 6.4] and the remainder of this proof.

By [5, Section 5], the quantum group O_N^+ is monoidally equivalent to $SU_q(2)$ for $N = q + q^{-1}, q \in (0,1)$, and so we conclude from Theorems 2.4 and 4.5 and Corollary 3.5. Similarly, S_N^+ is monoidally equivalent to $SO_q(3)$ for $N = q^2 + 1 + q^{-2}$; this follows for instance from [30, Theorem 5.11], together with the observation that the dual of $SO_q(3)$ has no quantum subgroups. By [40, Section 4] and [50, Propositions 5.1 and 5.2], we have

identifications as compact quantum groups $S'^{+}_N \simeq S^+_N \times \mathbb{Z}_2$, $B^+_N \simeq O^+_{N-1}$, $B'^{+}_N \simeq O^+_{N-1} \times \mathbb{Z}_2$ and $B^{\#+}_N \simeq O^+_{N-1} * \mathbb{Z}_2$, so that our results follow from the cases of O^+_N and S^+_N and Theorem 2.6. The only remaining case, H^+_N, was covered in Corollary 5.4. □

Remark 5.6. The cases of O^+_N, S^+_N, B^+_N, S'^{+}_N, B'^{+}_N and $B^{\#+}_N$ in Theorem 5.5 were already covered in [26].

We also state the following theorem for completeness, though here we do not give applications in the non-Kac case. We refer to [10] for such examples. We mention that it is an open problem whether a theorem of this form holds under the assumption of the W*CBAP only instead of the W*CCAP.

Theorem 5.7. *Let $\mathbb{G}$ be a compact quantum group. Suppose that $L_\infty\left(\mathbb{G} \wr_* S^+_N\right)$ is solid and has the W*CCAP. If $\mathbb{G}$ carries a QMS of central multipliers that is approximately linear with almost commuting intertwiners that is immediately L_2-compact, then the free wreath product $\mathbb{G} \wr_* S^+_N, N \geq 5$, is strongly solid.*

Proof. It follows from [30, Theorem 6.4 and Remark 6.6] and [21] that $\mathbb{G}$ has the W*CCAP. Then we conclude by Theorem 5.1 and Corollary 3.6. □

6. Noncommutative Riesz transforms and the Akemann–Ostrand property

The aim of this section is to show that the methods in this paper also show that the von Neumann algebras we consider satisfy the Akemann–Ostrand property. The proof is the same as [12, Section 5], but the setting presented there is too narrow for the current setup. Essentially we need to replace the filtrations considered in [12] by more general fusion rules. Let us first recall the definition of the Akemann–Ostrand property from [27]:

Definition 6.1. A von Neumann algebra M satisfies the Akemann–Ostrand property (briefly called AO$^+$) if there exists a σ-weakly dense unital C*-subalgebra $A \subseteq M$ such that

(1) A is locally reflexive [9, Section 9] and
(2) there exists a unital completely positive map $\theta : A \otimes_{\min} A^{\mathrm{op}} \to B(L_2(M))$ such that $\theta(a \otimes b^{\mathrm{op}}) - ab^{\mathrm{op}}$ is compact for all $a,b \in A$.

Now let $\mathbb{G}$ be a compact quantum group and let Φ be a QMS of central multipliers on $\mathbb{G}$ with generator Δ.

Definition 6.2. We say that Φ has *subexponential growth* if Δ has compact resolvent and for every $\alpha,\gamma \in \mathrm{Irr}(\mathbb{G})$ we have

$$\lim_{\beta\to\infty} \sup_{\substack{\beta' \subseteq \alpha\otimes\beta\otimes\gamma \\ \beta' \in \mathrm{Irr}(\mathbb{G})}} \left| \frac{\Delta_{\beta'}}{\Delta_\alpha} - 1 \right| = 0.$$

Here the limit $\lim_{\alpha\to\infty} c_\alpha = c$ is defined as saying that for every $\epsilon > 0$, there exists a compact set $K \subseteq \mathrm{Irr}(\mathbb{G})$ such that for all $\alpha \in \mathrm{Irr}(\mathbb{G}) \backslash K$ we have $|c_\alpha - c| < \epsilon$.

Remark 6.3. Formula (2.2) implies that Φ has subexponential growth.

Remark 6.4. The subexponential growth condition should be compared to the amenability results from [18] and [10, Appendix]. These results show that if the eigenvalues of Δ grow fast, then the von Neumann algebra must be amenable. As a rule of thumb, many semigroups on nonamenable von Neumann algebras will have subexponential growth.

The aim of this section is to state the following theorem. By Remark 6.3 it applies to all QMSs that are approximately linear with almost commuting intertwiners and for which the generator has compact resolvent; in particular it applies to the examples in this paper.

Theorem 6.5. *Let $\mathbb{G}$ be a compact quantum group of Kac type. Let Φ be a QMS of central multipliers that is immediately gradient-$\mathcal{S}_2$ and has subexponential growth.*[1] *Assume that $C_r(\mathbb{G})$ is locally reflexive. Then $L_\infty(\mathbb{G})$ satisfies* AO^+.

Proof sketch. The proof is a straightforward adaptation of the arguments in [12, Section 5], with the following considerations taken into account. The idea is to consider an $L_\infty(\mathbb{G})$-$L_\infty(\mathbb{G})$-bimodule H_∇ (called the gradient bimodule or the *carré du champ*) together with an isometry

$$S := \partial \Delta^{-\frac{1}{2}} : L_2(\mathbb{G}) \to H_\nabla$$

(called the Riesz transform). We refer to [12, Eqn. (5.1)] for their definitions, which make perfect sense in the current context. By [10, Proposition 3.8 and Proposition 4.4] and the fact that Φ is immediately gradient-$\mathcal{S}_2$, we see that H_∇ is weakly contained in the coarse bimodule of $L_\infty(\mathbb{G})$. We must then prove a suitable replacement of [12, Theorem 5.12] stating that for every $x,y \in \mathrm{Pol}(\mathbb{G})$ (and hence for every $x,y \in C_r(\mathbb{G})$), the map

$$T_{x,y} : L_2(\mathbb{G}) \to H_\nabla : \xi \mapsto S(x\xi y) - xS(\xi)y \tag{6.1}$$

is compact. Then a standard argument yields the condition AO^+, for which we refer to [12, Proposition 5.2], finishing the proof.

The most important part is thus that we must prove that [12, Theorem 5.12] still holds in the current context, meaning that formula (6.1) is compact. In [12, Theorem 5.12] the von Neumann algebra is assumed to be filtered, which is not the case in the setting of Theorem 6.5. However, we can still make the following observation. For $\alpha \in \mathrm{Irr}(\mathbb{G})$ we set the space of matrix coefficients

$$\mathcal{A}(\alpha) = \{(\iota \otimes \omega)(\alpha) \mid \omega \in M_{n_\alpha}(\mathbb{C})^*\}.$$

Then for $\alpha,\beta \in \mathrm{Irr}(\mathbb{G})$ we have

$$\mathcal{A}(\alpha)\mathcal{A}(\beta) \subseteq \oplus_{\gamma \subseteq \alpha\otimes\beta} \mathcal{A}(\gamma),$$

which replaces the filtered condition from [12]. With this observation in mind and with the current notion of subexponential growth (Definition 6.2), the proof of [12, Theorem 5.12] translates literally to the current setting. □

[1] The immediately gradient-$\mathcal{S}_2$ condition can be replaced by the weaker gradient coarse condition from [10, Definition 4.1].

Essentially, Theorem 6.5 applies to all the examples mentioned in Section 5. For instance, we get the following result, which was already known from [26], except for the case of H_N^+:

Theorem 6.6. *The seven series of free orthogonal easy quantum groups classified in [50] under the names* $O_{N_3}^+$, $S_{N_5}^+$, $H_{N_5}^+$, $B_{N_4}^+$, $S_{N_5}'^+$, $B_{N_4}'^+$ *and* $B_{N_4}^{\#+}$ *satisfy* AO^+ *for* $N_3 \geq 3$, $N_4 \geq 4$ *and* $N_5 \geq 5$.

Finally, it should be mentioned that strong solidity results can also be obtained through condition AO^+ using results from [27].

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