The Mathematical Analysis of Problems Describing the Dynamics of Pipes Conveying Fluids

by

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Contents

1 Abstract

In this Masters thesis, the dynamics of pipes conveying pulsating flow are investigated. The initial-boundary value problem associated with the linear beam equations of motions governing the pipe system is derived using the principles of Lagrangian mechanics. In this thesis, the fluid flow is assumed to have a small velocity with harmonic time dependence $V(t) = \varepsilon(V_0 + V_1 \sin(\Omega t))$, which allows us to investigate the effects of different pulsation frequencies on the pipe system. For certain Ω frequencies, the pipe system is observed to be exposed to more complex dynamical behaviours. By using the multiple time scale perturbation method, comprehensive insights into the stability and the dynamic behaviour of pipe systems are achieved.

The study focuses on investigating the primary resonance frequencies and understanding how pulsation frequencies near those resonance frequencies impact the stability of the system. Furthermore, we elaborate on special resonance cases where multiple oscillatory modes interact leading to even more complicated dynamics.

By building upon existing literature this research enhances our understanding of stability and dynamic behaviors under various flow pulsation frequencies. This study makes an important contribution to the present literature by exploring scenarios where multiple resonant modes interact, due to coinciding primary resonance frequencies, which has not been extensively discussed in the literature. Our findings suggests scepticism on the relevance of the existing solution methods and results in the literature for certain parameter values.

2 Introduction

In numerous industries and infrastructural systems, pipe structures that convey fluids are essential to various operational processes. Whether in the applications of urban water distribution, the transportation and extraction of oil and gas in refineries and offshore platforms, or the fuel supply mechanisms of engine systems, pipes stand as the backbone of numerous industrial applications. The fluctuations in the fluid velocity can lead to vibrations in the pipe structures. This can be due to the operation of equipment like reciprocating pumps or inherent fluid excitation fluctuations. These vibrations present potential challenges in terms of system integrity, safety, and longevity. Understanding the mechanisms behind such systems and their real-world implications is crucial for efficient system design and operation.

The study of the dynamics of pipes conveying fluids has a rich history dating back to 1885 when Brillouin first began investigating the subject. However, the first written study on the topic did not appear until the late 1930s, when Bourrieres published his work [5]. In his work in 1939, Bourrieres treated the pipe as a string-like structure and made conclusions on the stability of the cantilevered system. The research topic was re-initiated in 1950 by Ashley and Haviland, who further developed this model, during their research on the Trans Arabic pipeline, for cantilevered pipes, treating the pipe as a beam-like structure and considering its bending stiffness [3]. In the following years, Feodos'ev [8], Housner [12], and Niordson [16] made significant contributions to the field, by studying the dynamics of pipes supported at both ends, as they were among the first to independently derive correct linear equations of motion using different methods, and reached the same conclusions about the stability of the system [19]. In 1976, Païdoussis and Laithier were the first ones that considered the shear deformations and modelled the system as a Timoshenko beam, which is a higher order and more accurate model [21].

The dynamics of pipes conveying fluid can be explored using equations from two principal branches of analytical mechanics: Newtonian and Lagrangian mechanics. Both methods lead to the same governing differential equations. However, the distinct advantage of the Lagrangian approach lies in its boundary condition treatment. While the Newtonian approach requires careful assumptions on the determination of boundary conditions, the Lagrangian method naturally provides these conditions derived from scalar quantities like kinetic and potential energy. This advantage of the Lagrangian method simplifies the problem formulation, making it the preferred method for our study.

Benjamin (1961) presented one of the first complete Lagrangian derivations of the linear equations of motion in the literature, considering the pipe as an Euler-Bernoulli beam with constant fluid velocity and neglecting viscoelastic damping and gravity [4]. In 1971, Chen was the first to derive the equations for pulsating fluid flow [7], and later in 1974, Païdoussis and Issid corrected this model for pulsating fluid flow, and improved it by including the Kelvin-Voigt viscoelastic effects of the pipe material [20]. In 1994, Semler and Païdoussis derived the nonlinear equations of motion by considering partial derivatives of longitudinal deflections and including higher-order terms for the curvature term in the expression for strain energy [26]. The effects of viscoelasticity have been studied in detail for accelerating beams by Chen, Tang, Zhang, and others [6,31,33].

In 2005, Païdoussis and his colleagues, and Kuiper and Metrekine revisited the direction of mean flow velocity at the inlet and derived the non-classical free end boundary condition for aspirating pipes from Newtonian principles [13, 18]. Aldraihem obtained the same boundary conditions as Païdoussis et al. in 2007 using the variational approach and called it the "flow out release effect" [2]. In his study, Aldraihem also adjusted the free-end boundary conditions for Kelvin-Voigt viscoelastic pipes, which were previously not considered. Furthermore, the effects of viscoelasticity on natural boundary conditions have also been studied for accelerated beam problems, that are similar to problems of pipes conveying fluid, by Chen, Tang, Zhang, and others [6, 31, 33]. Furthermore, van Horssen, Gaiko, Sandilo and Akkaya studied dashpot-spring-mass non-classical boundary conditions for string-like problems [1, 9, 25].

The Galerkin truncation method, established by Gregory and Païdoussis in 1966 as a solution approach for the existing equations of pipes conveying fluids [10], provides a truncated solution of the equations that consist of often, up to four modes of oscillations. This method has been widely used as the primary tool to solve equations related to pipes conveying fluids. In their later study in 1991, Païdoussis and his colleagues highlighted some quantitative differences between existing solutions obtained by the truncation method and experimental results for certain parameter values, as reported in [22].

In their studies, Suweken, Ponomareva, and van Horssen have shown that for conveyor belt problems, which are governed by string-like equations similar to those of pipes conveying fluids, the truncation method may not provide accurate approximations of the solutions on long-time scales for certain parameter values [23, 24, 28–30]. This is due to the presence of internal resonances, in which all modes interact. Considering only the first N modes and truncating the higher-order modes avoids the appearance of interactions of higher-order modes and consequently, neglects their contribution to the actual system. In these studies, authors have made improvements for the solutions of conveyor belt problems for various boundary conditions. In their study, Gaiko and van Horssen (2015) also highlighted that, for string-like equations, with the absence of bending stiffness, the Galerking truncation is only applicable for $t \sim \mathscr{O}(1)$ [9].

In string-like equations, van Horssen and Ponomareva proposed the solution method by Laplace transform as an alternative solution method to eigenfunction expansion [32]. In 2015, Malookani and van Horssen compared the solution obtained by the method of multiple scales and Laplace transform with Galerkin's truncation method to axially moving strings with varying axial velocity [14]. In the context of beam-like equations, Suweken and van Horssen employed the method of multiple time scales for conveyor belt problems, highlighting potential interactions with higher-order modes for specific parameter values [29]. In a subsequent study [30], they used this approach to examine the stability of beam structures governed by nonlinear equations of motion.

Despite the advancements in the literature on pipes conveying fluids, certain aspects remain to be addressed more thoroughly. Where the main solution method in the literature is the Galerkin truncation method, our study aims to analyze the initial boundary value problem with multiple time scales methods, particularly in the context of flow-induced vibrations. The primary questions guiding this research include: When do internal resonances arise? How do the various oscillatory modes interact? And, what are the limitations of the Galerkin truncation method?

To address the outlined research questions, the equations of motion will be derived using mathematical modelling and the asymptotic approximation of the solution of this problem will be constructed. The thesis is structured as follows: Section 3.1 the mathematical derivations of the cantilevered pipe using the Lagrangian approach, focusing on the determination of energy and virtual work contributions is presented. Section 3.2 adapts these derivations to establish the equations of motion for a simply supported pipe and in section 3.3, the respective initial-boundary value problems are presented. Moving to Section 4.1, the multiple time scales perturbation method is applied to the initial-boundary value problem for the simply supported pipe system. Section 4.2 discusses the occurrence of various resonant cases, while Section 4.3 examines the non-resonant scenario. Sections 4.4 and 4.5 provide detailed analysis of the primary resonant cases and the resonance coincidence cases, respectively. In Section 5, we summarize and reflect on the findings of our study. Finally, Section 6 presents possible directions for future research.

3 Derivation of Equations of Motion

In this study, we will use the Lagrangian approach to obtain the linear equation of motion for a cantilevered pipe conveying fluid. We assume the transverse deflections are small and that the pipe material is sufficiently resistant to bending, which leads to the Euler-Bernoulli beam model. To simplify our analysis, we assume that the pipe has uniform material properties and cross-section in the longitudinal direction. To account for the viscoelastic damping effects of the pipe material, we will initially consider a more general derivation. In addition, we will consider the flow velocity inside the pipe to be small and time-dependent to study non-classical flow cases, such as pulsating flow represented by $V(t) = \varepsilon(V_0 + V_1 \sin(\Omega t))$, where $0 < \varepsilon \ll 1$.

3.1 Cantilevered Pipe

In this subsection, we will derive the equation of motion for a cantilevered pipe by building upon the works of Benjamin [4], Païdoussis and Issid [20], Semler and Païdoussis [26], and Chen and his colleagues [6]. We have adjusted the derivation of linear equations and natural boundary conditions, resulting in a more comprehensive model.

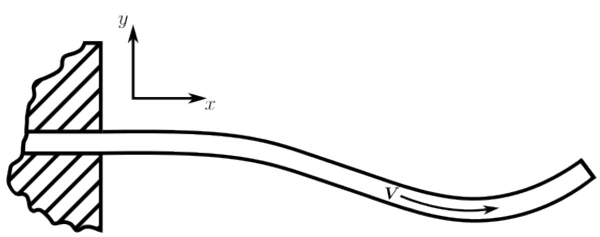

Figure 1: Schematic diagram of the cantilever pipe system.

Inextensibility Condition

For the cantilevered pipe, due to the absence of external axial tension, we assume that the pipe preserves its initial length.

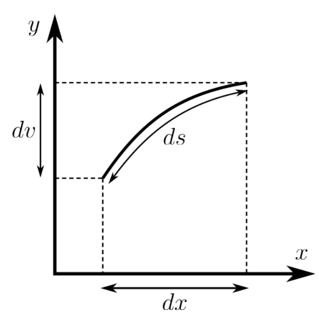

Figure 2: Infinitesimal element of pipe.

If we assume that the pipe is inextensible, we can write $(ds)^2 = (dx)^2 + (dv)^2$ and therefore, the horizontal displacement of the deflected pipe can be written as

$$u(x,t) = \int_0^x ds - dx = \int_0^x \sqrt{(dx)^2 + (dv)^2} - dx = \int_0^x \left(\sqrt{1+v_x^2} - 1\right) dx. \tag{1}$$

Due to the assumption of small deformation ($|v_x| \ll 1$), we can write the Taylor expansion of the square root term as

$$\sqrt{1+x^2} = 1 + \frac{x^2}{2} + \mathscr{O}(x^4).$$

Hence, we obtain the horizontal displacement of the pipe as

$$u(x,t) \simeq \int_0^x \frac{1}{2} v_x^2 dx. \tag{2}$$

Energy Method

The equations of motion will be derived using Hamilton's principle, often referred to as the energy method [26]. Hamilton's principle for the open system can be written as,

$$\delta \int_{t_1}^{t_2} \mathscr{L} dt + \int_{t_1}^{t_2} \delta \mathscr{W} dt = 0, \tag{3}$$

where δ denotes the variation of a function, $\mathscr{L}$ is the Lagrangian of the system and defined as $\mathscr{L} = \mathscr{T} - \mathscr{V}$, where $\mathscr{T}$ and $\mathscr{V}$ are the kinetic energy and potential energy respectively, and $\delta\mathscr{W}$ is the virtual work done by non-conservative forces that are not included in the Lagrangian [26].

Hamilton's principle states that among all possible paths between the end points, the motion of a system will occur along the path that gives an extreme value to the integral

$$I(\boldsymbol{u}) = \int_{t_1}^{t_2} F(\boldsymbol{u}) dt, \tag{4}$$

for arbitrary times t_1 and t_2. Assuming that $\boldsymbol{u}$ minimizes the Hamiltonian integral, let us consider the true evolution of the system as $\bar{\boldsymbol{u}} = \boldsymbol{u} + \varepsilon\boldsymbol{\mu}$, where $\varepsilon \in \mathbb{R}$ and $\boldsymbol{u} \in \mathbb{R}^n$, $\varepsilon\delta\boldsymbol{u}$ is the variation or perturbation of the function and $\boldsymbol{\mu}$ is the perturbation direction. In the literature, the perturbation direction is often denoted as $\boldsymbol{\mu} = \delta\boldsymbol{u}$, and in this study, we will also use the notation $\bar{\boldsymbol{u}} = \boldsymbol{u} + \varepsilon\delta\boldsymbol{u}$ to stay consistent with the notation in the literature. The function $\delta\boldsymbol{u}$ is arbitrary, differentiable, and vanishes at the end time points t_1 and t_2, that is, $\delta\boldsymbol{u}(\boldsymbol{x}, t_1) = \delta\boldsymbol{u}(\boldsymbol{x}, t_2) = \boldsymbol{0}$ [9].

Since the functional $I = I(\boldsymbol{u})$ has a minimum at $\boldsymbol{u}$, the functional $I(\boldsymbol{u} + \varepsilon\delta\boldsymbol{u})$ also has a minimum at $\varepsilon = 0$. Therefore, we have

$$\delta I(\boldsymbol{u}; \delta\boldsymbol{u}) = \frac{d}{d\varepsilon}(I(\boldsymbol{u} + \varepsilon\delta\boldsymbol{u}))\bigg|_{\varepsilon=0} = 0. \tag{5}$$

This definition is also known as the Gateaux derivative [27].

3.1.1 Kinetic Energy

The kinetic energy of the Lagrangian consists of two components, such that $\mathscr{T} = \mathscr{T}_p + \mathscr{T}_f$, where $\mathscr{T}_p$ refers to the kinetic energy of the pipe and $\mathscr{T}_f$ refers to the kinetic energy of the fluid.

Kinetic Energy of Pipe $\mathscr{T}_p$
The velocity vector of a pipe element is defined as

$$\boldsymbol{V}_p = u_t \boldsymbol{i} + v_t \boldsymbol{j}. \tag{6}$$

If we assume that the longitudinal velocity component (along the x-axis) of a pipe element is much smaller compared to the transversal velocity component, by neglecting the contribution of the longitudinal velocity component, the kinetic energy of the pipe can be written as

$$\mathscr{T}_p = \frac{1}{2} m \int_0^L |\boldsymbol{V}_p|^2 dx = \frac{1}{2} m \int_0^L \left(u_t^2 + v_t^2\right) dx \simeq \frac{1}{2} m \int_0^L v_t^2 dx, \tag{7}$$

where m is the mass per unit length of the pipe.

In order to find the kinetic energy component of the Lagrangian in (3), we apply the variational operator to the integral $I = \int_{t_1}^{t_2} (\mathscr{T}_p + \mathscr{T}_f) dt$. Hence, for the $\mathscr{T}_p$ component, we can write the Gateaux derivative of $I = \int_{t_1}^{t_2} \mathscr{T}_p dt$ as

$$\begin{aligned}
I(v + \varepsilon \delta v) =& \frac{1}{2} m \int_{t_1}^{t_2} \int_0^L (v_t + \varepsilon \delta v_t)^2 dx dt \\
=& \frac{1}{2} m \int_{t_1}^{t_2} \int_0^L (v_t^2 + 2\varepsilon v_t \delta v_t + \mathscr{O}(\varepsilon^2)) dx dt, \\
\Rightarrow \frac{d}{d\varepsilon} I(v + \varepsilon \delta v)\bigg|_{\varepsilon=0} =& \left(\frac{1}{2} m \int_{t_1}^{t_2} \int_0^L (2 v_t \delta v_t + \mathscr{O}(\varepsilon)) dx dt\right)\bigg|_{\varepsilon=0} \\
=& m \int_{t_1}^{t_2} \int_0^L v_t \delta v_t dx dt \\
=& m \int_0^L \underbrace{[v_t \delta v]_{t_1}^{t_2}}_{=0} dx - m \int_{t_1}^{t_2} \int_0^L v_{tt} \delta v dx dt \\
=& - m \int_{t_1}^{t_2} \int_0^L v_{tt} \delta v dx dt.
\end{aligned} \tag{8}$$

Kinetic Energy of Fluid $\mathscr{T}_f$
The velocity of the fluid element is the combination of the relative velocity of the fluid to pipe, $V\boldsymbol{\tau}$, and the velocity of the pipe (6). $V = V(t)$ is the amplitude of the average fluid velocity inside the pipe and $\boldsymbol{\tau}$ is the tangential vector of relative fluid velocity defined as $\boldsymbol{\tau} = \boldsymbol{i} \cos\theta + \boldsymbol{j} \sin\theta$. By assuming that the deflections are small, we will approximate $\sin\theta \simeq v_x + \mathscr{O}(v_x^3)$ and $\cos\theta \simeq 1 - v_x^2/2 + \mathscr{O}(v_x^4)$. Hence, we obtain the absolute fluid velocity as

$$\begin{aligned}
\boldsymbol{V}_f =& \boldsymbol{V}_p + V\boldsymbol{\tau} \\
=& (V \cos\theta - u_t)\boldsymbol{i} + (V \sin\theta + v_t)\boldsymbol{j} \\
\simeq& \left(V\left(1 - v_x^2/2\right) - u_t\right)\boldsymbol{i} + (V v_x + v_t)\boldsymbol{j}.
\end{aligned} \tag{9}$$

From fluid velocity (9), the kinetic energy of the enclosed fluid is expressed as

$$\mathscr{T}_f = \frac{1}{2}M\int_0^L |\boldsymbol{V}_f|^2 dx = \frac{1}{2}M\int_0^L \left(\left(V\left(1-v_x^2/2\right)-u_t\right)^2 + (Vv_x+v_t)^2\right)dx, \tag{10}$$

with M being the mass per unit length of the fluid. If the higher order terms are neglected, we can write (10) as

$$\mathscr{T}_f \simeq \frac{1}{2}M\int_0^L \left(V^2+v_t^2+2Vv_xv_t-2Vu_t\right)dx. \tag{11}$$

By using the definition for the longitudinal deflection term given in (2), the last term in equation (11) can be rewritten as

$$\begin{aligned} 2\int_0^L Vu_t dx &= \int_0^L V\frac{\partial}{\partial t}\left(\int_0^x v_x^2 dx\right)dxdt \\ &= \left\{\left[xV\frac{\partial}{\partial t}\left(\int_0^x v_x^2dx\right)\right]_0^L - \int_0^L xV\frac{\partial}{\partial t}(v_x^2)dx\right\}dt \\ &= \int_0^L (L-x)V\frac{\partial}{\partial t}(v_x^2)dxdt. \end{aligned} \tag{12}$$

By plugging (12) into (11) and integrating with respect to t from t_1 to t_2, we obtain

$$\int_{t_1}^{t_2}\mathscr{T}_f dt = M\int_{t_1}^{t_2}\int_0^L\left(\frac{1}{2}V^2+\frac{1}{2}v_t^2+Vv_xv_t-\frac{1}{2}(L-x)V(v_x^2)_t\right)dxdt. \tag{13}$$

Now, we can write the Gateaux derivative of $I=\int_{t_1}^{t_2}\mathscr{T}_f dt$, which results in

$$\begin{aligned} I(v+\varepsilon\delta v) =& M\int_{t_1}^{t_2}\int_0^L\left(\frac{1}{2}V^2+\frac{1}{2}(v_t+\varepsilon\delta v_t)^2+V(v_x+\varepsilon\delta v_x)(v_t+\varepsilon\delta v_t)\right. \\ &\left. -\frac{1}{2}(L-x)V((v_x+\varepsilon\delta v_x)^2)_t\right)dxdt, \\ \Rightarrow \frac{d}{d\varepsilon}I(v+\varepsilon\delta v)\bigg|_{\varepsilon=0} =& \left[M\int_{t_1}^{t_2}\int_0^L(v_t\delta v_t+V(v_x\delta v_t+v_t\delta v_x)-(L-x)V(v_x\delta v_x)_t+\mathscr{O}(\varepsilon))dxdt\right]\bigg|_{\varepsilon=0} \\ =& M\int_{t_1}^{t_2}\int_0^L(v_t\delta v_t+V(v_x\delta v_t+v_t\delta v_x)-(L-x)V(v_x\delta v_x)_t)dxdt. \end{aligned} \tag{14}$$

If we apply integration by parts, we obtain

$$\begin{aligned} \delta\int_{t_1}^{t_2}\mathscr{T}_f dt =& -M\int_{t_1}^{t_2}\int_0^L(v_{tt}+2Vv_{xt}+V_t(L-x)v_{xx})\delta v dxdt \\ &+M\int_{t_1}^{t_2}\left([Vv_t\delta]_0^L - MV_tLv_x(0,t)\delta v(0,t)\right)dt. \end{aligned} \tag{15}$$

3.1.2 Potential Energy

The potential energy of the system consists of the strain energy of the pipe $\mathcal{V}_s$ and the gravitational potential energy $\mathcal{V}_g$, and we write $\mathcal{V} = \mathcal{V}_s + \mathcal{V}_g$.

Strain Energy $\mathcal{V}_s$

We assume our pipe structure to remain in the linear (or elastic) deformation region under loading. Thus, the stress-strain relation is expressed as $\sigma = E\varepsilon$, known as Hooke's law, where σ and ε are stress and strain respectively. We consider the strain energy $\mathcal{V}_s$ in the form

$$\delta \int_{t_1}^{t_2} \mathcal{V}_s dt = \int_{t_1}^{t_2} \int_0^L \int_S (\sigma\delta\varepsilon) dS dx dt, \tag{16}$$

see [6,31,34]. From the definition of strain, $\varepsilon = y/r = y\kappa = yv_{xx}$, r being the radius of curvature and $\kappa = v_{xx}$ being the curvature, (16) can be given by

$$\begin{aligned} \int_{t_1}^{t_2} \int_0^L \int_A E\varepsilon(\delta\varepsilon) dA dx dt &= \int_{t_1}^{t_2} \int_0^L \int_A Ey(v_{xx})\delta(yv_{xx}) dA dx dt \\ &= \int_{t_1}^{t_2} \int_0^L \left(\int_A y^2 dA \right) E v_{xx} \delta v_{xx} dx dt, \end{aligned} \tag{17}$$

and where $I = \int_A y^2 dA$ is defined as the second moment of inertia, (17) becomes

$$\delta \int_{t_1}^{t_2} \mathcal{V}_s dt = \int_{t_1}^{t_2} \int_0^L EI v_{xx} \delta v_{xx} dx dt. \tag{18}$$

By applying integration by parts twice, we can obtain

$$\delta \int_{t_1}^{t_2} \mathcal{V}_s dt = \int_{t_1}^{t_2} \int_0^L EI v_{xxxx} \delta v dx dt + \int_{t_1}^{t_2} EI[v_{xx}\delta v_x]_0^L - \int_{t_1}^{t_2} EI[v_{xxx}\delta v]_0^L. \tag{19}$$

Gravitational Potential Energy $\mathcal{V}_g$

• *Horizontal Pipe*

The gravitational potential energy $\mathcal{V}_g$ when the pipe is horizontally placed (or perpendicular to the vector of gravitational acceleration) is given by

$$\mathcal{V}_g = \int_0^L (m+M) g v dx, \tag{20}$$

where g is the gravitational acceleration. Hence, we can write $I = -\int_{t_1}^{t_2} \int_0^L (m+M) g v dx dt$, and from the Gateaux derivative, we obtain

$$\delta \int_{t_1}^{t_2} \mathcal{V}_g dt = \frac{d}{d\varepsilon} I(v + \varepsilon\delta v)\bigg|_{\varepsilon=0} = \int_{t_1}^{t_2} \int_0^L (m+M) g \delta v dx dt. \tag{21}$$

• *Vertical Pipe*
For the case that the undeflected pipe is vertical, the gravitational potential energy is given by

$$\mathcal{V}_g = \int_0^L (m+M) g u dx = \frac{1}{2}(m+M) g \int_0^L \left(\int_0^x v_x^2 dx \right) dx. \tag{22}$$

Now if we apply integration by parts as in (12), (22) becomes

$$\mathcal{V}_g = \frac{1}{2}(m+M) g \int_0^L (L-x) v_x^2 dx. \tag{23}$$

From this expression, we can define the integral $I = \int_{t_1}^{t_2} \mathcal{V}_g dt$ and from the Gateaux derivative, we obtain

$$\delta \int_{t_1}^{t_2} \mathcal{V}_g dt = \frac{d}{d\varepsilon} I(v + \varepsilon \delta v) \bigg|_{\varepsilon=0} = \int_{t_1}^{t_2} \int_0^L (m+M) g (L-x) v_x \delta v_x dx dt \tag{24}$$

and from integration by parts, we get

$$\begin{aligned} \delta \int_{t_1}^{t_2} \mathcal{V}_g dt &= \int_{t_1}^{t_2} (m+M) g [(L-x) v_x \delta v]_0^L - \int_{t_1}^{t_2} \int_0^L (m+M) g (L-x) v_{xx} \delta v dx dt \\ &= \int_{t_1}^{t_2} (m+M) g L v_x(0,t) \delta v(0,t) dt - \int_{t_1}^{t_2} \int_0^L (m+M) g (L-x) v_{xx} \delta v dx dt. \end{aligned} \tag{25}$$

3.1.3 Virtual Work by Non-conservative Forces

Viscoelastic Damping
In this study, viscoelastic damping of the pipe is assumed to follow the Kelvin-Voigt model. The stress-strain relation of a Kelvin-Voigt viscoelastic material is proposed as

$$\sigma = E\varepsilon + \eta \varepsilon_t, \tag{26}$$

where η is the viscoelastic damping coefficient. Let $\sigma_{KV} = \eta \varepsilon_t$ be the Kelvin-Voigt viscoelastic component of stress. Again by using the definition of strain, we obtain

$$\sigma_{KV} = \eta y v_{txx}. \tag{27}$$

Hence, we can write the expression corresponding to the virtual work term as

$$\begin{aligned} \int_{t_1}^{t_2} \delta W_{KV} dt &= -\int_{t_1}^{t_2} \int_0^L \int_A (\sigma_{KV} \delta \varepsilon) dA dx dt \\ &= -\int_{t_1}^{t_2} \int_0^L \int_A \eta y v_{txx} \delta (y v_{xx}) dA dx dt \\ &= -\int_{t_1}^{t_2} \int_0^L \eta I v_{txx} \delta v_{xx} dx dt. \end{aligned} \tag{28}$$

If we apply integration by parts twice, we can obtain

$$\int_{t_1}^{t_2} \delta W_{KV} dt = -\int_{t_1}^{t_2}\int_0^L \eta I v_{txxxx}\delta v dx dt - \int_{t_1}^{t_2} \eta I [v_{txx}\delta v_x]_0^L dt + \int_{t_1}^{t_2} \eta I [v_{txxx}\delta v]_0^L dt. \tag{29}$$

Fluid Discharge From Outlet

The virtual work done by the discharge of fluid from the outlet, δW_d is expressed as the change of momentum times virtual displacement, hence $\delta W_d = -MV(\dot{\boldsymbol{R}} + U\boldsymbol{\tau})\cdot\delta\boldsymbol{R}$ [4, 15]. The dot-

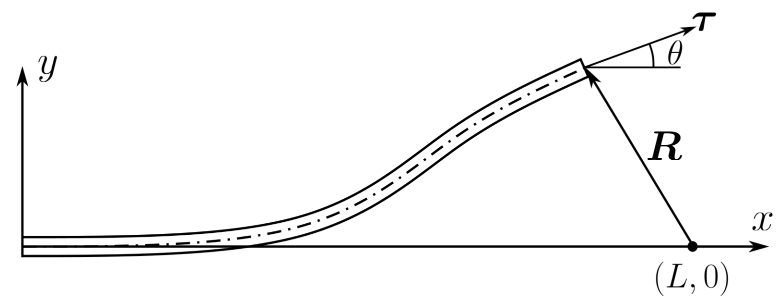

Figure 3: Schematic diagram of the cantilever pipe system.

dashed curve corresponds to the natural axis of the deflected pipe of length L, $\boldsymbol{\tau}$ represents the unit normal vector of the pipe outlet and the $\boldsymbol{R}$ is the displacement vector of the outlet from the undeflected pipe (see Figure 3). We define $\boldsymbol{R} := -u(L,t)\boldsymbol{i} + v(L,t)\boldsymbol{j}$ and from the previous definition of $\boldsymbol{\tau}$, $\boldsymbol{\tau}_L := (1 - v_x^2(L,t)/2)\boldsymbol{i} + v_x(L,t)\boldsymbol{j}$. From these definitions, the virtual work δW_d can be written as

$$\begin{aligned}\int_{t_1}^{t_2} \delta W_d dt &= -\int_{t_1}^{t_2}\left[MV\left(V - u_t - V\frac{1}{2}v_x^2\right)\delta u(L,t) - MV(v_t + Vv_x)\delta v(L,t)\right]dt \\ &\simeq -\int_{t_1}^{t_2}\big(MV^2\delta u(L,t) - MV(v_t + Uv_x)\delta v(L,t)\big)dt.\end{aligned} \tag{30}$$

If we use the longitudinal deflection expression from (2) for the first term in the integral in (30), we get

$$\int_{t_1}^{t_2} \delta W_d dt = -\int_{t_1}^{t_2}\left(MV^2\delta\left(\int_0^L \frac{1}{2}v_x^2 dx\right) - MV(v_t + Vv_x)\delta v(L,t)\right)dt. \tag{31}$$

If we apply the Gateaux derivative to the integral $\int_0^L \frac{1}{2}v_x^2 dx$, we get

$$\delta\left(\int_0^L \frac{1}{2}v_x^2 dx\right) = \frac{d}{d\varepsilon}\left(\int_0^L \frac{1}{2}(v_x + \varepsilon\delta v_x)^2 dx\right)\bigg|_{\varepsilon=0} = \int_0^L v_x\delta v_x dx. \tag{32}$$

And from integration by parts, we obtain

$$\int_0^L v_x\delta v_x dx = [v_x\delta v]_0^L - \int_0^L v_{xx}\delta v dx. \tag{33}$$

Thus (31) becomes

$$\int_{t_1}^{t_2} \delta W_d dt = \int_{t_1}^{t_2} \int_0^L MV^2 v_{xx} \delta v dx dt - \int_{t_1}^{t_2} \left(MV^2 [v_x \delta v]_0^L + MV(v_t + V v_x) \delta v(L,t) \right) dt. \quad (34)$$

3.1.4 Equation of Motion and Boundary Conditions

If we substitute, $L = \mathscr{T}_p + \mathscr{T}_f - \mathscr{V}_s - \mathscr{V}_g$ and $\delta W = \delta W_{KV} + \delta W_d$ into (3), then we obtain

$$\begin{aligned} &-\int_{t_1}^{t_2} \int_0^L \big(\eta I v_{txxxx} + EI v_{xxxx} + MV_t (L-x) v_{xx} + MV^2 v_{xx} \\ &\qquad + 2MV v_{xt} + (m+M) v_{tt} + (m+M) g \big) \delta v dx dt \\ &+ \int_{t_1}^{t_2} \big(M[MV v_t \delta v]_0^L - MV_t L v_x(0,t) \delta v(0,t) - EI[v_{xx} \delta v_x]_0^L \\ &\qquad + EI[v_{xxx} \delta v]_0^L - \eta I [v_{txx} \delta v_x]_0^L + \eta I [v_{txxx} \delta v]_0^L \\ &\qquad + MV^2 [v_x \delta v]_0^L - MV(v_t(L,t) + V v_x(L,t)) \delta v(L,t) \big) \, dt = 0. \end{aligned} \quad (35)$$

The equation of motion for a horizontally cantilevered pipe is obtained from the first integral in (35) and is given by

$$\eta I v_{txxxx} + EI v_{xxxx} + \left(MV_t (L-x) + MV^2 \right) v_{xx} + 2MV v_{xt} + (m+M) v_{tt} + (m+M) g = 0. \quad (36)$$

The second integral in (35) leads to the natural (or force) boundary conditions and can be written as

$$x = 0 \Rightarrow \begin{cases} MV(v_t + V v_x) + MV_t L v_x + EI v_{xxx} + \eta v_{txxx} = 0, \\ EI v_{xx} + \eta I v_{txx} = 0, \end{cases} \quad (37a)$$

$$x = L \Rightarrow \begin{cases} EI v_{xxx} + \eta v_{txxx} = 0, \\ EI v_{xx} + \eta I v_{txx} = 0. \end{cases} \quad (37b)$$

The terms in (36) represent viscoelasticity, bending strain, acceleration effect of the fluid, centrifugal effect, Coriolis effect, local transverse acceleration and gravitational force respectively. It must be highlighted that the first boundary condition in (37a) is the same as the inflow condition proposed by Païdoussis [18], Kuiper and Metrikine [13] if we assume V =const. and $\eta = 0$. Also, the difference between the left and right boundary condition is termed the outflow release effect by Aldraihem [2].

3.1.5 Dimensional Analysis

• *Horizontal Cantilevered Pipe*

Before we study the partial differential equation with boundary conditions that are obtained

(36) and (37a) & (37b), we must first convert the equation in dimensionless form and we will use the dimensionless groups to determine a scaling [11].

$$\begin{aligned}[E] &= \frac{M}{LT^2}, \quad [\eta] = \frac{M}{LT}, \quad [I] = L^4, \quad [g] = \frac{L}{T^2}, \quad [V] = \frac{L}{T}, \\ [M] &= [m+M] = \frac{M}{L}, \quad [x] = L, \quad [v] = L, \quad [t] = T.\end{aligned} \tag{38}$$

And now, in order to nondimensionalize the problem, we introduce the change of variables

$$t = t_c t^*, \quad x = x_c x^*, \quad v = v_c v^*, \tag{39}$$

where the asterisk superscript denotes the dimensionless variable and the c subscript denotes the characteristic value of the given variable, which is also constant. By using the chain rule, we substitute the dimensionless variables (39) into the equation (36). Each term of the equation is in units of $\frac{M}{T^2}$. Thus, we multiply each term by $\frac{t_c^2}{(m+M)v_c}$ in order to make each term dimensionless. We obtain

$$\begin{aligned}&\frac{\eta I t_c}{(m+M)x_c^4} v^*_{txxxx} + \frac{EIt_c^2}{(m+M)x_c^4} v^*_{xxxx} + \frac{MV_t(L - x_c x^*)t_c}{(m+M)x_c^2} v^*_{xx} \\ &+ \frac{MV^2 t_c^2}{(m+M)x_c^2} v^* + \frac{2MVt_c}{(m+M)x_c t_c} v^*_{xt} + v^*_{tt} + \frac{g t_c^2}{v_c} = 0.\end{aligned} \tag{40}$$

We choose $x_c = v_c = L$, which is the constant geometrical parameter. To determine the characteristic value for time, denoted by t_c, we follow existing literature and notation conventions, such as [17, 20], by selecting the coefficient of the second term in (40) and setting it equal to one. Once we have determined t_c in this manner, we can obtain the dimensionless parameters as follows:

$$\begin{aligned}x^* &= \frac{x}{L}, \quad v^* = \frac{v}{L}, \quad t^* = \sqrt{\frac{EI}{m+M}}\frac{t}{L^2}, \quad \alpha = \sqrt{\frac{I}{E(m+M)}}\frac{\eta}{L^2} \\ \beta &= \frac{M}{m+M}, \quad V^* = \sqrt{\frac{M}{EI}} VL, \quad \gamma = \frac{m+M}{EI} L^3 g.\end{aligned} \tag{41}$$

For convenience, we will not use the asterisk notation, and with the dimensionless parameters defined in (41), we obtain the dimensionless problem (36) with (37a) & (37b) as

$$\alpha v_{txxxx} + v_{xxxx} + \left(V^2 + \sqrt{\beta} V_t (1-x)\right) v_{xx} + 2\sqrt{\beta} V v_{xt} + v_{tt} + \gamma = 0, \tag{42}$$

with the natural boundary conditions,

$$x = 0 \Rightarrow \begin{cases} V(\sqrt{\beta} v_t + V v_x) + \sqrt{\beta} V_t v_x + v_{xxx} + \alpha v_{txxx} = 0, \\ v_{xx} + \alpha v_{txx} = 0, \end{cases} \tag{43a}$$

$$x = 1 \Rightarrow \begin{cases} v_{xxx} + \alpha v_{txxx} = 0, \\ v_{xx} + \alpha v_{txx} = 0. \end{cases} \tag{43b}$$

• *Vertical Cantilevered Pipe*
If we follow the same steps with the potential energy of a vertical pipe, we obtain

$$\alpha v_{txxxx} + v_{xxxx} + \left(V^2 + \left(\sqrt{\beta} V_t - \gamma\right)(1-x)\right) v_{xx} + 2\sqrt{\beta} V v_{xt} + v_{tt} = 0, \tag{44}$$

with the natural boundary conditions,

$$x = 0 \Rightarrow \begin{cases} V\sqrt{\beta} v_t + \left(V^2 + \sqrt{\beta} V_t + \gamma\right) v_x + v_{xxx} + \alpha v_{txxx} = 0, \\ v_{xx} + \alpha v_{txx} = 0, \end{cases} \tag{45a}$$

$$x = 1 \Rightarrow \begin{cases} v_{xxx} + \alpha v_{txxx} = 0, \\ v_{xx} + \alpha v_{txx} = 0. \end{cases} \tag{45b}$$

3.2 Simply Supported Pipe

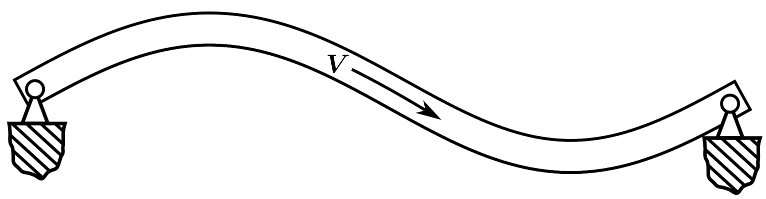

Figure 4: Schematic diagram of the simply supported pipe system.

In this subsection, we will derive the equation of motion for a simply supported pipe conveying fluid by introducing adjustments to the cantilevered pipe. For this case, the inextensibility condition is no longer valid due to the fact that both ends of the pipe are pinned, and a vertical deflection results in a change in the length of the pipe. Furthermore, we can no longer consider virtual work due to fluid discharge, hence $\delta W_d = 0$.

We consider the elongation ϵ of the pipe due to transversal deflection as the same expression for the inextensibility condition given in (2), namely

$$\epsilon = \frac{ds}{dx} - 1 = \sqrt{1 + v_x^2} - 1 = \frac{1}{2} v_x^2 + \mathcal{O}(v_x^4). \tag{46}$$

In contrast to the cantilevered pipe, the fluid velocity V will get a contribution from the elongation of the pipe [17]. With the elongation of the pipe and conservation of mass (also conservation of volume due to the assumption of incompressible fluid flow),

$$S_0 dx = S_1 (1 + \epsilon) dx, \tag{47}$$

where S_0 is the undeflected cross-section and S_1 is the cross-section area of the pipe after elongation. And from the incompressible fluid inside the pipe, we have

$$V_0 S_0 = V_1 S_1, \tag{48}$$

V_0 and V_1 are the velocities before and after the elongation. Therefore, we obtain

$$V_1 = \frac{S_0}{S_1} V_0 = (1+\epsilon) V_0 = (1 + v_x^2/2) V_0 \tag{49}$$

[17]. For a simply supported beam, if we neglect the axial velocity of the beam, we can write the adjusted kinetic energy of the fluid element as

$$\mathscr{T}_f = \frac{1}{2} M \int_0^L \left(V\left(1+v_x^2/2\right)\left(1-v_x^2/2\right)\right)^2 + (v_t + V v_x)^2 dx, \tag{50}$$

If the higher order terms are neglected, we can write (50) as

$$\mathscr{T}_f \simeq \frac{1}{2} M \int_0^L V^2 + v_t^2 + 2V v_x v_t + V^2 v_x^2 dx. \tag{51}$$

From the kinetic energy of the fluid in (51) the contribution of the fluid kinetic energy to the Lagrangian becomes

$$\begin{aligned} \delta \int_{t_1}^{t_2} \mathscr{T}_f dt = &- M \int_{t_1}^{t_2} \int_0^L (v_{tt} + 2V v_{xt} + V_t v_x + V^2 v_{xx}) \delta v dx dt \\ &+ \int_{t_1}^{t_2} [MV v_t \delta v + MV^2 v_x \delta v]_0^L dt. \end{aligned} \tag{52}$$

It should be noted that even in the absence of virtual work done by the fluid discharge from the free end, the $MV^2 v_{xx}$ term remains in the equation due to the contribution of the elongation to the fluid velocity [17], as also obtained in (52).

Another adjustment that is necessary when comparing simply supported pipes to cantilevered pipes is accounting for the contribution of potential energy resulting from external axial tension. Potential energy due to external axial tension P can be defined as follows:

$$\mathscr{V}_t := \int_0^L P\epsilon dx = \int_0^L P \frac{1}{2} v_x^2 dx. \tag{53}$$

For its contribution to the Lagrangian, we get

$$\delta \int_{t_1}^{t_2} \mathscr{V}_t dt = - \int_{t_1}^{t_2} \int_0^L P v_{xx} \delta v dx dt + \int_{t_1}^{t_2} [P v_x \delta v]_0^L dt. \tag{54}$$

3.2.1 Equation of Motion and Boundary Conditions

From the changes above, we obtain the equation of motion and the natural boundary conditions for the simply supported beam in the form:

$$\eta I v_{txxxx} + EI v_{xxxx} + \left(MV^2 - P\right) v_{xx} + 2MV v_{xt} + (m+M) v_{tt} + MV_t v_x + (m+M) g = 0, \tag{55}$$

with the natural boundary conditions,

$$x=0 \Rightarrow \begin{cases} MV(v_t + V v_x) + P v_x + EI v_{xxx} + \eta v_{txxx} = 0, \\ EI v_{xx} + \eta I v_{txx} = 0, \end{cases} \tag{56a}$$

$$x=L \Rightarrow \begin{cases} MV(v_t + V v_x) + P v_x + EI v_{xxx} + \eta v_{txxx} = 0, \\ EI v_{xx} + \eta I v_{txx} = 0. \end{cases} \tag{56b}$$

3.2.2 Dimensional Analysis

We choose the characteristic value for variables $x_c = v_c = L$, and $t_c = \sqrt{\frac{(m+M)}{P}}L$. Following the change of variables, we introduce the dimensionless parameters

$$
\begin{gathered}
x^* = \frac{x}{L}, \quad v^* = \frac{v}{L}, \quad t^* = \sqrt{\frac{P}{m+M}}\frac{t}{L}, \quad \alpha = \frac{\eta I}{\varepsilon L^3 \sqrt{(m+M)P}} \\
\mu = \frac{EI}{PL^2}, \quad \beta = \frac{M}{m+M}, \quad V^* = \sqrt{\frac{M}{P}}V, \quad \gamma = \frac{m+M}{P}Lg,
\end{gathered} \tag{57}
$$

where the dimensionless parameter ε indicates that the viscoelastic damping coefficient η is small. By plugging these dimensionless parameters to the equation (55) and the boundary conditions (56a) and (56b), we obtain

$$
\varepsilon \alpha v_{txxxx} + \mu v_{xxxx} + \left(V^2 - 1\right) v_{xx} + 2\sqrt{\beta} V v_{xt} + v_{tt} + \sqrt{\beta} V_t v_x + \gamma = 0, \tag{58}
$$

with the dimensionless natural boundary conditions

$$
x = 0 \Rightarrow \begin{cases} V(\sqrt{\beta} v_t + V v_x) + v_x + \mu v_{xxx} + \varepsilon \alpha v_{txxx} = 0, \\ \mu v_{xx} + \alpha v_{txx} = 0, \end{cases} \tag{59a}
$$

$$
x = 1 \Rightarrow \begin{cases} V(\sqrt{\beta} v_t + V v_x) + v_x + \mu v_{xxx} + \varepsilon \alpha v_{txxx} = 0, \\ \mu v_{xx} + \alpha v_{txx} = 0. \end{cases} \tag{59b}
$$

3.3 Problem Definition

For physical relevance, we will focus on two specific pipe configurations: a vertical cantilevered pipe and a simply supported horizontal pipe. In both cases, we assume that the fluid velocity inside the pipe has a harmonic variation around the small mean velocity, and we consider it:

$$
V(t) = \varepsilon(V_0 + V_1 \sin(\Omega t)), \tag{60}
$$

where $|V_1| < |V_0|$. This equation characterizes a unidirectional flow, with no reverse flow occurring. We assume that there are no transversal external forces applying to the pipe system. By considering the essential geometrical boundary conditions, we can present the equations of motion along with their corresponding initial and boundary conditions as shown below.

It can be observed that we have maintained the natural boundary conditions for the free-end boundary ($x = 1$) in (62), which corresponds to the classical free-end boundary condition for beam equations when we neglect the viscoelastic damping term. On the other hand, for the left boundary conditions, we have applied the clamped boundary conditions, i.e. $v(0,t) = v_x(0,t) = 0$. These are the essential (or geometrical) boundary conditions that correspond to the clamped beam.

The dimensionless partial differential equation that governs the transversal motion of a vertical cantilevered pipe is represented by:

$$\alpha v_{txxxx} + v_{xxxx} + \left(\varepsilon^2 (V_0 + V_1 \sin(\Omega t))^2 + \left(\varepsilon\sqrt{\beta} V_1 \Omega \cos(\Omega t) - \gamma\right)(1-x)\right) v_{xx} + 2\varepsilon\sqrt{\beta}(V_0 + V_1 \sin(\Omega t)) v_{xt} + v_{tt} = 0, \tag{61}$$

for $0 < x < 1$ and $t > 0$, with the boundary conditions:

$$\begin{aligned} &v(0,t) = v_x(0,t) = 0, \\ &v_{xxx}(1,t) + \alpha v_{txxx}(1,t) = v_{xx}(1,t) + \alpha v_{txx}(1,t) = 0, \end{aligned} \tag{62}$$

for $t > 0$, and with the initial conditions:

$$\begin{aligned} v(x,0) &= \phi(x), \\ v_t(x,0) &= \psi(x), \end{aligned} \tag{63}$$

for $0 < x < 1$.

For a simply supported beam, we also consider the essential boundary conditions for pinned at both ends (65), and we obtain the initial boundary problem.

The dimensionless partial differential equation that governs the transversal motion of a simply supported horizontal pipe is represented by:

$$\varepsilon\alpha v_{txxxx} + \mu v_{xxxx} + \left(\varepsilon^2 (V_0 + V_1 \sin(\omega t))^2 - 1\right) v_{xx} + 2\varepsilon\sqrt{\beta}(V_0 + V_1 \sin(\omega t)) v_{xt} + v_{tt} + \varepsilon\sqrt{\beta} V_1 \omega \cos(\omega t) v_x + \gamma = 0, \tag{64}$$

for $0 < x < 1$, $t > 0$, and $\varepsilon \ll 1$, with the initial and boundary conditions:

$$\begin{aligned} v(0,t) &= v_{xx}(0,t) = 0, \\ v(1,t) &= v_{xx}(1,t) = 0, \end{aligned} \tag{65}$$

for $t > 0$, and with the initial conditions:

$$\begin{aligned} v(x,0) &= \phi(x), \\ v_t(x,0) &= \psi(x), \end{aligned} \tag{66}$$

for $0 < x < 1$.

4 Perturbation Analysis

In this study, we will focus on solving a horizontal simply supported pipe governed by the equation (64) and with the initial-boundary conditions (65) and (66). In order to solve the Eq. (64), the naïve perturbation expansion will be applied. Therefore, we assume that the solution is in the form

$$v(x,t) = v_0(x,t) + \varepsilon v_1(x,t) + \mathcal{O}(\varepsilon^2). \tag{67}$$

$\mathcal{O}(1)$ **Equation**

By substituting (67) into (64), we obtain the $\mathcal{O}(1)$ problem

$$\mathcal{O}(1): \quad \mu\partial_x^4 v_0 - \partial_x^2 v_0 + \partial_t^2 v_0 + \gamma = 0, \tag{68}$$

with the initial and boundary conditions:

$$\begin{aligned} &\text{BC's:} \quad v(0,t) = \partial_x^2 v_0(0,t) = 0, \quad v(1,t) = \partial_x^2 v_0(1,t) = 0, \\ &\text{IC's:} \quad v_0(x,0) = \phi(x), \qquad\qquad\quad \partial_t v_0(x,0) = \psi(x), \end{aligned} \tag{69}$$

for $0 < x < 1$. Where the Eq. (68) is nonhomogeneous, we assume the solution to be in the form

$$v(x,t) = u(x,t) + U(x), \tag{70}$$

with the boundary and initial conditions:

$$\begin{aligned} &\text{BC's:} \quad u(0,t) + U(0) = \partial_x^2 u(0,t) + U(0) = 0, \quad u(1,t) + U(1) = \partial_x^2 u(1,t) + U(1) = 0, \\ &\text{IC's:} \quad u(x,0) + U(x) = \phi(x), \qquad\qquad\qquad\quad \partial_t u(x,0) = \psi(x). \end{aligned} \tag{71}$$

Thus, if $U(x)$ is the solution of the problem,

$$\begin{gathered} \mu U'''' - U'' + \gamma = 0, \\ U(0) = U(1) = U''(0) = U''(1) = 0, \end{gathered} \tag{72}$$

then $u(x,t)$ must satisfy

$$\begin{aligned} &\mu\partial_x^4 u - \partial_x^2 u + \partial_t^2 u = 0, \\ \text{BC's:} \quad &u(0,t) = \partial_x^2 u(0,t) = 0, \quad u(1,t) = \partial_x^2 u(1,t) = 0, \\ \text{IC's:} \quad &u(x,0) = \phi(x) - U(x), \quad \partial_t u(x,0) = \psi(x). \end{aligned} \tag{73}$$

Now, in order to solve Eq. (72), we introduce a variable $\xi(x) := U''(x)$, the homogeneous equation can be written as

$$\mu\xi'' - \xi = 0. \tag{74}$$

And if we assume the solution in the form $\xi(x) = e^{kx}$, we can obtain

$$\left(k^2 - 1/\mu\right)e^{kx} = 0 \iff k = \pm\sqrt{1/\mu} \Rightarrow \xi(x) = c_1 e^{\sqrt{1/\mu}x} + c_2 e^{-\sqrt{1/\mu}x}. \tag{75}$$

From (75), we can write the nonhomogeneous solution as

$$U''(x) = c_1 e^{\sqrt{1/\mu}x} + c_2 e^{-\sqrt{1/\mu}x} + \frac{\gamma}{\mu}. \tag{76}$$

And by integrating this expression twice, we obtain

$$U(x) = c_1 \mu e^{\sqrt{1/\mu}x} + c_2 \mu e^{-\sqrt{1/\mu}x} + \frac{\gamma}{2\mu}x^2 + c_3 x + c_4 \tag{77}$$

and from the boundary conditions in Eq. (72), we get the coefficients as

$$c_1 = -\frac{\gamma}{\mu(e^{1/\sqrt{\mu}} + 1)}, \quad c_2 = -\frac{\gamma e^{1/\sqrt{\mu}}}{\mu(e^{1/\sqrt{\mu}} + 1)}, \quad c_3 = -\frac{\gamma}{2\mu}, \quad c_4 = \gamma. \tag{78}$$

From these coefficients, Eq. (77) becomes

$$U(x) = -\gamma \operatorname{sech}\left(\frac{1}{2\sqrt{\mu}}\right)\cosh\left(\frac{1-2x}{2\sqrt{\mu}}\right) + \frac{\gamma}{2\mu}\left(x^2 - x + 2\mu\right). \tag{79}$$

Now we assume that the solution of Eq. (73) is in the form $u(x,t) = X(x)T(t)$, hence we can write

$$\begin{aligned} &\mu X'''' T - X'' T + X\ddot{T} = 0 \\ \Rightarrow &\frac{X''''}{X} - \frac{1}{\mu}\frac{X''}{X} + \frac{1}{\mu}\frac{\ddot{T}}{T} = 0 \\ \Rightarrow &\frac{1}{\mu}\frac{\ddot{T}}{T} = \frac{1}{\mu}\frac{X''}{X} - \frac{X''''}{X} = k. \end{aligned} \tag{80}$$

If we first assume the spatial component be in the form $X(x) = e^{rx}$, we get

$$\left(r^4 - \frac{1}{\mu}r^2 + k\right)e^{rx} = 0 \Rightarrow r = \pm\sqrt{\frac{1 \pm \sqrt{1 - 4\mu^2 k}}{2\mu}}. \tag{81}$$

This leads to the following possibilities for k:

$$\begin{aligned} &\textcircled{1}: \quad k > \frac{1}{4\mu^2}, \\ &\textcircled{2}: \quad k = \frac{1}{4\mu^2}, \\ &\textcircled{3.1}: \quad \frac{1}{4\mu^2} > k > 0, \quad \textcircled{3.2}: \quad \frac{1}{4\mu^2} > k = 0, \quad \textcircled{3.3}: \quad \frac{1}{4\mu^2} > 0 > k. \end{aligned} \tag{82}$$

The only interval for k that leads to non-trivial solution is (3.3) : $k < 0$ and choosing $k := -(n\pi)^4 - (n\pi)^2/\mu$ leads to the solution

$$X_n(x) = \tilde{A}_n \sin(n\pi x). \tag{83}$$

If we also define

$$\omega_n := n\pi\sqrt{1+\mu n^2\pi^2}, \tag{84}$$

we obtain the temporal component of the solution as

$$T_n(t) = A_n \sin(\omega_n t) + B_n \cos(\omega_n t). \tag{85}$$

If we substitute (83) and (85) into (70), we obtain the general form of the $\mathcal{O}(1)$ solution as

$$v_0(x,t) = U(x) + \sum_{n=1}^{\infty} [A_n \sin(\omega_n t) + B_n \cos(\omega_n t)] \sin(n\pi x). \tag{86}$$

$\mathcal{O}(\varepsilon)$ **Equation**

From the naïve expansion, we obtain the $\mathcal{O}(\varepsilon)$ Eq. as:

$$\begin{aligned}\mathcal{O}(\varepsilon):\quad & \alpha\partial_x^4\partial_t v_0 + 2\sqrt{\beta}(V_0 + V_1\sin(\omega t_0))\partial_x\partial_t v_0 + \sqrt{\beta}V_1\omega\cos(\omega t_0)\partial_x v_0 \\ & + \mu\partial_x^4 v_1 - \partial_x^2 v_1 + \partial_t^2 v_1 = 0.\end{aligned} \tag{87}$$

Since, (83) is odd and 2-periodic for our spatial domain, Eq. (87) must also be expanded odd and 2-periodic in space by multiplying even terms (i.e. $\partial_x\partial_t v_0$ and $\partial_x v_0$) by [28]

$$\mathcal{H}(x) = \sum_{j=0}^{\infty} \frac{4\sin\big((2j+1)\pi x\big)}{(2j+1)\pi} = \begin{cases} 1, & 0 < x < 1, \\ -1, & -1 < x < 0. \end{cases} \tag{88}$$

This gives us

$$\begin{aligned}
&\mu\partial_x^4 v_1 - \partial_x^2 v_1 + \partial_t^2 v_1 = \\
&-\alpha\pi^4 \sum_{n=1}^{\infty} n^4 \omega_n \sin(n\pi x)[A_n\cos(\omega_n t) + B_n \sin(\omega_n t)] \\
&-4\sqrt{\beta}V_0 \sum_{n=1}^{\infty}\sum_{j=0}^{\infty} \frac{n\omega_n}{2j+1}\left[\sin\big((2j+1+n)\pi x\big) + \sin\big((2j+1-n)\pi x\big)\right] \\
&\times [A_n\cos(\omega_n t) + B_n\sin(\omega_n t)] \\
&-2\sqrt{\beta}V_1 \sum_{n=1}^{\infty}\sum_{j=0}^{\infty} \frac{n\omega_n}{2j+1}\left[\sin\big((2j+1+n)\pi x\big) + \sin\big((2j+1-n)\pi x\big)\right] \\
&\times \{A_n[\sin((\omega_n+\Omega)t) + \sin((\omega_n-\Omega)t)] + B_n[\cos((\omega_n+\Omega)t) - \cos((\omega_n-\Omega)t)]\} \\
&-\sqrt{\beta}V_1 \sum_{n=1}^{\infty}\sum_{j=0}^{\infty} \frac{n}{2j+1}\left[\sin\big((2j+1+n)\pi x\big) + \sin\big((2j+1-n)\pi x\big)\right] \\
&\times \{A_n[\sin((\omega_n+\Omega)t) + \sin((\omega_n-\Omega)t)] + B_n[\cos((\omega_n+\Omega)t) + \cos((\omega_n-\Omega)t)]\} \\
&+\frac{4}{\pi}\sqrt{\beta}V_1\Omega\cos(\Omega t)\sum_{j=0}^{\infty}\frac{1}{2j+1}\sin\big((2j+1)\pi x\big)\partial_x U.
\end{aligned} \tag{89}$$

However, the expression obtained from the naïve expansion leads to secular terms. Therefore, a multiple time scales approach will be used to obtain an approximation of the solution which is $\mathcal{O}(\varepsilon)$ accurate for $t \sim \mathcal{O}\left(\frac{1}{\varepsilon}\right)$, and $\varepsilon \to 0$.

4.1 Method of Multiple Scales

In order to avoid the secular terms in v_1, we introduce a two-time-scales perturbation methods. Therefore, we introduce the variables

$$t_0 = t \quad \text{and} \quad t_1 = \varepsilon t. \tag{90}$$

As a consequence of these time variables, the first and second order time derivatives become

$$\frac{d}{dt} \to \frac{dt_0}{dt}\frac{\partial}{\partial t_0} + \frac{dt_1}{dt}\frac{\partial}{\partial t_1} = \partial_{t_0} + \varepsilon\partial_{t_1} \text{ and } \frac{d^2}{dt^2} \to \partial_{t_0}^2 + 2\varepsilon\partial_{t_0}\partial_{t_1} + \varepsilon^2\partial_{t_1}^2. \tag{91}$$

If we substitute Eq. (91) into Eq. (87), we obtain the $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ equations as

$$\mathcal{O}(1): \quad \mu\partial_x^4 v_0 - \partial_x^2 v_0 + \partial_{t_0}^2 v_0 + \gamma = 0, \tag{92}$$

$$\begin{aligned}
\mathcal{O}(\varepsilon): \quad &\alpha\partial_x^4\partial_{t_0} v_0 + 2\sqrt{\beta}(V_0 + V_1\sin(\Omega t_0))\partial_x\partial_{t_0} v_0 + \sqrt{\beta}V_1\Omega\cos(\Omega t_0)\partial_x v_0 \\
&+ 2\partial_{t_0}\partial_{t_1} v_0 + \mu\partial_x^4 v_1 - \partial_x^2 v_1 + \partial_{t_0}^2 v_1 = 0.
\end{aligned} \tag{93}$$

We know from Eq. (86) that the Eq. (92) has the solution in the form

$$v_0(x, t_0, t_1) = U(x) + \sum_{n=1}^{\infty}[A_n(t_1)\sin(\omega_n t_0) + B_n(t_1)\cos(\omega_n t_0)]\sin(n\pi x). \tag{94}$$

By substituting Eq. (94) into Eq. (93), the $\mathcal{O}(\varepsilon)$ equation is obtained in the form

$$Lv_1 = \mu\partial_x^4 v_1 - \partial_x^2 v_1 + \partial_{t_0}^2 v_1 = f, \tag{95}$$

with f representing the RHS terms occurring from (93). We assume that $v_1 = \sum_{k=1}^{\infty} u_k(t_0, t_1)\phi_k(x)$, with $\phi_k(x) = \sin(k\pi x)$ as obtained in Eq. (83). Thus, Eq. (95) can be written as

$$Lv_1 = \sum_{k=1}^{\infty}\left(\mu u_k\phi_k'''' - u_k\phi_k'' + \partial_{t_0}^2 u_k\phi_k\right) = f. \tag{96}$$

We multiply both sides of (96) by the eigenfunction ϕ_k and integrate over the spatial domain, such as $\int_0^1 (Lv_1)\phi_k dx = \int_0^1 f\phi_k dx$, and that leads to:

$$\begin{aligned}
&\partial_{t_0}^2 u_k + \omega_k^2 u_k \\
&= -\alpha\pi^4 k^4\omega_k[A_k\cos(\omega_k t_0) - B_k\sin(\omega_k t_0)] - 2\omega_k\left[\dot{A}_k\cos(\omega_k t_0) - \dot{B}_k\sin(\omega_k t_0)\right] \\
&\quad - \textstyle\sum_n^* \sqrt{\beta}V_1\dfrac{2kn}{n^2-k^2}\Big\{(\Omega+2\omega_n)[A_n\sin((\omega_n+\Omega)t_0) + B_n\cos((\omega_n+\Omega)t_0)] \\
&\qquad\qquad + (\Omega-2\omega_n)[A_n\sin((\omega_n-\Omega)t_0) + B_n\cos((\omega_n-\Omega)t_0)]\Big\} \\
&\quad - \frac{8}{\pi}\sqrt{\beta}V_1\Omega\cos(\Omega t_0)\sum_{j=0}^{\infty}\frac{1}{2j+1}\int_0^1(\partial_x U)\sin\big((2j+1)\pi x\big)\sin(k\pi x)dx,
\end{aligned} \tag{97}$$

where $\sum_n^* := \sum_{2j+1+n=k} + \sum_{2j+1-n=k} - \sum_{2j+1-n=-k}$ for $n+k$ is odd, and zero otherwise. For simplicity, we define $C_k := \sum_{j=0}^{\infty}\frac{1}{2j+1}\int_0^1 U'(x)\sin\big((2j+1)\pi x\big)\sin(k\pi x)dx$, where

$$\begin{aligned}
&\int_0^1 U'\sin\big((2j+1)\pi x\big)\sin(k\pi x)dx \\
&= \begin{cases} 4\gamma k\frac{\mu^2\pi^4(2j+1-k)^2(2j+1+k)^2 - [1+(2j+1-k)^2\mu\pi^2][1+(2j+1+k)^2\mu\pi^2]}{\mu\pi^2(2j+1-k)^2(2j+1+k)^2[1+(2j+1-k)^2\mu\pi^2][1+(2j+1+k)^2\mu\pi^2]}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.}\end{cases}
\end{aligned} \tag{98}$$

It is observed that C_k is a constant and nonzero only for k is even. Exact calculation of the coefficient C_K is not in the scope of this study, however, the convergence of C_k is proven in Appendix A. Hence Eq. (97) can be written as

$$\begin{aligned}
&\partial_{t_0}^2 u_k + \omega_k^2 u_k \\
&= -\alpha\pi^4 k^4\omega_k[A_k\cos(\omega_k t_0) - B_k\sin(\omega_k t_0)] - 2\omega_k\left[\dot{A}_k\cos(\omega_k t_0) - \dot{B}_k\sin(\omega_k t_0)\right] \\
&\quad - \textstyle\sum_n^* \sqrt{\beta}V_1\dfrac{2kn}{n^2-k^2}\Big\{(\Omega+2\omega_n)[A_n\sin((\omega_n+\Omega)t_0) + B_n\cos((\omega_n+\Omega)t_0)] \\
&\qquad\qquad + (\Omega-2\omega_n)[A_n\sin((\omega_n-\Omega)t_0) + B_n\cos((\omega_n-\Omega)t_0)]\Big\} \\
&\quad - \frac{8}{\pi}\sqrt{\beta}V_1\Omega C_k\cos(\Omega t_0).
\end{aligned} \tag{99}$$

4.2 Resonant Cases

In this section, we will determine and study the cases where different values of fluid pulsation frequencies (Ω) lead to secular terms in the solution.

It can be observed that $\sin(\omega_k t_0)$ and $\cos(\omega_k t_0)$ are in the kernel of $\partial_{t_0}^2 u_k + \omega_k^2 u_k = 0$, thus the existence of these terms in the RHS of Eq. (99) will lead to secular terms in the solution u_k. In order to avoid the secular terms, the coefficients $A_k(t_1)$ and $B_k(t_1)$ need to be chosen to eliminate these terms.

From Eq. (99) it can be seen that the excitation frequencies Ω, that leads to resonance are ω_K, $\omega_K - \omega_N$ and $\omega_K + \omega_N$. These frequencies are called primary resonance frequencies. Where the natural frequencies are defined as in (84), given primary resonant frequencies may coincide for critical parameter values μ. These special resonant cases are presented in Table 1.

Ω	ω_K	$\omega_K - \omega_N$	$\omega_K + \omega_N$
$\omega_{\tilde{K}}$		Case-1	Case-2
$\omega_{\tilde{K}} - \omega_{\tilde{N}}$		Case-3	Case-4
$\omega_{\tilde{K}} + \omega_{\tilde{N}}$			Case-5

Table 1: Special cases where different resonant frequencies coincide

By considering primary resonance frequencies and coincided resonance frequencies, the cases studied in this section are presented in Figure 5

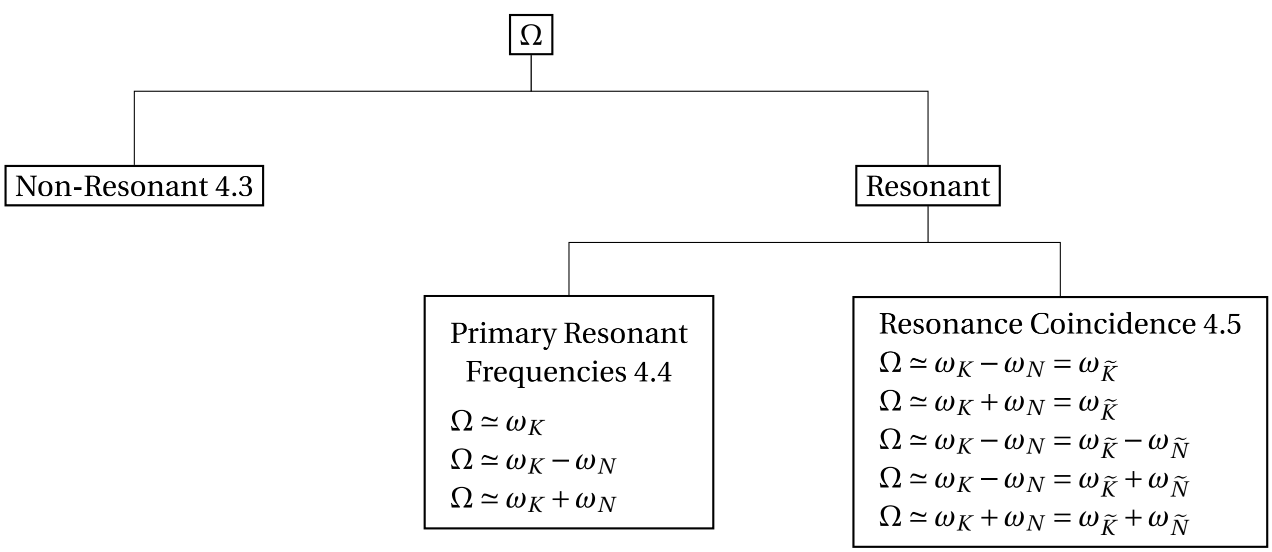

Figure 5: Different resonant cases occuring due to excitation frequency Ω

4.3 The Case Ω Being Non-Resonant

Before studying resonant frequencies Ω, we will examine the scenario where Ω does not lead to resonances. When Ω is not in an order ε neighborhood of ω_K or $\omega_K \pm \omega_N$ with $K+N$ is odd, Eq. (99) becomes

$$\begin{aligned}\partial_{t_0}^2 u_k + \omega_k^2 u_k = &-\alpha\pi^4 k^4 \omega_k [A_k \cos(\omega_k t_0) - B_k \sin(\omega_k t_0)] \\ &- 2\omega_k \left[\dot{A}_k \cos(\omega_k t_0) - \dot{B}_k \sin(\omega_k t_0)\right] + \text{n.r.t.}.\end{aligned} \tag{100}$$

where "n.r.t." stands for non-resonant terms. In this case, the secular terms in $u_k(t_0, t_1)$ can be avoided if

$$\dot{A}_k + \frac{\alpha\pi^4 k^4}{2} A_k = 0 \quad \text{and} \quad \dot{B}_k + \frac{\alpha\pi^4 k^4}{2} B_k = 0 \tag{101}$$

holds. Thus $A_k(t_1)$ and $B_k(t_1)$ must satisfy

$$A_k(t_1) = A_k(0) e^{-\frac{\alpha\pi^4 k^4}{2} t_1} \quad \text{and} \quad B_k(t_1) = B_k(0) e^{-\frac{\alpha\pi^4 k^4}{2} t_1}. \tag{102}$$

Hence, it is observed that the solution, Eq. (94), converges to the steady state solution $U(x)$ as $t \to \infty$.

4.4 Primary Resonant Cases

For various values of Ω, internal resonances can be observed in Eq. (99). This phenomenon occurs when the frequency in the right-hand side of Eq. (99) matches a natural frequency ω_K. The frequencies at which fluid fluctuation Ω can lead to internal resonances are order ε neighbourhood of ω_K or $\omega_K \pm \omega_N$, where K and N are arbitrary integers satisfying the condition that $K+N$ is odd (due to the indexing of $\sum_n^*$). In this section, we will study these primary resonance frequencies, without and with a small detuning $\varepsilon\varphi$.

4.4.1 The Case $\Omega \simeq \omega_K$

The first primary resonant case arises when $\Omega \simeq \omega_K$. In this section, the fluid frequency $\Omega = \omega_K$ and $\Omega = \omega_K + \varepsilon\varphi$ will be studied.

4.4.1.1 The Pure Resonance Case $\Omega = \omega_K$

One can easily see that $\Omega = \omega_K$ causes resonance in the Eq. (99) for the index value $k = K$ with K even (see Eq. (98)), and by separating the terms that give rise to secular terms in $u_K(t_0, t_1)$, Eq. (99) becomes

$$\begin{aligned}\partial_{t_0}^2 u_K + \omega_K^2 u_K = &-\alpha\pi^4 K^4 \omega_K [A_K \cos(\omega_K t_0) - B_K \sin(\omega_K t_0)] \\ &- 2\omega_K \left[\dot{A}_K \cos(\omega_K t_0) - \dot{B}_K \sin(\omega_K t_0)\right] \\ &- \frac{8}{\pi}\sqrt{\beta} V_1 \omega_K C_K \cos(\omega_K t_0) + \text{n.r.t.}\ .\end{aligned} \tag{103}$$

In this case, the secular terms can be prevented if

$$\dot{A}_K + \frac{\alpha\pi^4 K^4}{2} A_K + \frac{4}{\pi}\sqrt{\beta} V_1 C_K = 0, \quad \text{and} \quad \dot{B}_K + \frac{\alpha\pi^4 K^4}{2} B_K = 0. \tag{104}$$

Therefore, $A_K(t_1)$ and $B_K(t_1)$ must satisfy

$$A_K(t_1) = A_K(0) e^{-\frac{\alpha\pi^4 K^4}{2} t_1} - \frac{4}{\pi}\sqrt{\beta} V_1 C_K \quad \text{and} \quad B_K(t_1) = B_K(0) e^{-\frac{\alpha\pi^4 K^4}{2} t_1}. \tag{105}$$

As time approaches infinity, we note that A_K converges to $-\frac{4}{\pi}\sqrt{\beta} V_1 C_K$ while B_K tends towards zero. Consequently, instead of decaying to the steady solution $U(x)$, Eq. (94)evolves towards a solution that exhibits oscillations around $U(x)$.

4.4.1.2 The Case $\Omega = \omega_K + \varepsilon\varphi$

In practical applications, fluid velocity fluctuations often deviate from their natural frequency. To account for this deviation, we will investigate the resonant scenario with a slight detuning parameter $\varepsilon\varphi$.

$$\begin{aligned} \partial_{t_0}^2 u_K + \omega_K^2 u_K = &- \alpha\pi^4 K^4 \omega_K [A_K \cos(\omega_K t_0) - B_K \sin(\omega_K t_0)] \\ &- 2\omega_K \left[\dot{A}_K \cos(\omega_K t_0) - \dot{B}_K \sin(\omega_K t_0)\right] \\ &- \frac{8}{\pi}\sqrt{\beta} V_1 \Omega C_K \cos\left(\omega_K t_0 + \varphi t_1\right) + \text{n.t.r.}. \end{aligned} \tag{106}$$

From the relation $\cos\left(\omega_K t_0 + \varphi t_1\right) = \cos(\omega_K t_0)\cos\left(\varphi t_1\right) - \sin(\omega_K t_0)\sin\left(\varphi t_1\right)$, Eq. (106) becomes

$$\begin{aligned} \partial_{t_0}^2 u_K + \omega_K^2 u_K = &\cos(\omega_K t_0)\left[-\alpha\pi^4 K^4 \omega_K A_K - 2\omega_K \dot{A}_K - \frac{8}{\pi}\sqrt{\beta} V_1 \Omega C_K \cos\left(\varphi t_1\right)\right] \\ &+ \sin(\omega_K t_0)\left[\alpha\pi^4 K^4 \omega_K B_K + 2\omega_K \dot{B}_K + \frac{8}{\pi}\sqrt{\beta} V_1 \Omega C_K \sin\left(\varphi t_1\right)\right] \\ &+ \text{n.t.r.}. \end{aligned} \tag{107}$$

For the sake of simplicity, let us introduce the parameters:

$$a := \frac{\alpha\pi^4 K^4}{2}, \quad c := \frac{4}{\pi}\sqrt{\beta} V_1 C_K. \tag{108}$$

After employing this new notation and considering the orthogonality of sine and cosine functions, the equations that $A_K(t_1)$ and $B_K(t_1)$ must satisfy in order to avoid secular terms can be derived as

$$\begin{aligned} \dot{A}_K &= - a A_K - c\cos\left(\varphi t_1\right), \\ \dot{B}_K &= - a B_K - c\sin\left(\varphi t_1\right). \end{aligned} \tag{109}$$

By using method of integrating factor, one can solve the Eq. (109) as

$$\begin{aligned} A_K(t_1) &= - \frac{c}{a^2 + \varphi^2}\left(a\cos\left(\varphi t_1\right) + \varphi\sin\left(\varphi t_1\right)\right) + \left(A_K(0) + \frac{ac}{a^2 + \varphi^2}\right) e^{-a t_1}, \\ B_K(t_1) &= - \frac{c}{a^2 + \varphi^2}\left(a\sin\left(\varphi t_1\right) - \varphi\cos\left(\varphi t_1\right)\right) + \left(B_K(0) - \frac{\varphi c}{a^2 + \varphi^2}\right) e^{-a t_1}. \end{aligned} \tag{110}$$

For the detuned case, as time progress, the exponential terms dissipate, leaving both A_K and B_K to exhibit oscillations in t_1. This leads, Eq. (94) to transition into a solution that oscillates around steady hanging position $U(x)$ with a slow phase shift.

4.4.2 The Case $\Omega \simeq \omega_K - \omega_N$

As the second primary resonant case, in this section, we will study the fluid velocity fluctuation frequency $\Omega = \omega_K - \omega_N$ and $\Omega = \omega_K - \omega_N + \varepsilon\varphi$.

4.4.2.1 The Pure Resonance Case $\Omega = \omega_K - \omega_N$

In this case, where the fluid velocity fluctuation is equal to difference of two natural frequencies, the terms that lead to secular terms in $u_k(t_0, t_1)$ are

$$\begin{aligned}\partial_{t_0}^2 u_K + \omega_K^2 u_K = &- \alpha\pi^4 K^4 \omega_K [A_K \cos(\omega_K t_0) - B_K \sin(\omega_K t_0)]\\ &- 2\omega_K \left[\dot{A}_K \cos(\omega_K t_0) - \dot{B}_K \sin(\omega_K t_0)\right]\\ &- \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2}(\omega_K + \omega_N)[A_N \sin(\omega_K t_0) + B_N \cos(\omega_K t_0)] + \text{n.r.t.},\end{aligned} \tag{111a}$$

$$\begin{aligned}\partial_{t_0}^2 u_N + \omega_N^2 u_N = &- \alpha\pi^4 N^4 \omega_N [A_N \cos(\omega_N t_0) - B_N \sin(\omega_N t_0)]\\ &- 2\omega_N \left[\dot{A}_N \cos(\omega_N t_0) - \dot{B}_N \sin(\omega_N t_0)\right]\\ &- \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2}(\omega_K + \omega_N)[A_K \sin(\omega_N t_0) + B_K \cos(\omega_N t_0)] + \text{n.r.t.}.\end{aligned} \tag{111b}$$

The terms that lead to resonance can be rewritten as

$$\begin{aligned}\partial_{t_0}^2 u_K + \omega_K^2 u_K = &\cos(\omega_K t_0)\left[-\alpha\pi^4 K^4 \omega_K A_K - 2\omega_K \dot{A}_K - \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2}(\omega_K + \omega_N) B_N\right]\\ &+ \sin(\omega_K t_0)\left[\alpha\pi^4 K^4 \omega_K B_K + 2\omega_K \dot{B}_K - \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2}(\omega_K + \omega_N) A_N\right]\\ &+\text{n.r.t.},\end{aligned} \tag{112a}$$

$$\begin{aligned}\partial_{t_0}^2 u_N + \omega_N^2 u_N = &\cos(\omega_N t_0)\left[-\alpha\pi^4 N^4 \omega_N A_N - 2\omega_N \dot{A}_N - \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2}(\omega_K + \omega_N) B_K\right]\\ &+ \sin(\omega_N t_0)\left[\alpha\pi^4 N^4 \omega_N B_N + 2\omega_N \dot{B}_N - \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2}(\omega_K + \omega_N) A_K\right]\\ &+\text{n.r.t.}.\end{aligned} \tag{112b}$$

In order to avoid secular terms in $u_k(t_0,t_1)$ and $u_N(t_0,t_1)$, the coefficients $A_K(t_1)$, $B_K(t_1)$, $A_N(t_1)$ and $B_N(t_1)$ must satisfy

$$\begin{aligned}
\dot{A}_K &= -\frac{\alpha\pi^4K^4}{2}A_K - \sqrt{\beta}V_1\left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_K}\right)B_N,\\
\dot{B}_K &= -\frac{\alpha\pi^4K^4}{2}B_K + \sqrt{\beta}V_1\left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_K}\right)A_N,\\
\dot{A}_N &= -\frac{\alpha\pi^4N^4}{2}A_N - \sqrt{\beta}V_1\left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_N}\right)B_K,\\
\dot{B}_N &= -\frac{\alpha\pi^4N^4}{2}B_N + \sqrt{\beta}V_1\left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_N}\right)A_K.
\end{aligned} \tag{113}$$

We define the parameters:

$$\begin{gathered}
a := \frac{\alpha\pi^4K^4}{2}, \quad b := \frac{\alpha\pi^4N^4}{2},\\
p := \sqrt{\beta}V_1\left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_K}\right), \quad q := \sqrt{\beta}V_1\left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_N}\right)
\end{gathered} \tag{114}$$

for simplicity, and rewrite Eq. (113)

$$\begin{pmatrix}\dot{A}_K\\ \dot{B}_K\\ \dot{A}_N\\ \dot{B}_N\end{pmatrix} = \begin{pmatrix}-a & 0 & 0 & -p\\ 0 & -a & p & 0\\ 0 & -q & -b & 0\\ q & 0 & 0 & -b\end{pmatrix}\begin{pmatrix}A_K\\ B_K\\ A_N\\ B_N\end{pmatrix} \tag{115}$$

or equivalently,

$$\begin{pmatrix}\dot{A}_K\\ \dot{B}_N\end{pmatrix} = \begin{pmatrix}-a & -p\\ q & -b\end{pmatrix}\begin{pmatrix}A_K\\ B_N\end{pmatrix}, \quad \begin{pmatrix}\dot{B}_K\\ \dot{A}_N\end{pmatrix} = \begin{pmatrix}p & -a\\ -b & -q\end{pmatrix}\begin{pmatrix}B_K\\ A_N\end{pmatrix}. \tag{116}$$

The systems in Eq. (116) have identical characteristic polynomials, and they are stable if

$$\begin{aligned}
\operatorname{tr}\begin{pmatrix}-a & -p\\ q & -b\end{pmatrix} &= -(a+b) = -\frac{\alpha\pi^4(K^4+N^4)}{2} < 0,\\
\det\begin{pmatrix}-a & -p\\ q & -b\end{pmatrix} &= ab+pq = \frac{\alpha^4\pi^8K^4N^4}{4} + \beta V_1^2\frac{K^2N^2}{(N^2-K^2)^2}\frac{(\omega_K+\omega_N)^2}{\omega_K\omega_N} > 0.
\end{aligned} \tag{117}$$

It can be observed that for $\alpha > 0$, or equivalently assuming that viscoelastic damping effects are present in the system, stability of the given system is guaranteed. And the eigenvalues can be written as

$$\lambda_{1,2} = -\frac{a+b}{2} \pm \frac{\sqrt{(a-b)^2-4pq}}{2}. \tag{118}$$

Hence, for $\Omega = \omega_K - \omega_N$, the pipe system gradually approaches the equilibrium state over time.

4.4.2.2 The Detuned Resonance Case $\Omega = \omega_K - \omega_N + \varepsilon\varphi$

Now we study the case where fluid excitation deviates from difference of two natural frequencies by a detuning parameter $\varepsilon\varphi$. If $\Omega = \omega_K - \omega_N + \varepsilon\varphi$, Eq. (99) can be written as

$$\begin{aligned}
&\partial_{t_0}^2 u_K + \omega_K^2 u_K \\
&= -\alpha\pi^4 K^4 \omega_K [A_K \cos(\omega_K t_0) - B_K \sin(\omega_K t_0)] \\
&\quad - 2\omega_K \left[\dot{A}_K \cos(\omega_K t_0) - \dot{B}_K \sin(\omega_K t_0)\right] \\
&\quad - \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2} (\omega_K + \omega_N) \Big\{ A_N \left[\sin(\omega_K t_0)\cos(\varphi t_1) + \cos(\omega_K t_0)\sin(\varphi t_1)\right] \\
&\qquad\qquad + B_N \left[\sin(\omega_K t_0)\cos(\varphi t_1) + \cos(\omega_K t_0)\sin(\varphi t_1)\right] \Big\} \\
&\quad + \text{n.r.t.},
\end{aligned} \tag{119a}$$

$$\begin{aligned}
&\partial_{t_0}^2 u_N + \omega_N^2 u_N \\
&= -\alpha\pi^4 N^4 \omega_N [A_N \cos(\omega_N t_0) - B_N \sin(\omega_N t_0)] \\
&\quad - 2\omega_N \left[\dot{A}_N \cos(\omega_N t_0) - \dot{B}_N \sin(\omega_N t_0)\right] \\
&\quad - \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2} (\omega_K + \omega_N) \Big\{ A_K \left[\sin(\omega_N t_0)\cos(\varphi t_1) - \cos(\omega_N t_0)\sin(\varphi t_1)\right] \\
&\qquad\qquad + B_K \left[\cos(\omega_N t_0)\cos(\varphi t_1) + \sin(\omega_N t_0)\sin(\varphi t_1)\right] \Big\} \\
&\quad + \text{n.r.t.}.
\end{aligned} \tag{119b}$$

By rearranging the above equations (119), we obtain

$$\begin{aligned}
&\partial_{t_0}^2 u_K + \omega_K^2 u_K \\
&= \cos(\omega_K t_0) \left\{ -\alpha\pi^4 K^4 \omega_K A_K - 2\omega_K \dot{A}_K - \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2} (\omega_K + \omega_N) \left[A_N \sin(\varphi t_1) + B_N \cos(\varphi t_1)\right] \right\} \\
&+ \sin(\omega_K t_0) \left\{ \alpha\pi^4 K^4 \omega_K B_K + 2\omega_K \dot{B}_K - \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2} (\omega_K + \omega_N) \left[A_N \cos(\varphi t_1) - B_N \sin(\varphi t_1)\right] \right\} \\
&+ \text{n.r.t.},
\end{aligned} \tag{120a}$$

$$\begin{aligned}
&\partial_{t_0}^2 u_N + \omega_N^2 u_N \\
&= \cos(\omega_N t_0) \left\{ -\alpha\pi^4 N^4 \omega_N A_N - 2\omega_N \dot{A}_N - \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2} (\omega_K + \omega_N) \left[-A_K \sin(\varphi t_1) + B_K \cos(\varphi t_1)\right] \right\} \\
&+ \sin(\omega_N t_0) \left\{ \alpha\pi^4 N^4 \omega_N B_N + 2\omega_N \dot{B}_N - \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2} (\omega_K + \omega_N) \left[A_K \cos(\varphi t_1) + B_K \sin(\varphi t_1)\right] \right\} \\
&+ \text{n.r.t.}.
\end{aligned} \tag{120b}$$

With the parameters defined for pure resonance case (114), we obtain the set of differential equations that must be satisfied for $u_K(t_0, t_1)$ and $u_N(t_0, t_1)$ to avoid secular terms as

$$
\begin{aligned}
\dot{A}_K &= -aA_K - p\sin(\varphi t_1)A_N - p\cos(\varphi t_1)B_N,\\
\dot{B}_K &= -aB_K + p\cos(\varphi t_1)A_N - p\sin(\varphi t_1)B_N,\\
\dot{A}_N &= -bA_N + q\sin(\varphi t_1)A_K - q\cos(\varphi t_1)B_K,\\
\dot{B}_N &= -bB_N + q\cos(\varphi t_1)A_K + q\sin(\varphi t_1)B_K.
\end{aligned}
\tag{121}
$$

We can observe that, contrary to Eq. (115), Eq. (121) is a coupled system that cannot be reduced to two smaller systems. The intermediate steps are presented in the Appendix B. We obtain from the Eq. (121)

$$
\begin{aligned}
&\ddot{A}_K + (a+b)\dot{A}_K + (ab+pq)A_K + \varphi(\dot{B}_K + aB_K) = 0,\\
&\ddot{B}_K + (a+b)\dot{B}_K + (ab+pq)B_K - \varphi(\dot{A}_K + aA_K) = 0,
\end{aligned}
\tag{122}
$$

or

$$
\begin{aligned}
&A_K^{(4)} + 2(a+b)\dddot{A}_K + \left(\varphi^2 + (a+b)^2 + 2(ab+pq)\right)\ddot{A}_K\\
&+ \left(2a\varphi^2 + 2(a+b)(ab+pq)\right)\dot{A}_K + \left(\varphi^2 a^2 + (ab+pq)^2\right)A_K = 0
\end{aligned}
\tag{123}
$$

which have identical characteristic polynomials. And the corresponding characteristic equation is

$$
\begin{aligned}
&\lambda^4 + 2(a+b)\lambda^3 + \left(\varphi^2 + (a+b)^2 + 2(ab+pq)\right)\lambda^2\\
&+ \left(2a\varphi^2 + 2(a+b)(ab+pq)\right)\lambda + \left(\varphi^2 a^2 + (ab+pq)^2\right) = 0.
\end{aligned}
\tag{124}
$$

From the quartic polynomial Eq. (124), the eigenvalues of Eq. (122) can be obtained as

$$
\begin{aligned}
\lambda_1 &= -\frac{(a+b)}{2} + \frac{\varphi}{2}i - \frac{1}{2}\sqrt{2\varphi(a-b)i + (a-b)^2 - 4pq - \varphi^2},\\
\lambda_2 &= -\frac{(a+b)}{2} + \frac{\varphi}{2}i + \frac{1}{2}\sqrt{2\varphi(a-b)i + (a-b)^2 - 4pq - \varphi^2},\\
\lambda_3 &= -\frac{(a+b)}{2} - \frac{\varphi}{2}i - \frac{1}{2}\sqrt{-2\varphi(a-b)i + (a-b)^2 - 4pq - \varphi^2},\\
\lambda_4 &= -\frac{(a+b)}{2} - \frac{\varphi}{2}i + \frac{1}{2}\sqrt{-2\varphi(a-b)i + (a-b)^2 - 4pq - \varphi^2}.
\end{aligned}
\tag{125}
$$

For known parameter values, one can determine the roots and the solutions of Eq. (122) respectively. For known A_K and B_K, one can calculate the A_N and B_N as:

$$
\begin{aligned}
A_N &= -\frac{1}{p}\left[\sin(\varphi t_1)\left(\dot{A}_K + aA_K\right) - \cos(\varphi t_1)\left(\dot{B}_K + aB_K\right)\right],\\
B_N &= -\frac{1}{p}\left[\cos(\varphi t_1)\left(\dot{A}_K + aA_K\right) + \sin(\varphi t_1)\left(\dot{B}_K + aB_K\right)\right].
\end{aligned}
\tag{126}
$$

In order to define the stability of Eq. (122) and consequently (121) for arbitrary parameter values, we study the real part of the eigenvalues obtained in (125). Where the square root of a complex number is in the form:

$$z = \sqrt{A+Bi} = \pm\left(\sqrt{\frac{\sqrt{A^2+B^2}+A}{2}} + i\frac{B}{|B|}\sqrt{\frac{\sqrt{A^2+B^2}-A}{2}}\right),$$

the eigenvalue with the greatest real component must satisfy the condition such that its real part meets the following condition:

$$-\frac{(a+b)}{2} + \frac{1}{2}\sqrt{\frac{\sqrt{\left[(a-b)^2-\varphi^2-4pq\right]^2+4\varphi^2(a-b)^2}+(a-b)^2-4pq-\varphi^2}{2}} < 0. \tag{127}$$

This leads to the inequality

$$\varphi^2(a-b)^2 < \left(\varphi^2+4(ab+pq)\right)(a+b)^2. \tag{128}$$

Where $a, b, pq > 0$, it is clear that $\varphi^2(a-b)^2 < \varphi^2(a+b)^2 < \left(\varphi^2+4(ab+pq)\right)(a+b)^2$, hence the inequality (128) is true for all parameter values. Therefore, we can conclude that the case $\Omega = \omega_K - \omega_N + \varepsilon\varphi$ is stable for all detuning frequencies.

4.4.3 The Case $\Omega \simeq \omega_K + \omega_N$

As the third and final primary resonance frequency, we study the fluid pulsation frequency is equal to sum of two natural frequencies.

4.4.3.1 The Pure Resonance Case $\Omega = \omega_K + \omega_N$

If the frequency of fluid velocity fluctuation is the sum of two natural frequencies, Eq. (99) with resonant forcing terms can be written as

$$\begin{aligned}\partial_{t_0}^2 u_K + \omega_K^2 u_K =& -\alpha\pi^4 K^4\omega_K[A_K\cos(\omega_K t_0) - B_K\sin(\omega_K t_0)]\\ &-2\omega_K\left[\dot{A}_K\cos(\omega_K t_0) - \dot{B}_K\sin(\omega_K t_0)\right]\\ &-\sqrt{\beta}V_1\frac{2KN}{N^2-K^2}(\omega_K-\omega_N)[-A_N\sin(\omega_K t_0)+B_N\cos(\omega_K t_0)] + \text{n.r.t.},\end{aligned} \tag{129a}$$

$$\begin{aligned}\partial_{t_0}^2 u_N + \omega_N^2 u_N =& -\alpha\pi^4 N^4\omega_N[A_N\cos(\omega_N t_0) - B_N\sin(\omega_N t_0)]\\ &-2\omega_N\left[\dot{A}_N\cos(\omega_N t_0) - \dot{B}_N\sin(\omega_N t_0)\right]\\ &-\sqrt{\beta}V_1\frac{2KN}{N^2-K^2}(\omega_K-\omega_N)[-A_K\sin(\omega_N t_0)+B_K\cos(\omega_N t_0)] + \text{n.r.t.}.\end{aligned} \tag{129b}$$

By collecting terms with $\sin(\omega_K t_0)$ and $\cos(\omega_K t_0)$, we obtain

$$\begin{aligned}
\partial_{t_0}^2 u_K + \omega_K^2 u_K = &\cos(\omega_K t_0)\left[-\alpha\pi^4 K^4 \omega_K A_K - 2\omega_K \dot{A}_K - \sqrt{\beta} V_1 \frac{2KN}{N^2-K^2}(\omega_K-\omega_N) B_N\right] \\
&+ \sin(\omega_K t_0)\left[\alpha\pi^4 K^4 \omega_K B_K + 2\omega_K \dot{B}_K + \sqrt{\beta} V_1 \frac{2KN}{N^2-K^2}(\omega_K-\omega_N) A_N\right] \\
&+\text{n.r.t.},
\end{aligned} \tag{130a}$$

$$\begin{aligned}
\partial_{t_0}^2 u_N + \omega_N^2 u_N = &\cos(\omega_N t_0)\left[-\alpha\pi^4 N^4 \omega_N A_N - 2\omega_N \dot{A}_N - \sqrt{\beta} V_1 \frac{2KN}{N^2-K^2}(\omega_K-\omega_N) B_K\right] \\
&+ \sin(\omega_N t_0)\left[\alpha\pi^4 N^4 \omega_N B_N + 2\omega_N \dot{B}_N + \sqrt{\beta} V_1 \frac{2KN}{N^2-K^2}(\omega_K-\omega_N) A_K\right] \\
&+\text{n.r.t.}.
\end{aligned} \tag{130b}$$

In order to avoid secular terms in $u_K(t_0,t_1)$ and $u_N(t_0,t_1)$, coefficients $A_K(t_1)$, $B_K(t_1)$, $A_N(t_1)$ and $B_N(t_1)$ must satisfy

$$\begin{aligned}
\dot{A}_K &= -\frac{\alpha\pi^4 K^4}{2} A_K - \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K-\omega_N}{\omega_K}\right) B_N \\
\dot{B}_K &= -\frac{\alpha\pi^4 K^4}{2} B_K - \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K-\omega_N}{\omega_K}\right) A_N \\
\dot{A}_N &= -\frac{\alpha\pi^4 N^4}{2} A_N - \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K-\omega_N}{\omega_N}\right) B_K \\
\dot{B}_N &= -\frac{\alpha\pi^4 N^4}{2} B_N - \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K-\omega_N}{\omega_N}\right) A_K
\end{aligned} \tag{131}$$

Given the parameters:

$$\begin{aligned}
a := \frac{\alpha\pi^4 K^4}{2}, \quad b := \frac{\alpha\pi^4 N^4}{2}, \\
p := \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K-\omega_N}{\omega_K}\right), \quad q := \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K-\omega_N}{\omega_N}\right),
\end{aligned} \tag{132}$$

Eq. (131) can be expressed as:

$$\begin{pmatrix}\dot{A}_K \\ \dot{B}_N\end{pmatrix} = \begin{pmatrix}-a & -p \\ -q & -b\end{pmatrix}\begin{pmatrix}A_K \\ B_N\end{pmatrix}, \quad \begin{pmatrix}\dot{A}_N \\ \dot{B}_K\end{pmatrix} = \begin{pmatrix}-b & -q \\ -p & -a\end{pmatrix}\begin{pmatrix}A_N \\ B_K\end{pmatrix}. \tag{133}$$

The systems in Eq. (133) have identical characteristic polynomials. Their stability is determined by the inequalities.

$$\begin{aligned}
\text{tr}\begin{pmatrix}-a & -p \\ -q & -b\end{pmatrix} &= -(a+b) = -\frac{\alpha\pi^4(K^4+N^4)}{2} < 0, \\
\det\begin{pmatrix}-a & -p \\ -q & -b\end{pmatrix} &= ab - pq = \frac{\alpha^2\pi^8 K^4 N^4}{4} - \beta V_1^2 \frac{K^2N^2}{(N^2-K^2)^2}\frac{(\omega_K-\omega_N)^2}{\omega_K\omega_N} > 0.
\end{aligned} \tag{134}$$

Thus, the stability of the coefficients A_K, B_K, A_N and B_N is determined by

$$\frac{\alpha^2\pi^8K^2N^2}{4} > \frac{\beta V_1^2}{(N^2-K^2)^2}\frac{(\omega_K-\omega_N)^2}{\omega_K\omega_N}. \tag{135}$$

The eigenvalues of the system can be represented as:

$$\lambda_{1,2} = -\frac{a+b}{2} \pm \frac{\sqrt{(a-b)^2+4pq}}{2}. \tag{136}$$

4.4.3.2 The Case $\Omega = \omega_K + \omega_N + \varepsilon\varphi$

Now we will study the case where excitation deviates from sum of two frequencies by a detuning parameter $\varepsilon\varphi$.

$$\begin{aligned}
&\partial_{t_0}^2 u_K + \omega_K^2 u_K \\
&= -\alpha\pi^4K^4\omega_K[A_K\cos(\omega_K t_0) - B_K\sin(\omega_K t_0)] \\
&- 2\omega_K\left[\dot{A}_K\cos(\omega_K t_0) - \dot{B}_K\sin(\omega_K t_0)\right] \\
&- \sqrt{\beta}V_1\frac{2KN}{N^2-K^2}(\omega_K-\omega_N)\Big\{-A_N\left[\sin(\omega_K t_0)\cos\left(\varphi t_1\right) + \cos(\omega_K t_0)\sin\left(\varphi t_1\right)\right] \\
&\qquad\qquad + B_N\left[\sin(\omega_K t_0)\cos\left(\varphi t_1\right) - \cos(\omega_K t_0)\sin\left(\varphi t_1\right)\right]\Big\} \\
&+ \text{n.r.t.}
\end{aligned} \tag{137a}$$

$$\begin{aligned}
&\partial_{t_0}^2 u_N + \omega_N^2 u_N \\
&= -\alpha\pi^4N^4\omega_N[A_N\cos(\omega_N t_0) - B_N\sin(\omega_N t_0)] \\
&- 2\omega_N\left[\dot{A}_N\cos(\omega_N t_0) - \dot{B}_N\sin(\omega_N t_0)\right] \\
&- \sqrt{\beta}V_1\frac{2KN}{N^2-K^2}(\omega_K-\omega_N)\Big\{-A_K\left[\sin(\omega_N t_0)\cos\left(\varphi t_1\right) + \cos(\omega_N t_0)\sin\left(\varphi t_1\right)\right] \\
&\qquad\qquad + B_K\left[\cos(\omega_N t_0)\cos\left(\varphi t_1\right) - \sin(\omega_N t_0)\sin\left(\varphi t_1\right)\right]\Big\} \\
&+ \text{n.r.t.}
\end{aligned} \tag{137b}$$

By collecting terms with $\sin(\omega_K t_0)$ and $\cos(\omega_K t_0)$, we obtain

$$\begin{aligned}
&\partial_{t_0}^2 u_K + \omega_K^2 u_K \\
&= \cos(\omega_K t_0)\left\{-\alpha\pi^4 K^4 \omega_K A_K - 2\omega_K \dot{A}_K + \sqrt{\beta} V_1 \frac{2KN}{N^2-K^2}(\omega_K - \omega_N)\left[A_N \sin(\varphi t_1) - B_N \cos(\varphi t_1)\right]\right\} \\
&+ \sin(\omega_K t_0)\left\{\alpha\pi^4 K^4 \omega_K B_K + 2\omega_K \dot{B}_K + \sqrt{\beta} V_1 \frac{2KN}{N^2-K^2}(\omega_K - \omega_N)\left[A_N \cos(\varphi t_1) + B_N \sin(\varphi t_1)\right]\right\} \\
&+ \text{n.r.t.},
\end{aligned} \tag{138a}$$

$$\begin{aligned}
&\partial_{t_0}^2 u_N + \omega_N^2 u_N \\
&= \cos(\omega_N t_0)\left\{-\alpha\pi^4 N^4 \omega_N A_N - 2\omega_N \dot{A}_N + \sqrt{\beta} V_1 \frac{2KN}{N^2-K^2}(\omega_K - \omega_N)\left[A_K \sin(\varphi t_1) - B_K \cos(\varphi t_1)\right]\right\} \\
&+ \sin(\omega_N t_0)\left\{\alpha\pi^4 N^4 \omega_N B_N + 2\omega_N \dot{B}_N + \sqrt{\beta} V_1 \frac{2KN}{N^2-K^2}(\omega_K - \omega_N)\left[A_K \cos(\varphi t_1) + B_K \sin(\varphi t_1)\right]\right\} \\
&+ \text{n.r.t.}.
\end{aligned} \tag{138b}$$

With the parameters defined for the pure resonance case (132), we obtain the system of equations for coefficients $A_K(t_1)$, $B_K(t_1)$, $A_N(t_1)$ and $B_N(t_1)$ must satisfy in order to avoid secular terms in $u_K(t_0, t_1)$ and $u_N(t_0, t_1)$,

$$\begin{aligned}
\dot{A}_K &= -aA_K + p\sin(\varphi t_1)A_N - p\cos(\varphi t_1)B_N, \\
\dot{B}_K &= -aB_K - p\cos(\varphi t_1)A_N - p\sin(\varphi t_1)B_N, \\
\dot{A}_N &= -bA_N + q\sin(\varphi t_1)A_K - q\cos(\varphi t_1)B_K, \\
\dot{B}_N &= -bB_N - q\cos(\varphi t_1)A_K - q\sin(\varphi t_1)B_K.
\end{aligned} \tag{139}$$

After following similar steps as presented in the Appendix B, the time independent systems for A_K and B_K can be obtain from the Eq. (139) becomes

$$\begin{aligned}
\ddot{A}_K + (a+b)\dot{A}_K + (ab - pq)A_K + \varphi(\dot{B}_K + aB_K) &= 0, \\
\ddot{B}_K + (a+b)\dot{B}_K + (ab - pq)B_K - \varphi(\dot{A}_K + aA_K) &= 0.
\end{aligned} \tag{140}$$

The corresponding characteristic equation is

$$\begin{aligned}
&\lambda^4 + 2(a+b)\lambda^3 + \left(\varphi^2 + (a+b)^2 + 2(ab-pq)\right)\lambda^2 \\
&+ \left(2a\varphi^2 + 2(a+b)(ab-pq)\right)\lambda + \left(\varphi^2 a^2 + (ab-pq)^2\right) = 0
\end{aligned} \tag{141}$$

and we find the solution of Eq (141) to be

$$\begin{aligned}
\lambda_1 &= -\frac{(a+b)}{2} + \frac{\varphi}{2} i - \frac{1}{2}\sqrt{2\varphi(a-b)i + (a-b)^2 + 4pq - \varphi^2},\\
\lambda_2 &= -\frac{(a+b)}{2} + \frac{\varphi}{2} i + \frac{1}{2}\sqrt{2\varphi(a-b)i + (a-b)^2 + 4pq - \varphi^2},\\
\lambda_3 &= -\frac{(a+b)}{2} - \frac{\varphi}{2} i - \frac{1}{2}\sqrt{-2\varphi(a-b)i + (a-b)^2 + 4pq - \varphi^2},\\
\lambda_4 &= -\frac{(a+b)}{2} - \frac{\varphi}{2} i + \frac{1}{2}\sqrt{-2\varphi(a-b)i + (a-b)^2 + 4pq - \varphi^2}.
\end{aligned} \tag{142}$$

For known parameter values, one can determine the roots and the solutions of Eq. (122) respectively. For known A_K and B_K, one can calculate the A_N and B_N as:

$$\begin{aligned}
A_N &= \frac{1}{p}\left[\sin(\varphi t_1)\left(\dot{A}_K + aA_K\right) - \cos(\varphi t_1)\left(\dot{B}_K + aB_K\right)\right],\\
B_N &= -\frac{1}{p}\left[\cos(\varphi t_1)\left(\dot{A}_K + aA_K\right) + \sin(\varphi t_1)\left(\dot{B}_K + aB_K\right)\right].
\end{aligned} \tag{143}$$

In order to define the stability of Eq. (140), we study the real part of the eigenvalues (142) and we obtain the condition for stability as

$$\varphi^2 > (a+b)^2\left(\frac{pq}{ab} - 1\right) \tag{144}$$

or explicitly

$$\varphi^2 > (K^4 + N^4)^2\left(\left(\frac{\beta V_1^2(\omega_K - \omega_N)^2}{K^2N^2(N^2-K^2)^2\omega_K\omega_N}\right) - \frac{\alpha^2\pi^8}{4}\right). \tag{145}$$

4.5 Resonance Coincidence Cases

In Sections 4.3 and 4.4, we investigated the occurrence of resonance associated with the frequencies $\Omega = \omega_K$, $\omega_K - \omega_N$ and $\omega_K + \omega_N$, where $\omega_n = n\pi\sqrt{1 + n^2\pi^2\mu}$ and μ is the parameter corresponding to bending stiffness.

However, for critical values of μ, two primary resonance frequencies may coincide (see Table 1). We will investigate the given resonant frequency coincidence for the described cases in Table 1. Each case will be studied for both pure resonance and detuned resonance frequencies. Coinciding resonance frequencies may also have a common frequency, e.g. $\Omega = \omega_K - \omega_N = \omega_N - \omega_{\tilde{N}}$, and these scenarios will be discussed further in detail. The resonance coincidence cases occurring for Case-1 and Case-2, $\Omega = \omega_K \pm \omega_N = \omega_{\tilde{K}}$, where all indices are different can be found as

$$\Omega = \omega_K \pm \omega_N = \omega_{\tilde{K}} \Rightarrow \omega_K^2 + \omega_N^2 \pm 2\omega_K\omega_N = \omega_{\tilde{K}}^2 \Rightarrow \left(\omega_K^2 + \omega_N^2 - \omega_{\tilde{K}}^2\right)^2 = 4\omega_K^2\omega_N^2. \tag{146}$$

Now by substituting $\omega_n = n\pi\sqrt{1+n^2\pi^2\mu}$ for each resonant frequency with corresponding index values and divide the whole expression by π^4, we obtain the quadratic equation in μ for arbitrary values of K, N and $\widetilde{K}$ as:

$$\begin{aligned}&\pi^4\big(K^2+\widetilde{K}^2-N^2\big)\big(K^2-\widetilde{K}^2+N^2\big)\big(K^2+\widetilde{K}^2+N^2\big)\big(K^2-\widetilde{K}^2-N^2\big)\mu^2\\&+2\pi^2\big(K^6+\widetilde{K}^6+N^6-K^4\widetilde{K}^2-K^4N^2-K^2\widetilde{K}^4-K^2N^4-\widetilde{K}^4N^2-\widetilde{K}^2N^4\big)\mu\\&+\big(K-\widetilde{K}+N\big)\big(K+\widetilde{K}+N\big)\big(K-\widetilde{K}-N\big)\big(K+\widetilde{K}-N\big)=0.\end{aligned} \tag{147}$$

When the Case-1 occurs with a common frequency, i.e. $\Omega = \omega_K - \omega_N = \omega_N$, we obtain the expression of μ for arbitrary K and N as:

$$\Omega = \omega_K - \omega_N = \omega_N \Rightarrow \omega_K^2 - 4\omega_N^2 = 0 \Rightarrow \mu = \frac{4N^2-K^2}{\pi^2\big(K^4-4N^4\big)} \tag{148}$$

and there exists a positive real μ value that leads to coincidence as long as $2N > K > \sqrt{2}N$ holds true.

The resonance coincidence cases occurring with 4 different natural frequencies are investigated numerically, by searching the zero of

$$\begin{aligned}&\text{Case-3:}\quad \Omega = \omega_K - \omega_N = \omega_{\widetilde{K}} - \omega_{\widetilde{N}} \Rightarrow \omega_K - \omega_N - \omega_{\widetilde{K}} + \omega_{\widetilde{N}} = 0, &(149a)\\&\text{Case-4:}\quad \Omega = \omega_K - \omega_N = \omega_{\widetilde{K}} + \omega_{\widetilde{N}} \Rightarrow \omega_K - \omega_N - \omega_{\widetilde{K}} - \omega_{\widetilde{N}} = 0, &(149b)\\&\text{Case-5:}\quad \Omega = \omega_K + \omega_N = \omega_{\widetilde{K}} + \omega_{\widetilde{N}} \Rightarrow \omega_K + \omega_N - \omega_{\widetilde{K}} - \omega_{\widetilde{N}} = 0 &(149c)\end{aligned}$$

or equivalently searching the zero's of

$$\begin{aligned}F_1(\mu) &= K\sqrt{1+\mu K^2\pi^2} - N\sqrt{1+\mu N^2\pi^2} - \widetilde{K}\sqrt{1+\mu\widetilde{K}^2\pi^2} + \widetilde{N}\sqrt{1+\mu\widetilde{N}^2\pi^2}, &(150a)\\F_2(\mu) &= K\sqrt{1+\mu K^2\pi^2} - N\sqrt{1+\mu N^2\pi^2} - \widetilde{K}\sqrt{1+\mu\widetilde{K}^2\pi^2} - \widetilde{N}\sqrt{1+\mu\widetilde{N}^2\pi^2}, &(150b)\\F_3(\mu) &= K\sqrt{1+\mu K^2\pi^2} + N\sqrt{1+\mu N^2\pi^2} - \widetilde{K}\sqrt{1+\mu\widetilde{K}^2\pi^2} - \widetilde{N}\sqrt{1+\mu\widetilde{N}^2\pi^2} &(150c)\end{aligned}$$

respectively. It is observed that, a common frequency can occur only for the Case-3 and Case-4. It is clear that for Case-5, $\Omega = \omega_K + \omega_N = \omega_{\widetilde{K}} + \omega_N$ implies that $\omega_K = \omega_{\widetilde{K}}$, thus, $K = \widetilde{K}$. The resonance coincidence cases with a common frequency occurring for Case-3 and Case-4, $\Omega = \omega_K - \omega_N = \omega_{\widetilde{K}} \pm \omega_N$, can be found as

$$\Omega = \omega_K - \omega_N = \omega_N \pm \omega_{\widetilde{N}} \Rightarrow \omega_K^2 + \omega_{\widetilde{N}}^2 \pm 2\omega_K\omega_{\widetilde{N}} = 4\omega_N^2 \Rightarrow \left(4\omega_N^2 - \omega_K^2 - \omega_{\widetilde{N}}^2\right)^2 = 4\omega_K^2\omega_{\widetilde{N}}^2. \tag{151}$$

By substituting $\omega_n = n\pi\sqrt{1+n^2\pi^2\mu}$ with corresponding indices, we obtain the quadratic expression for μ that satisfies the coincidence of Case-3 and Case-4 as

$$\begin{aligned}&\pi^4\big(K^2-\widetilde{N}^2+2N^2\big)\big(K^2+\widetilde{N}^2-2N^2\big)\big(K^2-\widetilde{N}^2-2N^2\big)\big(K^2+\widetilde{N}^2+2N^2\big)\mu^2\\&+2\pi^2\big(K^2+\widetilde{N}^2-2N^2\big)\big(K^4-2K^2\widetilde{N}^2-2K^2N^2+\widetilde{N}^4-2\widetilde{N}^2N^2-8N^4\big)\mu\\&+\big(K-\widetilde{N}-2N\big)\big(K-\widetilde{N}+2N\big)\big(K+\widetilde{N}-2N\big)\big(K+\widetilde{N}+2N\big)=0.\end{aligned} \tag{152}$$

In this section, we will delve into the study of these special cases of primary resonance frequency coincidences. A similar study can be found in [29, 30] where a number of these cases were studied for definite modes for equations of conveyor belt problems, which are similar to the equations that are being studied.

In Figure 6, the number of cases with respect to μ values for different resonant cases are presented for $K, N, \widetilde{K}, \widetilde{N} \leq 50$. The vertical values of each horizontal interval indicate the number of instances that primary resonance frequencies coincide for μ being in this interval.

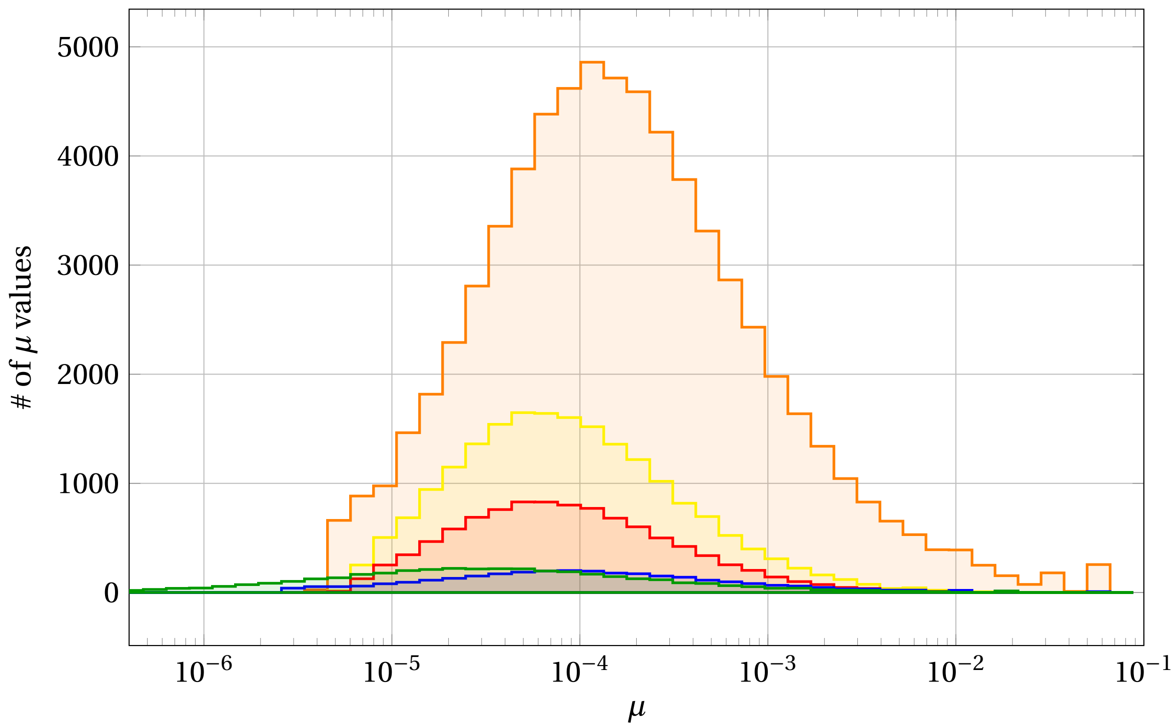

Figure 6: Number of μ values within the horizontal interval with —— for $\omega_K - \omega_N = \omega_{\widetilde{K}}$, —— for $\omega_K + \omega_N = \omega_{\widetilde{K}}$, —— for $\omega_K - \omega_N = \omega_{\widetilde{K}} - \omega_{\widetilde{N}}$, —— for $\omega_K - \omega_N = \omega_{\widetilde{K}} + \omega_{\widetilde{N}}$ and —— for $\omega_K + \omega_N = \omega_{\widetilde{K}} + \omega_{\widetilde{N}}$

It can be concluded that the instances, where $\omega_K - \omega_N = \omega_{\widetilde{K}} + \omega_{\widetilde{N}}$, occur more frequently for almost all μ intervals, which shows the abundance of occurrence of this special resonant case. Additionally, it must be noted that the horizontal intervals illustrated by Figure 6 use a logarithmic scale. This highlights that until a certain μ value, the number of instances that leads to resonance coincidences increases expeditiously as μ decreases. However, for μ smaller or larger than certain thresholds, we do not observe any instances of resonance coincidences.

4.5.1 Case-1: $\Omega \simeq \omega_K - \omega_N = \omega_{\widetilde{K}}$

We begin this section by exploring a resonant scenario in which a natural frequency coincides with the difference between two other natural frequencies. Initially, the scenario where all indices are different will be studied. Table 2 displays some of the coincidence cases with the respective μ values.

Table 2: Instances of coinciding frequencies: $\omega_K - \omega_N = \omega_{\widetilde{K}}$ along with their respective μ values

Ω	μ	Ω	μ	Ω	μ
$\omega_4 - \omega_3 = \omega_2$	0.010935	$\omega_6 - \omega_5 = \omega_2$	0.003376	$\omega_8 - \omega_5 = \omega_4$	0.000718
$\omega_5 - \omega_2 = \omega_4$	0.005484	$\omega_7 - \omega_2 = \omega_6$	0.002310	$\omega_8 - \omega_5 = \omega_6$	0.015623
$\omega_5 - \omega_4 = \omega_2$	0.005484	$\omega_7 - \omega_6 = \omega_2$	0.002310	$\omega_8 - \omega_7 = \omega_2$	0.001688
$\omega_6 - \omega_3 = \omega_4$	0.001992	$\omega_8 - \omega_3 = \omega_6$	0.000891	$\cdots$	$\cdots$

Figure 7 illustrates the relation between the number of primary resonance coincidences and decreasing μ values. It can be observed that, as μ decreases, we observe an increasing density of resonance frequencies concerning the higher order modes ω_n.

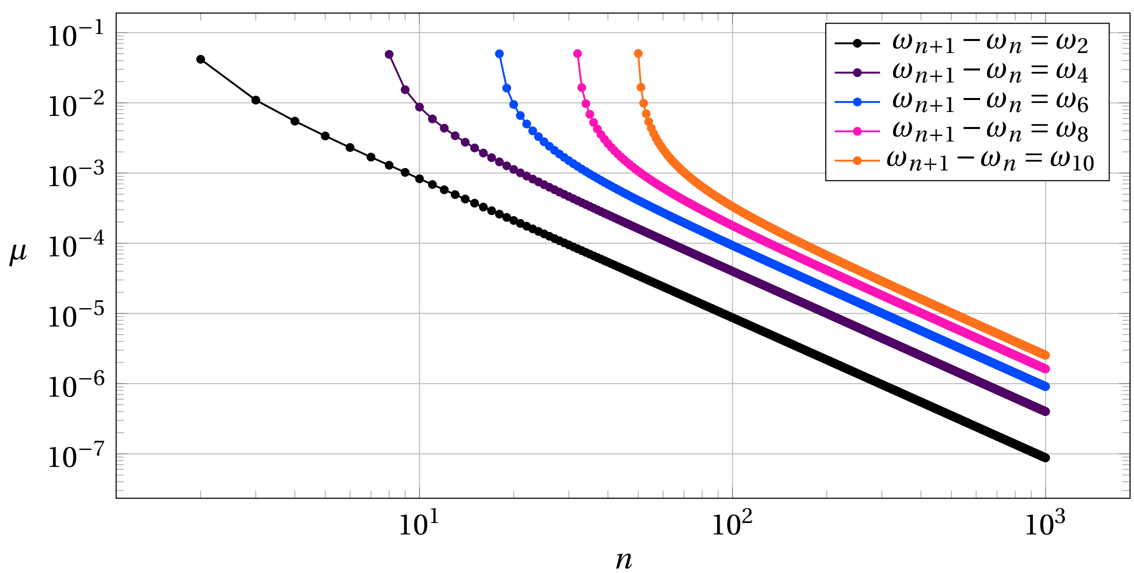

Figure 7: Values of μ corresponding to the condition $\omega_{n+1} - \omega_n = \omega_k$ for varying indices n and k

In Figure 8, the first 100 interactions occurring between the lower order modes, their μ values and the Ω pulsation frequency where the interactions occur are presented. At the top, the index values K, N and $\widetilde{K}$ are presented for each instances, i.e. values on horizontal axis. In the second and third plots, the μ and Ω values are given, for their respective index values.

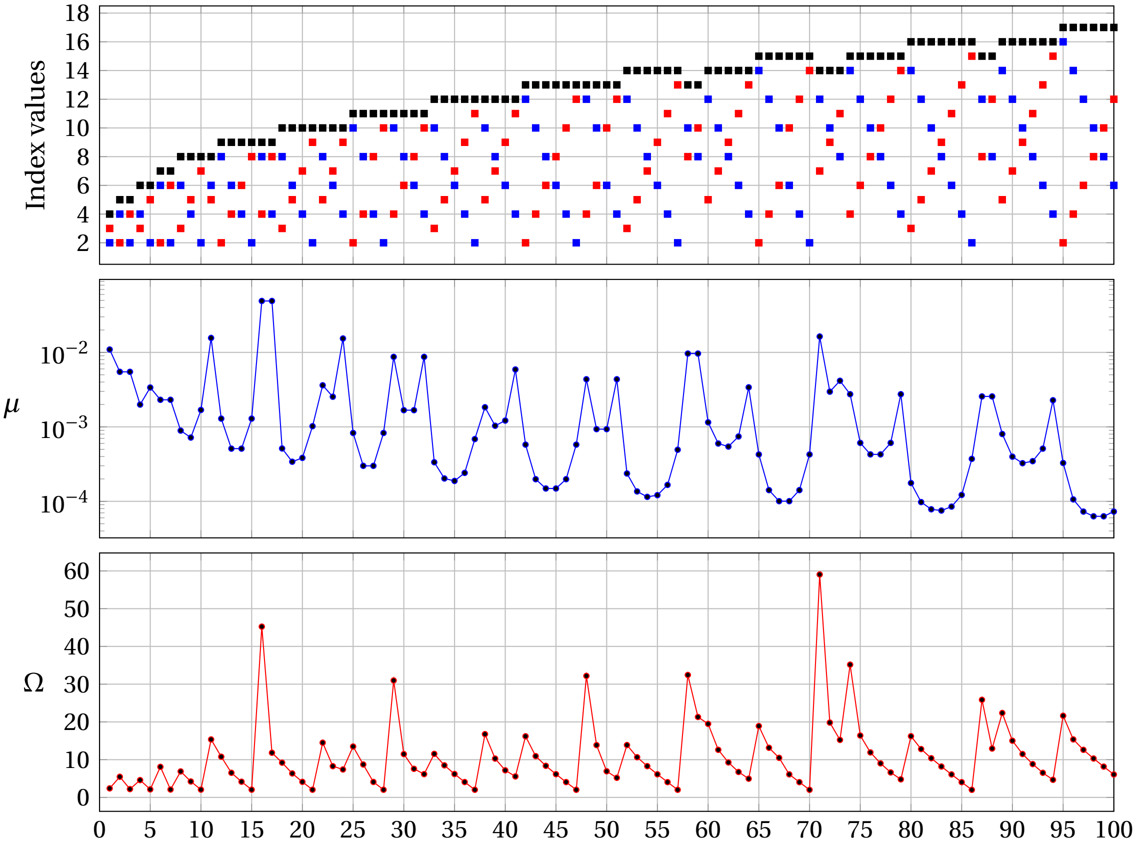

Figure 8: Index values for the first 100 coincidence cases: $\omega_{\blacksquare} - \omega_{\blacksquare} = \omega_{\blacksquare}$ and corresponding —•— μ and —•— Ω values respectively.

For the first 100 instances, the μ values associated with the coincidences tend to decrease slightly, with all being less than 0.1. However, for many instances, we do not observe significant differences between their μ values. Similarly, excitation frequencies Ω remain relatively close. This means that a selected fluid fluctuation frequency Ω, for a given μ value, can potentially fall within the order ε neighbourhood of other resonance frequencies and consequently can excite respective other modes as well.

For the situation where all indices in $\Omega = \omega_K - \omega_N = \omega_{\widetilde{K}}$ are distinct, it's observed that $\omega_K > \omega_N, \omega_{\widetilde{K}}$ and $\omega_N \neq \omega_{\widetilde{K}}$. This translates to conditions on their indices, namely $K > N, \widetilde{K}$ with $N \neq \widetilde{K}$. Under these constraints, Eq. (99) takes the form:

$$\begin{aligned}\partial_{t_0}^2 u_K + \omega_K^2 u_K = &-\alpha\pi^4 K^4 \omega_K [A_K \cos(\omega_K t_0) - B_K \sin(\omega_K t_0)] \\ &-2\omega_K \left[\dot{A}_K \cos(\omega_K t_0) - \dot{B}_K \sin(\omega_K t_0)\right] \\ &-\sqrt{\beta} V_1 \frac{2KN}{N^2-K^2} (\omega_K + \omega_N) [A_N \sin(\omega_K t_0) + B_N \cos(\omega_K t_0)] + \text{n.r.t.},\end{aligned} \tag{153a}$$

$$\begin{aligned}\partial_{t_0}^2 u_N + \omega_N^2 u_N = &-\alpha\pi^4 N^4 \omega_N [A_N \cos(\omega_N t_0) - B_N \sin(\omega_N t_0)] \\ &-2\omega_N \left[\dot{A}_N \cos(\omega_N t_0) - \dot{B}_N \sin(\omega_N t_0)\right] \\ &-\sqrt{\beta} V_1 \frac{2KN}{N^2-K^2} (\omega_K + \omega_N) [A_K \sin(\omega_N t_0) + B_K \cos(\omega_N t_0)] + \text{n.r.t.}.\end{aligned} \tag{153b}$$

$$\begin{aligned}\partial_{t_0}^2 u_{\widetilde{K}} + \omega_{\widetilde{K}}^2 u_{\widetilde{K}} = &-\alpha\pi^4 \widetilde{K}^4 \omega_{\widetilde{K}} \left[A_{\widetilde{K}} \cos\left(\omega_{\widetilde{K}} t_0\right) - B_{\widetilde{K}} \sin\left(\omega_{\widetilde{K}} t_0\right)\right] \\ &-2\omega_{\widetilde{K}} \left[\dot{A}_{\widetilde{K}} \cos\left(\omega_{\widetilde{K}} t_0\right) - \dot{B}_{\widetilde{K}} \sin\left(\omega_{\widetilde{K}} t_0\right)\right] \\ &-\frac{8}{\pi}\sqrt{\beta} V_1 \omega_{\widetilde{K}} C_{\widetilde{K}} \cos\left(\omega_{\widetilde{K}} t_0\right) + \text{n.r.t.}.\end{aligned} \tag{153c}$$

It can be seen that these equations are identical to the individual equations for $\Omega = \omega_K - \omega_N$ and $\Omega = \omega_{\widetilde{K}}$, in Section 4.4.2 and 4.4.1 respectively. In order to avoid the secular terms in $u_k(t_0, t_1)$, $k = K, N, \widetilde{K}$, coefficients must satisfy

$$\begin{aligned}\dot{A}_K &= -\frac{\alpha\pi^4 K^4}{2} A_K - \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_K}\right) B_N, \\ \dot{B}_K &= -\frac{\alpha\pi^4 K^4}{2} B_K + \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_K}\right) A_N, \\ \dot{A}_N &= -\frac{\alpha\pi^4 N^4}{2} A_N - \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_N}\right) B_K, \\ \dot{B}_N &= -\frac{\alpha\pi^4 N^4}{2} B_N + \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_N}\right) A_K.\end{aligned} \tag{154a}$$

$$\begin{aligned}\dot{A}_{\widetilde{K}} &= -\frac{\alpha\pi^4 \widetilde{K}^4}{2} A_{\widetilde{K}} - \frac{4}{\pi}\sqrt{\beta} V_1 C_{\widetilde{K}}, \\ \dot{B}_{\widetilde{K}} &= -\frac{\alpha\pi^4 \widetilde{K}^4}{2} B_{\widetilde{K}}.\end{aligned} \tag{154b}$$

Hence the solution of the equations will be identical to the individual systems (104) and (113). Similarly, the detuned case leads to systems (109) and (121) with corresponding index values.

4.5.1.1 The Pure Resonance of Case-1 With a Common Frequency: $\Omega = \omega_K - \omega_N = \omega_N$

A further special case occurs when the coinciding frequencies have a common natural frequency. In our case, the only possibility is that $N = \widetilde{K}$, thus $\Omega = \omega_K - \omega_N = \omega_N$. Some of the associated coinciding frequencies and the μ values for the coincidence are presented in Table 3.

Table 3: Instances of coinciding frequencies: $\omega_K - \omega_N = \omega_N$ along with their respective μ values

Ω	μ	Ω	μ	Ω	μ
$\omega_3 - \omega_2 = \omega_2$	0.041720	$\omega_{13} - \omega_8 = \omega_8$	0.000724	$\omega_{17} - \omega_{12} = \omega_{12}$	0.050397
$\omega_7 - \omega_4 = \omega_4$	0.001104	$\omega_{15} - \omega_8 = \omega_8$	0.000092	$\omega_{19} - \omega_{10} = \omega_{10}$	0.000044
$\omega_9 - \omega_6 = \omega_6$	0.004636	$\omega_{15} - \omega_{10} = \omega_{10}$	0.001669	$\omega_{19} - \omega_{12} = \omega_{12}$	0.000460
$\omega_{11} - \omega_6 = \omega_6$	0.000246	$\omega_{17} - \omega_{10} = \omega_{10}$	0.000258	$\cdots$	$\cdots$

With the Ω considered, resonant terms in Eq. (99) can be written as

$$\begin{aligned}
\partial_{t_0}^2 u_K + \omega_K^2 u_K = &- \alpha\pi^4 K^4 \omega_K [A_K \cos(\omega_K t_0) - B_K \sin(\omega_K t_0)] \\
&- 2\omega_K \big[\dot{A}_K \cos(\omega_K t_0) - \dot{B}_K \sin(\omega_K t_0)\big] \\
&- \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2} (\omega_K + \omega_N) [A_N \sin(\omega_K t_0) + B_N \cos(\omega_K t_0)] + \text{n.r.t.},
\end{aligned} \tag{155a}$$

$$\begin{aligned}
\partial_{t_0}^2 u_N + \omega_N^2 u_N = &- \alpha\pi^4 N^4 \omega_N [A_N \cos(\omega_N t_0) - B_N \sin(\omega_N t_0)] \\
&- 2\omega_N \big[\dot{A}_N \cos(\omega_N t_0) - \dot{B}_N \sin(\omega_N t_0)\big] \\
&- \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2} (\omega_K + \omega_N) [A_K \sin(\omega_N t_0) + B_K \cos(\omega_N t_0)] \\
&- \frac{8}{\pi} \sqrt{\beta} V_1 \omega_N C_N + \text{n.r.t.}
\end{aligned} \tag{155b}$$

Rearranging (155) results in

$$\begin{aligned}
&\partial_{t_0}^2 u_K + \omega_K^2 u_K \\
&= \cos(\omega_K t_0) \left[-\alpha\pi^4 K^4 \omega_K A_K - 2\omega_K \dot{A}_K - \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2} (\omega_K + \omega_N) B_N \right] \\
&+ \sin(\omega_K t_0) \left[\alpha\pi^4 K^4 \omega_K B_K + 2\omega_K \dot{B}_K - \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2} (\omega_K + \omega_N) A_N \right] + \text{n.r.t.},
\end{aligned} \tag{156a}$$

$$\begin{aligned}
&\partial_{t_0}^2 u_N + \omega_N^2 u_N \\
&= \cos(\omega_N t_0) \left[-\alpha\pi^4 N^4 \omega_N A_N - 2\omega_N \dot{A}_N - \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2} (\omega_K + \omega_N) B_K - \frac{8}{\pi} \sqrt{\beta} V_1 \omega_N C_N \right] \\
&+ \sin(\omega_N t_0) \left[\alpha\pi^4 N^4 \omega_N B_N + 2\omega_N \dot{B}_N - \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2} (\omega_K + \omega_N) A_K \right] + \text{n.r.t.}.
\end{aligned} \tag{156b}$$

In order to avoid secular terms in $u_K(t_0, t_1)$ and $u_N(t_0, t_1)$, coefficients $A_K(t_1)$, $B_K(t_1)$, $A_N(t_1)$ and $B_N(t_1)$ must satisfy

$$\begin{aligned}
\dot{A}_K &= -\frac{\alpha\pi^4 K^4}{2} A_K - \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_K}\right) B_N, \\
\dot{B}_K &= -\frac{\alpha\pi^4 K^4}{2} B_K + \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_K}\right) A_N, \\
\dot{A}_N &= -\frac{\alpha\pi^4 N^4}{2} A_N - \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_N}\right) B_K - \frac{4}{\pi}\sqrt{\beta} V_1 C_N, \\
\dot{B}_N &= -\frac{\alpha\pi^4 N^4}{2} B_N + \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_N}\right) A_K,
\end{aligned} \tag{157}$$

From the intermediate variables defined in (108) (with the index N) and (114), we rewrite Eq. (157) as

$$\begin{pmatrix} \dot{A}_K \\ \dot{B}_N \end{pmatrix} = \begin{pmatrix} -a & -p \\ q & -b \end{pmatrix} \begin{pmatrix} A_K \\ B_N \end{pmatrix}, \quad \begin{pmatrix} \dot{A}_N \\ \dot{B}_K \end{pmatrix} = \begin{pmatrix} -b & -q \\ p & -a \end{pmatrix} \begin{pmatrix} A_N \\ B_K \end{pmatrix} - \begin{pmatrix} c \\ 0 \end{pmatrix}. \tag{158}$$

The equilibrium of the Eq. (158) is $A_K = B_N = 0$, $A_N = -\frac{ac}{ab+pq}$, $B_K = -\frac{pc}{ab+pq}$ and the stability of the system is identical to the system 116, so is stable.

4.5.2 Case-2: $\Omega \simeq \omega_K + \omega_N = \omega_{\widetilde{K}}$

Now, we study the case where the sum of two natural frequencies is equal to a natural frequency. It is observed that Case-2 is only possible if $K > \widetilde{K} > N$, by considering $K > N$ to avoid repeating the scenarios considered. Since all the indices are different, the solution of pure resonance and detuned resonance are identical to systems $\Omega = \omega_K + \omega_N$ (131) and $\Omega = \omega_{\widetilde{K}}$ (104) and can be solved respectively. Thus we will not further study this case.

Furthermore, occurrences of Case-2 with the corresponding μ values are presented in the Table 4

Table 4: Instances of coinciding frequencies: $\omega_K + \omega_N = \omega_{\widetilde{K}}$ along with their respective μ values

Ω	μ	Ω	μ	Ω	μ
$\omega_3+\omega_2=\omega_4$	0.010935	$\omega_5+\omega_4=\omega_8$	0.000718	$\omega_6+\omega_5=\omega_9$	0.001438
$\omega_4+\omega_3=\omega_6$	0.001992	$\omega_6+\omega_3=\omega_7$	0.010140	$\omega_6+\omega_5=\omega_{10}$	0.000340
$\omega_5+\omega_2=\omega_6$	0.003376	$\omega_6+\omega_3=\omega_8$	0.000891	$\omega_7+\omega_2=\omega_8$	0.001688
$\omega_5+\omega_4=\omega_7$	0.004570	$\omega_6+\omega_5=\omega_8$	0.015623	$\cdots$	$\cdots$

In order to provide a general view on the interacted modes, μ values and Ω frequencies are presented in Figure 9 for the first 100 interactions in terms of index values.

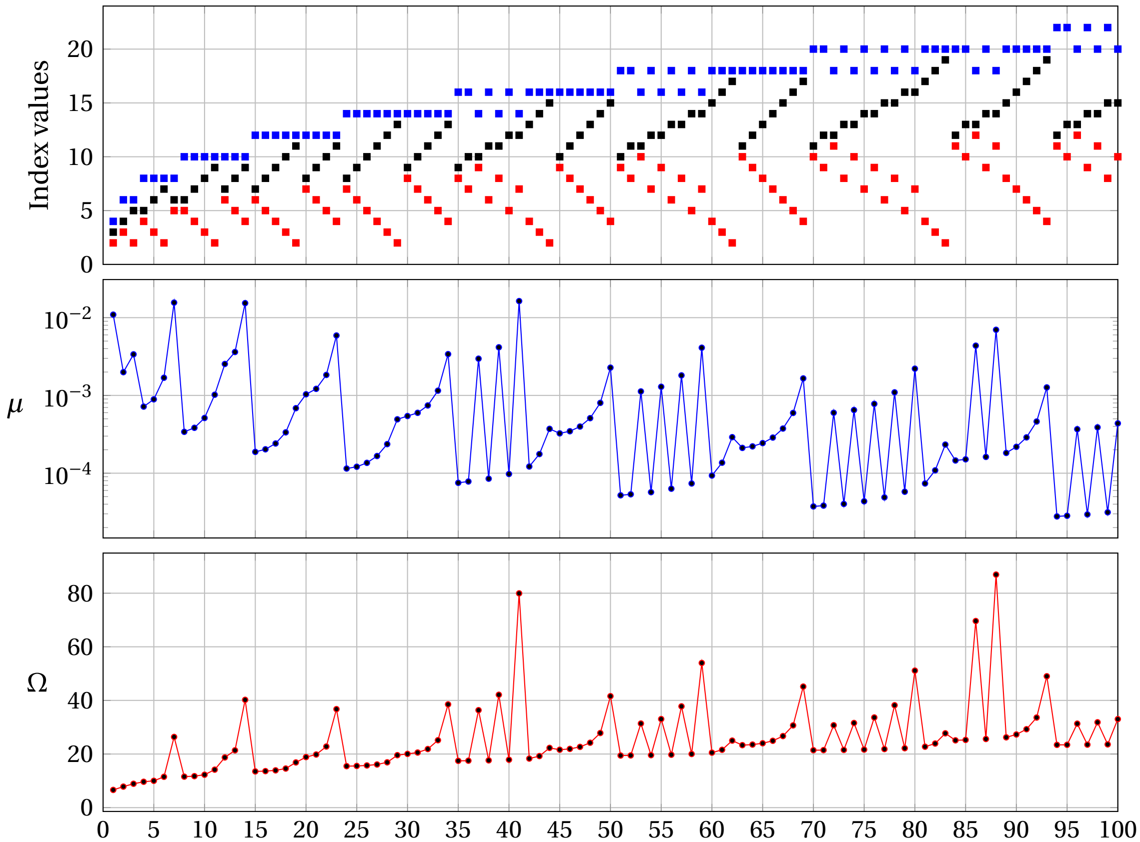

Figure 9: Index values for the first 100 coincidence cases: $\omega_{\blacksquare} + \omega_{\blacksquare} = \omega_{\blacksquare}$ and corresponding —•— μ and —•— Ω values respectively.

It can be observed that, as interacted mode numbers increase, the corresponding μ values tend to decrease. On the other hand we do not observe drastic changes for excitation frequency Ω.

4.5.3 Case-3: $\Omega \simeq \omega_K - \omega_N = \omega_{\tilde{K}} - \omega_{\tilde{N}}$

We will now examine the scenario where the difference of two natural frequencies coincides for a specific parameter value μ. There are two possibilities for this coincidence. First, all index values are distinct, i.e., $\Omega = \omega_K - \omega_N = \omega_{\tilde{K}} - \omega_{\tilde{N}}$. Alternatively, the subtractions involve a common frequency. In this case, it takes the form $\Omega = \omega_K - \omega_N = \omega_{\tilde{K}} - \omega_K$, as the only possibility. These two situations yield different physical outcomes, and we will explore these distinctions in the subsequent sections. Table 5 provides some occurrences of Case-3 for different index values.

Ω	μ	Ω	μ	Ω	μ
$\omega_8 - \omega_5 = \omega_6 - \omega_1$	0.005733	$\omega_{10} - \omega_7 = \omega_6 - \omega_1$	0.001368	$\omega_{11} - \omega_6 = \omega_9 - \omega_2$	0.002055
$\omega_9 - \omega_4 = \omega_8 - \omega_1$	0.010323	$\omega_{10} - \omega_9 = \omega_4 - \omega_1$	0.005471	$\omega_{11} - \omega_8 = \omega_6 - \omega_1$	0.000939
$\omega_9 - \omega_8 = \omega_4 - \omega_1$	0.010323	$\omega_{11} - \omega_4 = \omega_{10} - \omega_1$	0.003627	$\omega_{11} - \omega_8 = \omega_7 - \omega_2$	0.001413
$\omega_{10} - \omega_5 = \omega_8 - \omega_1$	0.001844	$\omega_{11} - \omega_6 = \omega_8 - \omega_1$	0.000939	$\cdots$	$\cdots$

Table 5: Instances of coinciding frequencies: $\omega_K - \omega_N = \omega_{\widetilde{K}} - \omega_{\widetilde{N}}$ along with their respective μ values

In Figure 10, interactions for Case-3 with all possible index values for the first 100 coincidences are presented.

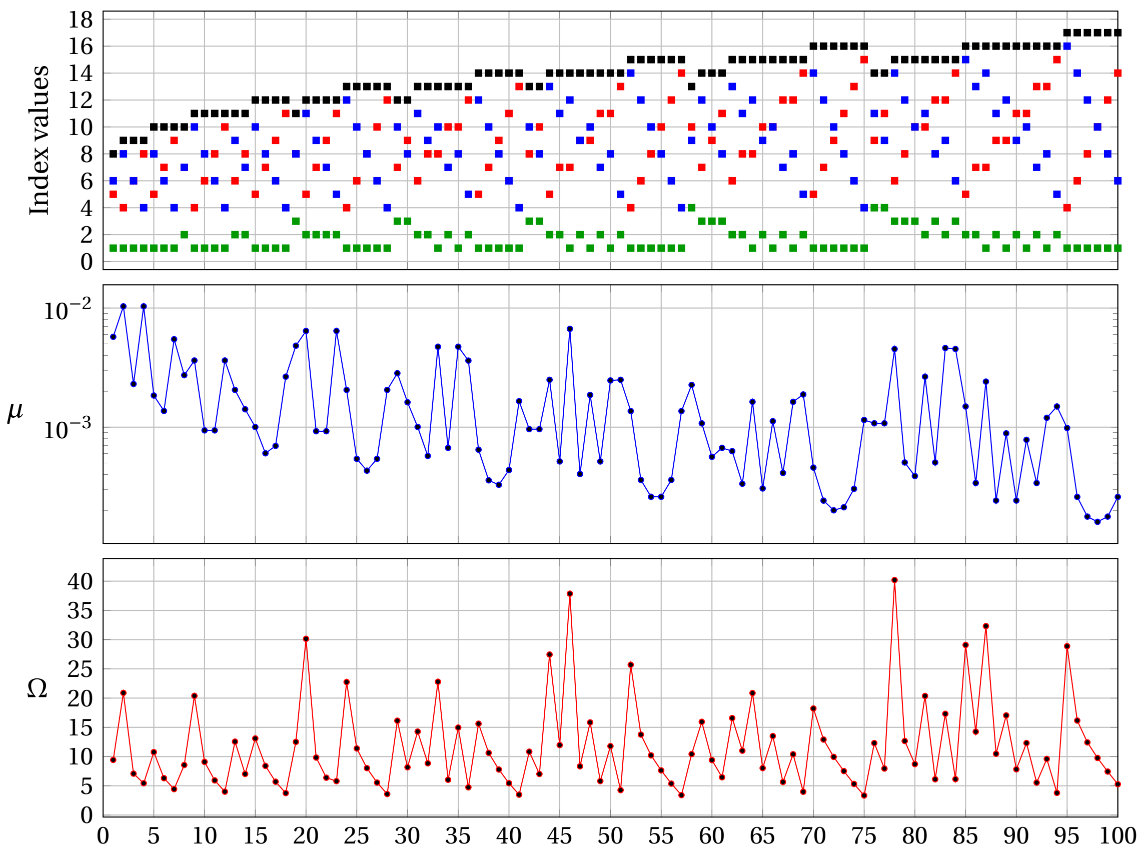

Figure 10: Index values for the first 100 coincidence cases: $\omega_{\blacksquare} - \omega_{\blacksquare} = \omega_{\blacksquare} - \omega_{\blacksquare}$ and corresponding —•— μ and —•— Ω values respectively.

For the scenario where all indices are distinct from each other, similarly we obtain two distinct 4 dimensional systems of (113) with respective indices. As we learned in Section 4.4.2, both systems are unconditionally stable.

4.5.3.1 The Pure Resonance of Case-3 With a Common Frequency: $\Omega = \omega_K - \omega_N = \omega_{\widetilde{K}} - \omega_K$

We now consider the case where both resonance frequencies have a common natural frequency. It can be observed that, this is only possible if $\widetilde{N} = K$, hence, $\Omega = \omega_K - \omega_N = \omega_{\widetilde{K}} - \omega_K$. Table 6 presents some instances where such coincidences occur for special μ values. With defined

Table 6: Instances of coinciding frequencies: $\omega_K - \omega_N = \omega_{\widetilde{K}} - \omega_K$ along with their respective μ values

Ω	μ	Ω	μ	Ω	μ
$\omega_9 - \omega_6 = \omega_6 - \omega_1$	0.002303	$\omega_{14} - \omega_9 = \omega_9 - \omega_2$	0.000404	$\omega_{16} - \omega_{11} = \omega_{11} - \omega_4$	0.000382
$\omega_{10} - \omega_7 = \omega_7 - \omega_2$	0.002729	$\omega_{15} - \omega_{10} = \omega_{10} - \omega_1$	0.001122	$\omega_{17} - \omega_{10} = \omega_{10} - \omega_1$	0.000160
$\omega_{11} - \omega_8 = \omega_8 - \omega_3$	0.004826	$\omega_{15} - \omega_{10} = \omega_{10} - \omega_3$	0.000389	$\omega_{17} - \omega_{12} = \omega_{12} - \omega_1$	0.012271
$\omega_{13} - \omega_8 = \omega_8 - \omega_1$	0.000431	$\omega_{16} - \omega_{11} = \omega_{11} - \omega_2$	0.001287	$\cdots$	$\cdots$

excitation frequency $\Omega = \omega_K - \omega_N = \omega_{\widetilde{K}} - \omega_K$, resonant terms in Eq. (99) can be written as

$$\begin{aligned}\partial_{t_0}^2 u_K + \omega_K^2 u_K = &- \alpha\pi^4 K^4 \omega_K [A_K \cos(\omega_K t_0) - B_K \sin(\omega_K t_0)] \\ &- 2\omega_K \left[\dot{A}_K \cos(\omega_K t_0) - \dot{B}_K \sin(\omega_K t_0)\right] \\ &- \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2} (\omega_K + \omega_N) [A_N \sin(\omega_K t_0) + B_N \cos(\omega_K t_0)] \\ &- \sqrt{\beta} V_1 \frac{2K\widetilde{K}}{\widetilde{K}^2 - K^2} (\omega_K + \omega_{\widetilde{K}}) \left[A_{\widetilde{K}} \sin(\omega_K t_0) + B_{\widetilde{K}} \cos(\omega_K t_0)\right] + \text{n.r.t.},\end{aligned} \tag{159a}$$

$$\begin{aligned}\partial_{t_0}^2 u_N + \omega_N^2 u_N = &- \alpha\pi^4 N^4 \omega_N [A_N \cos(\omega_N t_0) - B_N \sin(\omega_N t_0)] \\ &- 2\omega_N \left[\dot{A}_N \cos(\omega_N t_0) - \dot{B}_N \sin(\omega_N t_0)\right] \\ &- \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2} (\omega_K + \omega_N) [A_K \sin(\omega_N t_0) + B_K \cos(\omega_N t_0)] + \text{n.r.t.},\end{aligned} \tag{159b}$$

$$\begin{aligned}\partial_{t_0}^2 u_{\widetilde{K}} + \omega_{\widetilde{K}}^2 u_{\widetilde{K}} = &- \alpha\pi^4 \widetilde{K}^4 \omega_{\widetilde{K}} \left[A_{\widetilde{K}} \cos\left(\omega_{\widetilde{K}} t_0\right) - B_{\widetilde{K}} \sin\left(\omega_{\widetilde{K}} t_0\right)\right] \\ &- 2\omega_{\widetilde{K}} \left[A'_{\widetilde{K}} \cos\left(\omega_{\widetilde{K}} t_0\right) - B'_{\widetilde{K}} \sin\left(\omega_{\widetilde{K}} t_0\right)\right] \\ &+ \sqrt{\beta} V_1 \frac{2K\widetilde{K}}{\widetilde{K}^2 - K^2} (\omega_K + \omega_{\widetilde{K}}) \left[A_K \sin\left(\omega_{\widetilde{K}} t_0\right) + B_K \cos\left(\omega_{\widetilde{K}} t_0\right)\right] + \text{n.r.t.}.\end{aligned} \tag{159c}$$

The terms that leads to resonance can be rewritten as

$$
\begin{aligned}
\dot{A}_K &= -\frac{\alpha\pi^4 K^4}{2} A_K - \sqrt{\beta} V_1 \left(\frac{KN}{N^2 - K^2}\right)\left(\frac{\omega_K + \omega_N}{\omega_K}\right) B_N + \sqrt{\beta} V_1 \left(\frac{K\widetilde{K}}{\widetilde{K}^2 - K^2}\right)\left(\frac{\omega_K + \omega_{\widetilde{K}}}{\omega_K}\right) B_{\widetilde{K}},\\
\dot{B}_K &= -\frac{\alpha\pi^4 K^4}{2} B_K + \sqrt{\beta} V_1 \left(\frac{KN}{N^2 - K^2}\right)\left(\frac{\omega_K + \omega_N}{\omega_K}\right) A_N - \sqrt{\beta} V_1 \left(\frac{K\widetilde{K}}{\widetilde{K}^2 - K^2}\right)\left(\frac{\omega_K + \omega_{\widetilde{K}}}{\omega_K}\right) A_{\widetilde{K}},\\
\dot{A}_N &= -\frac{\alpha\pi^4 N^4}{2} A_N - \sqrt{\beta} V_1 \left(\frac{KN}{N^2 - K^2}\right)\left(\frac{\omega_K + \omega_N}{\omega_N}\right) B_K,\\
\dot{B}_N &= -\frac{\alpha\pi^4 N^4}{2} B_N + \sqrt{\beta} V_1 \left(\frac{KN}{N^2 - K^2}\right)\left(\frac{\omega_K + \omega_N}{\omega_N}\right) A_K,\\
\dot{A}_{\widetilde{K}} &= -\frac{\alpha\pi^4 \widetilde{K}^4}{2} A_{\widetilde{K}} + \sqrt{\beta} V_1 \left(\frac{K\widetilde{K}}{\widetilde{K}^2 - K^2}\right)\left(\frac{\omega_K + \omega_{\widetilde{K}}}{\omega_{\widetilde{K}}}\right) B_K,\\
\dot{B}_{\widetilde{K}} &= -\frac{\alpha\pi^4 \widetilde{K}^4}{2} B_{\widetilde{K}} - \sqrt{\beta} V_1 \left(\frac{K\widetilde{K}}{\widetilde{K}^2 - K^2}\right)\left(\frac{\omega_K + \omega_{\widetilde{K}}}{\omega_{\widetilde{K}}}\right) A_K.
\end{aligned}
\tag{160}
$$

If we introduce the intermediate parameters:

$$
\begin{gathered}
a := \frac{\alpha\pi^4 K^4}{2}, \quad b := \frac{\alpha\pi^4 N^4}{2}, \quad c := \frac{\alpha\pi^4 \widetilde{K}^4}{2}\\
p := \sqrt{\beta} V_1 \left(\frac{KN}{N^2 - K^2}\right)\left(\frac{\omega_K + \omega_N}{\omega_K}\right), \quad q := \sqrt{\beta} V_1 \left(\frac{KN}{N^2 - K^2}\right)\left(\frac{\omega_K + \omega_N}{\omega_N}\right),\\
s := \sqrt{\beta} V_1 \left(\frac{K\widetilde{K}}{\widetilde{K}^2 - K^2}\right)\left(\frac{\omega_K + \omega_{\widetilde{K}}}{\omega_K}\right), \quad r := \sqrt{\beta} V_1 \left(\frac{K\widetilde{K}}{\widetilde{K}^2 - K^2}\right)\left(\frac{\omega_K + \omega_{\widetilde{K}}}{\omega_{\widetilde{K}}}\right),
\end{gathered}
\tag{161}
$$

Eq. (160) becomes

$$
\begin{pmatrix} \dot{A}_k \\ \dot{B}_{\widetilde{K}} \\ \dot{B}_N \end{pmatrix} = \begin{pmatrix} -a & s & -p \\ -r & -c & 0 \\ q & 0 & -b \end{pmatrix} \begin{pmatrix} A_K \\ B_{\widetilde{K}} \\ B_N \end{pmatrix}, \quad \begin{pmatrix} \dot{B}_k \\ \dot{A}_{\widetilde{K}} \\ \dot{A}_N \end{pmatrix} = \begin{pmatrix} -a & -s & p \\ r & -c & 0 \\ -q & 0 & -b \end{pmatrix} \begin{pmatrix} B_K \\ A_{\widetilde{K}} \\ A_N \end{pmatrix}.
\tag{162}
$$

The characteristic polynomial for both system in Eq. (162) are identical and that is

$$
r^3 + (a + b + c)r^2 + (ab + ac + bc + pq + rs)r + (abc + cpq + brs) = 0.
\tag{163}
$$

One can define the roots of the characteristic polynomial by usin the Cardano's formula for given parameter values. In order to obtain a manageable explicit expression for stability of system (162), we apply the Routh-Hurwitz criterion to the characteristic polynomial (163). For a general, cubic polynomial $d_3 r^3 + d_2 r^2 + d_1 r^1 + d_0 = 0$, Routh-Hurwitz stability criterion is satisfied if $T_0, T_1, T_2, T_3 > 0$ where

$$
T_0 = d_3, \quad T_1 = d_2, \quad T_2 = \det\begin{pmatrix} d_2 & d_3 \\ d_0 & d_1 \end{pmatrix}, \quad T_3 = \begin{pmatrix} d_2 & d_3 & 0 \\ d_0 & d_1 & d_2 \\ 0 & 0 & d_0 \end{pmatrix}.
\tag{164}
$$

If we substitute the characteristic polynomial obtained in Eq. (163), we obtain

$$
\begin{aligned}
T_0 = 1 =, \quad T_1 = a+b+c, \quad T_2 = (a+b)(b+c)(a+c)+(a+b)pq+(a+c)rs \\
T_3 = (abc+ars+cpq)\big[(a+b)(b+c)(a+c)+(a+b)pq+(b+c)rs\big].
\end{aligned}
\tag{165}
$$

Parameters a, b, and c correspond to structural damping terms and are positive. Similarly, pq and rs are positive as well. Therefore, it is evident that T_0, T_1, T_2 and T_3 are all greater than zero for all parameter values and vibration modes. As a result, we can conclude that the coefficients are stable for **Case-1**.

4.5.3.2 Detuned Case-3: $\Omega = \omega_K - \omega_N + \varepsilon\varphi = \omega_{\widetilde{K}} - \omega_K + \varepsilon\varphi$

Now we study the case with a detuning. If we substitute $\Omega = \omega_K - \omega_N + \varepsilon\varphi = \omega_{\widetilde{K}} - \omega_K + \varepsilon\varphi$ into the Eq. (99), we obtain

$$
\begin{aligned}
\dot{A}_K &= -\frac{\alpha\pi^4 K^4}{2} A_K - \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_K}\right)\left[\sin(\varphi t_1) A_N + \cos(\varphi t_1) B_N\right], \\
&\quad - \sqrt{\beta} V_1 \left(\frac{K\widetilde{K}}{\widetilde{K}^2-K^2}\right)\left(\frac{\omega_K+\omega_{\widetilde{K}}}{\omega_K}\right)\left[\sin(\varphi t_1) A_{\widetilde{K}} - \cos(\varphi t_1) B_{\widetilde{K}}\right], \\
\dot{B}_K &= -\frac{\alpha\pi^4 K^4}{2} B_K + \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_K}\right)\left[\cos(\varphi t_1) A_N - \sin(\varphi t_1) B_N\right], \\
&\quad - \sqrt{\beta} V_1 \left(\frac{K\widetilde{K}}{\widetilde{K}^2-K^2}\right)\left(\frac{\omega_K+\omega_{\widetilde{K}}}{\omega_K}\right)\left[\cos(\varphi t_1) A_{\widetilde{K}} + \sin(\varphi t_1) B_{\widetilde{K}}\right], \\
\dot{A}_N &= -\frac{\alpha\pi^4 N^4}{2} A_N + \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_N}\right)\left[\sin(\varphi t_1) A_K - \cos(\varphi t_1) B_K\right], \\
\dot{B}_N &= -\frac{\alpha\pi^4 N^4}{2} B_N + \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_N}\right)\left[\cos(\varphi t_1) A_K + \sin(\varphi t_1) B_K\right], \\
\dot{A}_{\widetilde{K}} &= -\frac{\alpha\pi^4 \widetilde{K}^4}{2} A_{\widetilde{K}} + \sqrt{\beta} V_1 \left(\frac{K\widetilde{K}}{\widetilde{K}^2-K^2}\right)\left(\frac{\omega_K+\omega_{\widetilde{K}}}{\omega_{\widetilde{K}}}\right)\left[\sin(\varphi t_1) A_K + \cos(\varphi t_1) B_K\right], \\
\dot{B}_{\widetilde{K}} &= -\frac{\alpha\pi^4 \widetilde{K}^4}{2} B_{\widetilde{K}} - \sqrt{\beta} V_1 \left(\frac{K\widetilde{K}}{\widetilde{K}^2-K^2}\right)\left(\frac{\omega_K+\omega_{\widetilde{K}}}{\omega_{\widetilde{K}}}\right)\left[\cos(\varphi t_1) A_K - \sin(\varphi t_1) B_K\right].
\end{aligned}
\tag{166}
$$

From the intermediate parameters defined fore the pure resonant case in (161), the linear system can be obtained as

$$
\begin{aligned}
\dot{A}_K &= -aA_K - p\left[\sin(\varphi t_1) A_N + \cos(\varphi t_1) B_N\right] - s\left[\sin(\varphi t_1) A_{\widetilde{K}} - \cos(\varphi t_1) B_{\widetilde{K}}\right], \\
\dot{B}_K &= -aB_K + p\left[\cos(\varphi t_1) A_N - \sin(\varphi t_1) B_N\right] - s\left[\cos(\varphi t_1) A_{\widetilde{K}} + \sin(\varphi t_1) B_{\widetilde{K}}\right], \\
\dot{A}_N &= -bA_N + q\left[\sin(\varphi t_1) A_K - \cos(\varphi t_1) B_K\right], \\
\dot{B}_N &= -bB_N + q\left[\cos(\varphi t_1) A_K + \sin(\varphi t_1) B_K\right], \\
\dot{A}_{\widetilde{K}} &= -cA_{\widetilde{K}} + r\left[\sin(\varphi t_1) A_K + \cos(\varphi t_1) B_K\right], \\
\dot{B}_{\widetilde{K}} &= -cB_{\widetilde{K}} - r\left[\cos(\varphi t_1) A_K - \sin(\varphi t_1) B_K\right].
\end{aligned}
\tag{167}
$$

It can be observed that the detuned case leads to a time dependent linear dynamical system. Where the intermediate calculations are presented in the Appendix B.2, we obtain the time independent system for A_K and B_K as below:

$$\begin{aligned}\dddot{A}_K =&-(a+b+c)\ddot{A}_K-(ab+ac+bc+pq+rs)\dot{A}_K-(abc+brs+cpq)A_K\\&-\varphi\big[-(b-c)\dot{B}_K-(ab-ac+pq-rs)B_K\big]-\varphi^2(\dot{A}_K+aA_K),\\ \dddot{B}_K =&-(a+b+c)\ddot{B}_K-(ab+ac+bc+pq+rs)\dot{B}_K-(abc+brs+cpq)B_K,\\&-\varphi\big[(b-c)\dot{A}_K+(ab-ac+pq-rs)A_K\big]-\varphi^2(\dot{B}_K+aB_K).\end{aligned} \tag{168}$$

The above equation leads to a sextic characteristic polynomial for its eigenvalues. Therefore we can neither analytically solve it nor obtain a manageable Routh-Hurwitz stability criterion for arbitrary parameter values or frequency numbers. Hence, a case with definite index values of natural frequencies will be studied as an example.

4.5.3.3 Case-3 Example: $\Omega = \omega_6 - \omega_1 + \varepsilon\varphi = \omega_9 - \omega_6 + \varepsilon\varphi$

As an example, we study the case where $K = 6$, $N = 1$ and $\widetilde{K} = 9$, which is the case that satisfies $\omega_K - \omega_N = \omega_{\widetilde{K}} - \omega_N$ with smallest indices ($K + N + \widetilde{K}$ is the smallest). With this choice of indices and from (84), we obtain that $\mu = \frac{1}{44\pi^2} \simeq 0.0023028$.

If we substitute $\mu = \frac{1}{44\pi^2}$, $K = 6$, $N = 1$ and $\widetilde{K} = 9$, and by introducing variables $\widetilde{\alpha} := \alpha\pi^4/2$ and $\widetilde{\beta} := \sqrt{\beta}V_1$ we obtain

$$\begin{gathered}\omega_1 = \frac{\pi 3\sqrt{5}}{\sqrt{44}}, \qquad \omega_6 = \frac{\pi 24\sqrt{5}}{\sqrt{44}}, \qquad \omega_9 = \frac{\pi 45\sqrt{5}}{\sqrt{44}},\\ a = 1296\widetilde{\alpha}, \qquad b = \widetilde{\alpha}, \qquad c = 6561\widetilde{\alpha},\\ p = -\frac{27\widetilde{\beta}}{140}, \quad q = -\frac{54\widetilde{\beta}}{35}, s = \frac{69\widetilde{\beta}}{20}, \quad r = \frac{46\widetilde{\beta}}{25}.\end{gathered} \tag{169}$$

If we substitute these into Eq. (168), we obtain the characteristic equation in the form

$$d_6\lambda^6 + d_5\lambda^5 + \cdots + d_1\lambda + d_0 = 0 \tag{170}$$

where the coefficients of $d_i, i = 1, \ldots, 6$ with respect to $\widetilde{\alpha}$, $\widetilde{\beta}$ and φ are obtained to be:

$$
\begin{aligned}
d_6 = 1, \quad d_5 = 15716\tilde{\alpha}, \quad d_4 = 78769990\tilde{\alpha}^2 + \frac{81408}{6125}\tilde{\beta}^2 + 2\varphi^2,& \\
d_3 = 1337745148 20\tilde{\alpha}^3 + \frac{663696672}{6125}\tilde{\alpha}\tilde{\beta}^2 + 18308\tilde{\alpha}\varphi^2,& \\
d_2 = 725692741216 65\tilde{\alpha}^4 + \frac{881390319168}{6125}\tilde{\alpha}^2\tilde{\beta}^2 + \frac{1656815616}{37515625}\tilde{\beta}^4& \\
+\left(80423362\tilde{\alpha}^2 + \frac{81408}{6125}\tilde{\beta}^2\right)\varphi^2 + \varphi^4,& \\
d_1 = 1447375397700256\tilde{\alpha}^5 + \frac{204891216113952}{6125}\tilde{\alpha}^3\tilde{\beta}^2 + \frac{976595116032}{37515625}\tilde{\alpha}\tilde{\beta}^4& \\
+\left(133620383808\tilde{\alpha}^3 + \frac{615711456}{6125}\tilde{\alpha}\tilde{\beta}^2\right)\varphi^2 + 2592\tilde{\alpha}\varphi^4,& \\
d_0 = 723019613339136\tilde{\alpha}^6 + \frac{204010489410048}{6125}\tilde{\alpha}^4\tilde{\beta}^2 + \frac{143911309660416}{37515625}\tilde{\alpha}^2\tilde{\beta}^4& \\
+\left(72301963018752\tilde{\alpha}^4 + \frac{661227867648}{6125}\tilde{\alpha}^2\tilde{\beta}^2 + \frac{1373369481}{37515625}\tilde{\beta}^4\right)\varphi^2 + 1679616\tilde{\alpha}^2\varphi^4.&
\end{aligned} \tag{171}
$$

The Routh-Hurwitz stability criterion is satisfied for a sixth order system by $T_i > 0$, $i = 0, 1, \ldots, 6$ where T_i are defined as

$$
\begin{aligned}
&T_0 := d_6, \quad T_1 := d_5, \\
&T_2 := \det\begin{pmatrix} d_5 & d_6 \\ d_3 & d_4 \end{pmatrix}, \quad T_3 := \det\begin{pmatrix} d_5 & d_6 & 0 \\ d_3 & d_4 & d_5 \\ d_1 & d_2 & d_3 \end{pmatrix}, \quad T_4 := \det\begin{pmatrix} d_5 & d_6 & 0 & 0 \\ d_3 & d_4 & d_5 & d_6 \\ d_1 & d_2 & d_3 & d_4 \\ 0 & d_0 & d_1 & d_2 \end{pmatrix}, \\
&T_5 := \det\begin{pmatrix} d_5 & d_6 & 0 & 0 & 0 \\ d_3 & d_4 & d_5 & d_6 & 0 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ 0 & d_0 & d_1 & d_2 & d_3 \\ 0 & 0 & 0 & d_0 & d_1 \end{pmatrix}, \quad T_6 := \det\begin{pmatrix} d_5 & d_6 & 0 & 0 & 0 & 0 \\ d_3 & d_4 & d_5 & d_6 & 0 & 0 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ 0 & d_0 & d_1 & d_2 & d_3 & d_4 \\ 0 & 0 & 0 & d_0 & d_1 & d_2 \\ 0 & 0 & 0 & 0 & 0 & d_0 \end{pmatrix}.
\end{aligned} \tag{172}
$$

By substituting the coefficients obtained in (171) into (172), we obtain $T_i > 0$ for $i = 0, 1, \ldots, 6$. Consequently, we can conclude that $A_6(t_1)$ and $B_6(t_1)$ are stable for all α, β, V_1 parameters and all small frequency deviations $\varepsilon\varphi$ from $\Omega = \omega_6 - \omega_1 = \omega_9 - \omega_6$. This will lead from Appendix C.1 and C.2 that the other coefficients are also stable. The same procedure can be followed for other coincidence cases, such as in table 6.

4.5.4 Case-4: $\Omega \simeq \omega_K - \omega_N = \omega_{\tilde{K}} + \omega_{\tilde{N}}$

Now we study the case where a difference of natural frequencies is equal to a sum of natural frequencies for a critical μ value. One possibility for this case is that all indices are distinct, which is equivalent the systems $\Omega = \omega_K - \omega_N$ (113) and $\Omega = \omega_{\tilde{K}} + \omega_{\tilde{N}}$ (131) individually and will not be discussed further. A number of instances where Case-4 occurs for different modes are presented in Table 7 and in Figure 11, first 100 instances for all possible index values, their respective μ and Ω values are illustrated.

Table 7: Instances of coinciding frequencies: $\omega_K - \omega_N = \omega_{\tilde{K}} + \omega_{\tilde{N}}$ along with their respective μ values

Ω	μ	Ω	μ	Ω	μ
$\omega_4 - \omega_1 = \omega_3 + \omega_2$	0.039017	$\omega_6 - \omega_1 = \omega_4 + \omega_3$	0.004940	$\omega_7 - \omega_2 = \omega_4 + \omega_3$	0.002136
$\omega_4 - \omega_3 = \omega_2 + \omega_1$	0.039017	$\omega_6 - \omega_1 = \omega_5 + \omega_2$	0.009380	$\omega_7 - \omega_2 = \omega_5 + \omega_4$	0.021868
$\omega_5 - \omega_2 = \omega_4 + \omega_1$	0.016282	$\omega_6 - \omega_3 = \omega_4 + \omega_1$	0.004940	$\omega_7 - \omega_2 = \omega_6 + \omega_1$	0.006211
$\omega_5 - \omega_4 = \omega_2 + \omega_1$	0.016282	$\omega_6 - \omega_5 = \omega_2 + \omega_1$	0.009380	$\cdots$	$\cdots$

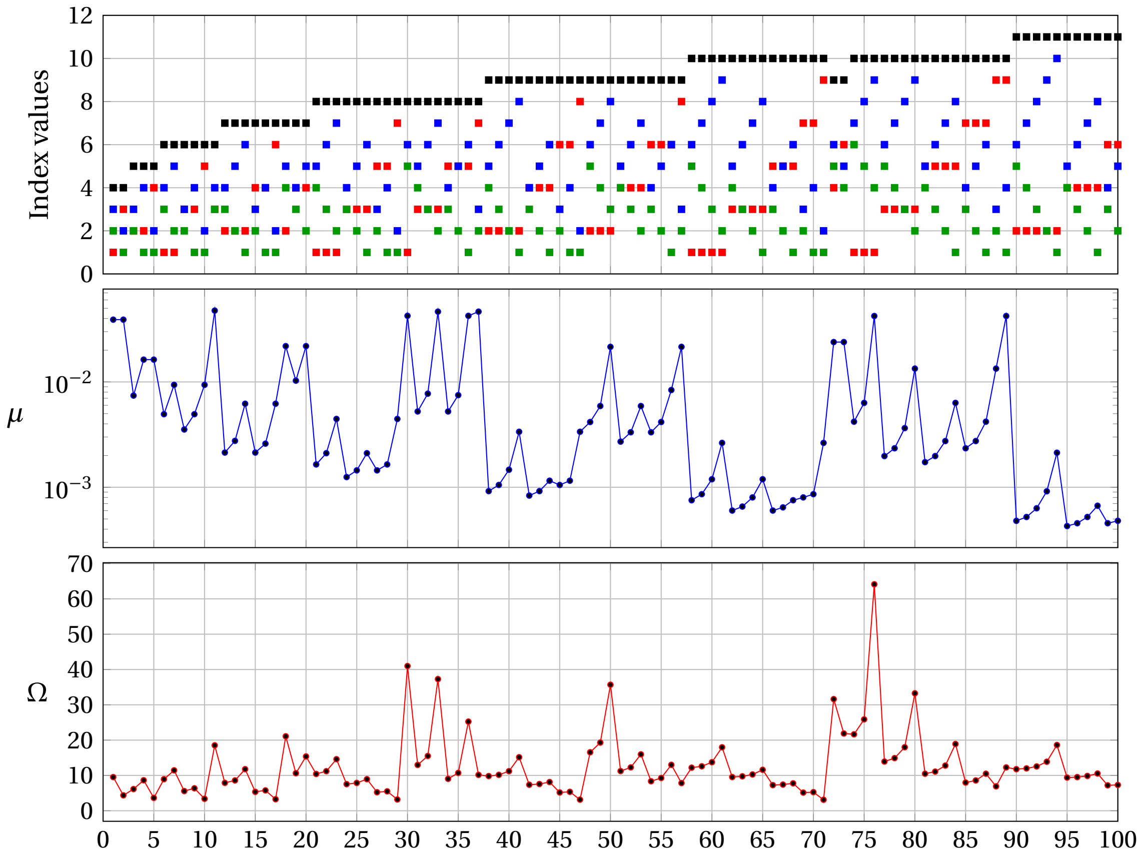

Figure 11: Index values for the first 100 coincidence cases: $\omega_{\blacksquare} - \omega_{\blacksquare} = \omega_{\blacksquare} + \omega_{\blacksquare}$ and corresponding μ and Ω values respectively.

4.5.4.1 The Pure Resonance of Case-4 With a Common Frequency: $\Omega = \omega_K - \omega_N = \omega_{\widetilde{K}} + \omega_N$

A more special case occurs when the difference and sum of natural frequencies coincide with a common natural frequency. Some instances leading to this coincidence case are shown in table 8. If we substitute $\Omega = \omega_K - \omega_N = \omega_{\widetilde{K}} + \omega_N$ into the Eq. (99) we get

Table 8: Instances of coinciding frequencies: $\omega_K - \omega_N = \omega_{\widetilde{K}} + \omega_N$ along with their respective μ values

Ω	μ	Ω	μ	Ω	μ
$\omega_5 - \omega_2 = \omega_3 + \omega_2$	0.007420	$\omega_7 - \omega_4 = \omega_4 + \omega_1$	0.002595	$\omega_8 - \omega_5 = \omega_5 + \omega_2$	0.007494
$\omega_6 - \omega_3 = \omega_3 + \omega_2$	0.003537	$\omega_7 - \omega_4 = \omega_4 + \omega_3$	0.010286	$\omega_9 - \omega_2 = \omega_7 + \omega_2$	0.001467
$\omega_6 - \omega_3 = \omega_4 + \omega_3$	0.047544	$\omega_8 - \omega_3 = \omega_4 + \omega_3$	0.001250	$\omega_9 - \omega_4 = \omega_4 + \omega_3$	0.000834
$\omega_7 - \omega_2 = \omega_5 + \omega_2$	0.002758	$\omega_8 - \omega_3 = \omega_6 + \omega_3$	0.007738	$\cdots$	$\cdots$

$$
\begin{aligned}
\partial_{t_0}^2 u_K + \omega_K^2 u_K = &- \alpha\pi^4 K^4 \omega_K [A_K \cos(\omega_K t_0) - B_K \sin(\omega_K t_0)] \\
&- 2\omega_K \left[\dot{A}_K \cos(\omega_K t_0) - \dot{B}_K \sin(\omega_K t_0)\right] \\
&- \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2} (\omega_K + \omega_N) [A_N \sin(\omega_K t_0) + B_N \cos(\omega_K t_0)] + \text{n.r.t.},
\end{aligned} \tag{173a}
$$

$$
\begin{aligned}
\partial_{t_0}^2 u_N + \omega_N^2 u_N = &- \alpha\pi^4 N^4 \omega_N [A_N \cos(\omega_N t_0) - B_N \sin(\omega_N t_0)] \\
&- 2\omega_N \left[\dot{A}_N \cos(\omega_N t_0) - \dot{B}_N \sin(\omega_N t_0)\right] \\
&- \sqrt{\beta} V_1 \frac{2KN}{N^2 - K^2} (\omega_K + \omega_N) [A_K \sin(\omega_N t_0) + B_K \cos(\omega_N t_0)] \\
&- \sqrt{\beta} V_1 \frac{2\widetilde{K}N}{N^2 - \widetilde{K}^2} (\omega_{\widetilde{K}} - \omega_N) \left[-A_{\widetilde{K}} \sin(\omega_N t_0) + B_{\widetilde{K}} \cos(\omega_N t_0)\right] + \text{n.r.t.},
\end{aligned} \tag{173b}
$$

$$
\begin{aligned}
\partial_{t_0}^2 u_{\widetilde{K}} + \omega_{\widetilde{K}}^2 u_{\widetilde{K}} = &- \alpha\pi^4 \widetilde{K}^4 \omega_{\widetilde{K}} \left[A_{\widetilde{K}} \cos\left(\omega_{\widetilde{K}} t_0\right) - B_{\widetilde{K}} \sin\left(\omega_{\widetilde{K}} t_0\right)\right] \\
&- 2\omega_{\widetilde{K}} \left[A'_{\widetilde{K}} \cos\left(\omega_{\widetilde{K}} t_0\right) - B'_{\widetilde{K}} \sin\left(\omega_{\widetilde{K}} t_0\right)\right] \\
&+ \sqrt{\beta} V_1 \frac{2\widetilde{K}N}{N^2 - \widetilde{K}^2} (\omega_{\widetilde{K}} - \omega_N) \left[A_N \sin\left(\omega_{\widetilde{K}} t_0\right) + B_N \cos\left(\omega_{\widetilde{K}} t_0\right)\right] + \text{n.r.t.}.
\end{aligned} \tag{173c}
$$

The terms that leads to resonance can be rewritten as

$$\begin{aligned}
\partial_{t_0}^2 u_K + \omega_K^2 u_K =& \cos(\omega_K t_0)\left[-\alpha\pi^4 K^4\omega_K A_K - 2\omega_K \dot{A}_K - \sqrt{\beta}V_1\frac{2KN}{N^2-K^2}(\omega_K+\omega_N)B_N\right]\\
&+\sin(\omega_K t_0)\left[\alpha\pi^4 K^4\omega_K B_K + 2\omega_K \dot{B}_K - \sqrt{\beta}V_1\frac{2KN}{N^2-K^2}(\omega_K+\omega_{\widetilde{K}})A_N\right]\\
&+\text{n.r.t.},
\end{aligned} \tag{174a}$$

$$\begin{aligned}
\partial_{t_0}^2 u_N + \omega_N^2 u_N =& \cos(\omega_N t_0)\left[-\alpha\pi^4 N^4\omega_N A_N - 2\omega_N \dot{A}_N - \sqrt{\beta}V_1\frac{2KN}{N^2-K^2}(\omega_K+\omega_N)B_K\right.\\
&\left. - \sqrt{\beta}V_1\frac{2N\widetilde{K}}{N^2-\widetilde{K}^2}(\omega_{\widetilde{K}}-\omega_N)B_{\widetilde{K}}\right]\\
&+\sin(\omega_N t_0)\left[\alpha\pi^4 N^4\omega_N B_N + 2\omega_N \dot{B}_N - \sqrt{\beta}V_1\frac{2KN}{N^2-K^2}(\omega_K+\omega_N)A_K\right.\\
&\left. + \sqrt{\beta}V_1\frac{2N\widetilde{K}}{N^2-\widetilde{K}^2}(\omega_{\widetilde{K}}-\omega_N)A_{\widetilde{K}}\right]\\
&+\text{n.r.t.},
\end{aligned} \tag{174b}$$

$$\begin{aligned}
\partial_{t_0}^2 u_{\widetilde{K}} + \omega_{\widetilde{K}}^2 u_{\widetilde{K}} =& \cos\left(\omega_{\widetilde{K}} t_0\right)\left[-\alpha\pi^4 \widetilde{K}^4\omega_{\widetilde{K}} A_{\widetilde{K}} - 2\omega_{\widetilde{K}} \dot{A}_{\widetilde{K}} + \sqrt{\beta}V_1\frac{2\widetilde{K}N}{N^2-\widetilde{K}^2}(\omega_{\widetilde{K}}-\omega_N)B_K\right]\\
&+\sin\left(\omega_{\widetilde{K}} t_0\right)\left[\alpha\pi^4 \widetilde{K}^4\omega_{\widetilde{K}} B_{\widetilde{K}} + 2\omega_{\widetilde{K}} \dot{B}_{\widetilde{K}} + \sqrt{\beta}V_1\frac{2\widetilde{K}N}{N^2-\widetilde{K}^2}(\omega_{\widetilde{K}}-\omega_N)A_K\right]\\
&+\text{n.r.t.}.
\end{aligned} \tag{174c}$$

This leads to the system:

$$\begin{aligned}
\dot{A}_K &= -\frac{\alpha\pi^4 K^4}{2}A_K - \sqrt{\beta}V_1\left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_K}\right)B_N,\\
\dot{B}_K &= -\frac{\alpha\pi^4 K^4}{2}B_K + \sqrt{\beta}V_1\left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_K}\right)A_N,\\
\dot{A}_N &= -\frac{\alpha\pi^4 N^4}{2}A_N - \sqrt{\beta}V_1\left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_N}\right)B_K - \sqrt{\beta}V_1\left(\frac{K\widetilde{K}}{\widetilde{K}^2-K^2}\right)\left(\frac{\omega_{\widetilde{K}}-\omega_K}{\omega_K}\right)B_{\widetilde{K}},\\
\dot{B}_N &= -\frac{\alpha\pi^4 N^4}{2}B_N + \sqrt{\beta}V_1\left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_N}\right)A_K - \sqrt{\beta}V_1\left(\frac{K\widetilde{K}}{\widetilde{K}^2-K^2}\right)\left(\frac{\omega_{\widetilde{K}}-\omega_K}{\omega_K}\right)A_{\widetilde{K}},\\
\dot{A}_{\widetilde{K}} &= -\frac{\alpha\pi^4 \widetilde{K}^4}{2}A_{\widetilde{K}} - \sqrt{\beta}V_1\left(\frac{\widetilde{K}N}{N^2-\widetilde{K}^2}\right)\left(\frac{\omega_{\widetilde{K}}-\omega_N}{\omega_{\widetilde{K}}}\right)B_N,\\
\dot{B}_{\widetilde{K}} &= -\frac{\alpha\pi^4 \widetilde{K}^4}{2}B_{\widetilde{K}} - \sqrt{\beta}V_1\left(\frac{\widetilde{K}N}{N^2-\widetilde{K}^2}\right)\left(\frac{\omega_{\widetilde{K}}-\omega_N}{\omega_{\widetilde{K}}}\right)A_N.
\end{aligned} \tag{175}$$

For simplicity, we introduce the parameters

$$
\begin{aligned}
a &:= \frac{\alpha\pi^4 K^4}{2}, \quad b := \frac{\alpha\pi^4 N^4}{2}, \quad c := \frac{\alpha\pi^4 \widetilde{K}^4}{2} \\
p &:= \sqrt{\beta} V_1 \left(\frac{KN}{N^2 - K^2}\right)\left(\frac{\omega_K + \omega_N}{\omega_K}\right), \quad q := \sqrt{\beta} V_1 \left(\frac{KN}{N^2 - K^2}\right)\left(\frac{\omega_K + \omega_N}{\omega_N}\right), \\
r &:= \sqrt{\beta} V_1 \left(\frac{\widetilde{K}N}{\widetilde{K}^2 - N^2}\right)\left(\frac{\omega_{\widetilde{K}} - \omega_N}{\omega_{\widetilde{K}}}\right), \quad s := \sqrt{\beta} V_1 \left(\frac{\widetilde{K}N}{\widetilde{K}^2 - N^2}\right)\left(\frac{\omega_{\widetilde{K}} - \omega_N}{\omega_N}\right),
\end{aligned}
\tag{176}
$$

which leads to the linear systems

$$
\begin{pmatrix} \dot{A}_K \\ \dot{B}_{\widetilde{K}} \\ \dot{B}_N \end{pmatrix} = \begin{pmatrix} -a & 0 & p \\ 0 & -c & -r \\ -q & -s & -b \end{pmatrix} \begin{pmatrix} A_K \\ B_{\widetilde{K}} \\ B_N \end{pmatrix}, \quad \begin{pmatrix} \dot{B}_K \\ \dot{A}_{\widetilde{K}} \\ \dot{A}_N \end{pmatrix} = \begin{pmatrix} -a & 0 & -p \\ 0 & -c & -r \\ q & -s & -b \end{pmatrix} \begin{pmatrix} B_K \\ A_{\widetilde{K}} \\ A_N \end{pmatrix}. \tag{177}
$$

The characteristic polynomial for both system in Eq. (177) are identical and that is

$$
\lambda^3 + (a+b+c)\lambda^2 + (ab + ac + bc + pq - rs)\lambda + (abc - ars + cpq) = 0. \tag{178}
$$

In order to obtain an explicit expression for stability of system (177), we apply the Routh-Hurwitz criterion to the characteristic polynomial (178). For a general, cubic polynomila $d_3\lambda^3 + d_2\lambda^2 + d_1\lambda^1 + d_0 = 0$, Routh-Hurwitz stability criterion is satisfied if $T_0, T_1, T_2, T_3 > 0$ where

$$
T_0 = d_3, \quad T_1 = d_2, \quad T_2 = \det\begin{pmatrix} d_2 & d_3 \\ d_0 & d_1 \end{pmatrix}, \quad T_3 = \begin{pmatrix} d_2 & d_3 & 0 \\ d_0 & d_1 & d_2 \\ 0 & 0 & d_0 \end{pmatrix}. \tag{179}
$$

If we substitute the characteristic polynomial obtained in Eq. (178), we get

$$
\begin{aligned}
T_0 = 1, \quad T_1 = a + b + c, \quad T_2 &= (a+b)(b+c)(a+c) + (a+b)pq - (b+c)rs \\
T_3 &= (abc - ars + cpq)\big[(a+b)(b+c)(a+c) + (a+b)pq - (b+c)rs\big].
\end{aligned}
\tag{180}
$$

where $a, b, c > 0$, we obtain the stability conditions as:

$$
rs < (a+b)(a+c) + \frac{(a+b)}{(b+c)} pq, \quad rs < bc + \frac{c}{a} pq. \tag{181}
$$

4.5.4.2 Detuned Case-4: $\Omega = \omega_K - \omega_N + \varepsilon\varphi = \omega_{\widetilde{K}} + \omega_{\widetilde{N}} + \varepsilon\varphi$

Now we study the Case-4 with a small detuning. By substituting $\Omega = \omega_K - \omega_N + \varepsilon\varphi = \omega_{\widetilde{K}} + \omega_{\widetilde{N}} + \varepsilon\varphi$

into Eq. (99), we obtain the system of equations:

$$
\begin{aligned}
\dot{A}_K &= -\frac{\alpha\pi^4 K^4}{2} A_K - \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_K}\right)\left[\sin(\varphi t_1) A_N + \cos(\varphi t_1) B_N\right],\\
\dot{B}_K &= -\frac{\alpha\pi^4 K^4}{2} B_K + \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_K}\right)\left[\cos(\varphi t_1) A_N - \sin(\varphi t_1) B_N\right],\\
\dot{A}_N &= -\frac{\alpha\pi^4 N^4}{2} A_N + \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_N}\right)\left[\sin(\varphi t_1) A_K - \cos(\varphi t_1) B_K\right],\\
&\qquad + \sqrt{\beta} V_1 \left(\frac{\widetilde{K}N}{N^2-\widetilde{K}^2}\right)\left(\frac{\omega_{\widetilde{K}}-\omega_N}{\omega_N}\right)\left[\sin(\varphi t_1) A_{\widetilde{K}} - \cos(\varphi t_1) B_{\widetilde{K}}\right],\\
\dot{B}_N &= -\frac{\alpha\pi^4 N^4}{2} B_N + \sqrt{\beta} V_1 \left(\frac{KN}{N^2-K^2}\right)\left(\frac{\omega_K+\omega_N}{\omega_N}\right)\left[\cos(\varphi t_1) A_K + \sin(\varphi t_1) B_K\right],\\
&\qquad - \sqrt{\beta} V_1 \left(\frac{\widetilde{K}N}{N^2-\widetilde{K}^2}\right)\left(\frac{\omega_{\widetilde{K}}-\omega_N}{\omega_N}\right)\left[\cos(\varphi t_1) A_{\widetilde{K}} + \sin(\varphi t_1) B_{\widetilde{K}}\right],\\
\dot{A}_{\widetilde{K}} &= -\frac{\alpha\pi^4 \widetilde{K}^4}{2} A_{\widetilde{K}} + \sqrt{\beta} V_1 \left(\frac{\widetilde{K}N}{N^2-\widetilde{K}^2}\right)\left(\frac{\omega_{\widetilde{K}}-\omega_N}{\omega_{\widetilde{K}}}\right)\left[\sin(\varphi t_1) A_N + \cos(\varphi t_1) B_N\right],\\
\dot{B}_{\widetilde{K}} &= -\frac{\alpha\pi^4 \widetilde{K}^4}{2} B_{\widetilde{K}} - \sqrt{\beta} V_1 \left(\frac{\widetilde{K}N}{N^2-\widetilde{K}^2}\right)\left(\frac{\omega_{\widetilde{K}}-\omega_N}{\omega_{\widetilde{K}}}\right)\left[\cos(\varphi t_1) A_N - \sin(\varphi t_1) B_N\right].
\end{aligned}
\tag{182}
$$

with the parameters defined in (176), we obtain the time dependent linear system as

$$
\begin{aligned}
\dot{A}_K &= -aA_K - p\left[\sin(\varphi t_1) A_N + \cos(\varphi t_1) B_N\right],\\
\dot{B}_K &= -aB_K + p\left[\cos(\varphi t_1) A_N - \sin(\varphi t_1) B_N\right],\\
\dot{A}_N &= -bA_N + q\left[\sin(\varphi t_1) A_K - \cos(\varphi t_1) B_K\right] + s\left[\sin(\varphi t_1) A_{\widetilde{K}} - \cos(\varphi t_1) B_{\widetilde{K}}\right],\\
\dot{B}_N &= -bB_N + q\left[\cos(\varphi t_1) A_K + \sin(\varphi t_1) B_K\right] - s\left[\cos(\varphi t_1) A_{\widetilde{K}} + \sin(\varphi t_1) B_{\widetilde{K}}\right],\\
\dot{A}_{\widetilde{K}} &= -cA_{\widetilde{K}} + r\left[\sin(\varphi t_1) A_K - \cos(\varphi t_1) B_K\right],\\
\dot{B}_{\widetilde{K}} &= -cB_{\widetilde{K}} - r\left[\cos(\varphi t_1) A_K + \sin(\varphi t_1) B_K\right].
\end{aligned}
\tag{183}
$$

After some manipulations similar as Appendix B.2, we obtain the time independent equations below.

$$
\begin{aligned}
\dddot{A}_K &= -(a+b+c)\ddot{A}_K - (ab+ac+bc+pq+rs)\dot{A}_K - (abc+brs+cpq)A_K\\
&\quad - \varphi\left[-(b-c)\dot{B}_K - (ab-ac+pq-rs)B_K\right] - \varphi^2(\dot{A}_K + aA_K),\\
\dddot{B}_K &= -(a+b+c)\ddot{B}_K - (ab+ac+bc+pq+rs)\dot{B}_K - (abc+brs+cpq)B_K,\\
&\quad - \varphi\left[(b-c)\dot{A}_K + (ab-ac+pq-rs)A_K\right] - \varphi^2(\dot{B}_K + aB_K).
\end{aligned}
\tag{184}
$$

Due to the complications of solving this system, we do not further study the given case for arbitrary resonant modes.

4.5.4.3 Case-4: $\Omega = \omega_5 - \omega_2 = \omega_3 + \omega_2$

Due to studying Case-4 with arbitrary modes is not feasible, we study the case occurring with index values $K = 5$, $N = 2$ and $\widetilde{K} = 3$. With this choice of indices and from (84), we obtain that $\mu = \frac{13+5\sqrt{65}}{728\pi^2} \simeq 0.0074197$. Substituting these into (176) and by introducing variables $\widetilde{\alpha} := \alpha\pi^4/2$ and $\widetilde{\beta} := \sqrt{\beta}V_1$ we obtain the corresponding natural frequencies and parameter values as

$$\begin{gathered}\omega_2 = 2\pi\sqrt{\frac{195+5\sqrt{65}}{182}}, \quad \omega_3 = 3\pi\sqrt{\frac{845+45\sqrt{65}}{728}}, \quad \omega_5 = 5\pi\sqrt{\frac{1053+125\sqrt{65}}{728}} \\ a = 625\widetilde{\alpha}, \quad b = 16\widetilde{\alpha}, \quad c = 81\widetilde{\alpha}, \\ p = \frac{(\sqrt{65}-25)\widetilde{\beta}}{28}, q = -\frac{5(\sqrt{65}+17)\widetilde{\beta}}{56}, r = \frac{(\sqrt{65}-13)\widetilde{\beta}}{10}, s = -\frac{3(3\sqrt{65}+19)\widetilde{\beta}}{40}.\end{gathered} \tag{185}$$

With the parameters determined, we obtain the characteristic polynomial

$$d_6\lambda^6 + d_5\lambda^5 + \cdots + d_1\lambda + d_0 = 0 \tag{186}$$

where

$$\begin{aligned}
d_6 = 1, \quad d_5 = 1444\widetilde{\alpha}, \quad d_4 = 645126\widetilde{\alpha}^2 + \frac{1857-305\sqrt{65}}{1225}\widetilde{\beta}^2 + 2\varphi^2, \\
d_3 = 910339 24\widetilde{\alpha}^3 + \frac{971379-444835\sqrt{65}}{1225}\widetilde{\alpha}\widetilde{\beta}^2 + 1476\widetilde{\alpha}\varphi^2, \\
d_2 =& 5003850241\widetilde{\alpha}^4 - \frac{151701453+181065155\sqrt{65}}{1225}\widetilde{\alpha}^2\widetilde{\beta}^2 + \frac{4747537-566385\sqrt{65}}{3001250}\widetilde{\beta}^4 \\
&+\left(442882\widetilde{\alpha}^2 + \frac{1857-305\sqrt{65}}{1225}\widetilde{\beta}^2\right)\varphi^2 + \varphi^4, \\
d_1 =& 100312020000\widetilde{\alpha}^5 - \frac{854715975+566242185\sqrt{65}}{49}\widetilde{\alpha}^3\widetilde{\beta}^2 + \frac{15069045-1217877\sqrt{65}}{12005}\widetilde{\alpha}\widetilde{\beta}^4 \\
&+\left(-40598528\widetilde{\alpha}^3 + \frac{1710129+4415\sqrt{65}}{1225}\widetilde{\alpha}\widetilde{\beta}^2\right)\varphi^2 + 32\widetilde{\alpha}\varphi^4, \\
d_0 =& 656100000000\widetilde{\alpha}^6 + \frac{11967750000-7277850000\sqrt{65}}{49}\widetilde{\alpha}^4\widetilde{\beta}^2 + \frac{2732882625+132753375\sqrt{65}}{4802}\widetilde{\alpha}^2\widetilde{\beta}^4 \\
&+\left(1676310625\widetilde{\alpha}^4 - \frac{6359400+842510\sqrt{65}}{49}\widetilde{\alpha}^2\widetilde{\beta}^2 + \frac{6554081+810120\sqrt{65}}{1500625}\widetilde{\beta}^4\right)\varphi^2 + 256\widetilde{\alpha}^2\varphi^4.
\end{aligned} \tag{187}$$

From the coefficients of the characteristic polynomial, we can compute the Routh-Hurwitz determinants in order to determine the stability of A_2 and B_2. The stability of A_2 and B_2 leads to the stability of the whole system from Appendix C.1 and C.2. These determinants defined in 172, and one can observe that T_0, T_1 and T_2 are positive for $a, b, c > 0$ and for all other parameter values. The remaining T_3, T_4, T_5 and T_6 correspond to algebraic plane curves in $\widetilde{\alpha}$ and $\widetilde{\beta}$ and φ is the constant parameter. In Figure 12, the plane algebraic curves $T_i = 0$, $i = 3, \ldots, 6$ and the intersection region where $T_i(\widetilde{\alpha}, \widetilde{\beta}) > 0$ are presented. Note that the analytical expressions could not be given due to the fact that, them being too lengthy to be presented in the format of thesis.

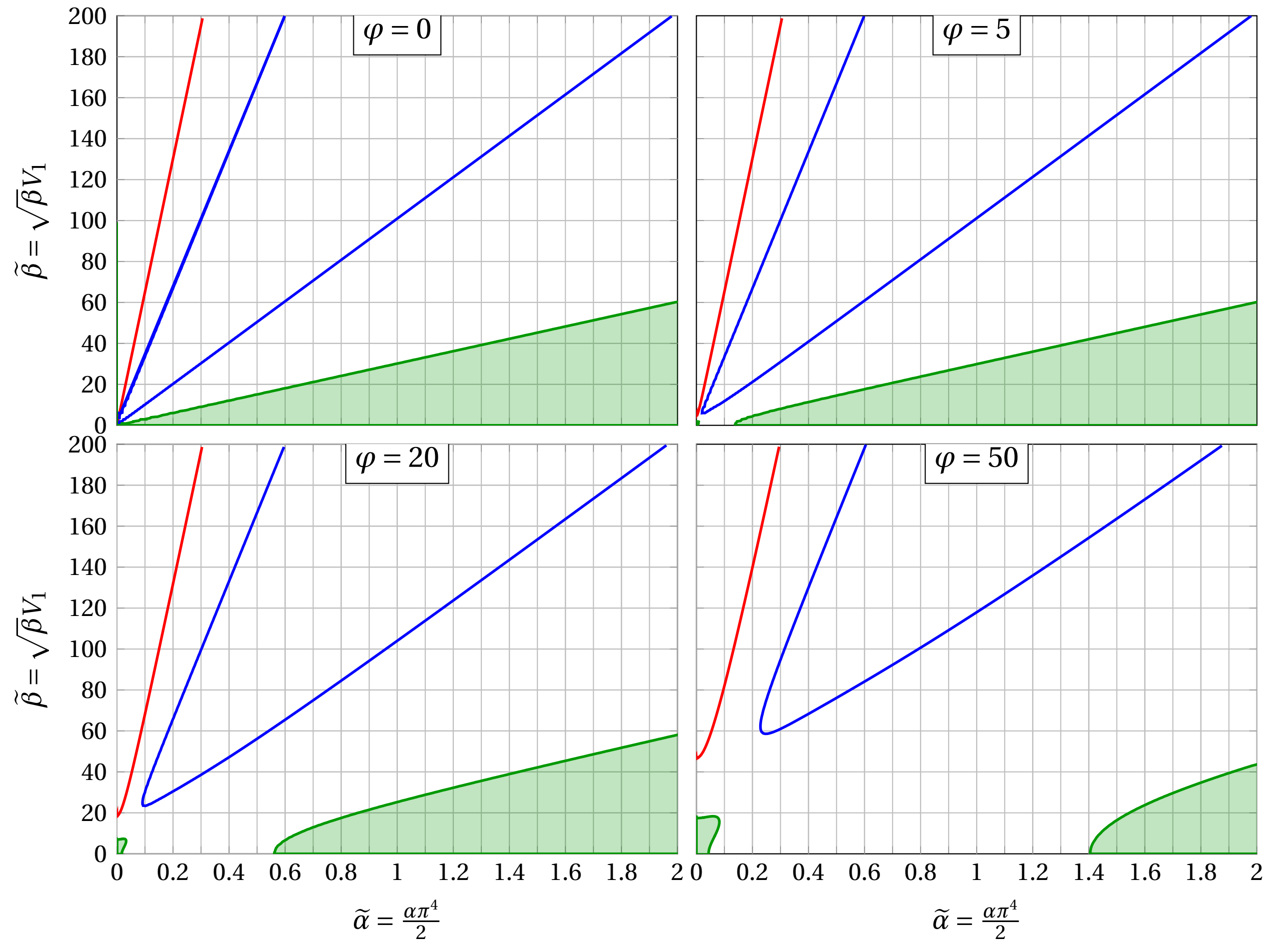

Figure 12: —— $T_3 = 0$, —— $T_4 = 0$ and —— $T_5 = T_6 = 0$ curves in $(\tilde{\alpha}, \tilde{\beta})$ pane. The domain where $T_3, T_4, T_5, T_6 > 0$ is shown with green region.

Figure 12 shows that the intersection of regions $T_i > 0$, $i = 0, 1, \ldots, 6$ are equivalent to the region defined by $T_5 > 0$ or $T_6 > 0$. It is also observed that for $\varphi = 0$ we have a single region defining the stability of the equilibrium. The domain of stability evolves into two distinct regions with nonzero detuning parameter $f s$. As φ increases the stable region surrounding the origin expands and while the second region shifts in the positive $\tilde{\alpha}$ direction.

4.5.5 Case-5: $\Omega \simeq \omega_K + \omega_N = \omega_{\tilde{K}} + \omega_{\tilde{N}}$

Finally, we study the case where sum of two natural frequencies coincides with another sum of two natural frequencies. It can be observed that this case is only possible if all the indices $K, N, \tilde{K}, \tilde{N}$ are different. As it is previously studied, this leads to two 4-dimensional decoupled systems that can be solved by the system (131) for corresponding modes. The instances where Case-5 is emerged are presented in Table 9 and Figure 13.

Table 9: Instances of coinciding frequencies: $\omega_K + \omega_N = \omega_{\tilde{K}} + \omega_{\tilde{N}}$ along with their respective μ values

Ω	μ	Ω	μ	Ω	μ
$\omega_8 + \omega_1 = \omega_6 + \omega_5$	0.005733	$\omega_{11} + \omega_2 = \omega_8 + \omega_7$	0.001413	$\omega_{12} + \omega_1 = \omega_{10} + \omega_5$	0.001001
$\omega_{10} + \omega_1 = \omega_7 + \omega_6$	0.001368	$\omega_{11} + \omega_2 = \omega_9 + \omega_6$	0.002055	$\omega_{12} + \omega_1 = \omega_{11} + \omega_4$	0.002653
$\omega_{10} + \omega_1 = \omega_8 + \omega_5$	0.001844	$\omega_{12} + \omega_1 = \omega_8 + \omega_7$	0.000603	$\omega_{12} + \omega_3 = \omega_9 + \omega_8$	0.001618
$\omega_{10} + \omega_1 = \omega_9 + \omega_4$	0.005471	$\omega_{12} + \omega_1 = \omega_9 + \omega_6$	0.000696	$\cdots$	$\cdots$

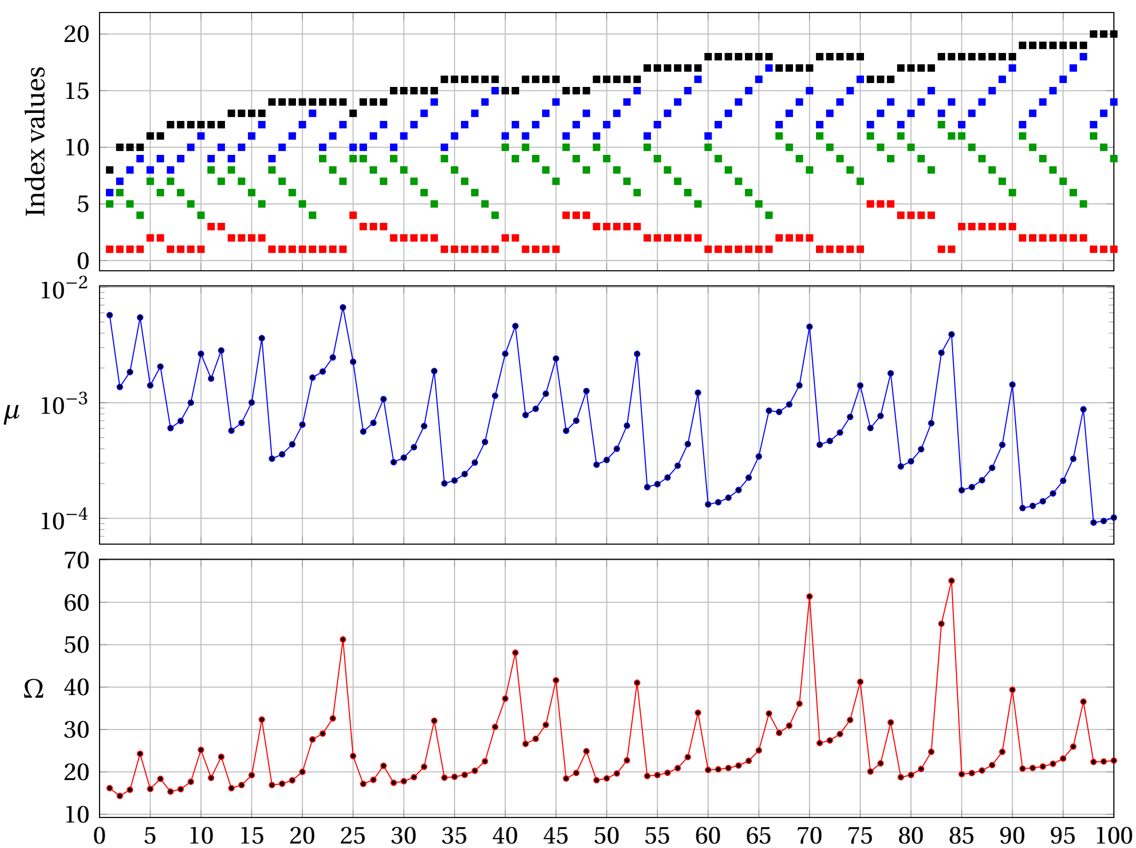

Figure 13: Index values for the first 100 coincidence cases: $\omega_{■} + \omega_{■} = \omega_{■} + \omega_{■}$ and corresponding —•— μ and —•— Ω values respectively.

5 Conclusion

In this study, we explore the dynamics of pipes transporting fluid with a pulsating flow. We derive the initial-boundary value problem for the linear beam equations that govern the dynamics of pipes conveying fluid with a pulsating axial velocity, using the principles of Lagrangian mechanics. These equations are then studied using the multiple time scale perturbation method. Our research is aimed to provide a comprehensive insight into modelling problems of pipes conveying fluid and to expand the knowledge in the current literature on stability and dynamical behaviour of these systems under various flow pulsation frequencies, building upon and refining previous works in the field.

In our investigation, we assume the fluid velocity inside the pipe to be represented by $V(t) = \varepsilon(V_0 + V_1 \sin(\Omega t))$, where V_0, V_1, Ω, and ε are constants with $|V_1| < |V_0|$ and $0 < \varepsilon \ll 1$. Our analysis examines the system's stability under various flow pulsation frequencies Ω and arbitrary parameter values. Specifically, we evaluate whether a given pulsation frequency is close to any resonance frequencies, and how this influences the system's stability.

When Ω is not near any resonance frequencies, the pipe system remains stable. This implies that, from an initial condition, the system reaches its steady state, resting under the influence of gravity. However, when the excitation frequency Ω aligns with a natural frequency, a difference of two natural frequencies, or a sum of two natural frequencies, the behaviour of the pipe system becomes more complicated. These specific frequencies are termed primary resonance frequencies. Moreover, the system exhibits even more complicated dynamical behaviours when two primary resonance frequencies coincide.

When $\Omega = \omega_K$ for K is any natural number, the equilibrium of the coefficients A_K and B_K is no longer at the origin. Where B_K goes to zero, A_K goes to a nonzero constant. This indicates that the solution $v_0(x,t)$ approaches to an oscillatory state with constant amplitude and frequency. In the detuned scenario where $\Omega = \omega_K + \varepsilon\varphi$, as t_1 goes to infinity, A_K and B_N retain only oscillatory terms, for the slow time variable t_1.This suggests that the pipe system undergoes oscillations with a constant amplitude and slow phase shift, due to terms such as $\sin(\omega_K t_1)\sin(\omega_K t_0)$, around its steady hanging position. For this resonance case, no interaction among different vibration modes occurs.

For the scenario where $\Omega = \omega_K - \omega_N$ with $K > N$ and $K + N$ being odd, the vibration modes K and N interact. As a result, instead of a 2nd order system, we encounter a 4th order system, which is unconditional stable with the presence of viscoelasticity of the pipe structure. This ensures that the pipe transitions to a steady state as t approaches infinity. Furthermore, when the excitation frequency deviates from the resonance frequency, given by $\Omega = \omega_K - \omega_N + \varepsilon\varphi$, the system's stability remains unaffected, and it remains stable.

When $\Omega = \omega_K + \omega_N$, the modes K and N interact, resulting in a 4th order system. The system's stability is met when the condition $\frac{\alpha^4\pi^8}{4} > \frac{\beta V_1^2}{K^2N^2(N^2-K^2)^2}\frac{(\omega_K+\omega_N)^2}{\omega_K\omega_N}$ holds. Therefore, if the viscoelasticity parameter α is not sufficiently large, the system becomes unstable. Additionally, for the detuned situation, the pipe system is found to be stable if the following expression is satisfied $\varphi^2 > (K^4+N^4)^2\left(\left(\frac{\beta V_1^2(\omega_K-\omega_N)^2}{K^2N^2(N^2-K^2)^2\omega_K\omega_N}\right) - \frac{\alpha^2\pi^8}{4}\right)$. Hence, when α is small, the excitation frequency Ω must deviate from the resonance frequency for the system to exhibit stable behaviour.

If the detuning parameter φ is not large enough, the pipe system becomes unstable.

For particular values of μ, a coincidence of primary resonance cases arises. If the coinciding frequencies, such as $\Omega = \omega_K \pm \omega_N = \omega_{\widetilde{K}} \pm \omega_{\widetilde{N}}$, don't share any common frequencies, thus all indices are distinct, we obtain two separate systems for each primary resonance case, exhibiting no interaction. Consequently, both systems occur simultaneously, and the solutions or stability conditions for each are assessed independently.

We also identified unique scenarios where coinciding primary resonance frequencies share a common natural frequency, i.e. $\Omega = \omega_K - \omega_N = \omega_N$, $\Omega = \omega_K - \omega_N = \omega_N - \omega_{\widetilde{K}}$, and $\Omega = \omega_K - \omega_N = \omega_N + \omega_{\widetilde{K}}$.

For the case of $\Omega = \omega_K - \omega_N = \omega_N$, the system evolves into a 4th order system where modes K and N interact. The equilibrium shifts from the origin and is determined to be unconditionally stable across all modes and parameter values.

In the scenario of $\Omega \simeq \omega_K - \omega_N = \omega_{\widetilde{K}} - \omega_K$, the system is characterized by an interaction of modes K, N, and $\widetilde{K}$, giving rise to a 6-dimensional system. The size of the system leads to complications in determining stability conditions for arbitrary resonant modes. As a specific example, we examine the case $\Omega = \omega_6 - \omega_1 + \varepsilon\varphi = \omega_9 - \omega_6 + \varepsilon\varphi$ and find the system to be stable across all parameter values of α, β, and V_1. Additionally, changes in the detuning parameter φ do not alter the system's stability.

In the case of $\Omega \simeq \omega_K - \omega_N = \omega_{\widetilde{K}} + \omega_N$, we observe a system where modes K, N, and $\widetilde{K}$ interact, forming a 6-dimensional system. Given the challenges of defining stability conditions for such systems with arbitrary resonance modes, we examined the specific case of $\Omega = \omega_5 - \omega_2 + \varepsilon\varphi = \omega_3 + \omega_2 + \varepsilon\varphi$. The stability of this system is determined within specific regions of α and $\sqrt{\beta}V_1$. When the viscoelasticity parameter α is relatively much smaller than $\sqrt{\beta}V_1$, the system becomes unstable. Additionally, we discuss how the stability criterion changes with variations in the detuning parameter associated with the fluid pulsation frequency.

In examining cases such as $\Omega \simeq \omega_5 - \omega_2 = \omega_3 + \omega_2$, it becomes evident that truncating the eigenvalue expansion after the 4th mode results in the negligence of interactions associated with the 5th mode. Similarly, for scenarios like $\Omega \simeq \omega_9 - \omega_6 = \omega_6 - \omega_1$, restricting the analysis to only the first six modes erroneously neglects contributions from the 9th mode. Consequently, the solutions derived for these specific conditions fail to be $\mathcal{O}(\varepsilon)$ accurate for the time scale $t \sim \mathcal{O}\left(\frac{1}{\varepsilon}\right)$. It is expected that the special resonance cases $\Omega = \omega_K - \omega_N = \omega_N - \omega_{\widetilde{K}}$ and $\Omega = \omega_K - \omega_N = \omega_N + \omega_{\widetilde{K}}$ can be solved for other resonance modes.

Moreover, our study has discovered numerous instances where multiple modes interact due to overlapping primary resonance frequencies, particularly for relatively small bending stiffness parameters μ. For special resonant cases, a chosen fluid pulsation frequency Ω with a given μ value may fall in the $\mathcal{O}(\varepsilon)$ neighbourhood of other coincidences of resonance frequencies. Potentially, this can lead to the excitation of other resonant modes. Such interactions among special resonant cases might be overlooked when using the truncation method for smaller μ parameters.

6 Further Research

In this paper, we focused on the linear equations of motion for a simply supported pipe conveying pulsating fluid flow, taking into account various parameter values and Ω pulsation frequencies. From our findings, several potential paths for further exploration emerge:

Special Resonance Cases: It might be worthwhile to study conditions whether the excitation frequency is equal to coincidence of three (or more) primary resonant case with common frequencies, such as

$$\Omega = \omega_K - \omega_N = \omega_N - \omega_M = \omega_M + \omega_L, \tag{188a}$$

$$\Omega = \omega_K - \omega_N = \omega_N + \omega_M = \omega_M - \omega_L, \tag{188b}$$

$$\Omega = \omega_K - \omega_N = \omega_N - \omega_M = \omega_M - \omega_L. \tag{188c}$$

This would mean the interaction of 5 modes instead of 3 and lead to even more complicated dynamics. Another, and more likely case is when the excitation frequency is close to other resonance frequencies. In particular,

$$\Omega = \omega_K \pm \omega_N = \omega_{\tilde{K}_1} \pm \omega_{\tilde{N}_1} + \varepsilon\theta_1 = \cdots = \omega_{\tilde{K}_m} \pm \omega_{\tilde{N}_m} + \varepsilon\theta_m. \tag{189}$$

Such conditions could lead to more interactions within the system.

Different Boundary Conditions: Exploring other boundary conditions can be an area of interest. For example, we can delve into cantilevered (see Eq. (62)) or mass-spring-dashpot boundary conditions.

Incorporating Additional Physical Effects: This study centered on a horizontally placed pipe. However, examining vertically oriented pipe systems may provide insights into simulating real-world pipe structures used in applications (see Eq. (61)). Additionally, the current mathematical model can be expanded by adding more physical parameters. One possibility is looking into the external forces brought about by wake oscillations from cross-flow perpendicular to the pipe. To capture the effects of vortex shedding, the simply periodic or van der Pol-type models might be integrated.

Exploring Nonlinear Equations: A logical next step could involve studying nonlinear equations of motion. This would mean factoring in aspects like axial deflections and considering higher-order terms when defining curvature.

Considering these topics for further study can help in deepening our understanding and contribute to expanding the current research.

Appendices

Appendix A

In this appendix, the convergence of

$$\sum_{j=0}^{\infty} \frac{1}{2j+1} \int_0^1 U'(x) \sin\big((2j+1)\pi x\big) \sin(k\pi x)\, dx \tag{190}$$

will be proved. Substituting Eq. (98) into Eq. (190) results in

$$\begin{aligned} C_k &= \sum_{j=0}^{\infty} \frac{1}{2j+1} \int_0^1 U' \sin\big((2j+1)\pi x\big) \sin(k\pi x)\, dx \\ &= \sum_{j=0}^{\infty} 4\gamma k \frac{\mu^2\pi^4(2j+1-k)^2(2j+1+k)^2 - \big(1+(2j+1-k)^2\mu\pi^2\big)\big(1+(2j+1+k)^2\mu\pi^2\big)}{\mu\pi^2(2j+1-k)^2(2j+1+k)^2\big(1+(2j+1-k)^2\mu\pi^2\big)\big(1+(2j+1+k)^2\mu\pi^2\big)} \\ &= \underbrace{\sum_{j=0}^{\infty} \frac{4\gamma k\mu\pi^2}{\big(1+(2j+1-k)^2\mu\pi^2\big)\big(1+(2j+1+k)^2\mu\pi^2\big)}}_{=:S_1} - \underbrace{\sum_{j=0}^{\infty} \frac{4\gamma k}{\mu\pi^2(2j+1-k)^2(2j+1+k)^2}}_{=:S_2}. \end{aligned} \tag{191}$$

Let's focus on S_1. Where $2j+1$ corresponding to odd natural numbers, we first study the series sum with the index set as all natural numbers with k being excluded. Thus, we define the series sum as:

$$\widetilde{S}_1 = \left(\sum_{i=0}^{k-1} + \sum_{i=k+1}^{\infty}\right) \frac{4\gamma k\mu\pi^2}{\big(1+(i-k)^2\mu\pi^2\big)\big(1+(i+k)^2\mu\pi^2\big)}. \tag{192}$$

We consider the series element a_i as:

$$a_i = \frac{4\gamma k\mu\pi^2}{\big(1+(i-k)^2\mu\pi^2\big)\big(1+(i+k)^2\mu\pi^2\big)}. \tag{193}$$

Now, we split the series $\widetilde{S}_1$ into two parts, for $i < k$ and $i > k$:

$$\widetilde{S}_1 = \sum_{i=0}^{k-1} a_i + \sum_{i=k+1}^{\infty} a_i, \tag{194}$$

where the series $\sum_{i=0}^{k-1} a_i$ is a finite sum and hence bounded.

Next, we study whether if $\sum_{i=k+1}^{\infty} a_i$ is bounded. To simplify the summation, we introduce the variable $n = i + k$ so that the series becomes:

$$\sum_{i=k+1}^{\infty} a_i = \sum_{n=1}^{\infty} \frac{4\gamma k\mu\pi^2}{\big(1+n^2\mu\pi^2\big)\big(1+(n+2k)^2\mu\pi^2\big)} \tag{195}$$

and Eq. (195) can be bounded as

$$\left|\sum_{i=k+1}^{\infty} a_i\right| \le \sum_{i=k+1}^{\infty} |a_i| < \sum_{n=1}^{\infty} \frac{4\gamma k\mu\pi^2}{\left(1+n^2\mu\pi^2\right)^2} < \sum_{n=1}^{\infty} \frac{4\gamma k}{\mu\pi^2 n^4} = \frac{4\gamma k}{\mu\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{4\gamma k}{\mu\pi^2}\frac{\pi^4}{90} = \frac{2\gamma k\pi^2}{45\mu}. \tag{196}$$

This concludes that $\widetilde{S}_1$ is absolutely convergent. Since, the subseries of an absolutely convergent series is also convergent, S_1 is also convergent.

Now, we focus on S_2. Similarly, we define the infinite sum on the index set of natural numbers with $i \neq k$,

$$\widetilde{S}_2 = \left(\sum_{i=0}^{k-1} + \sum_{i=k+1}^{\infty}\right)\frac{4\gamma k}{\mu\pi^2(2j+1-k)^2(2j+1+k)^2}. \tag{197}$$

We consider the series element a_i with $i \neq k$:

$$b_i = \frac{4\gamma k}{\mu\pi^2(i-k)^2(i+k)^2}. \tag{198}$$

Now, we split the series $\widetilde{S}_1$ into two parts, for $i < k$ and $i > k$:

$$\sum_{i=0}^{k-1} b_i + \sum_{i=k+1}^{\infty} b_i, \tag{199}$$

the series $\sum_{i=0}^{k-1} a_i$ is a finite sum and hence bounded. For the part$\sum_{i=k+1}^{\infty} a_i$, in order to simplify the summation, we introduce the variable $n = i + k$ so that the series yields to

$$\sum_{i=k+1}^{\infty} b_i = \sum_{n=1}^{\infty} \frac{4\gamma k}{\mu\pi^2 n^2(n+2k)^2}. \tag{200}$$

and Eq. (200) can be bounded as

$$\left|\sum_{i=k+1}^{\infty} b_n\right| \le \sum_{i=k+1}^{\infty} |b_n| < \sum_{n=1}^{\infty} \frac{4\gamma k}{\mu\pi^2 n^4} < \sum_{n=1}^{\infty} \frac{4\gamma k}{\mu\pi^2 n^4} = \frac{4\gamma k}{\mu\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{4\gamma k}{\mu\pi^2}\frac{\pi^4}{90} = \frac{2\gamma k\pi^2}{45\mu}. \tag{201}$$

Thus, we have shown that $\widetilde{S}_2$ is absolutely convergent, which implies that S_2 is also convergent.

Since both S_1 and S_2 are convergent, their difference $C_k = S_1 - S_2$ is also convergent. Therefore, we can conclude that C_k is a bounded constant.

Appendix B

B.1

In this appendix, the intermediate steps between of Eq. (121) and (123) will be provided. We start from (121)

$$\begin{aligned}
\dot{A}_K &= -aA_K - p\sin(\varphi t_1)A_N - p\cos(\varphi t_1)B_N,\\
\dot{B}_K &= -aB_K + p\cos(\varphi t_1)A_N - p\sin(\varphi t_1)B_N,\\
\dot{A}_N &= -bA_N + q\sin(\varphi t_1)A_K - q\cos(\varphi t_1)B_K,\\
\dot{B}_N &= -bB_N + q\cos(\varphi t_1)A_K + q\sin(\varphi t_1)B_K.
\end{aligned} \tag{202}$$

where a, b, p and q are given by (114), by differentiating the first term of (202), we obtain

$$
\begin{aligned}
\ddot{A}_K =& - a\dot{A}_K - p\sin\big(\varphi t_1\big)\dot{A}_N - p\cos\big(\varphi t_1\big)\dot{B}_N + \varphi\big(p\cos\big(\varphi t_1\big)A_N + p\sin\big(\varphi t_1\big)B_N\big)\\
=& - a\dot{A}_K - \varphi(\dot{B}_K + aB_K) + p\sin\big(\varphi t_1\big)\big[-bA_N + q\sin\big(\varphi t_1\big)A_K - q\cos\big(\varphi t_1\big)B_K\big]\\
& - p\cos\big(\varphi t_1\big)\big[-bA_N + q\sin\big(\varphi t_1\big)A_K - q\cos\big(\varphi t_1\big)B_K\big]\\
=& - a\dot{A}_K - \varphi(\dot{B}_K + aB_K) - bp\sin\big(\varphi t_1\big)(\varphi t_1)A_N + pq\sin^2(\varphi t_1)A_K\\
& - pq\sin\big(\varphi t_1\big)\cos\big(\varphi t_1\big)B_K + bp\cos\big(\varphi t_1\big)B_N + pq\cos^2(\varphi t_1)A_K\\
& + pq\sin\big(\varphi t_1\big)\cos\big(\varphi t_1\big)B_K\\
=& - (a+b)\dot{A}_K - abA_K + pqA_K - \varphi\big(\dot{B}_K + aB_K\big).
\end{aligned}
\tag{203}
$$

Similarly from the differentiation of the second term results in

$$
\ddot{B}_K = -(a+b)\dot{B}_K - abB_K + pqB_K + \varphi\big(\dot{A}_K + aA_K\big).
\tag{204}
$$

As it can be seen that, already two systems that are not time dependent are obtained after these manipulations. However, if we want to elliminate the second variable B_K from this system, we can continue this procedure and differentiating the Eq. (203), resulting in

$$
\begin{aligned}
\dddot{A}_K =& - (a+b)\dot{A}_K - (ab+pq)\dot{A}_K - \varphi(\ddot{B}_K + a\dot{B}_K)\\
=& - (a+b)\ddot{A}_K - (\varphi^2 ab + pq)\dot{A}_K - a\varphi^2 A_K + \varphi b\dot{B}_K + \varphi(ab+pq)B_K.
\end{aligned}
\tag{205}
$$

By substituting Eq. (204) into (205) and differentiating one final time leads to

$$
A_K^{(4)} = - (a+b)\dddot{A}_K - (\varphi^2 ab + pq)\ddot{A}_K - a\varphi^2\dot{A}_K + \varphi b\ddot{B}_K + \varphi(ab+pq)\dot{B}_K.
\tag{206}
$$

Lastly, after substituting Eq. (204) into Eq. (206), we obtain

$$
\begin{aligned}
& A_K^{(4)} + 2(a+b)\dddot{A}_K + \big(\varphi^2 + (a+b)^2 + 2(ab+pq)\big)\ddot{A}_K\\
& + \big(2a\varphi^2 + 2(a+b)(ab+pq)\big)\dot{A}_K + \big(\varphi^2 a^2 + (ab+pq)^2\big)A_K = 0.
\end{aligned}
\tag{207}
$$

B.2

In this appendix, the intermediate steps between the system (167) and (168) are presented. Where we have Eq. (167) given as

$$
\begin{aligned}
\dot{A}_K &= - aA_K - p\big[\sin\big(\varphi t_1\big)A_N + \cos\big(\varphi t_1\big)B_N\big] - s\big[\sin\big(\varphi t_1\big)A_{\widetilde{K}} - \cos\big(\varphi t_1\big)B_{\widetilde{K}}\big],\\
\dot{B}_K &= - aB_K + p\big[\cos\big(\varphi t_1\big)A_N - \sin\big(\varphi t_1\big)B_N\big] - s\big[\cos\big(\varphi t_1\big)A_{\widetilde{K}} + \sin\big(\varphi t_1\big)B_{\widetilde{K}}\big],\\
\dot{A}_N &= - bA_N + q\big[\sin\big(\varphi t_1\big)A_K - \cos\big(\varphi t_1\big)B_K\big],\\
\dot{B}_N &= - bB_N + q\big[\cos\big(\varphi t_1\big)A_K + \sin\big(\varphi t_1\big)B_K\big],\\
\dot{A}_{\widetilde{K}} &= - cA_{\widetilde{K}} + r\big[\sin\big(\varphi t_1\big)A_K + \cos\big(\varphi t_1\big)B_K\big],\\
\dot{B}_{\widetilde{K}} &= - cB_{\widetilde{K}} - r\big[\cos\big(\varphi t_1\big)A_K - \sin\big(\varphi t_1\big)B_K\big].
\end{aligned}
\tag{208}
$$

differentiating the first equation of (208) yields to

$$\begin{aligned}\ddot{A}_K =& -a\dot{A}_K - p\left[\sin(\varphi t_1)\dot{A}_N + \cos(\varphi t_1)\dot{B}_N\right] - s\left[\sin(\varphi t_1)\dot{A}_{\widetilde{K}} - \cos(\varphi t_1)\dot{B}_{\widetilde{K}}\right]\\ &- \varphi\{p\left[\cos(\varphi t_1)A_N - \sin(\varphi t_1)B_N\right] + s\left[\cos(\varphi t_1)A_{\widetilde{K}} + \sin(\varphi t_1)B_{\widetilde{K}}\right]\}\\ =& -a\dot{A}_K - pqA_K + pb\left[\sin(\varphi t_1)A_N + \cos(\varphi t_1)B_N\right] + cs\left[\sin(\varphi t_1)A_{\widetilde{K}} - \cos(\varphi t_1)B_{\widetilde{K}}\right]\\ &- rsA_K - \varphi\{p\left[\cos(\varphi t_1)A_N - \sin(\varphi t_1)B_N\right] + s\left[\cos(\varphi t_1)A_{\widetilde{K}} + \sin(\varphi t_1)B_{\widetilde{K}}\right]\}.\end{aligned} \tag{209}$$

Similarly, $\ddot{B}_K$ can be obtained as

$$\begin{aligned}\ddot{B}_k =& -a\dot{B}_K - pqB_K - pb\left[\cos(\varphi t_1)A_N - \sin(\varphi t_1)B_N\right] + cs\left[\cos(\varphi t_1)A_{\widetilde{K}} + \sin(\varphi t_1)B_{\widetilde{K}}\right]\\ &- rsA_K - \varphi\{p\left[\sin(\varphi t_1)A_N + \cos(\varphi t_1)B_N\right] - s\left[\sin(\varphi t_1)A_{\widetilde{K}} - \cos(\varphi t_1)B_{\widetilde{K}}\right]\}.\end{aligned} \tag{210}$$

Now differentiating Eq. (209) once again and using Eq. (210) results in

$$\begin{aligned}\dddot{A}_K =& -a\ddot{A}_K - pq\dot{A}_K + pb\left[\sin(\varphi t_1)\dot{A}_N + \cos(\varphi t_1)\dot{B}_N\right] + cs\left[\sin(\varphi t_1)\dot{A}_{\widetilde{K}} - \cos(\varphi t_1)\dot{B}_{\widetilde{K}}\right]\\ &- rs\dot{A}_K - \varphi^2\{p\left[\sin(\varphi t_1)A_N + \cos(\varphi t_1)B_N\right] + s\left[-\sin(\varphi t_1)A_{\widetilde{K}} + \cos(\varphi t_1)B_{\widetilde{K}}\right]\}\\ &- \varphi\{p\left[\cos(\varphi t_1)\dot{A}_N - \sin(\varphi t_1)\dot{B}_N\right] + s\left[\cos(\varphi t_1)\dot{A}_{\widetilde{K}} + \sin(\varphi t_1)\dot{B}_{\widetilde{K}}\right]\\ &+ pb\left[\cos(\varphi t_1)A_N - \sin(\varphi t_1)B_N\right] + cs\left[\cos(\varphi t_1)A_{\widetilde{K}} + \sin(\varphi t_1)B_{\widetilde{K}}\right]\}\end{aligned} \tag{211}$$

$$\begin{aligned}=& -a\ddot{A}_K - (pq+rs)\dot{A}_K + (bpq+crs)A_K - b^2p\left[\sin(\varphi t_1)A_N + \cos(\varphi t_1)B_N\right]\\ &- c^2s\left[\sin(\varphi t_1)A_{\widetilde{K}} - \cos(\varphi t_1)B_{\widetilde{K}}\right] - \varphi^2\left(\dot{B}_K + aB_K\right) + \varphi\{(pq-rs)B_K\\ &+ 2bp\left[\cos(\varphi t_1)A_N - \sin(\varphi t_1)B_N\right] + 2cs\left[\cos(\varphi t_1)A_{\widetilde{K}} + \sin(\varphi t_1)B_{\widetilde{K}}\right]\}.\end{aligned} \tag{212}$$

Since, the detuned case must overlap with the pure resonance case for $\varphi = 0$, from comparing Eq. (212) with Eq. (163), it is observed that the missing terms are $-(b+c)\left[\ddot{A}_K + a\dot{A}_K + (pq+rs)A_K\right] - bc(\dot{A}_K + aA_K)$. If we substitute Eq. (209) into the expression for missing terms, we obtain

$$\begin{aligned}& -(b+c)\left[\ddot{A}_K + a\dot{A}_K + (pq+rs)A_K\right] - bc(\dot{A}_K + aA_K)\\ =& -(b+c)\{-(pq+rs)A_K + pb\left[\sin(\varphi t_1)A_N + \cos(\varphi t_1)B_N\right]\\ &+ cs\left[\sin(\varphi t_1)A_{\widetilde{K}} - \cos(\varphi t_1)B_{\widetilde{K}}\right] + (pq+rs)A_K\\ &- \varphi\{p\left[\cos(\varphi t_1)A_N - \sin(\varphi t_1)B_N\right] + s\left[\cos(\varphi t_1)A_{\widetilde{K}} + \sin(\varphi t_1)B_{\widetilde{K}}\right]\}\\ &- bc\{-p\left[\sin(\varphi t_1)A_N + \cos(\varphi t_1)B_N\right] - s\left[\sin(\varphi t_1)A_{\widetilde{K}} - \cos(\varphi t_1)B_{\widetilde{K}}\right]\}\end{aligned} \tag{213}$$

$$\begin{aligned}\Rightarrow& -b^2p\left[\sin(\varphi t_1)A_N + \cos(\varphi t_1)B_N\right] - c^2s\left[\sin(\varphi t_1)A_{\widetilde{K}} - \cos(\varphi t_1)B_{\widetilde{K}}\right]\\ =& -(b+c)\left[\ddot{A}_K + a\dot{A}_K + (pq+rs)A_K\right] - bc(\dot{A}_K + aA_K)\\ &- \varphi\{bp\left[\cos(\varphi t_1)A_N - \sin(\varphi t_1)B_N\right] + cp\left[\cos(\varphi t_1)A_N - \sin(\varphi t_1)B_N\right]\\ &+ bs\left[\cos(\varphi t_1)A_{\widetilde{K}} + \sin(\varphi t_1)B_{\widetilde{K}}\right] + cs\left[\cos(\varphi t_1)A_{\widetilde{K}} + \sin(\varphi t_1)B_{\widetilde{K}}\right]\}.\end{aligned} \tag{214}$$

Substituting Eq. (214) into Eq. (212) leads to

$$\begin{aligned}
\dddot{A}_K =& -(a+b+c)\ddot{A}_K - (ab+ac+bc+pq+rs)\dot{A}_K - (abc+cpq+brs)A_K \\
&+ \varphi\{(pq-rs)B_K + 2bp[\cos(\varphi t_1)A_N - \sin(\varphi t_1)B_N]\} \\
&- bc\{-p[\sin(\varphi t_1)A_N + \cos(\varphi t_1)B_N] - s[\sin(\varphi t_1)A_{\tilde{K}} - \cos(\varphi t_1)B_{\tilde{K}}]\} \\
&+ 2cs[\cos(\varphi t_1)A_{\tilde{K}} + \sin(\varphi t_1)B_{\tilde{K}}] - bp[\sin(\varphi t_1)A_{\tilde{K}} - \cos(\varphi t_1)B_{\tilde{K}}] \\
&- cs[\cos(\varphi t_1)A_{\tilde{K}} + \sin(\varphi t_1)B_{\tilde{K}}] - cp[\sin(\varphi t_1)A_{\tilde{K}} - \cos(\varphi t_1)B_{\tilde{K}}] \\
&- bs[\cos(\varphi t_1)A_{\tilde{K}} + \sin(\varphi t_1)B_{\tilde{K}}]\} - \varphi^2(\dot{A}_K + aA_K) \\
=& -(a+b+c)\ddot{A}_K - (ab+ac+bc+pq+rs)\dot{A}_K - (abc+cpq+brs)A_K \\
&+ \varphi\{(b-c)\dot{B}_K + (ab-ac+pq-rs)B_K\} - \varphi^2(\dot{A}_K + aA_K).
\end{aligned} \tag{215}$$

Similarly, $\dddot{B}_K$ can be obtained as

$$\begin{aligned}
\dddot{B}_K =& -(a+b+c)\ddot{B}_K - (ab+ac+bc+pq+rs)\dot{B}_K - (abc+cpq+brs)B_K \\
&- \varphi\{(b-c)\dot{A}_K + (ab-ac+pq-rs)A_K\} - \varphi^2(\dot{B}_K + aB_K).
\end{aligned} \tag{216}$$

Appendix C

C.1

In this section, we study the boundedness of the solution of (167). We assume that A_K and B_K are stable and remaining variables A_N, B_N, $A_{\tilde{K}}$ and $B_{\tilde{K}}$ satisfies

$$\begin{aligned}
\dot{A}_N &= -bA_N + q[\sin(\varphi t_1)A_K - \cos(\varphi t_1)B_K], \\
\dot{B}_N &= -bB_N + q[\cos(\varphi t_1)A_K + \sin(\varphi t_1)B_K], \\
\dot{A}_{\tilde{K}} &= -cA_{\tilde{K}} + r[\sin(\varphi t_1)A_K + \cos(\varphi t_1)B_K], \\
\dot{B}_{\tilde{K}} &= -cB_{\tilde{K}} - r[\cos(\varphi t_1)A_K - \sin(\varphi t_1)B_K].
\end{aligned} \tag{217}$$

We initially consider the first equation in Eq. (217). Using method of integration factor, with the integrator factor being e^{bt_1}, results in

$$\begin{aligned}
A_N(t_1) =& A_N(0)e^{-bt_1} + e^{-bt_1}\int_0^{t_1} qe^{bs}(\sin(\varphi s)A_K(s) - \cos(\varphi s)B_K(s))ds \\
\Rightarrow |A_N(t_1)| \le& |A_N(0)|e^{-bt_1} + e^{-bt_1}\left|\int_0^{t_1} qe^{bs}\sin(\varphi s)A_K(s)ds\right| \\
&+ e^{-bt_1}\left|\int_0^{t_1} qe^{bs}\cos(\varphi s)B_K(s)ds\right| \\
\le& |A_N(0)|e^{-bt_1} + e^{-bt_1}M\left|\int_0^{t_1} qe^{bs}ds\right| \\
=& |A_N(0)|e^{-bt_1} + \frac{M|q|}{b}\left(1 - e^{-bt_1}\right) \\
\le& |A_N(0)| + \frac{M|q|}{b}
\end{aligned} \tag{218}$$

for $t_1 \geq 0$, with $\frac{M}{2} := \max\left\{\sup_{s\in[0,t]}(A_K(s)), \sup_{s\in[0,t]}(B_K(s))\right\}$. Hence, we can conclude that the solution $A_N(t_1)$ is bounded.

One can easily observe that, the same steps can be applied to B_N, $A_{\widetilde{K}}$ and $B_{\widetilde{K}}$ as well. Similarly as for $A_N(t_1)$, B_N, $A_{\widetilde{K}}$ and $B_{\widetilde{K}}$ are bounded for all $t_1 \geq 0$.

C.2

From the fact that A_K and B_K are solutions of a linear system with eigenvalues λ_i, $i = 1, \cdots, n$ such that $\mathrm{Re}(\lambda_i) < 0$, we consider the equation

$$\dot{A}_N = -bA_N + q\left[\sin\left(\varphi t_1\right)A_K - \cos\left(\varphi t_1\right)B_K\right] \tag{219}$$

in the form $\dot{A}_N(t_1) = -bA_N(t_1) + C(t_1)$ with $|C(t_1)| \leq ke^{-\mu t_1}$ with b, k, μ being positive constants, and $\mu < |\lambda_i|$. We can apply the method of integrating factor and obtain

$$A_N(t_1) = A_N(0)e^{-bt_1} + e^{-bt_1}\int_0^{t_1} e^{bs}C(s)ds. \tag{220}$$

If $\widetilde{b} \neq \widetilde{\mu}$ we can bound the solution by

$$\begin{aligned} |A_N(t_1)| \leq& |A_N(0)|e^{-bt_1} + e^{-bt}\left|\int_0^{t_1} e^{bs}C(s)ds\right| \\ \leq& |A_N(0)|e^{-bt_1} + e^{-bt}\left|\int_0^{t_1} ke^{(b-\mu)s}ds\right| \\ \leq& |A_N(0)|e^{-bt_1} + \frac{k}{|b-\mu|}\left|e^{-\mu t_1} - e^{-bt_1}\right| \end{aligned} \tag{221}$$

and as $t_1 \to \infty$, $|A_N(t_1)| \to 0$. If $b = \mu$ we can bound the solution by

$$\begin{aligned} |A_N(t_1)| \leq& |A_N(0)|e^{-bt_1} + e^{-bt_1}\left|\int_0^{t_1} kds\right| \\ \leq& (|A_N(0)| + kt_1)e^{-bt_1} \end{aligned} \tag{222}$$

similarly, as $t_1 \to \infty$, $|A_N(t_1)| \to 0$. Thus, we can conclude that $A_N(t_1)$ is stable. The stability of other variables in Eq. (217) can be easily shown similarly.

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