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**DOI**
10.1016/j.jsv.2021.116356

**Publication date**
2021

**Document Version**
Final published version

**Published in**
Journal of Sound and Vibration

Journal of Sound and Vibration 512 (2021) 116356

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

# On resonances in transversally vibrating strings induced by an external force and a time-dependent coefficient in a Robin boundary condition

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A R T I C L E  I N F O

*Keywords:*
Robin boundary condition
Time-dependent coefficient
Interior layer analysis
Multiple-timescales perturbation method
Resonance zone

A B S T R A C T

In this paper an initial–boundary value problem on a bounded, fixed interval is considered for a one-dimensional and forced string equation subjected to a Dirichlet boundary condition at one end of the string and a Robin boundary condition with a slowly varying time-dependent coefficient at the other end of the string. This problem may serve as a simplified model describing transverse or longitudinal vibrations as well as resonances in axially moving cables for which the length changes in time. By introducing an adapted version of the method of separation of variables, by using averaging and singular perturbation techniques, and by finally using a three time-scales perturbation method, resonances in the problem are detected and accurate, analytical approximations of the solutions of the problem are constructed. It will turn out that small order $\varepsilon$ excitations can lead to order $\sqrt{\varepsilon}$ responses when the frequency of the external force satisfies certain conditions. Finally, numerical simulations are presented, which are in full agreement with the obtained analytical results.

## 1. Introduction

In the last few decades, high-rise buildings have become increasingly popular. The higher buildings rise, the more vulnerable they become to wind and earthquake influences. At the same time, it poses particular design challenges to the vertical transportation systems such as high-rise elevators. Not only can external wind and earthquake forces cause building sway, it can also damage the elevator cables. If a frequency of the external wind force coincides with one of the natural frequencies of the elevator cable, large oscillations can occur and damage can be caused. This phenomenon is called resonance. In most cases resonance finally leads to failures. In order to prevent such failures, it is important to understand the nature of these vibrations. There is a lot of research on these types of problems. Kaczmarczyk [1] analysed resonance in a catenary-vertical cable with slowly varying length under a periodic excitation. Fajans et al. [2] introduced auto-resonant (nonstationary) excitation of the diocotron mode in non-neutral plasmas. Chen and Yang [3] considered the stability in parametric resonance of axially moving viscoelastic beams with time-dependent speed. Friedland et al. [4] proposed auto-resonant phase-space holes in plasmas. Kimura et al. [5] proposed forced vibration analysis of an elevator rope with both ends excited by wind-induced displacement sway of the building. Sandilo and van Horssen [6] studied auto-resonance phenomena in a space–time-varying mechanical system. Gaiko and van Horssen [7] considered lateral vibrations of a

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https://doi.org/10.1016/j.jsv.2021.116356
Received 16 February 2021; Received in revised form 24 June 2021; Accepted 11 July 2021
Available online 11 August 2021

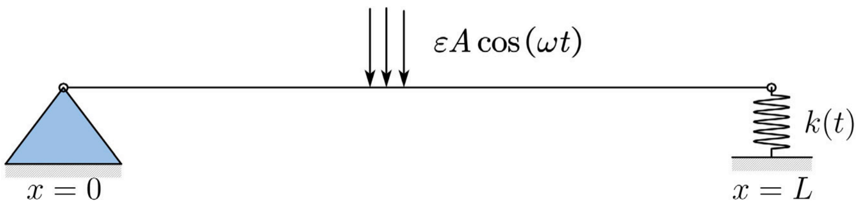


**Fig. 1.** The transverse vibrating string with a time-varying spring-stiffness support at $x = L$, and an external force $\varepsilon A\cos(\omega t)$.

vertically moving string with in time harmonically varying length, and in [8] the authors further discussed resonances and vibrations in an elevator cable system due to boundary sway. Kaczmarczyk [9] studied how simulation models and control strategies can be deployed to mitigate the effects of resonance conditions induced by wind loads and long-period seismic excitations. Zhu and Wu [10] studied the transverse vibration of the translating string with sinusoidally varying velocities.

In this paper we are motivated by resonance phenomena occurring in a transversally vibrating string (see Fig. 1), where one end of the string is fixed, and the other one is attached to a spring for which the stiffness properties change in time (due to fatigue, temperature change, and so on). Mathematically, we will show how to (approximately) solve an initial–boundary value problem for a nonhomogeneous wave equation on a bounded, fixed interval with a Dirichlet type of boundary condition at one endpoint, and a Robin type of boundary condition with a time-dependent coefficient at the other end. Actually, the Robin boundary condition is an interesting one to study from the application and mathematical point of view. The wave equations involving a Robin type of boundary condition with a time-varying coefficient can be regarded as simple models for vibrations of elevator or mining cables in the study of axially moving strings with time-varying lengths. Chen et al. [11] considered an analytical vibration response in the time domain for an axially translating and laterally vibrating string with mixed boundary conditions. Further Chen et al. [12] investigated the exchange of vibrational energy of a finite length translating tensioned string model with mixed boundary conditions applying d'Alembert's principle and the reflection properties. Wang et al. [13] designed an output feedback controller to regulate the state of a wave equation on a time-varying spatial interval with an unknown boundary disturbance. Wang et al. [14] studied a typical flexible hoisting system and proposed an absorber with artificial intelligence optimization to reduce system vibrations. For more information on initial–boundary value problems for axially moving continua, the reader is referred to [15–21]. Also in other fields, Robin boundary conditions play an important role, and are sometimes called impedance boundary conditions in electromagnetic problems or convective boundary conditions in heat transfer problems.

Usually the method of separation of variables (SOV), or the (equivalent) Laplace transform method is used to solve initial value problem for a wave equation on a bounded interval for various types of boundary conditions with constant coefficients. However, when a Robin boundary condition with a time-dependent coefficient is involved in the problem, the afore-mentioned methods are not applicable. For this reason, van Horssen and Wang in [21] employed the method of d'Alembert to solve a homogeneous wave equation involving Robin type of boundary conditions with time-dependent coefficients. In this method, the time domain can be divided into finite intervals of length 2, so that the initial conditions extension procedure for each interval coincides with the previous ones. Accordingly, one can obtain an analytical expression in a rather straightforward way for the solution on the time-interval $[0, 2n]$ with $n = 1, 2, 3, \ldots, N$, and $N$ not too large. But, one will encounter computational issues for large N. When $t$ is relatively large, the amount of calculations is very large and the calculation process is complicated. Therefore, in this paper we want to construct accurate approximations of the solutions of these problems on long timescales by using a different procedure.

This paper is organized as follows. In Section 2, the problem is formulated and a short motivation is given for the methods which are used in this paper. In Section 3, an adapted version of the method of separation of variables is introduced. This approach allows us to define a new slow variable $\tau = \varepsilon t$, and to separate $u(x,t)$ as $T(t,\tau)X(x,\tau)$. In Section 4, averaging and singular perturbation techniques are used to detect resonance zones, and to determine the scalings which are presented in the problem. By using these scalings, a three time-scales perturbation method is used in Section 5 to construct accurate, analytical approximations of the solutions of the initial–boundary value problem. In Section 6 numerical simulations are presented, which are in full agreement with the obtained, analytical approximations. As numerical method a standard finite difference method is used in this paper for simplicity, but of course also more advanced methods such as the finite element method (as has been used in [22]) can be applied. Finally, in Section 7 some concluding remarks will be made.

## 2. Formulation of the problem

By using Hamilton's principle, the governing equation of motion to describe the transversal vibration of a string as shown in Fig. 1 can be derived, and is given by:

$$\rho u_{tt}(x,t) - P u_{xx}(x,t) = \varepsilon A\cos(\omega t), \quad \omega > 0, \quad 0 < x < L, \quad t > 0, \tag{1}$$

where $\rho$ is the mass density, $P$ is the axial tension (which is assumed to be constant), $L$ is the distance between the supports, and $u$ describes the lateral displacement of the string. The term $\varepsilon A\cos(\omega t)$ in (1) is a small external force acting on the whole string, where $\varepsilon$, $\omega$ and $A$ are constants with $0 < \varepsilon \ll 1$, $\omega > 0$ and $A \in \mathbb{R}$. The boundary conditions and initial conditions are given by:

$$u(0,t) = 0, \quad P u_x(L,t) + k(t)u(L,t) = 0, \quad k(t) = 1 + \varepsilon t, \quad t \geq 0, \tag{2}$$

$$u(x,0) = \varepsilon u_0(x), \quad u_t(x,0) = \varepsilon u_1(x), \quad 0 < x < L, \tag{3}$$

where $k(t)$ is the time-varying stiffness of the spring at $x = L$. The boundary condition at $x = 0$ is a Dirichlet type of boundary condition, and the boundary condition at $x = L$ is a Robin type of boundary condition with a time-dependent coefficient $k(t)$. In this example, the choice of $k(t)$ leads to a spring which becomes stiffer and stiffer in time.

For simplicity, based on the Buckingham Pi theorem, the following dimensionless parameters are used:

$$\bar{x} = \frac{x}{L}, \quad \bar{u} = \frac{u}{L}, \quad \bar{t} = \frac{t}{L}\sqrt{\frac{P}{\rho}}, \quad \bar{k} = \frac{L}{P}k, \quad \bar{\varepsilon} = \varepsilon L\sqrt{\frac{\rho}{P}},$$
$$\bar{A} = \frac{A}{\sqrt{\rho P}}, \quad \bar{\omega} = L\omega\sqrt{\frac{\rho}{P}}, \quad \bar{u}_0 = \sqrt{\frac{P}{\rho}}\frac{u_0}{L^2}, \quad \bar{u}_1 = \frac{u_1}{L},$$

by which, the governing equation (1), the boundary conditions (2), and the initial conditions (3) can be rewritten into the following non-dimensional form:

$$u_{tt}(x,t) - u_{xx}(x,t) = \varepsilon A\cos(\omega t), \quad \omega > 0, \quad 0 < x < 1, \quad t > 0, \tag{4}$$
$$u(0,t) = 0, \quad t \geq 0,$$
$$u_x(1,t) + k(t)u(1,t) = 0, \quad k(t) = 1 + \varepsilon t, \quad t \geq 0, \tag{5}$$
$$u(x,0) = \varepsilon u_0(x), \quad 0 < x < 1, \quad u_t(x,0) = \varepsilon u_1(x), \quad 0 < x < 1, \tag{6}$$

respectively, where the overbar notations are omitted for convenience.

Due to the Robin boundary condition with the time-dependent coefficient $k(t)$, traditional, analytical methods, such as the method of separation of variables (SOV), and the (equivalent) Laplace transform method, can usually not be used. By putting $k(t) = 1+\varepsilon t$ an additional difficulty is introduced: for $t < O(\frac{1}{\varepsilon})$, $\varepsilon t$ is a small term, while for $t = O(\frac{1}{\varepsilon})$, $\varepsilon t$ is not a small term. So, we need to analyse this problem from a new view-point. Firstly, since the coefficient changes slowly in time, we study the problem in Section 3 by an adapted version of the method of separation of variables, in which an extra independent slow time variable $\tau = \varepsilon t$ is defined, and $u(x,t)$ can be separated as $T(t,\tau)X(x,\tau)$. Then, by using the boundary conditions, the original partial differential equation can be transformed into linear ordinary differential equations with slowly varying (prescribed) frequencies. Unexpectedly (or not), the slow variation leads to a singular perturbation problem. By applying an interior layer analysis in the averaging procedure in Section 4 a resonance manifold is found. Three different scalings turn out to be present in the problem. For that reason, a three-timescales perturbation method is used in Section 5 to construct explicit approximations of the solutions of the initial–boundary value problem (4)–(6).

## 3. An adapted version of the method of separation of variables

First of all, in the method of separation of variables we consider the homogeneous part of Eq. (4) subject to the homogeneous boundary conditions (5):

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, \quad 0 < x < 1, \quad t \geq 0, \\ u(0,t) = 0, \quad u_x(1,t) + k(t)u(1,t) = 0, \quad k(t) = 1 + \varepsilon t, \quad t \geq 0. \end{cases} \tag{7}$$

Note that the coefficient $k(t)$ in the Robin boundary condition is slowly varying in time. So, in order to derive a solution of problem (7), we define an extra slow time variable $\tau = \varepsilon t$, which will be treated independently from the variable $t$. Hence $u(x,t)$ becomes a new function $\bar{u}(x,t,\tau)$ and further problem (7) becomes

$$\bar{u}_{tt}(x,t,\tau) + 2\varepsilon\bar{u}_{t\tau}(x,t,\tau) + \varepsilon^2\bar{u}_{\tau\tau}(x,t,\tau) - \bar{u}_{xx}(x,t,\tau) = 0, \quad 0 < x < 1, \quad t > 0, \quad \tau > 0,$$
$$\bar{u}(0,t,\tau) = 0, \quad \bar{u}_x(1,t,\tau) + (1+\tau)\bar{u}(1,t,\tau) = 0, \quad t \geq 0, \quad \tau \geq 0. \tag{8}$$

By looking for a nontrivial solution $\bar{u}(x,t,\tau)$ in the form $T(t,\tau)X(x,\tau)$, the governing equations of (8) can be approximately written as

$$X(x,\tau)T_{tt}(t,\tau) + 2\varepsilon X(x,\tau)T_{t\tau}(t,\tau) + 2\varepsilon X_\tau(x,\tau)T_t(t,\tau) - X_{xx}(x,\tau)T(t,\tau) + O(\varepsilon^2) = 0,$$

or equivalently as

$$\frac{T_{tt}(t,\tau)}{T(t,\tau)} + O(\varepsilon) = \frac{X_{xx}(x,\tau)}{X(x,\tau)}, \quad 0 < x < 1, \quad t \geq 0, \quad \tau \geq 0. \tag{9}$$

The $O(1)$ part of the left-hand side of Eq. (9) is a function of $t$ and $\tau$, and the right-hand side is a function of $x$ and $\tau$. To be equal, both sides need to be equal to a function of $\tau$. Let this function be $-\lambda^2(\tau)$ (which will be defined later), so we get

$$\frac{T_{tt}(t,\tau)}{T(t,\tau)} = \frac{X_{xx}(x,\tau)}{X(x,\tau)} = -\lambda^2(\tau), \quad 0 < x < 1, \quad t \geq 0, \quad \tau \geq 0,$$

implying:

$$X_{xx}(x,\tau) + \lambda^2(\tau)X(x,\tau) = 0, \quad 0 < x < 1, \quad \tau \geq 0,$$

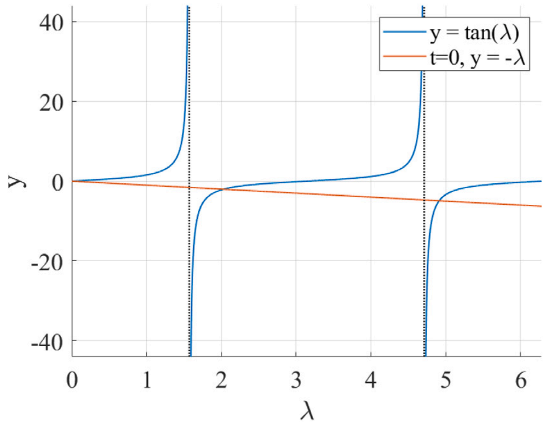


**Fig. 2.** For $t = 0$, intersection points of $y = \tan\lambda$ and $y = -\lambda$ are giving $\lambda_n(0)$.

$$T_{tt}(t,\tau) + \lambda^2(\tau)T(t,\tau) = 0, \quad t \geq 0, \quad \tau \geq 0. \tag{10}$$

From the boundary condition (8), we obtain

$$\begin{aligned} T(t,\tau)X(0,\tau) = 0 \quad &\Rightarrow \quad X(0,\tau) = 0, \\ T(t,\tau)X_x(1,\tau) + (1+\tau)T(t,\tau)X(1,\tau) = 0 \quad &\Rightarrow \quad X_x(1,\tau) + (1+\tau)X(1,\tau) = 0. \end{aligned} \tag{11}$$

In accordance with the first equation for $X(x,\tau)$ in (10), a nontrivial solution $X_n(x,\tau)$ (satisfying (11)) is

$$X_n(x,\tau) = B_n(\tau)\sin(\lambda_n(\tau)x), \tag{12}$$

where $B_n(\tau)$ is a function of $\tau$ only, and $\lambda_n(\tau)$ is the $n$th positive root of

$$\tan(\lambda_n(\tau)) = -\frac{\lambda_n(\tau)}{1+\tau}. \tag{13}$$

For $\tau = 0$ it is indicated in Fig. 2 how $\lambda_n(0)$ can be obtained. In should be observed that the eigenfunctions $X_n(x,\tau)$ are orthogonal on $0 < x < 1$. And so, the general solution of (4)–(6) can be expanded in the following form:

$$u(x,t) = \bar{u}(x,t,\tau) = \sum_{n=1}^{\infty} T_n(t,\tau)\sin(\lambda_n(\tau)x), \tag{14}$$

where the boundary conditions (5) are automatically satisfied.

Substituting Eq. (14) into Eq. (4), and into Eq. (6) yields

$$\begin{aligned} &\sum_{n=1}^{\infty} [(T_{n,tt} + 2\varepsilon T_{n,t\tau} + \lambda_n^2(\tau)T_n)\sin(\lambda_n(\tau)x) + 2\varepsilon x\frac{d\lambda_n(\tau)}{d\tau}T_{n,t}\cos(\lambda_n(\tau)x)] = \varepsilon A\cos(\omega t) + O(\varepsilon^2), \\ &\sum_{n=1}^{\infty} [T_n(0,0)\sin(\lambda_n(0)x)] = \varepsilon u_0(x), \\ &\sum_{n=1}^{\infty} [(T_{n,t}(0,0) + \varepsilon T_{n,\tau}(0,0))\sin(\lambda_n(0)x) + \varepsilon T_n(0,0)\frac{d\lambda_n(0)}{d\tau}x\cos(\lambda_n(0)x)] \\ &\qquad = \varepsilon u_1(x). \end{aligned} \tag{15}$$

Now, by multiplying the first equation in (15) with $\sin(\lambda_k(\tau)x)$, and the second and third equations in (15) with $\sin(\lambda_k(0)x)$, by integrating the so-obtained equation from $x = 0$ to $x = 1$, and by using the orthogonality properties of the sin-functions on $0 < x < 1$, we obtain the following differential equations for $k = 1,2,3,\ldots$, and $t > 0, \tau > 0$:

$$\begin{cases} T_{k,tt} + \lambda_k^2(\tau)T_k = \varepsilon[-2T_{k,t\tau} - 2\sum_{n=1}^{\infty}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)T_{n,t} + Ad_k(\tau)\cos(\omega t)] + O(\varepsilon^2), \quad t \geq 0, \quad \tau \geq 0, \\ T_k(0,0) = \varepsilon\frac{\int_0^1 u_0(\xi)\sin(\lambda_k(0)\xi)d\xi}{\int_0^1 \sin(\lambda_k(0)\xi)\sin(\lambda_k(0)\xi)d\xi} = \varepsilon F_k, \\ T_{k,t}(0,0) + \varepsilon T_{k,\tau}(0,0) = \varepsilon\frac{\int_0^1 u_1(\xi)\sin(\lambda_k(0)\xi)d\xi}{\int_0^1 \sin(\lambda_k(0)\xi)\sin(\lambda_k(0)\xi)d\xi} \\ \qquad - \varepsilon\sum_{n=1}^{\infty} T_n(0,0)\frac{d\lambda_n(0)}{d\tau}\frac{\int_0^1 \xi\cos(\lambda_n(0)\xi)\sin(\lambda_k(0)\xi)d\xi}{\int_0^1 \sin(\lambda_k(0)\xi)\sin(\lambda_k(0)\xi)d\xi} \\ \qquad = \varepsilon G_k. \end{cases} \tag{16}$$

where $c_{n,k}(\tau)$ and $d_k(\tau)$ are functions of $\tau$, and are given by:

$$\begin{aligned} c_{n,k}(\tau) &= \frac{\int_0^1 x\cos(\lambda_n(\tau)x)\sin(\lambda_k(\tau)x)dx}{\int_0^1 \sin^2(\lambda_k(\tau)x)dx}, \\ c_{k,k}(\tau) &= \frac{\int_0^1 x\cos(\lambda_k(\tau)x)\sin(\lambda_k(\tau)x)dx}{\int_0^1 \sin^2(\lambda_k(\tau)x)dx}, \\ d_k(\tau) &= \frac{\int_0^1 \sin(\lambda_k(\tau)x)dx}{\int_0^1 \sin^2(\lambda_k(\tau)x)dx} = \frac{4(1-\cos(\lambda_k(\tau)))}{2\lambda_k(\tau)-\sin(2\lambda_k(\tau))}. \end{aligned} \tag{17}$$

To simplify the formula, we define a new dependent variable: $\tilde{T}_k(t) = T_k(t,\tau)$, yielding

$$\begin{cases} \tilde{T}_{k,tt} + \lambda_k^2(\tau)\tilde{T}_k = \varepsilon[-2\Sigma_{n=1}^{\infty}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)\tilde{T}_{n,t} + Ad_k(\tau)\cos(\omega t)] + O(\varepsilon^2), \\ \tilde{T}_k(0) = \varepsilon F_k, \\ \tilde{T}_{k,t}(0) = \varepsilon G_k, \end{cases} \tag{18}$$

where $\tau = \varepsilon t$, $t \geq 0$. In the next section we will use the averaging method to detect resonance zones in problem (18), and to determine time-scales which describe the solution of (18) accurately.

## 4. Averaging and resonance zones

The linear ordinary differential equation (18) with the slowly varying frequency $\lambda_k(\tau)$ as given by (13), can be analysed by making use of the averaging method. In this section it will be shown that an interior layer analysis (including a rescaling and balancing procedure) leads to a description of an (un-)expected resonance manifold and leads to time-scales which describe the solution of (18) sufficiently accurately. To apply the method of averaging to (18) the following standard transformations are introduced:

$$\phi_k(t) = \int_0^t \lambda_k(\varepsilon s)ds \quad \text{and} \quad \Phi = \omega t, \tag{19}$$

and $\tilde{T}_k(t), \tilde{T}_{k,t}(t)$ are described by $A_k(t), B_k(t)$ in the following way:

$$\begin{aligned} \tilde{T}_k(t) &= A_k(t)\sin(\phi_k(t)) + B_k(t)\cos(\phi_k(t)), \\ \tilde{T}_{k,t}(t) &= \lambda_k(\tau)A_k(t)\cos(\phi_k(t)) - \lambda_k(\tau)B_k(t)\sin(\phi_k(t)). \end{aligned} \tag{20}$$

Problem (18) can now be rewritten in the following problem:

$$\begin{cases} \dot{A}_k = \varepsilon[-\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{\lambda_k(\tau)} + 2c_{k,k}(\tau))A_k\cos^2(\phi_k(t)) + \frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + c_{k,k}(\tau))B_k\sin(2\phi_k(t)) \\ \quad -2\Sigma_{n\neq k}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)A_n\cos(\phi_n(t))\cos(\phi_k(t)) + 2\Sigma_{n\neq k}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)B_n\sin(\phi_n(t))\cos(\phi_k(t)) \\ \quad + \frac{Ad_k(\tau)}{2\lambda_k(\tau)}(\cos(\Phi+\phi_k(t)) + \cos(\Phi-\phi_k(t)))], \\ \dot{B}_k = \varepsilon[\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + c_{k,k}(\tau))A_k\sin(2\phi_k(t)) - \frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{\lambda_k(\tau)} + 2c_{k,k}(\tau))B_k\sin^2(\phi_k(t)) \\ \quad +2\Sigma_{n\neq k}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)A_n\cos(\phi_n(t))\sin(\phi_k(t)) - 2\Sigma_{n\neq k}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)B_n\sin(\phi_n(t)t)\sin(\phi_k(t)) \\ \quad - \frac{Ad_k(\tau)}{2\lambda_k(\tau)}(\sin(\Phi+\phi_k(t)) - \sin(\Phi-\phi_k(t)))], \\ \dot{\tau} = \varepsilon, \\ \dot{\Phi} = \omega, \\ \dot{\phi}_k = \lambda_k(\tau). \end{cases} \tag{21}$$

Resonance in (21), due to the external forcing with frequency $\omega$, can be expected when $\dot{\Phi} - \dot{\phi}_k \approx 0$, or $\dot{\Phi} + \dot{\phi}_k \approx 0$. But since $\omega > 0$ and $\lambda_k(\tau) > 0$, resonance only will occur when

$$\omega \approx \lambda_k(\tau) \Leftrightarrow \tau \approx -\frac{\omega}{\tan\omega} - 1 \Leftrightarrow t \approx \frac{1}{\varepsilon}(-\frac{\omega}{\tan\omega} - 1). \tag{22}$$

Since $\lambda_k(\tau)$ satisfies (13), that is, $\tan(\lambda_k(\tau)) = -\frac{\lambda_k(\tau)}{1+\tau}$, it follows (see also Fig. 2) that when t increases, then the value of $\lambda_k(\tau)$ increases. Besides, for t tending to infinity, $\lambda_k(\tau)$ tends to $k\pi$ (with $k = 1, 2, \ldots$). Therefore, $\lambda_k(\tau)$ is increasing in time and

$$0 < \lambda_k(0) \leq \lambda_k(\tau) < k\pi, \qquad \tan(\lambda_k(0)) = -\lambda_k(0). \tag{23}$$

From (23) we can then conclude that:

1. When the external force frequency $\omega$ is bounded away from $\lambda_k(0)$ by a constant and satisfies $(k-1)\pi \leq \omega < \lambda_k(0)$, then no resonance will occur;
2. When the external force frequency $\omega$ satisfies $0 < \lambda_k(0) \leq \omega < k\pi$, then resonance will occur around $t = \frac{1}{\varepsilon}(-\frac{\omega}{\tan\omega} - 1)$. Moreover, for $-\frac{\omega}{\tan\omega} - 1 > 0$, and $-\frac{\omega}{\tan\omega} - 1 = O(\varepsilon)$, the resonance time zone is around $\tilde{t}$ with $\tilde{t} = O(1)$; and when $-\frac{\omega}{\tan\omega} - 1 = O(1)$, the resonance time zone is around $\tilde{t}$ with $\tilde{t} = O(\frac{1}{\varepsilon})$.

As long as we stay out of the resonance time zone (or equivalently, the resonance manifold), the variables $A_k$ and $B_k$ are slowly varying in time. For that reason we can average the right-hand side of the equations in (21) over $\phi_k$ and $\Phi$ while keeping $A_k$ and $B_k$ constant. The averaged equation for $A_k$ and $B_k$ now become

$$\begin{cases} \dot{A}_k^a = -\varepsilon \frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + c_{k,k}(\tau))A_k^a, \\ \dot{B}_k^a = -\varepsilon \frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + c_{k,k}(\tau))B_k^a, \end{cases} \tag{24}$$

where the upper index $a$ indicates that this is the averaged function. From the expression for $c_{k,k}$ in (17), we then obtain

$$\begin{aligned} \dot{A}_k^a &= -\varepsilon \frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + \frac{\int_0^1 x\cos(\lambda_k(\tau)x)\sin(\lambda_k(\tau)x)dx}{\int_0^1 \sin^2(\lambda_k(\tau)x)dx})A_k^a \\ &= -\frac{\varepsilon}{2}\frac{d\lambda_k(\tau)}{d\tau}(\frac{d(\ln(\lambda_k(\tau)))}{d\lambda_k(\tau)} + \frac{d(\ln(\int_0^1 \sin^2(\lambda_k(\tau)x)dx))}{d\lambda_k(\tau)})A_k^a \\ &= -\frac{\varepsilon}{2}\frac{d\lambda_k(\tau)}{d\tau}(\frac{d(\ln(\lambda_k(\tau)))}{d\lambda_k(\tau)} + \frac{d(\ln(\frac{1}{2} - \frac{\sin(2\lambda_k(\tau))}{4\lambda_k(\tau)}))}{d\lambda_k(\tau)})A_k^a \\ &= -\frac{1}{2}(\frac{d(\ln(\frac{\lambda_k(\tau)}{2} - \frac{\sin(2\lambda_k(\tau))}{4}))}{dt})A_k^a, \end{aligned} \tag{25}$$

which implies that

$$A_k^a = \frac{C_1}{\sqrt{2\lambda_k(\tau) - \sin(2\lambda_k(\tau))}}, \quad B_k^a = \frac{C_2}{\sqrt{2\lambda_k(\tau) - \sin(2\lambda_k(\tau))}}, \tag{26}$$

with

$$C_1 = \frac{\varepsilon G_k\sqrt{2\lambda_k(0) - \sin(2\lambda_k(0))}}{\lambda_k(0)}, \quad C_2 = \varepsilon F_k\sqrt{2\lambda_k(0) - \sin(2\lambda_k(0))}, \tag{27}$$

where $G_k$ and $F_k$ are given in (16).

Hence, outside the resonance manifold the solution of system (18) is given by

$$\begin{aligned} \tilde{T}_k(t) = &\frac{\varepsilon G_k\sqrt{2\lambda_k(0) - \sin(2\lambda_k(0))}}{\lambda_k(0)\sqrt{2\lambda_k(\tau) - \sin(2\lambda_k(\tau))}}\sin(\phi_k(t)) \\ &+ \frac{\varepsilon F_k\sqrt{2\lambda_k(0) - \sin(2\lambda_k(0))}}{\sqrt{2\lambda_k(\tau) - \sin(2\lambda_k(\tau))}}\cos(\phi_k(t)), \end{aligned} \tag{28}$$

where $\phi_k(t)$ is defined in (19).

When $\omega$ satisfies $\lambda_k(0) \le \omega < k\pi$ for a certain $k$ (with $k = 1, 2, \dots$) then a resonance zone will occur. We introduce

$$\psi = \Phi(t) - \phi_k(t), \tag{29}$$

and rescale $\tau - \tau_k = \delta(\varepsilon)\bar{\tau}$ with $\bar{\tau} = O(1)$ and $\tau_k = -\frac{\omega}{\tan\omega} - 1$. System (21) then becomes:

$$\begin{cases} \dot{A}_k = \varepsilon[-\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{\lambda_k(\tau)} + 2c_{k,k}(\tau))A_k\cos^2(\phi_k(t)) + \frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + c_{k,k}(\tau))B_k\sin(2\phi_k(t)) \\ \quad -2\Sigma_{n\neq k}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)A_n\cos(\phi_n(t))\cos(\phi_k(t)) + 2\Sigma_{n\neq k}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)B_n\sin(\phi_n(t))\cos(\phi_k(t)) \\ \quad + \frac{Ad_k(\tau)}{2\lambda_k(\tau)}\cos(\Phi + \phi_k(t)) + \cos(\psi)], \\ \dot{B}_k = \varepsilon[\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + c_{k,k}(\tau))A_k\sin(2\phi_k(t)) - \frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{\lambda_k(\tau)} + 2c_{k,k}(\tau))B_k\sin^2(\phi_k(t)) \\ \quad +2\Sigma_{n\neq k}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)A_n\cos(\phi_n(t)t)\sin(\phi_k(t)) - 2\Sigma_{n\neq k}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)B_n\sin(\phi_n(t))\sin(\phi_k(t)) \\ \quad - \frac{Ad_k(\tau)}{2\lambda_k(\tau)}\sin(\Phi + \phi_k(t)) - \sin(\psi)], \\ \dot{\tau} = \varepsilon, \\ \dot{\Phi} = \omega, \\ \dot{\bar{\tau}} = \frac{\varepsilon}{\delta(\varepsilon)}, \\ \dot{\psi} = \omega - \lambda_k(\tau_k + \delta(\varepsilon)\bar{\tau}). \end{cases} \tag{30}$$

To simplify (30) it should be observed that for $\tau = \tau_k + \delta(\varepsilon)\bar{\tau}$ we have

$$\begin{aligned} \dot{\psi} &= \omega - \lambda_k(\tau_k + \delta(\varepsilon)\bar{\tau}) = \lambda_k(\tau_k) - \lambda_k(\tau_k + \delta(\varepsilon)\bar{\tau}) \\ &= \lambda_k(\tau_k) - (\lambda_k(\tau_k) + \delta(\varepsilon)\bar{\tau}\frac{d\lambda_k}{d\tau}|_{\tau=\tau_k} + O(\delta^2(\varepsilon))) \\ &= -\delta(\varepsilon)\bar{\tau}\frac{d\lambda_k}{d\tau}|_{\tau=\tau_k} + O(\delta^2(\varepsilon)). \end{aligned} \tag{31}$$

By differentiating (13), that is, $\tan(\lambda_k(\tau)) = -\frac{\lambda_k(\tau)}{1+\tau}$ with respect to $\tau$, we obtain

$$\frac{1}{\cos^2\lambda_k}\frac{d\lambda_k}{d\tau} = \frac{-1}{1+\tau}\frac{d\lambda_k}{d\tau} + \frac{\lambda_k}{(1+\tau)^2}. \tag{32}$$

And so, it follows from (32) that

$$\begin{aligned}\frac{d\lambda_k}{d\tau}|_{\tau=\tau_k} &= \frac{\lambda_k}{(1+\tau)}\cdot\frac{\cos^2\lambda_k}{1+\tau+\cos^2\lambda_k}|_{\tau=\tau_k}\\ &= -\frac{\sin 2\omega}{2(1+\tau_k+\cos^2\omega)} = \frac{\sin^2\omega}{\omega-\sin\omega\cos\omega},\end{aligned} \tag{33}$$

which implies for $\dot{\psi}$ (see (31)):

$$\dot{\psi} = -\frac{\sin^2\omega}{\omega-\sin\omega\cos\omega}\delta(\varepsilon)\bar{\tau} + O(\delta^2(\varepsilon)). \tag{34}$$

From (33) and from $\lambda_k(0) \le \omega < k\pi$, we obtain $\frac{\sin^2\omega}{\omega-\sin\omega\cos\omega} \neq 0$.

Based on (34) it now follows from (34) that a balance in system (30) occurs when $\frac{\varepsilon}{\delta(\varepsilon)} = \delta(\varepsilon)$, and this implies for the averaging procedure in the resonance zone that $\delta(\varepsilon) = \sqrt{\varepsilon}$. So, together with $\tau - \tau_k = \delta(\varepsilon)\bar{\tau}$, it follows from (30) that

$$\bar{\tau} = \sqrt{\varepsilon}(t-t_k), \qquad t_k = \frac{\tau_k}{\varepsilon}. \tag{35}$$

Further, from (34), we obtain

$$\psi(t) = \psi(t_k) - \frac{1}{2}\alpha\varepsilon(t-t_k)^2, \quad \alpha = \frac{\sin^2\omega}{\omega-\sin\omega\cos\omega}. \tag{36}$$

Hence, in the resonance zone, we can write

$$\cos(\psi(t)) = \cos(-\frac{1}{2}\alpha\varepsilon(t-t_k)^2 + \omega t_k - \phi_k(t_k)), \quad t_k = -\frac{1}{\varepsilon}(\frac{\omega}{\tan\omega}+1). \tag{37}$$

So taking into account (35), let us average system (30) over the fast variables. Then, the averaged equations for $A_k$ and $B_k$ become

$$\begin{cases}\dot{A_k^a} = -\varepsilon\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + c_{k,k}(\tau))A_k^a + \varepsilon\frac{Ad_k(\tau)}{2\lambda_k(\tau)}\cos(\psi),\\ \dot{B_k^a} = -\varepsilon\frac{d\lambda_k(\tau)}{d\tau}(\frac{1}{2\lambda_k(\tau)} + c_{k,k}(\tau))B_k^a + \varepsilon\frac{Ad_k(\tau)}{2\lambda_k(\tau)}\sin(\psi),\end{cases} \tag{38}$$

where the upper index $a$ indicates that this is the averaged function.

It follows from (37) and (38) that $A_k^a$ can be written as

$$A_k^a = \frac{C_1}{l_k(\varepsilon t)} + \frac{A\varepsilon}{l_k(\varepsilon t)}\int_0^t h_k(\varepsilon\bar{t})\cos[-\frac{1}{2}\alpha\varepsilon(\bar{t}-t_k)^2 + \omega t_k - \phi_k(t_k)]d\bar{t}, \tag{39}$$

where $C_1$ is given by (27) and

$$l_k(\varepsilon t) = \sqrt{2\lambda_k(\varepsilon t) - \sin(2\lambda_k(\varepsilon t))}, \tag{40}$$

$$h_k(\varepsilon\bar{t}) = \frac{l_k(\varepsilon\bar{t})}{2\lambda_k(\varepsilon\bar{t})}d_k(\varepsilon\bar{t}) = \frac{2(1-\cos(\lambda_k(\varepsilon\bar{t})))}{\lambda_k(\varepsilon\bar{t})\sqrt{2\lambda_k(\varepsilon\bar{t}) - \sin(2\lambda_k(\varepsilon\bar{t}))}}. \tag{41}$$

For $\bar{t} = t_k + O(\frac{1}{\sqrt{\varepsilon}})$, $\tau_k = \varepsilon t_k$,

$$\begin{aligned}h_k(\varepsilon\bar{t}) &= h_k(\varepsilon t_k + O(\sqrt{\varepsilon})) = h_k(-\frac{\omega}{\tan\omega} - 1 + O(\sqrt{\varepsilon}))\\ &= h_k(-\frac{\omega}{\tan\omega} - 1) + O(\sqrt{\varepsilon})\cdot\frac{dh_k(a)}{da}|_{a=\tau_k} + \text{h.o.t},\end{aligned} \tag{42}$$

where $\frac{dh_k(a)}{da}|_{a=\tau_k}$ is bounded due to (33). Then,

$$\begin{aligned}A_k^a &= \frac{C_1}{l_k(\varepsilon t)} + \frac{\varepsilon A h_k(-\frac{\omega}{\tan\omega}-1)}{l_k(\varepsilon t)}\int_0^t \cos[-\frac{1}{2}\alpha\varepsilon(\bar{t}-t_k)^2 + \omega t_k - \phi_k(t_k)]d\bar{t}\\ &\quad + \text{h.o.t},\end{aligned} \tag{43}$$

where $C_1$ is given by (27). We know that $\frac{C_1}{l_k(\varepsilon t)} = O(\varepsilon)$ and $\frac{Ah_k(-\frac{\omega}{\tan\omega}-1)}{l_k(\varepsilon t)} = O(1)$, and so it is important to consider the order of

$$\varepsilon\int_0^t \cos[-\frac{1}{2}\alpha\varepsilon(t-t_k)^2 + \omega t_k - \phi_k(t_k)]d\bar{t}. \tag{44}$$

By setting $u = \sqrt{\frac{1}{2}\alpha\varepsilon}(t-t_k)$, we obtain

$$\varepsilon\int_0^t \cos[-\frac{1}{2}\alpha\varepsilon(t-t_k)^2 + \omega t_k - \phi_k(t_k)]d\bar{t}$$

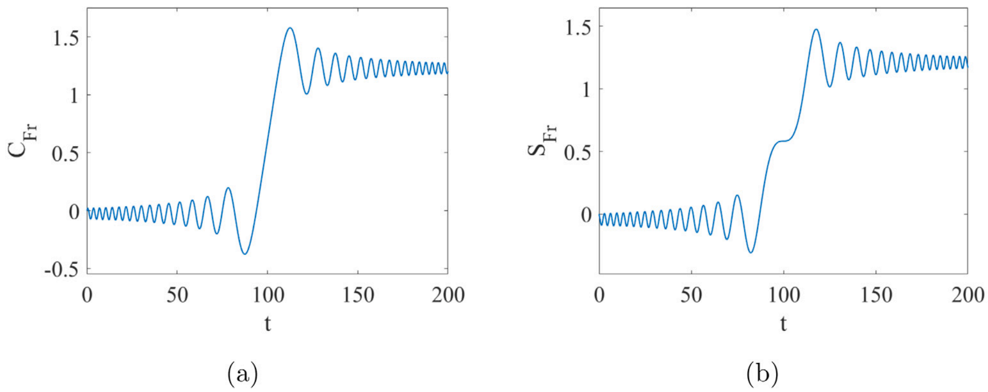


**Fig. 3.** $C_{Fr}(t)$ (a) and $S_{Fr}(t)$ (b) have a resonance jump from $O(\sqrt{\varepsilon})$ to $O(1)$ around $t = 100$.

$$= \sqrt{\frac{2\varepsilon}{\alpha}} \int_{\sqrt{\frac{\alpha\varepsilon}{2}}(-t_k)}^{\sqrt{\frac{\alpha\varepsilon}{2}}(t-t_k)} \cos(-u^2 + \omega t_k - \phi_k(t_k))du$$

$$= \sqrt{\frac{2\varepsilon}{\alpha}} \cos(\omega t_k - \phi_k(t_k))C_{Fr}(t) + \sqrt{\frac{2\varepsilon}{\alpha}} \sin(\omega t_k - \phi_k(t_k))S_{Fr}(t), \tag{45}$$

where

$$C_{Fr}(t) = \int_{\sqrt{\frac{\alpha\varepsilon}{2}}(-t_k)}^{\sqrt{\frac{\alpha\varepsilon}{2}}(t-t_k)} \cos(u^2)du,\ S_{Fr}(t) = \int_{\sqrt{\frac{\alpha\varepsilon}{2}}(-t_k)}^{\sqrt{\frac{\alpha\varepsilon}{2}}(t-t_k)} \sin(u^2)du,\ t_k = \frac{\tau_k}{\varepsilon}. \tag{46}$$

When $0 \le t_k \le O(\frac{1}{\sqrt{\varepsilon}})$,

$$\begin{cases} 0 \le C_{Fr}(t) \le O(1), & 0 \le t \le t_k + O(\frac{1}{\sqrt{\varepsilon}}); \\ C_{Fr}(t) = O(1), & t > t_k + O(\frac{1}{\sqrt{\varepsilon}}). \end{cases} \tag{47}$$

When $t_k > O(\frac{1}{\sqrt{\varepsilon}})$,

$$\begin{cases} C_{Fr}(t) = O(\sqrt{\varepsilon}), & 0 \le t < t_k - O(\frac{1}{\sqrt{\varepsilon}}); \\ O(\sqrt{\varepsilon}) \le C_{Fr}(t) \le O(1), & t_k - O(\frac{1}{\sqrt{\varepsilon}}) \le t \le t_k + O(\frac{1}{\sqrt{\varepsilon}}); \\ C_{Fr}(t) = O(1), & t > t_k + O(\frac{1}{\sqrt{\varepsilon}}). \end{cases} \tag{48}$$

In the same way, $S_{Fr}(t, \hat{\alpha})$ also satisfies (47) and (48). So, the presence of the functions $C_{Fr}(t)$ and $C_{Fr}(t)$ causes resonance jumps in the system. $C_{Fr}(t)$ and $S_{Fr}(t)$ are plotted for $\varepsilon = 0.01$, $\omega = 2.2889$, and so $t_k = 100$ in Fig. 3.

Above all, it follows from (43) that

$$A_k = \frac{2\sqrt{2\varepsilon}A(1-\cos(\lambda_k(\varepsilon t)))}{\sqrt{\alpha}\lambda_k(\varepsilon t)(2\lambda_k(\varepsilon t) - \sin(2\lambda_k(\varepsilon t)))}\Big|_{t=-\frac{\omega}{\varepsilon tan\omega}-\frac{1}{\varepsilon}}$$

$$\cdot [\cos(\omega t_k - \phi_k(t_k))C_{Fr}(t) + \sin(\omega t_k - \phi_k(t_k))S_{Fr}(t)] + O(\varepsilon), \tag{49}$$

and

$$\begin{cases} A_k = O(\varepsilon), & 0 \le t < t_k - O(\frac{1}{\sqrt{\varepsilon}}); \\ O(\varepsilon) \le A_k \le O(\sqrt{\varepsilon}), & t_k - O(\frac{1}{\sqrt{\varepsilon}}) \le t \le t_k + O(\frac{1}{\sqrt{\varepsilon}}); \\ A_k = O(\sqrt{\varepsilon}), & t > t_k + O(\frac{1}{\sqrt{\varepsilon}}). \end{cases} \tag{50}$$

Similarly, $B_k$ also satisfies (50). So, in the resonance zone,

$$\tilde{T}_k(t)$$
$$= \sqrt{\varepsilon}M_1[(\cos(\omega t_k - \phi_k(t_k))C_{Fr}(t) + \sin(\omega t_k - \phi_k(t_k))S_{Fr}(t))\sin(\phi_k(t))$$
$$+ (\sin(\omega t_k - \phi_k(t_k))C_{Fr}(t) - \cos(\omega t_k - \phi_k(t_k))S_{Fr}(t))\cos(\phi_k(t))] + O(\varepsilon)$$

$$= \sqrt{\varepsilon}M_1[(\cos(\omega t_k)C_{Fr}(t)+\sin(\omega t_k)S_{Fr}(t))\sin(\int_{t_k}^{t}\lambda_k(\varepsilon\bar{t})d\bar{t})$$
$$+ (\sin(\omega t_k)C_{Fr}(t)-\cos(\omega t_k)S_{Fr}(t))\cos(\int_{t_k}^{t}\lambda_k(\varepsilon\bar{t})d\bar{t})]+O(\varepsilon), \tag{51}$$

where $C_{Fr}(t)$ and $S_{Fr}(t)$ are given in (46), and

$$M_1 = \frac{2\sqrt{2}A(1-\cos(\lambda_k(\varepsilon t)))}{\sqrt{\alpha}\lambda_k(\varepsilon t)(2\lambda_k(\varepsilon t)-\sin(2\lambda_k(\varepsilon t)))}\Big|_{t=-\frac{\omega}{\varepsilon\tan\omega}-\frac{1}{\varepsilon}}. \tag{52}$$

Hence, when the external force frequency $\omega$ is bounded away from $\lambda_k(0)$ by a constant and satisfies $(k-1)\pi \le \omega < \lambda_k(0)$, for all $k = 1, 2, \dots$, (where $\lambda_k(0)$ satisfies (23)), then no resonance will occur, and for an $O(\varepsilon)$ external excitation there is only an $O(\varepsilon)$ response, which is described in detail in (28). When the external force frequency $\omega$ satisfies $\lambda_k(0) \le \omega < k\pi$, for a certain $k$ (with $k = 1, 2, \dots$) and $\lambda_k(0)$ satisfies (23), then a resonance will occur for $t$ near $-\frac{\omega}{\varepsilon\tan(\omega)}-\frac{1}{\varepsilon}$, and for the $O(\varepsilon)$ external excitation an $O(\sqrt{\varepsilon})$ amplitude response will occur, which is described in detail in (51).

In the next section, the occurrence of the (un)expected timescales will be used to construct accurate approximation of the solution for problem (15) and for the original problem (7) when a resonance zone exists. When a resonance zone does not exist then the solution of problem (7) will remain $O(\varepsilon)$ for $t = O(\varepsilon^{-1})$.

## 5. Three-timescales perturbation method

In the previous section, it was shown that (under certain condition on the external frequency $\omega$) resonance can occur around time $t = \frac{1}{\varepsilon}(-\frac{\omega}{\tan\omega}-1)$. For this reason, we rescale $t$ by defining $t = \tilde{t} + \frac{1}{\varepsilon}(-\frac{\omega}{\tan\omega}-1)$, and $\tau = \varepsilon\tilde{t} - \frac{\omega}{\tan\omega} - 1$. Thus, problem (18) can be rewritten in $\tilde{t}$ as follows

$$\begin{cases}\tilde{T}_{k,\tilde{t}\tilde{t}} + \lambda_k^2(\tau)\tilde{T}_k = \varepsilon[-2\Sigma_{n=1}^{n=\infty}\frac{d\lambda_n(\tau)}{d\tau}c_{n,k}(\tau)\tilde{T}_{n,\tilde{t}} \\ \qquad + Ad_k(\tau)\cos(\omega\tilde{t}+\frac{\omega}{\varepsilon}(-\frac{\omega}{\tan\omega}-1))] + O(\varepsilon^2), \\ \tilde{T}_k(\frac{1}{\varepsilon}(\frac{\omega}{\tan\omega}+1)) = \varepsilon F_k, \\ \tilde{T}_{k,\tilde{t}}(\frac{1}{\varepsilon}(\frac{\omega}{\tan\omega}+1)) = \varepsilon G_k.\end{cases} \tag{53}$$

In this section, we study problem (53) in detail under the assumption that $\omega$ is such that a resonance zone exits for the $k$th oscillation mode. The application of the straightforward expansion method to solve (53) will result in the occurrence of so-called secular terms which cause the approximations of the solutions to become unbounded on long timescales. For this reason, to remove secular terms, we introduce three timescales $t_0 = \tilde{t}$, $t_1 = \sqrt{\varepsilon}\tilde{t}$, $t_2 = \varepsilon\tilde{t}$. The time-scale $t_1 = \sqrt{\varepsilon}\tilde{t}$ is introduced because of the size of the resonance zone which has been found in the previous section, and the other two time-scales are the natural scalings for weakly nonlinear equations such as (53). By using the three timescales perturbation method, the function $\tilde{T}_k(\tilde{t};\sqrt{\varepsilon})$ is supposed to be a function of $t_0$, $t_1$ and $t_2$,

$$\tilde{T}_k(\tilde{t};\sqrt{\varepsilon}) = w_k(t_0,t_1,t_2;\sqrt{\varepsilon}). \tag{54}$$

By substituting (54) into (53), we obtain the following equations up to $O(\varepsilon\sqrt{\varepsilon})$:

$$\begin{cases}\frac{\partial^2 w_k}{\partial t_0^2} + \lambda_k^2(t_2)w_k + 2\sqrt{\varepsilon}\frac{\partial^2 w_k}{\partial t_0\partial t_1} + \varepsilon(2\frac{\partial^2 w_k}{\partial t_0\partial t_2} + \frac{\partial^2 w_k}{\partial t_1^2}) + 2\varepsilon\sqrt{\varepsilon}\frac{\partial^2 w_k}{\partial t_1\partial t_2} \\ = \varepsilon[-2\Sigma_{n=1}^{n=\infty}\frac{d\lambda_n(t_2)}{dt_2}c_{n,k}(t_2)\frac{\partial w_n}{\partial t_0} + Ad_k(t_2)\cos(\omega(t_0-a))] \\ \quad -2\varepsilon\sqrt{\varepsilon}[\Sigma_{n=1}^{n=\infty}\frac{d\lambda_n(t_2)}{dt_2}c_{n,k}(t_2)\frac{\partial w_n}{\partial t_1}], \\ w_k(a,b,c;\sqrt{\varepsilon}) = \varepsilon F_k, \\ \frac{\partial w_k}{\partial t_0}(a,b,c;\sqrt{\varepsilon}) + \sqrt{\varepsilon}\frac{\partial w_k}{\partial t_1}(a,b,c;\sqrt{\varepsilon}) + \varepsilon\frac{\partial w_k}{\partial t_2}(a,b,c;\sqrt{\varepsilon}) = \varepsilon G_k,\end{cases} \tag{55}$$

where

$$a = \frac{1}{\varepsilon}(\frac{\omega}{\tan(\omega)}+1), \quad b = \frac{1}{\sqrt{\varepsilon}}(\frac{\omega}{\tan(\omega)}+1), \quad c = \frac{\omega}{\tan(\omega)}+1. \tag{56}$$

By using a three-timescales perturbation method, $w_k(t_0,t_1,t_2;\sqrt{\varepsilon})$ will be approximated by the formal asymptotic expansion

$$w_k(t_0,t_1,t_2;\sqrt{\varepsilon}) = \sqrt{\varepsilon}w_{k,0}(t_0,t_1,t_2;\sqrt{\varepsilon}) + \varepsilon w_{k,1}(t_0,t_1,t_2;\sqrt{\varepsilon})$$
$$+\varepsilon\sqrt{\varepsilon}w_{k,2}(t_0,t_1,t_2;\sqrt{\varepsilon}) + O(\varepsilon^2). \tag{57}$$

It is reasonable to assume this solution form since the function $w_k(t_0,t_1,t_2;\sqrt{\varepsilon})$ analytically depends on $\sqrt{\varepsilon}$, and we are interested in approximations of the solution of Eqs. (4)–(6), when the initial conditions and the external excitation are of $O(\varepsilon)$. By substituting (57) into problem (55), and after equating the coefficients of like powers in $\sqrt{\varepsilon}$, we obtain as:
the $O(\sqrt{\varepsilon})$-problem:

$$\frac{\partial^2 w_{k,0}}{\partial t_0^2} + \lambda_k^2(t_2)w_{k,0} = 0, \quad w_{k,0}(a,b,c) = 0, \quad \frac{\partial w_{k,0}}{\partial t_0}(a,b,c) = 0, \tag{58}$$

the $O(\varepsilon)$-problem:

$$\frac{\partial^2 w_{k,1}}{\partial t_0^2} + \lambda_k^2(t_2) w_{k,1} = -2\frac{\partial^2 w_{k,0}}{\partial t_0 \partial t_1} + A d_k(t_2)\cos(\omega(t_0 - a)),$$
$$w_{k,1}(a,b,c) = F_k, \quad \frac{\partial w_{k,1}}{\partial t_0}(a,b,c) = -\frac{\partial w_{k,0}}{\partial t_1}(a,b,c) + G_k, \tag{59}$$

and the $O(\varepsilon\sqrt{\varepsilon})$-problem:

$$\begin{aligned}\frac{\partial^2 w_{k,2}}{\partial t_0^2} + \lambda_k^2(t_2) w_{k,2} &= -2\frac{\partial^2 w_{k,1}}{\partial t_0 \partial t_1} - 2\frac{\partial^2 w_{k,0}}{\partial t_0 \partial t_2} - \frac{\partial^2 w_{k,0}}{\partial t_1^2} \\ &\quad - 2\Sigma_{n=1}^{\infty} \frac{d\lambda_n(t_2)}{d\tau} c_{n,k}(t_2)\frac{\partial w_{n,0}}{\partial t_0}, \\ w_{k,2}(a,b,c) = 0, \quad \frac{\partial w_{k,2}}{\partial t_0}(a,b,c) &= -\frac{\partial w_{k,0}}{\partial t_2}(a,b,c) - \frac{\partial w_{k,1}}{\partial t_1}(a,b,c).\end{aligned} \tag{60}$$

The $O(\sqrt{\varepsilon})$– problem has as solution

$$w_{k,0}(t_0,t_1,t_2;\sqrt{\varepsilon}) = C_{k,1}(t_1,t_2)\sin(\lambda_k(t_2)t_0) + C_{k,2}(t_1,t_2)\cos(\lambda_k(t_2)t_0), \tag{61}$$

where $C_{k,1}$ and $C_{k,2}$ are still unknown functions of the slow variables $t_1$ and $t_2$, and they can be determined by avoiding secular terms in the solutions of the $O(\varepsilon)$– and $O(\varepsilon\sqrt{\varepsilon})$– problems. By using the initial conditions (58), it follows that $C_{k,1}(b,c) = C_{k,2}(b,c) = 0$. Now, we shall solve the $O(\varepsilon)$– problem (59).

This problem (outside as well as inside the resonance manifold) can be written as

$$\begin{aligned}\frac{\partial^2 w_{k,1}}{\partial t_0^2} + \lambda_k^2(t_2) w_{k,1} &= -2\lambda_k(t_2)[\frac{\partial C_{k,1}}{\partial t_1}\cos(\lambda_k(t_2)t_0) - \frac{\partial C_{k,2}}{\partial t_1}\sin(\lambda_k(t_2)t_0)] \\ &\quad + A d_k(t_2)\cos(\omega(t_0 - a)), \\ w_{k,1}(a,b,c) = F_k, &\qquad \frac{\partial w_{k,1}}{\partial t_0}(a,b,c) = -\frac{\partial w_{k,0}}{\partial t_1}(a,b,c) + G_k.\end{aligned} \tag{62}$$

By introducing the transformation $(w_{k,1}, w_{k,1,t_0}) \to (D_{k,1}, D_{k,2})$ with

$$\begin{aligned}&w_{k,1}(t_0,t_1,t_2;\sqrt{\varepsilon}) \\ &= D_{k,1}(t_0,t_1,t_2;\sqrt{\varepsilon})\sin(\lambda_k(t_2)t_0) + D_{k,2}(t_0,t_1,t_2;\sqrt{\varepsilon})\cos(\lambda_k(t_2)t_0), \\ &w_{k,1,t_0}(t_0,t_1,t_2;\sqrt{\varepsilon}) \\ &= \lambda_k(t_2)[D_{k,1}(t_0,t_1,t_2;\sqrt{\varepsilon})\cos(\lambda_k(t_2)t_0) - D_{k,2}(t_0,t_1,t_2;\sqrt{\varepsilon})\sin(\lambda_k(t_2)t_0)],\end{aligned}$$

it follows that the partial differential equation in (62) is equivalent with the system

$$\begin{cases}\dot{D}_{k,1} = -\frac{\partial C_{k,1}}{\partial t_1}[\cos(2\lambda_k(t_2)t_0) + 1] + \frac{\partial C_{k,2}}{\partial t_1}\sin(2\lambda_k(t_2)t_0) \\ \quad + \frac{A d_k(t_2)}{2\lambda_k(t_2)}[\cos(\omega t_0 - \omega a + \lambda_k(t_2)t_0) + \cos(\omega t_0 - \omega a - \lambda_k(t_2)t_0)], \\ \dot{D}_{k,2} = \frac{\partial C_{k,1}}{\partial t_1}\sin(2\lambda_k(t_2)t_0) - \frac{\partial C_{k,2}}{\partial t_1}[1 - \cos(2\lambda_k(t_2)t_0)] \\ \quad - \frac{A d_k(t_2)}{2\lambda_k(t_2)}[\sin(\omega t_0 - \omega a + \lambda_k(t_2)t_0) - \sin(\omega t_0 - \omega a - \lambda_k(t_2)t_0)],\end{cases} \tag{63}$$

where the overdot represents differentiation with respect to $t_0$, that is, $\dot{} = \frac{\partial}{\partial t_0}$.

Outside of the resonance zone, whether it exists or not it should be observed that the last terms in both equations in (63) do not give rise to secular terms in $D_{k,1}$ and $D_{k,2}$. To avoid secular terms, $C_{k,1}$ and $C_{k,2}$ have to satisfy the following conditions

$$\frac{\partial C_{k,1}}{\partial t_1} = 0, \quad \frac{\partial C_{k,2}}{\partial t_1} = 0. \tag{64}$$

Then, $C_{k,1}(t_1,t_2) = \bar{C}_{k,1}(t_2)$, and $C_{k,2}(t_1,t_2) = \bar{C}_{k,2}(t_2)$.

Inside the resonance zone, we observe that $\cos(\omega t_0 - \omega a - \lambda_k(t_2)t_0)$ and $\sin(\omega t_0 - \omega a - \lambda_k(t_2)t_0)$ might cause secular terms. In accordance with the resonance timescale of $O(\frac{1}{\sqrt{\varepsilon}})$, see (34), it is convenient to rewrite the arguments of the cos – and sin – function as

$$\omega t_0 - \omega a - \lambda_k(t_2)t_0 = -\frac{\alpha}{2}t_1^2 - \omega a,$$

where $\alpha$ is given by (36). Accordingly, the solution of $w_{k,1}$ in Eq. (62) has unbounded terms in $t_0$ unless

$$-2\lambda_k(t_2)\frac{\partial C_{k,1}}{\partial t_1} + A d_k(t_2)\cos[-\frac{\alpha}{2}t_1^2 - \omega a] = 0,$$

$$2\lambda_k(t_2)\frac{\partial C_{k,2}}{\partial t_1} - Ad_k(t_2)\sin[-\frac{\alpha}{2}t_1^2 - \omega a] = 0, \tag{65}$$

which implies that

$$\begin{aligned} C_{k,1}(t_1,t_2) &= \bar{C}_{k,1}(t_2) + \frac{A\cos(\omega a)d_k(t_2)}{\sqrt{2\alpha}\lambda_k(t_2)}\bar{C}_{Fr}(t_1) - \frac{A\sin(\omega a)d_k(t_2)}{\sqrt{2\alpha}\lambda_k(t_2)}\bar{S}_{Fr}(t_1), \\ C_{k,2}(t_1,t_2) &= \bar{C}_{k,2}(t_2) - \frac{A\sin(\omega a)d_k(t_2)}{\sqrt{2\alpha}\lambda_k(t_2)}\bar{C}_{Fr}(t_1) - \frac{A\cos(\omega a)d_k(t_2)}{\sqrt{2\alpha}\lambda_k(t_2)}\bar{S}_{Fr}(t_1), \end{aligned} \tag{66}$$

where

$$\bar{C}_{Fr}(t) = \int_{\sqrt{\frac{\alpha}{2}}b}^{\sqrt{\frac{\alpha}{2}}t}\cos(x^2)dx, \quad \text{and} \quad \bar{S}_{Fr}(t) = \int_{\sqrt{\frac{\alpha}{2}}b}^{\sqrt{\frac{\alpha}{2}}t}\sin(x^2)dx, \tag{67}$$

and which are the well-known Fresnel integrals. Thus, it follows from (63) that

$$\begin{aligned} D_{k,1}(t_0,t_1,t_2) &= \bar{D}_{k,1}(t_1,t_2) + \frac{Ad_k(t_2)}{2\lambda_k(t_2)(\omega+\lambda_k(t_2))}\sin(\omega t_0 - \omega a + \lambda_k(t_2)t_0) \\ &\quad + \frac{Ad_k(t_2)}{2\lambda_k(t_2)(\omega-\lambda_k(t_2))}\sin(\omega t_0 - \omega a - \lambda_k(t_2)t_0), \\ D_{k,2}(t_0,t_1,t_2) &= \bar{D}_{k,2}(t_1,t_2) + \frac{Ad_k(t_2)}{2\lambda_k(t_2)(\omega+\lambda_k(t_2))}\cos(\omega t_0 - \omega a + \lambda_k(t_2)t_0) \\ &\quad - \frac{Ad_k(t_2)}{2\lambda_k(t_2)(\omega-\lambda_k(t_2))}\cos(\omega t_0 - \omega a - \lambda_k(t_2)t_0). \end{aligned} \tag{68}$$

where $\bar{D}_{k,1}$ and $\bar{D}_{k,2}$ are still unknown functions of the slow variables $t_1$ and $t_2$. The undetermined behaviour with respect to $t_1$ and $t_2$ can be used to avoid secular terms in the $O(\varepsilon\sqrt{\varepsilon})-$ problem (60), and in the high order problems.

Taking into account the secularity conditions, the general solution of the $O(\varepsilon)$ equation is given by

$$w_{k,1}(t_0,t_1,t_2;\sqrt{\varepsilon}) = D_{k,1}(t_0,t_1,t_2)\sin(\lambda_k(t_2)t_0) + D_{k,2}(t_0,t_1,t_2)\cos(\lambda_k(t_2)t_0), \tag{69}$$

where $D_{k,1}(t_0,t_1,t_2)$ and $D_{k,1}(t_0,t_1,t_2)$ are given by (68) and

$$D_{k,2}(a,b,c) = F_k, \qquad D_{k,1}(a,b,c) = -\frac{\partial w_{k,0}}{\partial t_1}(a,b,c) + G_k. \tag{70}$$

The $O(\varepsilon\sqrt{\varepsilon})-$ problem (60) outside and inside the resonance manifold can be written as

$$\begin{aligned} \frac{\partial^2 w_{k,2}}{\partial t_0^2} + \lambda_k^2(t_2)w_{k,2} &= -2\lambda_k(t_2)[\frac{\partial D_{k,1}}{\partial t_1}\cos(\lambda_k(t_2)t_0) - \frac{\partial D_{k,2}}{\partial t_1}\sin(\lambda_k(t_2)t_0)] \\ &\quad -2\lambda_k(t_2)[\frac{\partial C_{k,1}}{\partial t_2}\cos(\lambda_k(t_2)t_0) - \frac{\partial C_{k,2}}{\partial t_2}\sin(\lambda_k(t_2)t_0)] \\ &\quad +[\frac{\partial^2 C_{k,1}}{\partial t_1^2}\sin(\lambda_k(t_2)t_0) + \frac{\partial^2 C_{k,2}}{\partial t_1^2}\cos(\lambda_k(t_2)t_0)] \\ &\quad -2\Sigma_{n=1}^{n=\infty}\frac{d\lambda_n(t_2)}{dt_2}c_{n,k}(t_2)\lambda_k(t_2)[C_{n,1}\cos(\lambda_k(t_2)t_0) \\ &\quad -C_{n,2}\sin(\lambda_k(t_2)t_0)], \\ w_{k,2}(a,b,c) = 0, &\quad \frac{\partial w_{k,2}}{\partial t_0}(a,b,c) = -\frac{\partial w_{k,0}}{\partial t_2}(a,b,c) - \frac{\partial w_{k,1}}{\partial t_1}(a,b,c). \end{aligned} \tag{71}$$

To avoid secular terms in the solution $w_{k,2}$ in Eq. (71), the following conditions have to be imposed

$$\begin{aligned} &-2\lambda_k(t_2)\frac{\partial D_{k,1}}{\partial t_1} - 2\lambda_k(t_2)\frac{\partial C_{k,1}}{\partial t_2} + \frac{\partial^2 C_{k,2}}{\partial t_1^2} - 2c_{k,k}(t_2)\lambda_k(t_2)\frac{d\lambda_k(t_2)}{dt_2}C_{k,1} = 0, \\ &2\lambda_k(t_2)\frac{\partial D_{k,2}}{\partial t_1} + 2\lambda_k(t_2)\frac{\partial C_{k,2}}{\partial t_2} + \frac{\partial^2 C_{k,1}}{\partial t_1^2} + 2c_{k,k}(t_2)\lambda_k(t_2)\frac{d\lambda_k(t_2)}{dt_2}C_{k,2} = 0. \end{aligned} \tag{72}$$

Next, we analyse this Eq. (72) inside and outside the resonance manifold.

Inside the resonance zone, substituting (66) and (68) into (72), we obtain the following secularity conditions:

$$\begin{aligned} &-2\frac{\partial \bar{D}_{k,1}}{\partial t_1} - 2\frac{d\bar{C}_{k,1}}{dt_2} - 2c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\bar{C}_{k,1} \\ &-\frac{d(\frac{A\cos(\omega a)d_k(t_2)}{\lambda_k(t_2)})}{dt_2}(\bar{C}_{Fr}(t_1) - \bar{S}_{Fr}(t_1)) + \frac{Ad_k(t_2)}{\lambda_k(t_2)}\alpha t_1\sin[-\alpha t_1^2 - \omega a] \end{aligned}$$

$$+\ c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\frac{A\cos(\omega a)d_k(t_2)}{\lambda_k(t_2)}(\bar{S}_{Fr}(t_1)-\bar{C}_{Fr}(t_1))=0, \tag{73}$$

and

$$\begin{aligned} &2\frac{\partial\bar{D}_{k,2}}{\partial t_1}+2\frac{d\bar{C}_{k,2}}{dt_2}+2c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\bar{C}_{k,2}\\ &-\frac{d(\frac{A\cos(\omega a)d_k(t_2)}{\lambda_k(t_2)})}{dt_2}(\bar{C}_{Fr}(t_1)+\bar{S}_{Fr}(t_1))-\frac{Ad_k(t_2)}{\lambda_k(t_2)}\alpha t_1\cos[-\alpha t_1^2-\omega a]\\ &-\ c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\frac{A\cos(\omega a)d_k(t_2)}{\lambda_k(t_2)}(\bar{S}_{Fr}(t_1)+\bar{C}_{Fr}(t_1))=0. \end{aligned} \tag{74}$$

Solving (73) and (74) for $\bar{D}_{k,1}$ and $\bar{D}_{k,2}$, we observe that the solution will be unbounded in $t_1$, due to terms which are only depending on $t_2$. Therefore, to have secular-free solutions for $\bar{D}_{k,1}$ and $\bar{D}_{k,2}$, the following conditions have to be imposed independently

$$\frac{\partial\bar{C}_{k,1}}{\partial t_2}+c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\bar{C}_{k,1}=0,\qquad \frac{\partial\bar{C}_{k,2}}{\partial t_2}+c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\bar{C}_{k,2}=0, \tag{75}$$

together with

$$\begin{aligned} &-2\frac{\partial\bar{D}_{k,1}}{\partial t_1}-\frac{d(\frac{A\cos(\omega a)d_k(t_2)}{\lambda_k(t_2)})}{dt_2}(\bar{C}_{Fr}(t_1)-\bar{S}_{Fr}(t_1))+\frac{Ad_k(t_2)}{\lambda_k(t_2)}\alpha t_1\sin[-\alpha t_1^2-\omega a]\\ &+\ c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\frac{A\cos(\omega a)d_k(t_2)}{\lambda_k(t_2)}(\bar{S}_{Fr}(t_1)-\bar{C}_{Fr}(t_1))=0, \end{aligned} \tag{76}$$

and

$$\begin{aligned} &2\frac{\partial\bar{D}_{k,2}}{\partial t_1}+\frac{d(\frac{A\cos(\omega a)d_k(t_2)}{\lambda_k(t_2)})}{dt_2}(\bar{C}_{Fr}(t_1)-\bar{S}_{Fr}(t_1))-\frac{Ad_k(t_2)}{\lambda_k(t_2)}\alpha t_1\cos[-\alpha t_1^2-\omega a]\\ &-\ c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\frac{A\cos(\omega a)d_k(t_2)}{\lambda_k(t_2)}(\bar{S}_{Fr}(t_1)+\bar{C}_{Fr}(t_1))=0. \end{aligned} \tag{77}$$

where $\bar{D}_{k,1}$ and $\bar{D}_{k,2}$ can be found by integration of (76) and (77), but we omit the details because of cumbersome expressions.

Next, from (75) we obtain the functions $\bar{C}_{k,1}(t_2)$ and $\bar{C}_{k,2}(t_2)$:

$$\bar{C}_{k,1}(t_2)=m_1e^{-\int_c^{t_2}c_{kk}(s)\frac{d\lambda_k(s)}{ds}ds},\quad \bar{C}_{k,2}(t_2)=m_2e^{-\int_c^{t_2}c_{kk}(s)\frac{d\lambda_k(s)}{ds}ds}, \tag{78}$$

where $m_1$ and $m_2$ are constants. Since $C_{k,1}(b,c)=0$ and $C_{k,2}(b,c)=0$, together with (66), this implies that $\bar{C}_{k,1}(t_2),\bar{C}_{k,2}(t_2)$ are both identically equal to zero.

So, in the resonance zone,

$$\begin{aligned} &w_{k,0}(t_0,t_1,t_2;\sqrt{\varepsilon})\\ =&\ [\frac{A\cos(\omega a)d_k(t_2)}{\sqrt{2\alpha}\lambda_k(t_2)}\bar{C}_{Fr}(t_1)-\frac{A\sin(\omega a)d_k(t_2)}{\sqrt{2\alpha}\lambda_k(t_2)}\bar{S}_{Fr}(t_1)]\sin(\lambda_k(t_2)t_0)\\ &-[\frac{A\cos(\omega a)d_k(t_2)}{\sqrt{2\alpha}\lambda_k(t_2)}S_{Fr}(t_1)+\frac{A\sin(\omega a)d_k(t_2)}{\sqrt{2\alpha}\lambda_k(t_2)}\bar{C}_{Fr}(t_1)]\cos(\lambda_k(t_2)t_0)\\ =&\ M_1[(\cos(\omega a)\bar{C}_{Fr}(t_1)-\sin(\omega a)\bar{S}_{Fr}(t_1))\sin(\lambda_k(t_2)t_0)\\ &-(\cos(\omega a)\bar{S}_{Fr}(t_1)+\sin(\omega a)\bar{C}_{Fr}(t_1))\cos(\lambda_k(t_2)t_0)]+h.o.t, \end{aligned} \tag{79}$$

where $M_1$ is given by (52). Outside the resonance zone, it follows from (64) and (72) that

$$\frac{\partial\bar{C}_{k,1}}{\partial t_2}+c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\bar{C}_{k,1}=0,\qquad \frac{\partial\bar{C}_{k,2}}{\partial t_2}+c_{k,k}(t_2)\frac{d\lambda_k(t_2)}{dt_2}\bar{C}_{k,2}=0,$$

which implies that

$$\bar{C}_{k,1}(t_2)=m_1e^{-\int_c^{t_2}c_{kk}(s)\frac{d\lambda_k(s)}{ds}ds}\quad \bar{C}_{k,2}(t_2)=m_2e^{-\int_c^{t_2}c_{kk}(s)\frac{d\lambda_k(s)}{ds}ds}, \tag{80}$$

where $m_1$ and $m_2$ are constants. Since $C_{k,1}(b,c)=0$, $C_{k,2}(b,c)=0$, this implies that $\bar{C}_{k,1}(t_2)$, and $\bar{C}_{k,2}(t_2)$ are identically equal to zero outside the resonance zone.

Now we can solve the $O(\varepsilon\sqrt{\varepsilon})-$ problem, where

$$w_{k,2}(t_0,t_1,t_2;\sqrt{\varepsilon})=E_{k,1}(t_0,t_1,t_2)\sin(\lambda_k(t_2)t_0)+E_{k,2}(t_0,t_1,t_2)\cos(\lambda_k(t_2)t_0), \tag{81}$$

where $E_{k,1}$ and $E_{k,2}$ are still unknown functions of the variable $t_0$, and the slow variables $t_1$, $t_2$, and they can be obtained by avoiding secular terms from higher order problems. Moreover,

$$E_{k,2}(a,b,c)\ =\ 0,$$

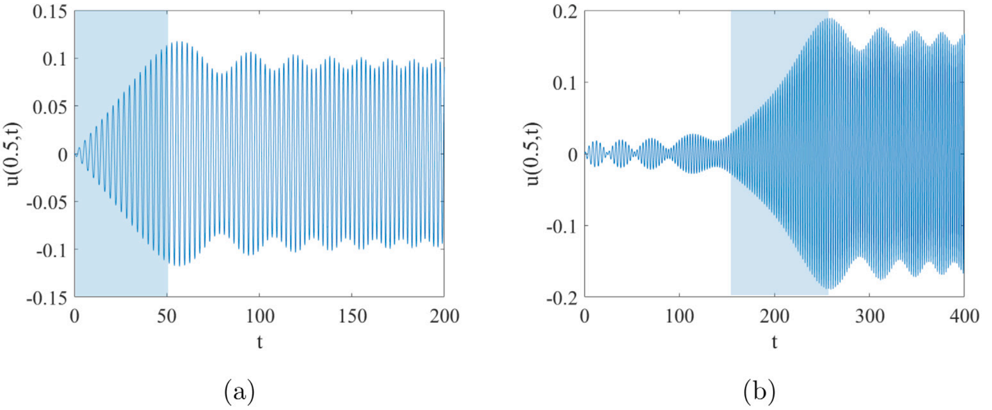


**Fig. 4.** (a) The solution $u(0.5,t)$ of the system with $\omega = 2.0917$, and the resonance time $t \approx 20$, $\lambda_1(\varepsilon \cdot 20) = \omega$. (b) The solution $u(0.5,t)$ of the system with $\omega = 2.4556$, and the resonance time $t \approx 200$, $\lambda_1(\varepsilon \cdot 200) = \omega$.

$$E_{k,1}(a,b,c) = -\frac{\partial w_{k,0}}{\partial t_2}(a,b,c) - \frac{\partial w_{k,1}}{\partial t_1}(a,b,c). \tag{82}$$

Note that in Eqs. (69) and (81), $D_{k,1}, D_{k,2}, E_{k,1}, E_{k,2}$ are yet also undetermined functions. All these unknown functions can be determined from the $O(\varepsilon^2)$– problem and $O(\varepsilon^2\sqrt{\varepsilon})$– problem. At this moment, only the first term in the expansion of the solution for the string problem is important from the physical point of view. We are not interested in high-order approximations; that is why we take

$$\begin{aligned} \bar{D}_{k,1}(t_1,t_2) &= \bar{D}_{k,1}(b,c), \quad \bar{D}_{k,2}(t_1,t_2) = \bar{D}_{k,2}(b,c), \\ E_{k,1}(t_0,t_1,t_2) &= E_{k,1}(a,b,c), \quad E_{k,2}(t_0,t_1,t_2) = E_{k,2}(a,b,c). \end{aligned} \tag{83}$$

Thus, an approximation of the solution of Eqs. (4)–(6) is given by

$$\begin{aligned} &u(x,t) \\ &= \sum_{n=1}^{\infty}[\sqrt{\varepsilon} w_{n,0}(t_0,t_1,t_2;\sqrt{\varepsilon}) + \varepsilon w_{n,1}(t_0,t_1,t_2;\sqrt{\varepsilon}) \\ &+ \varepsilon\sqrt{\varepsilon} w_{n,2}(t_0,t_1,t_2;\sqrt{\varepsilon})]\sin(\lambda_n(\tau)x) + O(\varepsilon^2), \end{aligned} \tag{84}$$

where $w_{k,0}$, $w_{k,1}$ and $w_{k,2}$ are given by (61), (69) and (81).

- When the external force frequency $\omega$ is bounded away from $\lambda_k(0)$ by a constant and satisfies $(k-1)\pi \le \omega < \lambda_k(0)$, with $\lambda_k(0)$ given by (23), for all $k = 1,2,\ldots$, it follows from (64) and (80) that $w_{k,0}(t_0,t_1,t_2;\sqrt{\varepsilon}) = 0$. Further from (54) and (57), we obtain $\tilde{T}_k(t) = O(\varepsilon)$, i.e., for the $O(\varepsilon)$ external excitation, there is an $O(\varepsilon)$ response. This case can be referred to as the non-resonant case.
- When the external force frequency $\omega$ satisfies $\lambda_k(0) \le \omega < k\pi$, with $\lambda_k(0)$ given by (23), for a fixed $k = 1,2,\ldots$, it follows from (79) that $w_{n,0}(t_0,t_1,t_2;\sqrt{\varepsilon})_{n\neq k} = 0$ and

$$\begin{aligned} &w_{k,0}(t_0,t_1,t_2;\sqrt{\varepsilon}) \\ &= M_1[(\cos(\omega a)\bar{C}_{Fr}(t_1) - \sin(\omega a)\bar{S}_{Fr}(t_1))\sin(\lambda_k(t_2)t_0) \\ &- (\cos(\omega a)\bar{S}_{Fr}(t_1) + \sin(\omega a)\bar{C}_{Fr}(t_1))\cos(\lambda_k(t_2)t_0)], \end{aligned} \tag{85}$$

  where $M_1$ satisfies (52). For the solution $u(x,t)$ of the original problem (69) this implies a resonance jump from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$ around $t = \frac{1}{\varepsilon}(-\frac{\omega}{\tan\omega} - 1)$ in the $k$th oscillation mode.

It should be observed that $\tilde{T}_k(t)$ as calculated by using a three-timescales perturbation method agrees well with the approximation as calculated by using the averaging method. Further, by using the first term $w_{k,0}(t_0,t_1,t_2;\sqrt{\varepsilon})$ as $u(x,t) = [\sqrt{\varepsilon} w_{k,0} + O(\varepsilon)]\sin(\lambda_k(\tau)x)$ with $\lambda_k(0) \le \omega < k\pi$ and $\lambda_k(0)$ given by (23), $u(0.5,t)$ is plotted for $\varepsilon = 0.01$, $A = 1$, $\omega = 2.0917$, and so $t_k = 20$ in Fig. 4(a) and $u(0.5,t)$ is plotted for $\varepsilon = 0.01$, $A = 1$, $\omega = 2.4556$, and so $t_k = 200$ in Fig. 4(b).

## 6. Numerical results

In this section the finite difference method is used to present numerical approximations of the vibration response and energy of the string. The computations are performed by using the parameters $\varepsilon = 0.01$, $A = 1$. Let us assume that the initial displacement is prescribed by $f = \sin(1.7155x)$, and the initial velocity $g = 0$ for $0 \le x \le 1$. There will be different behaviour in the amplitude

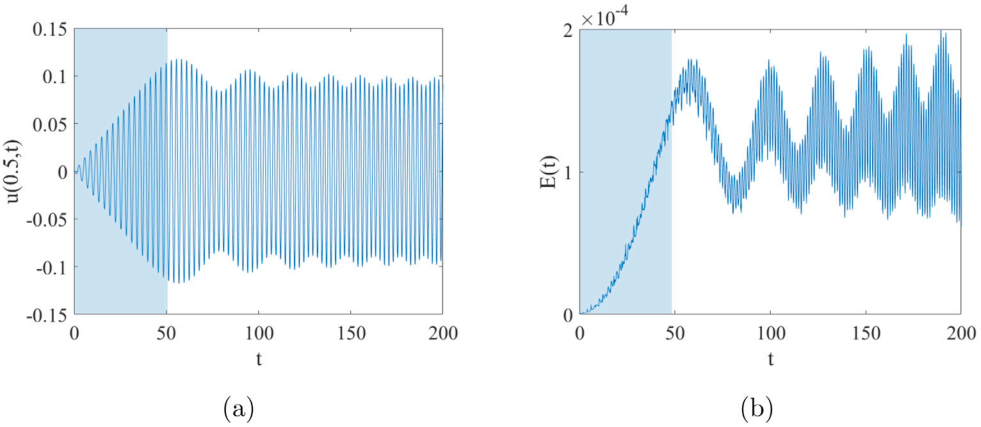


**Fig. 5.** The solution $u(0.5, t)$ (a) and the energy $E(t)$ (b) of the system with $\omega = 2.0917$, and the resonance time $t \approx 20$, $\lambda_1(\varepsilon \cdot 20) = \omega$.

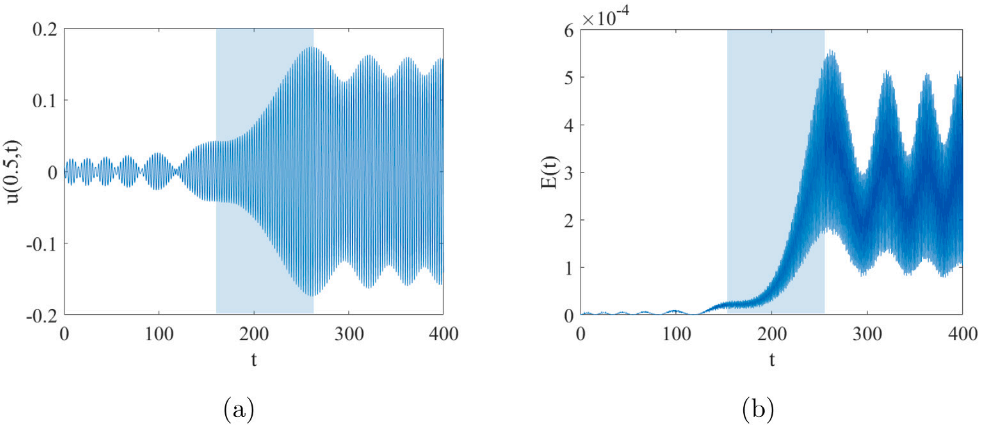


**Fig. 6.** The solution $u(0.5, t)$ (a) and the energy $E(t)$ (b) of the system with $\omega = 2.4556$, and the resonance time $t \approx 200$, $\lambda_1(\varepsilon \cdot 200) = \omega$.

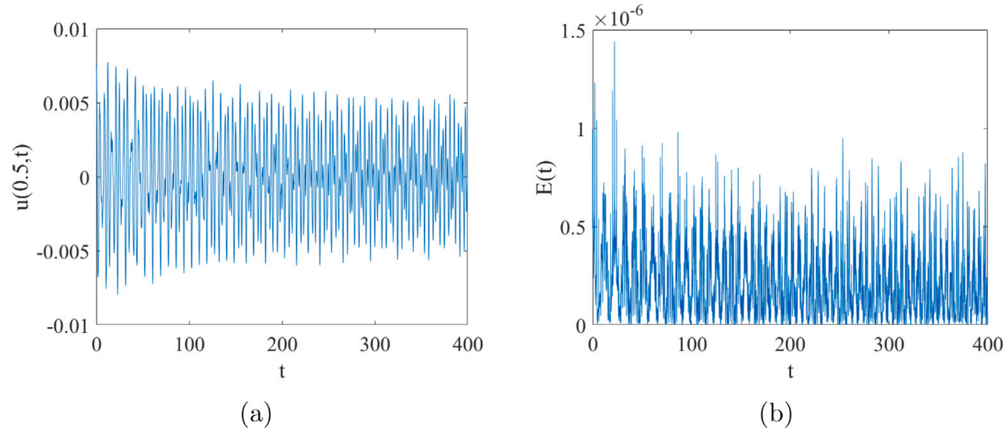


**Fig. 7.** The solution $u(0.5, t)$ (a) and the energy $E(t)$ (b) of the system with $\omega = 1.5$, there is no resonance.

response of the solution for different choices of the parameter $\omega$. Note that the following numerical results are computed based on $O(\varepsilon)$ approximations of the equations. Higher order terms in the equations are neglected due to their unimportant contribution in the solution. By using (22), and according to our analytical results, resonance occurs around times

$$t = -\frac{\omega}{\varepsilon \cdot \tan \omega} - \frac{1}{\varepsilon}. \tag{86}$$

When the external force frequency $\omega$ is bounded away from $\lambda_k(0)$ by a constant and satisfies $(k-1)\pi \le \omega < \lambda_k(0)$, with $\lambda_k(0)$ given by (23), for all $k = 1, 2, \ldots$ Fig. 5 shows the displacement at $x = 0.5$ and the vibratory energy of the system, respectively, for times up to$t = 200$ for $\omega = 2.0917$. We observe that for $t \approx 20$ the response amplitudes of the vibration become of order $\sqrt{\varepsilon}$ from order $\varepsilon$ at $t = 0$. Similarly, Fig. 6 shows the displacement and vibratory energy of the system for times up to $t = 400$ for $\omega = 2.4556$. Again we observe that the response amplitudes of the vibration become of order $\sqrt{\varepsilon}$ but now for $t \approx 200$. When the external force frequency $\omega$ is bounded away from $\lambda_k(0)$ by a constant and satisfies $(k-1)\pi \le \omega < \lambda_k(0)$, with $\lambda_k(0)$ given by (23), for all $k = 1, 2, \ldots$, Fig. 7 shows the displacement and vibratory energy of the system for times up to$t = 400$ for $\omega = 1.58$, but now there is no resonance and the response amplitudes of the vibration are still of order $\varepsilon$. These numerical results are in agreement with our results as presented in Section 4 and in Section 5. Moreover, in these figures, the shadowed bands represent the resonance zones, which have the size of $(O(\frac{1}{\sqrt{\varepsilon}}))$ as was also obtained analytically. Therefore, from Figs. 5–7, we can conclude that the general dynamic behaviour of the solution as approximated numerically is in complete agreement with the analytic approximations as obtained in Section 4 and in Section 5.

## 7. Conclusions

In this paper resonances in a transversally vibrating string are studied. A small, externally applied and harmonic force with frequency $\omega$ is acting on the whole string. The string is fixed at one end, and at the other end a spring is attached for which the stiffness slowly varies in time. In this paper one choice is made for the slowly varying stiffness, but also other choices can be made and a similar analysis as presented in this paper can be given. By assuming that the small external force is of order $\varepsilon$ and by assuming that the initial values are also small and of order $\varepsilon$, it is shown in this paper that resonances can occur for certain values of $\omega$, and that the amplitude response (in case of resonance) is of order $\sqrt{\varepsilon}$. To obtain these results an adapted version of the method of separation of variables is introduced, and perturbation methods, (such as averaging methods, singular perturbation techniques, and multiple timescales perturbation methods) are used. Not only the interval for $\omega$ for which resonance occurs is determined, but also the times and time-intervals are found for which this resonance occurs. Furthermore, explicit, and accurate approximations of the solution of the initial–boundary value problem are constructed. All approximations are valid on time-scales of order $\varepsilon^{-1}$. Also a finite difference method is applied to construct numerical approximations of the solution of the initial–boundary value problem. These numerical approximations are in full agreement with the analytically obtained approximations.

## CRediT authorship contribution statement

**Jing Wang:** Methodology, Implementation of the computer code and supporting algorithms, Writing – original draft. **Wim T. van Horssen:** Ideas, Formulation of overarching research goals and aims, Writing – review & editing. **Jun-Min Wang:** Writing – review & editing.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Acknowledgements

This research is supported by the National Natural Science Foundation of China under Grant 62073037. The first author Jing Wang would like to thank Delft Instituted of Applied Mathematics (DIAM), Delft University of Technology, for providing such an excellent research environment, and would like to thank the financial support from the China Scholarship Council (CSC).

## Appendix. Numerical approximation

Firstly, we introduce a uniform mesh $\Delta x$, a constant discretization time $\Delta t$, and a rectangular mesh consisting of points $(x_i, t_j)$ with $x_i = i\Delta x$ and $t_j = j\Delta t$, where $i = 1, 2, 3, \ldots, N$, $j = 1, 2, \ldots$, with $N\Delta x = 1$. Following the finite difference method and by using the Taylor series expansion, the second order space and time derivatives can be discretized by

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{(\Delta x)^2} + O((\Delta x)^2), \tag{87}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})}{(\Delta x)^2} + O((\Delta t)^2). \tag{88}$$

Substituting the finite difference formulae into Eqs. (4)–(6), and rearranging the terms, we end up with the linear iterative system

$$u_{i,j+1} = \sigma^2 u_{i+1,j} + 2(1-\sigma^2)u_{i,j} + \sigma^2 u_{i-1,j} - u_{i,j-1} + \varepsilon A\cos(\omega t_j), \tag{89}$$

where $\sigma = \frac{\Delta t}{\Delta x}$. From the boundary condition (5) it follows that

$$u_{0,j} = 0, \quad u_{n,j} = \frac{u_{n-1,j}}{1 + k(t_j)\Delta x}. \tag{90}$$

Let us introduce the following vector: $U^{(j)} = [u_{1,j}, u_{2,j}, \dots, u_{n-2,j}, u_{n-1,j}]^T$, $S^{(j)} = [\underbrace{\bar{s}_j, \bar{s}_j, \dots, \bar{s}_j}_{n-1 \text{ times}}]^T$, $\bar{s}_j = \varepsilon A \cos(\omega t_j)$,

$$B = \begin{pmatrix} 2(1-\sigma^2) & \sigma^2 & 0 & \cdots & \cdots & 0 \\ \sigma^2 & 2(1-\sigma^2) & \sigma^2 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \sigma^2 & 2(1-\sigma^2) & \sigma^2 \\ 0 & \cdots & \cdots & 0 & \sigma^2 & 2(1-\sigma^2) + \frac{\sigma^2}{1+k(t_j)\Delta x} \end{pmatrix},$$

then the iteration process can be rewritten in the following matrix form

$$U^{(j+1)} = BU^{(j)} - U^{(j-1)} + S^{(j)}, \tag{91}$$

where the initial conditions imply:

$$u_{i,0} = f_i = f(x_i), \quad u_{i,1} = \frac{1}{2}\sigma^2 f_{i+1} + (1-\sigma^2) f_i + \frac{1}{2}\sigma^2 f_{i-1} + \Delta t g_i. \tag{92}$$

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