# On linear positive operators in approximation theory

# ON LINEAR POSITIVE OPERATORS IN APPROXIMATION THEORY

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# INTRODUCTION

In 1953 KOROVKIN ([9], [7])* introduced the notion "linear positive operator" in approximation theory, although a corresponding idea already occurs in 1952 in a paper by BOHMAN [3].

DEFINITION

*Let $V$ denote the set of all real functions $f(x)$, defined on a set $D$ of points of the real $x$-axis. Let $L$ be an operator which maps each element $f$ of $V$ into a set $W$ of real functions $g(x) = L(f(t); x)$ which are defined on a set $E$ of points of the real $x$-axis.*

*Then $L$ is said to be positive if and only if for each $f \in V$ with $f(x) \geqslant 0$ on $D$ we have*

$$g(x) \geqslant 0 \text{ on } E.$$

EXAMPLE

$$g(x) = L(f; x) = f(-1)\,(x-x^2)+f(0)(1-x^2)+f(1)(x^2+x).$$

*Here $D$ consists of the three points $-1$, $0$ and $1$ of the $x$-axis and $E$ is the interval $[0, 1]$.*

In this introduction we will first give a brief exposition of results from which we start in this thesis. Next the Chapters 1–4 are summarized.

*We remark that from now on the word operator means linear positive operator.*

## 1. **Survey of the contents of this thesis**

### 1.1. PRELIMINARIES

For the formulation of a number of theorems concerning operators, we introduce a definition.

DEFINITION

*The set of functions which are bounded on the whole real axis and which are continuous on the interval $[a,b]$, continuous to the left in $x = a$ and continuous to the right in $x = b$, is denoted by $Q(a,b)$.*

*The set of functions which satisfy the above conditions and, moreover, are periodic with period $2\pi$ will be denoted by $Q_{2\pi}(a,b)$ in case $b-a > 2\pi$.*

* Numbers in brackets refer to the references at the end of this thesis.

The following theorem* can be considered as one of the basic theorems about operators in approximation theory.

THEOREM I (BOHMAN [3], KOROVKIN [7])

*Let $f(x) \in Q(a,b)$ and let $\{L_n\}(n = 1,2,\ldots)$ be a sequence of operators defined on $Q(a,b)$. If this sequence satisfies on the interval $[a,b]$ the conditions*

$$\begin{aligned} L_n(1;x) &= 1+\alpha_n(x) \\ L_n(t;x) &= x+\beta_n(x) \qquad (n = 1,2,\ldots) \\ L_n(t^2;x) &= x^2+\gamma_n(x), \end{aligned}$$

*where $\alpha_n(x)$, $\beta_n(x)$ and $\gamma_n(x)$ tend uniformly to zero on the interval $[a,b]$ when $n \to \infty$, then the sequence $\{L_n(f;x)\}(n = 1,2,\ldots)$ converges uniformly on this interval to $f(x)$.*

Using this theorem KOROVKIN ([7], p. 28–30) has given a new proof of the well-known approximation theorem of WEIERSTRASS.

The trigonometric analogon of theorem I reads as follows.

THEOREM II (KOROVKIN [7])

*Let $f(x) \in Q_{2\pi}(a,b)$ and let $\{L_n\}(n = 1,2,\ldots)$ be a sequence of operators defined on $Q_{2\pi}(a,b)$. If this sequence satisfies on the interval $[a,b]$ the conditions*

$$\begin{aligned} L_n(1;x) &= 1+\alpha_n(x) \\ L_n(\cos t;x) &= \cos x + \beta_n(x) \qquad (n = 1,2,\ldots) \\ L_n(\sin t;x) &= \sin x + \gamma_n(x), \end{aligned}$$

*where $\alpha_n(x)$,$\beta_n(x)$ and $\gamma_n(x)$ tend uniformly to zero on the interval $[a,b]$ when $n \to \infty$, then the sequence $\{L_n(f;x)\}(n = 1,2,\ldots)$ converges uniformly on this interval to $f(x)$.***

Theorems I and II give sufficient conditions for a sequence of operators to converge to a continuous function $f(x)$.*** Moreover, KOROVKIN [7] shows that a number of algebraic and trigonometric polynomials, suitable for the approximation of functions, can indeed be defined by means of operators.

BASKAKOV [1] gives a fairly general form of a sequence of operators $\{L_n\}(n = 1,2,\ldots)$ which is generated by a sequence of functions $\{\varphi_n(x)\}$

* For a particular sequence of operators the germ of the theorem can already be found in a paper by POPOVICIU [21a].

** In 1957 VOLKOV [30] has generalized theorem I in case of the approximation of functions of many variables, whereas MOROZOV [17] in 1958 extended theorem II for this purpose (cf. Chapter 2).

*** The three functions 1, $x$, $x^2$ respectively 1, $\cos x$, $\sin x$ cannot be replaced by any two functions as KOROVKIN proved in [7], § 5. But theorems I and II do hold if the conditions of the theorems are satisfied for the functions $f_0(x)$, $f_1(x)$, $f_2(x)$ assuming this system $f_k(x)$ $(k = 0, 1, 2)$ is a Chebyshev system on [a, b]. This is also a necessary condition (cf. [7], § 5).

$(n = 1,2,\ldots)$ (cf. [5]). It is proved that these operators satisfy the conditions of theorem I. Moreover an estimation for the difference $|L_n(f;x) - f(x)|$ is derived in which the modulus of continuity $\omega(\delta)$ plays a rôle. Because this notion appears frequently in the following pages, we will give here its definition.

DEFINITION

*If $f(x)$ is defined on an interval $[a,b]$, then for every $\delta > 0$*

$$\omega(\delta) = \sup_{|x'-x''| \leqslant \delta} |f(x')-f(x'')| \qquad (x', x'' \in [a,b]).$$

The modulus of continuity has the property [28] that

$$\omega(\lambda\delta) \leqslant (1+]\lambda[)\omega(\delta) \quad (\lambda > 0), \tag{1}$$

where $]\lambda[$ is the largest integer being smaller than $\lambda$.

In [1] an asymptotic approximation formula is derived for a sequence of operators belonging to a particular sequence of functions $\{\varphi_n(x)\}(n = 1,2,\ldots)$, i.e. information is given about the speed with which the difference

$$L_n(f;x) - f(x) \tag{2}$$

tends to zero when $n \to \infty$.

By means of a suitable choice of the sequence $\{\varphi_n(x)\}$ the well-known Bernstein polynomials

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k} \quad (x \in [0,1])$$

are generated.

MAMEDOV [12] investigates the speed with which the difference (2) converges to zero in a certain point $x_0 \in [a,b]$ when $n \to \infty$. We will state here his most important result.* First we give a definition.

DEFINITION

*With $Q^{(2)}(x_0)$ we denote the subset of $Q(a,b)$ of functions for which the second derivative at the fixed point $x_0 \in [a,b]$ exists.*

THEOREM III (MAMEDOV [12])

*Let $f(x) \in Q^{(2)}(x_0)$. Assume that the sequence of operators $\{L_n\}(n = 1,2,\ldots)$ has the property that in $x_0$ we have*

$$L_n(1;x_0) = 1 + o\left(\frac{1}{\varphi(n)}\right)$$

* [12] contains three theorems without proof. MAMEDOV does not notice that the first two of them are particular cases of the third one.

$$L_n(t;\, x_0) = x_0 + \frac{\psi_1(x_0)}{\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right)$$

$$L_n(t^2;\, x_0) = x_0^2 + \frac{\psi_2(x_0)}{\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right),$$

*where* $\varphi(n) \neq 0$ *and* $\varphi(n) \to \infty$ *when* $n \to \infty$. *If there exists a positive integer* $m$ *such that*

$$L_n\{(t-x_0)^{2m+2};\, x_0\} = o\left(\frac{1}{\varphi(n)}\right) \qquad (n \to \infty),$$

*then we have*

$$L_n(f;\, x_0) - f(x_0) = \frac{2\psi_1(x_0)f'(x_0) + \{\psi_2(x_0) - 2x_0\psi_1(x_0)\}f''(x_0)}{2\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right).$$

Applications of theorem III were given by MAMEDOV for the derivation of some familiar asymptotic approximation formulae.*

## 1.2. **Survey of the contents of Chapters 1—4** **

### CHAPTER 1

Theorem 1 is an extension of theorem I of BOHMAN-KOROVKIN. It is a generalization in two different ways; first the uniform convergence is replaced by pointwise convergence and in the second place the class of functions for which theorem I is valid, is extended. In fact, theorem I deals with the class of real functions which are bounded on the whole real axis and which are continuous on a finite interval $[a, b]$. However, Korovkin's proof of theorem I is given ([7], p. 14–15) with the aid of two functions $f(x) = x$ and $f(x) = x^2$, which themselves are not bounded on the real axis. Therefore we consider a class of functions, called $H(x_0)$, containing both $x$ and $x^2$ and for which we prove that the results of theorem I hold.

Theorem 2 is a somewhat modified version of theorem III in the sense that the class of functions for which theorem III is valid, is extended. A full proof is given, because paper [12] furnishes no proof at all. In view of the approximation of $2\pi$-periodic functions, we state theorem 2 in a way appropriate to this case. This yields theorem 3.

Next a method is evaluated for the construction of a sequence of operators with a sequence of functions $\{\varphi_n(x)\}$ $(n = 1, 2, \ldots)$. Theorem 4 states that the considered sequence of operators satisfies the conditions of theorem 1.

* In [12] the formula given for the Fejér operators is false.

** Some parts of this thesis have been published before in the author's papers [23], [24], [25], [26], [27].

The influence of the function $f(x)$ on the approximation is given by theorem 5 which gives an estimation of the difference

$$|L_n(f; x_0) - f(x_0)|.$$

Some applications of the theorems are given. We consider a particular case of the so defined sequence of operators and write down some sequences of functions $\{\varphi_n(x)\}$ $(n = 1, 2, \ldots)$ suitable for the generation of operators.

Theorems 6 and 7 are derived with the aid of theorem 2. They show the influence of the sequence $\{\varphi_n(x)\}$ on the asymptotic approximation.

Korovkin has derived a theorem concerning the asymptotic approximation for a special type of operators. We state his result here.

Theorem IV (Korovkin [8])

*The operators $L_n$ are defined by*

$$L_n(f; x) = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x+t) \left\{\frac{1}{2} + \sum_{k=1}^{n} \varrho_k^{(n)} \cos kt\right\} dt \tag{3}$$

*with*

$$\frac{1}{2} + \sum_{k=1}^{n} \varrho_k^{(n)} \cos kt \geqslant 0 \qquad (-\pi \leqslant t \leqslant +\pi).$$

*In order that for a function $f(x) \in Q^{(2)}(x_0)$ the relation*

$$L_n(f; x_0) - f(x_0) = (1-\varrho_1^{(n)}) f''(x_0) + o(1-\varrho_1^{(n)})$$

*is valid, a necessary and sufficient condition is that*

$$\lim_{n \to \infty} \frac{1-\varrho_2{}^{(n)}}{1-\varrho_1{}^{(n)}} = 4. \tag{4}$$

At the end of the Chapter, theorems 3 and IV are used to get asymptotic approximation formulae for some well-known operators such as Jackson's.

Chapter 2

Some results about the approximation of real functions of $m$ real variables are formulated. For this purpose a generalization (theorem 8) of a theorem of Volkov [30] is given. Theorem 9 is in the same way an extension of theorem 2.

A general method is given for the construction of sequences of operators suitable for the approximation of functions of many variables (cf. [2], [31]). Generalizations of some theorems of Chapter 1 are stated without proof.

As an example we give some sequences of functions with the corresponding operators and the region in which they are defined. Some already known results (a.o. a generalized theorem of VORONOVSKAJA) are derived in another way and in particular we pay attention to a constant $\varkappa$ which appears in the inequality

$$|B_n(f; X) - f(X)| \leqslant \varkappa\, \omega(n^{-1/2}, \ldots, n^{-1/2}), \quad X = (x_1, \ldots, x_m),$$

where $B_n(f; X)$ are generalized Bernstein polynomials defined on a certain simplex. Here $\omega(\delta_1, \ldots, \delta_m)$ is the generalized modulus of continuity of the function $f(X)$ in the domain of $f(X)$. This domain is denoted by $K$.

DEFINITION

*If the function $f(X)$ is continuous in a convex domain $K$ in $R_m$, then the generalized modulus of continuity is for every $\delta_j > 0$ defined by*

$$\omega(\delta_1, \ldots, \delta_m) = \sup_{\substack{|x_j' - x_j''| \leqslant \delta_j \\ (j=1, \ldots, m)}} |f(X') - f(X'')| \quad (X', X'' \in K). \tag{5}$$

STANCU [29] has estimated the constant $\varkappa$. We are improving his result. To that end we first prove a theorem about the generalized modulus of continuity by sharpening inequality

$$\omega(\lambda_1\delta_1, \ldots, \lambda_m\delta_m) \leqslant \left(1 + \sum_{k=1}^{m} \lambda_k\right) \omega(\delta_1, \ldots, \delta_m) \quad (\lambda_k > 0;\ k = 1, \ldots, m).$$

Use is also made of a result of SIKKEMA [28].

Finally a theorem of STANCU is derived in another way and asymptotic approximation formulae are deduced for some operators.

CHAPTER 3

The operators of FEJÉR and JACKSON are considered in Chapter 1. These operators are particular cases of the general sequence of operators of the form

$$L_{np-p}(f; x) = \frac{1}{A_{np-p}} \int_{-\pi}^{+\pi} f(x+t) \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt \tag{6}$$

with

$$A_{np-p} = \int_{-\pi}^{+\pi} \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt \qquad (n = 1, 2, \ldots).$$

Here $p$ is an arbitrary but fixed positive integer; the subscript $np-p$ denotes that $\left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p}$ is an (even) trigonometric polynomial of degree $np-p$.

The operators (6) are the object of Chapter 3. The sequence of the form (6) is a special case of the sequence of operators (3) because

$$\left(\frac{\sin {}^1/_2nt}{\sin {}^1/_2t}\right)^{2p} = \frac{1}{6}\varrho_0^{(np-p)} + \frac{1}{3}\sum_{k=1}^{np-p}\varrho_k^{(np-p)}\cos kt \geqslant 0 \; (-\pi \leqslant t \leqslant +\pi). \tag{7}$$

A study of (3) was made by Korovkin ([7], [8], [9]) and from his investigations it follows that the coefficient $\varrho_1^{(n)}$ plays an important rôle in the behaviour of the approximation.

We first establish that the sequence $\{L_{np-p}(f; x)\}$ satisfies the conditions of theorem 1.

In order to derive formulae for the operators $L_{np-p}$, we have to consider the integrals $A_{np-p}(p = 1,2,\ldots)$. The value of the integral $A_{np-p}$, $p$ fixed, is completely determined by the coefficient $\varrho_0^{(np-p)}$ in expansion (7). With regard to the asymptotic approximation we have to know (theorem IV) the coefficients $\varrho_1^{(np-p)}$ and $\varrho_2^{(np-p)}$. The calculation of these coefficients $\varrho_k^{(np-p)}$ $(k = 0,1,2)$ can be established in an elementary way, one must, however, have at one's disposal the coefficients

$$\varrho_k^{n(p-1)-(p-1)} \; (0 \leqslant k \leqslant n(p-1)-(p-1)).$$

The coefficients $\varrho_k^{(2n-2)}$ $(0 \leqslant k \leqslant 2n-2)$ were first given by Safronova [22]; this is sufficient for the calculation of the operators $L_{np-p}$ up to and including $p = 4$.

In cases $p = 1$ and $p = 2$ we have the well-known Fejér- and Jackson operators. The formulae in cases $p = 3$ and $p = 4$ were given by Matsuoka ([14], [15]), and independently, by the author [26] (cf. [6]). For the sake of completeness we state the results.*

$$L_{n-1}(f; x) = \frac{1}{2\pi n}\int_{-\pi}^{+\pi} f(x+t)\left(\frac{\sin {}^1/_2nt}{\sin {}^1/_2t}\right)^2 dt,$$

$$L_{2n-2}(f; x) = \frac{3}{2\pi(2n^3+n)}\int_{-\pi}^{+\pi} f(x+t)\left(\frac{\sin {}^1/_2nt}{\sin {}^1/_2t}\right)^4 dt,$$

$$L_{3n-3}(f; x) = \frac{10}{\pi(11n^5+5n^3+4n)}\int_{-\pi}^{+\pi} f(x+t)\left(\frac{\sin {}^1/_2nt}{\sin {}^1/_2t}\right)^6 dt,$$

$$L_{4n-4}(f; x) = \frac{315}{\pi(302n^7+140n^5+98n^3+90n)}\int_{-\pi}^{+\pi} f(x+t)\left(\frac{\sin {}^1/_2nt}{\sin {}^1/_2t}\right)^8 dt.$$

* One has a check on the results because $A_{np-p} = 2\pi$ for $n = 1$ and $p$ arbitrary.

Theorem 16 states formulae for the coefficients $\varrho_k^{(3n-3)} (0 \leqslant k \leqslant 3n-3)$ in the expansion (7) in case $p = 3$. Using these results we derive in theorem 17 the form of operators $L_{5n-5}$ and $L_{6n-6}$. There is every appearance that the considered method is not so suitable for the further derivation of the form of the operators $L_{np-p}$ $(p = 7, 8, \ldots)$.

A remarkable property is shown by the form of the above operators. The values of the integrals $A_{np-p}$ $(p = 1, 2, 3, 4)$ consist only of odd powers of $n$. It turns out that this also holds in cases $p = 5$ and $p = 6$.

In order to state the results which follow, we quote a few definitions from mathematical literature.

Definitions

1) *$Z$ is the class of all functions $f(x) \in C_{2\pi}$ satisfying the inequality*

$$|f(x+t) - 2f(x) + f(x-t)| \leqslant 2|t|$$

*for all fixed $x$ and all $t$ on the real axis;*

2) *$Z_2$ is the class of all functions $f(x) \in C_{2\pi}$ satisfying the inequality*

$$|f(x+t) - 2f(x) + f(x-t)| \leqslant t^2$$

*for all fixed $x$ and all $t$ on the real axis;*

3) *when $f(x) \in C_{2\pi}$ we define the norm of $f(x)$ as*

$$\|f\| = \max \{|f(x)|; -\pi \leqslant t \leqslant +\pi\}.$$

Petrov [21] established a theorem on the degree of approximation for functions $f(x) \in Z$ by the operators $L_{2n-2}$. He puts

$$b_n = 2n \sup_{(f)} \{\|L_{2n-2}(f) - f\|; f(x) \in Z\}$$

and proves that

$$\lim_{n \to \infty} b_n = \frac{12 \ln 2}{\pi}.$$

Concerning the degree of approximation for functions belonging to the class $Z_2$, Matsuoka [15] proved the following theorem and derived numerical results for the operators $L_{3n-3}$ and $L_{4n-4}$.

Theorem V (Matsuoka)

*If $L_n$ is an operator of type (3) and if condition (4) is satisfied, then*

$$\sup_{(f)} \{\|L_n(f) - f\|; f(x) \in Z_2\} \backsimeq 1 - \varrho_1^{(n)},$$

*where $a_n \backsimeq b_n$ means*

$$\lim_{n \to \infty} a_n / b_n = 1.$$

Following Petrov's result, theorem 19 is concerned with the degree of approximation for functions belonging to $Z$ by means of the operators $L_{3n-3}(n=1,2,...)$. Moreover some applications of Matsuoka's theorem are given.

Theorem 21 gives the asymptotic approximation formulae for the operators $L_{5n-5}$ and $L_{6n-6}$, using theorem IV (cf. [6], [14], [26]). The order of approximation is equal to $1/n^2$; KOROVKIN [10] proved that, concerning the order, it is in general not possible to get a better approximation with operators.

An asymptotic approximation formula for the operators $L_{np-p}$ for arbitrary $p$ is derived in theorem 22 by comparing the integrals

$$\int_{-\pi}^{+\pi} t^2 \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt \qquad \text{and} \qquad \int_{-\pi}^{+\pi} \sin^2 t \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt.$$

Several of the results of this Chapter can be generalized to functions of many variables. An example of such a generalization concludes this Chapter.

## CHAPTER 4

We further investigate the method of Chapter 1, § 2 for the generation of operators by means of sequences of functions. In particular we pay attention to the *construction* of suitable sequences $\{\varphi_n(x)\}(n=1,2,\ldots)$. One of the conditions which every element of the sequence $\{\varphi_n(x)\}$ has to satisfy, viz.

$$-\varphi_n^{(k)}(x) = n\varphi_{m_n}^{(k-1)}(x)(1+\alpha_{k,n}(x)) \quad (k=1,2,\ldots), \tag{8}$$

where $\alpha_{k,n}(x)$ is a correction term and

$$\lim_{n\to\infty} \frac{m_n}{n} = 1,$$

is taken as the central point.

We restrict ourselves to the case $m_n = n-c$, $c$ being an integer, and $\alpha_{k,n}(x) \equiv 0$ $(k,n=1,2,\ldots)$. Using the concept of starting-function a constructive method is developed for deriving sequences of functions suitable for the generation of operators.

The second part of the Chapter is devoted to the case where the $k^{th}$-derivative of $\varphi_n(x)$ in (8) is a certain combination of all the derivatives of lower order. Theorem 25 shows that the corresponding sequence of operators satisfies the conditions of theorem 1.

Cases $m_n = n-1$, $m_n = n$ and $m_n = n+1$ serve as an illustration.

We particularly consider a sequence of functions of the form

$$\varphi_n(x) = e^{1/p(1-x)^p - 1/p}(1-x)^{n-1} \quad (n=1,2,\ldots)$$

($p$ being an arbitrary, but fixed, positive integer) corresponding to case $m_n = n-1$. This sequence generates a sequence of operators defined on the interval [0, 1]. Theorem 26 states that these operators all have the formula of VORONOVSKAJA as asymptotic approximation formula.

CHAPTER 1

# BASIC THEOREMS WITH APPLICATIONS

This Chapter starts with a generalization of a theorem of BOHMAN [3] – KOROVKIN [7]. Then we prove theorem III of MAMEDOV in a somewhat modified form. Another theorem of this type is given. Next we introduce a sequence of operators $\{L_n\}(n = 1,2,\ldots)$, for which we derive some theorems. Applications are given with particular examples.

## 1 Theorems on the approximation of functions with operators

1.1 We will generalize here theorem I of the Introduction.

DEFINITION

*By $H(x_0)$ we denote the class of all real functions $f(x)$ which are defined on the whole real axis and which have the properties*

$$1)\ f(x) \text{ is continuous for } x = x_0;$$

$$2)\ f(x) = O(x^2) \text{ when } |x| \to \infty.$$

THEOREM 1

*Let $f(x) \in H(x_0)$ and let $\{L_n\}$ $(n = 1,2,\ldots)$ be a sequence of operators defined on $H(x_0)$. If we write*

$$\begin{aligned} L_n(1;x) &= 1+\alpha_n(x) \\ L_n(t;x) &= x+\beta_n(x) \qquad (n = 1,2,\ldots) \\ L_n(t^2;x) &= x^2+\gamma_n(x) \end{aligned}$$

*and if $\alpha_n(x)$, $\beta_n(x)$ and $\gamma_n(x)$ have the property that*

$$\lim_{n\to\infty} \alpha_n(x_0) = \lim_{n\to\infty} \beta_n(x_0) = \lim_{n\to\infty} \gamma_n(x_0) = 0, \tag{9}$$

*then*

$$\lim_{n\to\infty} L_n(f;x_0) = f(x_0).$$

PROOF

Since $f \in H(x_0)$, there exists to each $\varepsilon > 0$ a positive number $\delta \leqslant 1$ such that the inequality

$$|f(t)-f(x_0)| \leqslant {}^1\!/_3\varepsilon$$

holds for each $t$ with $|t-x_0| \leqslant \delta$.

Also there exists a positive number $A$ such that for each $t$ with $|t-x_0| > \delta$

$$|f(t)-f(x_0)| \leqslant A(t-x_0)^2$$

holds.

The above two inequalities imply

$$|f(t)-f(x_0)| \leqslant {}^1/_3\varepsilon + \frac{A(t-x_0)^2}{\delta^2} \qquad t \in (-\infty, +\infty).$$

In view of the monotony of the operators $L_n$ we can write

$$|L_n\{f(t)-f(x_0);\, x_0\}| \leqslant L_n\left({}^1/_3\varepsilon + \frac{A(t-x_0)^2}{\delta^2};\, x_0\right).$$

Using the conditions of theorem 1 and the fact that the operators $L_n$ are linear we have

$$L_n\left({}^1/_3\varepsilon + \frac{A(t-x_0)^2}{\delta^2};\, x_0\right) =$$

$$= {}^1/_3\varepsilon L_n(1;\, x_0) + \frac{A}{\delta^2}\{L_n(t^2;\, x_0) - 2x_0 L_n(t;\, x_0) + x_0{}^2 L_n(1;\, x_0)\} =$$

$$= {}^1/_3\varepsilon(1+\alpha_n(x_0)) + \frac{A}{\delta^2}\{\gamma_n(x_0) - 2x_0\beta_n(x_0) + x_0{}^2\alpha_n(x_0)\}.$$

From this and from (9), it follows that there exists a positive integer $N_1(\varepsilon)$ such that for each $n \geqslant N_1$

$$\left|{}^1/_3\varepsilon\alpha_n(x_0) + \frac{A}{\delta^2}\{\gamma_n(x_0) - 2x_0\beta_n(x_0) + x_0{}^2\alpha_n(x_0)\}\right| \leqslant {}^1/_3\varepsilon.$$

Because of the fact that

$$\begin{aligned} L_n\{f(t)-f(x_0);\, x_0\} &= L_n(f(t);\, x_0) - f(x_0) L_n(1;\, x_0) = \\ &= L_n(f(t);\, x_0) - f(x_0) - f(x_0)\alpha_n(x_0), \end{aligned}$$

we have

$$|L_n(f(t);\, x_0) - f(x_0)| \leqslant {}^2/_3\varepsilon + |f(x_0)||\alpha_n(x_0)| \qquad (n \geqslant N_1).$$

On account of (9) there exists a positive integer $N_2(\varepsilon)$ such that for each $n \geqslant N_2$

$$|f(x_0)||\alpha_n(x_0)| \leqslant {}^1/_3\varepsilon.$$

Thus

$$|L_n(f(t);\, x_0) - f(x_0)| \leqslant \varepsilon$$

for each $n \geqslant \max\,(N_1, N_2)$.

This proves theorem 1.

1.2 As is the case with theorem I of the Introduction, the class of functions for which theorem III is valid can be extended.

Definition

*By $H^{(2)}(x_0)$ we denote the class of all functions $\subset H(x_0)$ of which the second derivative exists at the point $t = x_0$.*

THEOREM 2

*Let $f(x) \in H^{(2)}(x_0)$ and let $\{L_n\}(n = 1, 2, \dots)$ be a sequence of operators defined on $H(x_0)$. If the operators $L_n(n = 1, 2, \dots)$ have the property that in a fixed point $x_0$*

$$\begin{cases} L_n(1;\, x_0) = 1 + o\left(\dfrac{1}{\varphi(n)}\right) \\ L_n(t;\, x_0) \;= x_0 + \dfrac{\psi_1(x_0)}{\varphi(n)} + o\left(\dfrac{1}{\varphi(n)}\right) \\ L_n(t^2;\, x_0) = x_0{}^2 + \dfrac{\psi_2(x_0)}{\varphi(n)} + o\left(\dfrac{1}{\varphi(n)}\right), \end{cases} \tag{10}$$

*where $\varphi(n) \neq 0$ and $\varphi(n) \to \infty$ when $n \to \infty$. If there exists a positive integer $m$ such that*

$$L_n\{(t-x_0)^{2m+2};\, x_0\} = o\left(\frac{1}{\varphi(n)}\right) \qquad (n \to \infty), \tag{11}$$

*then we have*

$$L_n(f;\, x_0) - f(x_0) = \frac{2\psi_1(x_0)f'(x_0) + \{\psi_2(x_0) - 2x_0\psi_1(x_0)\}f''(x_0)}{2\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right).$$

PROOF

The operators $L_n(n = 1, 2, \dots)$ satisfy the conditions of theorem 1, hence the sequence $\{L_n(f;\, x_0)\}$ converges to $f(x_0)$. When $f(x) \in H^{(2)}(x_0)$, we can write

$$f(t) - f(x_0) - f'(x_0)(t-x_0) = {}^1\!/_2\{f''(x_0) + h(t-x_0)\}(t-x_0)^2 \tag{12}$$

with $h(0) = 0$, $h(t-x_0)$ is continuous in $t = x_0$ and $h(t-x_0)$ is bounded in $t$ for $-\infty < t < +\infty$. Hence there exists to each $\varepsilon > 0$ a $\delta > 0$ such that

$$|h(t-x_0)| < \varepsilon \text{ whenever } |t-x_0| < \delta,$$

and also there exists a positive number $M$ such that

$$|h(t-x_0)| \leqslant M \;\; (-\infty < t < +\infty).$$

Now we introduce the function $\lambda_\delta(t)$ defined by

$$\begin{cases} \lambda_\delta(t) = 1 \text{ whenever } |t-x_0| \geqslant \delta \\ \lambda_\delta(t) = 0 \text{ whenever } |t-x_0| < \delta. \end{cases} \tag{13}$$

Then the inequality

$$|h(t-x_0)| \leqslant \varepsilon + M\lambda_\delta(t) \tag{14}$$

holds everywhere on the real $t$-axis.

Since the operators $L_n$ are linear it follows from (12) that

$$L_n(f;\, x_0) - f(x_0)L_n(1;\, x_0) - f'(x_0)L_n(t-x_0;\, x_0) - {}^1\!/_2 f''(x_0)L_n\{(t-x_0)^2;\, x_0\} =$$
$$= {}^1\!/_2 L_n\{(t-x_0)^2 h(t-x_0);\, x_0\} \qquad (n = 1, 2, \dots).$$

Using formulae (10), (14) and the fact that $L_n$ is monotone, we have the inequality

$$\left| L_n(f;\, x_0) - f(x_0) - \frac{2f'(x_0)\psi_1(x_0) + \{\psi_2(x_0) - 2x_0\psi_1(x_0)\}f''(x_0)}{2\varphi(n)} \right| \leqslant$$

$$\leqslant {}^1/_2 L_n\{(t-x_0)^2(\varepsilon + M\lambda_\delta(t));\, x_0\} + \frac{b_n(x_0)}{\varphi(n)}, \tag{15}$$

where $b_n(x_0) \geqslant 0$ and $\lim\limits_{n\to\infty} b_n(x_0) = 0$.

Using now formulae (10), (11), (13) we can write

$$^1/_2 L_n\{(t-x_0)^2(\varepsilon + M\lambda_\delta(t));\, x_0\} + \frac{b_n(x_0)}{\varphi(n)} =$$

$$= {}^1/_2 \varepsilon L_n\{(t-x_0)^2;\, x_0\} + \frac{M}{2} L_n\{(t-x_0)^2\lambda_\delta(t);\, x_0\} + \frac{b_n(x_0)}{\varphi(n)} \leqslant$$

$$\leqslant \varepsilon\, \frac{\psi_2(x_0) - 2x_0\psi_1(x_0)}{2\varphi(n)} + \frac{M}{2\delta^{2m}} L_n\{(t-x_0)^{2m+2};\, x_0\} + \frac{\gamma_n(x_0)}{\varphi(n)},$$

where $m$ is the positive integer occurring in (11) and $\lim\limits_{n\to\infty} \gamma_n(x_0) = 0$. Because of (11), it follows in connection with (15) that

$$L_n(f;\, x_0) - f(x_0) = \frac{2f'(x_0)\psi_1(x_0) + \{\psi_2(x_0) - 2x_0\psi_1(x_0)\}f''(x_0)}{2\varphi(n)} + o\left(\frac{1}{\varphi(n)}\right)$$

which was to be proved.

The corresponding theorem for the trigonometric case reads as follows.

THEOREM 3

*Let $f(x)$ be a bounded, $2\pi$-periodic function which is twice differentiable at the point $t = x_0 \in [-\pi, +\pi]$. Further suppose that on the interval $[-\pi, +\pi]$ the sequence $\{L_n(f;\, x)\}$ $(n = 1,2,\ldots)$ of operators $L_n$ possesses the properties*

$$\begin{cases} L_n(1;\, x_0) = 1 + o\left(\dfrac{1}{\varphi(n)}\right) \\ L_n(\sin kt;\, x_0) = \sin kx_0 + \dfrac{\psi_{1,k}(x_0)}{\varphi(n)} + o\left(\dfrac{1}{\varphi(n)}\right) \qquad (k = 1,2) \\ L_n(\cos kt;\, x_0) = \cos kx_0 + \dfrac{\psi_{2,k}(x_0)}{\varphi(n)} + o\left(\dfrac{1}{\varphi(n)}\right), \end{cases}$$

*where $\varphi(n) \neq 0$ and $\varphi(n) \to \infty$ when $n \to \infty$.*

*If there exists a positive integer $m$ such that for $x_0 \in [-\pi, +\pi]$*

$$L_n\{(t-x_0)^{2m+2};\, x_0\} = o\left(\frac{1}{\varphi(n)}\right) \quad (n \to \infty), \tag{16}$$

*then we have*

$$L_n(f;\, x_0) - f(x_0) =$$

$$= \frac{2f'(x_0)\{\cos x_0\, \psi_{1,1}(x_0) - \sin x_0\, \psi_{2,1}(x_0)\} - f''(x_0)\{\cos 2x_0\, \psi_{2,2}(x_0) + \sin 2x_0 \psi_{1,2}(x_0)\}}{4\varphi(n)} +$$

$$+ o\left(\frac{1}{\varphi(n)}\right).$$

The proof will be omitted since it can be given in a way similar to that of theorem 2.

## 2 **A method for constructing operators**

2.1 In order to construct sequences of operators with the aid of sequences of functions, we state the following

CONDITIONS:

We assume that in the sequence

$$\{\varphi_n(x)\} \quad (n = 1, 2, \ldots)$$

of real functions $\varphi_n(x)$ of the real variable $x$, each function has the following properties on the interval $[0, b]$ $(b > 0)$.

I: $\varphi_n(0) = 1$;

II: $\varphi_n(x)$ is infinitely differentiable on $[0, b]$ and $(-1)^k \varphi_n^{(k)}(x) \geqslant 0$ $(k = 0, 1, 2, \ldots)$;

III: there exists a positive integer $m_n$ not depending on $k$, such that $-\varphi_n^{(k)}(x) = \psi(n, x) \varphi_{m_n}^{(k-1)}(x)\{1 + \alpha_{k,n}(x)\} \quad (k = 1, 2, \ldots)$

where

III$_1$: $\alpha_{k,n}(x)$ converges to zero uniformly in $k$ when $n \to \infty$;

III$_2$: $\psi(n, x)$ is such that there exists a positive function $\chi(n)$, monotonically increasing to infinity when $n \to \infty$, with the property that

$$\lim_{n \to \infty} \frac{\psi(n, x)}{\chi(n)} = 1, \tag{17}$$

$$\lim_{n \to \infty} \frac{\psi(m_n, x)}{\chi(n)} = 1. \tag{18}$$

The sequence $\{\varphi_n(x)\}$ generates in the following way a sequence of operators. Since

$$(-1)^k \varphi_n^{(k)}(x) \geqslant 0 \quad (k = 0, 1, 2, \ldots;\ x \in [0, b]),$$

there follows that $\varphi_n(x)$ is analytic for $|x - b| \leqslant b$.* Expanding the function $\varphi_n(x + h)$ with $x \in [0, b]$ and $|h| \leqslant x$ in a Taylor's series

* Such a function is called completely monotonic.

$$\varphi_n(x+h) = \sum_{k=0}^{\infty} \frac{\varphi_n^{(k)}(x)h^k}{k!}$$

and taking $h = -x$, we have by I

$$1 = \sum_{k=0}^{\infty} \frac{(-1)^k\varphi_n^{(k)}(x)x^k}{k!}. \tag{19}$$

We define for $n = 1,2,\ldots$ the operator $L_n$ by

$$L_n(f;x) = \sum_{k=0}^{\infty} \frac{(-1)^k\varphi_n^{(k)}(x)x^k}{k!} f\left(\frac{k}{\chi(n)}\right), \tag{20}$$

which is obviously linear. These operators have a meaning for each function $f(x)$ which is bounded for $x \geqslant 0$ * and they are positive on $[0,b]$ in view of II.

THEOREM 4

*If $f(x) \in H(x_0)$ and if the sequence $\{\varphi_n(x)\}$ satisfies the conditions I–III, then the sequence $\{L_n(f;x_0)\}$ $(n = 1,2,\ldots)$ defined in* (20), *converges to $f(x_0)$ when $n \to \infty$.*

*If, moreover, in $III_1$ $\alpha_{k,n}(x)$ converges also to zero uniformly in $x$ on $[0,b]$ and if the relations* (17) *and* (18) *hold uniformly in $x$ on $[0,b]$, then the sequence $\{L_n(f;x)\}$ $(n = 1,2,\ldots)$ converges uniformly on $[0,b]$ to $f(x)$ assuming $f(x) \in H(x_0)$ $(0 \leqslant x_0 \leqslant b)$.*

PROOF

We show that the sequence $\{L_n(f;x_0)\}$ $(n = 1,2,\ldots)$ converges to $f(x_0)$ when $n \to \infty$ in the three cases $f(t) \equiv 1$, $f(t) \equiv t$ and $f(t) \equiv t^2$.

The convergence of $\{L_n(f;x_0)\}$ to 1 in case $f(t) \equiv 1$ follows from (19). When $f(t) \equiv t$, we consider the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k\varphi_n^{(k)}(x_0){x_0}^k}{k!} \frac{k}{\chi(n)}.$$

Using III we have for $k > 0$

$$\frac{(-1)^k\varphi_n^{(k)}(x_0)x_0^k}{k!} \frac{k}{\chi(n)} = (-1)^{k-1} \frac{\psi(n,x_0)\varphi_{m_n}^{(k-1)}(x_0)x_0^k}{\chi(n)(k-1)!} (1+\alpha_{k,n}(x_0)) =$$

$$= x_0 \frac{(-1)^{k-1}\varphi_{m_n}^{(k-1)}(x_0)x_0^{k-1}}{(k-1)!} + x_0\left(\frac{\psi(n,x_0)}{\chi(n)} - 1\right) \frac{(-1)^{k-1}\varphi_{m_n}^{(k-1)}(x_0)x_0^{k-1}}{(k-1)!} +$$

$$+ x_0 \frac{\psi(n,x_0)}{\chi(n)} \frac{(-1)^{k-1}\varphi_{m_n}^{(k-1)}(x_0)x_0^{k-1}}{(k-1)!} \alpha_{k,n}(x_0). \tag{21}$$

* This condition is obviously sufficient. Perhaps it can be weakened, but that depends on the behaviour of the sequence $\{\varphi_n(x)\}$.

Summation over $k$ gives

$$L_n(t;\, x_0) - x_0 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\varphi_{m_n}^{(k-1)}(x_0)x_0^{k-1}}{(k-1)\,!} =$$

$$= x_0\left(\frac{\psi(n, x_0)}{\chi(n)} - 1\right)\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\varphi_{m_n}^{(k-1)}(x_0)x_0^{k-1}}{(k-1)\,!} +$$

$$+ x_0\,\frac{\psi(n, x_0)}{\chi(n)} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\varphi_{m_n}^{(k-1)}(x_0)x_0^{k-1}}{(k-1)\,!}\,\alpha_{k,n}(x_0).$$

Using II and (19) we thus have

$$\left| L_n(t;\, x_0) - x_0 \right| \leqslant x_0 \left| \frac{\psi(n,x_0)}{\chi(n)} - 1 \right| + x_0 \left| \frac{\psi(n, x_0)}{\chi(n)} \right| |\alpha_{k,n}(x_0)|.$$

On account of $III_1$ and $III_2$ there exists to each positive $\varepsilon$ a positive integer $N(\varepsilon, x_0)$ such that for each $n \geqslant N(\varepsilon, x_0)$

$$\left| \frac{\psi(n, x_0)}{\chi(n)} - 1 \right| < \frac{\varepsilon}{3b}$$

and

$$|\alpha_{k,n}(x_0)| < \frac{\varepsilon}{3b} \quad (k = 1, 2, \ldots).$$

We assume that $\varepsilon$ satisfies the inequality

$$\varepsilon < \min\,(3b,\, 1).$$

Then we have

$$|L_n(t;\, x_0) - x_0| \leqslant x_0\,\frac{\varepsilon}{3b} + x_0\left(1 + \frac{\varepsilon}{3b}\right)\frac{\varepsilon}{3b} \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\left(1 + \frac{\varepsilon}{3b}\right) < \varepsilon$$

when $n \geqslant N(\varepsilon, x_0)$ and hence

$$\lim_{n\to\infty} \quad L_n(t;\, x_0) = x_0.$$

When $f(t) \equiv t^2$ we arrive at the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k\varphi_n^{(k)}(x_0)x_0^k}{k\,!}\,\frac{k^2}{\chi^2(n)}.$$

Using III two times ($k \geqslant 2$), we can write the $k^{th}$-term of it in the form

$$\frac{(-1)^k\varphi_n^{(k)}(x_0)x_0^k}{k\,!}\,\frac{k^2}{\chi^2(n)} =$$

$$= {x_0}^2\,\frac{\psi(n, x_0)\psi(m_n, x_0)}{\chi^2(n)}\,\frac{(-1)^{k-2}\varphi_{m_{m_n}}^{(k-2)}(x_0)x_0^{k-2}}{(k-2)\,!}\,(1+\alpha_{k,n}(x_0))\,(1+\alpha_{k-1,m_n}(x_0)) +$$

$$+ x_0\,\frac{\psi(n, x_0)}{\chi^2(n)}\,\frac{(-1)^{k-1}\varphi_{m_n}^{(k-1)}(x_0)x_0^{k-1}}{(k-1)\,!}\,(1+\alpha_{k,n}(x_0)), \tag{22}$$

from which we easily deduce that

$$\lim_{n\to\infty} L_n(t^2; x_0) = x_0{}^2.$$

According to theorem 1 this establishes the first part of theorem 4. Under the conditions imposed on $\alpha_{k,n}(x)$, $\psi(n,x)$ and $\psi(m_n,x)$ in the second part of theorem 4, the proof of that part follows from the above proof.

## 3 An estimation of $|\mathbf{L_n(f; x_0) - f(x_0)}|$

### 3.1 Theorem 5

*If the conditions of the first part of theorem 4 are satisfied and if in property III $\alpha_{k,n}(x)$ is independent of k, then the following estimation holds in each fixed point $x_0$ of $[0, b]$*

$$|L_n(f; x_0) - f(x_0)| \leqslant \omega(\delta)\left\{1 + \frac{x_0\varepsilon_1(n,x_0) + x_0{}^2\varepsilon_2(n,x_0)}{\delta^2}\right\}. \tag{23}$$

*$\delta$ is an arbitrarily chosen positive number and $\varepsilon_1(n,x_0)$, $\varepsilon_2(n,x_0)$ are defined in (27) and (28).*

Proof

By II and (19) we have

$$|L_n(f; x_0) - f(x_0)| \leqslant \sum_{k=0}^{\infty} |f(k/\chi(n)) - f(x_0)| \frac{(-1)^k\varphi_n^{(k)}(x_0)x_0^k}{k!}.$$

The convergence of the series on the right follows from the proof of theorem 4. Let $\delta$ be an arbitrarily chosen positive number. Then we split up the sum $\sum\limits_{k=0}^{\infty}$ into two new sums

$$\sum_{k=0}^{\infty} = \sum_{|k/\chi(n)-x_0|\leqslant\delta} + \sum_{|k/\chi(n)-x_0|>\delta}.$$

Using property (1) of the modulus of continuity, we get

$$|L_n(f; x_0) - f(x_0)| \leqslant$$

$$\leqslant \omega(\delta) \sum_{|k/\chi(n)-x_0|\leqslant\delta} \frac{(-1)^k\varphi_n^{(k)}(x_0)x_0^k}{k!} + \omega(\delta) \sum_{|k/\chi(n)-x_0|>\delta} \left\{\frac{|k/\chi(n)-x_0|}{\delta} + 1\right\} \frac{(-1)^k\varphi_n^{(k)}(x_0)x_0^k}{k!} =$$

$$= \omega(\delta)\left[1 + \sum_{|k/\chi(n)-x_0|>\delta} \frac{|k/\chi(n)-x_0|}{\delta} \frac{(-1)^k\varphi_n^{(k)}(x_0)x_0^k}{k!}\right] \leqslant$$

$$\leqslant \omega(\delta)\left[1 + \sum_{|k/\chi(n)-x_0|>\delta} \frac{(k/\chi(n)-x_0)^2}{\delta^2} \frac{(-1)^k\varphi_n^{(k)}(x_0)x_0^k}{k!}\right] \leqslant$$

$$\leqslant \omega(\delta)\left[1 + \frac{1}{\delta^2}\sum_{k=0}^{\infty} (k/\chi(n)-x_0)^2 \frac{(-1)^k\varphi_n^{(k)}(x_0)x_0^k}{k!}\right]. \tag{24}$$

Since the $\alpha_{k,n}(x)$ do not depend on $k$, we will write $\alpha_n(x)$ instead of $\alpha_{k,n}(x)$ $(k = 1,2,\ldots)$ and we have then by (19), (21) and (22)

$$\sum_{k=0}^{\infty} (k/\chi(n)-x_0)^2 \frac{(-1)^k \varphi_n^{(k)}(x_0) x_0{}^k}{k!} = x_0 \frac{\psi(n,x_0)}{\chi^2(n)} (1+\alpha_n(x_0)) + \tag{25}$$
$$+ x_0{}^2 \left(1-2\frac{\psi(n,x_0)}{\chi(n)}(1+\alpha_n(x_0)) + \frac{\psi(n,x_0)\psi(m_n,x_0)}{\chi^2(n)}(1+\alpha_n(x_0))(1+\alpha_{m_n}(x_0))\right).$$

Putting

$$\frac{\psi(n,x_0)}{\chi(n)} = 1+\beta(n,x_0),\ \frac{\psi(m_n,x_0)}{\chi(n)} = 1+\gamma(n,x_0) \tag{26}$$

we have because of (17) and (18)

$$\lim_{n\to\infty} \beta(n,x_0) = \lim_{n\to\infty} \gamma(n,x_0) = 0.$$

Denoting by $\varepsilon_1(n,x_0)$ and $\varepsilon_2(n,x_0)$ the coefficients of $x_0$ and $x_0{}^2$ resp. in formula (25), we see that

$$\varepsilon_1(n,x_0) = \frac{(1+\beta(n,x_0))(1+\alpha_n(x_0))}{\chi(n)} \tag{27}$$

and

$$\begin{aligned}\varepsilon_2(n,x_0) = {} & -\alpha_n(x_0)-\beta(n,x_0)-\alpha_n(x_0)\beta(n,x_0)+\alpha_{m_n}(x_0)+\alpha_n(x_0)\alpha_{m_n}(x_0)+\\ & +\beta(n,x_0)\alpha_{m_n}(x_0)+\beta(n,x_0)\alpha_n(x_0)\alpha_{m_n}(x_0)+\gamma(n,x_0)+\\ & +\gamma(n,x_0)\alpha_n(x_0)+\gamma(n,x_0)\alpha_{m_n}(x_0)+\gamma(n,x_0)\alpha_n(x_0)\alpha_{m_n}(x_0)+\\ & +\beta(n,x_0)\gamma(n,x_0)+\beta(n,x_0)\gamma(n,x_0)\alpha_n(x_0)+\\ & +\beta(n,x_0)\gamma(n,x_0)\alpha_{m_n}(x_0)+\alpha_n(x_0)\alpha_{m_n}(x_0)\beta(n,x_0)\gamma(n,x_0).\end{aligned} \tag{28}$$

Obviously $\varepsilon_1(n,x_0)$ and $\varepsilon_2(n,x_0)$ converge to zero when $n \to \infty$. With these notations (25) can be written in the form (23).

Some applications of theorem 5 are given in section 4.2 of this Chapter.

### 3.2 Some remarks

1) It is always possible to choose $\delta = \delta(n, x_0)$ in such a way that

$$\lim_{n\to\infty} \delta(n,x_0) = 0$$

and, moreover, that

$$\lim_{n\to\infty} \frac{x_0\varepsilon_1(n,x_0)+x_0{}^2\varepsilon_2(n,x_0)}{\delta^2} = 0.$$

It is also possible to choose $\delta$ such that

$$\limsup_{n\to\infty} \left| \frac{x_0\varepsilon_1(n,x_0)+x_0{}^2\varepsilon_2(n,x_0)}{\delta^2} \right| = \infty.$$

These two choices are of little interest to us. In many cases, however, it is possible to choose $\delta = \delta(n, x_0)$ such that

$$a(x_0) = \lim_{n\to\infty} \frac{x_0\varepsilon_1(n,x_0)+x_0{}^2\varepsilon_2(n,x_0)}{\delta^2}$$

exists, where $a(x_0)$ is finite and not identical to zero. (See e.g. section 4.2 of this Chapter). Sometimes such a choice may be

$$\delta = \sqrt{\varepsilon^*(n,x_0)}$$

where

$$\varepsilon^*(n,x_0) = \max\big(|\varepsilon_1(n,x_0)|, |\varepsilon_2(n,x_0)|\big).$$

2) In some particular cases the estimation

$$|L_n(f;\, x_0)-f(x_0)| \leqslant \omega(\delta) \left\{1 + \frac{1}{\delta}\sqrt{x_0\varepsilon_1(n,x_0)+x_0{}^2\varepsilon_2(n,x_0)}\right\} \tag{29}$$

yields a better result than (23). It can be derived in a similar way as (23), except that in deducing (24) we now write

$$|L_n(f;\, x_0)-f(x_0)| \leqslant \omega(\delta) \left\{1 + \frac{1}{\delta} \sum_{|k/\chi(n)-x_0|>\delta} |k/\chi(n)-x_0| \, \frac{(-1)^k\varphi_n^{(k)}(x_0)x_0{}^k}{k!}\right\} \leqslant$$

$$\leqslant \omega(\delta) \left\{1 + \frac{1}{\delta} \sum_{k=0}^{\infty} |k/\chi(n)-x_0| \, \frac{(-1)^k\varphi_n^{(k)}(x_0)x_0{}^k}{k!}\right\}.$$

Application of Cauchy-Schwarz' inequality then yields the estimation

$$|L_n(f;\, x_0)-f(x_0)| \leqslant \omega(\delta) \left[1 + \frac{1}{\delta}\left\{\sum_{k=0}^{\infty} (k/\chi(n)-x_0)^2 \, \frac{(-1)^k\varphi_n^{(k)}(x_0)x_0{}^k}{k!}\right\}^{1/2}\right].$$

An example will be given in section 4.2 of this Chapter.

3) In some cases it may be useful to push one step beyond (24) in the following way.

$$|L_n(f;\, x_0)-f(x_0)| \leqslant \omega(\delta) \left\{1 + \frac{1}{\delta^2} \sum_{|k/\chi(n)-x_0|>\delta} (k/\chi(n)-x_0)^2 \, \frac{(-1)^k\varphi_n^{(k)}(x_0)x_0{}^k}{k!}\right\} \leqslant$$

$$\leqslant \omega(\delta) \left\{1 + \frac{1}{\delta^4} \sum_{|k/\chi(n)-x_0|>\delta} (k/\chi(n)-x_0)^4 \, \frac{(-1)^k\varphi_n^{(k)}(x_0)x_0{}^k}{k!}\right\} \leqslant$$

$$\leqslant \omega(\delta)\left\{1+\frac{1}{\delta^4}\sum_{k=0}^{\infty}(k/\chi(n)-x_0)^4\,\frac{(-1)^k\varphi_n^{(k)}(x_0)x_0{}^k}{k!}\right\}. \quad (30)$$

An example is given in section 4.2 of this Chapter.

## 4 **Applications**

4.1 If $\chi(n) = n$, the operators (20) have the form

$$L_n(f;x)=\sum_{k=0}^{\infty}\frac{(-1)^k\varphi_n^{(k)}(x)x^k}{k!}f\left(\frac{k}{n}\right)\quad (n=1,2,\ldots). \quad (31)$$

We also suppose that

$$\psi(n,x)=n,\ \psi(m_n,x)=m_n$$

where according to $\mathrm{III}_2$ we will have

$$\lim_{n\to\infty}\frac{m_n}{n}=1.$$

Assuming these conditions, we shall give three sequences of functions $\{\varphi_n(x)\}$ and the form of the corresponding operators.

a) $\varphi_n(x) = (1-x)^{n+p}$, where $x \in [0, 1]$ and $p$ is a positive integer or zero. It is easily verified that this sequence has the properties I–III. The operators generated by this sequence of functions have the form

$$L_n(f;x)=(1-x)^p\sum_{k=0}^{n+p}f\left(\frac{k}{n}\right)\binom{n+p}{k}x^k(1-x)^{n-k}\quad (x\in[0,1]). \quad (32)$$

For $p = 0$ we have the well-known Bernstein polynomials

$$B_n(f;x)=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}\quad (x\in[0,1].$$

b) $\varphi_n(x) = e^{-(n+p)x}$; $(x \in [0,b](b < \infty), p \geqslant 0)$.

This sequence has the properties I–III and the corresponding operators have the form

$$L_n(f;x)=e^{-(n+p)x}\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)\frac{(n+p)^kx^k}{k!}\quad (x\in[o,b]). \quad (33)$$

c) $\varphi_n(x) = (1+x)^{-(n+p)}$; $(x \in [0, b](b < \infty), p \geqslant 0)$.

The operators generated by this sequence of functions have the form

$$L_n(f;x)=$$
$$(1+x)^{-(n+p)}\sum_{k=0}^{\infty}\frac{(n+p)(n+p+1)\ldots(n+p+k-1)}{k!}\left(\frac{x}{1+x}\right)^kf\left(\frac{k}{n}\right)(x\in[0,b]). \quad (34)$$

4.2 In case of the operators $L_n$ of the form (32), we easily deduce that $m_n = n-1$ and $\alpha_{k,n}(x) = p/n$. Consequently $\beta(n,x) \equiv 0$ and $\gamma(n,x) \equiv -1/n$. Hence we get from (27) and (28) that

$$\varepsilon_1(n,x_0) = \frac{1}{n} + \frac{p}{n^2},$$

$$\varepsilon_2(n,x_0) = -\frac{1}{n} - \frac{p(1-p)}{n^2}.$$

According to remark 1 of section 3.1, a suitable choice for $\delta$ in formula (23) is here

$$\delta = \frac{1}{\sqrt{n}}.$$

We have then by theorem 5

$$|L_n(f;\, x_0) - f(x_0)| \leqslant \omega\left(\frac{1}{\sqrt{n}}\right)\left\{1 + x_0(1-x_0) + \frac{px_0}{n}(1 - x_0(1-p))\right\}.$$

For the Bernstein polynomials we get from this inequality

$$|B_n(f;\, x) - f(x)| \leqslant \frac{5}{4}\,\omega\left(\frac{1}{\sqrt{n}}\right) \qquad (n = 1, 2, \ldots) \tag{35}$$

because

$$\max_{0 \leqslant x \leqslant 1} x(1-x) = {}^1/_4. \tag{36}$$

This result is due to LORENTZ ([11], p. 20).

Using the inequality (30) in remark 3 of section 3.1, we arrive after some elementary calculations at

$$|B_n(f;\, x_0) - f(x_0)| \leqslant \omega(\delta)\left[1 + \frac{1}{n^4\delta^4}\{3n(n-2)x_0^2(1-x_0)^2 + nx_0(1-x_0)\}\right].$$

For $n \geqslant 2$ the right-hand member takes on its largest value for $x_0 = {}^1/_2$, yielding the inequality

$$|B_n(f;\, x) - f(x)| \leqslant \omega(\delta)\left\{1 + \frac{1}{n^2\delta^4}\left(\frac{3}{16} - \frac{1}{8n}\right)\right\},$$

which holds on [0, 1] and it is valid for $n = 1, 2, \ldots$.

Taking in particular $\delta = \dfrac{1}{\sqrt{n}}$, we get

$$|B_n(f;\, x) - f(x)| \leqslant \left(1\frac{3}{16} - \frac{1}{8n}\right)\omega\left(\frac{1}{\sqrt{n}}\right) < 1\frac{3}{16}\,\omega\left(\frac{1}{\sqrt{n}}\right),$$

which is sharper than Lorentz' result.

For the operators (23) we have that $m_n = n$, $\alpha_{k,n} \equiv p/n$, $\beta(n,x) \equiv 0$ and $\gamma(n,x) \equiv 0$. Consequently

$$\varepsilon_1(n,x_0) = \frac{1}{n} + \frac{p}{n^2},$$

$$\varepsilon_2(n,x_0) = \frac{p^2}{n^2}.$$

In case $b > 1$ and $x_0 \in [1, b]$, it is advantageous to use (29) instead of (23) in deriving an estimation for the difference

$$|L_n(f;\, x_0) - f(x_0)|.$$

In fact, with $\delta = 1/\sqrt{n}$ we then get

$$|L_n(f;\, x_0) - f(x_0)| \leqslant \omega\left(\frac{1}{\sqrt{n}}\right)\left\{1 + \sqrt{x_0 + \frac{x_0 p + x_0{}^2 p^2}{n}}\right\}.$$

Similar estimations can be given for the operators $L_n$ of the form (34).

4.3 We assume that $f(x) \in H^{(2)}(x_0)$, that the $\alpha_{k,n}(x)$ are independent of $k$ and that $m_n = n + c$ where $c$ is an integer independent of $n$. With (19), (21) and (22) there follows

$$\begin{cases} L_n(1;\, x_0) = 1, \\ L_n(t;\, x_0) = x_0(1+\alpha_n(x_0)), \\ L_n(t^2;\, x_0) = x_0{}^2\left(1 + \dfrac{c}{n}\right)(1+\alpha_n(x_0))(1+\alpha_{n+c}(x_0)) + \dfrac{x_0}{n}(1+\alpha_n(x_0)). \end{cases} \tag{37}$$

An asymptotic approximation formula for the operators of the form (31) is then given by

Theorem 6

*If the $\alpha_n(x)$ possess the property that in a fixed point $x_0 \in [0,b]$*

$$\alpha_n(x_0) = \frac{\varrho(x_0)}{n} + o\left(\frac{1}{n}\right), \tag{38}$$

*then we have*

$$L_n(f;\, x_0) - f(x_0) = \frac{2x_0\varrho(x_0)f'(x_0) + f''(x_0)(x_0 + cx_0{}^2)}{2n} + o\left(\frac{1}{n}\right). \tag{39}$$

Proof

In (11) we take $m = 1$. Then we have

$$L_n\{(t-x_0)^4; x_0\} =$$

$$= x_0^4 \frac{(n+c)(n+2c)(n+3c)}{n^3} (1+\alpha_n(x_0))(1+\alpha_{n+c}(x_0))(1+\alpha_{n+2c}(x_0))(1+\alpha_{n+3c}(x_0)) +$$

$$+ \left(\frac{6x_0^3}{n} - 4x_0^4\right) \frac{(n+c)(n+2c)}{n^2} (1+\alpha_n(x_0))(1+\alpha_{n+c}(x_0))(1+\alpha_{n+2c}(x_0)) +$$

$$- \left(\frac{12x_0^3}{n} - 6x_0^4\right) \frac{(n+c)}{n} (1+\alpha_n(x_0))(1+\alpha_{n+c}(x_0)) +$$

$$+ \left(\frac{6x_0^3}{n} - 4x_0^4\right) (1+\alpha_n(x_0)) + x_0^4 + o\left(\frac{1}{n}\right). \tag{40}$$

Hence

$$L_n\{(t-x_0)^4; x_0\} = x_0^4 \Big(-\alpha_n(x_0) + 3\alpha_{n+c}(x_0) - 3\alpha_{n+2c}(x_0) + \alpha_{n+3c}(x_0)\Big) + o\left(\frac{1}{n}\right).$$

From this and from (38) it follows that (11) is satisfied. Application of theorem 2 and (37) completes the proof of theorem 6.

In a similar way we can prove

THEOREM 7

*If the $\alpha_n(x)$ possess the property that in a fixed point $x_0 \in [0, b]$*

$$\alpha_n(x_0) = \frac{\varrho(x_0)}{\tau(n)} + o\left(\frac{1}{\tau(n)}\right),$$

*where $\tau(n)$ has the properties* 1) $\tau(n) \neq 0$; 2) $\lim\limits_{n\to\infty} \tau(n) = \infty$; 3) $\tau(n) = o(n)$, *then we have*

$$L_n(f; x_0) - f(x_0) = \frac{x_0 \varrho(x_0) f'(x_0)}{\tau(n)} + o\left(\frac{1}{\tau(n)}\right). \tag{41}$$

4.4 Next we consider the operators of DE LA VALLÉE-POUSSIN $V_n$, which are defined by *

$$V_n(f; x) = \frac{(2n)!!}{2\pi(2n-1)!!} \int_{-\pi}^{+\pi} f(x+t) \cos^{2n} t/2 \, dt \ (n = 1, 2, \ldots; f(x) \in C_{2\pi}).$$

They possess the property that for each bounded, $2\pi$-periodic function which is twice differentiable at the point $t = x_0$ the asymptotic relation

$$V_n(f; x_0) - f(x_0) = \frac{f''(x_0)}{n} + o\left(\frac{1}{n}\right) \tag{42}$$

holds.

* The expression $n!!$ denotes the product of all positive integers, which are not greater than $n$ and which are at the same time even or odd when $n$ is even or odd.

Proof

We will prove this relation by applying theorem 3.

Making use of the identity ([18], p. 197)

$$\cos^{2n} t/2 = \frac{(2n-1)!!}{(2n)!!}\left\{1+2\sum_{k=1}^{n}\frac{(n!)^2}{(n-k)!\,(n+k)!}\cos kt\right\},$$

we can write

$$V_n(f;x) = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x+t)\left\{\frac{1}{2}+\sum_{k=1}^{n}\varrho_k^{(n)}\cos kt\right\}dt$$

with

$$\varrho_k^{(n)} = \frac{(n!)^2}{(n-k)!\,(n+k)!} \quad (k=1,\dots,n). \tag{43}$$

It is readily verified that

$$\begin{aligned}
&V_n(1;x_0) = 1,\\
&V_n(\sin t;x_0) = \sin x_0 - \frac{\sin x_0}{n} + o\left(\frac{1}{n}\right),\\
&V_n(\sin 2t;x_0) = \sin 2x_0 - \frac{4\sin 2x_0}{n} + o\left(\frac{1}{n}\right),\\
&V_n(\cos t;x_0) = \cos x_0 - \frac{\cos x_0}{n} + o\left(\frac{1}{n}\right),\\
&V_n(\cos 2t;x_0) = \cos 2x_0 - \frac{4\cos 2x_0}{n} + o\left(\frac{1}{n}\right).
\end{aligned}$$

Further we have to establish the existence of a positive integer $m$ such that

$$V_n\{(t-x_0)^{2m+2};x_0\} = o\left(\frac{1}{n}\right) \quad (n\to\infty).$$

Taking $m=1$ we have

$$V_n\{(t-x_0)^4;x_0\} = \frac{(2n)!!}{2\pi(2n-1)!!}\int_{-\pi}^{+\pi} t^4\cos^{2n} t/2\,dt =$$

$$= \frac{(2n)!!}{2\pi(2n-1)!!}\left[\int_0^{\pi/2}\sin^4 t\cos^{2n} t\,dt + \int_0^{\pi/2}(t^4-\sin^4 t)\cos^{2n} t\,dt\right]. \tag{44}$$

Using the fact that ([18], p. 9)

$$\int_{-\pi}^{+\pi}\cos^{2n} t/2\,dt = \frac{(2n-1)!!}{(2n)!!}2\pi$$

and the identity

$$\sin^4 t = 1-2\cos^2 t + \cos^4 t,$$

we can write

$$\int_0^{\pi/2} \sin^4\cos^{2n}t\,dt = \frac{\pi}{2}\left\{\frac{(2n-1)!!}{(2n)!!} - 2\,\frac{(2n+1)!!}{(2n+2)!!} + \frac{(2n+3)!!}{(2n+4)!!}\right\} = \frac{3}{2}\pi\,\frac{(2n-1)!!}{(2n+4)!!}. \tag{45}$$

For the second integral in the right-hand side of (44) we have

$$\int_0^{\pi/2} (t^4-\sin^4 t)\cos^{2n} t\,dt = \int_0^{\pi/2} (t^2+\sin^2 t)(t+\sin t)(t-\sin t)\cos^{2n} t\,dt. \tag{46}$$

Now the expression $(t^2+\sin^2 t)(t+\sin t)$ is bounded on $[0, \pi/2]$ and in connection with (cf. [18], p. 187)

$$\int_0^{\pi/2} (t-\sin t)\cos^{2n} t\,dt = O(1/n^2),$$

$$\frac{(2n)!!}{(2n-1)!!} = O(n^{1/2}) \quad (n\to\infty),$$

we conclude from (44), (45) and (46) that

$$V_n\{(t-x_0)^4; x_0\} = o(1/n).$$

Hence we can apply theorem 3 and thus we get relation (42). This formula is due to NATANSON * ([18], p. 187).

4.5 Jackson's operators are defined by

$$L_{2n-2}(f;x) = \frac{3}{2\pi n(2n^2+1)}\int_{-\pi}^{+\pi} f(x+t)\left(\frac{\sin^1/_2 nt}{\sin^1/_2 t}\right)^4 dt \quad (n=1,2,\ldots; f(x)\in C_{2\pi}).$$

These operators possess the property that

$$L_{2n-2}(f;x_0) - f(x_0) = \frac{3f''(x_0)}{2n^2} + o\left(\frac{1}{n^2}\right), \tag{47}$$

where $f(x)$ is a bounded, $2\pi$-periodic function which is twice differentiable at the point $t = x_0 \in [-\pi, +\pi]$.

For the proof of (47) with the help of theorem 3 ,we need the following

---

* A somewhat easier way to prove relation (42) is to use theorem IV of KOROVKIN. This is allowed because by (43) we have

$$1-\varrho_1^{(n)} = \frac{1}{n+1} \quad\text{and}\quad 1-\varrho_2^{(n)} = \frac{4n+2}{(n+1)(n+2)}.$$

Hence condition (4) is satisfied.

LEMMA (SAFRONOVA [22])

*The coefficients* $\varrho_k^{(2n-2)}(k = 0, 1, \ldots, 2n-2)$ *in the expansion*

$$\left(\frac{\sin {}^1\!/_2 nt}{\sin {}^1\!/_2 t}\right)^4 = \frac{1}{6}\varrho_0^{(2n-2)} + \frac{1}{3}\sum_{k=1}^{2n-2} \varrho_k^{(2n-2)} \cos kt \tag{48}$$

*are given by the formulae*

$$\varrho_k^{(2n-2)} = \begin{cases} 3k^3 - 6nk^2 - 3k + 4n^3 + 2n & (0 \leqslant k \leqslant n+1) \quad (49) \\ -k^3 + 6nk^2 - (12n^2 - 1)k + 8n^3 - 2n & (n-1 \leqslant k \leqslant 2n). \quad (50) \end{cases}$$

PROOF OF (47)

Using Safronova's result and ([18], p. 75)

$$\int_{-\pi}^{+\pi} \left(\frac{\sin {}^1\!/_2 nt}{\sin {}^1\!/_2 t}\right)^4 dt = \frac{2\pi n(2n^2+1)}{3},$$

we establish

$$\begin{aligned} &L_{2n-2}(1; x_0) = 1, \\ &L_{2n-2}(\sin t; x_0) = \sin x_0 - \frac{3}{2n^2} \sin x_0 + o\left(\frac{1}{n^2}\right), \\ &L_{2n-2}(\sin 2t; x_0) = \sin 2x_0 - \frac{6}{n^2} \sin 2x_0 + o\left(\frac{1}{n^2}\right), \\ &L_{2n-2}(\cos t; x_0) = \cos x_0 - \frac{3}{2n^2} \cos x_0 + o\left(\frac{1}{n^2}\right), \\ &L_{2n-2}(\cos 2t; x_0) = \cos 2x_0 - \frac{6}{n^2} \cos 2x_0 + o\left(\frac{1}{n^2}\right). \end{aligned}$$

For the application of theorem 3, it is sufficient to prove that

$$L_{2n-2}\{(t-x_0)^4; x_0\} = o\left(\frac{1}{n^2}\right) \quad (n \to \infty).$$

In fact, we have

$$\begin{aligned} L_{2n-2}\{(t-x_0)^4; x_0\} &= \frac{3}{2\pi n(2n^2+1)} \int_{-\pi}^{+\pi} t^4 \left(\frac{\sin {}^1\!/_2 nt}{\sin {}^1\!/_2 t}\right)^4 dt = \\ &= \frac{96}{\pi n(2n^2+1)} \left[\int_0^{\pi/2n} t^4 \left(\frac{\sin nt}{\sin t}\right)^4 dt + \int_{\pi/2n}^{\pi/2} t^4 \left(\frac{\sin nt}{\sin t}\right)^4 dt\right]. \end{aligned}$$

Because

$$|\sin nt| < n|\sin t| \quad (-\infty < t < +\infty) \tag{51}$$

and

$$\sin t \geqslant \frac{2}{\pi} t \quad \left(0 \leqslant t \leqslant \frac{\pi}{2}\right), \tag{52}$$

the right-hand side of the above equality is less than

$$\frac{48}{\pi n^3}\left[n^4\int_0^{\pi/2n} t^4 dt + \frac{\pi^4}{16}\int_{\pi/2n}^{\pi/2} dt\right] = o\left(\frac{1}{n^2}\right).$$

Applying theorem 3 we get the asymptotic approximation formula (47).

Relation (47) is also due to NATANSON [19]. We remark that theorem IV can also be used in deriving (47).

4.6 The Fejér operators are defined by

$$L_{n-1}(f;x) = \frac{1}{2\pi n}\int_{-\pi}^{+\pi} f(x+t)\left(\frac{\sin {}^1\!/_2 nt}{\sin {}^1\!/_2 t}\right)^2 dt \quad (n = 1,2,\ldots; f(x) \in C_{2\pi}).$$

We establish that in this case theorems 2, 3 and IV cannot be applied.

PROOF

Using the identity ([18], p. 73)

$$\left(\frac{\sin {}^1\!/_2 nt}{\sin {}^1\!/_2 t}\right)^2 = n + 2\sum_{k=1}^{n-1}(n-k)\cos kt, \tag{53}$$

we see that in theorems 2 and 3 we have $\varphi(n) = n$.

We now prove that there does not exist a positive integer $m$ with the property

$$L_{n-1}\{(t-x_0)^{2m+2}; x_0\} = o\left(\frac{1}{n}\right) \quad (n \to \infty).$$

For we have

$$\frac{1}{2\pi n}\int_{-\pi}^{+\pi} t^{2m+2}\left(\frac{\sin {}^1\!/_2 nt}{\sin {}^1\!/_2 t}\right)^2 dt = \frac{2^{2m+3}}{\pi n}\int_0^{\pi/2} u^{2m+2}\left(\frac{\sin nu}{\sin u}\right)^2 du.$$

But

$$\frac{2^{2m+3}}{\pi n}\int_0^{\pi/2} u^{2m+2}\left(\frac{\sin nu}{\sin u}\right)^2 du > \frac{2^{2m+3}}{\pi n}\int_{\pi/4}^{\pi/2} u^{2m+2}\left(\frac{\sin nu}{\sin u}\right)^2 du >$$

$$> \frac{2^{2m+3}}{\pi n}\left(\frac{\pi}{4}\right)^{2m+2}\int_{\pi/4}^{\pi/2}\sin^2 nu\, du = \frac{1}{n}\left(\frac{\pi}{2}\right)^{2m+1}\left(\frac{\pi}{8} + \frac{1}{4n}\sin\frac{n\pi}{2}\right).$$

The expression on the right side is of the order $O(1/n)$ no matter what positive integer $m$ we choose. Also theorem IV is not applicable because (4) does not hold, which follows from (53).

4.7 In the theory of Bernstein polynomials we encounter the sum ([18], p. 6)

$$\sum_{|k/n-x_0|\geqslant\delta}\binom{n}{k} x_0^k(1-x_0)^{n-k} \quad (x_0 \in [0,1]), \tag{54}$$

of which it is proved that it is $o(1/n)$.

Here we consider a sum of a more general type of which (54) is a particular case. In fact we prove that

$$\sum_{|k/n-x_0|\geqslant\delta} \frac{(-1)^k\varphi_n^{(k)}(x_0)x_0{}^k}{k!} = o\left(\frac{1}{n}\right) \quad (x_0 \in [0,b]), \tag{55}$$

where $\delta$ is an arbitrary, but fixed, positive number. Moreover we assume that $m_n = n+c$, $c$ being an integer independent of $n$, and moreover, that

$$\alpha_{k,n}(x) \equiv 0 \quad (k,n = 1,2,\ldots).$$

Proof

When $f(x) \in H^{(2)}(x_0)$ we have

$$f(t) = f(x_0)+f'(x_0)(t-x_0)+\left\{\frac{f''(x_0)}{2}+{}^1/_2\beta(t-x_0)\right\}(t-x_0)^2, \tag{56}$$

where $\lim\limits_{t\to x_0} \beta(t-x_0) = 0$ and $\beta(t-x_0)$ is bounded on the real $t$-axis. Consequently by using (37)

$$L_n(t;x_0)-f(x_0) = f''(x_0)\,\frac{x_0+cx_0{}^2}{2n}+{}^1/_2L_n\{(t-x_0)^2\beta(t-x_0);\ x_0\},$$

where $L_n$ stands for the operator defined by (31). From this and (39) follows

$$L_n\{(t-x_0)^2\beta(t-x_0);\ x_0\} = o\left(\frac{1}{n}\right) \quad (n\to\infty)$$

and this is equal to

$$\sum_{k=0}^{\infty} \beta\left(\frac{k}{n}-x_0\right)\left(\frac{k}{n}-x_0\right)^2 \frac{(-1)^k\varphi_n^{(k)}(x_0)x_0{}^k}{k!} = o\left(\frac{1}{n}\right) \quad (n\to\infty). \tag{57}$$

Now we choose a function $f(x)$ such that $\beta(t-x_0) > 0$ when $t \neq x_0$ and $\beta(t-x_0) > {}^1/_2$ when $|t-x_0| \geqslant \delta$. If we have moreover

$$f''(x_0) = -1,\ f'(x_0) = 0$$

we get from (56)

$$f(t) = f(x_0)+{}^1/_2(t-x_0)^2(-1+\beta(t-x_0)). \tag{58}$$

Taking in particular

$$\left.\begin{aligned} \beta(t-x_0) &= \frac{1}{2\delta^2}(t-x_0)^2 && \text{whenever} \quad |t-x_0| \leqslant \delta \\ \beta(t-x_0) &= 1-{}^1/_2e^{-(t-x_0)^2+\delta^2} && \text{whenever} \quad |t-x_0| > \delta, \end{aligned}\right\} \tag{59}$$

then we see that $\beta(t-x_0)$ is bounded and continuous on the whole real $t$-axis and it satisfies the prescribed conditions. Because of (59) and II each partial sum of (57) is also of the order $o(1/n)$ and therefore

$$\sum_{|k/n-x_0|\geqslant\delta} \beta\left(\frac{k}{n}-x_0\right)\left(\frac{k}{n}-x_0\right)^2 \frac{(-1)^k \varphi_n^{(k)}(x_0) x_0{}^k}{k!} = o\left(\frac{1}{n}\right). \tag{60}$$

Using

$$\sum_{|k/n-x_0|\geqslant\delta} \beta\left(\frac{k}{n}-x_0\right)\left(\frac{k}{n}-x_0\right)^2 \frac{(-1)^k \varphi_n^{(k)}(x_0) x_0{}^k}{k!} \geqslant {}^1/_2\delta^2 \sum_{|k/n-x_0|\geqslant\delta} \frac{(-1)^k \varphi_n^{(k)}(x_0) x_0{}^k}{k!},$$

the result (55) follows from (60).

CHAPTER 2

# APPROXIMATION OF FUNCTIONS OF MANY VARIABLES

Some of the results of Chapter 1 are generalized to functions of more than one variable. Further an inequality due to STANCU [29] concerning generalized Bernstein polynomials is investigated, the constant occurring in it is improved. Some applications are given.

## 1 Formulation of two basic theorems

1.1 We will denote with $X = X(x_1,\ldots,x_m)$ a point of the $m$-dimensional space $R_m$. For a function $f(x_1,\ldots,x_m)$ defined on a set in $R_m$ we will write $f(X)$.

DEFINITION

*By $H(X_0)$ we denote the class of all real functions $f(X)$ which are defined in $R_m$ and which have the properties*

1) *$f(X)$ is continuous for $X = X_0$;*

2) $f(X) = O\left(\sum_{j=1}^{m} x_j^2\right)$ *when* $|x_j| \to \infty$ $(j = 1,\ldots,m)$.

First a generalization of a theorem of VOLKOV [30] is given.

THEOREM 8

*Let $f(X) \in H(X_0)$ and let $\{L_n\}$ $(n = 1,2,\ldots)$ be a sequence of operators defined on $H(X_0)$. If we write*

$$L_n(1;\, X) = 1+\alpha_n(X)$$

$$L_n(t_j;\, X) = x_j+\beta_{n,j}(X) \qquad (j = 1,\ldots,m;\ n = 1,2,\ldots)$$

$$L_n\Big(\sum_{j=1}^{m} t_j^2;\, X\Big) = \sum_{j=1}^{m} x_j^2+\gamma_n(X)$$

*and if $\alpha_n(X)$, $\beta_{n,j}(X)$ and $\gamma_n(X)$ have the property that*

$$\lim_{n\to\infty} \alpha_n(X_0) = \lim_{n\to\infty} \beta_{n,j}(X_0) = \lim_{n\to\infty} \gamma_n(X_0) = 0 \quad (j = 1,\ldots,m),$$

*then we have*

$$\lim_{n\to\infty} L_n(f;\, X_0) = f(X_0).$$

## 1.2 Definitions

a) *The sequence* $(n_1, \ldots, n_m)$ *of positive integers* $n_1, \ldots, n_m$ *will be denoted by* $N$;

b) *By* $H^{(2)}(X_0)$ *we denote the class of all real functions* $f(X) \in H(X_0)$ *of which all the second derivatives exist at the point* $X_0$.

Theorem 9

*Let* $f(X) \in H^{(2)}(X_0)$ *and let* $\{L_N\}$ *be a sequence of operators defined on* $H(X_0)$. *If the operators* $L_N$ *have the property that in a fixed point* $X_0$

$$\begin{aligned} L_N(1; X_0) &= 1 + o\left(\frac{1}{\varphi(n_j)}\right) \\ L_N(t_j; X_0) &= x_{0_j} + \frac{\psi_{1;j}(X_0)}{\varphi(n_j)} + o\left(\frac{1}{\varphi(n_j)}\right) \qquad (k,j = 1, \ldots, m;\ k \neq j) \\ L_N(t_j^2; X_0) &= x_{0_j}^2 + \frac{\psi_{2;j,j}(X_0)}{\varphi(n_j)} + o\left(\frac{1}{\varphi(n_j)}\right) \\ L_N(t_k t_j; X_0) &= x_{0_k} x_{0_j} + \frac{\psi_{2;k,j}(X_0)}{\varphi(n_k)} + \frac{\psi_{2;j,k}(X_0)}{\varphi(n_j)} + o\left(\frac{1}{\varphi(n_j)}\right) + o\left(\frac{1}{\varphi(n_k)}\right), \end{aligned}$$

*where* $\varphi(n_j) \neq 0$ *and* $\varphi(n_j) \to \infty$ *when* $n_j \to \infty$ $(j = 1, \ldots, m)$. *If there exist positive integers* $p_j (j = 1, \ldots, m)$ *such that*

$$L_N\{(t_j - x_{0_j})^{2p_j+2}; X_0\} = o\left(\frac{1}{\varphi(n_j)}\right) \quad (n_j \to \infty;\ j = 1, \ldots, m),$$

*hen we have*

$$L_N(f; X_0) - f(X_0) = \sum_{j=1}^{m} \frac{[\psi_{1;j} f'_{x_j} + {}^1/_2(\psi_{2;j,j} - 2x_{0_j}\psi_{1;j}) f''_{x_j x_j}]}{\varphi(n_j)} +$$

$$+ {}^1/_2 \sum_{\substack{j,k=1 \\ j \neq k}}^{m} \left\{ \frac{\psi_{2;k,j} - x_{0_j}\psi_{1;k}}{\varphi(n_k)} + \frac{\psi_{2;j,k} - x_{0_k}\psi_{1;j}}{\varphi(n_j)} \right\} f''_{x_k x_j} + \sum_{j=1}^{m} o\left(\frac{1}{\varphi(n_j)}\right),$$

*where the values of all functions* $\psi, f'$ *and* $f''$ *are taken in* $X_0$.

## 2 **A generalization of sequence (20)**

2.1 For the sake of shortness we will abbreviate

$$\begin{aligned} &\frac{\partial^{i_1 + \ldots + i_m}}{\partial x^{i_1} \ldots \partial x_m^{i_m}} \varphi_{n_1, \ldots, n_m}(x_1, \ldots, x_m) && \text{to } \varphi_N^I(X), \\ &n_1 - 1, n_2, \ldots, n_m && \text{to } N - E_1, \\ &i_1, i_2 - 1, i_3, \ldots, i_m && \text{to } I - E_2 \quad \text{and so on.} \end{aligned}$$

Let the operators be defined in a domain $K$ of the first hyperquadrant of $R_m$.

We consider (cf. Chapter 1, § 2) a sequence of real functions

$$\{\varphi_N(X)\} \quad (n_j = 1,2,\ldots;\ j = 1,\ldots,m),$$

where each function has the following properties in the domain $K$.

I : $\varphi_N(X)$ can be expanded in a Taylor's series in a closed region $K_1$. $K_1$ is the sum (when $X$ runs through $K$) of the closed spheres with center in $X$ and radius $|X|$;

II : $\varphi_N(0,\ldots,0) = 1$;

III : $(-1)^{i_1+\ldots+i_m}\varphi_N^I(X) \geqslant 0 \quad (i_1,\ldots,i_m = 0,1,2,\ldots)$;

IV : there exist positive integers $\bar{n}_j (j = 1,\ldots,m)$ not depending on $i_1,\ldots,i_m$ such that

$$-\varphi_N^I(X) = \psi_j(n_j,X)\varphi_{N-(n_j-\bar{n}_j)E_j}^{I-E_j}(X)\{1+\alpha_{i_j,n_j}(X)\} \quad (j = 1,\ldots,m)$$

where

IV₁: $\alpha_{i_j,n_j}(X)$ $(j = 1,\ldots,m)$ converges to zero uniformly in $i_j$ when $n_j \to \infty$;

IV₂: there exist positive functions $\chi_j(n_j) (j = 1,\ldots,m)$, monotonically increasing to infinity when $n_j \to \infty$, with the property

$$\lim_{n_j\to\infty} \frac{\psi_j(n_j, X)}{\chi_j(n_j)} = 1 \tag{61}$$

$$(j = 1,\ldots,m)$$

$$\lim_{n_j\to\infty} \frac{\psi_j(\bar{n}_j, X)}{\chi_j(n_j)} = 1. \tag{62}$$

Proceeding in a similar way as in Chapter 1, § 2, the sequence $\{\varphi_N(X)\}$ generates a sequence of operators $\{L_N\}$ of the form *

$$L_N(f;X) = \sum_{i_1=0}^{\infty}\cdots\sum_{i_m=0}^{\infty} \frac{(-1)^{i_1+\ldots+i_m}\varphi_N^I(X)x_1^{i_1}\ldots x_m^{i_m}}{i_1!\ldots i_m!} f\left(\frac{i_1}{\chi_1(n_1)},\ldots,\frac{i_m}{\chi_m(n_m)}\right). \tag{63}$$

As an analogue to theorem 4 we have

Theorem 10

*If $f(X) \in H(X_0)$ and if the sequence $\{\varphi_N(X)\}$ satisfies the above conditions I–IV, then the sequence $\{L_N(f;X_0)\}$ $(n_j = 1,2,\ldots; j = 1,\ldots,m)$ defined in* (63) *converges to $f(X_0)$ when $n_j \to \infty$.*

*If, moreover, in $IV_1$ $\alpha_{i_j,n_j}(X)$ converges also to zero uniformly in $X$ in $K$ and if the*

* These operators are defined by means of a sequence of functions depending on $m$ parameters $n_1, \ldots, n_m$. It is possible to construct a sequence of operators, suitable for the approximation of functions of many variables, which is generated by a sequence of functions depending on only one parameter $n$. In section 3 we will give an example to this case.

*relations* (61) *and* (62) *hold uniformly in* $X$ *in* $K$, *then the sequence* (63) *converges uniformly in* $K$ *to* $f(X)$ *assuming* $f(X) \in H(X_0)$ ($X_0$ *runs through* $K$).

Concerning an estimation of the difference $|L_N(f;\, X_0)-f(X_0)|$ we have as a generalization of theorem 5

THEOREM 11

*If the conditions of the first part of theorem 10 are satisfied and if in property IV* $\alpha_{i_j,n_j}(X)$ *is independent of* $i_j$ (*we write* $\alpha_{i_j,n_j}(X) = \alpha_{n_j}(X)$), *then the following estimations hold in each fixed point* $X_0$ *of* $K$.

$$|L_N(f;\, X_0)-f(X_0)| \leqslant \omega(\delta_1,\ldots,\delta_m)\left\{1+\sum_{j=1}^{m}\frac{1}{\delta_j{}^2}\{x_{0j}\varepsilon_{j,1}(n_j,X_0)+x_{0j}^2\varepsilon_{j,2}(n_j,X_0)\}\right\} \tag{64}$$

$$|L_N(f;\, X_0)-f(X_0)| \leqslant \omega(\delta_1,\ldots,\delta_m)\left\{1+\sum_{j=1}^{m}\frac{1}{\delta_j}\{x_{0j}\varepsilon_{j,1}(n_j,X_0)+x_{0j}^2\varepsilon_{j,2}(n_j,X_0)\}^{1/2}\right\}. \tag{65}$$

$\delta_j(j = 1,\ldots,m)$ *are arbitrarily chosen positive numbers; the quantities* $\varepsilon_{j,1}(n_j,X_0)$ *and* $\varepsilon_{j,2}(n_j,X_0)$ *are respectively defined as* (*cf. formulae* (27) *and* (28))

$$\varepsilon_{j,1}(n_j,\, X_0) = \frac{(1+\beta_j(n_j,X_0))\,(1+\alpha_{n_j}(X_0))}{\chi_j(n_j)}, \tag{66}$$

$$\begin{aligned}\varepsilon_{j,2}(n_j,X_0) = {} & -\alpha_{n_j}(X_0)-\beta_j(n_j,X_0)\ldots\\ & \ldots+\alpha_{n_j}(X_0)\alpha_{\bar{n}_j}(X_0)\beta_j(n_j,X_0)\gamma_j(n_j,X_0).\end{aligned} \tag{67}$$

## 3 Examples of sequences of functions $\varphi$ and some applications

3.1 We restrict ourselves to the particular case in which for $N(n_1,\ldots,n_m)$ where $n_j = 1,2,\ldots\,(j = 1,\ldots,m)$, we have

$$\bar{n}_j = n_j+c_j \qquad (c_j \text{ is an integer}),$$

$$\alpha_{i_j,n_j}(X) = \alpha_{n_j}(X),\ \psi_j(n_j,\, X) = n_j,\ \chi_j(n_j) = n_j.$$

Under these conditions the operators (63) have the form

$$L_N(f;\, X) = \sum_{i_1=0}^{\infty}\cdots\sum_{i_m=0}^{\infty}\frac{(-1)^{i_1+\cdots+i_m}\varphi_N^I(X)x_1^{i_1}\ldots x_m^{i_m}}{i_1!\ldots i_m!}f\left(\frac{i_1}{n_1},\ldots,\frac{i_m}{n_m}\right). \tag{68}$$

The following examples 1⁰)–4⁰) of sequences of functions satisfy the conditions I–IV of section 2 of this Chapter and they have their origin in Chapter 1. We give the form of the corresponding operators.

1⁰) $\varphi_N(X) = \prod_{j=1}^{m}(1-x_j)^{n_j+a_j}\left(a_j(j = 1,\ldots,m)\text{ are positive integers or zero}\right).$

They give rise to

$$L_N(f;X) = \sum_{i_1=0}^{n_1+a_1} \cdots \sum_{i_m=0}^{n_m+a_m} \prod_{j=1}^{m} \binom{n_j+a_j}{i_j} x_j^{i_j}(1-x_j)^{n_j+a_j-i_j} f\left(\frac{i_1}{n_1},\ldots,\frac{i_m}{n_m}\right).$$

These operators $L_N$ are defined on the cube $0 \leqslant x_j \leqslant 1$ $(j = 1,\ldots,m)$. In the particular case $a_j \equiv 0$ $(j = 1,\ldots,m)$, we get the generalized Bernstein polynomials

$$B_N(f;X) = \sum_{i_1=0}^{n_1} \cdots \sum_{i_m=0}^{n_m} \prod_{j=1}^{m} \binom{n_j}{i_j} x_j^{i_j}(1-x_j)^{n_j-i_j} f\left(\frac{i_1}{n_1},\ldots,\frac{i_m}{n_m}\right). \tag{69}$$

(cf. LORENTZ [11], p. 51).

2⁰) $\varphi_n(X) = (1-x_1\ldots-x_m)^{n+a}$ ($a$ is a positive integer or zero).
This is a sequence of functions depending on one parameter $n$. The corresponding operators $L_n$ have the form

$$L_n(f;X) =$$
$$= \sum_{i_1=0}^{n+a} \sum_{i_2=0}^{n+a-i_1} \cdots\cdots \sum_{i_m=0}^{n+a-i_1\ldots-i_{m-1}} \frac{(n+a)!\, x_1^{i_1}\ldots x_m^{i_m}(1-x_1-\ldots-x_m)^{n+a-i_1-\ldots-i_m}}{i_1!\ldots i_m!\,(n+a-i_1-\ldots-i_m)!} f\left(\frac{i_1}{n},\ldots,\frac{i_m}{n}\right).$$

The region $K$ is defined by the simplex

$$\begin{cases} \sum_{j=1}^{m} x_j \leqslant 1 \\ x_j \geqslant 0 \quad (j = 1,\ldots,m). \end{cases} \tag{70}$$

Taking $a \equiv 0$, we get another kind of generalized Bernstein polynomials, defined on the simplex (70), viz.

$$B_n(f;X) = \sum_{i_1=0}^{n} \cdots\cdots \sum_{i_m=0}^{n-i_1-\ldots-i_{m-1}} \frac{n!\, x_1^{i_1}\ldots x_m^{i_m}(1-x_1\ldots-x_m)^{n-i_1-\ldots-i_m}}{i_1!\ldots i_m!\,(n-i_1\ldots-i_m)!} f\left(\frac{i_1}{n},\ldots,\frac{i_m}{n}\right). \tag{71}$$

(cf. LORENTZ [11], p. 51 and STANCU [29]).

3⁰) $\varphi_N(X) = e^{-\sum_{j=1}^{m} n_j x_j}$.
Then we have

$$L_N(f;X) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \prod_{j=1}^{m} \frac{n_j^{i_j} e^{-n_j x_j} x_j^{i_j}}{i_j!} f\left(\frac{i_1}{n_1},\ldots,\frac{i_m}{n_m}\right). \tag{72}$$

These operators are defined in the region $x_j \geqslant 0$ $(j = 1,\ldots,m)$ and they are a generalization of the Mirakjan operator (cf. [31]).

4⁰) $\varphi_N(X) = \prod_{j=1}^{m} (1+x_j)^{-n_j}$,

$$L_N(f;X) = \sum_{i_1=0}^{\infty}\cdots\sum_{i_m=0}^{\infty}\prod_{j=1}^{m}\frac{n_j(n_j+1)\ldots(n_j+i_j-1)}{i_j!}(1+x_j)^{-(n_j+i_j)}x_j^{i_j}f\left(\frac{i_1}{n_1},\ldots,\frac{i_m}{n_m}\right). \tag{73}$$

$K$ is defined by $x_j \geqslant 0$ $(j = 1,\ldots,m)$.

Other sequences of functions that can be used for the construction of sequences of operators are

$$\varphi_N(X) = e^{-(n+\gamma n)x_1}\prod_{j=2}^{m}(1-x_j)^n, \tag{74}$$

$$\varphi_N(X) = e^{-n_1x_1}(1+x_2)^{-n_2}\prod_{j=3}^{m}(1-x_j)^{n_j},$$

$$\varphi_N(X) = \prod_{j=1}^{k}(1-x_j)^{n_j}\prod_{j=k+1}^{m}(1+x_j)^{-n_j} \quad \text{when } k < m,$$

$$\varphi_N(X) = (1+x_1+\ldots+x_m)^{-(n+a)} \quad (a \text{ is a positive integer or zero}). \tag{75}$$

For an application of theorem 11 we return to the generalized Bernstein polynomials (69). It is easily deduced that in this case we have $\bar{n}_j = n_j - 1$ and $\alpha_{n_j}(X_0) \equiv 0$. Consequently $\beta_j(n_j, X_0) \equiv 0$ and $\gamma_j(n_j, X_0) = -1/n_j$ $(j = 1,\ldots,m)$. Hence we get from (66) and (67)

$$\varepsilon_{j,1}(n_j, X_0) = \frac{1}{n_j},$$

$$\varepsilon_{j,2}(n_j, X_0) = -\frac{1}{n_j},$$

and according to remark 1 of section 3, Chapter 1, a suitable choice for $\delta_j$ in theorem 11 is here

$$\delta_j = \frac{1}{\sqrt{n_j}} \quad (j = 1,\ldots,m).$$

Using these results and (36), the second estimation (65) of theorem 11 gives

$$|B_N(f;X) - f(X)| \leqslant \left(1+\frac{m}{2}\right)\omega\left(\frac{1}{\sqrt{n_1}},\ldots,\frac{1}{\sqrt{n_m}}\right) \quad (n_j = 1,2,\ldots).$$

For $m = 2$ this result was derived by IPATOV (cf. [31]). The first inequality of theorem 11 has the advantage that it gives in this case a better estimation. From (64) we get

$$|B_N(f;X)-f(X)| \leqslant \left(1+\frac{m}{4}\right)\omega\left(\frac{1}{\sqrt{n_1}},\ldots,\frac{1}{\sqrt{n_m}}\right) \quad (n_j = 1,2,\ldots).$$

When $m = 1$, this reduces to (35).
In the same way we derive for the Bernstein polynomials (71), defined on the simplex (70), the inequalities

$$|B_n(f;X)-f(X)| \leqslant \left(1+\frac{m}{2}\right)\omega\left(\frac{1}{\sqrt{n}},\ldots,\frac{1}{\sqrt{n}}\right), \tag{76}$$

$$(n = 1,2,\ldots)$$

$$|B_n(f;X)-f(X)| \leqslant \left(1+\frac{m}{4}\right)\omega\left(\frac{1}{\sqrt{n}},\ldots,\frac{1}{\sqrt{n}}\right). \tag{77}$$

Inequality (76) is due to STANCU [29].

## 4 Strengthening of estimation (77)

4.1 Now we will prove that inequality (77) can be sharpened further [24]. To that purpose we first derive a property of the generalized modulus of continuity. We have

THEOREM 12

*Defining for a function $f(X)$, continuous in a convex domain of $R_m$, the generalized modulus of continuity $\omega$ as in (5) we have*

$$\omega(\lambda_1\delta_1,\ldots,\lambda_m\delta_m) \leqslant (1+\max_{(k)}]\lambda_k[)\,\omega(\delta_1,\ldots,\delta_m), \quad (\lambda_k > 0).$$

*Here $]\lambda[$ stands for the largest integer being smaller than $\lambda$.*

PROOF

Let $X'(x_1',\ldots,x_m')$ and $X''(x_1'',\ldots,x_m'')$ be two points of the region in which $f(X)$ is defined with the property that

$$|x_k'-x_k''| \leqslant \lambda_k\delta_k \quad (k = 1,\ldots,m).$$

Then we have

$$|x_k'-x_k''| \leqslant \lambda_\sigma\delta_k$$

if

$$\lambda_\sigma = \max_{(k)} \lambda_k.$$

This means that the linesegment $X'X''$ can be covered by at most $]\lambda_\sigma[+1$ congruent closed $m$-dimensional parallelepipeds with sides parallel to the $x_1,\ldots,x_m$ axis and with length $\delta_1,\ldots,\delta_m$ respectively. In each of these parallelepipeds the difference between two values of $f(X)$ does not exceed $\omega(\delta_1,\ldots,\delta_m)$. Hence we have

$$|f(X')-f(X'')| \leqslant (1+]\lambda_\sigma[)\omega(\delta_1,\ldots,\delta_m),$$

and as this holds for each couple of points $X'$ and $X''$ with the above property, theorem 12 is established.

Theorem 13

*$f(X)$ is a real continuous function defined on the simplex* (70). *We associate with it the generalized Bernstein polynomials* (71). *Then the following inequality is true.*

$$|B_n(f;X)-f(X)| \leqslant \{1+(\varkappa-1)m\}\,\omega\left(\frac{1}{\sqrt{n}},\ldots,\frac{1}{\sqrt{n}}\right) \quad (n = 1,2,\ldots),$$

*where*

$$\varkappa = \frac{4306+837\sqrt{6}}{5832} = 1{,}0898873\ldots \tag{78}$$

Proof

For the sake of brevity we write

$$P = \frac{n!\,x_1^{i_1}\ldots x_m^{i_m}}{i_1!\ldots i_m!\,(n-i_1-\ldots-i_m)!}\,(1-x_1-\ldots-x_m)^{n-i_1-\ldots-i_m}.$$

Because of properties II and III of this Chapter we have

$$|f(X)-B_n(f;X)| \leqslant \sum_{i_1=0}^{n}\cdots\sum_{i_m=0}^{n-i_1-\ldots-i_{m-1}}\left|f(X)-f\left(\frac{i_1}{n},\ldots,\frac{i_m}{n}\right)\right| P.$$

Choosing arbitrary, fixed, positive numbers $\delta_j$ $(j = 1,\ldots,m)$ and a fixed point $X$ of the simplex (70), we split the sum $\sum_{i_1=0}^{n}\ldots\sum_{i_m=0}^{n-i_1\ldots-i_{m-1}}$ into $2^m$ parts in the following way.

$$\sum_{i_1=0}^{n}\cdots\sum_{i_m=0}^{n-i_1-\ldots-i_{m-1}} = \sum_{\substack{|x_k-i_k/n|\leqslant\delta_k\\(k=1,\ldots,m)}}\!\!\cdots\sum + \sum_{\substack{|x_1-i_1/n|>\delta_1\\|x_k-i_k/n|\leqslant\delta_k\\(k=2,\ldots,m)}}\!\!\cdots\sum + \ldots\ldots \sum_{\substack{|x_m-i_m/n|>\delta_m\\|x_k-i_k/n|\leqslant\delta_k\\(k=1,\ldots,m-1)}}\!\!\cdots\sum +$$

$$+ \sum_{\substack{|x_j-i_j/n|>\delta_j\\(j=1,2)\\|x_k-i_k/n|\leqslant\delta_k\\(k=3,\ldots,m)}}\!\!\cdots\sum + \sum_{\substack{|x_j-i_j/n|>\delta_j\\(j=1,3)\\|x_k-i_k/n|\leqslant\delta_k\\(k=2,4,\ldots,m)}}\!\!\cdots\sum + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \sum_{\substack{|x_j-i_j/n|>\delta_j\\(j=1,\ldots,m)}}\!\!\cdots\sum.$$

Using theorem 12 in a somewhat modified form*, we get

$$|f(X)-B_n(f;X)| \leqslant \omega(\delta_1,\ldots,\delta_m)\Bigg[\sum_{\substack{|x_k-i_k/n|\leqslant\delta_k\\(k=1,\ldots,m)}}\!\!\cdots\sum P+\sum_{\substack{|x_1-i_1/n|>\delta_1\\|x_k-i_k/n|\leqslant\delta_k\\(k=2,\ldots,m)}}\!\!\cdots\sum\left\{\left]\frac{x_1-i_1/n}{\delta_1}\right[+1\right\}P+$$

$$\cdots\sum_{\substack{|x_m-i_m/n|>\delta_m\\|x_k-i_k/n|\leqslant\delta_k\\(k=1,\ldots,m-1)}}\!\!\cdots\sum\left\{\left]\frac{x_m-i_m/n}{\delta_m}\right[+1\right\}P+\sum_{\substack{|x_j-i_j/n|>\delta_j\\(j=1,2)\\|x_k-i_k/n|\leqslant\delta_k\\(k=3,\ldots,m)}}\!\!\cdots\sum\left\{\left]\frac{x_1-i_1/n}{\delta_1}\right[+\left]\frac{x_2-i_2/n}{\delta_2}\right[+1\right\}P+\ldots$$

$$\ldots\ldots\ldots+\sum_{\substack{|x_j-i_j/n|>\delta_j\\(j=1,\ldots,m)}}\!\!\cdots\sum\left\{\left]\frac{x_1-i_1/n}{\delta_1}\right[+\ldots\ldots\ldots\left]\frac{x_m-i_m/n}{\delta_m}\right[+1\right\}P\Bigg]\leqslant$$

$$\leqslant \omega(\delta_1,\ldots,\delta_m)\left\{1+\frac{1}{\delta_1}\sum_{\substack{i_2,\ldots,i_m\\|x_1-i_1/n|>\delta_1}}\!\!\cdots\sum\left|x_1-\frac{i_1}{n}\right|P+\ldots\frac{1}{\delta_m}\sum_{\substack{i_1,\ldots,i_{m-1}\\|x_m-i_m/n|>\delta_m}}\!\!\cdots\sum\left|x_m-\frac{i_m}{n}\right|P\right\}\leqslant$$

$$\leqslant \omega(\delta_1,\ldots,\delta_m)\left\{1+\frac{1}{\delta_1}\sum_{|x_1-i_1/n|>\delta_1}\left|x_1-\frac{i_1}{n}\right|\binom{n}{i_1}x_1^{i_1}(1-x_1)^{n-i_1}+\ldots\right.$$

$$\left.\ldots+\frac{1}{\delta_m}\sum_{|x_m-i_m/n|>\delta_m}\left|x_m-\frac{i_m}{n}\right|\binom{n}{i_m}x_m^{i_m}(1-x_m)^{n-i_m}\right\}.$$

Choosing $\delta_1=\delta_2=\ldots\delta_m=\dfrac{1}{\sqrt{n}}$, we get

$$|f(X)-B_n(f;X)|\leqslant$$

$$\leqslant \omega\left(\frac{1}{\sqrt{n}},\ldots,\frac{1}{\sqrt{n}}\right)\left\{1+\sum_{k=1}^{m}\frac{1}{\sqrt{n}}\sum_{|x_k-i_k/n|>1/\sqrt{n}}\left|x_k-\frac{i_k}{n}\right|\binom{n}{i_k}x_k^{i_k}(1-x_k)^{n-i_k}\right\}.$$

Sikkema [28] proved that the inequality

$$\frac{1}{\sqrt{n}}\sum_{|x-i/n|>1/\sqrt{n}}\left|x-\frac{i}{n}\right|\binom{n}{i}x^i(1-x)^{n-i}\leqslant \varkappa-1$$

holds for $n=1,2,\ldots$ and $x\in[0,1]$, where $\varkappa$ is given by (78). This proves theorem 13.

---

* We actually use

$$\omega(\lambda_1\delta_1,\ldots,\lambda_m\delta_m)\leqslant(1+\sum_i\,]\lambda_i[\,)\,\omega(\delta_1,\ldots,\delta_m),$$

where the sum is taken over those $\lambda$'s for which $\lambda_i>1$.

## 5 Asymptotic approximation

5.1 In order to derive asymptotic approximation formulae for the operators (63) we use theorem 9. First we need the following results.

$$
\begin{aligned}
L_N(1; X_0) &= 1,\\
L_N(t_j; X_0) &= x_{0_j}(1+\alpha_{n_j}(X_0)),\\
L_N(t_j^2; X_0) &= x_{0_j}^2\left(1+\frac{c_j}{n_j}\right)(1+\alpha_{n_j}(X_0))(1+\alpha_{n_j+c_j}(X_0)) + \frac{x_{0_j}}{n_j}(1+\alpha_{n_j}(X_0)),\\
L_N(t_k t_j; X_0) &= x_{0_k}x_{0_j}(1+\alpha_{n_k}(X_0))(1+\alpha_{n_j}(X_0)),\ (j,k=1,\ldots,m;\, j\neq k),
\end{aligned}
$$

which can be derived in a similar way as formulae (21), (22).

As a generalization of theorems 6 and 7 we have

THEOREM 14

a) *Let $f(X)\in H^{(2)}(X_0)$ and let the $\alpha_{n_j}(X)$ possess the property that in a fixed point $X_0\in K$*

$$\alpha_{n_j}(X_0) = \frac{\varrho(X_0)}{n_j} + o\left(\frac{1}{n_j}\right),$$

*then we have*

$$L_N(f; X_0) - f(X_0) = \sum_{j=1}^{m} \frac{x_{0_j}\varrho(X_0)f'_{x_j}(X_0) + {}^1\!/_2(x_{0_j}+c_j x_{0_j}^2)f''_{x_jx_j}(X_0)}{n_j} + \sum_{j=1}^{m} o\left(\frac{1}{n_j}\right).$$

b) *Let $f(X)\in H^{(2)}(X_0)$. If*

$$\alpha_{n_j}(X_0) = \frac{\varrho(X_0)}{\tau(n_j)} + o\left(\frac{1}{\tau(n_j)}\right),$$

*where $\tau(n_j)$ $(j=1,\ldots,m)$ has the properties* 1) $\tau(n_j)\neq 0$; 2) $\lim\limits_{n_j\to\infty}\tau(n_j)=\infty$; 3) $\tau(n_j)=o(n_j)$, $n_j\to\infty$, *then*

$$L_N(f; X_0) - f(X_0) = \sum_{j=1}^{m} \frac{x_{0_j}\varrho(X_0)f'_{x_j}(X_0)}{\tau(n_j)} + \sum_{j=1}^{m} o\left(\frac{1}{\tau(n_j)}\right).$$

5.2 Finally we give some applications of this theorem. For the Bernstein polynomials (69) we have $c_j=-1$ $(j=1,\ldots,m)$ and thus

$$B_N(f; X_0) - f(X_0) = \sum_{j=1}^{m} \frac{x_{0_j}(1-x_{0_j})}{2n_j} f''_{x_jx_j}(X_0) + \sum_{j=1}^{m} o\left(\frac{1}{n_j}\right).$$

This is a generalization of the well-known formula of VORONOVSKAJA (cf. [31]).

In the same way we derive that the asymptotic approximation formula for the operators (72) ($c_j\equiv 0$, $j=1,\ldots,m$) has the form

$$L_N(f;X_0)-f(X_0)=\sum_{j=1}^{m}\frac{x_{0j}f''_{x_jx_j}(X_0)}{2n_j}+\sum_{j=1}^{m}o\left(\frac{1}{n_j}\right),$$

whereas for the operators (73) ($c_j=1, j=1,\ldots,m$) we have

$$L_N(f;X_0)-f(X_0)=\sum_{j=1}^{m}\frac{x_{0j}(1+x_{0j})}{2n_j}f''_{x_jx_j}(X_0)+\sum_{j=1}^{m}o\left(\frac{1}{n_j}\right).$$

Now we consider the sequence of operators corresponding to the sequence of functions (74). Here we have an application of the second part of theorem 14 and the asymptotic approximation formula for these operators has the simple form

$$L_N(f;X_0)-f(X_0)=\frac{f'_{x_1}(X_0)}{n^{1/2}}+o\left(\frac{1}{n^{1/2}}\right).$$

5.3 With theorem 9, used in a slightly modified form*, we can derive asymptotic approximation formulae for the Bernstein polynomials (71) and for the operators $L_n$ corresponding to the sequence (75).

We have

$$B_n(f;X_0)-f(X_0)=\frac{\sum\limits_{j=1}^{m}x_{0j}f''_{x_jx_j}(X_0)-\sum\limits_{j,k=1}^{m}x_{0j}x_{0k}f''_{x_jx_k}(X_0)}{2n}+o\left(\frac{1}{n}\right)\qquad(79)$$

and

$$L_n(f;X_0)-f(X_0)=\frac{\sum\limits_{j=1}^{m}x_{0j}\{2af'_{x_j}(X_0)+f''_{x_jx_j}(X_0)\}+\sum\limits_{j,k=1}^{m}x_{0j}x_{0k}f''_{x_jx_k}(X_0)}{2n}+o\left(\frac{1}{n}\right).$$

Formula (79) is due to Stancu [29].

* The Bernstein polynomials (70) and the operators belonging to the sequence of functions (75) depend on only one parameter $n$. Hence the sequence $N=(n_1,\ldots,n_m)$ in theorem 9 is replaced by $n$.

CHAPTER 3

# ON A PARTICULAR SEQUENCE OF OPERATORS

In this Chapter we will prove a theorem on the approximation of functions belonging to $C_{2\pi}$ by means of the operators $L_{np-p}$ $(n = 1,2,\ldots)$ defined by

$$L_{np-p}(f;x) = \frac{1}{A_{np-p}} \int_{-\pi}^{+\pi} f(x+t) \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt \tag{80}$$

with

$$A_{np-p} = \int_{-\pi}^{+\pi} \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt, \tag{81}$$

and where $p$ is an arbitrary, but fixed, positive integer.

Moreover, we consider a number of particular examples of sequence (80) of which we will give some numerical results. For this purpose we use theorems IV and V of the Introduction.

## 1 A general theorem

### 1.1 Theorem 15

*The sequence* $\{L_{np-p}(f;x)\}(n = 1,2,\ldots)$ *defined by* (80) *converges uniformly on* $[-\pi, +\pi]$ *to the function* $f(x) \in C_{2\pi}$.

Proof

We may assume that $p$ is a fixed positive integer $\geqslant 3$ (the cases $p = 1$ and $p = 2$ have already been treated in sections 4.5 and 4.6 of Chapter 1) and we show that the sequence $\{L_{np-p}(f;x)\}$ converges on the interval $[-\pi, +\pi]$ uniformly to $f(x)$ if $n \to \infty$ in the three cases $f(t) \equiv 1, f(t) \equiv t$ and $f(t) \equiv t^2$.

The uniform convergence of $L_{np-p}(f;x)$ to 1 if $f(t) \equiv 1$ follows from (81). Further we have

$$L_{np-p}(t;x) = x \quad (n = 1,2,\ldots)$$

because

$$\int_{-\pi}^{+\pi} t \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt = 0,$$

and

$$L_{np-p}(t^2;x) = x^2 + \frac{1}{A_{np-p}} \int_{-\pi}^{+\pi} t^2 \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt.$$

There remains to prove that the expression $\dfrac{1}{A_{np-p}}\displaystyle\int_{-\pi}^{+\pi} t^2\left(\frac{\sin {}^1/_2nt}{\sin {}^1/_2t}\right)^{2p} dt$ converges to zero when $n \to \infty$.

We have

$$\int_{-\pi}^{+\pi} t^2\left(\frac{\sin {}^1/_2nt}{\sin {}^1/_2t}\right)^{2p} dt = 16\left[\int_0^{\pi/2n} t^2\left(\frac{\sin nt}{\sin t}\right)^{2p} dt + \int_{\pi/2n}^{\pi/2} t^2\left(\frac{\sin nt}{\sin t}\right)^{2p} dt\right].$$

Using (51), (52) there follows

$$\int_{-\pi}^{+\pi} t^2\left(\frac{\sin {}^1/_2nt}{\sin {}^1/_2t}\right)^{2p} dt \leqslant 2\pi^3\left(\frac{1}{3}+\frac{1}{2p-3}\right) n^{2p-3}. \tag{82}$$

The integral $A_{np-p}$ is estimated in the following way.

$$\int_{-\pi}^{+\pi} \left(\frac{\sin {}^1/_2nt}{\sin {}^1/_2t}\right)^{2p} dt > 4\int_0^{\pi/2n} \left(\frac{\sin nt}{\sin t}\right)^{2p} dt.$$

In view of the inequalities

$$\left.\begin{aligned} &\sin nt \geqslant nt - \frac{(nt)^3}{3!} \\ &\sin\ t \leqslant t, \end{aligned}\right\}\ t \in [0, \pi/2n]$$

we have

$$\int_0^{\pi/2n} \left(\frac{\sin nt}{\sin t}\right)^{2p} dt > \int_0^{\pi/2n} n^{2p}\left(1-\frac{n^2t^2}{6}\right)^{2p} dt = n^{2p}\int_0^{\pi/2n} \sum_{k=0}^{2p} (-1)^k\binom{2p}{k}\left(\frac{n^2t^2}{6}\right)^k dt.$$

Hence

$$\int_{-\pi}^{+\pi} \left(\frac{\sin {}^1/_2nt}{\sin {}^1/_2t}\right)^{2p} dt > 2\pi\, n^{2p-1}\sum_{k=0}^{2p} \frac{(-1)^k}{2k+1}\binom{2p}{k}\left(\frac{\pi^2}{24}\right)^k = C_p n^{2p-1}. \tag{83}$$

The constant $C_p$ is positive because $\displaystyle\int_0^{\pi/2n}\left(1-\frac{n^2t^2}{6}\right)^{2p} dt > 0$. Combination of (82) and (83) gives

$$0 < \frac{\displaystyle\int_{-\pi}^{+\pi} t^2\left(\frac{\sin {}^1/_2nt}{\sin {}^1/_2t}\right)^{2p} dt}{A_{np-p}} < \frac{2\pi^3\left(\dfrac{1}{3}+\dfrac{1}{2p-3}\right)}{C_p n^2}.$$

Hence for $p = 3, 4, \ldots$ we have

$$\lim_{n\to\infty} \frac{\int\limits_{-\pi}^{+\pi} t^2 \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt}{A_{np-p}} = 0,$$

and theorem I of the Introduction proves theorem 15.*

## 2 Some properties of the operators $\mathbf{L_{3n-3}}$, $\mathbf{L_{4n-4}}$, $\mathbf{L_{5n-5}}$ and $\mathbf{L_{6n-6}}$

2.1 The relation of Safronova can be used to calculate the form of the operators $L_{3n-3}$ and $L_{4n-4}$. In order to do the same for the operators $L_{5n-5}$ and $L_{6n-6}$, we will push one step further and we will derive formulae for the coefficients $\varrho_k^{(3n-3)}$ $(k = 0,\ldots,3n-3)$ in the expansion (7) for $p = 3$.

Theorem 16

*The coefficients $\varrho_k^{(3n-3)}$ in the expansion*

$$\left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^6 = {}^1/_6 \varrho_0^{(3n-3)} + {}^1/_3 \sum_{k=1}^{3n-3} \varrho_k^{(3n-3)} \cos kt \tag{84}$$

*are given by the formulae*

$$\varrho_k^{(3n-3)} = \begin{cases} 3^3/_{10} n^5 + 1^1/_2 n^3 + 1^1/_5 n - 2k + k^2(-3n^3 - 4^1/_2 n) + 2^1/_2 k^3 + 1^1/_2 nk^4 - {}^1/_2 k^5 \\ \qquad\qquad (0 \leqslant k \leqslant n+1) \qquad (85) \\ 2^{11}/_{20} n^5 + 5^1/_4 n^3 - 1^4/_5 n + k(3^3/_4 n^4 - 11^1/_4 n^2 + 1) + k^2(-10^1/_2 n^3 + 6^3/_4 n) + \\ + k^3(7^1/_2 n^2 - 1^1/_4) - 2^1/_4 nk^4 + {}^1/_4 k^5 \quad (n-1 \leqslant k \leqslant 2n+1) \qquad (86) \\ 12^3/_{20} n^5 - 6^3/_4 n^3 + {}^3/_5 n + k(-20^1/_4 n^4 + 6^3/_4 n^2 - {}^1/_5) + k^2(13^1/_2 n^3 - 2^1/_4 n) + \\ + k^3(-4^1/_2 n^2 + {}^1/_4) + {}^3/_4 nk^4 - {}^1/_{20} k^5 \quad (2n-2 \leqslant k \leqslant 3n). \qquad (87) \end{cases}$$

Proof

We have

$$\left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^6 = \left\{n + 2\sum_{k=1}^{n-1} (n-k) \cos kt\right\}\left\{{}^1/_6 \varrho_0^{(2n-2)} + {}^1/_3 \sum_{k=1}^{2n-2} \varrho_k^{(2n-2)} \cos kt\right\},$$

where the coefficients $\varrho_k^{(2n-2)} (k = 0,\ldots,2n-2)$ are given by (49) and (50). It turns out that the calculation of the coefficients $\varrho_k^{(3n-3)} (k = 0,\ldots,3n-3)$ breaks

* Considering theorem II of Korovkin, a consequence of the above result is that the expressions

$$\frac{\varrho_k^{(np-p)}}{\varrho_0^{(np-p)}} \quad (k = 1, \ldots, np-p),$$

derived from expansion (7), tend to 1 when $n \to \infty$.

up into three parts. By performing the above multiplication, we get the following expressions for $\varrho_k^{(3n-3)}$.

$$\begin{aligned}\varrho_k^{(3n-3)} &= n\varrho_k^{(2n-2)}+(n-1)\varrho_{k-1}^{(2n-2)}+\;\ldots..\;+(n-k)\varrho_0^{(2n-2)}\;+\\ &+\;(n-1)\varrho_{k+1}^{(2n-2)}+(n-2)\varrho_{k+2}^{(2n-2)}+\;\ldots..\;+\;1\cdot\varrho_{n+k-1}^{(2n-2)}\;+\\ &+\;(n-k-1)\varrho_1^{(2n-2)}+(n-k-2)\varrho_2^{(2n-2)}+\;\ldots..\;+\;1\cdot\varrho_{n-k-1}^{(2n-2)},\end{aligned}$$

where $0 \leqslant k \leqslant n-1$;

$$\begin{aligned}\varrho_k^{(3n-3)} &= n\varrho_k^{(2n-2)}+(n-1)\varrho_{k-1}^{(2n-2)}+\;\ldots..\;+\;1\cdot\varrho_{k-n+1}^{(2n-2)}\;+\\ &+\;(n-1)\varrho_{k+1}^{(2n-2)}+(n-2)\varrho_{k+2}^{(2n-2)}+\;\ldots..\;+\;(k-n+2)\varrho_{2n-2}^{(2n-2)},\end{aligned}$$

where $n-1 \leqslant k \leqslant 2n-1$;

$$\varrho_k^{(3n-3)} = (3n-k-2)\varrho_{2n-2}^{(2n-2)}+(3n-k-3)\varrho_{2n-3}^{(2n-2)}+\;\ldots..\;+\;1\cdot\varrho_{k-n+1}^{(2n-2)},$$

where $2n-1 \leqslant k \leqslant 3n-3$.

These sums have to be calculated where, depending on the value of $k$, the corresponding formula for the $\varrho_k^{(2n-2)}$ of the lemma of SAFRONOVA has to be taken. According to this, the calculation of the sum

$$(n-1)\varrho_{k+1}^{(2n-2)}+(n-2)\varrho_{k+2}^{(2n-2)}+\;\ldots..\;+\;1\cdot\varrho_{n+k-1}^{(2n-2)} \tag{88}$$

is split up into two parts.

For the first $n-k-1$ terms we have

$$\varrho_k^{(2n-2)} = 3k^3-6nk^2-3k+4n^3+2n,$$

whereas for the remaining $k$ terms in (88) formula (50) can be applied.

We will only calculate the coefficients $\varrho_k^{(3n-3)}$ $(0 \leqslant k \leqslant n-1)$, noticing that the cases with $n-1 \leqslant k \leqslant 2n-1$ and $2n-1 \leqslant k \leqslant 3n-3$ can be treated in a similar way.

In case $0 \leqslant k \leqslant n-1$ we have

$$\begin{aligned}\varrho_k^{(3n-3)} &= (n-k)(4n^3+2n)+(n-k+1)(4n^3-6n\cdot 1^2+3\cdot 1^3+2n-3\cdot 1)+\ldots..\\ &+(n-k+k)(4n^3-6nk^2+3k^3+2n-3k)+[n+k-(k+1)]\{4n^3-6n(k+1)^2+\\ &+3(k+1)^3+2n-3(k+1)\}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\ldots..+[n+k-(n-1)]\{4n^3-6n(n-1)^2+3(n-1)^3+2n-3(n-1)\}+\\ &+(n+k-n)(8n^3-12n^2\cdot n+6n\cdot n^2-n^3-2n+n)+\ldots\ldots\ldots\ldots\\ &+[n+k-(n+k-1)]\{8n^3-12n^2(n+k-1)+\ldots+(n+k-1)\}+\\ &+(n-k-1)(4n^3-6n\cdot 1^2+3\cdot 1^3+2n-3\cdot 1)+\ldots\ldots\ldots\ldots\ldots\\ &+[n-k-(n-k-1)]\{4n^3-6n(n-k-1)^2+\ldots-3(n-k-1)\} =\\ &= (n-k)\left[(4n^3+2n)(k+1)-3\sum_{i=1}^{k}i-6n\sum_{i=1}^{k}i^2+3\sum_{i=1}^{k}i^3\right]\\ &\qquad+(4n^3+2n)\sum_{i=1}^{k}i-3\sum_{i=1}^{k}i^2-6n\sum_{i=1}^{k}i^3+3\sum_{i=1}^{k}i^4\end{aligned}$$

$$+ (n+k)\left[(4n^3+2n)(n-k-1)-3\sum_{i=k+1}^{n-1} i-6n\sum_{i=k+1}^{n-1} i^2+3\sum_{i=k+1}^{n-1} i^3\right]+$$

$$-(4n^3+2n)\sum_{i=k+1}^{n-1} i+3\sum_{i=k+1}^{n-1} i^2+6n\sum_{i=k+1}^{n-1} i^3-3\sum_{i=k+1}^{n-1} i^4+$$

$$+ (n+k)\left[(8n^3-2n)k-(12n^2-1)\sum_{i=n}^{n+k-1} i+6n\sum_{i=n}^{n+k-1} i^2-\sum_{i=n}^{n+k-1} i^3\right]+$$

$$-(8n^3-2n)\sum_{i=n}^{n+k-1} i+(12n^2-1)\sum_{i=n}^{n+k-1} i^2-6n\sum_{i=n}^{n+k-1} i^3+\sum_{i=n}^{n+k-1} i^4+$$

$$+ (n-k)\left[(4n^3+2n)(n-k-1)-3\sum_{i=1}^{n-k-1} i-6n\sum_{i=1}^{n-k-1} i^2+3\sum_{i=1}^{n-k-1} i^3\right]+$$

$$-(4n^3+2n)\sum_{i=1}^{n-k-1} i+3\sum_{i=1}^{n-k-1} i^2+6n\sum_{i=1}^{n-k-1} i^3-3\sum_{i=1}^{n-k-1} i^4.$$

By using summation formulae for sums of the type $\sum_{i=1}^{n} i^r$ $(r = 1,2,3,4)$, we arrive at (85).*

2.2 Next we will derive the form of the operators $L_{5n-5}$ and $L_{6n-6}$. Using (48) and (84) we deduce that

$$A_{5n-5} = \frac{\pi}{9}\left(1/2\varrho_0^{(2n-2)}\cdot\varrho_0^{(3n-3)} + \sum_{k=1}^{2n-2}\varrho_k^{(2n-2)}\cdot\varrho_k^{(3n-3)}\right),$$

$$\varrho_1^{(5n-5)} = \frac{\pi}{18A_{5n-5}}\sum_{k=0}^{2n-2}\left(\varrho_k^{(2n-2)}\cdot\varrho_{k+1}^{(3n-3)}+\varrho_{k+1}^{(2n-2)}\varrho_k^{(3n-3)}\right),$$

$$\varrho_2^{(5n-5)} = \frac{\pi}{18A_{5n-5}}\left\{\varrho_1^{(2n-2)}\cdot\varrho_1^{(3n-3)} + \sum_{k=0}^{2n-2}\{\varrho_k^{(2n-2)}\cdot\varrho_{k+2}^{(3n-3)}+\varrho_{k+2}^{(2n-2)}\cdot\varrho_k^{(3n-3)}\}\right\},$$

$$A_{6n-6} = \frac{\pi}{9}\left(1/2\varrho_0^{2(3n-3)} + \sum_{k=1}^{3n-3}\varrho_k^{2(3n-3)}\right),$$

$$\varrho_1^{(6n-6)} = \frac{\pi}{9A_{6n-6}}\sum_{k=0}^{3n-4}\varrho_k^{(3n-3)}\cdot\varrho_{k+1}^{(3n-3)},$$

* We have a check on the results because formulae (85), (86) and (87) of theorem 16 must yield zero if $n = 1$ and $k = 1$.

$$\varrho_2^{(6n-6)} = \frac{\pi}{9A_{6n-6}}\left(1/2\varrho_1^{2(3n-3)} + \sum_{k=0}^{3n-5} \varrho_k^{(3n-3)} \cdot \varrho_{k+2}^{(3n-3)}\right),$$

where the $\varrho_k^{(2n-2)}$ and $\varrho_k^{(3n-3)}$ are given by (49), (50), (85), (86) and (87). Using summation formulae for the sums $\sum_{i=1}^{n} i^r$ $(r = 1, \ldots, 10)$, we obtain after some elementary calculations the following results.

$$A_{5n-5} = \frac{\pi}{18144} \sum_{i=1}^{5} c_{2i-1} n^{2i-1} \tag{89}$$

with

$$\begin{aligned} c_1 &= 4032 \\ c_3 &= 4100 \\ c_5 &= 5187 \\ c_7 &= 7350 \\ c_9 &= 15619, \end{aligned}$$

$$\varrho_1^{(5n-5)} = \frac{\pi(15619n^9 - 1470n^7 + o(n^7))}{18144A_{5n-5}}, \tag{90}$$

$$\varrho_2^{(5n-5)} = \frac{\pi(15619n^9 - 27930n^7 + o(n^7))}{18144A_{5n-5}}; \tag{91}$$

$$A_{6n-6} = \frac{\pi}{831600} \sum_{i=1}^{6} c_{2i-1} n^{2i-1} \tag{92}$$

with

$$\begin{aligned} c_1 &= 151200 \\ c_3 &= 147532 \\ c_5 &= 175835 \\ c_7 &= 221991 \\ c_9 &= 311465 \\ c_{11} &= 655177, \end{aligned}$$

$$\varrho_1^{(6n-6)} = \frac{\pi(655177n^{11} + o(n^9))}{831600A_{6n-6}}, \tag{93}$$

$$\varrho_2^{(6n-6)} = \frac{\pi(655177n^{11} - 934395n^9 + o(n^9))}{831600A_{6n-6}}. \tag{94}$$

We notice that the value of the integrals $A_{5n-5}$ and $A_{6n-6}$ is equal to $2\pi$ when $n = 1$; the coefficients $\varrho_1^{(5n-5)}$, $\varrho_2^{(5n-5)}$, $\varrho_1^{(6n-6)}$ and $\varrho_2^{(6n-6)}$ satisfy the necessary condition that they converge to 1 when $n \to \infty$.

In view of (89) and (92) we have the following theorem.

THEOREM 17

$$L_{5n-5}(f;x) =$$

$$= \frac{18144}{\pi(15619n^9+7350n^7+5187n^5+4100n^3+4032n)} \int_{-\pi}^{+\pi} f(x+t) \left(\frac{\sin {}^1/_2nt}{\sin {}^1/_2t}\right)^{10} dt,$$

$$L_{6n-6}(f;x) =$$

$$= \frac{831600}{\pi(655177n^{11}+311465n^9+221991n^7+175835n^5+147532n^3+151200n)} \times$$

$$\times \int_{-\pi}^{+\pi} f(x+t) \left(\frac{\sin {}^1/_2nt}{\sin {}^1/_2t}\right)^{12} dt.$$

With the aid of a theorem of KOROVKIN * ([7], p. 72), we can easily derive an estimation for the difference $|L_{np-p}(f;x)-f(x)|$ in cases $p = 3, 4, 5, 6$. In fact, we have

THEOREM 18

*If* $f(x) \in C_{2\pi}$, *then*

$$|L_{np-p}(f;x)-f(x)| \leqslant \left(1+\frac{\pi}{\sqrt{2}}\right)\omega\left(\frac{1}{n}\right) \quad (n = 1,2,\ldots; p = 3,4,5,6).$$

PROOF

Using (85), (90), (93) and a result of MATSUOKA,** the proof of theorem 18 is immediate.

2.3 In the Introduction we noticed that PETROV [21] established a theorem on the degree of approximation for functions belonging to the class $Z$ by the operator $L_{2n-2}$ (Jackson's operator). As an application of theorem 16 we will give here an analogous result for the operator $L_{3n-3}$.

---

* It reads:
If $f(x) \in C_{2\pi}$ and $L_n$ is an operator of type (3), then

$$|L_n(f;x)-f(x)| \leqslant \omega\left(\frac{1}{m}\right)\left(1+\frac{m\pi}{\sqrt{2}}\sqrt{1-\varrho_1^{(n)}}\right) \quad (m > 0).$$

(cf. [16], p. 50).

** MATSUOKA proved [14] that

$$\varrho_1^{(4n-4)} = \frac{151n^6-35n^4-56n^2-60}{151n^6+70n^4+49n^2+45} \quad (n = 1, 2, \ldots).$$

THEOREM 19

*If we put*

$$b_n = 3n \sup_{(f)} \{\|L_{3n-3}(f)-f\|; f(x) \in Z\},$$

*where* $\|g\|$ *denotes the Chebyshev norm of* $g$, *then*

$$\lim_{n\to\infty} b_n = \frac{40}{11\pi}\,(10^1/_8 \ln 3 - 12 \ln 2).$$

PROOF

Because

$$L_{3n-3}(f; x) - f(x) = \frac{10}{\pi(11n^5+5n^3+4n)} \int_{-\pi}^{+\pi} \frac{f(x+t)-2f(x)+f(x-t)}{2} \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^6 dt \tag{95}$$

and $f(x) \in Z$, we have

$$\|L_{3n-3}(f)-f\| \leqslant \frac{10}{\pi(11n^5+5n^3+4n)} \int_{-\pi}^{+\pi} |t| \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^6 dt. \tag{96}$$

If in (95) we choose $f(t) = |t|$, then the sign of equality appears in formula (96). Consequently

$$b_n = \frac{30}{\pi(11n^4+5n^2+4)} \int_{-\pi}^{+\pi} |t| \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^6 dt,$$

and this expression can be written as

$$b_n = \frac{60}{\pi(11n^4+5n^2+4)} \int_0^{\pi} t \left\{{}^1/_6 \varrho_0^{(3n-3)} + {}^1/_3 \sum_{k=1}^{3n-3} \varrho_k^{(3n-3)} \cos kt\right\} dt,$$

where the $\varrho_k^{(3n-3)}$ are given by theorem 16. Performing the integration, we get

$$b_n = {}^3/_2 n\pi + \frac{20}{\pi(11n^4+5n^2+4)} \sum_{k=1}^{3n-3} \varrho_k^{(3n-3)} \frac{(-1)^k - 1}{k^2}.$$

This sum has to be split up into three parts. For the performance of the calculation we introduce the function ([20], p. 100–103)

$$g(x) = 2 \sum_{s=0}^{\infty} \frac{(-1)^s}{x+s}.$$

Then we have

$$g(x) = \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{4x^4} + O\left(\frac{1}{x^6}\right) \quad (x \to \infty)$$

and

$$g'(x) = -2\sum_{s=0}^{\infty}\frac{(-1)^s}{(x+s)^2} = -\frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{x^5} + O\left(\frac{1}{x^7}\right).$$

The sums $\sum_{k=n}^{\infty}\frac{(-1)^k}{k}, \sum_{k=n}^{\infty}\frac{(-1)^k}{k^2}, \sum_{k=2n-1}^{\infty}\frac{(-1)^k}{k}, \ldots..$ which we encounter, can be derived from these expressions. We find e.q.

$$\sum_{k=n}^{\infty}\frac{(-1)^k}{k} = (-1)^n\, {}^1/_2\{g(x)\}_{x=n},$$

$$\sum_{k=n}^{\infty}\frac{(-1)^k}{k^2} = (-1)^{n-1}\, {}^1/_2\{g'(x)\}_{x=n}.$$

Moreover, using formulae ([4], p. 305)

$$\sum_{k=1}^{n}(-1)^{k-1}k = \begin{cases} -{}^1/_2 n & \text{if } n \text{ is even} \\ {}^1/_2(n+1) & \text{if } n \text{ is odd,} \end{cases}$$

$$\sum_{k=1}^{n}(-1)^{k-1}k^2 = (-1)^{n-1}\, {}^1/_2(n^2+n),$$

$$\sum_{k=1}^{n}(-1)^{k-1}k^3 = \begin{cases} -({}^1/_2 n^3 + {}^3/_4 n^2) & \text{if } n \text{ is even} \\ {}^1/_2 n^3 + {}^3/_4 n^2 - {}^1/_4 & \text{if } n \text{ is odd,} \end{cases}$$

we get

$$b_n = \frac{40}{11\pi}(10\,{}^1/_8 \ln 3 - 12 \ln 2) + O\left(\frac{1}{n}\right).$$

This proves theorem 19.

2.4 Next we will give an application of Matsuoka's theorem V of the Introduction.

THEOREM 20

*If we put*

$$\beta_n = (5n-5)^2 \sup_{(f)} \{\|L_{5n-5}(f)-f\|; f(x) \in Z_2\}$$

*and*

$$\gamma_n = (6n-6)^2 \sup_{(f)} \{\|L_{6n-6}(f)-f\|; f(x) \in Z_2\},$$

*then*

$$\lim_{n\to\infty} \beta_n = \frac{220500}{15619},$$

$$\lim_{n\to\infty} \gamma_n = \frac{11212740}{655177}.$$

PROOF

Using (90), (91), (93) and (94), we can easily show that the operators $L_{5n-5}$ and $L_{6n-6}$ satisfy the conditions of Matsuoka's theorem. Then the above results are an easy consequence of his theorem.

## 3 On asymptotic approximation formulae

3.1 In this section we will give the asymptotic approximation formulae for the operators $L_{5n-5}$ and $L_{6n-6}$. Next we will derive such a formula in a more general form for the operators (80) in case of $p \neq 1,2$. Then it will turn out that in case of sequence (80), the conditions (11) and (16) in theorems 2 and 3 are equivalent with condition (4) of theorem IV.

THEOREM 21

*If $f(x) \in C_{2\pi}$ and if $f(x)$ is twice differentiable at the point $t = x_0 \in [-\pi, +\pi]$, then*

$$L_{5n-5}(f;\, x_0) - f(x_0) = \frac{8820}{15619}\frac{f''(x_0)}{n^2} + o\left(\frac{1}{n^2}\right),$$

$$L_{6n-6}(f;\, x_0) - f(x_0) = \frac{311465}{655177}\frac{f''(x_0)}{n^2} + o\left(\frac{1}{n^2}\right).$$

PROOF

On account of (90), (91), 93) and (94), it is very easy to verify that condition (4) of theorem IV is satisfied. Using again (90) and (93), the above results follow immediately from theorem IV of KOROVKIN.*

In order to derive an asymptotic approximation formula for the operators $L_{np-p} (p \geqslant 3)$, we first use theorem 2. For this purpose we need the following results, which can be deduced in a way similar to that of section 1 of this Chapter. We have with respect to $n \to \infty$

$$\frac{1}{n^{2p-1}} \int_{-\pi}^{+\pi} \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt = O(1),$$

$$\frac{1}{n^{2p-3}} \int_{-n}^{+\pi} t^2 \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt = O(1), \tag{97}$$

$$\frac{1}{n^{2p-5}} \int_{-\pi}^{+\pi} t^4 \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt = O(1). \tag{98}$$

* We notice that the results of theorem 21 can be refined with Komleva's theorem ([6], [14]), but this demands a complete calculation of several coefficients $\varrho_k^{(5n-5)}$ and $\varrho_k^{(6n-6)}$, the number of which depends on $n$.

Consequently condition (11) is satisfied, and, moreover, the difference $L_{np-p}(f; x_0) - f(x_0)$ is clearly of order $O(1/n^2)$ when $n$ tends to infinity.
Evaluating

$$\frac{1}{A_{np-p}} \int_{-\pi}^{+\pi} t^2 \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt,$$

we have

$$\int_{-\pi}^{+\pi} t^2 \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt = 16 \int_0^{\pi/2} t^2 \left(\frac{\sin nt}{\sin t}\right)^{2p} dt = 16 \int_0^{\pi/2} (\sin^2 t + t^2 - \sin^2 t) \left(\frac{\sin nt}{\sin t}\right)^{2p} dt =$$

$$= 8 \int_0^{\pi/2} (1 - \cos 2t) \left\{{}^1/_6 \varrho_0^{(np-p)} + {}^1/_3 \sum_{k=1}^{np-p} \varrho_k^{(np-p)} \cos kt\right\} dt + 16 \int_0^{\pi/2} (t^2 - \sin^2 t) \left(\frac{\sin nt}{\sin t}\right)^{2p} dt =$$

$$= {}^2/_3 (\varrho_0^{(np-p)} - \varrho_1^{(np-p)}) + 16 \int_0^{\pi/2} (t - \sin t)(t + \sin t) \left(\frac{\sin nt}{\sin t}\right)^{2p} dt.$$

Because $t + \sin t$ is bounded on $[0, \pi/2]$ and using

$$0 \leqslant t - \sin t \leqslant t^3/6,$$

we have when $n \to \infty$

$$16 \int_0^{\pi/2} (t - \sin t)(t + \sin t) \left(\frac{\sin nt}{\sin t}\right)^{2p} dt \leqslant 8 \int_0^{\pi/2} t^3 \left(\frac{\sin nt}{\sin t}\right)^{2p} dt = O(n^{2p-4}).$$

Consequently

$$\frac{1}{n^{2p-3}} \{\varrho_0^{(np-p)} - \varrho_1^{(np-p)}\} = O(1)$$

and thus

$$\frac{1}{A_{np-p}} \int_{-\pi}^{+\pi} t^2 \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt = 2\left(1 - \frac{\varrho_1^{(np-p)}}{\varrho_0^{(np-p)}}\right) + o\left(\frac{1}{n^2}\right),$$

where

$$1 - \frac{\varrho_1^{(np-p)}}{\varrho_0^{(np-p)}} = O\left(\frac{1}{n^2}\right) \quad (n \to \infty).$$

Hence we have

THEOREM 22

*If $f(x) \in C_{2\pi}$ and if $f(x)$ is twice differentiable at the point $t = x_0 \in [-\pi, +\pi]$,* then

$$L_{np-p}(f; x_0) - f(x_0) = \left(1 - \frac{\varrho_1^{(np-p)}}{\varrho_0^{(np-p)}}\right) f''(x_0) + o\left(\frac{1}{n^2}\right) \quad (p = 2, 3, \ldots)$$

*and the $\varrho_0^{(np-p)}$ and $\varrho_1^{(np-p)}$ are defined by* (7).

The formula (98) makes an application of theorem 3 possible. In this case we get in connection with theorem 22

$$L_{np-p}(f;x_0) - f(x_0) = {}^1/_4\left(1 - \frac{\varrho_2^{(np-p)}}{\varrho_0^{(np-p)}}\right) f''(x_0) + o\left(\frac{1}{n^2}\right).$$

From this and theorem 22 follows

$$\frac{1 - \dfrac{\varrho_2^{(np-p)}}{\varrho_0^{(np-p)}}}{1 - \dfrac{\varrho_1^{(np-p)}}{\varrho_0^{(np-p)}}} \to 4 \qquad \text{when } n \to \infty,$$

in accordance with theorem IV of KOROVKIN. Moreover, by writing the integral in (98) as

$$\int_{-\pi}^{+\pi} t^4 \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt = \int_{-\pi}^{+\pi} \sin^4 t \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt + \int_{-\pi}^{+\pi} (t^4 - \sin^4 t) \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt,$$

we easily deduce that in case $m = 1$ conditions (11) and (16) of the theorems 2 and 3 are equivalent to condition (4) of theorem IV.

3.2 Several of the results of this Chapter can be generalized for the approximation of functions of many variables. We have for instance

THEOREM 23

*Let $f(X)$ be a periodic function of $m$ variables which is continuous in the domain $-\pi \leqslant t_j \leqslant +\pi$ $(j = 1,\ldots,m)$. Let $\{L_{3n-3}\}$ $(n = 1,2,\ldots)$ be a sequence of operators defined by*

$$L_{3n-3}(f;X) = \left(\frac{10}{\pi(11n^5+5n^3+4n)}\right)^m \int_{-\pi}^{+\pi} \cdots\cdots \int_{-\pi}^{+\pi} f(X+T) \prod_{j=1}^{m} \left(\frac{\sin {}^1/_2 nt_j}{\sin {}^1/_2 t_j}\right)^6 dt_1 \ldots dt_m.$$

*Then $L_{3n-3}(f;X)$ converges uniformly in this domain to the function $f(X)$ when $n \to \infty$.*

*Moreover, if $f(X)$ has all derivatives of second order in a fixed point $X_0$, the corresponding asymptotic approximation formula has the form*

$$L_{3n-3}(f;X_0) - f(X_0) = \frac{10 \sum_{j=1}^{m} f''_{x_j x_j}(X_0)}{11n^2} + o\left(\frac{1}{n^2}\right).$$

CHAPTER 4

# ON SEQUENCES OF FUNCTIONS GENERATING SEQUENCES OF OPERATORS

In Chapter 1, § 2 we considered a general method for the construction of sequences of operators. Here we recall that the essential point of the method is to find sequences of functions suitable for the construction of sequences of operators. Such a sequence of functions we will call a *suitable sequence*. Furthermore we have given in section 4.1 of Chapter 1 three sequences of such functions.

In this Chapter we investigate in more detail the possibilities of constructing suitable sequences of functions. The method of section 2.1 of Chapter 1 is generalized.

## 1 Suitable sequences as solution of a system of differential equations

1.1 The proporties I-III of section 2.1 of Chapter 1 are important for the construction of suitable sequences. If we want to construct such sequences we may first choose the coefficient $m_n$ in property III, which property can be considered as representing a recursive system of infinitely many differential equations.

The investigations in this paragraph are devoted to the case $m_n = n-c$ where $c$ is an integer. In the following discussion property III is taken as starting point, assuming

$$\begin{cases} \psi(n,x) = n & (n = 1,2,\ldots) \\ \alpha_{k,n}(x) \equiv 0 & (k,n = 1,2,\ldots). \end{cases}$$

In view of these assumptions, it will turn out that it is sufficient to consider the system of differential equations only for $k = 1$. Treating this system of differential equations, we distinguish between the cases $c > 0$, $c = 0$ and $c < 0$. Here case $c = 0$ is easy to handle (cf. section 1.3). In case $c$ is a positive integer, property I can be satisfied during the process but it is more difficult to meet property II. When $c$ is a negative integer, property II causes no difficulties but in general we cannot meet property I (cf. section 1.4).

In what follows, we restrict ourselves to the operators of the form (31) and we particularly investigate the cases 1) $m_n = n-1$, 2) $m_n = n$ and 3) $m_n = n+1$.

1.2 We first consider the case $m_n = n-c$, $c$ being an arbitrary but fixed positive integer. From property III we have

$$-\varphi_n^{(k)}(x) = n\varphi_{n-c}^{(k-1)}(x) \quad (n = c+1, c+2, \ldots; k = 1, 2, \ldots).$$

For the infinitely differentiable functions $\varphi_1(x), \varphi_2(x), \ldots$, these relations are identical with the following independent $c$ systems of differential equations

$$(\mathrm{III}_1) \begin{cases} -\varphi'_{c+1}(x) = (c+1)\varphi_1(x) \\ -\varphi'_{2c+1}(x) = (2c+1)\varphi_{c+1}(x) \\ -\varphi'_{3c+1}(x) = (3c+1)\varphi_{2c+1}(x) \\ \vdots \\ \vdots \end{cases} \quad \begin{matrix} \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \\ \vdots \\ \vdots \end{matrix} \quad (\mathrm{III}_c) \begin{cases} -\varphi'_{2c}(x) = 2c\varphi_c(x) \\ -\varphi'_{3c}(x) = 3c\varphi_{2c}(x) \\ -\varphi'_{4c}(x) = 4c\varphi_{3c}(x) \\ \vdots \\ \vdots \end{cases} .$$

We introduce the concept of *starting-function.* This (these) is (are) a (the) function(s) $\varphi$ from which the remaining functions $\varphi$ of the sequence $\{\varphi_n(x)\}$ are derived by means of the differential equations $(\mathrm{III}_1)$–$(\mathrm{III}_c)$ and properties I, II of section 2.1 of Chapter 1. According to these properties a starting-function $\varphi$ has to satisfy the following conditions on a certain interval $[0, R]$ $(R > 0)$

$$\begin{cases} \varphi(0) = 1 \\ (-1)^k \varphi^{(k)}(x) \geqslant 0 \quad (k = 0, 1, 2, \ldots;\ x \in [0, R]). \end{cases} \tag{99}$$

In the case which is being considered here, $\varphi_j(x)$ $(j = 1, \ldots, c)$ can be chosen such that they are starting-functions. The remaining functions $\varphi_n(x)$ $(n = c+1, c+2, \ldots)$ can be determined successively from equations $(\mathrm{III}_1)$–$(\mathrm{III}_c)$, where in accordance with property I the constants of integration $c_n$ are taken such that $\varphi_n(0) = 1$.

We have the following general theorem.

THEOREM 24

*Let $\varphi_j(x)$ $(j = 1, \ldots, c)$ be starting-functions on the intervals $[0, R_j]$ $(R_j > 0;$ $j = 1, \ldots, c)$. Then the general element of the sequence $\{\varphi_n(x)\}$ has the form*

$$\varphi_{cn+j}(x) = (1-cx)^{n+j/c} + \frac{(-1)^n}{(n-1)!} \prod_{k=1}^{n} (ck+j) \int_0^x (x-t)^{n-1}\{\varphi_j(t) - (1-ct)^{j/c}\}dt \tag{100}$$
$$(j = 1, \ldots, c;\ n = 1, 2, \ldots).$$

*The corresponding sequence of operators can be defined* on the interval $[0, R]$ where*

$$R = \min_{1 \leqslant j \leqslant c} \left(\frac{1}{3c},\ R_j\right).$$

PROOF

Using property I, it follows by integration of $(\mathrm{III}_1)$—$(\mathrm{III}_c)$ that we have for $j = 1, \ldots, c$

* We notice that the interval $[0, R]$ need not be the largest interval on which the operators $L_n$ of the form (31) can be defined. Cf. some examples following theorem 24.

$$\varphi_{c+j}(x) = -(c+j)\int_0^x \varphi_j(t)\,dt+1,$$

$$\varphi_{2c+j}(x) = (c+j)(2c+j)\iint_0^x \varphi_j(t)\,dt-(2c+j)x+1,$$

$$\vdots$$

$$\varphi_{cn+j}(x) = (-1)^n\prod_{k=1}^{n}(ck+j)\underbrace{\int_0^x\ldots\int}_{(n)}\varphi_j(t)\,dt+\sum_{k=1}^{n}\prod_{i=k+1}^{n}\frac{(ci+j)(-1)^{n-k}x^{n-k}}{(n-k)\,!}, \quad (101)$$

where

$$\underbrace{\int_0^x\ldots\int}_{(i)}\varphi_j(t)\,dt \equiv \int_0^x\int_0^{t_i}\int_0^{t_{i-1}}\ldots\int_0^{t_2}\varphi_j(t_1)\,dt_1\ldots dt_i \quad (i=1,\ldots,n)$$

and by definition $\prod\limits_{\substack{i=r\\ r>s}}^{s} = 1.$

Using a theorem of Cauchy we have

$$\underbrace{\int_0^x\ldots\int}_{(n)}\varphi_j(t)\,dt = \frac{1}{(n-1)\,!}\int_0^x (x-t)^{n-1}\varphi_j(t)\,dt \quad (j=1,\ldots,c),$$

while the second part of formula (101) consists of the first $n$ terms in the expansion of $(1-cx)^{n+j/c}$ into powers of $x$. This proves the first part of theorem 24.

Now we have to find an interval $[0, R]\subseteq[0, 1/c]$ on which each function of the sequence $\{\varphi_n(x)\}$ possesses property II. Using the systems $(\mathrm{III}_1)$–$(\mathrm{III}_c)$, it follows that if there exists such an interval in case $k=0$, the functions $\varphi_1(x), \varphi_2(x),\ldots$ will also have property II on this interval in cases $k=1,2,\ldots$. Hence we restrict ourselves to the case $k=0$ and it remains to determine an interval $[0, R]$ on which we have

$$\varphi_n(x) \geqslant 0 \qquad (n=1,2,\ldots).$$

In formula (100) let $n$ first be odd. We get

$$\varphi_{2mc+c+j}(x) = (1-cx)^{2m+1+j/c}-\frac{1}{(2m)\,!}\prod_{k=1}^{2m+1}(ck+j)\int_0^x (x-t)^{2m}\{\varphi_j(t)-(1-ct)^{j/c}\}\,dt \geqslant \qquad (m=0,1,2,\ldots)$$

$$\geqslant (1-cx)^{2m+1+j/c}-\frac{1}{(2m)\,!}\prod_{k=1}^{2m+1}(ck+j)\int_0^x (x-t)^{2m}\{1-(1-ct)^{j/c}\}\,dt, \quad (102)$$

because in view of formula (99) the choice

$$\varphi_j(x) \equiv 1 \quad (j = 1, \ldots, c)$$

is possible.

Using (102), we further have

$$\varphi_{2mc+c+j}(x) \geqslant (1-cx)^{2m+2} - c^{2m+2}(2m+1)(2m+2)\int_0^x (x-t)^{2m} t\, dt \qquad (103)$$
$$= (1-cx)^{2m+2} - (cx)^{2m+2} \geqslant 0$$

when $cx \in [0, {}^1/_2]$ and hence $x \in [0, 1/2c]$.

In deriving (103) from (102), the sign of equality only holds in case $j = c$.

From the above calculations there follows that

$$\varphi_{2mc+c+j}(x) \geqslant 0 \ (m = 0, 1, 2, \ldots;\ j = 1, \ldots, c)$$

on the interval $[0, R^*]$ where

$$R^* = \min_{1 \leqslant j \leqslant c} \left(\frac{1}{2c}, R_j\right). \qquad (104)$$

When in (100) $n$ is even we have

$$\varphi_{2mc+j}(x) = (1-cx)^{2m+j/c} + \frac{1}{(2m-1)!}\prod_{k=1}^{2m}(ck+j)\int_0^x (x-t)^{2m-1}\{\varphi_j(t) - (1-ct)^{j/c}\}dt$$
$$(m = 1, 2, \ldots)$$

and hence

$$\varphi_{2mc+j}(x) > (1-cx)^{2m+j/c} - \frac{1}{(2m-1)!}\prod_{k=1}^{2m}(ck+j)\int_0^x (x-t)^{2m-1}(1-ct)^{j/c}\, dt \geqslant$$
$$\geqslant (1-cx)^{2m+j/c} - c^{2m}(2m)(2m+1)\int_0^x (x-t)^{2m-1}(1-ct)^{j/c}\, dt \geqslant$$
$$\geqslant (1-cx)^{j/c-1}\{(1-cx)^{2m+1} - c^{2m}(2m)(2m+1)\int_0^x (x-t)^{2m-1}(1-ct)dt\} =$$
$$= (1-cx)^{j/c-1}\{(1-cx)^{2m+1} - (2m+1)(cx)^{2m} + (cx)^{2m+1}\}.$$

Now it is easily verified that for every arbitrary but fixed positive integer $m$ the function

$$\psi_{2m+1}(x) = (1-cx)^{2m+1} - (2m+1)(cx)^{2m} + (cx)^{2m+1}$$

is decreasing on the interval $[0, 1/2c]$. Moreover, we have

$$\psi_{2m+1}\left(\frac{1}{3c}\right) = \left(\frac{1}{3}\right)^{2m+1}(2^{2m+1} - 6m - 2) =$$
$$= 2\left(\frac{1}{3}\right)^{2m+1}\{(1+3)^m - 3m - 1\} \geqslant 0 \quad (m = 1, 2, \ldots)$$

with equality only for $m = 1$.

From this and from (104) we get

$$\varphi_n(x) \geqslant 0 \quad \text{on} \quad [0, R]$$

where

$$R = \min_{1 \leqslant j \leqslant c} \left(\frac{1}{3c}, R_j\right).$$

On this interval $[0, R]$ the sequence $\{\varphi_n(x)\}$ $(n = 1,2,\ldots)$ generates a sequence of operators $\{L_n\}$ $(n = 1,2,\ldots)$ of the form (31). This proves theorem 24.

The case $m_n = n-1$ will now be considered in more detail. Formula (100) then takes the simple form

$$\varphi_{n+1}(x) = (1-x)^{n+1}+(-1)^n n(n+1) \int_0^x (x-t)^{n-1}\{\varphi_1(t)-(1-t)\}dt \quad (n = 1,2,\ldots). \tag{105}$$

**Applications**

a) The function

$$\varphi_1(x) = 1-x$$

is according to (99) a starting-function on the interval $[0, 1]$. In view of (105) we have

$$\varphi_{n+1}(x) = (1-x)^{n+1} \geqslant 0 \quad (n = 0,1,2,\ldots)$$

on the interval $[0, 1]$.

The sequence $\{\varphi_n(x)\}$ generates on $[0, 1]$ the well-known Bernstein operators $B_n$ which have the form

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k} \qquad (x \in [0,1]).$$

Remark 1

The choice

$$\varphi_1(x) = 1-x$$

is for two reasons the best one*. Because the integral expression in formula (105) vanishes, the form of the sequence $\{\varphi_n(x)\}$ has a simple character and this is also true for the corresponding sequence of operators.

---

* When $c \neq 1$, it is not so easy to determine the best choice of the starting-functions $\varphi_j(x)$ $(j = 1, \ldots, c)$ with regard to the size of the interval on which the corresponding operators can be defined. Now it is not possible to make the integral expression vanish in formula (100) in cases $n = 1, 2, \ldots$ and $j = 1, \ldots, c$ by means of a suitable choice of the starting-functions. In fact, of all the functions

$$(1-cx)^{j/c} \quad (j = 1, \ldots, c),$$

there is only one which can be taken as a starting-function and this one we get when $j = c$.

In the second place we thus get the largest interval on which operators can be defined in case $m_n = n-1$.

b) More generally we can take as a starting-function

$$\varphi_1(x) = 1-\alpha x \quad (\alpha \geqslant 0,\ \alpha \neq 1).$$

The interval of maximal length on which the corresponding operators can be defined will be shown to depend on $\alpha$.

Using formula (105) we deduce that

$$\varphi_n(x) = (1-x)^n+(-1)^{n-1}x^n(1-\alpha) \quad (n = 1,2,\ldots).$$

Examining this sequence, we distinguish between some cases.

1) $\alpha = 0$.
The corresponding operators can be defined on the interval $[0, {}^1/_2]$.

2) $0 < \alpha < 1$.

$$\varphi_{2m+1}(x) = (1-x)^{2m+1}+x^{2m+1}(1-\alpha) > 0$$

when $x \in [0, 1]$,

$$\varphi_{2m}(x) = (1-x)^{2m}-x^{2m}(1-\alpha).$$

Obviously $\varphi_{2m}(x) > 0$ when $x \in [0, {}^1/_2]$. This interval cannot be enlarged. In fact, assuming the converse, let $\varphi_{2m}(x) \geqslant 0$ on $[0, {}^1/_2+\delta]$, $\delta > 0$.

Then we would have

$$({}^1/_2-\delta)^{2m} > ({}^1/_2+\delta)^{2m}(1-\alpha),$$

which is obviously not true for large $m$.

3) $1 < \alpha \leqslant 2$.

$$\varphi_{2m}(x) = (1-x)^{2m}+x^{2m}(\alpha-1) > 0$$

when $x \in [0, \infty)$,

$$\varphi_{2m+1}(x) = (1-x)^{2m+1}-x^{2m+1}(\alpha-1) \geqslant 0$$

when $x \in [0, {}^1/_2]$.

The corresponding operators can be defined on the interval $[0, {}^1/_2]$. It is easily seen that this interval cannot be enlarged.

---

The other ones are not suitable (cf. (99)). As a consequence of this we have that in case $c \neq 1$ the interval $[0, R]$ on which the operators can be defined will be contained in the interval $[0, 1/c]$.

We finally remark that when choosing

$$\varphi_j(x) \equiv 1 \quad (j = 1, \ldots, c)$$

the corresponding operators can be defined on the interval $[0, 1/2c]$. This follows from the proof of theorem 24.

4) $\alpha > 2$.

$$\varphi_{2m}(x) = (1-x)^{2m} + x^{2m}(\alpha-1) > 0$$

when $x \in [0, \infty)$,

$$\varphi_{2m+1}(x) = x^{2m+1}\left\{\left(\frac{1}{x}-1\right)^{2m+1} - (\alpha-1)\right\} \geqslant 0$$

when $x \in [0, 1/\alpha]$; the sign of equality occurs when $m = 0$ and $x = 1/\alpha$.

The corresponding operators can be defined on the interval $[0, 1/\alpha]$.

REMARK 2

If $0 \leqslant \alpha < 3$ theorem 24 gives an interval $[0, R]$ for the operators which is not of maximal length. But when $\alpha \geqslant 3$, theorem 24 gives the best result for the interval $[0, R]$.

REMARK 3

If $\alpha \neq 1$, then the influence of the integral expression in formula (105) can be considered as leading to a disturbance of the Bernstein operators.

c) In our third example we take as a starting-function

$$\varphi_1(x) = e^{-ax} \quad (a > 0 \text{ and fixed}).$$

The general element $\varphi_n(x)$ of the sequence $\{\varphi_n(x)\}$ $(n = 1,2,\ldots)$ can be written in the form (105). Another representation is

$$\varphi_n(x) = \frac{n!}{a^{n-1}}\, e^{-ax} + \sum_{k=2}^{n} \binom{n}{k}\left(1 - \frac{k!}{a^{k-1}}\right)(-x)^{n-k} \quad (n = 2,3,\ldots),$$

which also can be derived from system $\mathrm{III}_1$ by using mathematical induction.

As a consequence of theorem 24 the corresponding operators can be defined on the interval $[0, 1/3]$.

REMARK

The above choice of the starting-function for large values of $a$ shows agreement with the choice of the functions $\varphi_j(x)$ $(j = 1,\ldots,c)$ which we considered in the second part of the proof of theorem 24.

Consequently $[0, 1/3]$ is the largest interval on which the operators can be defined for an arbitrary but fixed value of $a$.

1.3 In this section we consider the case $m_n = n$.

The corresponding system of differential equations has the form

$$-\varphi_n^{(k)}(x) = n\varphi_n^{(k-1)}(x) \qquad (n, k = 1,2,\ldots).$$

For the infinitely differentiable functions $\varphi_n(x)$ $(n = 1,2,\ldots)$ this is identical with the system

$$\begin{cases} -\varphi_1'(x) = \varphi_1(x) \\ -\varphi_2'(x) = 2\varphi_2(x) \\ -\varphi_3'(x) = 3\varphi_3(x) \\ \qquad\vdots \end{cases}.$$

Obviously the equations of this system are mutually independent. The solution is

$$\varphi_n(x) = c_n e^{-nx} \quad (n = 1,2,\ldots),$$

where $c_n \equiv 1$ according to the condition $\varphi_n(0) = 1$ $(n = 1,2,\ldots)$.

The corresponding sequence of operators can be defined on the interval $[0, \infty)$.

1.4 The case $m_n = n+c$ ($c$ is an arbitrary but fixed positive integer) will be treated in this section.

In view of property III we have

$$-\varphi_n^{(k)}(x) = n\varphi_{n+c}^{(k-1)}(x) \qquad (n, k = 1,2,\ldots).$$

For the infinitely differentiable functions $\varphi_n(x)$ $(n = 1,2,\ldots)$ this is identical with the $c$ systems

$$(\mathrm{III}_1') \begin{cases} -\varphi'_1(x) = \varphi_{c+1}(x) \\ -\varphi'_{c+1}(x) = (c+1)\varphi_{2c+1}(x) \\ -\varphi'_{2c+1}(x) = (2c+1)\varphi_{3c+1}(x) \\ \qquad\vdots \end{cases} \quad \begin{matrix} \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \vdots \end{matrix} \quad (\mathrm{III}_c') \begin{cases} -\varphi_c'(x) = c\varphi_{2c}(x) \\ -\varphi'_{2c}(x) = 2c\varphi_{3c}(x) \\ -\varphi'_{3c}(x) = 3c\varphi_{4c}(x) \\ \qquad\vdots \end{cases}.$$

These systems of differential equations are in general not suitable for a constructive approach using starting-functions $\varphi_j(x)$ $(j = 1,\ldots,c)$. In fact, $\varphi_{c+1}(x)$ is completely determined by the choice of $\varphi_1(x)$ and in general we have

$$\varphi_{c+1}(0) \neq 1.$$

Thus property I cannot be satisfied during the process. Considering the structure of the systems $(\mathrm{III}_1')$–$(\mathrm{III}_c')$, it is possible to take $\varphi_{cn_j+j}(x)$ $(j = 1, \ldots,c)$ as starting-functions, where $n_j$ $(j = 1,\ldots,c)$ are arbitrary but fixed positive integers with $n_j > 1$ $(j = 1,\ldots,c)$. Then it is always possible to choose the constants of integration such that $\varphi_{cn+j}(0) = 1$ $(n = 1,2,\ldots,n_j; j = 1,\ldots,c)$.

However, the following remark must be made. First we are choosing $\varphi_{cn_j+j}(x)$; then the other functions $\varphi_{cn+j}(x)$ $(n = n_j-1,\ldots, 1; j = 1,\ldots,c)$ are determined according to the corresponding equations in systems $(\mathrm{III}_1')$–$(\mathrm{III}_c')$. Because of the fact that the approximation becomes better when $n$ increases*, the determination of these functions does not make any sense.

* Under the conditions $\chi(n) = n$, $\psi(n,x) = n$, $m_n = n-c$, $\alpha_{k,n}(x) \equiv 0$, this follows from theorems 5 and 6 of Chapter 1.

The first choice of the starting-functions is at once the best one, so this procedure of approximation does not do, only with the exception of those cases where it is possible to start with $\varphi_j(x)$ $(j = 1,\ldots,c)$, where $\varphi_j(x)$ satisfy (99). Then it is necessary that the condition $\varphi_n(0) = 1$ $(n = c+1, c+2,\ldots)$ is automatically satisfied during the process. Property II causes no difficulties and the corresponding sequence of operators can then be defined on the interval $[0, R]$ where

$$R = \min_{1\leqslant j\leqslant c} R_j.$$

We conclude this section by giving an example to this case.

Example

If we take as starting-functions

$$\varphi_j(x) = (1+cx)^{-j/c} \quad (j = 1,\ldots,c),$$

then the condition $\varphi_n(0) = 1$ $(n = c+1, c+2,\ldots)$ is automatically satisfied during the differentiation process of systems $(\mathrm{III}_1')$–$(\mathrm{III}_c')$.

The suitable sequence of functions has the form

$$\varphi_n(x) = (1+cx)^{-n/c} \quad (n = 1,2,\ldots)$$

and the corresponding sequence of operators can be defined on $[0, \infty)$.

Remark

In case $m_n = n+1$ we get the third example of section 4.1 of Chapter 1, if we take there $p = 0$.

## 2 A generalization of the method of Chapter 1, § 2 with some applications

2.1 As we have seen the systems of differential equations which we have been using in § 1 of this Chapter originate from property III of Chapter 1, § 2. This property is a relation between two derivatives of two functions from the sequence $\{\varphi_n(x)\}$ $(n = 1,2,\ldots)$. This relation can be generalized in the sense that from onwards a certain positive integer $n_0$ the $k^{\text{th}}$-derivative of $\varphi_n(x)$ $(n = n_0, n_0+1,\ldots)$ is a certain linear combination of the preceding $k-1$ derivatives of some other function from the sequence $\{\varphi_n(x)\}$.

Again, a sequence of operators can be adjoined to the sequence $\{\varphi_n(x)\}$ in such a way that it satisfies the conditions of theorem 1. Moreover, we construct a general sequence of functions of which the corresponding sequence of operators has Voronovskaja's formula as asymptotic approximation formula.

2.2 We assume that in the sequence

$$\{\varphi_n(x)\} \quad (n = 1,2,\ldots)$$

of real functions of the real variable $x$, each function $\varphi_n(x)$ has the following properties on the interval $[0, b]$ $(b > 0)$.

I′: $\varphi_n(0) = 1$;

II′: $\varphi_n(x)$ is infinitely differentiable on $[0, b]$ and $(-1)^k \varphi_n^{(k)}(x) \geqslant 0$ $(k = 0, 1, 2, \ldots)$;

III′: there exists a sequence of functions $\{g_n(x)\}$ $(n = 1, 2, \ldots)$ and a positive integer $m_n$ not depending on $k$, such that

$$-\varphi_n^{(k)}(x) = \sum_{i=0}^{k-1} \binom{k-1}{i} g_n^{(i)}(x) \varphi_{m_n}^{(k-i-1)}(x) \quad (k = 1, 2, \ldots), \tag{106}$$

where we assume that

$$(-1)^k g_n^{(k)}(x) \geqslant 0 \;\; (k = 0, 1, 2, \ldots,; \; n = 1, 2, \ldots); \tag{107}$$

$$\lim_{n \to \infty} \frac{g_n(0)}{n} = 1; \tag{108}$$

$$\lim_{n \to \infty} \frac{g_n^{(i)}(0)}{n} = 0 \quad (i = 1, 2, 3); \tag{109}$$

$$\lim_{n \to \infty} \frac{m_n}{n} = 1. \tag{110}$$

We adjoin to the sequence $\{\varphi_n(x)\}$ of functions $\varphi_n(x)$ $(n = 1, 2, \ldots)$ a sequence of operators $\{L_n\}$ $(n = 1, 2, \ldots)$ defined by

$$L_n(f; x) = \sum_{k=0}^{\infty} \frac{(-1)^k \varphi_n^{(k)}(x) x^k}{k!} f\left(\frac{k}{n}\right). \tag{111}$$

Theorem 25

*If $f(x) \in H(x_0)$ and if the sequence $\{\varphi_n(x)\}$ satisfies the conditions I′–III′, then the sequence $\{L_n(f; x_0)\}$ $(n = 1, 2, \ldots)$ defined in* (111) *converges to $f(x_0)$ when $n \to \infty$.*

*If $f(x) \in H(x_0)$ $(0 \leqslant x_0 \leqslant b)$, then the convergence is uniform on the interval $[0, b]$.*

Proof

We show that the sequence $\{L_n(f; x_0)\}$ $(n = 1, 2, \ldots)$ converges to $f(x_0)$ when $n \to \infty$ in the three cases $f(t) \equiv 1$, $f(t) \equiv t$ and $f(t) \equiv t^2$.

In case $f(t) \equiv 1$ the convergence of $\{L_n(f; x_0)\}$ to 1 follows from I′.

When $f(t) \equiv t$ we have

$$L_n(t; x_0) = \sum_{k=0}^{\infty} \frac{k}{n} \frac{(-1)^k \varphi_n^{(k)}(x_0) x_0^k}{k!}.$$

Using III′ we get*

$$\sum_{k=0}^{\infty} \frac{k}{n} \frac{(-1)^k \varphi_n^{(k)}(x_0) x_0^k}{k!} = \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \frac{(-1)^{k-1} \binom{k-1}{i} g_n^{(i)}(x_0) \varphi_{m_n}^{(k-i-1)}(x_0) x_0^k}{(k-1)!\, n} =$$

$$= \frac{x_0 g_n(x_0)}{n} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \varphi_{m_n}^{(k-1)}(x_0) x_0^{k-1}}{(k-1)!} + \frac{(-1)^1 x_0^2 g_n'(x_0)}{1!\, n} \sum_{k=2}^{\infty} \frac{(-1)^{k-2} \varphi_{m_n}^{(k-2)}(x_0) x_0^{k-2}}{(k-2)!} + \ldots\ldots$$

In view of I′ and Taylor's theorem we get

$$L_n(t; x_0) = \frac{x_0}{n} \sum_{i=0}^{\infty} \frac{(-1)^i g_n^{(i)}(x_0) x_0^i}{i!} = \frac{x_0 g_n(0)}{n}. \tag{112}$$

According to (108) we have

$$L_n(t; x_0) \to x_0 \text{ when } n \to \infty.$$

When $f(t) \equiv t^2$ we have to consider the series

$$\sum_{k=0}^{\infty} \frac{k^2}{n^2} \frac{(-1)^k \varphi_n^{(k)}(x_0) x_0^k}{k!}.$$

Using III′ we get

$$\sum_{k=0}^{\infty} \frac{k^2}{n^2} \frac{(-1)^k \varphi_n^{(k)}(x_0) x_0^k}{k!} = \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \frac{k \binom{k-1}{i} g_n^{(i)}(x_0) \varphi_{m_n}^{(k-i-1)}(x_0) x_0^k}{(k-1)!\, n^2} =$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} k g_n(x_0) \varphi_{m_n}^{(k-1)}(x_0) x_0^k}{(k-1)!\, n^2\, 0!} + \sum_{k=2}^{\infty} \frac{(-1)^{k-1} k g_n'(x_0) \varphi_{m_n}^{(k-2)}(x_0) x_0^k}{(k-2)!\, n^2\, 1!} + \ldots =$$

$$= \sum_{k=2}^{\infty} \frac{(-1)^{k-1} g_n(x_0) \varphi_{m_n}^{(k-1)}(x_0) x_0^k}{(k-2)!\, n^2\, 0!} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} g_n(x_0) \varphi_{m_n}^{(k-1)}(x_0) x_0^k}{(k-1)!} +$$

---

* If

$$\sum_{k=0}^{\infty} \frac{k}{n} \frac{(-1)^k \varphi_n{}^{(k)}(x_0) x_0{}^k}{k!} \tag{†}$$

is a convergent series, it is obviously an absolutely convergent series in view of property II′. We have altered the order of summation and still deal with sequences with positive terms according to (107). Because the series in formula (112) converges, the original series (†) also converges.

A similar remark does hold for the series

$$\sum_{k=0}^{\infty} \frac{k^2}{n^2} \frac{(-1)^k \varphi_n{}^{(k)}(x_0) x_0{}^k}{k!}, \ldots, \sum_{k=0}^{\infty} \frac{k^4}{n^4} \frac{(-1)^k \varphi_n{}^{(k)}(x_0) x_0{}^k}{k!}.$$

$$+\sum_{k=3}^{\infty}\frac{(-1)^{k-1}g_n'(x_0)\varphi_{m_n}^{(k-2)}(x_0)x_0^k}{(k-3)!\,n^2\,1!}+2\sum_{k=2}^{\infty}\frac{(-1)^{k-1}g_n'(x_0)\varphi_{m_n}^{(k-2)}(x_0)x_0^k}{(k-2)!\,n^2\,1!}+$$

$$+\quad\cdots\cdots\cdots\cdots\cdots\cdots+\quad\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\vdots\qquad\qquad\qquad\qquad\vdots\qquad .$$

Summation of the second column gives

$$\sum_{i=0}^{\infty}\frac{(-1)^i(i+1)g_n^{(i)}(x_0)x_0^{i+1}}{n^2\,i!}=\frac{x_0g_n(0)}{n^2}-\frac{x_0^2g_n'(0)}{n^2}. \tag{113}$$

Again using III′ in the first column, we have

$$\sum_{k=2}^{\infty}\sum_{i=0}^{k-2}\frac{(-1)^{k-2}\binom{k-2}{i}g_{m_n}^{(i)}(x_0)g_n(x_0)\varphi_{m_{m_n}}^{(k-i-2)}(x_0)x_0{}^k}{(k-2)!\,n^2\,0!}+$$

$$+\sum_{k=3}^{\infty}\sum_{i=0}^{k-3}\frac{(-1)^{k-2}\binom{k-3}{i}g_{m_n}^{(i)}(x_0)g_n'(x_0)\varphi_{m_{m_n}}^{(k-i-3)}(x_0)x_0^k}{(k-3)!\,n^2\,1!}+\ldots=$$

$$=\sum_{k=2}^{\infty}\frac{(-1)^{k-2}g_{m_n}(x_0)g_n(x_0)\varphi_{m_{m_n}}^{(k-2)}(x_0)x_0^k}{(k-2)!\,n^2\,0!\,0!}+\sum_{k=3}^{\infty}\frac{(-1)^{k-2}g_{m_n}'(x_0)g_n(x_0)\varphi_{m_{m_n}}^{(k-3)}(x_0)x_0^k}{(k-3)!\,n^2\,0!\,1!}+\ldots$$

$$+\sum_{k=3}^{\infty}\frac{(-1)^{k-2}g_{m_n}(x_0)g_n'(x_0)\varphi_{m_{m_n}}^{(k-3)}(x_0)x_0^k}{(k-3)!\,n^2\,1!\,0!}+\sum_{k=4}^{\infty}\frac{(-1)^{k-2}g_{m_n}'(x_0)g_n'(x_0)\varphi_{m_{m_n}}^{(k-4)}(x_0)x_0^k}{(k-4)!\,n^2\,1!\,1!}+\ldots$$

$$+\quad\cdots\cdots\cdots\cdots\cdots\cdots+\quad\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\vdots\qquad\qquad\qquad\qquad\vdots$$

$$=\sum_{i=0}^{\infty}\frac{(-1)^ig_n(x_0)g_{m_n}^{(i)}(x_0)x_0^{i+2}}{n^2\,0!\,i!}+\sum_{i=0}^{\infty}\frac{(-1)^{i+1}g_n'(x_0)g_{m_n}^{(i)}(x_0)x_0{}^{i+3}}{n^2\,1!\,i!}+\ldots=$$

$$=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^{i+j}g_n^{(j)}(x_0)g_{m_n}^{(i)}(x_0)x_0^{i+j+2}}{n^2\,i!\,j!}=$$

$$=\frac{x_0^2g_n(0)g_{m_n}(0)}{n^2}.$$

Taking into account formula (113) we have

$$L_n(t^2;x_0)=\frac{x_0^2g_n(0)g_{m_n}(0)}{n^2}+\frac{x_0g_n(0)}{n^2}-\frac{x_0^2g_n'(0)}{n^2}. \tag{114}$$

Using (107), (108), (109) and (110) there follows

$$L_n(t^2;x_0)\to x_0^2\quad\text{when}\quad n\to\infty.$$

According to theorem 1 theorem 25 is established. When $f(x) \in H(x_0)$ $(0 \leqslant x_0 \leqslant b)$, the uniform convergence follows from formulae (112) and (114).

2.3 Concerning the construction of suitable sequences of functions $\{\varphi_n(x)\}$ $(n = 1,2,\ldots)$, we will restrict ourselves to the case $m_n = n-c$, where $c$ is an integer.

If $c$ is a negative integer, similar remarks can be made as in section 1.4 of this Chapter. Then no constructive approach with starting- functions is in general available, because condition I′ cannot be satisfied during the differentiation process of system (106).

Using $c$ starting-functions in case $m_n = n-c$, $c$ being a positive integer, we can, after first choosing the appropriate functions $g_n(x)$ $(n = 1,2,\ldots)$, construct a suitable sequence satisfying (106). Obviously property II′ is then equivalent with

$$\begin{cases} (-1)^k \varphi_i^{(k)}(x) \geqslant 0 \ (i = 1,\ldots,c;\ k = 0,1,2,\ldots;\ x \in [0,b]) & (115) \\ \varphi_i(x) \geqslant 0 \ (i = c+1,\ c+2,\ldots;\ x \in [0,b]). & (116) \end{cases}$$

A constructive approach is thus possible in this case.

2.4 Finally we will give an application in cases $m_n = n-1$, $m_n = n$ and $m_n = n+1$.

a) $m_n = n-1$.

The function

$$\varphi_1(x) = e^{1/p(1-x)^p - 1/p},$$

where $p$ is an arbitrary but fixed positive integer, is a starting-function on the interval $[0,1]$.

When we assume

$$g_n(x) = n-1+(1-x)^p \quad (n = 1,2,\ldots), \tag{117}$$

then it is easy to verify that conditions (107)–(109) are satisfied on $[0,1]$. Solving system (106), we get

$$\varphi_n(x) = e^{1/p(1-x)^p - 1/p}(1-x)^{n-1} \quad (n = 1,2,\ldots)$$

and these functions are not negative on the interval $[0,1]$. Consequently a sequence of operators $\{L_{n,p}\}$ $(n = 1,2,\ldots)$ is generated which can be defined on $[0,1]$.

In view of (112), (114) and (117) we have

$$\begin{cases} L_{n,p}(1;x_0) = 1, & \\ L_{n,p}(t;x_0) = x_0, & (n = 1,2,\ldots;\ p \text{ fixed}) \\ L_{n,p}(t^2;x_0) = x_0^2 + \dfrac{x_0(1-x_0)}{n} + \dfrac{p x_0^2}{n^2}. & \end{cases} \tag{118}$$

In deducing an asymptotic approximation formula for this sequence of operators we use theorem 2. The condition (11), sufficient for applying this theorem, is satisfied in case $m = 1$, for in a way similar to the proof of theorem 25 we derive

$$L_n(t^3; x_0) = \frac{x_0{}^3 g_n(0) g_{m_n}(0) g_{m_{m_n}}(0)}{n^3} + \frac{3x_0{}^2 g_n(0) g_{m_n}(0)}{n^3} + o\left(\frac{1}{n}\right),$$

$$L_n(t^4; x_0) = \frac{x_0{}^4 g_n(0) g_{m_n}(0) g_{m_{m_n}}(0) g_{m_{m_{m_n}}}(0)}{n^4} + \frac{6x_0{}^3 g_n(0) g_{m_n}(0) g_{m_{m_n}}(0)}{n^4} + o\left(\frac{1}{n}\right).$$

Using this and formulae (112), (114), (108), (109), (110) we have

$$L_n\{(t-x_0)^4; x_0\} = \sum_{k=0}^{\infty} \left(\frac{k}{n} - x_0\right)^4 \frac{(-1)^k \varphi_n^{(k)}(x_0) x_0^k}{k!} = o\left(\frac{1}{n}\right) \quad (n \to \infty),$$

hence theorem 2 can be applied.
We then get

THEOREM 26

*Let $f(x) \in H^{(2)}(x_0)$. The sequence of operators $\{L_{n,p}\}$ $(n = 1,2,\ldots)$, defined on the interval $[0,1]$ and generated by the sequence of functions of the form*

$$\varphi_n(x) = e^{1/p(1-x)^p - 1/p}(1-x)^{n-1} \quad (n = 1,2,\ldots \text{ and } p \text{ is a fixed positive integer})$$

*has the property*

$$L_{n,p}(f; x_0) - f(x_0) = \frac{x_0(1-x_0)}{2n} f''(x_0) + o\left(\frac{1}{n}\right) \quad (x_0 \in [0,1]).$$

REMARK

*Voronovskaja's asymptotic approximation formula is thus valid for this class of sequences of operators.*

PROOF

Using theorem 2 the proof follows at once from (118).

b) $m_n = n$.
Assuming

$$g_n(x) = n - 2x + x^2 \quad (n = 1,2,\ldots),$$

we derive from (106) the suitable sequence $\{\varphi_n(x)\}$ with

$$\varphi_n(x) = e^{-nx + x^2 - x^3/3} \quad (n = 1,2,\ldots).$$

The corresponding operators can be defined on the interval $[0,1]$ and there yields

$$\begin{cases} L_n(1;\, x_0) &= 1, \\ L_n(t;\, x_0) &= x_0, \\ L_n(t^2;\, x_0) &= x_0{}^2 + \dfrac{x_0}{n} + \dfrac{2x_0{}^2}{n^2}. \end{cases}$$

Using this and theorem 2 an asymptotic approximation formula can be given.

c) $m_n = n+1$.

The sequence of functions $\{g_n(x)\}$ of the form

$$g_n(x) = n-1+(1+x)^{-p} \quad (n = 1, 2, \ldots),$$

$p$ being an arbitrary but fixed positive integer, satisfies conditions (107)–(109).

Starting with

$$\varphi_1(x) = e^{1/p(1+x)^{-p}-1/p} \quad (x \in [0, \infty))$$

in system (106), condition I′ is automatically satisfied and a suitable sequence can thus be constructed. We derive

$$\varphi_n(x) = e^{1/p(1+x)^{-p}-1/p}(1+x)^{-n+1} \quad (n = 1, 2, \ldots).$$

The corresponding operators can be defined on the interval $[0, \infty)$ and we have

$$\begin{cases} L_{n,p}(1;\, x_0) &= 1, \\ L_{n,p}(t;\, x_0) &= x_0, \\ L_{n,p}(t^2;\, x_0) &= x_0{}^2 + \dfrac{x_0(1+x_0)}{n} + \dfrac{px_0{}^2}{n^2}. \end{cases}$$

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# SAMENVATTING VAN DE INHOUD

Dit proefschrift handelt over lineaire positieve operatoren in de approximatietheorie.

In de inleiding wordt een lineaire positieve operator gedefinieerd, daarna vermelden wij enkele resultaten waarvan in dit proefschrift wordt uitgegaan.

In hoofdstuk 1 worden enkele fundamentele stellingen afgeleid, waarna wij ingaan op een methode om lineaire positieve operatoren $L_n$ te construeren. Voor een aldus gevormde rij $\{L_n\}$ $(n = 1,2,\ldots)$ wordt vervolgens een schatting gegeven van het verschil

$$|L_n(f; x_0) - f(x_0)|.$$

Verder wordt een aantal methoden om asymptotische approximatieformules af te leiden met elkaar vergeleken.

Hoofdstuk 2 is gewijd aan de approximatie van functies van meer veranderlijken. Hier vindt men een aantal resultaten van hoofdstuk 1 gegeneraliseerd. Daar wordt ook een schatting gegeven van een constante die optreedt in de theorie van gegeneraliseerde Bernstein polynomen.

Hoofdstuk 3 heeft betrekking op lineaire positieve operatoren $L_{np-p}$ van de vorm

$$L_{np-p}(f; x) = \frac{1}{\displaystyle\int_{-\pi}^{+\pi} \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt} \int_{-\pi}^{+\pi} f(x+t) \left(\frac{\sin {}^1/_2 nt}{\sin {}^1/_2 t}\right)^{2p} dt \quad (n = 1,2,\ldots),$$

die van belang zijn bij de approximatie van continue periodieke functies met periode $2\pi$. In de gevallen $p = 3,4,5,6$ wordt een aantal eigenschappen van deze operatoren afgeleid.

In hoofdstuk 4 wordt het onderzoek betreffende het genereren van lineaire positieve operatoren, zoals uitgevoerd in hoofdstuk 1, voortgezet. Uitgaande van een stelsel recurrente differentiaalvergelijkingen worden met behulp van z.g. startfuncties rijen van functies geconstrueerd die rijen van lineaire positieve operatoren voortbrengen.

In de hoofdstukken 1–4 zijn tal van toepassingen van de hier bewezen stellingen opgenomen.