A Cramér-Rao Lower Bound for Complex Parameters

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***Abstract*—An expression is derived for a Cramér-Rao lower bound on the variance of unbiased estimators of complex parameters.**

I. Introduction

In signal processing, the Cramér-Rao lower bound (CRLB) on the variance of unbiased estimators is widely used as a measure of attainable precision of parameter estimates from a given set of observations. It is observed that up to now, most of its applications are concerned with the estimation of real parameters. However, recent publications such as [1] show the usefulness of extending the CRLB to include the estimation of complex parameters. The results described in [1] are specialized to the CRLB for the parameters of superimposed signals corrupted by normally distributed errors, and the complex parameters are the amplitudes of these signals. The purpose of this correspondence is to derive a CRLB for complex parameters applicable to any distribution and model of the observations. The derivation is analogous to the derivation of the CRLB for real parameters in [2].

II. Derivation of the Cramér-Rao Lower Bound for Complex Parameters

Suppose that a vector of possibly complex scalar observations is available described by $w = (w_1 \cdots w_N)^T$. In addition, suppose that the probability density function (pdf) of the observations is $g(\omega; \alpha, \beta)$, where the elements of the $N \times 1$ vector ω correspond to those of the vector w. The elements of the $K \times 1$ vectors α and β are the real and imaginary parts of the elements $\gamma_k = \alpha_k + j\beta_k$ of a vector of complex parameters γ with $j = \sqrt{-1}$.

The pdf $g(\omega; \alpha, \beta)$ may be transformed into a reparameterized version $f(\omega; \theta)$ with

$$\theta = (\gamma_1\, \gamma_1^* \ldots \gamma_K\, \gamma_K^*)^T \tag{1}$$

by substituting $(\gamma + \gamma^*)/2$ and $-j(\gamma - \gamma^*)/2$ for α and β, respectively, where γ^* is the complex conjugate of γ. It will be assumed that $f(\omega; \theta)$ is analytic with respect to the elements of θ. Next, define the $L \times 1$ vector ρ as

$$\rho = (\rho_1(\theta) \cdots \rho_L(\theta))^T \tag{2}$$

where the possibly complex functions $\rho_\ell(\theta)$ are assumed analytic with respect to the elements of θ. If $r(w)$ is any unbiased estimator of ρ, then

$$\int_\Omega r(\omega)\, f(\omega; \theta)\, d\omega = \rho \tag{3}$$

where Ω is the sample space. It is easy to see that under suitable regularity conditions

$$\int_\Omega r(w)\, \frac{\partial f(w;\theta)}{\partial \theta^T} d\omega = E\left[r(w)\, \frac{\partial \ln f(\omega;\theta)}{\partial \theta^T}\right] = \frac{\partial \rho}{\partial \theta^T}. \tag{4}$$

In this expression, the elements of the $1 \times 2K$ vector $\partial f/\partial \theta^T$ are equal to $\partial f/\partial \theta_k$, where $f = f(\omega; \theta)$. Furthermore, in (4), the (ℓ, k)th element of the $L \times 2K$ matrix $\partial \rho/\partial \theta^T$ is equal to $\partial \rho_\ell/\partial \theta_k$. By definition

$$\int_\Omega f(\omega; \theta)\, d\omega = 1 \tag{5}$$

and hence

$$\int_\Omega \frac{\partial f(\omega;\theta)}{\partial \theta^T}\, d\omega = E\left[\frac{\partial \ln f(\omega;\theta)}{\partial \theta^T}\right] = 0 \tag{6}$$

where 0 is the 1×2 K null vector. From (4) and (6), it is concluded that

$$\frac{\partial \rho}{\partial \theta^T} = E\left[r(w)\, \frac{\partial \ln f}{\partial \theta^T}\right] \tag{7}$$

is equal to the covariance matrix of the $L \times 1$ vector $r(w)$ and the 2K $\times$ 1 vector $(\partial \ln f/\partial \theta^T)^H$, where H denotes the complex conjugate transpose. Therefore, the autocovariance matrix of the partitioned $(L + 2\text{ K}) \times 1$ vector

$$\begin{bmatrix} r \\ \left[\dfrac{\partial \ln f}{\partial \theta^T}\right]^H \end{bmatrix} \tag{8}$$

is equal to

$$\begin{bmatrix} \operatorname{cov}(r, r) & \dfrac{\partial \rho}{\partial \theta^T} \\ \left[\dfrac{\partial \rho}{\partial \theta^T}\right]^H & E\left[\left[\dfrac{\partial \ln f}{\partial \theta^T}\right]^H \dfrac{\partial \ln f}{\partial \theta^T}\right] \end{bmatrix} \tag{9}$$

where $\operatorname{cov}(r, r)$ is the autocovariance matrix of $r = r(w)$.

Since the matrix (9) is an autocovariance matrix, it is positive semidefinite. Hence, so is

$$\left[I \;-\; \frac{\partial \rho}{\partial \theta^T} M^{-1}\right] \begin{bmatrix} \operatorname{cov}(r, r) & \dfrac{\partial \rho}{\partial \theta^T} \\ \left[\dfrac{\partial \rho}{\partial \theta^T}\right]^H & M \end{bmatrix} \left[I \;-\; \frac{\partial \rho}{\partial \theta^T} M^{-1}\right] \tag{10}$$

where

$$M = E\left[\left[\frac{\partial \ln f}{\partial \theta^T}\right]^H \frac{\partial \ln f}{\partial \theta^T}\right] \tag{11}$$

and I is the identity matrix of order L. Carrying out the multiplications in (10) shows that

$$\operatorname{cov}(r, r) - \frac{\partial \rho}{\partial \theta^T}\, M^{-1} \left[\frac{\partial \rho}{\partial \theta^T}\right]^H \tag{12}$$

is positive semidefinite. Therefore, the CRLB for unbiased estimation of functions ρ is described by

$$\frac{\partial \rho}{\partial \theta^T}\, M^{-1} \left[\frac{\partial \rho}{\partial \theta^T}\right]^H. \tag{13}$$

From this expression, it follows that the CRLB for unbiased estimation of the complex parameters θ is equal to M^{-1} since in this case, $\partial \rho/\partial \theta^T$ is the identity matrix.

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References

[1] S. F. Yau and Y. Bresler, "A compact Cramér-Rao bound expression for parametric estimation of superimposed signals," *IEEE Trans. Signal Processing*, vol. 40, pp. 1226–1230, May 1992.

[2] P. J. Dhrymes, *Econometrics: Statistical Foundations and Applications.* New York: Harper and Row, 1970.

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