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BOUNDARY SINGULARITIES AND CHARACTERISTICS OF HAMILTON–JACOBI EQUATION

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Abstract. Boundary-value problems for first order PDEs are locally considered, when the classical sufficient condition for the solution existence does not hold, but a solution still exists, possibly defined on one or both sides of the boundary surface. We note three situations when such a surface (locally) arises: (1) the part of the boundary surface with the given boundary value; (2) the part of the boundary surface with no value initially specified on it, while such a value arises during the constructions; (3) a singular surface arising during the constructions in the internal part of the domain. In the latter two cases, which are typical for the problems of optimal control and differential games, the solution value on the surface is specified due to singular characteristics. Although the character of the singularity of the solution is well known, we show in this paper that it is completely determined by the signs of two multiple Poisson brackets naturally arising in the equations of singular characteristics. We recall the notion of regular and singular characteristics and formulate a new sufficiency condition for the existence and uniqueness in the irregular case. This condition is in invariant form based on Poisson (Jacobi) bracket, which is convenient in applications. We give several examples with irregular solutions and boundary characteristics.

1. Introduction

In many problems of optimal control, differential games, and mathematical physics, the following boundary-value problem arises in terms of the first-order (Hamilton–Jacobi, Bellman–Isaacs) PDE and the scalar unknown function $u(x)$ [7, 2, 4, 5]:

$$F(x, u(x), p(x)) = 0, \quad x \in \Omega, \quad p(x) = \frac{\partial u(x)}{\partial x}, \tag{1}$$
$$u(x) = w(x), \quad x \in M \subset \partial\Omega = M + M_0.$$

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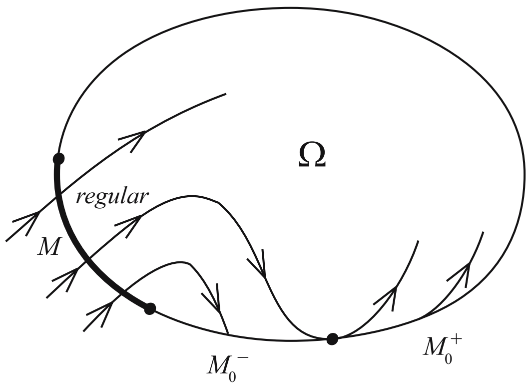

Fig. 1

Generally, to obtain a unique solution in the whole Ω, one needs to specify the boundary conditions only on a part $M \subset \partial\Omega$ of the boundary $\partial\Omega$ (see Fig. 1). In [7], such a surface M is called the usable part of the boundary. The part M_0 is free of boundary conditions, or the limit of the solution from Ω to M_0 does not coincide with the boundary value specified.

During the constructions (say, by means of characteristics), another new boundary subproblem may arise in the vicinity of a surface $M_0^+ \subset M_0$. Such a subproblem may be regular or irregular and has generic character, as show the last two examples of this paper.

1.1. **Classical characteristics.** In the case of smooth functions F and $w(x)$ and the surface M (at least of the class C^2), the construction of the smooth solution $u(x) \in C^2(\Omega)$ is known to be reduced to the integration of the classical (regular) characteristic ODE system [4, 5, 10]:

$$\dot{x} = F_p, \quad \dot{u} = \langle p, F_p \rangle, \quad \dot{p} = -F_x - pF_u, \quad p = \frac{\partial u}{\partial x}. \tag{2}$$

Here we differentiate with respect to some independent variable t, $\dot{x} = dx/dt$, not necessarily having the sense of time.

The classical characteristics admit a certain generalization, which is useful for the construction of nonsmooth solutions with possibly nonsmooth left-hand side F (see Sec. 3).

1.2. **Initial strip.** To integrate the characteristic system on some interval $[t_0, t_1]$, one needs the initial conditions for the variables (x, u, p). One can set $x(t_0) = x^*$, $u(t_0) = w(x^*)$, $x^* \in M$, and needs to find $p(t_0)$. To do this, one can differentiate the boundary condition $u(x) - w(x) = 0$, $x \in M$, in the directions $y_j(x)$ tangent to M and combining with the PDE itself come to the system of n equations with respect to $p = (p_1, p_2, \dots, p_n)$:

$$\begin{aligned} F(x, w(x), p) = 0, \quad q(x) = \frac{\partial w(x)}{\partial x}, \\ \langle p - q(x), y_j(x) \rangle = 0, \quad j = 1, \dots, n-1, \end{aligned} \tag{3}$$

where $y_j(x) \in \mathbb{R}^n$, $j = 1, \dots, n-1$, is a basis of the tangent space $T_x M$ at a point $x \in M$. The manifold Σ defined in the (x, u, p)-space by Eqs. (3) is called an initial strip (see [4]).

1.3. **Regularity condition.** We assume that the Jacobian of system (3) for the vector p is nonzero,

$$\Delta(x^*, p^*) = \det |F_p, y_1, \dots, y_{n-1}| \neq 0,$$

for some (x^*, p^*) satisfying the system. Then owing to the implicit-function theorem, there exists a C^1-solution $p = r(x)$ of system (3), $p^* = r(x^*)$. Parameterizing the surface M by a proper $(n-1)$-dimensional parameter s, $x = \varphi(s)$, $s \in S_0 \subset \mathbb{R}^{n-1}$ (when one can take $y_j = \partial\varphi/\partial s_j$, $j = 1, \dots, n-1$), one can obtain the initial conditions:

$$x(t_0) = \varphi(s), \quad u(t_0) = w(\varphi(s)), \quad p(t_0) = r(\varphi(s)), \quad s \in S_0,$$

and the corresponding solution of system (3) in the form

$$x = X(s, t), \quad u = U(s, t), \quad p = P(s, t), \quad s \in S_0, \quad |t - t_0| \leq \varepsilon.$$

The Jacobian of the equation $x = X(s, t)$ at $t = t_0$, $s = s^*$, where $\varphi(s^*) = x^*$, is $\Delta(x^*, p^*) \neq 0$. Thus, one can solve it for (s, t) to obtain $t = T(x)$ and $s = S(x)$ and form the solution:

$$u(x) = U(S(x), T(x)), \quad p(x) = P(S(x), T(x)) = \frac{\partial u}{\partial x}.$$

The solution $u(x)$ here is of the class C^2 since $p(x)$ is at least C^1.

One can see that the (regularity) condition $\Delta(x, p) \neq 0$ is important and guarantees two things: the existence of an initial strip

$$\Sigma = \{z = (x, u, p) \in \mathbb{R}^{2n+1} : u = w(x),\ p = r(x),\ x \in M\}$$

and the reverse of the equation $x = X(s, t)$.

Note another form of the regularity condition which will be used in the sequel. In case where the surface M is given by an equation $g(x) = 0$ with a scalar g, while $g_x \neq 0$, one has $\langle g_x(x), y_j(x) \rangle = 0$, $j = 1, \dots, n-1$, since g_x is a normal to M, as the vector $p - q$ is, so that $g_x = \lambda(p - q)$ with some scalar λ. The regularity condition means that the vector F_p is not in the linear subspace spanned on y_j; thus,

$$\langle F_p, g_x(x) \rangle \neq 0.$$

This inequality, the regularity condition, guarantees the local existence of a C^2-solution $u(x)$ defined on both sides of the surface M.

If one prefers the set Ω be given as $g(x) \le 0$, then $\langle F_p, g_x(x)\rangle \le 0$ for an initial-value problem and $\langle F_p, g_x(x)\rangle \ge 0$ for a terminal-value problem, depending the characteristic flow goes inside or outside of Ω (see [12]).

The aim of this paper is to compare irregular problems locally related to the following surfaces: (1) the part of the boundary surface with the given boundary value [7, 2, 4, 5]; (2) the part of the boundary surface with no value initially specified on it, while such a value arises during the constructions [11]; (3) a singular surface arising during the constructions in the internal part of the domain [10], and based on it to formulate a new sufficiency condition for the existence and uniqueness of the irregular solution. For this purpose, we exploit the two multiple Poisson (Jacobi) brackets naturally arising in the equations of singular characteristics. The signs of these brackets determine the character of the singularity. Such conditions have invariant nature and are convenient in applications.

2. Irregular problem and boundary conditions on M

Assume that the regularity condition is not fulfilled identically:

$$\langle F_p(x, w(x), r(x)), g_x(x)\rangle = 0$$

for all $x \in M$. Such a problem will be called an irregular problem. To analyze an irregular problem, we assume that an initial strip Σ is given in advance:

$$\Sigma = \{z = (x, u, p) \in \mathbb{R}^{2n+1} : u = w(x),\ p = r(x),\ x \in M\}, \qquad (4)$$

where the functions $w(x)$ and $r(x)$ satisfy the strip conditions (3) for $x \in M = \{x \in \mathbb{R}^n : g(x) = 0\}$.

Although one has the initial conditions for the integration of the characteristic system, the above algorithm for the construction of $u(x)$ fails for an irregular problem because the equation $x = X(s, t)$, generally, cannot be solved for (s, t). But irregular problem still may have solutions under certain assumptions.

To formulate these assumptions, denote $z = (x, u, p)$ and introduce the vector-field (right-hand side of the characteristic equation):

$$\xi_F(z) = (F_p, \langle p, F_p\rangle, -F_x - pF_u) \in \mathbb{R}^{2n+1}.$$

We will distinguish between the following two cases.

The irregular characteristic problem. An irregular problem for which the vector F_p is tangent to M in $\mathbb{R}^n$ and the vector ξ_F is tangent to Σ in $\mathbb{R}^{2n+1}$:

$$\langle F_p(z), g_x(x)\rangle = 0, \quad \xi_F \in T_z\Sigma, \quad z \in \Sigma, \quad x \in M,$$

is called an irregular characteristic problem.

The irregular noncharacteristic problem. In the case where only the first tangency condition holds,

$$\langle F_p(z), g_x(x)\rangle = 0, \quad \xi_F \notin T_z\Sigma, \quad z \in \Sigma, \quad x \in M,$$

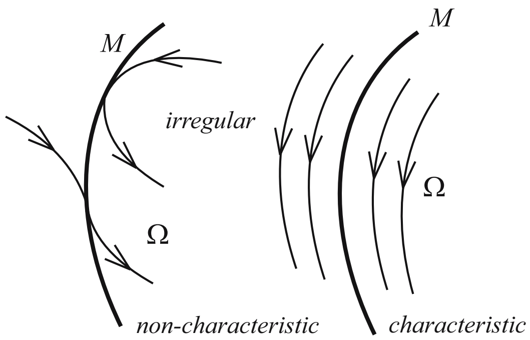

Fig. 2

the problem is called said to be noncharacteristic. Figure 2 illustrates the behavior of characteristics in the x-space.

The irregular characteristic problem has infinitely many C^2-solutions [4, 10]. The noncharacteristic problem may have a unique solution. We formulate a sufficiency condition for this.

2.1. **Initial strip for irregular noncharacteristic problem.** One may think to substitute the first equation $F(x, w(x), r(x)) = 0$ in the strip system by the following one:

$$\langle F_p(z), g_x(x)\rangle = 0,$$

while a further singularity may exist. To formulate the problem, introduce the functions

$$F^i = \left\langle g_x, \frac{\partial}{\partial p} \right\rangle^i F, \quad i = 0, 1, 2, \dots, \quad F^0 = F.$$

Now we consider the system with respect to unknown p:

$$\begin{gathered} F^{k-1}(x, w(x), p) = 0, \quad \langle p - w_x(x), y^j(x)\rangle = 0, \\ j = 1, \dots, n-1, \quad x \in M. \end{gathered} \tag{5}$$

Assume that the system is satisfied for some point $z^* = (x^*, w(x^*), p^*)$, while

$$F^k(z^*) \neq 0.$$

This condition is equivalent to the nonvanishing of the Jacobian of the system. Thus, there exists a unique solution $p = r(x)$, $x \in M$, defining a manifold Σ. If, in addition, this solution also satisfies the following $k - 1$

equations:

$$\begin{aligned}
&F(x, w(x), r(x)) = 0,\\
&F^1(x, w(x), r(x)) = \langle g_x, F\rangle = 0,\\
&\vdots\\
&F^{k-2}(x, w(x), r(x)) = \left\langle g_x, F^{k-3}\right\rangle = 0,
\end{aligned} \tag{6}$$

where $x \in M$, we will say that an irregular noncharacteristic problem is given with the order $k-1$ of irregularity and with the initial strip Σ. These $k-1$ equations describe the overdetermination of Σ. Thus, one has a generic position only for regular case of $k = 1$. The noncharactericity of such a problem will be shown in the sequel, as well as the inequality, say, $F^k(z^*) \geq 0$, if one wishes to have $g(x) \geq 0$ for Ω.

For the investigation of the irregular noncharacteristic problem, the notion of singular characteristics can be useful.

3. Method of singular characteristics

Next, we provide a brief description of the method of singular characteristics for Eq. (1). If $u(x)$ and $F(x, u, p)$ are smooth, the local construction of a solution of problem (1) is known to be reduced to integration of system (2) of classical characteristics. This integration generates an n-dimensional strip Σ_n containing the initial strip, $\Sigma \subset \Sigma_n$, while the solution $u(x)$ is the projection of Σ_n to the (x, u)-space. If u and/or F are nonsmooth, system (2) breaks down. Singularities typically lie along smooth manifolds. Singular characteristics provide a means to describe such manifolds. The differential-geometric descriptions of the regular and singular characteristics are similar [1]. In the regular case, the equation $F(x, u, p) = 0$ defines a $2n$-dimensional surface W_1 in the $(2n+1)$-dimensional space of (x, u, p). The classical characteristics (2) define a tangent field on W_1, which generates integral manifolds Σ_n of the standard 1-form $du - \langle p, dx\rangle$ (i.e., preserves the strip property of $\Sigma \subset \Sigma_n$). For a given manifold W_1, a field with these properties is defined uniquely up to a scaling homogeneity factor.

Similar tangent characteristic fields can be defined on a surface W_k of odd codimension $k = 1, 3, \ldots$ under some general requirements to W_k. The integral curves of this characteristic field on a surface of codimension 3 or higher are called singular characteristics. In many cases, by using both regular and singular characteristics, one can construct the desired solution $u(x)$ even if it and/or the Hamiltonian is nonsmooth [10]. The corresponding construction technique is called the *method of singular characteristics* (MSC). Regular and singular characteristics together are referred to as generalized characteristics.

The differential-geometric description of the characteristic field on an even-dimensional manifold is given in [1]. To write down the corresponding analytical formulas, assume that the singularities of u lie along a surface $\Gamma \subset \Omega$, $\dim \Gamma = n-1$, which is associated with a surface W_3 of codimension 3 in the (x,u,p)-space (see the note after Theorem 1). Generically, W_3 is locally described by three equations:

$$W_3: \qquad F_1(x,u,p)=0, \quad F_0(x,u,p)=0, \quad F_{-1}(x,u,p)=0. \tag{7}$$

The choice of F_i is determined by the type of singularity and properties of the solution, such as viscosity solution inequalities or matching conditions across singular surfaces Γ. One of the F_i may represent the PDE itself, say $F_0 \equiv F$. The vector p in (7) is the limiting value of the gradient of the function u from an appropriate side of the singular hypersurface $\Gamma \subset \mathbb{R}^n$. The corresponding tangent field (the system of singular characteristics) on W_3 can be written using so-called *singular Hamiltonian* H^σ:

$$\mu H^\sigma = \{F_1 F_0\} F_{-1} + \{F_0 F_{-1}\} F_1 + \{F_{-1} F_1\} F_0, \tag{8}$$

where $\mu = \mu(z)$ is a nonzero homogeneity factor, a smooth function, and $\{\,\cdot\,\}$ is the Jacobi bracket:

$$\{FG\} = \langle F_x + pF_u, G_p\rangle - \langle G_x + pG_u, F_p\rangle .$$

This expression becomes the Poisson bracket if there is no dependence on u: $F_u = 0$, $G_u = 0$.

Theorem 1. *Let $\xi_{H^\sigma}(z^*) \neq 0$ for some $z^* \in W_3$. Then the system* (*of the regular characteristics in terms of H^σ*)

$$\dot z = \xi_{H^\sigma}(z), \quad z \in W_3, \quad \xi_{H^\sigma}(z) = \left(H_p^\sigma, \langle p, H_p^\sigma\rangle, -H_x^\sigma - pH_u^\sigma\right) \tag{9}$$

defines locally the unique (*singular*) *characteristic field on W_3.*

This theorem follows from [10, Theorem 1.3]. Note that for the construction of the singular manifold Γ, the system (9) is integrated subject to some initial $(n-2)$-dimensional strip Σ_{n-2} to get an $(n-1)$-dimensional strip Σ_{n-1}, and the projection of the latter strip to (x,u)-space determines the manifold Γ and the corresponding value of u.

It is also shown in [10] how the typical types of singular hypersurfaces arising in differential games (universal, equivocal and focal; see Isaacs [7]) are described by singular characteristics.

3.1. **Equivocal singular characteristics in differential games.** For illustration, consider a singular surface of equivocal type (see [7]). Regular characteristics approach an equivocal surface Γ on one side and depart from it on the other, while the gradient $p(x)$ is discontinuous across Γ. For the case of smooth Hamiltonian $F \in C^2$, this leads to the following set of necessary conditions, defining the manifold W_3:

$$W_3: \quad F_0 = F = 0, \quad F_1(x,u) = u - v(x) = 0,$$
$$F_{-1} = \{F_1 F\} = \langle F_p, p - q\rangle = 0, \quad q = \frac{\partial v}{\partial x}.$$

The last equality is the tangency condition, which follows from viscosity solution requirements, and $v(x) \in C^2$ is the solution of (1) on the "arrival side" of Γ. The MSC approach leads to the following result.

Theorem 2. *Let $\mu(z) = \{\{F_1 F\} F_1\}$, and for some $z^* \in W_3$, let $\mu(z^*) \neq 0$ and $\{\{F F_1\} F\} \neq 0$. Then the initial strip* (4) *is in W_3, $\Sigma \subset W_3$, and is an invariant manifold of the following system of singular characteristics:*

$$\dot{x} = F_p, \quad \dot{u} = \langle p, F_p\rangle, \quad \dot{p} = -F_x - pF_u - \frac{\{\{F F_1\} F\}}{\{\{F_1 F\} F_1\}}(p - q). \tag{10}$$

The proof of Theorem 2 follows from the proofs of [10, Theorem 1.5, Lemma 2.3].

System (10) allows one to construct the surface Γ. For the construction of the solution $u(x)$ from the "departure" side one has an irregular boundary value problem due to tangency condition.

If F is nonsmooth, the situation is different. A common type of nonsmooth Hamiltonian is the minimum of two (or more) smooth H_i: $F(x,u,p) = \min[H_0(x,u,p), H_1(x,u,p)]$. Now the conditions (7) are applied using

$$W_3: \quad F_0 = H_0, \quad F_{-1} = H_1, \quad F_1(x,u) = u - v(x).$$

The corresponding system of singular characteristics is as follows:

$$\begin{gathered}
\dot{x} = \lambda_0 H_{0p} + \lambda_1 H_{1p}, \quad \dot{u} = \lambda_0 \langle p, H_{0p}\rangle + \lambda_1 \langle p, H_{1p}\rangle, \\
\dot{p} = -\lambda_0 (H_{0x} + pH_{0u}) - \lambda_1 (H_{1x} + pH_{1u}) - \frac{\{H_1 H_0\}}{\mu}(p - q(x)), \\
\lambda_0 + \lambda_1 = 1, \quad \lambda_0 = \frac{\{F_1 H_1\}}{\mu}, \quad \mu = \{F_1 H_1\} + \{H_0 F_1\}.
\end{gathered} \tag{11}$$

In this case, the construction of the solution $u(x)$ on the other side of Γ one, generally, has a regular boundary value problem.

3.2. **Boundary singular characteristics.** The notion of boundary singular characteristics allows one to simplify the analysis of the problem with irregular boundary conditions, comparing to that of given in [10].

In this section, we consider the part $M_0 = M_0^+ + M_0^-$ of the boundary with no solution value specified on it, or, if specified, do not coincide with the solution limit from inside of Ω. We assume that the characteristic flow goes outside of Ω through M_0^-, while on M_0^+ the characteristics are not approaching the boundary.

Assume that in the vicinity of M_0^+ the domain Ω locally is given by the inequality

$$\Omega = \{x \in \mathbb{R}^n \mid g(x) < 0\},$$

where $g(x)$ is a smooth scalar function, so that the surface M_0^+ is described by $g(x) = 0$. Thus, the above assumption about M_0^+ means that

$$M_0^+ : \quad \langle F_p, g_x \rangle \leq 0,$$

where g_x provides an exterior normal vector to $\partial\Omega$. The vector $p(x) = u_x(x)$ at a point x on the boundary $g(x) = 0$ denotes the continuous extension of the gradient from the open domain $g(x) < 0$. Thus, the following manifold Σ^+ is defined satisfying the strip condition (since one has $F(x, u(x), u_x(x)) = 0$):

$$\Sigma^+ = \{z = (x, u, p) \in \mathbb{R}^{2n+1} : u = u(x),\ p = u_x(x),\ x \in M_0^+\}.$$

The solution of the original boundary value problem is understood in the viscosity sense (see, e.g., [5, 10, 6]). As shown in [6], the so-called supersolution property must be fulfilled up to the surface M_0. Based on this, it is proved in [11] that in the case of smooth Hamiltonian $F(x, u, p)$ the following conditions are fulfilled on M_0^+:

$$\langle F_p, g_x \rangle \geq 0, \quad \langle F_{pp} g_x, g_x \rangle = -\{\{gF\}g\} \geq 0.$$

Comparing with the previous inequality on M_0^+, this leads to the equality

$$\langle F_p, g_x \rangle = \{gF\} = 0.$$

Now one can observe that the three conditions

$$W_3 : \quad F_0 = F = 0, \quad F_1(x) = g(x) = 0, \quad F_{-1} = \{gF\} = \langle F_p, g_x \rangle = 0$$

are fulfilled on the boundary. These conditions generate a system of singular characteristics similar to the equivocal system (10).

Theorem 3. *Let $\mu(z) = \{\{gF\}g\}$, and for some $z^* \in W_3$, let $\mu(z^*) \neq 0$ and $\{\{Fg\}F\} \neq 0$. Then the initial strip Σ^+ is in $\in W_3$, $\Sigma^+ \subset W_3$, and is an invariant manifold of the following system of singular characteristics*:

$$\dot{x} = F_p, \quad \dot{u} = \langle p, F_p \rangle, \quad \dot{p} = -F_x - pF_u + \frac{\{\{Fg\}F\}}{\langle F_{pp} g_x, g_x \rangle} g_x. \tag{12}$$

The proof of Theorem 3 is similar to that of Theorem 2 (see also [11]).

In the sequel it will be shown that the numerator here also is nonnegative, $\{\{Fg\}F\} \geq 0$.

In the case of a nonsmooth Hamiltonian $F = \min[H_0, H_1]$ one has $F_i = 0$ on the surface, where

$$W_3 : \quad F_0 = H_0, \quad F_1(x) = g(x), \quad F_{-1} = H_1.$$

This leads to the following system of characteristic equations similar to (11):

$$\dot{x} = \lambda_0 H_{0p} + \lambda_1 H_{1p}, \quad \dot{u} = \lambda_0 \langle p, H_{0p}\rangle + \lambda_1 \langle p, H_{1p}\rangle,$$
$$\dot{p} = -\lambda_0 (H_{0x} + pH_{0u}) - \lambda_1 (H_{1x} + pH_{1u}) - \frac{\{H_1 H_0\}}{\mu} g_x, \tag{13}$$
$$\lambda_0 + \lambda_1 = 1, \quad \lambda_0 = \frac{\{gH_1\}}{\mu}, \quad \mu = \{gH_1\} + \{H_0 g\}.$$

In this case, regular characteristics depart the boundary surface transversally as in a regular problem. Generally, tangential departure also is possible if the solution locally is determined by one of the branches H_0 or H_1, in which case instead of (13) one has a system of singular characteristics similar to (12) with H_0 or H_1 as F.

3.3. **Higher order singularity.** In case of smooth Hamiltonian further singularity may occur when one has $\{g\{gF\}\} = 0$ and $\{g\{g\{gF\}\}\} \neq 0$ on Σ^+. The next step is also possible. To describe this, we use the functions F^i introduced earlier, writing their other equivalent representation in terms of the Jacobi bracket:

$$F^i = \{gF^{i-1}\} = \left\langle g_x, \frac{\partial}{\partial p} \right\rangle^i F, \quad i = 1, 2, \ldots, \quad F^0 = F.$$

Assume that for some $k \geq 1$ one has $F^i = 0$, $i = 0, 1, \ldots, k-1$, and $\{F^{k-1}F\} \neq 0$ on Σ^+. Consider the manifold

$$W_3: \quad F_0 = F = 0, \quad F_1(x) = g(x) = 0, \quad F_{-1} = F^{k-1} = 0$$

with the singular characteristics on it:

$$\dot{x} = F_p, \quad \dot{u} = \langle p, F_p\rangle, \quad \dot{p} = -F_x - pF_u - \frac{\{F^{k-1}F\}}{\{gF^{k-1}\}} g_x.$$

In the sequel, we will see that from the condition $\{F^{k-1}F\} \neq 0$ it follows also the inequality $F^k = \{gF^{k-1}\} \neq 0$.

4. Comparison of boundary conditions on M and M_0^+

Taking into account the irregularity conditions, like $\langle F_p, g_x\rangle = 0$, on the boundary surface M, one can see many similarities in the conditions on M and M_0^+. With both surfaces, M and M_0^+, a manifold W_3 is associated:

for M: $F = 0$, $u - w(x) = 0$, $\langle F_p, p - w_x\rangle = 0$,

for M_0^+: $F = 0$, $g(x) = 0$, $\langle F_p, g_x\rangle = 0$

with the characteristic field ξ_H on it:

$$\xi_H(z) = (H_p, \langle p, H_p\rangle, -H_x - pH_u) \in \mathbb{R}^{2n+1},$$

where H is a singular Hamiltonian. On M_0^+, we need the characteristic field to construct (to obtain) the boundary value for $u(x)$, and we need the

x-component H_p of ξ_H to be nonzero, while on M such a value is given. In addition, on M the characteristic field generates some new boundary value, extending it from some $(n-2)$-dimensional submanifold $M_s \subset M$ to the whole M. The extended value should be equal to the existing one. And this is the case indeed, since the function $u - w(x)$ is a first integral of the field ξ_H:

$$\frac{d}{dt}(u - w(x)) = \dot{u} - \langle w_x, \dot{x} \rangle = \langle H_p, p - w_x \rangle = 0$$

since $p - w_x$ is a normal vector to M: $g_x = \lambda(p - w_x)$.

So, more important the characteristic field is for the surface M_0^+. Observing the characteristic equations of the form

$$\dot{x} = F_p, \quad \dot{u} = \langle F_p, p \rangle, \quad \dot{p} = -F_x - pF_u - \frac{A}{B}(p - q),$$

one can guess that sufficient conditions for the system to be effective could be $A \neq 0$ and $B \neq 0$, which also leads to the right choice of the manifold W_3. In the next section, we show that the condition $B \neq 0$ follows from $A \neq 0$.

An interesting situation one has in the regular case where the following manifold can be associated with the boundary conditions

$$W_3 : \quad F(z) = 0, \quad g(x) = 0, \quad u - w(x) = 0$$

with the corresponding characteristic field $\xi_H(z)$ on it. One actually does not need this characteristic field to construct the boundary value, since it is given. To understand the role of the field $\xi_H(z)$ let us introduce the notion of a characteristic point [1]. A point $z \in W_3$ for which $\xi_H(z) = 0$ is called a characteristic point of the manifold W_3.

Lemma 1. *The initial surface Σ for a regular problem consists of the characteristic points of the manifold W_3 defined by the equation and initial conditions.*

Proof. The characteristic field of the above manifold W_3 has the form

$$\xi_H(z) = (0, 0, \langle F_p, p - w_x \rangle\, g_x - \langle F_p, g_x \rangle\, (p - w_x)).$$

This $(2n+1)$-vector has only n last components that are not identically zero. The characteristicity condition of a point $z = (x, u, p) \in W_3$ takes now the form

$$\langle F_p, p - w_x \rangle\, g_x - \langle F_p, g_x \rangle\, (p - w_x) = 0.$$

Let $y^1(x), \dots, y^{n-1}(x) \in \mathbb{R}^n$ be a basis of the tangent space $T_x M$, i.e., in particular,

$$\left\langle g_x, y^j \right\rangle = 0.$$

Multiplying scalarly the n-vector equation above by y^j we obtain

$$\langle F_p, g_x \rangle \left\langle y^j, p - w_x \right\rangle = 0, \quad j = 1, \dots, n-1.$$

Note that similar multiplying by F_p gives identically zero, which means that among the n nontrivial equations in $\xi_H(z) = 0$ only $n - 1$ are independent. For the regular case under consideration one has

$$\langle F_p, g_x \rangle \neq 0.$$

Now the above equations together with Eq. (1) give system (3) for the initial strip which completes the proof of the lemma. □

Thus, the initial strip in a regular problem consists of the stationary points of the characteristic field $\xi_H(z)$ on the manifold W_3.

5. Sufficient conditions for the existence and uniqueness

In Theorem 4 formulated later in this section, we consider local solutions possibly defined to one or both sides of the boundary surface. A solution of the irregular noncharacteristic problem must be searched for in more wide class of functions. Such a class, specified by a given surface Γ, will be denoted as $K(\Gamma)$. Let a neighborhood D of a point $x^* \in \Gamma \subset \mathbb{R}^n$ be given and let a smooth hypersurface Γ divide the domain D into two open subdomains:

$$D = D_0 + \Gamma + D_1.$$

The functions of the class $K(\Gamma)$ are defined either in the whole D or only in one of the domains $D_0 + \Gamma$ and $D_1 + \Gamma$. We require the following differentiability properties:

$$u(x) \in C^1(D), \quad u(x) \in C^2(D_i), \quad i = 0, 1,$$

for the functions defined in D, and

$$u(x) \in C^1(D_i + \Gamma), \quad u(x) \in C^2(D_i)$$

for the functions defined in D_i. The inclusion $u(x) \in C^1(D_i + \Gamma)$ means that the gradient of $u(x)$ has a continuous extension from D_i to Γ. This allows the classical interpretation of the solution of the equation

$$F\left(x, u(x), \frac{\partial u}{\partial x}\right) = 0$$

in the domain $D_i + \Gamma$. Thus, we do not expect the continuous second derivatives for $u(x) \in K(\Gamma)$, although $C^2(D) \subset K(\Gamma)$.

The role of the set Ω in (1) may play D_0 or D_1.

5.1. **Formulation of the simple problem.** To simplify technically the proof of Theorem 4, we seek the appropriate change of variables. Since the important conditions of Theorem 4 are expressed in terms of the Jacobi brackets, one can try contact transformations of the variables which preserve the value of the brackets and the form of the characteristic equations.

Using such transformation, one can reduce the original problem to so-called simple problem with the following boundary surface, boundary condition and the initial strip (see [10]):

$$M = \{x \in \mathbb{R}^n : x_1 = 0\}, \quad u(0, x_2, \dots, x_n) = 0,$$
$$\Sigma = \{(x, u, p) \in \mathbb{R}^{2n+1} : x_1 = 0,\ u = 0,\ p_1 = 1,\ p_2 = 0,\ \dots,\ p_n = 0\}.$$

The value $p_1 = 1$ satisfies the following equation and inequality:

$$F^{k-1}(0, x_2, \dots, x_n, 0, p_1, 0, \dots, 0) = 0, \quad \frac{\partial F^{k-1}}{\partial p_1} = F^k \neq 0$$

while the additional $k-1$ equations overdetermining Σ areas follows:

$$F = F^0(0, x_2, \dots, x_n, 0, p_1, 0, \dots, 0) = 0,$$
$$F^i(0, x_2, \dots, x_n, 0, p_1, 0, \dots, 0) = \frac{\partial F^{i-1}}{\partial p_1} = 0, \quad i = 1, \dots, k-2.$$

The function $g(x)$ for the simple problem has the form $g(x) = x_1$.

The simple problem can be considered for $n = 2$, when $x = (x_1, x_2)$; the case $n > 2$ is considered similarly. The surface M in 2D simple problem is the coordinate axis $x_1 = 0$ with zero boundary condition on it:

$$M = \{(x_1, x_2) : x_1 = 0,\ x_2 \in \mathbb{R}^1\}, \quad u(0, x_2) = 0, \quad x_2 \in \mathbb{R}^1.$$

Local considerations actually will be restricted to some segment $|x_2| \leq \delta$, the reference point being the origin: $x^* = (0, 0)$.

Thus, the initial strip, the surface Σ, is the following one-dimensional straight line in the 5-dimensional space:

$$\Sigma = \{(x_1, x_2, u, p_1, p_2) : x_1 = 0,\ x_2 = s,\ u = 0,\ p_1 = 1,\ p_2 = 0,\ s \in \mathbb{R}^1\}$$

and the following values vanish on Σ: $x_1 = 0$, $u = 0$, $p_2 = 0$, and $F_{p_1} = 0$. Some additional vanishing values, like $F_{x_2} = 0$ and $F_{p_1 x_2} = 0$, can be found by differentiating the overdetermining equalities with respect to x_2.

Corollary 2 of Theorem 4 is related to the simple problem.

5.2. **Formulation of the theorem.** Different combinations of multiple brackets will be used in further considerations. Introduce the new notation $F_0 = F$ a $F_1 = g$ for the functions $F(z)$ and $g(x)$ and the multiple brackets of the following structure:

$$G_{ms} = \{\dots\{F_0 F_1\} F_{j_3} \dots F_{j_{m+1}}\}, \quad m = 1, 2, \dots, \tag{14}$$

where j_ν are equal either 0 or 1; s is the number of 1s in the vector $(j_1, j_2, j_3, \dots, j_{m+1})$ with $j_1 = 0$, $j_2 = 1$, and thus $s \leq m$. The brackets with the same m will be called the brackets of the level m.

We need also special notation for the two extremal cases where all j_ν, $\nu \geq 2$, are the same:

$$F_0^{i+1} = \{gF_0^i\} = \left\langle g_x, \frac{\partial}{\partial p} \right\rangle^{i+1} F,$$

$$F_0^0 = F, \quad F_1^{i+1} = \{F_1^i F\}, \quad F_1^0 = g(x), \quad i = 0, 1, \ldots,$$

where F_0^i coincides with F^i introduced earlier.

The functions F and w and the surface $M \subset D$ are assumed to be sufficiently smooth.

Theorem 4. *The irregular noncharacteristic problem with the given initial surface Σ is solvable in the class of functions $K(M)$ if for some integer $k \geq 2$ the following conditions hold*:

$$F_0^i(z) = 0, \quad z \in \Sigma, \quad i = 0, 1, \ldots, k-1,$$

$$\{F_0^{k-1} F_0\} \neq 0 \quad \textit{for } z^* \in \Sigma.$$

For odd k, the solution is unique and defined in the whole D; the characteristic flow is in the direction g_x (direction $-g_x$) when $F_1^k > 0$ (when $F_1^k < 0$).

For even k, the brackets F_1^k and $\{F_0^{k-1} F_0\}$ have the same sign, and there exist two solutions (or one double-valued solution) defined, for the case of $F_1^k > 0$, in that half-neighborhood of M to which the vector g_x is directed, and for the case of $F_1^k < 0$, in the opposite half-neighborhood. The solutions, both for k even and odd, have unbounded second derivatives at the points of M.

Corollary 1. *All brackets generated by $F_0 = F$ and $F_1 = g$ of the level $0, 1, \ldots, k-1$ vanish identically on Σ. All brackets of the level k are nonzero; those involving an odd number of the functions F_1 have for k even the same sign as $\{F_0^{k-1} F_0\}$. This sign is invariant under the substitution $F \to -F$.*

Corollary 2. *The irregular simple problem has a solution of the class $K(M)$ if for some $k \geq 2$ the following conditions hold*:

$$\frac{\partial F}{\partial p_1} = 0, \ldots, \frac{\partial^{k-1} F}{\partial p_1^{k-1}} = 0, \quad \frac{\partial^k F}{\partial p_1^k} \neq 0, \quad F_{x_1} + p_1 F_u \neq 0$$

at the points $(0, x_2, 0, 1, 0) \in \Sigma$. For even k, two solutions are defined in the half-plane $x_1 \geq 0$, if the product $(-F_{x_1} - p_1 F_u) \dfrac{\partial^k F}{\partial p_1^k}$ is positive, and in the half-plane $x_1 \leq 0$, if the product is negative.

Remark 1. In the preliminary considerations, in the formulation of the sufficiency statement in Theorem 4 and in its proof we follow the scheme of [10, Theorem 1.6]. The only difference is that instead of the notion of singular characteristics (10) we use here the notion of boundary characteristics

(12), which were developed later in [11]. This gives some simplification of the theorem conditions and its proof.

Remark 2. Theorem 4 allows one to determine the type of the solution of the irregular noncharacteristic problem by only computing certain Jacobi brackets at the points of the initial surface Σ. The Jacobi bracket does not change under the transformation in the x-space. Since arbitrary initial smooth surface M can be transformed to the plane $x_1 = 0$, the conditions of Theorem 4 are invariant and do not depend on the particular coordinate system.

6. Examples

6.1. **One-dimensional illustrative example.** For the illustrative purposes we consider an example with minimal possible dimension one, which Theorem 4 embraces as well.

Consider the following Cauchy problem:

$$F = F(x,p) = p^m - x = 0, \quad x \in \mathbb{R}^1, \quad p = \frac{du}{dx}, \quad u(0) = 0,$$

where x, u, and p are scalars and the initial (zero-dimensional) surface M is the point $x = 0$. The initial strip Σ is also zero-dimensional and the parameter p on it satisfies the condition $F(0,p) = p^m = 0$. Thus, $\Sigma = \{(0,0,0)\}$, i.e., consists of one point.

The problem is irregular noncharacteristic since one has

$$\frac{\partial F}{\partial p} = 0, \quad \ldots, \quad \frac{\partial^{m-1} F}{\partial p^{m-1}} = 0, \quad \frac{\partial^m F}{\partial p^m} \neq 0, \quad F_x + pF_u = -1 \neq 0.$$

The characteristic system with corresponding initial conditions on Σ has the form:

$$\begin{aligned} &\dot{x} = F_p = mp^{m-1}, && \dot{u} = pF_p = mp^m, && \dot{p} = -F_x = 1, \\ &x(0) = 0, && u(0) = 0, && p(0) = 0. \end{aligned}$$

The unique tangent vector of Σ and the characteristic vector ξ_F on Σ have the form $\eta = (0,0,0)$ and $\xi_F = (0,0,1)$ and are not collinear, which also means that the considered problem is noncharacteristic.

The solution of the characteristic system is

$$x(t) = t^m, \quad u(t) = \frac{m}{m+1} t^{m+1}, \quad p(t) = t.$$

For odd m, the first two equations here give

$$t = x^{1/m}, \quad u(x) = \frac{m}{m+1} x^{(m+1)/m}.$$

For even m, $x(t)$ is positive and the inverse function for $x = t^m$ has two branches:

$$t = x^{1/m}, \quad t = -x^{1/m}, \quad x \geq 0.$$

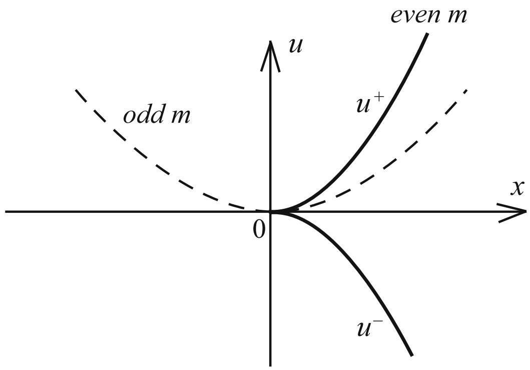

Fig. 3

Substituting into $u = u(t)$ gives two solutions:

$$u^+(x) = \frac{m}{m+1}x^{(m+1)/m}, \quad u^-(x) = -\frac{m}{m+1}x^{(m+1)/m}, \quad x \geq 0,$$

defined only for $x \geq 0$ (see Fig. 3). The function $u(x)$ in all cases possesses at $x = 0$ the first derivative; the second is infinite for $m > 1$ and finite for $m = 1$, which corresponds to the regular problem.

6.2. **A differential game with state-constraint.** Consider a boundary value problem for $u(x,t)$ with $x, t \in \mathbb{R}$:

$$F(x,t,p,q) = q + \sqrt{a^2 + p^2} - t\sqrt{b^2 + p^2} = 0, \quad (x,t) \in \Omega,$$
$$p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial t},$$
$$\Omega = \{(x,t) : t > 0,\ -x + \alpha t < 0\}, \quad g(x,t) = -x + \alpha t.$$

We will be interested in the lateral side of the boundary $g(x,t) = -x + \alpha t = 0$. The boundary condition are given only for the x-semiaxis:

$$u(x,0) = kx, \quad a, b, k = \text{const},\ b > a > 0,\ k < 0.$$

Note that we consider this problem as two-dimensional one with $x_1 = x$, $x_2 = t$, $p_1 = p$, and $p_2 = q$. Actually, the formulated problem is the HJBI equation in inverse time (according to the technique in [7]) for some differential game with an integral pay-off and terminal set $t = 0$, $x \geq 0$, subject to state constraint. We omit the discussions of these details.

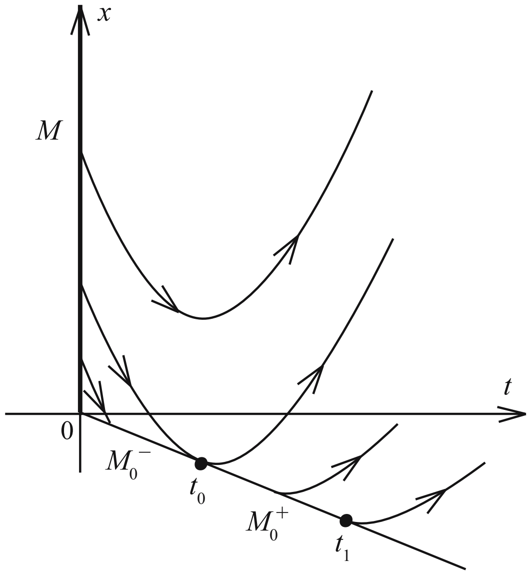

Fig. 4

One can integrate the following equations of regular characteristics with an appropriate initial conditions on that semiaxis:

$$\dot{x} = F_p = H_p = \frac{p}{\sqrt{a^2+p^2}} - \frac{tp}{\sqrt{b^2+p^2}},$$
$$\dot{u} = pF_p + qF_q = pH_p - H, \quad \dot{p} = -F_x = 0,$$
$$x(0) = s, \quad u(0) = ks, \quad p(0) = k, \quad s \in \mathbb{R}^+.$$

The equations for t and q variables are omitted here, since t serves as independent variable and from $F = q + H(t,p) = 0$ one has $q = -H(t,p)$. Here

$$H(t,p) = \sqrt{a^2+p^2} - t\sqrt{b^2+p^2}.$$

Projections of characteristics on (x,t)-plane are parabolas

$$x - C = \frac{kt}{\sqrt{a^2+k^2}} - \frac{kt^2}{2\sqrt{b^2+k^2}}, \quad C = \text{const},$$

defining the following solution in a part of Ω:

$$u(x,t) = kx + \frac{t^2}{2}\sqrt{b^2+k^2} - t\sqrt{a^2+k^2}.$$

There is the last parabola with $C = C_0$ touching the lateral side $g(x,t) = 0$ at the point $t = t_0$ (see Fig. 4):

$$t_0 = \sqrt{\frac{b^2 + k^2}{a^2 + k^2}} - \frac{\alpha\sqrt{b^2 + k^2}}{k}, \quad C_0 = \frac{t_0}{2}\left(\alpha - \frac{k}{\sqrt{a^2 + k^2}}\right),$$

after which the other part of Ω remains empty. To feel that empty part we first integrate equation of singular characteristics along the lateral boundary. For this subboundary $g(x,t) = -x + \alpha t = 0$ the tangency condition is

$$\{gF\} = -H_p + \alpha = 0.$$

Multiple brackets can be easily computed:

$$\{\{Fg\}F\} = -\frac{p}{\sqrt{b^2 + p^2}}, \quad \{\{gF\}g\} = -H_{pp}.$$

Singular characteristics along the boundary take the form

$$\dot{x} = H_p = \alpha, \quad \dot{p} = \frac{p}{H_{pp}\sqrt{b^2 + p^2}}, \quad \dot{u} = p\alpha - H.$$

Note that for regular characteristics one has $\dot{p} = 0$. Integrating the equations of singular characteristics, one get the boundary values for all variables on the free part of the lateral boundary, thus constructing an initial strip for an irregular noncharacteristic problem. Generally, this allows to integrate the regular characteristics inside Ω and fill up the whole domain. To be sure that the characteristics will fill up the expected side, one needs to check the sign of the bracket in Theorem 4 with even $k = 2$:

$$\{F^1F\} = \{\{gF\}F\} = -\{\{Fg\}F\} = \frac{p}{\sqrt{b^2 + p^2}} < 0.$$

Thus, two branches of the solution will fill the side opposite to the direction of the vector $(-1, \alpha)$ ($g_x = -1$, $g_t = \alpha$), which actually is needed for the problem. Among these two branches, we definitely choose the one which corresponds to the direct time t.

Note that singular characteristics extend on some interval $t_0 \leq t \leq t_1$. For positive $\alpha > 0$, one has $t_1 = \infty$, while for $\alpha \leq 0$ t_1 is finite, i.e., the singular characteristics end. For negative $\alpha < 0$ one can show that there remains an empty part of Ω, which means that form that part it is impossible to bring the state-vector to the terminal set $t = 0$, $x \geq 0$, without violating the state constraint. Mathematically, one can prescribe (nonuniquely) certain boundary values on the empty part of the line $g(x,t) = 0$ to obtain some solution. Thus in that sense the solution is not unique.

Note also, that for this particular low-dimensional problem one can find the value of p on the lateral boundary just by solving the tangency condition $-H_p + \alpha = 0$ without integrating the system of singular characteristics. For example, when $\alpha = 0$, one gets:

$$x = 0, \quad p = -a\sqrt{\frac{(b/a)^2 - t^2}{t^2 - 1}}, \quad 1 < \sqrt{\frac{b^2 + k^2}{a^2 + k^2}} = t_0 \le t \le t_1 = \frac{b}{a}.$$

Formally, the restriction on t comes from the requirement to have a positive expression under the root. Despite t_1 is finite, for the particular case of $\alpha = 0$ the domain is filled up, because the last regular characteristic, starting at $t = t_1$, $x = 0$, coincides with the t-axis $t \ge t_1$.

6.3. **A 2D problem with boundary characteristics.** This example illustrates the boundary characteristics in a problem with nonsmooth Hamiltonian, while the value function happens to be smooth. The problem is known as the dolichobrachistochrone problem; for the detailed formulation and the solution, see [7, 2, 3]. A modification of the problem is considered in [8].

The game is considered in the first quadrant of the (x_1, x_2)-plane with the dynamics:

$$\begin{aligned} \dot{x}_1 &= \sqrt{x_2}\cos u_1 + \frac{1}{2}w(u_2 + 1), \\ \dot{x}_2 &= \sqrt{x_2}\sin u_1 + \frac{1}{2}w(u_2 - 1), \end{aligned} \qquad |u_2| \le 1;$$

the terminal surface is $x_1 = 0$, $x_2 \ge 0$. The pay-off is the time elapsed, which is minimized by the first player and maximized by the second one.

As proved in [2], the usable part of the boundary is $x_2 > w^2$ and starts at the point

$$A : x_1 = 0, \ x_2 = w^2.$$

There are two more points on the x_2-axis,

$$\begin{aligned} &B : x_1 = 0, \ x_2 = b = \frac{2\pi^2 w^2}{(\pi + 2)^2} \approx 0.75w^2; \\ &C : x_1 = 0, \ x_2 = c = \frac{w^2}{2}, \end{aligned}$$

characterizing the solution (see Fig. 5). Through the segment AB the characteristic flow in inverse time goes outside of the first quadrant. The game can also terminate when starting near the segment BC with arbitrary small positive x_1. Motion along the segment BC is due to singular characteristics.

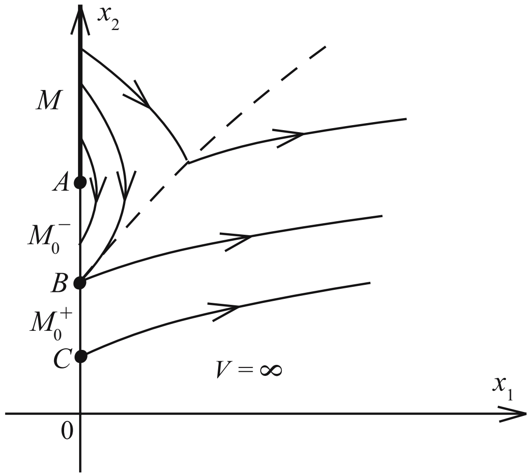

Fig. 5

The HJBI equation of the problem in terms of partials p_1 and p_2 of the game value $V(x_1, x_2)$ has the form

$$\min_{u_1} \max_{u_2} \left[p_1 \left(\sqrt{x_2} \cos u_1 + \frac{w(u_2+1)}{2} \right) + p_2 \left(\sqrt{x_2} \sin u_1 + \frac{w(u_2-1)}{2} \right) \right] + 1 = \max \left[R + \frac{w(p_1+p_2)}{2}, R - \frac{w(p_1+p_2)}{2} \right] = 0,$$

where

$$R = 1 + \frac{w(p_1 - p_2)}{2} - \sqrt{x_2}\sqrt{p_1^2 + p_2^2},$$

with optimal controls:

$$\cos u_1 = -\frac{p_1}{\sqrt{p_1^2 + p_2^2}}, \qquad \sin u_1 = -\frac{p_2}{\sqrt{p_1^2 + p_2^2}},$$

$$u_2 = \operatorname{sign}(p_1 + p_2).$$

Both smooth branches of the Hamiltonian must vanish on the segment PC

$$H_0 = R + \frac{w(p_1 + p_2)}{2} = 0,$$

$$H_1 = R - \frac{w(p_1 + p_2)}{2} = 0,$$

which is equivalent to the u_2-switching condition $p_1 + p_2 = 0$ together with $\mathbb{R} = 0$. This gives $p_1 = -p_2$. On the segment BC, one clearly has $p_2 \le 0$. Thus,

$$\cos u_1 = \frac{p_2}{\sqrt{2p_2^2}} = -\frac{1}{\sqrt{2}}, \quad \sin u_1 = -\frac{p_2}{\sqrt{2p_2^2}} = \frac{1}{\sqrt{2}}.$$

From the equality $R = 0$ one gets

$$p_2 = \frac{1}{w - \sqrt{2x_2}},$$

which is in consistency with the conjecture $p_2 \le 0$ on BC. Generally, to find the singular controls one needs to write the equations of singular characteristics, but in the considered low-dimensional problem it is sufficient to use the condition $\dot{x}_1 = 0$:

$$\dot{x}_1 = -\sqrt{x_2/2} + w(u_2 + 1) = 0, \quad u_2^\sigma = \sqrt{2x_2}/w - 1 > 1.$$

One also can verify that $u_2^\sigma \le 1$.

Note that in this problem there is a u_2-switching line (dashed line in Fig. 5) coming to the point B from inside of the first quadrant. So the segment BC can be viewed as a continuation of the switching line, where $p_1 + p_2 = 0$, with a corner point at B.

The above results coincide with the ones described in [2] based on different approach.

Singular characteristics in backward time supply the segment BC with the values p_1, p_2, and V. For this particular problem, one has actually to integrate only the scalar equation

$$\frac{\partial V(0, x_2)}{\partial x_2} = \frac{1}{w - \sqrt{2x_2}},$$

$$V(0, b) = V^* = \frac{\pi^2 w}{\pi + 2}, \quad b = \frac{2\pi^2 w^2}{(\pi + 2)^2},$$

to get

$$V = \sqrt{2b} - \sqrt{2x_2} + w \ln \frac{\sqrt{2b} - w}{\sqrt{2x_2} - w} + \frac{\pi^2 w}{\pi + 2},$$

$$c < x_2 \le b, \quad c = \frac{w^2}{2},$$

where V^* is the optimal time to go from B to the set M. Note that despite the value of $V(0, x_2)$ is specified as being zero for $x_2 \ge 0$, the solution $V(x_1, x_2)$ is nonzero and finite for $c < x_2 \le b$ as $x_1 \to +0$, and is infinite for $0 \le x_2 \le c$.

Now one can solve a boundary value problem with BC as the initial surface for the equation $H_1(x, p) = 0$. Unlike the previous example, this is a regular boundary value problem since on BC one has

$$\frac{dx_1}{d\tau} = -H_{1p_1} = \sqrt{\frac{x_2}{2}} > 0, \quad \tau = T - t.$$

On the same time, both examples demonstrate certain qualitative similarity. In both cases, the surface M is followed by a surface M_0^-, through which the characteristic flow goes outside, and then comes a surface M_0^+ with singular characteristics on it and with corresponding boundary-value problem. Such configuration also appear in many other problems (see, e.g., [9]).

7. Conclusions

Irregular boundary-value problems for nonlinear first order PDEs locally are considered in the vicinity of the following surfaces: (1) the part of the boundary surface with the given boundary value; (2) the part of the boundary surface with no value initially specified on it, while such a value arises during the constructions; (3) a singular surface arising during the constructions in the internal part of the domain.

Construction of the solution for the latter two cases can be carried out using the method of singular characteristics which allows to state that these two cases, unlike the first one, are in generic position.

It is shown that the solution singularity for an irregular problem is completely determined by the signs of two multiple Poisson (Jacobi) brackets naturally arising in the equations of singular characteristics. This allows to formulate a new sufficiency condition for the existence and uniqueness of the irregular solution. Such conditions have invariant nature and are convenient in applications.

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