

## The Thue–Morse Sequence in Base $3/2$

Dekking, F. M.

**Publication date**

2023

**Document Version**

Final published version

**Published in**

Journal of Integer Sequences

**Citation (APA)**

Dekking, F. M. (2023). The Thue–Morse Sequence in Base  $3/2$ . *Journal of Integer Sequences*, 26(2), Article 23.2.3.

**Important note**

To cite this publication, please use the final published version (if applicable). Please check the document version above.

**Copyright**

Other than for strictly personal use, it is not permitted to download, forward or distribute the text or part of it, without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license such as Creative Commons.

**Takedown policy**

Please contact us and provide details if you believe this document breaches copyrights. We will remove access to the work immediately and investigate your claim.



# The Thue–Morse Sequence in Base $3/2$

F. M. Dekking  
CWI Amsterdam and Delft University of Technology  
Faculty EEMCS  
P.O. Box 5031  
2600 GA Delft  
The Netherlands

[F.M.Dekking@math.tudelft.nl](mailto:F.M.Dekking@math.tudelft.nl)

## Abstract

We discuss the base  $3/2$  representation of the natural numbers. We prove that the sum-of-digits function of the representation is a fixed point of a 2-block substitution on an infinite alphabet, and that this implies that sum-of-digits function modulo 2 of the representation is a fixed point  $x_{3/2}$  of a 2-block substitution on  $\{0, 1\}$ . We prove that  $x_{3/2}$  is invariant for taking the binary complement, and present a list of conjectured properties of  $x_{3/2}$ , which we think will be hard to prove. Finally, we make a comparison with a variant of the base  $3/2$  representation, and give a general result on  $p$ - $q$ -block substitutions.

## 1 Introduction

A natural number  $N$  is written in base  $3/2$  if  $N$  has the form

$$N = \sum_{i \geq 0} d_i \left(\frac{3}{2}\right)^i, \tag{1}$$

with digits  $d_i = 0, 1$  or  $2$ .

Base  $3/2$  representations are also known as sesquinary representations of the natural numbers; see Propp [6]. We write these expansions as

$$\text{SQ}(N) = d_R(N) \cdots d_1(N) d_0(N) = d_R \cdots d_1 d_0.$$

We have, for example,  $\text{SQ}(7) = 211$ , since  $2 \cdot (9/4) + (3/2) + 1 = 7$ . See [A024629](#) for the continuation of Table 1. Ignoring leading 0's, the base  $3/2$  representation of a number  $N$  is unique (see Section 3).

$N$	0	1	2	3	4	5	6	7	8	9	10
$\text{SQ}(N)$	0	1	2	20	21	22	210	211	212	2100	2101

Table 1: Base 3/2 expansions for  $N = 1, \dots, 10$ .

For  $N \geq 0$  let

$$s_{3/2}(N) := \sum_{i=0}^{i=R} d_i(N)$$

be the sum-of-digits function of the base 3/2 expansions. We have (see [A244040](#))

$$s_{3/2} = 0, 1, 2, 2, 3, 4, 3, 4, 5, 3, 4, 5, 5, 6, 7, 4, 5, 6, 5, 6, 7, 7, 8, 9, 5, 6, 7, 5, 6, 7, 7, 8, 9, 8, 9, 10, \dots$$

In this note we study the base 3/2 analogue of the Thue–Morse sequence [A010060](#) (where the base equals 2), i.e., the sequence (see [A357448](#))

$$(x_{3/2}(N)) := (s_{3/2}(N) \bmod 2) = 0, 1, 0, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, \dots$$

The Thue Morse sequence is the fixed point starting with 0 of the substitution  $0 \rightarrow 01, 1 \rightarrow 10$ . This might be called a 1-2-block substitution.

Let  $p \leq q$  be two natural numbers. A  $p$ - $q$ -block substitution  $\kappa$  on an alphabet  $A$  is a map  $\kappa : A^p \rightarrow A^q$ . A  $p$ - $q$ -block substitution  $\kappa$  acts on  $(A^p)^*$  by defining

$$\kappa(w_1 w_2 \cdots w_{pm-1} w_{pm}) = \kappa(w_1 \cdots w_p) \cdots \kappa(w_{pm-p+1} \cdots w_{pm})$$

for  $w_1 w_2 \cdots w_{pm-1} w_{pm} \in (A^p)^*$  and  $m = 1, 2, \dots$ . Its action extends to infinite sequences  $x = x_0 x_1 \cdots$  by defining  $\kappa : x \mapsto y$  by  $y_{qm} \cdots y_{qm+q-1} = \kappa(x_{pm} \cdots x_{pm+p-1})$  for  $m = 0, 1, \dots$

**Theorem 1.** *The sequence  $x_{3/2}$  is a fixed point of the 2-3-block substitution*

$$\kappa : \begin{cases} 00 & \rightarrow 010 \\ 01 & \rightarrow 010 \\ 10 & \rightarrow 101 \\ 11 & \rightarrow 101 \end{cases}$$

Theorem 1 will be proved in Section 2.2.

## 2 Sum of digits function and Thue–Morse in base 3/2

### 2.1 Sum of digits function in base 3/2

Let  $s_{3/2} = (0, 1, 2, 2, 3, 4, 3, 4, 5, 3, 4, 5, 5, 6, 7, 4, 5, \dots)$  be the sum-of-digits function of the base 3/2 expansions. To describe this sequence, we extend the notion of a  $p$ - $q$ -block substitution to alphabets of infinite cardinality.

**Theorem 2.** *The sequence  $s_{3/2}$  is the fixed point starting with 0 of the 2-3-block substitution given by*

$$a, b \mapsto a, a + 1, a + 2 \quad \text{for } a = 0, 1, 2, \dots \text{ and } b = 0, 1, 2, \dots$$

*Proof.* We have  $d(0) = 0, d(1) = 1$  and from the uniqueness of the base  $3/2$  expansions it follows immediately that  $d(3N + r) = d(2N) + r$  for  $N \geq 0$  and  $r = 0, 1, 2$ .

Thus  $s_{3/2}(3N) = s_{3/2}(2N), s_{3/2}(3N + 1) = s_{3/2}(2N) + 1$ , and  $s_{3/2}(3N + 2) = s_{3/2}(2N) + 2$ . This gives the result.  $\square$

*Remark 3.* The base-4/3 version of this sequence is [A244041](#); the base-2 version is [A000120](#); the base-3 version is [A053735](#); the base-10 version is [A007953](#).

## 2.2 Thue–Morse in base $3/2$

*Proof of Theorem 1.* This follows directly from Theorem 2 by taking  $a$  and  $b$  modulo 2.  $\square$

Although iterates of  $\kappa : 00 \rightarrow 010, 01 \rightarrow 010, 10 \rightarrow 101, 11 \rightarrow 101$  are undefined, we can generate the fixed point  $x_{3/2}$  by iteration of a map  $\kappa'$  defined by  $\kappa'(w) = \kappa(w)$  if  $w$  has even length, and  $\kappa'(v) = \kappa(w)$  if  $v = w0$  or  $v = w1$  has odd length.

The fact that the iterates of  $\kappa$  are undefined causes difficulty in establishing properties of  $x_{3/2}$ . This is similar to the lack of progress in the last 25 years to prove the conjectures on the Kolakoski sequence, which is also a fixed point of a 2-block substitution (cf. the papers [2, 3]). Here is a property that is open for the Kolakoski sequence [A000002](#), but can be proved for  $x_{3/2}$ .

**Proposition 4.** *If a word  $w$  occurs in  $x_{3/2}$ , then its binary complement  $\bar{w}$  defined by  $\bar{0} = 1, \bar{1} = 0$ , also occurs in  $x_{3/2}$ .*

*Proof.* First one checks this for all 16 words of length 6 that occur in  $x_{3/2}$ . Note that then also  $\bar{w}$  occurs for all  $w$  with  $|w| \leq 6$ , where  $|w|$  denotes the length of  $w$ . Let  $u$  be a word of length  $m \geq 7$ . By adding at most 3 letters at the beginning and/or end of  $u$  one can obtain a word  $v$  with  $|v| = 3n$  that occurs in  $x_{3/2}$  at a position 0 modulo 3. But then Theorem 1 gives that  $v = \kappa(w)$  for at least one word  $w$  occurring in  $x_{3/2}$ . The length of  $w$  is  $|w| = 2n$ . Since  $\overline{\kappa(w)} = \kappa(\bar{w})$  the result follows by induction on  $m = |u|$ . For example, for  $|u| = m = 7$ , one has  $|v| = 9$ , and so  $|w| = 6$ .  $\square$

Here are some conjectured properties of  $x_{3/2}$ .

**Conjecture 5.**  $x_{3/2}$  is reversal invariant, i.e., if the word  $w = w_1 \cdots w_m$  occurs in  $x_{3/2}$  then  $\overleftarrow{w} = w_m \cdots w_1$  occurs in  $x_{3/2}$ .

**Conjecture 6.**  $x_{3/2}$  is uniformly recurrent, i.e., each word that occurs in  $x_{3/2}$  occurs infinitely often, with bounded gaps between consecutive occurrences.

**Conjecture 7.** The frequencies  $\mu[w]$  of the words  $w$  occurring in  $x_{3/2}$  exist. Two conjectured values:  $\mu[00] = 1/10, \mu[01] = 4/10$ .

**Conjecture 8.**  $\mu$  is invariant for binary complements, i.e.,  $\mu[w] = \mu[\bar{w}]$  for all words  $w$ .

**Conjecture 9.**  $\mu$  is reversal invariant, i.e.,  $\mu[w] = \mu[\overleftarrow{w}]$  for all words  $w$ .

**Conjecture 10.** (Shallit) The critical exponent (=largest number of repeated blocks) of  $x_{3/2}$  is 5.

### 3 Base $3/2$ and base $1/2 \cdot 3/2$

Many authors refer to the paper [1] from Akiyama, Frougny, and Sakarovitch for the properties of base  $3/2$  expansions (see, e.g., Propp [6] and Rigo and Stipulanti [7]). However, the  $q/p$  expansions studied in paper [1] are different from the  $3/2$  expansions that are usually considered as in Equation (1). In the paper [1]:

$$N = \sum_{i \geq 0} d_i \frac{1}{p} \left(\frac{q}{p}\right)^i, \quad (2)$$

with digits  $d_i = 0, 1$  or  $2$ . We write  $\text{AFS}(N)$  for the expansion of  $N$ .

*Remark 11.* There is a small notational problem here: Akiyama, Frougny, and Sakarovitch write about  $p/q$  expansions with  $p > q$ , but in this note we consider  $q/p$  expansions with  $p \leq q$ . This fits better with the  $p$ - $q$ -block substitutions, and with the order of  $p$  and  $q$  in the alphabet.

Here is the table given in the paper [1] for the case  $3/2$ :

$N$	0	1	2	3	4	5	6	7	8	9	10
$\text{AFS}(N)$	$\varepsilon$	2	21	210	212	2101	2120	2122	21011	21200	21202

Table 2: Base  $1/2 \cdot 3/2$  expansions for  $N = 1, \dots, 10$ .

These expansions will not even be found in the OEIS (at the moment).

The situation is clarified in the paper [5] by Frougny and Klouda. They consider both representations, called, respectively, base  $p/q$  and base  $1/q \cdot p/q$  representations. In the present note these are called respectively base  $q/p$  and base  $1/p \cdot q/p$  representations.

A combination of the results in [1] and [5] yields a proof of the uniqueness of the base  $3/2$  expansions ( $\text{QS}(N)$ ). There is also a direct proof of uniqueness in the paper by Edgar et al. [4]; see Theorem 1.1.

Note that  $\text{AFS}(N) = \text{QS}(2N)$  for  $N > 0$ . So uniqueness of the base  $3/2$  representation implies immediately uniqueness of the  $1/2 \cdot 3/2$  representation  $\text{AFS}(N)$ . This observation obviously extends to base  $q/p$ .

Next we consider the question whether also the sequence  $y_{3/2}$ , the sum-of-digits function modulo 2 of the base  $1/2 \cdot 3/2$  representation, is a fixed point of a 2-block substitution. This is indeed the case, and this 2-block substitution is given by Rigo and Stipulanti in [7].

**Theorem 12.** ([7])  $y_{3/2}$  is the fixed point with prefix 00 of the 2-3-block substitution

$$\kappa' : \begin{cases} 00 & \rightarrow 001 \\ 01 & \rightarrow 000 \\ 10 & \rightarrow 111 \\ 11 & \rightarrow 110 \end{cases}$$

In the paper [7] the proof of Theorem 12 is based on a generalization of Cobham's theorem to what are called  $\mathcal{S}$ -automatic sequences built on tree languages with a periodic labeled signature. Here we consider a more direct route, based on a simple closure property of  $p$ - $q$ -block substitutions. Recall that a coding is a letter to letter map from one alphabet to another.

**Theorem 13.** Let  $x = (x(N))$  be a fixed point of a  $p$ - $q$ -block substitution. Let  $r$  be a positive integer. Then the sequence  $(x(rN))$  is the fixed point of a coding of a  $p$ - $q$ -block substitution.

*Proof.* If  $x$  is a fixed point of a  $p$ - $q$ -block substitution, then  $x$  is also a fixed point of a  $pr$ - $qr$ -block substitution. As new alphabet, take the words of length  $r$  occurring in  $x$ . On this alphabet, the  $pr$ - $qr$ -block substitution induces a  $p$ - $q$ -block substitution in an obvious way. Mapping each word of length  $r$  to its first letter is a coding that gives the result.  $\square$

*Alternative proof for Theorem 12.* Apply Theorem 13 with  $r = 2$ . The 4-6-block substitution is given by

$$\begin{aligned} 0010 &\rightarrow 010101, & 0100 &\rightarrow 010010, & 0101 &\rightarrow 010010, & 0110 &\rightarrow 010101, \\ 1001 &\rightarrow 101010, & 1010 &\rightarrow 101101, & 1011 &\rightarrow 101101, & 1101 &\rightarrow 101010. \end{aligned}$$

Coding  $00 \mapsto a$ ,  $01 \mapsto b$ ,  $10 \mapsto c$ ,  $11 \mapsto d$ , this induces the 2-3-block substitution

$$ac \rightarrow bbb, ba \rightarrow bac, bb \rightarrow bac, bc \rightarrow bbb, cb \rightarrow ccc, cc \rightarrow cdb, cd \rightarrow cdb, db \rightarrow ccc.$$

If we code further  $a, b \mapsto 0$ , and  $c, d \mapsto 1$ , then we obtain  $\kappa'$  from Theorem 12.  $\square$

## 4 Acknowledgment

I am grateful to Jean-Paul Allouche for several useful comments. I also thank an anonymous referee for useful remarks.

## References

- [1] S. Akiyama, C. Frougny, and J. Sakarovitch, Powers of rationals modulo 1 and rational base number systems, *Israel J. Math.* **168** (2008), 53–91.

- [2] F. M. Dekking, What is the long range order in the Kolakoski sequence?, in *The Mathematics of Long-Range Aperiodic Order*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Vol. 489, Kluwer Acad. Publ., 1997, pp. 115–125.
- [3] F. M. Dekking and M. Keane, Two-block substitutions and morphic words, arxiv preprint arXiv:2202.13548 [math.CO], 2022. Available at <https://arxiv.org/abs/2202.13548>.
- [4] T. Edgar, H. Olafson, and J. Van Alstine, Some combinatorics of rational base representations, preprint, 2014. Available at <https://community.plu.edu/~edgartj/preprints/basepqarithmic.pdf>.
- [5] C. Frougny and K. Klouda, Rational base number systems for  $p$ -adic numbers, *RAIRO Theor. Inform. Appl.* **46** (2019), 87–106.
- [6] J. Propp, How do you write one hundred in base  $3/2$ ? Available at <https://tinyurl.com/2p9auyz6>. Accessed January 2023.
- [7] M. Rigo and M. Stipulanti, Automatic sequences: from rational bases to trees, *Disc. Math. Theor. Comput. Sci.* **24** (2022), #25.
- [8] N. J. A. Sloane et al., *On-Line Encyclopedia of Integer Sequences*, electronically available at <https://oeis.org>, 2023.

---

2020 *Mathematics Subject Classification*: Primary 11B85, Secondary 68R15

*Keywords*: Base  $3/2$ , Thue–Morse sequence, sum of digits, two-block substitution.

---

(Concerned with sequences [A000002](#), [A000120](#), [A007953](#), [A010060](#), [A024629](#), [A053735](#), [A244040](#), [A244041](#), and [A357448](#).)

---

Received February 9 2023; revised version received February 12 2023. Published in *Journal of Integer Sequences*, February 23 2023.

---

Return to [Journal of Integer Sequences home page](#).