Initial Data for the Cauchy Problem on a Lightcone via the Raychaudhuri Equation

Bachelor Thesis

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Abstract

The Cauchy problem on a lightcone differs from the one on a smooth spacelike surface in several ways; the initial surface cannot be smooth everywhere, and the initial surface being characteristic also has implications for the values the initial data may take. We show well-posedness of the vacuum Cauchy problem for a smooth spacelike initial surface in arbitrary dimension. The initial data for this problem is related to initial data for the characteristic Cauchy problem. We derive the Raychaudhuri equation for lightlike geodesics and use it to analyze initial data for the Cauchy problem on a lightcone. We show how to derive the extrinsic curvature of a lightcone using a field of connection coefficients and the metric. We discuss the Raychaudhuri equation as a constraint equation if one uses the metric as initial data. Alternatively, we show how to use the Raychaudhuri equation to complete initial data when a conformal class of metrics is used as initial data instead. In this case, we give an exact expression for the conformal factor associated with a representative of the conformal class.

Acknowledgments This thesis is written to obtain the title of Bachelor of applied physics and applied mathematics at the Delft university of technology. In the process of writing this thesis, I received supervision by Bas Janssens and Yaroslav Blanter. I want to take this space to express my gratitude for their direction and help in understanding the concepts and mathematics necessary to write this thesis.

-Luna Laven

Contents

Space tells matter how to move
Matter tells space how to curve

John Archibald Wheeler

1 Introduction

Einstein's theory of general relativity describes gravity as curvature in a unified understanding of space and time, called spacetime. The Einstein field equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

form the central system of partial differential equations for the curvature. The Einstein tensor $G_{\mu\nu}$ on the left hand side of this equation describes the curvature of spacetime. It can be expressed in terms of the metric tensor $g_{\mu\nu}$ which defines the inner product between vectors on each tangent space. The metric $g_{\mu\nu}$ is Lorentzian, which means that for any point in the spacetime, one can find a basis for the tangent vector space, $\{\partial_0, \dots, \partial_n\}$, such that

$$g(\partial_0, \partial_0) = -1, \quad \forall a \neq 0 : g(\partial_a, \partial_a) = 1, \quad \forall a \neq b : g(\partial_a, \partial_b) = 0.$$

On the right hand side of the Einstein field equations, the stress-energy-momentum tensor $T_{\mu\nu}$ captures the distribution and momentum of mass and energy in the spacetime. It serves as a generalization of the mass distribution one may find in Newtonian dynamics. The 'G' on the right hand side is Newton's gravitational constant. The Einstein tensor is expressed in terms of the metric, and as such, a solution of the field equations is a metric tensor field for which the equations hold. Note, however, that a solution of the Einstein field equations, $g_{\mu\nu}$, expressed in terms of coordinates on the spacetime is highly nonunique. This is because coordinates may be chosen freely on the manifold. That is, the Einstein field equations are invariant under coordinate transformations. In the literature, a certain choice of coordinates is called a gauge. This so-called gauge freedom is a physically significant property of the field equations. It may be interpreted as stating that, whereas curvature of the spacetime is innate, the names we give to the points in spacetime are a human invention.

Oftentimes, one wants to find the effect of gravity on a body moving through a part of spacetime where no matter is present. Two such examples are, the movement of the earth around the sun, and gravitational waves in the vacuum of space. In this case, the stress-energy-momentum tensor is identically zero in the considered region of spacetime. One then solves the vacuum Einstein field equations: $G_{\mu\nu} = 0$.

In an $n + 1$-dimensional spacetime, the (vacuum) Einstein field equations form a system of $(n + 1)(n + 2)/2$ independent coupled second order nonlinear partial differential equations[1] which makes finding exact solutions an infamously difficult endeavor. Only a handful of exact solutions is known.

[1]There are a total of $(n + 1)^2$ coupled second order nonlinear partial differential equations. However, since the Einstein tensor is symmetric, this reduces the number of independent equations to $(n + 1)(n + 2)/2$.

To solve problems in Newtonian dynamics, one finds the forces $\Sigma\vec{F}$ acting upon a body and solves the equation of motion $\Sigma\vec{F} = m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}$ where m is the mass of the body, $\vec{r}$ is its position, and t is time. That is, problems in Newtonian dynamics are expressed as (systems of) second order differential equations. To solve these equations, one needs to know the location and its first derivative of the studied bodies at a certain moment in time. Problems in Newtonian dynamics are expressed as well-posed initial value problems. By well-posedness, we refer to an existence-and-uniqueness result. Since Einsteinian and Newtonian predictions agree in the limit of weak gravity, one might expect the Einstein field equations to also admit a well-posed initial value formulation. However, if one is familiar with special relativity, one might know that the concept of time is not as clearly defined in general relativity as it is in Newtonian dynamics, two non accelerating observers in general relativity generally do not agree on the speed with which another observer travels through time.[2] To make matters worse, in arbitrary spacetimes they may not even agree on the direction of time.

These issues can be resolved by requiring that the studied spacetime adheres to a number of physically reasonable causality conditions. These conditions serve to ensure that cause and effect are well defined. One might, for example, require that causality between two events occurs only in one direction; if event 1 causes event 2, event 2 may not cause event 1. One might also require that two observers on a studied spacetime agree on the future timelike directions at any point. It can be shown that, if the studied spacetime satisfies a number of these physically reasonable causality conditions, that the vacuum Einstein field equations can be rewritten as a well-posed initial value problem with smooth initial data on a smooth spacelike hypersurface (Fourès-Bruhat, 1952). Such an initial value problem, where the initial data is prescribed on a (hyper)surface is called a Cauchy problem. The spacetime resulting from the initial data is called a Cauchy development, and the initial surface on which the data is prescribed is called a Cauchy surface. By well-posedness, we mean that such a physical system always admits a unique Cauchy development. This result is shown through the implementation of harmonic coordinates (i.e., coordinates (x^λ) such that $H^\lambda \equiv \nabla_\alpha\nabla^\alpha x^\lambda = 0$). Moreover, it can also be shown that this Cauchy problem admits a unique maximal Cauchy development (Choquet-Bruhat & Geroch, 1969). Ever since these results are known, physicists and mathematicians have tried to show well-posedness for the Cauchy problem where the initial data is prescribed on a characteristic Cauchy surface, instead of a spacelike one. A characteristic Cauchy surface is a surface whose normal vectors are tangent to it. This property means that the surface admits lightlike curves that are entirely contained in the hypersurface. Because of this, nonsmooth points must be introduced on the hypersurface if we want to have 'enough' initial data for our problem. Well-posedness of the characteristic Cauchy problem has not yet been proven, though significant progress has been made.

There are two general forms the characteristic Cauchy problem may take. Initial data may be prescribed on two transversally intersecting null hypersurfaces, or on a lightcone. Noteworthy results for the characteristic Cauchy problem on a lightcone are (Rendall, 1990) in which it is shown that a solution for the characteristic Cauchy problem in $3+1$ dimensions exists near the vertex of a lightcone. This

[2]Newtonian dynamics only really makes sense to use for inertial (i.e., non-accelerating) frames since Newton's laws only apply in inertial frames.

result has been generalized for arbitrary dimensions in (Choquet-Bruhat et al., 2011b). Existence of a solution in the neighborhood of the entire lightcone for $3+1$ dimensions was shown in (Luk, 2012). Luk's result for $3+1$ dimensions was later generalized to arbitrary dimensions in (Rodnianski & Shlapentokh-Rothman, 2018).

In this thesis, we introduce the causal structure of spacetimes. This is used to introduce the well-posed Cauchy problem on a spacelike hypersurface. We relate initial data for this problem to initial data for the characteristic Cauchy problem. We introduce conformal metrics. Which are metrics that agree on the causal structure of the studied spacetime. We derive the Raychaudhuri equation for lightlike geodesics in arbitrary dimension which describes the evolution of a continuous family of lightlike geodesics. We use the Raychaudhuri equation to analyze initial data on a lightcone. The initial data of the characteristic Cauchy problem in $n+1$ dimensions is studied in the following ways:

1. The free initial data takes the form of the triple $(\mathscr{N}, \overline{g}, \kappa)$ where $\mathscr{N}$ is a lightcone, $\overline{g}$ is a symmetric tensor of signature $(0, +, \ldots, +)$, and κ is a field of connection coefficients on the tangent vectors to the lightcone $\mathscr{N}$. The initial data $(\overline{g}, \kappa)$ are further required to satisfy the *Raychaudhuri equation*

$$\partial_r \theta - \kappa\theta = -\sigma^2 - \frac{\theta^2}{n-1}$$

where θ is the divergence and σ is the shear of the family of null curves that generate the lightcone $\mathscr{N}$.

2. The free initial data takes the form of the triple $(\mathscr{N}, [\gamma], \kappa)$ where $\mathscr{N}$ is an n-dimensional manifold. $[\gamma]$ is a conformal class of metrics of signature $(0, +, \ldots, +)$, and κ is a field of connections for the null vectors with respect to γ. Here the Raychaudhuri equation is used to find the conformal factor at different points.

The aim of this work is to analyze the occurrence of constraint equations for the initial data in the characteristic Cauchy problem on a lightcone. These equations restrict the values that initial data may take on a lightcone such that they are physically reasonable. Constraint equations are of importance because a solution of the vacuum field equations is uniquely associated with its initial data on a Cauchy surface. This means that any physically reasonable initial dataset carries the same information as its associated solution of the vacuum field equations.

In the second section, we introduce the causal structure and the Cauchy problem. Conformal metrics and the Penrose conformal compactification of Minkowski spacetime is introduced in the third section. In the forth section, we derive the Raychaudhuri equation for lightlike geodesics, we introduce conjugate points, and we give a lemma on conjugate points which is used in the following section. In section five, we study the different ways to apply the Raychaudhuri equation for initial data on a lightcone. Section six serves as an overview of this thesis in which we discuss the results.

2 Causal structure and the Cauchy problem

In this thesis, the characteristic Cauchy problem is studied. This section serves to give an understanding and introduction of the characteristic Cauchy problem. First, we introduce the causal structure on a spacetime manifold. This serves as preliminary knowledge for the introduction of the Cauchy problem. Second, we use the newly acquired language to introduce the Cauchy problem for a smooth spacelike Cauchy surface. Third, we discuss differences between the spacelike and characteristic Cauchy problem.

2.1 Causal structure

In order to introduce the Cauchy problem we first need to familiarize ourselves with the causal structure of spacetimes. Simply said, the causal structure is a set of conditions which physical spacetimes must adhere to in order to ensure that cause and effect are well defined. For this subsection we consider a spacetime $(\mathscr{M}, g)$ which does not necessarily satisfy the Einstein field equations.

A tangent vector $v \in T_p\mathscr{M}$ is defined to be *timelike* (*lightlike*) {*spacelike*} if $g(v, v)$ is smaller than (equal to) {larger than} zero.
A curve $\gamma : I \subseteq \mathbb{R} \to \mathscr{M}, s \mapsto \gamma(s)$ is said to be *timelike* (*lightlike*) {*spacelike*} if at every $p \in \gamma \subset \mathscr{M}$ the tangent vector $\dot{\gamma}$ is timelike (lightlike) {spacelike}. Sometimes *null* is used as a synonym for lightlike.

2.1.1 A Minkowskian lightcone

Let us consider the $n+1$ dimensional Minkowski spacetime $(\mathbb{M}^{n+1}, \eta)$. We may use the coordinates (x^μ) on the entire spacetime with the associated tangent vectors $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ such that the basis tangent vectors are orthogonal and

$$g(\partial_0, \partial_0) = -1, \quad \forall a \neq 0 : g(\partial_a, \partial_a) = 1.$$

That is to say, the ∂_0 direction is the only timelike direction which lets us consider x^0 as the time coordinate. From special relativity, one may remember the spacetime interval

$$\Delta s^2 = -(\Delta x^0)^2 + \sum_{i=0}^{n} (\Delta x^i)^2$$

which is independent of inertial frame. When one changes from special relativity to general relativity, this spacetime interval is instead expressed infinitesimally as the metric. We have the following expression for the Minkowskian metric η.

$$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + \sum_{i=1}^{n} (\mathrm{d}x^i)^2 \tag{2.1}$$

Note that in special relativity, one often writes $x^0 = ct$ or simply $x^0 = t$. The absence of the speed of light c in (2.1) shows that we use natural units $c \equiv 1$. A *lightcone through the origin in Minkowski spacetime* is the set of points that lightrays, emitted at the origin of our Minkowskian spacetime reaches. Since there is no curvature in Minkowski spacetime, the lightcone L, as a set of points, can be expressed in the (x^μ) coordinates by

$$L = \{(x^0, \dots, x^n) \mid x^0 = \sqrt{\sum_{i=1}^{n} (x^i)^2}\}. \tag{2.2}$$

This is indeed an expression for a cone through the origin. We remark that the choice of origin in Minkowski space is arbitrary since there is translational symmetry. If one is instead interested in the lightcone through the point $p = (y^0, \dots, y^n)$, one substitutes $x^\mu - y^\mu$ for each x^μ in our expression of the lightcone (2.2). Finally, we note that, given a lightlike curve which starts in the origin and moves through time by a constant velocity with respect to its parametrization, we find that this curve is contained in L.[3] This supports the idea that L consists of light rays emitted in the origin.

We may further make a distinction between two parts of the lightcone L. There are the points with positive x^0 component, and those with negative x^0 component. The subset of L with positive, and respectively negative x^0 coordinate are called the *future and past lightcone in Minkowski spacetime.*

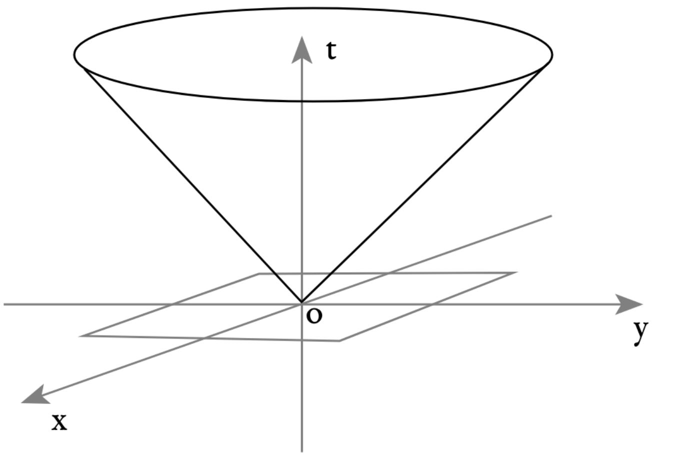

Figure 1: A future lightcone in $2 + 1$ dimensional Minkowski spacetime. The x^0 coordinate is renamed t, and the x^1 and respectively x^2 coordinates are renamed to x and y.

2.1.2 Continuation of the causal structure

We again consider an arbitrary spacetime $(\mathscr{M}, g)$. For all points $p \in \mathscr{M}$ the tangent space $T_p\mathscr{M}$ is isomorphic to Minkowski spacetime. The *lightcone passing through* $p \in \mathscr{M}$ is considered to be the lightcone passing through the origin in $T_p\mathscr{M}$. It must then be emphasized that a lightcone is a subset of the tangent plane $T_p\mathscr{M}$, not of the manifold $\mathscr{M}$. We define the *future and past lightcones* as is done above.[4] One must be cautious however; in a non-simply connected spacetime, there may not be

[3]If one considers $x^0 = ct$, one easily finds that a light ray emitted at the origin moves along the cone in a straight line with a speed of one lightsecond per second.

[4]Since this is defined for Minkowski spacetime and each tangent space is isomorphic to Minkowski space.

a continuous distinction of future and past lightcones. If a continuous choice can be made, we call the spacetime *time orientable.* Only in time-orientable spacetimes can we consistently distinguish between moving forwards and backwards through time.

It is evident that the physical spacetime is time orientable. For this reason we only consider time orientable spacetimes. In the spacetimes in this text, it is assumed that a continuous distinction of future and past has been made. We call timelike and null vectors in the future half of the lightcone *future directed.* Similarly, a curve $\gamma \subset \mathscr{M}$ is said to be *future directed* if at each $p \in \gamma$ the tangent vector $\dot{\gamma}$ is future directed. Here we may distinguish between *future directed null curves*, and *future directed timelike curves. Past directed curves* are defined analogously. We note that since a future, or past, directed curve admits only timelike or lightlike tangent vectors, a hypothetical observer traveling along such a curve may never exceed the speed of light. As we shall see later, we will postulate that nothing in the physical spacetime may exceed this speed limit. We then obtain that causality itself is inhibited by these curves. For this reason, we may also call a curve which is future directed, or past directed, a *causal curve.*

In a time orientable spacetime we are consistently able to differentiate between moving forwards and backwards through time. This can also be captured in the following way.

Lemma 2.1. *Let $(\mathscr{M}, g)$ be time orientable. Then there exists a (nonunique) smooth non-vanishing timelike vector field on $\mathscr{M}$.*

The proof of this lemma can be found in (Wald, 1984) wherein this is lemma 8.1.1. It can also be shown that the converse of this lemma holds.

The *chronological future of a point $p \in \mathscr{M}$*, denoted $\mathscr{I}^+(p)$ is the set of points in the spacetime that can be reached by a future directed timelike curve starting at p. We write

$$\mathscr{I}^+(p) = \left\{ q \in \mathscr{M} \;\middle|\; \begin{array}{l} \text{there exists a future directed timelike curve} \\ \gamma(t) \text{ such that } \gamma(0) = p \text{ and } \gamma(1) = q \end{array} \right\}.$$

Note that the curve being timelike means that its tangent vector may never be null or have zero norm. As such, usually $p \notin \mathscr{I}^+(p)$ holds. However, if there is a closed timelike loop, one could find some $p \in \mathscr{M}$ such that $p \in \mathscr{I}^+(p)$. Remark also that the chronological future of a point is open.

We define the *chronological future of a set $S \subseteq \mathscr{M}$* as the union of the chronological futures of the points in S. That is, $\mathscr{I}^+(S) = \bigcup_{p \in S} \mathscr{I}^+(p)$. Note that this is also open since it is the union of open sets.

The *causal future of $p \in \mathscr{M}$*, $\mathscr{J}^+(p)$ is defined in a similar fashion, changing 'future directed timelike curve' for 'future directed causal curve' in the definition. Again, we have $\mathscr{J}^+(S) = \bigcup_{p \in S} \mathscr{J}^+(p)$.

The *chronological* and *causal pasts* of points and sets in $\mathscr{M}$ are defined analogously to the definitions of the future sets.

We say that a set $S \subseteq \mathscr{M}$ is *achronal* if there do not exist $p, q \in S$ such that $q \in \mathscr{I}^+(p)$. That is, if $\mathscr{I}^+(S) \cap S = \emptyset$. An example of such an achronal set is the boundary of the chronological future. That is, let $S \subseteq \mathscr{M}$. Then $\dot{\mathscr{I}}^+(S)$ is achronal. $\dot{A}$ denotes the boundary of an arbitrary set $A \subseteq \mathscr{M}$. I.e.,

$$\dot{A} \equiv \left\{ p \in \mathscr{M} \;\middle|\; \begin{array}{l} \text{for every neighborhood } U \text{ of } p,\, U \text{ intersects} \\ A \text{ as well as the complement of } A \end{array} \right\}.$$

This claim can be understood by remarking that for $p, q \in \dot{\mathscr{I}}^+(S)$ the points p and q might be connected by a causal null curve but they may not be connected by a timelike curve. This is because, if they were connected by a timelike curve, we'd find that

$$q \in \mathscr{I}^+(p) \subseteq \mathscr{I}^+(\dot{\mathscr{I}}^+(S)) \subseteq \mathscr{I}^+(\overline{\mathscr{I}^+(S)}) = \mathscr{I}^+(\mathscr{I}^+(S)) = \mathscr{I}^+(S).$$

Here we use $\mathscr{I}^+(\overline{A}) = \mathscr{I}^+(A)$, and $\mathscr{I}^+(\mathscr{I}^+(A)) = \mathscr{I}^+(A)$ (Appendix: A.1, and A.2). Since $q \in \mathscr{I}^+(S)$, and the chronological future of S, $\mathscr{I}^+(S)$, is open, we find that $q \notin \dot{\mathscr{I}}^+(S)$. This is a clear contradiction which proves the claim.
This achronality of the edge of the chronological future will become important later since we'll prescribe initial data for our initial value problem on a generalized understanding of a lightcone. This generalized lightcone can be understood as the boundary of the chronological future of a point.

Note that a curve $\gamma : I \subseteq \mathbb{R} \to \mathscr{M}, \quad s \mapsto \gamma(s)$ may end because of different reasons. It might run into a hole or singularity in the chosen spacetime or it might simply not be defined for values larger than than a certain $s \in I$. We want to differentiate between these possibilities. First, we define a *future endpoint* of a curve γ to be a point $p \in \mathscr{M}$ such that for any neighborhood U of p there exists $s \in I$ such that $\gamma(s) \in U$. Moreover, the causal future of p may not intersect the curve γ. Now we can define a *future extendible curve* to be a curve which has a future endpoint. Conversely, a *future inextendible curve* has no future endpoint.

One important assumption in the spacetimes we consider is that of causality. As is done in much of physics, this text presupposes that two events cannot affect each other if they are so far apart that light does not have enough time to travel from the one to the other. That is to say, *nothing* may travel faster than light. This is precisely what went into the definition of causal curves, and the causal future and past. A trivial result of this notion is that an event may only precede or supersede another event; it cannot do both ($q \in \mathscr{I}^+(p) \implies p \notin \mathscr{I}^+(q)$). That is, an observer in a physical spacetime, starting at point $p \in \mathscr{M}$ may never revisit its starting point. This is equivalent to stating that $p \notin \mathscr{I}^+(p)$.

This causality postulate can be made to be mathematically rigorous through a formal definition. One such definition which is used in the literature is strong causality; a spacetime $(\mathscr{M}, g)$ is named *strongly causal* if for all $p \in \mathscr{M}$ and every neighborhood U of p there exists a neighborhood V of p contained in U such that no causal curve intersects V more than once. This may seem like a very strong notion of causality. However, one may find 'exotic' strongly causal spacetimes such that an arbitrarily small perturbation of the metric makes it such that the spacetime is no longer causal. A stronger version of causality is given by stable causality;

a spacetime $(\mathscr{M}, g)$ is said to be *stably causal* if there is no Lorentzian metric h sufficiently close to the spacetime metric g where h admits a closed timelike curve. To define 'sufficiently close', we consider a timelike vector t at $p \in \mathscr{M}$. We remark that the metric h defined by

$$h_{ab} = g_{ab} - t_a t_b$$

is also a Lorentzian metric at p. Furthermore, the lightcone corresponding to h is strictly larger than the lightcone corresponding to g.[5] We say that $(\mathscr{M}, g)$ is stably causal if there is a timelike vector t such that the spacetime $(\mathscr{M}, h)$, where h is defined as above, admits no closed timelike curve. It can be shown that stable causality implies strong causality (Wald, 1984; corollary to theorem 8.2.2).

Stable causality of a metric ensures that no timelike curves are arbitrarily close to being closed timelike loops. It can be shown that stable causality of a metric is equivalent to the existence of a global time function on the manifold[6] (Hawking, 1968). Since our (succesful) cosmological models are based upon this idea of a global time function. We expect physical spacetimes to be stably causal.

Now suppose we have a point $p \in \mathscr{M}$ at which we want to know relevant quantities, e.g., the metric tensor g. If we further assume that we know everything about the metric on a set of points S such that any causal curve through p intersects S, then we may naturally expect that the tensor at p can be completely reconstructed from the tensor on S. This is a sensible expectation since any causal information arriving at p is in a way 'recorded' in the set S. That is to say, if *nothing* may travel faster than light, not even information, then any information travelling on a causal curve which does not intersect S cannot arrive at the point p. This consideration leads us to the natural definition of the future domain of dependence. Let $S \subseteq \mathscr{M}$ then the *future domain of dependence of* S, denoted $\mathscr{D}^+(S)$, is defined in the following way:

$$\mathscr{D}^+(S) = \left\{ p \in \mathscr{M} \;\middle|\; \begin{array}{l} \text{every past inextendible curve} \\ \text{through } p \text{ intersects } S \end{array} \right\}$$

The past domain of dependence $\mathscr{D}^-(S)$ is defined analogously. We define the the *(full) domain of dependence* of S to be the set $\mathscr{D}(S) = \mathscr{D}^+(S) \cup \mathscr{D}^-(S)$. Note that the domain of dependence of a point is only the point itself.

[5]For any null vector ℓ with respect to g we find that $h_{ab}\ell^a\ell^b = g_{ab}\ell^a\ell^b - t_a t_b \ell^a \ell^b = -(t_a\ell^a)^2 < 0$.

[6]As one might suspect, this global time function is usually highly nonunique.

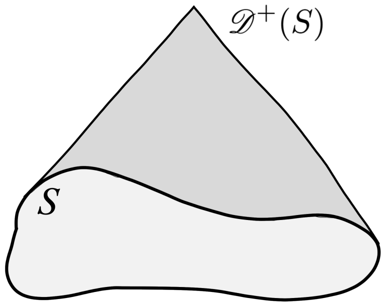

Figure 2: In dark grey: example of the future domain of dependence of a set S. The diagonal lines signify the boundary of $\mathscr{D}^+(S)$ which consist of (generally extendible) null geodesics. Note that the set S is trivially contained in $\mathscr{D}^+(S)$.

A closed achronal set Σ is called a *Cauchy surface* if its domain of dependence is the spacetime. That is, if $\mathscr{D}(\Sigma) = \mathscr{M}$. Since Σ is an achronal set, we can consider it to be a global instance in time, also called a *slice*. Any slice is an n dimensional embedded submanifold of the $n+1$ dimensional spacetime. A spacetime which possesses a Cauchy surface is called *globally hyperbolic*. This definition of global hyperbolicity is given in (Wald, 1984). It differs significantly from the definition given in (Hawking & Ellis, 1973) and the original formulation (Leray, 1952). However, it can be shown that all these definitions are equivalent. This is shown as a remark at the end of section 8.3 in (Wald, 1984).

Global hyperbolicity may also be used as an even stronger causality condition.

Lemma 2.2. *Any globally hyperbolic spacetime is stably causal*

A proof of this lemma can be found in (Wald, 1984) wherein this is lemma 8.3.14. Note that this also implies strong causality of the spacetime (Wald, 1984; corollary to theorem 8.2.2). Thus, by restricting ourselves to globally hyperbolic spacetimes, we ensure stable causality.

2.2 The Cauchy problem for a smooth spacelike Cauchy surface

The domain of dependence of a set is defined such that for a point in it, every past-inextendible causal curve containing the point must intersect the set. As we postulated, *nothing* may travel faster than light. That is to say, if we have a point $p \in \mathscr{D}(S)$, then one might expect to be able to reconstruct relevant information at the point p by information on S alone. In this subsection we give an overview of the initial value formulation for initial data on a spacelike hypersurface Σ. We will show how one arrives at a general well-posed Cauchy problem in the vacuum case ($G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$) of the Einstein field equations. As well-posedness is shown for initial data of a specific form, the knowledge of this form allows us to study arbitrary initial data in the later sections of this thesis. By well-posedness we refer to a number of properties of the problem. First, for arbitrary (smooth) initial data on a surface Σ, there exists a unique solution of the Einstein field equations on $\mathscr{D}(\Sigma)$, up to coordinate transformations. Second, the solution on $\mathscr{D}(\Sigma)$ depends continuously on the initial data, and third, a change on the initial data outside a set $S \subseteq \Sigma$ has no impact on the solution on $\mathscr{D}(S)$. This section serves to give us understanding of what 'good' initial data for the Cauchy problem is.

This section is a paraphrasing of the section on the initial value formulation in (Wald, 1984; chapter 10) in arbitrary dimension. It serves as preliminary knowledge for the examination of the characteristic Cauchy problem and its initial data later in this thesis. The spacetime we work in will have $n+1$ dimensions. Therefore, the Cauchy hypersurface will have n dimensions.

2.2.1 A useful theorem on well-posedness of initial value problems

To show well-posedness of the Cauchy problem for the vacuum Einstein equations, it will be useful to consider the inverse metric $g^{\alpha\beta}$ to be more fundamental than the physical metric. We make use of a theorem attributed to (Leray, 1952) on quasilinear differential equations. i.e., differential equations that are linear in the highest order derivative terms.

A system of m second-order partial differential equations for m unknown functions $\phi_1, \ldots, \phi_m$ on a manifold $\mathscr{M}$ is called a *quasilinear, diagonal, second order, hyperbolic system* if it may be written as

$$g^{\alpha\beta}(x;\phi_j;\nabla_\gamma\phi_j)\nabla_\alpha\nabla_\beta\phi_i = F_i(x;\phi_j;\nabla_\gamma\phi_j), \tag{2.3}$$

where ∇_α is any derivative operator. $g^{\alpha\beta}$ is a smooth Lorentzian metric and F_i is a smooth function in its variables. Here, $g^{\alpha\beta}$ depends on x, ϕ_j, and $\nabla_\gamma\phi_j$. Implicit notation will be used for brevity in the remainder of this section. We will try to rewrite the Einstein vacuum equations in the form (2.3), since this form can be shown to have a well-posed initial value formulation as we will see.

For us to consider the Cauchy problem to be well-posed, it needs to satisfy a number of properties. One of which is that the solution of the system of differential equations depends continuously on the initial data. For this continuity requirement, we define a topology on the space of initial data. We say that two functions f_1, and f_2 are 'close' if they and a finite number (or all) of their derivatives are close. That is, we define the distance between f_1 and f_2 by summing over the suprema of their difference, and the difference in their derivatives up to an order of k on the initial Cauchy surface Σ:

$$\|f_1 - f_2\| = \sup_{x\in\Sigma_0} |f_1(x) - f_2(x)| + \sum_{k_1,\ldots,k_n} \sup_{x\in\Sigma_0} |\frac{\partial^{k_1+\ldots+k_n}(f_1-f_2)}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}(x)|$$

Where $(x_1, \ldots, x_m)$ are the (spacelike) coordinates on the initial hypersurface Σ and $k_1, \ldots, k_m$ are such that $k_1 + \ldots + k_m \leq k$. We take open balls in this metric space as a basis for the topology of the initial data. In a similar fashion, the distance between solutions of the initial value problem may be taken to define the topology of the solution space. Note however that in the solution space, the sum is taken over all $n+1$ spacetime derivatives. We also need to be careful in ensuring that this distance exists. To guarantee existence of the distance between solutions we only consider compact subspaces of the spacetime. Continuity is then defined by the usual open set definition such that it holds for any compact subset of the obtained spacetime.

Given a solution $\Phi_0 \equiv ((\phi_0)_1, \ldots, (\phi_0)_m)$ of (2.3), we also require that there exists another solution of (2.3) for any initial data sufficiently close to that of Φ_0. By 'sufficiently close' to Φ_0 we mean that there exists a neighborhood of Φ_0 in the topology defined above such that a solution exists for all initial data in this neighborhood. By existence of a solution of (2.3), we mean that there exists a neighborhood of Σ, denoted by O, such that there is some $\Phi \equiv (\phi_1, \ldots, \phi_n)$ which satisfies (2.3) on $O \cap \mathscr{D}^+(\Sigma)$. Furthermore, $(O, g(x; \phi_j; \nabla_\gamma \phi_j))$ is globally hyperbolic with Σ as a Cauchy surface. We also require that the solution Φ is unique in O, given its initial data.[7] This requirement is the expected 'existence and uniqueness' requirement which one finds in most well-posedness definitions.

For well-posedness we also require causal propagation of information in the spacetime. By *causal propagation* we refer to the property that information may travel only on causal curves. More precisely, suppose two solutions of (2.3), Φ, and Φ' have initial data which agrees on a subset, S, of Σ. Then the solutions Φ, and Φ' must agree on $O \cap \mathscr{D}^+(S)$, where O is a neighborhood of Σ. This is the same as saying that all relevant information in the future domain of dependence $\mathscr{D}^+(S)$ may be reconstructed from information on S. This causal propagation property is actually a more specific uniqueness condition.

In summary, we define the initial value formulation (2.3) to be *well-posed* if the solution depends continuously on the initial data, the existence and uniqueness condition holds, and the causal propagation condition holds. Given this definition we introduce the following theorem (Wald, 1984; theorem 10.1.3):

Theorem 2.3. *Let $(\phi_1, \ldots, \phi_m)$ be any solution of the quasilinear hyperbolic system (2.3) on a manifold $\mathscr{M}$, and let $g^{\alpha\beta} = g^{\alpha\beta}(x; \phi_j; \nabla_\gamma \phi_j)$. Suppose $(\mathscr{M}, g)$ is a globally hyperbolic spacetime which has the smooth spacelike Cauchy surface Σ. Then the initial value formulation (2.3) is well posed in the sense given above this theorem.*

This theorem yields that if the Einstein vacuum equations can be rewritten in the form (2.3), then well-posedness must follow. The acquired spacetime as a solution of the Cauchy problem will be called a *Cauchy development* of the initial data.

2.2.2 An initial data ansatz

We shall now consider the vacuum Einstein field equations $R_{\mu\nu} = 0$ as an initial value problem. First, let us consider the nature of the field equations. For most field theories one solves for a field with respect to a background space(time). In general relativity, however, one solves for the spacetime itself. The distinctive property of the gravitational field is that it is self-interacting. That is, it defines the space over which it propagates. This makes the equations for which one solves nonlinear even in the absence of other fields.

We will show that for initial data of a certain kind, the vacuum equations may be rewritten in the form of (2.3). From this well-posedness follows by theorem 2.3.

[7] For the actual solution of the vacuum equations this is too strong a requirement since the solutions must be unique only up to a coordinate transformation. However, this is a good condition to have in the well-posedness definition since one works in a specific gauge choice. One may take a coordinate transformation of the solution if one has specific coordinates in mind.

Since the spacetime we consider, $(\mathscr{M}, g)$ is globally hyperbolic, it can be foliated by smooth spacelike Cauchy surfaces (Bernal & Sánchez, 2003).[8] By *foliated*, we mean that the spacetime admits a continuous family of hypersurfaces of dimension n such that these surfaces do not intersect. Since there is a timelike vector orthogonal to these surfaces one can think of each surface as being a moment in 'global time'. That is, there exists a global diffeomorphism between $\mathscr{M}$ and $\mathbb{R} \times \Sigma$. This foliation implies a global time function, t, on $\mathscr{M}$ such that for each t, the surface Σ_t is the Cauchy surface associated with that global time coordinate. We let ξ be the unit time vector orthogonal to the Σ_t hypersurfaces. The metric g then induces the Riemannian metric h on each Σ_t by

$$h_{\alpha\beta} = g_{\alpha\beta} + \xi_\alpha \xi_\beta. \tag{2.4}$$

More precisely, h is a tensor of signature $(0, +, \ldots, +)$[9] which acts on vectors in $T\mathscr{M}$, *not* in $T\Sigma_t$. However, we may define the Riemannian metric on Σ_t by considering the pullback of the tensor h along the embedding $f : \Sigma_t \to \mathscr{M}$ which maps points on the manifold Σ_t to points on Σ_t as a subset of the spacetime manifold $\mathscr{M}$. For this embedding, the vectors $x, y \in T\Sigma_t$ map to the vectors $\tilde{x}, \tilde{y} \in T\mathscr{M}$ for which $g(\tilde{x}, \xi) = g(\tilde{y}, \xi) = 0$. The inner product between x, and y in Σ_t is then given by the expression $h_{\mu\nu}\tilde{x}^\mu \tilde{y}^\nu$ in $\mathscr{M}$. This defines the Riemannian metric on the Σ_t hypersurface which is what we mean by stating that the metric g induces the Riemannian metric h on Σ_t by (2.4). When evaluating the metric h using a vector orthogonal to the hypersurface $\Sigma_t \subset \mathscr{M}$ one arrives at the result zero:

$$h_{\alpha\beta}\xi^\alpha = g_{\alpha\beta}\xi^\alpha + \xi_\alpha \xi_\beta \xi^\alpha = \xi_\beta - \xi_\beta = 0.$$

This further supports the interpretation of h as a Riemannian manifold on Σ_t because only vectors tangent to $\Sigma_t \subset \mathscr{M}$ are of interest when considering the metric h. Furthermore, for some $p \in \Sigma_t \subset \mathscr{M}$, the result of the metric h acting on vectors in $T_p\mathscr{M}$ is trivially the same as for the metric g as long as we consider vectors tangent to Σ_t. This holds since such a vector is orthogonal to ξ. Note also that the tensor $h^\alpha{}_\beta = g^{\gamma\alpha} h_{\alpha\beta}$ serves as a projection onto the Cauchy hypersurface. This follows immediately from the property $h_{\alpha\beta}\xi^\alpha = 0$ and the observation that the inner product using h equals the inner product using g for vectors tangential to the Cauchy surface Σ_t.[10]

In the Cauchy development associated with given initial data, we'll have introduced a time coordinate which may naturally be chosen to correspond with the global time function t. The only requirement for this coordinate will then be that one moves forward through time with one time unit per parameter unit of the ∂_t-integral curve. We consider a vector field σ on $\mathscr{M}$ which satisfies

$$\sigma^\alpha \nabla_\alpha t = 1. \tag{2.5}$$

This vector field can be thought of as representing the flow of time. That is, $\sigma = \partial_t$ in the resulting spacetime. The vector σ is decomposed in its parts orthogonal and tangential to Σ_t. The orthogonal part gives rise to the *lapse function*, n, and the tangential part defines the *shift vector*, N.

$$n = -\sigma^\alpha \xi_\alpha = (\xi^\alpha \nabla_\alpha t)^{-1} \tag{2.6}$$

[8]Note that this is an improvement upon Geroch's famous 'splitting theorem'.

[9]This follows from $h_{\alpha\beta}\xi^\alpha = 0$ which is shown later in this paragraph.

[10]If one has a vector v which is tangent to Σ_t, one obtains $v^\mu = h^\mu{}_\nu v^\nu$ since $h^\mu{}_\nu = \delta^\mu_\nu + \xi^\mu \xi_\nu$ and $v^\nu \xi_\nu = 0$.

$$N^\alpha = h^\alpha{}_\beta \sigma^\beta \tag{2.7}$$

The flow of time σ can then be expressed as

$$\sigma = n\xi + N. \tag{2.8}$$

If we start at a hypersurface Σ_0 and move forward by a parameter distance t along the σ-integral curve we arrive at the hypersurface Σ_t.[11]. Since this new n dimensional Cauchy surface Σ_t will also have a unit timelike vector orthogonal to it, we may express the induced metric on this hypersurface in the same way as (2.4). Along a σ-integral curve we then have a changing induced metric $h(t)$. Thus, we may view the Cauchy problem as the development of a Riemannian metric h on a fixed n-dimensional manifold. In this formulation, the Riemannian metric h serves as (part of) the initial data on the Cauchy surface Σ_0. One may expect, from having seen differential equations with boundary conditions in other settings, that the 'time derivative' of the metric h plays an important role in the solution of an arbitrary Cauchy problem.[12] This turns out to be true. In the remainder of this subsection, we will see that the 'time derivative' is actually the same as the extrinsic curvature of the Cauchy surface. This 'extrinsic curvature' will be rigorously defined. We will later show that the extrinsic curvature and the Riemannian metric h on Σ_0 constitute sufficient initial data for our problem.

Before we defined the metric h in expression (2.4), we introduced the unit normal timelike vector field ξ on the Σ_t hypersurfaces. Since the extension of ξ outside of our initial hypersurface Σ_0 is determined through a (nonunique) foliation of the spacetime $(\mathscr{M}, g)$, we have no information about it. That is, by only considering the initial surface Σ_0, and the metric h, we don't know ξ outside of Σ_0. Suppose we take another unit timelike vector field orthogonal to Σ_0, ζ, such that it is geodesic. We then have that $\xi = \zeta$ on Σ_0, but ξ, and ζ may be extended differently outside of the Cauchy hypersurface. Specifically, ζ can be constructed by letting it coincide with ξ on Σ_0 and having it satisfy the equation $\nabla_\zeta \zeta = 0$ on $\mathscr{M}$. ζ can then be understood as the tangent vector to a congruence of timelike geodesics through $\mathscr{M}$. By a *congruence* of timelike geodesics, we mean a family of timelike geodesics such that each point of $\mathscr{M}$ has precisely one such curve passing through it.

We shall now give a mathematical formulation for the extrinsic curvature through the use of the geodesic vector field ζ on the Cauchy surface Σ. We remark that a Cauchy surface Σ is an embedding in $\mathscr{M}$ where the Riemannian metric h on Σ can be understood to be induced by the Lorentzian metric g on $\mathscr{M}$. We may then describe the curvature of Σ in an extrinsic way. We define the *extrinsic curvature* of Σ, K with components $K_{\alpha\beta}$ to be the covariant derivative of the unit tangent vector to the congruence of timelike geodesics. That is:

$$K_{\alpha\beta} = \nabla_\alpha \zeta_\beta. \tag{2.9}$$

[11] The property $\sigma^\alpha \nabla_\alpha t = 1$ ensures that one moves through time at one unit time interval for each unit parameter distance of the global time function t.

[12] Two remarks: 1. The derivatives in the non timelike direction, that is, tangential to the hypersurface, don't serve as separate initial data since they can be reconstructed from the induced metric $h_{\alpha\beta}$. 2. If one has, for instance a 1-dimensional boundary/initial value problem for a differential equation of order m, then at the boundary, one needs the relevant derivatives up to and including order $m-1$. Since the Einstein field equations, and thus, the vacuum equations, form a system of second order partial differential equations in the components of the metric tensor, one expects that only the first order derivatives need be given. By the first remark in this footnote, one expects to only need the first order time derivative.

Note that K is purely spatial. By this we mean that the projection of K onto the Σ hypersurface is the same as K itself. This follows trivially from the identities $K_{\alpha\beta}\zeta^\alpha = K_{\alpha\beta}\zeta^\beta = 0$:

$$K_{\alpha\beta}\zeta^\alpha = \zeta^\alpha\nabla_\alpha\zeta_\beta = g_{\gamma\beta}\zeta^\alpha\nabla_\alpha\zeta^\gamma = 0$$
$$K_{\alpha\beta}\zeta^\beta = \zeta^\beta\nabla_\alpha\zeta_\beta = \nabla_\alpha\zeta^\beta\zeta_\beta - \zeta_\beta\nabla_\alpha\zeta^\beta = -\zeta^\beta\nabla_\alpha\zeta_\beta = 0$$

For the first equation, we use that ζ is geodesic. This implies that $\zeta^\alpha\nabla_\alpha\zeta^\gamma = 0$ since the ζ-integral curve has an affine parameter. For the second equation we use that $\zeta^\beta\zeta_\beta$ is everywhere -1 and thus vanishes in the covariant derivative. It can be shown that K is symmetric by analysis through the Raychaudhuri equation for timelike geodesics (Wald, 1984; section 9.3).

Since K is purely spatial, we have that

$$K_{\alpha\beta} = h_\alpha{}^\gamma K_{\gamma\beta} = h_\alpha{}^\gamma\nabla_\gamma\zeta_\beta.$$

We remark that this expression of the extrinsic curvature only considers derivatives tangential to the Cauchy hypersurface Σ. This means that it does not matter how the unit timelike vector field is extended off the Σ hypersurface. That is to say, we may take any (non)geodesic normal unit timelike vector instead of ζ in our description of the extrinsic curvature, as long as we project the first component of K onto Σ. Specifically, we have

$$K_{\alpha\beta} = h_\alpha{}^\gamma\nabla_\gamma\xi_\beta,$$

where ξ is the normal unit timelike vector from before. The extrinsic curvature of Σ can be considered to be the 'time derivative' of the Riemannian tensor h through the identity $K_{\alpha\beta} = \frac{1}{2}\pounds_\xi h_{\alpha\beta}$:

$$\begin{aligned}\pounds_\xi h_{\alpha\beta} &= \pounds_\zeta h_{\alpha\beta} = \pounds_\zeta g_{\alpha\beta} + \pounds_\zeta\zeta_\alpha\zeta_\beta \\ &= \zeta^\gamma\nabla_\gamma g_{\alpha\beta} + g_{\gamma\beta}\nabla_\alpha\zeta^\gamma + g_{\alpha\gamma}\nabla_\beta\zeta^\gamma + \zeta^\gamma\nabla_\gamma\zeta_\alpha\zeta_\beta + \zeta_\gamma\zeta_\beta\nabla_\alpha\zeta^\gamma + \zeta_\alpha\zeta_\gamma\nabla_\beta\zeta^\gamma \\ &= (g_{\gamma\beta} + \zeta_\gamma\zeta_\beta)\nabla_\alpha\zeta^\gamma + (g_{\alpha\gamma} + \zeta_\alpha\zeta_\gamma)\nabla_\beta\zeta^\gamma = h_{\gamma\beta}\nabla_\alpha\zeta^\gamma + h_{\alpha\gamma}\nabla_\beta\zeta^\gamma \\ &= h^\gamma{}_\beta\nabla_\alpha\zeta_\gamma + h_\alpha{}^\gamma\nabla_\beta\zeta_\gamma = 2K_{\alpha\beta}\end{aligned} \quad (2.10)$$

Where the last equality holds by symmetry of K. For the component expression of the Lie derivative of a tensor one is referred to (Appendix: A.4). If one is unfamiliar with the formal definition of the Lie derivative of a tensor, I refer to (4.2), later in this thesis.

Since we would generally take the time coordinate to agree with the 'flow of time' direction, σ, such that $\partial_t = \sigma$. It is worthwhile to express K in terms of the derivative of h in this direction. We obtain:

$$\begin{aligned}\pounds_\xi h_{\alpha\beta} &= \xi^\gamma\nabla_\gamma h_{\alpha\beta} + h_{\gamma\beta}\nabla_\alpha\xi^\gamma + h_{\alpha\gamma}\nabla_\beta\xi^\gamma \\ &= n^{-1}\left[n\xi^\gamma\nabla_\gamma h_{\alpha\beta} + h_{\gamma\beta}\nabla_\alpha(n\xi^\gamma) + h_{\alpha\gamma}\nabla_\beta(n\xi^\gamma)\right] = n^{-1}\pounds_{(\sigma-N)}h_{\alpha\beta} \\ &= n^{-1}h_\alpha{}^\gamma h_\beta{}^\delta\left(\pounds_\sigma h_{\gamma\delta} - \pounds_N h_{\gamma\delta}\right) = n^{-1}h_\alpha{}^\gamma h_\beta{}^\delta\pounds_\sigma h_{\gamma\delta} - n^{-1}h_\alpha{}^\gamma h_\beta{}^\delta\pounds_N h_{\gamma\delta}\end{aligned} \quad (2.11)$$

Where the second equality follows from the identity $h_{\gamma\beta}\xi^\gamma = 0$ since $h_{\gamma\beta}\nabla_\alpha(n\xi^\gamma) = nh_{\gamma\beta}\nabla_\alpha\xi^\alpha + h_{\gamma\beta}\xi^\gamma\nabla_\alpha n$. The third equality follows from (2.8) since the $n\xi^\gamma$ terms can be replaced by $\sigma^\gamma - N^\gamma$. The introduction of the projection onto the Cauchy surface in the second to last equality is not needed for the equality to hold since the extrinsic curvature, $\frac{1}{2}\pounds_\xi h_{\alpha\beta}$, has no normal component. However, the Lie derivative of the induced metric in the σ direction and the N direction could have exactly coinciding normal components. As stated previously, we examine the time evolution of a metric on a stationary Cauchy hypersurface. This is why the projection tensors in the last equality (2.11) are introduced. The projections ensure that each term in the last equality is well-defined on the considered manifold Σ.

From the above considerations, one might suspect that appropriate initial data should consist of a triple (Σ, h, K) where Σ is a three dimensional manifold, h is a Riemannian metric on Σ, and K is a symmetric tensor field on Σ. We will show that such initial data, when it satisfies a certain gauge, leads to vacuum equations that can be written as (2.3). Therefore, a globally hyperbolic spacetime $(\mathscr{M}, g)$ can be constructed for which Σ is a Cauchy surface. In the globally hyperbolic spacetime $(\mathscr{M}, g)$, h will be the induced metric on Σ, and K will be the extrinsic curvature of Σ, as defined in (2.9).

2.2.3 The Gauss-Codazzi equations

In order to write the vacuum equations in the form (2.3), we will first give some useful relations between the metrics, derivative operators, and curvature on the spacetime manifold $\mathscr{M}$, and the Cauchy surface Σ. These relations between the different curvatures are known as the Gauss-Codazzi equations. They will be used to derive the constraint equations which ensure that our initial data give rise to a solution of the vacuum field equations.

Since the induced Riemannian metric h on Σ is indeed a metric, there exists a unique natural derivative operator, denoted by D (Wald, 1984; theorem 3.1.1). By *natural derivative operator* we mean that it is metric preserving, i.e., $Dh = 0$.[13] The natural derivative on Σ defines the Riemann curvature tensor ${}^{(n)}R^\alpha{}_{\beta\gamma\delta}$ of the Cauchy surface. It can be shown that the natural derivative, D, for a tensor T in the full spacetime, written in components, is expressed as (Wald, 1984; Lemma 10.2.1)

$$D_\gamma T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_l} = h^{\alpha_1}{}_{\delta_1}\ldots h^{\alpha_k}{}_{\delta_k} h_{\beta_1}{}^{\epsilon_1}\ldots h_{\beta_l}{}^{\epsilon_l} h_\gamma{}^\zeta \nabla_\zeta T^{\delta_1\ldots\delta_k}{}_{\epsilon_1\ldots\epsilon_l}. \tag{2.12}$$

One can easily check that D is a derivative operator. Furthermore, we find that

$$\begin{aligned} D_\gamma h_{\alpha\beta} = h_\alpha{}^\delta h_\beta{}^\epsilon h_\gamma{}^\zeta \nabla_\zeta(g_{\delta\epsilon} + \xi_\delta\xi_\epsilon) &= h_\alpha{}^\delta h_\beta{}^\epsilon h_\gamma{}^\zeta \nabla_\zeta(\xi_\delta\xi_\epsilon) \\ &= h_\alpha{}^\delta \xi_\delta h_\beta{}^\epsilon h_\gamma{}^\zeta \nabla_\zeta \xi_\epsilon + h_\alpha{}^\delta h_\beta{}^\epsilon \xi_\epsilon h_\gamma{}^\zeta \nabla_\zeta \xi_\delta = 0 \end{aligned}$$

since $\nabla_\zeta g_{\delta\epsilon} = 0$, and $h_\alpha{}^\delta \xi_\delta = h_{\alpha\delta}\xi^\delta = 0$. Thus, D, as defined by (2.12) is the natural derivative which we were looking for.

We will now give an implicit expression of the curvature of the Cauchy surface Σ. This will be related to the Riemann curvature tensor of the spacetime manifold $\mathscr{M}$ such that it may be used to obtain the Ricci tensor which we'll need in the vacuum equations. We will later show that, if the vacuum equations hold on the

[13] Moreover, the derivative operator is also torsion free. This operator D is also known as the Levi-Cevita connection.

Cauchy surface Σ, that they will hold everywhere in its domain of dependence $\mathscr{D}^+(\Sigma)$. This section will be rather calculation heavy since we'll need to keep track of many components.

Using the natural derivative D as defined by (2.12), one may express the Riemann curvature tensor for Σ, ${}^{(n)}R^{\alpha}{}_{\delta\beta\gamma}$. Suppose we have a vector ω tangent to Σ. We write

$$ {}^{(n)}R^{\alpha}{}_{\delta\beta\gamma}\omega_\alpha = D_\beta D_\gamma \omega_\delta - D_\gamma D_\beta \omega_\delta. \tag{2.13}$$

We also find

$$\begin{aligned} D_\beta D_\gamma \omega_\delta &= D_\beta (h_\gamma{}^\epsilon h_\delta{}^\zeta \nabla_\epsilon \omega_\zeta) = h_\beta{}^\eta h_\gamma{}^\theta h_\delta{}^\iota \nabla_\eta (h_\theta{}^\epsilon h_\iota{}^\zeta \nabla_\epsilon \omega_\zeta) \\ &= h_\beta{}^\eta h_\gamma{}^\epsilon h_\delta{}^\zeta \nabla_\eta \nabla_\epsilon \omega_\zeta + h_\beta{}^\eta h_\gamma{}^\theta h_\delta{}^\iota \nabla_\eta (h_\theta{}^\epsilon h_\iota{}^\zeta) \nabla_\epsilon \omega_\zeta \end{aligned} \tag{2.14}$$

wherein we use Leibniz's rule to rewrite part of the last term

$$ h_\beta{}^\eta h_\gamma{}^\theta h_\delta{}^\iota \nabla_\eta (h_\theta{}^\epsilon h_\iota{}^\zeta) = h_\beta{}^\eta h_\gamma{}^\epsilon h_\delta{}^\iota \nabla_\eta h_\iota{}^\zeta + h_\beta{}^\eta h_\gamma{}^\theta h_\delta{}^\zeta \nabla_\eta h_\theta{}^\epsilon. \tag{2.15}$$

Notice that the two terms on the right hand side are remarkably similar. We concentrate on the first term.

$$\begin{aligned} h_\beta{}^\eta h_\gamma{}^\epsilon h_\delta{}^\iota \nabla_\eta h_\iota{}^\zeta &= h_\beta{}^\eta h_\gamma{}^\epsilon h_{\delta\iota} \nabla_\eta h^{\iota\zeta} = h_\beta{}^\eta h_\gamma{}^\epsilon h_{\delta\iota} \nabla_\eta \xi^\iota \xi^\zeta = h_\beta{}^\eta h_\gamma{}^\epsilon h_{\delta\iota} \xi^\zeta \nabla_\eta \xi^\iota \\ &= h_\beta{}^\eta h_\gamma{}^\epsilon h_\delta{}^\iota \xi^\zeta \nabla_\eta \xi_\iota = h_\beta{}^\eta h_\gamma{}^\epsilon h_\delta{}^\iota \xi^\zeta K_{\eta\iota} = h_\gamma{}^\epsilon \xi^\zeta K_{\beta\delta} \end{aligned} \tag{2.16}$$

Here, we used that K is purely spatial in the last equality. That is, projection onto Σ leaves it unaltered. Using the same steps, we find

$$ h_\beta{}^\eta h_\gamma{}^\theta h_\delta{}^\zeta \nabla_\eta h_\theta{}^\epsilon = h_\delta{}^\zeta \xi^\epsilon K_{\beta\gamma}. \tag{2.17}$$

We may then substitute (2.15),(2.16), and (2.17) in (2.14) to obtain

$$ D_\beta D_\gamma \omega_\delta = h_\beta{}^\eta h_\gamma{}^\epsilon h_\delta{}^\zeta \nabla_\eta \nabla_\epsilon \omega_\zeta + h_\delta{}^\zeta K_{\beta\gamma} \xi^\epsilon \nabla_\epsilon \omega_\zeta + h_\gamma{}^\epsilon K_{\beta\delta} \xi^\zeta \nabla_\epsilon \omega_\zeta. \tag{2.18}$$

The middle term in (2.18) disappears when taking the antisymmetric part over β and γ since K is symmetric. Moreover, we find

$$ h_\gamma{}^\epsilon \xi^\zeta \nabla_\epsilon \omega_\zeta = h_\gamma{}^\epsilon \nabla_\epsilon \xi^\zeta \omega_\zeta - h_\gamma{}^\epsilon \omega^\zeta \nabla_\epsilon \xi_\zeta = -h_\gamma{}^\epsilon \omega^\zeta \nabla_\epsilon \xi_\zeta = -\omega^\zeta K_{\gamma\zeta} = -\omega_\zeta K_\gamma{}^\zeta. \tag{2.19}$$

Taking twice the antisymmetric part of (2.18) over β and γ and using (2.19) in our expression for the Riemann curvature tensor for Σ, (2.13) yields

$$\begin{aligned} {}^{(n)}R^{\alpha}{}_{\delta\beta\gamma}\omega_\alpha &= h_\beta{}^\eta h_\gamma{}^\epsilon h_\delta{}^\zeta \nabla_\eta \nabla_\epsilon \omega_\zeta + h_\gamma{}^\epsilon K_{\beta\delta} \xi^\zeta \nabla_\epsilon \omega_\zeta \\ &\quad - h_\gamma{}^\eta h_\beta{}^\epsilon h_\delta{}^\zeta \nabla_\eta \nabla_\epsilon \omega_\zeta - h_\beta{}^\epsilon K_{\gamma\delta} \xi^\zeta \nabla_\epsilon \omega_\zeta \\ &= h_\beta{}^\eta h_\gamma{}^\epsilon h_\delta{}^\zeta (\nabla_\eta \nabla_\epsilon - \nabla_\epsilon \nabla_\eta)\omega_\zeta - K_{\beta\delta} K_\gamma{}^\zeta \omega_\zeta + K_{\gamma\delta} K_\beta{}^\zeta \omega_\zeta. \end{aligned}$$

Where we used (2.19) in the last equality. Notice that $(\nabla_\eta \nabla_\epsilon - \nabla_\epsilon \nabla_\eta)\omega_\zeta = R^\alpha{}_{\zeta\eta\epsilon}\omega_\alpha$. We conclude

$$ {}^{(n)}R^{\alpha}{}_{\beta\gamma\delta} = h_\beta{}^\eta h_\gamma{}^\epsilon h_\delta{}^\zeta h^\alpha{}_\theta R^\theta{}_{\eta\epsilon\zeta} - K_{\beta\delta} K_\gamma{}^\alpha + K_{\gamma\delta} K_\beta{}^\alpha \tag{2.20}$$

Note that we introduced the extra projection term $h^\alpha{}_\theta$. This is needed since our derivation depends on the vector ω being tangent to Σ. If we did this for an arbitrary vector we'd need to project it onto Σ.[14] Equation (2.20) is one of the two Gauss-Codazzi relations. We derive the other Gauss-Codazzi relation by evaluating the expression

$$D_\alpha K^\alpha{}_\beta - D_\beta K^\alpha{}_\alpha$$

using the components expression for D, (2.12), and the expression for the extrinsic curvature (2.9). This leads to the following calculation:

$$\begin{aligned}
D_\alpha K^\alpha{}_\beta - D_\beta K^\alpha{}_\alpha &= h^\alpha{}_\delta h_\beta{}^\gamma h_\alpha{}^\epsilon \nabla_\epsilon K^\delta{}_\gamma - h^\alpha{}_\delta h_\alpha{}^\gamma h_\beta{}^\epsilon \nabla_\epsilon K^\delta{}_\gamma \\
&= h_\beta{}^\gamma h_\delta{}^\epsilon \nabla_\epsilon K^\delta{}_\gamma - h_\delta{}^\gamma h_\beta{}^\epsilon \nabla_\epsilon K^\delta{}_\gamma = h_\beta{}^\gamma h_\delta{}^\epsilon \nabla_\epsilon K^\delta{}_\gamma - h_\beta{}^\gamma h_\delta{}^\epsilon \nabla_\gamma K^\delta{}_\epsilon \\
&= h_\beta{}^\gamma h^{\delta\epsilon}(\nabla_\epsilon K_{\delta\gamma} - \nabla_\gamma K_{\delta\epsilon}) = h_\beta{}^\gamma h^{\delta\epsilon}(\nabla_\epsilon \nabla_\gamma \xi_\delta - \nabla_\gamma \nabla_\epsilon \xi_\delta) \\
&= h_\beta{}^\gamma h_\delta{}^\epsilon(\nabla_\epsilon \nabla_\gamma \xi^\delta - \nabla_\gamma \nabla_\epsilon \xi^\delta) = h_\beta{}^\gamma h_\delta{}^\epsilon R^\delta{}_{\zeta\epsilon\gamma}\xi^\zeta = h_\beta{}^\gamma(\delta^\epsilon_\delta + \xi_\delta \xi^\epsilon) R^\delta{}_{\zeta\epsilon\gamma}\xi^\zeta \\
&= h_\beta{}^\gamma R_{\zeta\gamma}\xi^\zeta + h_\beta{}^\gamma R_{\delta\zeta\epsilon\gamma}\xi^\delta\xi^\zeta\xi^\epsilon = h_\beta{}^\gamma R_{\zeta\gamma}\xi^\zeta
\end{aligned}$$

Wherein the term $R_{\delta\zeta\epsilon\gamma}\xi^\delta\xi^\zeta\xi^\epsilon$ vanishes because the Riemann tensor is antisymmetric in δ, and ζ. Thus, we've shown that the identity

$$D_\alpha K^\alpha{}_\beta - D_\beta K^\alpha{}_\alpha = h_\beta{}^\gamma R_{\delta\gamma}\xi^\delta \tag{2.21}$$

holds. The expressions (2.20) and (2.21) form the Gauss-Codazzi equations which we will be using shortly to derive constraint equations for our initial value formulation.

2.2.4 Constraints on the initial data

We shall now move our focus to the vacuum Einstein field equations. We suppose that we have initial data of the form (h, K) on the n-dimensional manifold Σ. We try to construct a globally hyperbolic spacetime $(\mathscr{M}, g)$ for which Σ is a (smooth spacelike) Cauchy surface, h is the induced Riemannian metric on Σ, and K is the induced extrinsic curvature as defined by (2.9). Our approach is to write out the vacuum equations $G_{\mu\nu} = 0$ in terms of the spacetime metric $g_{\mu\nu}$.[15] The time coordinate t is chosen such that the $t = 0$ corresponds with the Σ hypersurface. We will choose a gauge which casts the vacuum field equations in the form (2.3). Then, theorem 2.3 gives that our initial value formulation is well-posed from which we conclude that the initial data (Σ, h, K) gives rise to a well-posed Cauchy problem for the vacuum Einstein field equations.

The Ricci tensor $R_{\mu\nu}$ can be written in terms of the components $g_{\mu\nu}$ and its derivatives. To make this calculation easier to keep track of we introduce an equivalence class. We say two expressions are 'equivalent' if they agree up to a function which depends on the metric and its first order derivatives. This equivalence is denoted by the "$\cong$"-sign. We are able to write the Ricci tensor in terms of the Christoffel coefficients:

$$R_{\mu\nu} \equiv R^\beta{}_{\mu\beta\nu} = \partial_\beta \Gamma^\beta_{\mu\nu} - \partial_\nu \Gamma^\beta_{\beta\mu} + \Gamma^\beta_{\beta\lambda}\Gamma^\lambda_{\nu\mu} - \Gamma^\beta_{\nu\lambda}\Gamma^\lambda_{\beta\mu} \cong \partial_\beta \Gamma^\beta_{\mu\nu} - \partial_\nu \Gamma^\beta_{\beta\mu}. \tag{2.22}$$

[14] Note that the extrinsic curvature is purely spacial. This is why we need not project K onto Σ for equation (2.20) to hold in general.

[15] This will be written in local coordinates. One may need numerous coordinate patches to cover the entire Cauchy surface Σ.

Here we used that the Christoffel coefficients are functions which depend on the first order derivative of the metric g and thus are equivalent to zero in our equivalence class. Using the expression

$$\Gamma^{\beta}_{\gamma\delta} = \frac{1}{2} g^{\beta\alpha} \left(\partial_\delta g_{\gamma\alpha} + \partial_\gamma g_{\delta\alpha} - \partial_\alpha g_{\gamma\delta} \right),$$

we may rewrite (2.22) as

$$\begin{aligned}
R_{\mu\nu} &\cong \partial_\beta \Gamma^{\beta}_{\mu\nu} - \partial_\nu \Gamma^{\beta}_{\beta\mu} \\
&= \frac{1}{2}\partial_\beta g^{\beta\alpha}(\partial_\nu g_{\mu\alpha} + \partial_\mu g_{\nu\alpha} - \partial_\alpha g_{\mu\nu}) - \frac{1}{2}\partial_\nu g^{\beta\alpha}(\partial_\mu g_{\beta\alpha} + \partial_\beta g_{\mu\alpha} - \partial_\alpha g_{\beta\mu}) \\
&\cong \frac{1}{2} g^{\alpha\beta}\partial_\beta(\partial_\nu g_{\mu\alpha} + \partial_\mu g_{\nu\alpha} - \partial_\alpha g_{\mu\nu}) - \frac{1}{2}\partial_\nu g^{\beta\alpha}\partial_\mu g_{\beta\alpha} \\
&= \frac{1}{2} g^{\alpha\beta}(2\partial_\beta \partial_{(\nu} g_{\mu)\alpha} - \partial_\beta\partial_\alpha g_{\mu\nu}) \qquad - \frac{1}{2} g^{\alpha\beta}\partial_\nu\partial_\mu g_{\beta\alpha} \\
&= -\frac{1}{2} g^{\alpha\beta}\left(-2\partial_\beta\partial_{(\nu} g_{\mu)\alpha} + \partial_\alpha\partial_\beta g_{\mu\nu} + \partial_\mu\partial_\nu g_{\alpha\beta}\right),
\end{aligned}$$

wherein the disappearance of the terms in the third equality follows from symmetry of the metric tensor; since we sum over all indices α, and β we find $g^{\beta\alpha}\partial_\beta g_{\mu\alpha} = g^{\beta\alpha}\partial_\alpha g_{\beta\mu}$. Without making reference to our equivalence class, we found

$$R_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta}\left(-2\partial_\beta\partial_{(\nu} g_{\mu)\alpha} + \partial_\alpha\partial_\beta g_{\mu\nu} + \partial_\mu\partial_\nu g_{\alpha\beta}\right) + F_{\mu\nu}(g, \partial g), \tag{2.23}$$

where $F_{\mu\nu}(g, \partial g)$ is a function which only contains the metric g and its first order derivatives. We want to work towards a similar expression of the Einstein tensor. To do this, we need to express the Ricci scalar in terms of the metric as well.

$$\begin{aligned}
R \equiv R^{\sigma}{}_{\sigma} &= g^{\rho\sigma} R_{\rho\sigma} \\
&\cong -\frac{1}{2} g^{\alpha\beta} g^{\rho\sigma}(-2\partial_\beta\partial_{(\sigma} g_{\rho)\alpha} + \partial_\alpha\partial_\beta g_{\rho\sigma} + \partial_\rho\partial_\sigma g_{\alpha\beta}) \\
&= -g^{\alpha\beta} g^{\rho\sigma}(\partial_\beta\partial_\rho g_{\sigma\alpha} + \partial_\alpha\partial_\beta g_{\rho\sigma})
\end{aligned}$$

From this, we find an expression for the Einstein tensor in terms of the metric:

$$\begin{aligned}
G_{\mu\nu} &\equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\
&= -\frac{1}{2} g^{\alpha\beta}\left(-2\partial_\beta\partial_{(\nu} g_{\mu)\alpha} + \partial_\alpha\partial_\beta g_{\mu\nu} + \partial_\mu\partial_\nu g_{\alpha\beta}\right) \\
&\quad + \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} g^{\rho\sigma}\left(-\partial_\beta\partial_\rho g_{\sigma\alpha} + \partial_\alpha\partial_\beta g_{\rho\sigma}\right) + \tilde{F}_{\mu\nu}(g, \partial g).
\end{aligned} \tag{2.24}$$

Here, $\tilde{F}_{\mu\nu}(g, \partial g)$ is another function which depends only on the metric and its first order derivatives. The right hand side of (2.24) is not yet of the form (2.3). This can be seen by considering the equations

$$G_{\mu\nu}\xi^\nu = 0. \tag{2.25}$$

We will find that these equations contain no second order time derivatives of the metric components. This will be shown by writing, these components of $G_{\mu\nu} = 0$ solely in terms of initial data on the Σ initial hypersurface. Since the equations

(2.25) must hold on Σ they will provide initial value constraints. They can be expressed in coordinate invariant form using the Gauss-Codazzi equations. From (2.21) we find that the part of (2.25) tangent to the Σ hypersurface can be rewritten in terms of initial data as follows.

$$0 = h^{\beta}{}_{\alpha} G_{\beta\gamma}\xi^{\gamma} = h^{\beta}{}_{\alpha} R_{\beta\gamma}\xi^{\gamma} = D_{\beta}K^{\beta}{}_{\alpha} - D_{\alpha}K^{\beta}{}_{\beta} \tag{2.26}$$

Moreover, we find that the part of (2.25) tangent to ξ implies

$$R_{\alpha\beta\gamma\delta}h^{\alpha\gamma}h^{\beta\delta} = 0, \tag{2.27}$$

since

$$\begin{aligned} R_{\alpha\beta\gamma\delta}h^{\alpha\gamma}h^{\beta\delta} &= R_{\alpha\beta\gamma\delta}(g^{\alpha\gamma} + \xi^{\alpha}\xi^{\gamma})(g^{\beta\delta} + \xi^{\beta}\xi^{\delta}) \\ &= R + 2R_{\alpha\gamma}\xi^{\alpha}\xi^{\gamma} \\ &= 2G_{\alpha\gamma}\xi^{\alpha}\xi^{\gamma}. \end{aligned}$$

Using (2.20), and (2.27), we obtain

$$\begin{aligned} {}^{(n)}R + (K^{\alpha}{}_{\alpha})^2 - K_{\alpha\beta}K^{\alpha\beta} &= {}^{(n)}R^{\alpha\beta}{}_{\alpha\beta} + K^{\alpha}{}_{\alpha}K^{\beta}{}_{\beta} - K_{\alpha\beta}K^{\alpha\beta} \\ = g^{\gamma\beta}\left({}^{(n)}R^{\alpha}{}_{\gamma\alpha\beta} + K^{\alpha}{}_{\alpha}K_{\gamma\beta} - K_{\alpha\beta}K^{\alpha}{}_{\gamma}\right) &= g^{\gamma\beta}h_{\gamma}{}^{\zeta}h_{\alpha}{}^{\delta}h_{\beta}{}^{\epsilon}h^{\alpha}{}_{\eta}R^{\eta}{}_{\zeta\delta\epsilon} \\ = g^{\gamma\beta}h_{\gamma}{}^{\zeta}h_{\beta}{}^{\epsilon}h^{\delta}{}_{\eta}R^{\eta}{}_{\zeta\delta\epsilon} &= h^{\beta\zeta}h_{\beta}{}^{\epsilon}h^{\delta}{}_{\eta}R^{\eta}{}_{\zeta\delta\epsilon} \\ = h^{\epsilon\zeta}h^{\delta}{}_{\eta}R^{\eta}{}_{\zeta\delta\epsilon} &= h^{\epsilon\zeta}h^{\delta\eta}R_{\eta\zeta\delta\epsilon} = 0, \end{aligned} \tag{2.28}$$

wherein the forth equality holds because $h_{\alpha}{}^{\delta}h^{\alpha}{}_{\eta} = h^{\delta}{}_{\eta}$, and the sixth equality holds because $h^{\beta\zeta}h_{\beta}{}^{\epsilon} = h^{\epsilon\zeta}$. Thus, equations (2.26), and (2.28) are the initial value constraint equations for the vacuum Cauchy problem. Notice that these equations are equivalent to (2.25), and since they consist of only initial data, the equations (2.25), contain no second order time derivatives.[16] They may be referred to as the *initial timelike vacuum equations* since we arrived at them through (2.25). We will show that if these equations hold on the initial hypersurface Σ, that they will hold everywhere in the spacetime $(\mathscr{M}, g)$. From standard results in differential geometry, we have the contracted Bianchi identity (Appendix; A.3):

$$\nabla^{\mu}G_{\mu\nu} = 0 \tag{2.29}$$

We suppose that the initial timelike vacuum equations (2.26), and (2.28) are satisfied on Σ. Moreover, we assume that the spatial components of Einstein's vacuum field equations are also satisfied. Since the vacuum equations in the spacelike coordinates hold on Σ, we only need to concern ourselves with the equations that have a timelike component in (2.29). That is, we consider the case where μ or ν is a time coordinate in (2.29). In the case where ν is the time coordinate and μ is a spatial coordinate, we automatically have that the vacuum equations are satisfied. This is because the Einstein tensor is identically zero on the initial Cauchy hypersurface, even in the timelike direction. This is ensured by the initial timelike vacuum equations (2.26), and (2.28). The case which we have not yet dealt with is the case where μ is the time coordinate. In that case, the equations (2.29) describe the time derivative of the Einstein tensor. Since the components of the Einstein

[16] K is initial data, and the spacelike Ricci scalar ${}^{(n)}R$ can be expressed purely in terms of it, and the other initial data, h, on the Cauchy hypersurface.

tensor are zero on Σ, and the (covariant) derivative is zero, we obtain that the components vanish everywhere on $\mathscr{M}$. Thus, a solution of the Cauchy problem for initial data which satisfies the spatial vacuum equations and the constraints (2.26), and (2.28) is a spacetime $(\mathscr{M}, g)$ for which the vacuum equations hold everywhere.

2.2.5 Harmonic coordinates and the reduced vacuum equations

Recall that a solution of the Einstein vacuum equations will remain a solution after a coordinate transformation. This means that we may choose arbitrary spatial and timelike coordinates. We will show that the choice of *harmonic coordinates*, that is coordinates (x^μ) which satisfy

$$H^\mu \equiv \nabla_\alpha \nabla^\alpha x^\mu = 0, \tag{2.30}$$

lead to a system of differential equations of the form (2.3). That is, a well-posed initial value formulation.

An important property of the harmonic gauge is that it will be satisfied on the resulting spacetime manifold $\mathscr{M}$ without loss of physical generality. This can be argued through a separate initial value problem for these coordinates (Wald, 1984; section 10.2).

In order to show cancellation of the unwanted terms in the Einstein tensor, we'll write the harmonic gauge condition in terms of the metric g. To do this, we make use of the following identity for the covariant derivative of a vector v:

$$\nabla_\alpha v^\alpha = \frac{1}{\sqrt{-\det g}} \partial_\alpha \left(\sqrt{-\det g} v^\alpha \right). \tag{2.31}$$

This identity follows from the calculation

$$\begin{aligned}
\nabla_\alpha v^\alpha &= \partial_\alpha v^\alpha + \Gamma^\alpha_{\alpha\lambda} v^\lambda = \partial_\alpha v^\alpha + \frac{1}{2} g^{\alpha\beta} \left(\partial_\alpha g_{\lambda\beta} + \partial_\lambda g_{\alpha\beta} - \partial_\beta g_{\alpha\lambda} \right) v^\lambda \\
&= \partial_\alpha v^\alpha + \frac{1}{2} g^{\alpha\beta} (\partial_\lambda g_{\alpha\beta}) v^\lambda = \partial_\alpha v^\alpha + \frac{1}{2} \frac{1}{\det g} (\partial_\lambda \det g) v^\lambda \\
&= \partial_\alpha v^\alpha + \frac{1}{2} \left(\partial_\alpha \log |\det g| \right) v^\alpha = \partial_\alpha v^\alpha + \left(\partial_\alpha \log \sqrt{|\det g|} \right) v^\alpha \\
&= \partial_\alpha v^\alpha + \frac{1}{\sqrt{-\det g}} \left(\partial_\alpha \sqrt{-\det g} \right) v^\alpha = \frac{1}{\sqrt{-\det g}} \left(\partial_\alpha \sqrt{-\det g} v^\alpha \right),
\end{aligned}$$

wherein we use the symmetry of the metric to get rid of the metric derivative terms in the third equality, and we use Jacobi's formula in the forth. Moreover, we use that $\det g < 0$ to rewrite $|\det g|$ to $-\det g$.[17]

[17] $\det g$ is smaller than zero because the metric at a point may always be written in terms of orthogonal vectors. This corresponds to a diagonal matrix for which one entry is negative. The negative entry corresponds with the timelike direction in such a coordinate expression.

Using (2.31), we may rewrite H^μ in the following way:

$$\begin{aligned}
H^\mu &= \nabla_\alpha \nabla^\alpha x^\mu = \nabla_\alpha g^{\alpha\beta} \nabla_\beta x^\mu = \nabla_\alpha g^{\alpha\beta} \partial_\beta x^\mu \\
&= \frac{1}{\sqrt{-\det g}} \partial_\alpha \left(\sqrt{-\det g} g^{\alpha\beta} \partial_\beta x^\mu \right) = \frac{1}{\sqrt{-\det g}} \partial_\alpha \left(\sqrt{-\det g} g^{\alpha\beta} \delta^\mu_\beta \right) \\
&= \frac{1}{\sqrt{-\det g}} \partial_\alpha \left(\sqrt{-\det g} g^{\alpha\mu} \right) = \partial_\alpha g^{\alpha\mu} + g^{\alpha\mu} \frac{1}{\sqrt{-\det g}} \partial_\alpha \sqrt{-\det g} \\
&= \partial_\alpha g^{\alpha\mu} + \frac{1}{2} g^{\alpha\mu} \frac{1}{-\det g} \partial_\alpha (-\det g) = \partial_\alpha g^{\alpha\mu} + \frac{1}{2} g^{\alpha\mu} g^{\rho\sigma} \partial_\alpha g_{\rho\sigma}
\end{aligned}$$

Where the last equality follows from Jacobi's formula. The harmonic gauge condition (2.30) may then be written in terms of the metric as

$$H^\mu = \partial_\alpha g^{\alpha\mu} + \frac{1}{2} g^{\alpha\mu} g^{\rho\sigma} \partial_\alpha g_{\rho\sigma} = 0. \tag{2.32}$$

As we know, the vacuum equations can be written as $R_{\mu\nu} = 0$. If our gauge condition (2.32) holds, we may add arbitrary extra terms to the left or right hand side as long as we multiply them with H^μ in the end. In particular, the *reduced Ricci tensor*

$$R^H_{\mu\nu} \equiv R_{\mu\nu} + g_{\alpha(\mu} \partial_{\nu)} H^\alpha. \tag{2.33}$$

in these coordinates is zero if and only if the (regular) Ricci tensor is zero. That is, the reduced Ricci tensor is zero if and only if the the vacuum equations are satisfied. In harmonic coordinates, we may then express the vacuum equations in terms of the metric g to obtain

$$\begin{aligned}
0 = R^H_{\mu\nu} &\equiv R_{\mu\nu} + g_{\alpha(\mu} \partial_{\nu)} H^\alpha \\
&= -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} + \hat{F}(g, \partial g),
\end{aligned} \tag{2.34}$$

where the function $\hat{F}(g, \partial g)$ depends only on the metric g and its first derivatives. The derivation of (2.34) is pretty much just a bookkeeping exercise. We use the following expression:

$$\begin{aligned}
0 &= \partial_\epsilon \partial_\zeta (g_{\mu\alpha} g^{\alpha\beta}) = \partial_\epsilon (g_{\mu\alpha} \partial_\zeta g^{\alpha\beta} + g^{\alpha\beta} \partial_\zeta g_{\mu\alpha}) \\
&= g_{\mu\alpha} \partial_\epsilon \partial_\zeta g^{\alpha\beta} + g^{\alpha\beta} \partial_\epsilon \partial_\zeta g_{\mu\alpha} + f(g, \partial g)
\end{aligned}$$

In our rewriting of the reduced vacuum equations, we only care to keep track of all the terms which are second order derivatives of g. With that in mind, we can write

$$g_{\mu\alpha} \partial_\epsilon \partial_\zeta g^{\alpha\beta} \cong -g^{\alpha\beta} \partial_\epsilon \partial_\zeta g_{\mu\alpha} \tag{2.35}$$

where we use the "$\cong$"-sign to denote that equality of the left and right hand side up to a function which depends only on g and its first derivatives. We shall now derive (2.34) by considering the equations $R^H_{\mu\nu} = 0$, and remembering (2.23) and (2.32):

$$\begin{aligned}
0 = R^H_{\mu\nu} &\equiv R_{\mu\nu} + g_{\alpha(\mu} \partial_{\nu)} H^\alpha \\
&\cong g^{\alpha\beta} \partial_\beta \partial_{(\nu} g_{\mu)\alpha} - \frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} - \frac{1}{2} g^{\alpha\beta} \partial_\mu \partial_\nu g_{\alpha\beta} + g_{\alpha(\mu} \partial_{\nu)} \partial_\beta g^{\beta\alpha} \\
&\qquad + \frac{1}{2} g_{\alpha(\mu} \partial_{\nu)} g^{\beta\alpha} g^{\rho\sigma} \partial_\beta g_{\rho\sigma}
\end{aligned} \tag{2.36}$$

We notice that the first and fourth term on the right hand side of this equation cancel with respect to our equivalence class. Indeed, we obtain

$$g^{\alpha\beta}\partial_\beta\partial_{(\nu}g_{\mu)\alpha} + g_{\alpha(\mu}\partial_{\nu)}\partial_\beta g^{\beta\alpha} = \frac{1}{2}(g^{\alpha\beta}\partial_\beta\partial_\nu g_{\mu\alpha} + g^{\alpha\beta}\partial_\beta\partial_\mu g_{\nu\alpha} \\ + g_{\alpha\mu}\partial_\nu\partial_\beta g^{\beta\alpha} + g_{\alpha\nu}\partial_\mu\partial_\beta g^{\beta\alpha}) \cong 0,$$

wherein the terms above each other cancel with respect to our equivalence class due to (2.35) and because the partial derivative operators ∂_ν, and ∂_β commute. Analysis of the last term in the right hand side of (2.36) yields the following result:

$$\frac{1}{2}g_{\alpha(\mu}\partial_{\nu)}g^{\beta\alpha}g^{\rho\sigma}\partial_\beta g_{\rho\sigma} = \frac{1}{4}(g_{\alpha\mu}\partial_\nu g^{\beta\alpha}g^{\rho\sigma}\partial_\beta g_{\rho\sigma} + g_{\alpha\nu}\partial_\mu g^{\beta\alpha}g^{\rho\sigma}\partial_\beta g_{\rho\sigma}) \\ \cong \frac{1}{4}(\delta^\beta_\mu g^{\rho\sigma}\partial_\nu\partial_\beta g_{\rho\sigma} + \delta^\beta_\nu g^{\rho\sigma}\partial_\mu\partial_\beta g_{\rho\sigma}) = \frac{1}{2}g^{\alpha\beta}\partial_\mu\partial_\nu g_{\alpha\beta}$$

In the last equality we used a relabeling of the indices over which a summation is taken. We see that this term cancels with the third term in (2.36) from which we obtain the result

$$R^H_{\mu\nu} \cong -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu}. \tag{2.37}$$

We remark that this equation only holds for the harmonic coordinates. Thus, the vacuum Einstein field equations are equivalent to the system (2.37), together with the harmonic coordinate condition (Fourès-Bruhat, 1952). The key point of our expression, (2.37), is that it is of the form (2.3). Indeed, the equations $R^H_{\mu\nu} = 0$ imply $g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} \cong 0$ which is equivalent to the system

$$g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} = F_{\mu\nu}(g, \partial g) \tag{2.38}$$

for some function of the metric and its first derivatives, F. Equation (2.38) is clearly of the form (2.3).

In the definition of well-posedness used in theorem 2.3, the existence condition stated that any solution Φ_0, must have a neighborhood of its initial data, as defined by the topology defined on the same page, such that any initial data in this neighborhood gives rise to a unique solution. That is, a solution for arbitrary initial data only has to exist if it is 'close' to another (known) solution of the initial value problem. Since we know Minkowski spacetime to be a solution of the vacuum equations we acquire that the Cauchy problem is well-posed for initial data sufficiently close to that of flat spacetime. It can be shown that this is enough to ensure well-posedness of the Cauchy problem for arbitrary initial data. This can be done by considering intersecting patches of the Cauchy surface which can be shrunk in order for the initial data to be 'sufficiently' close to that of Minkowski spacetime.[18] Each patch will have a unique Cauchy development in a neighborhood, given its initial data. These resulting Cauchy developments of the patches can then be joined together. In the resulting spacetime, one may choose a new Cauchy surface and continue inductively. Moreover, it can be shown that there exists a unique *maximal Cauchy development* (Choquet-Bruhat and Geroch, 1969),(Wald, 1984; section 10.2). Since this is not really relevant for the purposes of this thesis,

[18] For any point in a smooth spacetime, we may give a coordinate expression such that the metric there is equal to the Minkowski metric. Moreover, its first order derivatives may be chosen to vanish at that point.

we refer the curious reader to one of the sources above.

The considerations in this chapter suggest that appropriate initial data for the Cauchy problem with a smooth spacelike Cauchy surface should consist of a triple (Σ, h, K) where Σ is a three dimensional manifold, h is a Riemannian metric on Σ, and K is a symmetric tensor field on Σ. Using initial data of this form, we've shown that one arrives at a well-posed initial value formulation for the vacuum equations in general relativity. This was done by rewriting Einstein's vacuum field equations, with constraints expressed in our initial data, as a system of differential equations for which well-posedness is known.

2.3 The characteristic Cauchy problem

In the previous subsection we gave a well-posed initial value formulation for a smooth spacelike Cauchy surface. The observant reader may recall, from the subsection on causal structure, that a Cauchy surface need not be spacelike. More precisely, a Cauchy surface is defined to be an achronal set for which its domain of dependence equals the full spacetime manifold $\mathscr{M}$. Any spacelike surface is trivially achronal but the converse is not true; an achronal set is a set for which starting at a point contained in it, and traveling a nonzero parameter distance along any timelike curve, takes one out of the set. That is, an achronal set is a surface which admits no tangent timelike vectors.

In this section we discuss the Cauchy problem for a characteristic Cauchy surface. By a *characteristic Cauchy surface*, Σ, we refer to a Cauchy surface for which the orthogonal vectors at each point are tangent to the surface. This can be understood as the surface having a 'null direction'. This implies the existence of points $p, q \in \Sigma$ such that a causal null curve can be found connecting the points. That is, some points in the Cauchy hypersurface may communicate with each other. One may assume, then, that this has effect on the initial data on the hypersurface. Later in this thesis we will derive the Raychaudhuri equation which will be shown to serve as a constraint equation on the initial data, alternatively, it may be used to generate missing information of initial data.

Initial data for the characteristic Cauchy problem differs from data for the smooth spacelike Cauchy problem in three important ways. First, since a characteristic Cauchy surface Σ has a tangent null direction, the induced metric h on it is degenerate. This can be understood by considering a null vector which is tangent to Σ. We'll denote this vector by ℓ. Suppose v is any other tangent vector to the initial surface. Then $h(v, \ell) = g(v, \ell) = 0$ since ℓ is orthogonal to the surface, and $h(\ell, \ell) = g(\ell, \ell) = 0$ since ℓ is null. So the metric h has a nontrivial nullspace which shows its degeneracy. Second, the extrinsic curvature (also called the second fundamental form) on a null surface has vastly different properties to the one for a spacelike surface. This will be analyzed further in the section on constraints for the Cauchy problem on a lightcone. Third, initial data on a lightcone, or on two intersecting null surfaces determines the solution in one direction only, future or past (Choquet-Bruhat et al., 2011a).

There are two general shapes which the characteristic Cauchy problem takes. Initial data may be prescribed on two transversely intersecting characteristic surfaces, or it may be described on a lightcone. In this text, we concentrate on the Cauchy problem on a lightcone.

Recall that for the Cauchy problem with a smooth spacelike Cauchy surface, we found that the triple (Σ, h, K) gave rise to a well-posed initial value formulation. Here Σ is the Cauchy surface, h is the Riemannian metric on Σ, and K is the extrinsic curvature of Σ. In the case of the characteristic Cauchy problem, the triple $(\mathscr{N}, \overline{g}, \kappa)$ will be used instead. For this triple, $\mathscr{N}$ denotes the characteristic Cauchy surface, $\overline{g}$ denotes the metric on $\mathscr{N}$, and κ will be a connection coefficient for the null vectors tangent to the initial surface. We remark two differences with respect to the smooth spacelike Cauchy problem: First, note that $\overline{g}$ will not be Riemannian since the Cauchy surface admits a tangent null direction. That is, $\overline{g}$ is of the signature $(0, +, \dots, +)$. Second, we note that the connection κ is used as initial data instead of the extrinsic curvature K. This may seem to be quite a stark difference. However, as will be made more apparent later in this thesis, the connection coefficient κ corresponds uniquely with the extrinsic curvature of $\mathscr{N}$. This will be shown through analysis using the Raychaudhuri equation in the section on the constraints for the initial data.

Though we do not have a well-posedness theorem which is similarly strong to the one for the spacelike Cauchy surface, the use of the triple $(\mathscr{N}, \overline{g}, \kappa)$ can be used to show weaker results such as the existence of a solution to the vacuum field equations. Take for example (P. Chruściel, 2014), wherein existence of a solution in the neighborhood of the vertex of a lightcone is shown. It is for this reason that we will be using the triple $(\mathscr{N}, \overline{g}, \kappa)$ as initial data later in this thesis when lightcone-specific constraints are discussed.

3 Conformal metrics and compactifications

In the section on the causal structure and the Cauchy problem we saw that 'good' initial data for our initial value formulation of general relativity contained the metric as initial data on the Cauchy surface. As we'll see later in this thesis, in the case of the Cauchy problem on a lightcone, one may instead take a conformal class of metrics as initial data. For this reason, we introduce conformal metrics, and some of their properties in this section. Moreover, we introduce the Penrose conformal compactification for Minkowski space. This is a way of shrinking infinite flat spacetime and making it compact. In the process, we'll introduce a boundary to Minkowski spacetime.

3.1 Conformal compactifications, abstractly

A conformal compactification is a transformation which takes a general spacetime with noncompact manifold and turns its manifold into the interior of a new compact manifold. This kind of transformation changes metrics but keeps the causal structure of the spacetime intact. This way, the boundary of the new manifold maps to infinity in the original manifold. The study of asymptotic behavior on the original manifold is then tantamount to the study of the boundary of the new manifold. This section serves to formally define this idea and introduce Penrose-type conformal compactifications.

First, we introduce conformal metrics; let $\mathscr{M}$ be a manifold. The metrics g and $\tilde{g}$ are said to be *conformal* if there exists a nonzero scalar function Ω on $\mathscr{M}$ such that $\tilde{g} = \Omega^2 g$. We remark that this implies that the angle between vectors at a point $p \in \mathscr{M}$ is the same for both metrics. Indeed, the angle θ between the vectors $v, w \in T_p\mathscr{M}$ is defined by

$$\cos(\theta) = \frac{g(v,w)}{\sqrt{g(v,v)}\sqrt{g(w,w)}} = \frac{\hat{g}(v,w)}{\sqrt{\hat{g}(v,v)}\sqrt{\hat{g}(w,w)}}.$$

Another important property of conformal metrics is that they have the same causal structure. Recall that the definitions for the chronological past and future, causal past and future, and the past and future domain of dependence are only dependent on if there exist certain curves which are timelike, lightlike, or spacelike. Whether a curve is timelike, lightlike, or spacelike is in turn defined by the tangent vector to the curve. A tangent vector $v \in T_p\mathscr{M}$ which is timelike, lightlike, or respectively spacelike according to the metric g, is also timelike, lightlike, or spacelike with respect to the conformal metric $\tilde{g}$ since the sign of the norm of the vector is independent of the conformal factor Ω.[19] This implies that whether a curve is timelike, lightlike, or spacelike depends only on the conformal class of metrics. This does not mean that an arbitrary geodesic with respect to one metric is also geodesic with respect to a conformal metric. However, a null geodesic, considered as a point set with respect to a metric is also geodesic with respect to a conformal metric (Friedrich & Schmid, 1987). That is, given a null geodesic curve with respect to a metric g, if one instead considers the conformal metric $\tilde{g}$, one only needs to reparametrize the curve in order for it to become geodesic again.

[19] The sign of the norm of a vector v with respect to a metric g is given by $\frac{\|v\|_g}{|\|v\|_g|}$.

Now we are left to define compactifications. Intuitively, a compactification can be thought of as the addition of points to a noncompact space in order to make it compact. Let $\tilde{\mathscr{N}}$ and $\mathscr{N}$ be two spacetimes. Then $\tilde{\mathscr{N}}$ is called a *compactification* of $\mathscr{N}$, if $\tilde{\mathscr{N}}$ is a compact Hausdorff space and it equals the closure of $\mathscr{N}$.

The definition of conformal compactification is a natural result of these earlier definitions. That is, if a transformation between the spacetimes $(\mathscr{M}, g)$ and $(\overline{\mathscr{N}}, \tilde{g})$ given by $\phi : \mathscr{M} \to \mathscr{N}$, with $g = \Omega^2 \phi^* \tilde{g}$[20], $\phi(\mathscr{M}) = \mathscr{N}$ is the interior of $\overline{\mathscr{N}}$, and $\overline{\mathscr{N}}$ is a compactification of $\mathscr{N}$. Then this transformation is called a *conformal compactification.*

3.2 The Penrose conformal compactification of Minkowski spacetime

As a commonly used example, we may want to make a conformal compactification of Minkowski spacetime as is done in (Hawking and Ellis, 1973; chapter 5). The metric of 3+1 dimensional Minkowski spacetime, g, can be expressed as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

This metric can be re-expressed in advanced and retarded null-coordinates given by $v = t + r$ and $w = t - r$ respectively. Using $dv = dt + dr$, and $dw = dt - dr$, we obtain another expression for the metric.

$$\mathrm{d}s^2 = -\mathrm{d}v\mathrm{d}w + \frac{1}{4}(v - w)^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

Here we have $-\infty < w < v < \infty$. Making this space compact means that we have to find a way to add 'points at infinity'. To do this, we use another coordinate transformation to find a new expression for the same metric. We define $p = \arctan v$ and $q = \arctan w$. Note that we have $-\frac{\pi}{2} < q \leq p < \frac{\pi}{2}$. This yields the following expression for the metric g.

$$\mathrm{d}s^2 = \sec^2 p \sec^2 q(-\mathrm{d}p\mathrm{d}q + \frac{1}{4}\sin^2(p - q)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

We see that g is conformal to the metric $\hat{g}$, given by

$$\mathrm{d}\hat{s}^2 = -4\mathrm{d}p\mathrm{d}q + \sin^2(p - q)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2).$$

For this transformation, the function Ω from the definition of conformal metrics is given by $\Omega = 2\cos p \cos q$ ($\hat{g} = \Omega^2 g$). We can transform back to more intuitive coordinates by defining $t' = p + q$ and $r' = p - q$. The metric $\hat{g}$ is then expressed as

$$\mathrm{d}\hat{s}^2 = -(\mathrm{d}t')^2 + (\mathrm{d}r')^2 + \sin^2 r'(\mathrm{d}\theta^2 + \sin^2 \mathrm{d}\phi^2)$$

Thus, Minkowski space can be fully described by the space

$$-\pi < t' + r' < \pi, \quad -\pi < t' - r' < \pi, \quad r' \geq 0 \tag{3.1}$$

with

$$\mathrm{d}\hat{s}^2 = 4\cos^2(\frac{1}{2}(t' + r'))\cos^2(\frac{1}{2}(t' - r'))\mathrm{d}s^2$$

[20]that is, g is conformal to $\phi^*\tilde{g}$

wherein t and r are related to t' and r' by

$$\begin{aligned}
2t &= \tan(\frac{1}{2}(t'+r')) + \tan(\frac{1}{2}(t'-r')) \\
2r &= \tan(\frac{1}{2}(t'+r')) - \tan(\frac{1}{2}(t'-r'))
\end{aligned}$$

The space given by (3.1) can be made compact by adding points at $t'+r'=\pi$ and $t'-r'=-\pi$. Note that these added points are actually added spheres as θ and ϕ are free to take values in their respective intervals. These added points, or spheres, are the aforementioned 'points at infinity'.

The set of 'points at infinity' is called the *causal boundary of Minkowski spacetime*. We differentiate between subsets of this boundary by considering limits of geodesic curves with respect to the metric g. Suppose we consider a point $p \in \mathscr{M}$ and a future directed inextendible null geodesic $\gamma(s)$ that intersects p. Then, if we consider the parameter limit of this geodesic, we find that this limit corresponds with a point on the causal boundary. The *future boundary*, consists of the limit points of future directed curves. One may also show that a differentiation can be made between the future limit points of null, and timelike curves. The sets for the future null, and respectively timelike curves are called *future timelike infinity* and *future null infinity*. They are denoted by i^+, and respectively $\mathscr{I}^+$:

$$i^+ = \left\{ p \in \overline{\mathscr{N}} \;\middle|\; \begin{array}{l} \text{there exists a future directed timelike geodesic with} \\ \text{respect to the metric } g,\, \gamma(t) \text{ such that } \displaystyle\lim_{t\to\infty}\gamma(t) = p \end{array} \right\}$$

and

$$\mathscr{I}^+ = \left\{ p \in \overline{\mathscr{N}} \;\middle|\; \begin{array}{l} \text{there exists a future directed null} \\ \text{geodesic } \gamma(s) \text{ such that } \displaystyle\lim_{s\to\infty}\gamma(s) = p \end{array} \right\}.$$

Past null infinity, $\mathscr{I}^-$ and *past timelike infinity*, i^- are defined analogously. We further define *spacelike infinity* i^0 analogously to the definition of i^+ where we change 'future directed timelike geodesic' for 'spacelike geodesic'. It can then be shown that $\mathscr{I}^+$ consists of the null surface $p=\frac{\pi}{2}$, $\mathscr{I}^-$ consists of the null surface $q=-\frac{\pi}{2}$, $i^+ = \{(p,q) = (\frac{\pi}{2},\frac{\pi}{2})\}$, $i^- = \{(p,q) = (-\frac{\pi}{2},-\frac{\pi}{2})\}$, and $i^0 = \{(p,q) = (\frac{\pi}{2},-\frac{\pi}{2})\}$.

One can then consider any timelike geodesic to begin in i^- and end in i^+. Similarly, null geodesics can be thought of as beginning in $\mathscr{I}^-$ and ending in $\mathscr{I}^+$. We remark that non geodesic curves need not adhere to these observations. For example, one may consider the timelike curve in (t,r,θ,ϕ)-coordinates given by $\gamma(t) = (t, \sqrt{t^2+1}, 0, 0)$. This timelike curve begins in $\mathscr{I}^-$ and ends in $\mathscr{I}^+$.[21] Since the causal structure of the conformally completed spacetime remains unchanged, the curve $\gamma(t)$ remains timelike when each point is transformed to the spacetime $(\mathscr{N}, \hat{g})$.

[21] In (v,w)-coordinates this curve is expressed as $(t+\sqrt{t^2+1}, t-\sqrt{t^2+1})$, where we suppress the θ and ϕ coordinate. That is, the limit for $t\to\infty$ in (p,q) coordinates is given by $(\frac{\pi}{2},0) \in \mathscr{I}^+$ and the limit for $t\to-\infty$ is given by $(0,-\frac{\pi}{2}) \in \mathscr{I}^-$.

4 Null geodesics through a shared point

In this section, we derive the well known Raychaudhuri equation for lightlike geodesics in $n+1$ dimensional spacetime. Conjugate points are introduced and analyzed through this equation which will later serve to analyze smoothness of the space generated by the family of lightlike geodesics through a shared point in spacetime. That is, it will be used to analyze smoothness of a lightcone outside of the vertex.[22] We will see that a smooth lightcone[23] is a lightcone for which the null geodesics that generate it do not intersect. The Raychaudhuri equation will also serve as a physical constraint equation for the initial data, or alternatively, it will be used to complete initial data, depending on the form of data which is used.

4.1 The Raychaudhuri equation in arbitrary dimensions

This subsection contains a derivation of the Raychaudhuri equation in $n+1$ dimensions. This section is heavily based on (Hawking and Ellis, 1973; chapter 4) wherein the equation is derived in $3+1$ dimensions.

We consider a congruence of future directed lightlike geodesics. By a *congruence* of future directed lightlike geodesics, we mean a family of future directed null geodesic curves such that each point of $\mathscr{M}$ has precisely one such curve passing through it. We let k be the tangent vector field to the congruence of null curves. Suppose $\gamma(s)$ is a null geodesic in the congruence. We take a point $p \in \gamma$, and introduce a basis for the tangent space $T_p\mathscr{M}$, $\{e_0, e_1, \ldots, e_n\}$ wherein the vectors e_0, e_1 are null vectors such that $e_0 \equiv k$, e_1 is chosen such that it is lightlike, and $g(e_0, e_1) = -1$. Note then, that e_1 is also future directed, and for *any* linearly independent null direction to e_0 such a vector e_1 can be chosen.[24] The other basis vectors e_A are spacelike and are chosen such that they are normalized and orthogonal to every other basis vector. That is, $g(e_A, e_\alpha) = 0$, where $A \in \{2, 3, \cdots, n\}$ and $\alpha \in \{0, 1, \cdots, n\} \setminus \{A\}$. Such a basis is called a *pseudo-orthonormal basis.*[25] We have now introduced the pseudo-orthonormal basis for one point along the null geodesic γ. If we parallel transport the vectors $\{e_\mu\}$ along the geodesics, we obtain a pseudo-orthonormal basis everywhere along $\gamma(s)$.[26] We may therefore express the metric tensor $\hat{g}$ along one geodesic curve γ for this basis in matrix notation:

$$\hat{g} = \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{4.1}$$

We are interested in studying the way in which neighboring geodesics behave with respect to each other. Suppose $\lambda(t)$ is a spacelike curve through the lightlike

[22] A lightcone can only be smooth in the points that are not its vertex.

[23] That is, a lightcone which is smooth everywhere except for its vertex.

[24] If e_1 is null and it is not a scalar multiple of e_0 we have that $g(e_0, e_1) < 0$. Now define $\hat{e}_1$ by $\hat{e}_1 \equiv e_1 / -g(e_0, e_1)$. Then $g(e_0, \hat{e}_1) = -1$ and $\hat{e}_1$ trivially has the same direction as e_1.

[25] Note that an orthonormal basis cannot be chosen if one uses null vectors since they have zero arc-length and can therefore not be normalized.

[26] Moreover, since k is geodesic, we find that it remains the tangent vector to the null geodesic curve.

geodesics with tangent vector $z = \partial\lambda/\partial t$.[27] Then, one may construct a family of curves $\lambda(t,s)$ where $\lambda(t,s)$ is the curve $\lambda(t)$ for which every point has been moved a parameter distance s along each curve in the congruence it passes through. For such a separation vector z corresponding to a given value t in $\lambda(t)$, one may consider the vector $z(s)$ as the vector z which is transported along γ for a parameter distance s. z may be viewed as the separation between two of these neighboring null curves. Since we are only interested in the separation between neighboring integral curves of e_0, we are only considering the part of the separation vector z that has zero e_0 component. The tangent space in which z takes nonzero value is therefore spanned by the collection $\{e_1, \dots e_n\}$.

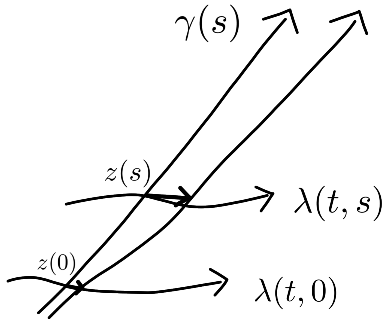

Figure 3: The construction of the separation z between a geodesic $\gamma(s)$ and a neighboring geodesic. Note that we only drew the $\gamma(s)$ curve and one neighboring geodesic, when in actuality we consider a congruence of lightlike geodesics.

4.1.1 The Lie derivative of a tensor

To study the evolution of the separation between null geodesics, we formally introduce the Lie derivative for tensors. Suppose one has a smooth nonvanishing vector field x. Then there exist a unique maximal integral curve $\phi(t)$ of x associated with each point (Burkill, 1956). As the derivative of this curve is smooth (the field x is smooth), the integral curve is smooth as well. Now let us define a map ϕ_t which takes a point p on the integral curve ϕ a parameter distance t along this curve $(p \mapsto \phi(\phi^{-1}(p)+t))$. Then the inverse of ϕ_t is trivially given by ϕ_{-t}. Both ϕ_t, and ϕ_{-t} are smooth. This implies that ϕ is a diffeomorphism. Thus, one can define the pullback of tensors along the function ϕ. We define the *Lie derivative* of a tensor T along the vector x at the point p in the following manner:

$$\pounds_x T|_p = \lim_{t\to 0} \frac{1}{t}(\phi_t^* T|_{\phi_t(p)} - T|_p) \tag{4.2}$$

The Lie derivative $\pounds$ of a vector v may also be expressed in coordinates in the following way (see appendix A.4).

$$\pounds_x v^a = v^a{}_{;b} x^b - x^a{}_{;b} v^b$$

The Lie derivative is a coordinate independent expression which makes it a natural choice to study the development of tensors along integral curves on the manifold.

[27]That is, the tangent vector along the curve $\lambda(t)$ in the positive t direction. It is sometimes written as $\dot{\lambda}$ in the literature. Note that t is simply the parameter of the λ-curve. It should not be interpreted as signifying time since the curve $\lambda(t)$ is spacelike.

If we shift our attention back to our construction with the γ curve and z separation vector, we may asses the expression for the change in separation along the integral curve to e_0, $\pounds_{e_0} z$.

$$\pounds_{e_0} z = \lim_{s\to 0} \frac{1}{s}(\gamma_s^* z(s) - z(0))$$

Here the diffeomorphism γ_s is analogous to ϕ_t in the lead up to the definition of the Lie derivative of a tensor. Note that $z(0) = \gamma_s^* z(s)$ which yields a vanishing Lie derivative: $\pounds_{e_0} z = 0$. Expressing this in coordinates gives us

$$\frac{\mathrm{D}}{\mathrm{d}s} z^\alpha = k^\alpha{}_{;\beta} z^\beta, \tag{4.3}$$

where $\frac{\mathrm{D}}{\mathrm{d}s}$ is the covariant derivative operator along $k \equiv e_0$. For this covariant derivative, the vector z is extended to a vector field on a neighborhood of γ.[28] We also obtain

$$\frac{\mathrm{D}^2}{\mathrm{d}s^2} z^\alpha = -R^\alpha{}_{\beta\gamma\delta} z^\gamma k^\beta k^\delta \tag{4.4}$$

as an equation for the second covariant derivative of the components of z along γ. This equality follows from (4.3) by the following calculation:

$$\begin{aligned}\frac{\mathrm{D}^2}{\mathrm{d}s^2} z^\alpha = \frac{\mathrm{D}}{\mathrm{d}s}\left(\frac{\mathrm{D}}{\mathrm{d}s} z^\alpha\right) = \frac{\mathrm{D}}{\mathrm{d}s}\left(k^\alpha{}_{;\beta} z^\beta\right) = \nabla_k \nabla_z k^\alpha = \nabla_k \nabla_z k^\alpha - \nabla_z \nabla_k k^\alpha - \nabla_{[k,z]} k \\ = -R^\alpha{}_{\beta\gamma\delta} z^\gamma k^\beta k^\delta\end{aligned}$$

Here we used $\nabla_k k^\alpha = 0$ since the basis vector k is geodesic. We also use $[k, z] \equiv \pounds_{e_0} z = 0$ which implies that $\nabla_{[k,z]} k = 0$.

Note that k is the tangent vector to a geodesic curve. This means that $k^\alpha{}_{;0}$ is zero.[29] As stated previously, we are interested in vectors z that have the component $z^0 = 0$. Having $k^\alpha{}_{;0} = 0$ ensures that such a z vector remains having zero e_0 component along the geodesics. This allows us to re-express equations (4.3) and (4.4) as systems of n ordinary differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}s} z^a = k^a{}_{;b} z^b \tag{4.5}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} z^a = -R^a{}_{0b0} z^b \tag{4.6}$$

where we use the convention that Latin indices take the values $1, 2, \ldots, n$. We zoom in on equation (4.5). Since the vector k is defined to be the tangent vector to a congruence of null geodesics on the entire spacetime $\mathscr{M}$ we see that $g(k, k) = 0$ everywhere, that is, $(g_{\alpha\beta} k^\alpha k^\beta)_{;\gamma} = 0$. From this, one finds that $k^1{}_{;\gamma} = 0$ since $k^\alpha \equiv \delta^\alpha_0$. Now consider z^1. Substituting 1 for a in (4.5) shows z^1 to be constant. One may interpret this as showing that two light rays, emitted at different times have

[28] The covariant derivative is defined as a map $\nabla : \mathrm{Vec}\mathscr{M} \times \mathrm{Vec}\mathscr{M} \to \mathrm{Vec}\mathscr{M}$ where $\mathrm{Vec}\mathscr{M}$ is the collection of vector fields on $\mathscr{M}$. A vector field on γ may not be defined on $\mathscr{M}$, however it can always be extended to a neighborhood of γ. Moreover, the extension chosen has no effect on the value of this derivative.

[29] k only has a component in the e_0 direction and this component is unity.

a constant time separation. We are therefore only left interested in the remaining $2(n-1)$ differential equations

$$\frac{\mathrm{d}}{\mathrm{d}s} z^A = k^A{}_{;B} z^B, \tag{4.7}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} z^A = -R^A{}_{0B0} z^B, \tag{4.8}$$

wherein we use the convention that capital Latin indices correspond to the separation in the spacelike directions; they take the values $2, 3, \ldots, n$.

4.1.2 Vorticity and expansion

Since (4.7), and (4.8) are linear differential equations, we may express the components z^A of the separation vector z at a parameter location s along γ as a linear combination of their values at some point q:

$$z^A(s) = M_{AB}(s) z^B|_q \tag{4.9}$$

Where M_{AB} is the corresponding $n-1 \times n-1$ matrix. This matrix must then satisfy the expressions[30]

$$\frac{\mathrm{d}}{\mathrm{d}s} M_{AB}(s) = k_{A;C} M_{CB}(s), \tag{4.10}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} M_{AB}(s) = -R_{A0C0} M_{CB}(s). \tag{4.11}$$

Wherein the sum over C is taken on the right hand side of the expressions. The matrix M may be written as the product of an orthogonal matrix O with positive determinant and a symmetric matrix S.

$$M = OS \iff M_{AB} = O_{AC} S_{CB}$$

Here the orthogonal matrix O can be thought of as representing the rotation of neighboring curves and the symmetric matrix S can be considered to represent the separation between neighboring curves. Both the matrix O and S are chosen to be the identity at $q \in \mathscr{M}$. At q we then have that $\frac{\mathrm{d}O}{\mathrm{d}s}$ is antisymmetric and $\frac{\mathrm{d}S}{\mathrm{d}s}$ is symmetric. Evaluating equation (4.10) at $s = \gamma^{-1}(q)$ lets us express the rotation of neighboring curves as the antisymmetric part of $k_{A;B}$ and the rate of change of their separation is given by the symmetric part. We define the *vorticity tensor* as

$$\omega_{AB} = k_{[A;B]},$$

and the *expansion tensor* as

$$\theta_{AB} = k_{(A;B)}. \tag{4.12}$$

We further define the *divergence* as the trace of the expansion tensor.

$$\theta = g^{AB}\theta_{AB} = k^A{}_{;A}$$

[30] $\left(\frac{\mathrm{d}}{\mathrm{d}s} M_{AB}\right) z^B|_q = \frac{\mathrm{d}}{\mathrm{d}s} M_{AB} z^B|_q = \frac{\mathrm{d}}{\mathrm{d}s} z^A \overset{(4.7)}{=} k^A{}_{;C} z^C = k^A{}_{;C} M_{CB} z^B|_q \implies \frac{\mathrm{d}}{\mathrm{d}s} M_{AB}(s) = k^A{}_{;C} M_{CB}(s)$, we may lower the A index as $g^{AA} \equiv 1$ which can be seen in (4.1). Expression (4.11) is found using the same steps as above except one uses equation (4.8) instead of (4.7).

The expansion and divergence are sometimes denoted by τ in the literature. The divergence shows the rate at which the geodesics diverge from each other. The *shear tensor* σ_{AB} is defined as the traceless part of θ_{AB}.

$$\sigma_{AB} = \theta_{AB} - \frac{1}{n-1}\theta g_{AB} \tag{4.13}$$

We may then express the covariant derivative of k in terms of these quantities.

$$k_{A;B} = \omega_{AB} + \sigma_{AB} + \frac{1}{n-1}\theta g_{AB}$$

Consider equation (4.10). The left and right hand side of this equation are $n-1 \times n-1$ matrices. That is, the right hand side is simply a matrix product between the covariant derivative matrix $k_{(\,\cdot\,);(\,\cdot\,)}$ and the matrix M. One may express the covariant derivatives as

$$k_{A;B} = \left[\left(\frac{\mathrm{d}}{\mathrm{d}s}M\right)M^{-1}\right]_{AB} = M^{-1}_{CB}\frac{\mathrm{d}}{\mathrm{d}s}M_{AC} \tag{4.14}$$

Using this, we may give a matrix expression of the vorticity and expansion.

$$\omega_{AB} = k_{[A;B]} = \frac{1}{2}M^{-1}_{CB}\frac{\mathrm{d}}{\mathrm{d}s}M_{AC} - \frac{1}{2}M^{-1}_{CA}\frac{\mathrm{d}}{\mathrm{d}s}M_{BC} = -M^{-1}_{C[A}\frac{\mathrm{d}}{\mathrm{d}s}M_{B]C} \tag{4.15}$$

and analogously

$$\theta_{AB} = k_{(A,B)} = M^{-1}_{C(A}\frac{\mathrm{d}}{\mathrm{d}s}M_{B)C}. \tag{4.16}$$

The divergence can then be expressed as

$$\theta = g^{AA}k_{A;A} \overset{(4.1)}{=} k_{A;A} = \operatorname{tr}\left[\left(\frac{\mathrm{d}}{\mathrm{d}s}M\right)M^{-1}\right] = \operatorname{tr}\left[M^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}s}M\right)\right]$$
$$= (\det M)^{-1}\frac{\mathrm{d}}{\mathrm{d}s}(\det M) \tag{4.17}$$

In equation (4.17) we keep the sum over A in the second equality.

Recall that we wanted to express the evolution of the separation between neighboring lightlike geodesics. Now that we introduced the vorticity, expansion, shear, and divergence as another way to view the separation, we wish to write differential equations for these quantities. We begin by considering the development of the vorticity ω_{AB} by considering equation (4.15). To analyze the development equations for the vorticity and expansion we make use of matrix notation. For a square matrix A let $A_{()} \equiv \frac{1}{2}(A + A^T)$, and respectively $A_{[]} \equiv \frac{1}{2}\left(A - A^T\right)$ denote the *symmetric*, and *antisymmetric* part of A. We remark that the derivative of the transpose of a matrix is the transpose of the derivative of the matrix, i.e., $\frac{\mathrm{d}}{\mathrm{d}s}\left(A^T\right) = \left(\frac{\mathrm{d}}{\mathrm{d}s}A\right)^T$. From this remark, it follows that the derivative of the (anti-)symmetric part of a matrix is the (anti-)symmetric part of the derivative of a matrix $\frac{\mathrm{d}}{\mathrm{d}s}\left(A_{()}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}s}A\right)_{()}$, $\frac{\mathrm{d}}{\mathrm{d}s}\left(A_{[]}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}s}A\right)_{[]}$. For the following derivations we use the hat-symbol, "^", to denote the corresponding matrix to the vorticity, expansion and shear tensor. Furthermore, let $\hat{R}$ denote the matrix of the tensor components of R_{A0B0} such that $\hat{R}_{AB} = R_{A0B0}$. If we take the derivative of

$\hat{\omega} = \left(\frac{\mathrm{d}}{\mathrm{d}s} M M^{-1}\right)_{[]}$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\hat{\omega} &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\left[\frac{\mathrm{d}}{\mathrm{d}s}M\right]M^{-1}\right)_{[]} = \left(\frac{\mathrm{d}M}{\mathrm{d}s}\frac{\mathrm{d}M^{-1}}{\mathrm{d}s}\right)_{[]} + \left(\frac{\mathrm{d}^2M}{\mathrm{d}s^2}M^{-1}\right)_{[]} \\ &= -\left(\frac{\mathrm{d}M}{\mathrm{d}s}M^{-1}\frac{\mathrm{d}M}{\mathrm{d}s}M^{-1}\right)_{[]} + \left(\frac{\mathrm{d}^2M}{\mathrm{d}s^2}M^{-1}\right)_{[]} = -\left[\left(\hat{\theta}+\hat{\omega}\right)\left(\hat{\theta}+\hat{\omega}\right)\right]_{[]} + \hat{R}_{[]}\end{aligned}$$

We use the identity $\frac{\mathrm{d}M^{-1}}{\mathrm{d}s} = -M^{-1}\frac{\mathrm{d}M}{\mathrm{d}s}M^{-1}$ in the third equality. We further use the matrix expression for the covariant derivative of the separation (4.14), to arrive at two terms in the final expression. The first term simplifies to $-\hat{\omega}\hat{\theta} - \hat{\theta}\hat{\omega}$ after multiplying and taking the antisymmetric part. In components notation this can be written as $2\omega_{C[A}\theta_{B]C}$. The second term vanishes by the interchange symmetry of the Riemann tensor ($R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$):

$$\hat{R}_{[]AB} = \hat{R}_{[AB]} = R_{[A0B]0} = 0$$

We are left with the following expression for the vorticity

$$\frac{\mathrm{d}}{\mathrm{d}s}\omega_{AB} = 2\omega_{C[A}\theta_{B]C}. \tag{4.18}$$

We see that the vorticity must be identically zero if it vanishes somewhere along the geodesic. This yields the following proposition:

Proposition 4.1. *Let γ be a lightlike geodesic in a congruence of lightlike geodesics such that $k = e_0$ is the tangent vector to γ. We use the pseudo-orthonormal basis $\{e_0, \ldots, e_n\}$. If the vorticity ω, defined by $\omega_{AB} = k_{[A;B]}$ ($A, B \in \{2, \ldots, n\}$), vanishes at a point along γ, then the vorticity is identically zero everywhere along the geodesic.*

Equation (4.18) is a linear homogeneous differential equation in ω_{AB}, so $\omega_{AB} = 0$ is a solution. Because this equation is a first order linear differential equation, the solution is uniquely determined by the value of ω_{AB} at a point. As ω_{AB} is assumed to vanish at some point along γ, it follows that $\omega_{AB} = 0$ is the only solution along the entire geodesic, which proves the proposition. □

We will use proposition 4.1 to show identical disappearance of the vorticity in the case where we concentrate on a family of intersecting geodesics. This will in turn be used to analyze the initial data for the characteristic Cauchy problem on a lightcone.

A similar analysis of the matrix associated with the expansion tensor yields:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\hat{\theta} &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\left[\frac{\mathrm{d}}{\mathrm{d}s}M\right]M^{-1}\right)_{()} = \left(\frac{\mathrm{d}M}{\mathrm{d}s}\frac{\mathrm{d}M^{-1}}{\mathrm{d}s}\right)_{()} + \left(\frac{\mathrm{d}^2M}{\mathrm{d}s^2}M^{-1}\right)_{()} \\ &= -\left(\frac{\mathrm{d}M}{\mathrm{d}s}M^{-1}\frac{\mathrm{d}M}{\mathrm{d}s}M^{-1}\right)_{()} + \left(\frac{\mathrm{d}^2M}{\mathrm{d}s^2}M^{-1}\right)_{()} = -\left[\left(\hat{\theta}-\hat{\omega}\right)\left(\hat{\theta}-\hat{\omega}\right)\right]_{()} + \hat{R}_{()}\end{aligned}$$

The first term in the final expression simplifies to $\hat{\theta}^2 + \hat{\omega}^2$ after multiplication and taking the symmetric part.[31] To analyze the components of the second term, we

[31] $\hat{\theta}$ and $\hat{\omega}$ are respectively symmetric and skew-symmetric.

recall equation (4.11) which yields the components $-R_{A0B0}$.[32] In components, this gives

$$\frac{\mathrm{d}}{\mathrm{d}s}\theta_{AB} = -R_{A0B0} - \theta_{AC}\theta_{CB} - \omega_{AC}\omega_{CB} \tag{4.19}$$

This equation captures the evolution of the expansion tensor along the geodesic. It is used to derive the evolution equation for the divergence. To arrive at the evolution equation for the divergence, we take the trace of (4.19). On the right hand side we have three terms. The first term may be rewritten as $-R_{A\mu B\nu}k^\mu k^\nu$. The trace of this expression gives us the coordinate independent Ricci tensor term $R_{\mu\nu}k^\mu k^\nu$. When taking the trace of the vorticity, and expansion tensor, we use expression (4.1) for the metric g,[33] and the definition of the shear (4.13) to rewrite the second term as

$$\begin{aligned} g^{AB}\theta_{AC}\theta_{CB} &= g^{AB}(\sigma_{AC} + \frac{\theta}{n-1}g_{AC})(\sigma_{CB} + \frac{\theta}{n-1}g_{CB}) \\ &= g^{AB}(\sigma_{AC}\sigma^C{}_B) + g^{AB}\frac{\theta^2}{(n-1)^2}g_{AB} = \sigma_{AB}\sigma^{AB} + \frac{\theta^2}{n-1}, \end{aligned}$$

where we use the symmetry of σ ($\sigma^{BA} = \sigma^{AB}$) in the second equality and relabel the indices in the third. We analyze the vorticity term from (4.19) in a similar manner. Using the skew-symmetry of the vorticity, $\omega_{BA} = -\omega_{AB}$, we find

$$g^{AB}\omega_{AC}\omega_{CB} = g^{AB}\omega_{AC}\omega^C{}_B = \omega_{BA}\omega^{AB} = -\omega_{AB}\omega^{AB},$$

wherein we again use expression (4.1) for the metric along γ to rewrite the matrix product $\omega_{AC}\omega_{CB}$ in components as the contraction $\omega_{AC}\omega^C{}_B$. We define $\sigma^2 \equiv \sigma_{AB}\sigma^{AB} \geq 0$ and $\omega^2 \equiv \omega_{AB}\omega^{AB} \geq 0$. Taking the trace of equation (4.19) thus yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\theta = -R_{\alpha\beta}k^\alpha k^\beta + \omega^2 - \sigma^2 - \frac{\theta^2}{n-1}. \tag{4.20}$$

This equation is known as the Raychaudhuri equation for a congruence of lightlike geodesics which may simply be called the Raychaudhuri equation for the purposes of this thesis. We remark that in the vacuum case ($R_{\alpha\beta} = 0$), the only term on the right hand side of equation (4.20) which may cause divergence of the collection of null geodesics is the vorticity tensor. That is, if the vorticity tensor is identically zero, one finds that the divergence of the congruence of geodesics must stay constant, slow down, or revert. Another important thing to remark is that the parameters in the Raychaudhuri equation depend on the choice of γ, and λ. However, the sign of the parameters is not curve dependent. It is the implication of a positive or negative sign of the parameters which will be analyzed in the remainder of this section.

4.2 The null convergence condition

The Einstein field equations $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$ consist of two parts. The left hand side is a nonlinear partial differential combination of components of the

[32] In the derivation of the ODE for the vorticity (4.18), we found that the antisymmetric part of this matrix must vanish. i.e., the matrix $\hat{R}$ is symmetric.

[33] The metric along the geodesic implies that for any arbitrary tensor τ with coordinates τ_{AB} we have $\tau_{AB} = \tau^A{}_B = \tau^{AB}$.

metric tensor whereas the right hand side captures the distribution of types of energy and momentum throughout the spacetime with the energy-momentum tensor. Through this equation one may hope to be able to qualitatively analyze the Ricci tensor by analysis of the energy-momentum tensor. In this section, we postulate a condition for the energy-momentum tensor which is in turn used to further analyze the development of the congruence of null geodesics. We paraphrase the weak energy condition and the null convergence condition from (Hawking and Ellis, 1973; chapter 4).

The *weak energy condition* is a condition on the energy-momentum tensor. It states that the measured energy density by any observer traveling through spacetime in a timelike direction[34] is nonnegative at any point $p \in \mathscr{M}$. Mathematically, this may be expressed as

$$T_{\alpha\beta}v^\alpha v^\beta \geq 0 \tag{4.21}$$

where v is any timelike vector in $T_p\mathscr{M}$.[35] If (4.21) holds for any timelike vector v, then, by continuity, it must also hold when v is null. Now if we let k be a null vector at a point $p \in \mathscr{M}$ we find that Einstein's field equations give us

$$R_{\mu\nu}k^\mu k^\nu = R_{\mu\nu}k^\mu k^\nu - \frac{1}{2}Rg_{\mu\nu}k^\mu k^\nu = 8\pi G T_{\mu\nu}k^\mu k^\nu \geq 0.$$

For the first equality we used that $g_{\mu\nu}k^\mu k^\nu = 0$ since k is null. The obtained expression

$$R_{\mu\nu}k^\mu k^\nu \geq 0 \tag{4.22}$$

is referred to as the *null convergence condition*. This is a fitting name as it ensures that the rate of change in the divergence of null geodesics may only decrease when the vorticity is identically zero.[36] Thus, if the weak energy condition holds, curvature in spacetime will have a converging (or more precisely non-diverging) effect on null geodesics.[37] This statement will be made more precise in the section on conjugate points.

4.3 Conjugate points of null geodesics

In the section on the Raychaudhuri equation we studied the behaviour of a congruence of null geodesics. In this section, we concentrate on a description of null geodesics which share a point with their neighboring curves. We found that the separation vector between a null geodesic and a neighboring curve satisfied the Jacobi equation (4.8):

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}z^A = -R^A{}_{0B0}z^B \tag{4.23}$$

A solution of this equation will be called a *Jacobi field* along the null geodesic $\gamma(s)$.

In the derivation of the Raychaudhuri equation, we considered a congruence of null geodesics. Recall that this means that for every point in the spacetime $(\mathscr{M}, g)$ there is a unique null geodesic intersecting it. Thus, the separation at any chosen

[34] i.e., when traveling slower than light.

[35] the local energy density to an observer with tangent vector v is given by $T_{\alpha\beta}v^\alpha v^\beta$.

[36] This follows directly from the Raychaudhuri equation (4.20).

[37] It is clear to see that the Ricci tensor has this effect from its occurrence in the Raychaudhuri equation. The other components to the curvature may also only have a nondiverging effect since they may influence the negative terms $-\theta^2$, and $-\sigma^2$ which could cause additional convergence.

point could be found through the separation at a given reference point. That is to say, we did not need to make reference to the derivative of z at the reference point in our matrix-vector-product-expression (4.9) of $z(s)$ since the derivative is determined by the congruence.

In the case where two infinitesimally separated null geodesics share a point, say $p \in \gamma \subset \mathscr{M}$, we have that the separation vector z disappears in the point p. In contrast, the separation vector is not everywhere zero since infinitesimally neighboring null geodesics of interest generally do not coincide everywhere. If we decide to choose p to be our reference point, and we want to find a quantity in p which differentiates between the neighboring geodesics, we find that the value of the separation vector z in p does not suffice. However, we may uniquely determine the neighboring geodesics by considering the derivative of the separation vector in p. A Jacobi field along $\gamma(s)$ which vanishes at a point $p \in \gamma$ can then be fully described by its derivatives at p. This field may then be written as

$$z^A(s) = M_{AB}(s) \frac{\mathrm{d}}{\mathrm{d}s} z^B|_p \tag{4.24}$$

where we earlier found

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} M_{AB} = -R_{A0C0} M_{CB}.$$

At p we have $z^A = 0$ which implies that the matrix M vanishes there. The vorticity, shear and divergence are defined the same way as in the section on the Raychaudhuri equation.

We recall that, as long as the null convergence condition (4.22) holds, that the vorticity ω is the only term in the Raychaudhuri equation (4.20) which may cause divergence of the lightlike geodesics. Moreover, by proposition 4.1, we saw that the vorticity must be identically zero if it disappears somewhere along a studied null geodesic. In the case of lightlike geodesics through a shared point, we can use proposition 4.1 to arrive at the following useful proposition.

Proposition 4.2. *Let $p \in \mathscr{M}$, and consider the family of lightlike geodesics through p. We take a pseudo-orthonormal basis $\{e_0, \ldots, e_n\}$ along these geodesics such that $k = e_0$ is the tangent vector to the geodesics. If the vorticity is defined by $\omega_{AB} = k_{[A;B]}$, where $A, B \in \{2, \ldots, n\}$, along these geodesics, then ω_{AB} identically vanishes along each geodesic curve.*

To prove this proposition, we consider the derivative of the expression $M^T \hat{\omega} M$ along one of the lightlike geodesics $\gamma(s)$ through $p \in \mathscr{M}$. If we define the $n-1 \times n-1$ matrix $\hat{k}_;$ to be the matrix with $k_{A;B}$ as its A, B-th entry we find by (4.14), (4.15), and (4.16) that we may express $\hat{k}_;$ as

$$\hat{k}_; = \left(\frac{\mathrm{d}}{\mathrm{d}s} M\right) M^{-1} = \hat{\theta} + \hat{\omega}.$$

We use these expressions to rewrite the $\frac{\mathrm{d}M}{\mathrm{d}s}$ terms in the derivative of $M^T \hat{\omega} M$ along

$\gamma(s)$. This yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\left(M^T\hat{\omega}M\right) &= \frac{\mathrm{d}M^T}{\mathrm{d}s}\hat{\omega}M + M^T\frac{\mathrm{d}\hat{\omega}}{\mathrm{d}s}M + M^T\hat{\omega}\frac{\mathrm{d}M}{\mathrm{d}s}\\ &= M^T\hat{k}_;^T\hat{\omega}M + M^T(\hat{\omega}\hat{\theta}+\hat{\theta}\hat{\omega})M + M^T\hat{\omega}\hat{k}_;M\\ &= M^T\left(\hat{k}_;^T\hat{\omega}+\hat{\omega}\hat{\theta}+\hat{\theta}\hat{\omega}+\hat{\omega}\hat{k}_;\right)M,\end{aligned}$$

wherein we also used the evolution of the vorticity (4.18) in the second equality. We analyze the middle factor in the final expression to obtain

$$\hat{k}_;^T\hat{\omega}-\hat{\omega}\hat{\theta}-\hat{\theta}\hat{\omega}+\hat{\omega}\hat{k}_; = \hat{\theta}\hat{\omega}+\hat{\omega}^T\hat{\omega}-\hat{\omega}\hat{\theta}-\hat{\theta}\hat{\omega}+\hat{\omega}\hat{\theta}+\hat{\omega}\hat{\omega}=0,$$

where we use the skew symmetry of $\hat{\omega}$ in the last equality. Thus we obtain $\frac{\mathrm{d}}{\mathrm{d}s}\left(M^T\hat{\omega}M\right)=0$ like we wanted. Recall that M is zero in the shared point of the neighboring geodesics since the separation vanishes at this point. We thus found that $M^T\hat{\omega}M=0$ everywhere along the geodesic. If we consider a point outside p for which the separation does not vanish, we have that M is nonsingular. This implies that we may multiply $M^T\hat{\omega}M$ by $(M^T)^{-1}$ on the left side and by M^{-1} on the right side to obtain $\hat{\omega}=0$ for all points that are not conjugate to p. Moreover, by proposition 4.1, we have that the vorticity must vanish everywhere if it vanishes at a point. This yields that the vorticity is zero, even at the point p. We therefore find that the vorticity vanishes everywhere along the geodesic γ, which finishes the proof. □

This proposition will be very useful in simplifying the Raychaudhuri equation in the case of initial data on a lightcone since a lightcone can be understood as the family of null geodesics through its vertex.

We define a point $q\in\gamma$ to be *conjugate to p along $\gamma(s)$* if there exists a Jacobi field which vanishes at p and q and which is not identically zero. Conjugate points, thus, are points where infinitesimally close geodesics intersect. Note that there need not actually be two separate geodesics that intersect for there to be conjugate points. As there exists a vector $\frac{\mathrm{d}}{\mathrm{d}s}z|_p$ for which $M(s)\frac{\mathrm{d}}{\mathrm{d}s}z|_p=0$ at q, M must have the eigenvalue zero at q. That is, the Jacobi fields along $\gamma(s)$, which vanish at q, are described by the matrix M which is singular at q. By (4.17), the divergence may be expressed as

$$\theta\equiv(\det M)^{-1}\frac{\mathrm{d}}{\mathrm{d}s}(\det M),$$

from which we find that the divergence must diverge to infinity as we approach the conjugate point q. Also, by the Raychaudhuri equation (4.20), the divergence must decrease along the geodesic if the vorticity is identically zero.

This realization that the divergence diverges to infinity for conjugate points, and proposition 4.2 will be used in the upcoming lemma.

The following lemma is a generalization of proposition 4.4.4 from (Hawking & Ellis, 1973) in arbitrary dimension.

Lemma 4.3. *Suppose we consider a family of lightlike geodesics through a shared point $q\in\mathscr{M}$. If $R_{\mu\nu}k^\mu k^\nu\geq 0$ everywhere (e.g., if the weak energy condition holds, or if we are in the vacuum case) and if the divergence θ takes negative value, say*

$\theta_1 < 0$, at some point $\gamma(s_1)$. Then there will be a point conjugate to q along $\gamma(s)$ between $\gamma(s_1)$ and $\gamma(s_1 + ((n-1)/-\theta_1))$ provided that $\gamma(s)$ can be extended to this parameter value.

The proof of this lemma is pretty simple. We remark that the expansion of the matrix M along $\gamma(s)$ obeys the Raychaudhuri equation with identically zero vorticity (4.20).

$$\frac{\mathrm{d}}{\mathrm{d}s}\theta = -R_{\mu\nu}k^{\mu}k^{\nu} - \sigma^2 - \frac{\theta^2}{n-1}$$

Since all terms on the right hand side are nonpositive we obtain that $\frac{\mathrm{d}}{\mathrm{d}s}\theta \leq -\frac{\theta^2}{n-1}$. From this, it follows that

$$\theta \leq \frac{n-1}{s - (s_1 + ((n-1)/-\theta_1)}$$

which implies that θ will become infinite, and thus that there is a conjugate point to p for some parameter value between s_1 and $s_1 + ((n-1)/-\theta_1)$ $\square$

The key takeaway from this lemma is that in order for a lightlike geodesic to admit no conjugate points in the physical spacetime, the divergence must be nonnegative everywhere along the geodesic. This will later be used to study smoothness of a lightcone outside of its vertex. Note, however, that this does not imply that two infinitesimally close lightrays cannot converge to each other as their parameter approaches infinity.[38]

[38] The analysis of conjugate points by the divergence and the Raychaudhuri equation relies on the eigenvalues of the matrix M being finite which does not generally hold in the limit of null infinity.

5 Constraints on a lightcone

This thesis considers initial data on a lightcone in $n+1$ dimensions. In this section, a coordinate system for that lightcone is introduced, the Raychaudhuri equation is reformulated to serve as a constraint equation for the metric tensor. Alternatively, if one considers a conformal class of metrics instead of a metric as initial data, then the Raychaudhuri equation is used to find an exact expression of the conformal factor Ω on the lightcone such that our initial data is physically reasonable. Smoothness of the lightcone outside the vertex is also briefly analyzed using the Raychaudhuri equation and lemma 4.3.

5.1 Adapted null coordinates

In this subsection, we introduce a coordinate system on a lightcone. This construction is the same as can be found in (Choquet-Bruhat et al., 2011a) and (P. T. Chruściel and Paetz, 2014). Later in this thesis, these coordinates will be used in order to express and analyze the Raychaudhuri equation for the initial data on the cone.

We consider an $n+1$ dimensional spacetime $(\mathscr{M}, g)$ which is diffeomorphic to $\mathbb{R}^{n+1}$. We consider the point $O \in \mathscr{M}$ and all null geodesics that intersect O. We may then consider the collection of points along these geodesics which we denote by C_O. Since this collection consists of all points that are reachable by lightlike geodesics originating in O, we may consider this set to be a generalized version of a lightcone. The set C_O may sometimes simply be called *the lightcone* to aid readability. We make the assumption that the geodesics in the family of lightlike geodesics through O admit no conjugate points. This is a restriction on the geometry of the spacetime for which it is well known that it does not hold in general.[39] This choice is made since it ensures smoothness of C_O outside of the vertex. We'll later see what this means for the initial data on the lightcone. Since the lightlike geodesics that generate the future lightcone admit no conjugate points, we see that the lightcone admits no null geodesics that reconverge after crossing in the vertex. This implies that the lightcone topologically resembles a cone.[40] A result of this assumption is that it allows us to work with one global set of coordinates on the lightcone. We may then describe coordinates (y^μ) on the manifold $\mathscr{M}$ such that C_O can be globally represented by the the equation of a Minkowskian lightcone. I.e.,

$$C_O \equiv \{(y^0, \ldots, y^n) \in \mathscr{M} \mid (y^0)^2 = \sum_{i=1}^{n} (y^i)^2\}.$$

One can think of this expression as a lightcone which is defined on the spacetime $(\mathscr{M}, g)$ instead of on the tangent space $T_O\mathscr{M}$. This expression for C_O is always possible for some neighborhood of the vertex in a differentiable Lorentzian spacetime, even in spacetimes where one may not be able to find coordinates such that C_O can be globally expressed by the expression for a Minkowskian lightcone. This is true because each such spacetime is locally Minkowskian. By *locally Minkowskian*,

[39] The observation of gravitational lensing is an example of the non-generality of this assumption since the paths of distinct lightrays emitted from a point reconverge. That is, there exist lightlike geodesics with conjugate points.

[40] Since there are no conjugate points, there will be no caustics. This will be important since this implies a diffeomorphic relation between the points in C_O and a cone.

we mean that, for any point in the manifold we may find coordinates such that the metric at that point is $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$, and $\partial_\rho g_{\mu\nu} = 0$. This implies that, if one needs a spacetime sufficiently close to Minkowskian spacetime, one may zoom in on any point, including the vertex O, until the spacetime one sees is sufficiently close to being flat.[41] This arbitrary closeness to Minkowskian space is important since the expression C_O is the same as the expression for a real, i.e., Minkowskian, lightcone.[42]

To further study the initial data on the lightcone we introduce a new set of coordinates $(x^\mu) \equiv (u, r, x^A)$. By (x^A) we refer to the $n-1$ coordinates unequal to u and r. These coordinates uniquely describe points on $C_O \setminus \{O\}$ when $u = 0$. The coordinate expressions for r and u are given by

$$r = \sqrt{\sum_{i=1}^{n} (y^i)^2}, \qquad u = r - y^0.$$

For each r we define $\Sigma_r \equiv \{(u, r, x^A) = (0, r, x^A) \in \mathscr{M}\} \cong S^{n-1}$. The coordinates (x^A) on Σ_r may be chosen arbitrarily.

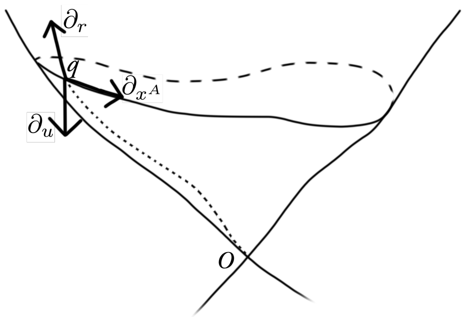

Figure 4: $2+1$ dimensional example of C_O with tangent vectors for $q \in \Sigma_r \subset C_O \subset \mathscr{M}$. The ellipsoidal cross section denotes the Σ_r hypersurface. The ∂_r and ∂_{x^A} tangent vectors are tangent to C_O whereas the vector ∂_u points off the cone. The dotted line is a ∂_r integral curve starting at the origin O. This dotted line denotes a null geodesic through O.

In the above coordinate choice, the r coordinate is chosen such that one approaches O as $r \to 0$. One has a metric g on $\mathscr{M}$. In this thesis we use an overline to denote quantities described on the lightcone C_O. That is, an overline denotes initial data in the characteristic Cauchy problem. The metric on C_O, $\overline{g}$ can be expressed in adapted null coordinates as

$$\mathrm{d}\overline{s}^2 = \overline{g}_{00}\mathrm{d}u^2 + 2\nu_0 \mathrm{d}u\mathrm{d}r + 2\nu_A \mathrm{d}u\mathrm{d}x^A + \mathrm{d}\check{s}^2 \qquad (5.1)$$

[41] More precisely, for an arbitrary closeness to Minkowski spacetime near O, one may find a neighborhood of O such that the metric here is close enough to η. *Closeness* to Minkowski spacetime may be defined by introducing the distance function between metrics: $\|g - \eta\| = \sup_{p \in O} \sum_{\mu,\nu} |g_{\mu\nu} - \eta_{\mu\nu}|$.

[42] A Minkowskian lightcone is a 'real' lightcone since we only defined lightcones for flat space.

Where we implicitly defined

$$\nu_0 \equiv \overline{g}_{01}, \quad \nu_A \equiv \overline{g}_{0A}$$

and where $\check{g}$ is expressed as

$$\mathrm{d}\check{s}^2 = \check{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B \equiv \overline{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B.$$

We have that the metric g induces a metric $\overline{g}$ on C_O and a metric $\check{g}$ on Σ_r. Whenever relevant we will use the notation $\check{g}_{\Sigma_r}$ to denote the relevant embedded subspace Σ_r for which the metric $\check{g}$ is defined. Note that the metric $\overline{g}$ is defined on a space with a null direction and thus has signature $(0, +, \cdots, +)$. The metric $\check{g}_{\Sigma_r}$ is Riemannian.

The components of the metric g_{00}, ν_0, and ν_A are coordinate-choice dependent. However, since ∂_r is chosen to be future directed and ∂_u past directed we find that $\nu_0 > 0$.[43] The restriction of the inverse of this metric to the lightcone can be written as.

$$\overline{g}^{\mu\nu}\partial_\mu\partial_\nu = 2\nu^0\partial_u\partial_r + \overline{g}^{11}\partial_r\partial_r + 2\overline{g}^{1A}\partial_r\partial_A + \overline{g}^{AB}\partial_A\partial_B \tag{5.2}$$

where

$$\nu^0 \equiv \overline{g}^{01} = (\nu_0)^{-1}, \quad \nu^A \equiv \overline{g}^{AB}\nu_B, \quad \overline{g}^{1A} = -\nu^0\nu^A, \quad \overline{g}^{11} = (\nu^0)^2(\nu^A\nu_A - \overline{g}_{00})$$

and where $\overline{g}^{AB}$ is the inverse of $\overline{g}_{AB}$.[44]

5.2 The Raychaudhuri equation as a constraint equation

The collection C_O is a lightcone in the sense that it is generated by the lightlike geodesics through $O \in \mathscr{M}$. Since these geodesics are indeed lightlike, one may analyze the evolution of their separation through use of the Raychaudhuri equation (4.20). In this subsection, the Raychaudhuri equation will be rewritten in terms of the initial data $(\overline{g}, \kappa)$, with $\overline{g}$, the metric on C_O, and κ a field of connections such that $\nabla_{\partial_r}\partial_r = \kappa\partial_r$. Through this reformulation, we'll see how this specified data is equivalent to the double (h, K) for which well-posedness for the vacuum equations on a smooth spacelike Cauchy surface was shown (section 2.2). For initial data $(\overline{g}, \kappa)$, the Raychaudhuri equation will serve as a constraint equation such that the initial data is a solution. We also mention how the Raychaudhuri equation may be used to analyze smoothness of the cone. This will serve as an additional constraint on the initial data.

The Raychaudhuri equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\theta = -R_{\alpha\beta}k^\alpha k^\beta + \omega^2 - \sigma^2 - \frac{\theta^2}{n-1},$$

as derived in section 4.1, is an equation for the development of the separation between neighboring lightlike geodesics. It is expressed in terms of the expansion,

[43] This is most easily seen in the original (y^μ) coordinates where $\partial_r = \partial_{y^0} +$ other terms, and $\partial_u = -\partial_{y^0}$ and thus $\nu_0 = \overline{g}(\partial_r, \partial_u) = \overline{g}(y^0, -y^0) > 0$ since ∂_{y^0} is timelike. The 'other terms' in the expression of ∂_r vanish when their inner product with ∂_{y^0} is taken as they are chosen to be orthogonal to ∂_{y^0}.

[44] i.e. $\overline{g}^{AC}\overline{g}_{BC} = \delta^A_B$.

and respectively vorticity tensor, θ_{AB}, and ω_{AB}. In the case of our problem, there are two properties which simplify the equation. First, we consider the vacuum Einstein equations. This simplifies the equation by setting $R_{\alpha\beta} = 0$. Second, we have the property that all ∂_r-integral curves intersect O. In section 4.3, we saw proposition 4.2 which showed that the vorticity must identically vanish for such curves. The above considerations allow us to simplify the Raychaudhuri equation to the following expression:[45]

$$\frac{\mathrm{d}}{\mathrm{d}s}\theta = -\sigma^2 - \frac{\theta^2}{n-1}. \tag{5.3}$$

We remark that the divergence θ, as well as the shear σ^2, may be expressed purely in terms of the expansion tensor θ_{AB}. That is, equation (5.3) is an equation in the expansion alone.

In this section, we'll show the analogy between the initial data for the characteristic Cauchy problem $(\overline{g}, \kappa)$ and the initial data for the Cauchy problem with a smooth spacelike initial surface (h, K). Similarly to the section on the Raychaudhuri equation, we concentrate on the tangent vector to the lightlike geodesics. This vector will be denoted by ℓ. However, it will be useful to consider tangent vectors which coincide with our coordinates on the lightcone. For this reason, we take $\ell \equiv \ell^\mu \partial_\mu \equiv \partial_r$. Remark that, while ℓ is tangent to the lightlike geodesic generators of C_O, it is not generally geodesic. This depends on the coordinate choice.

In our considerations in section 4.1 we saw that the part of the separation between neighboring null geodesics tangential to the direction of propagation of these rays was zero. Moreover, we derived that the part of the separation in the null direction of the pseudo-orthonormal basis unequal to the propagation direction remained constant. That is, the time separation between neighboring geodesics is constant. Since we have that all lightlike geodesics through O were once joined at the same point in spacetime, we see that these geodesics once had zero time separation. Since the time separation between neighboring geodesics is constant, the time separation between the null geodesics identically vanishes. In adapted null coordinates, this implies that the separation between neighboring null geodesics is expressed purely in the coordinates (x^A) on the $\Sigma_r \subset C_O$ hypersurfaces.

We may now define the expansion tensor θ_{AB}. Since the separation between neighboring geodesics is purely tangential to the Σ_r hypersurfaces, we define θ_{AB} by

$$\theta_{AB} \equiv \frac{1}{2}\nabla_B \ell_A + \frac{1}{2}\nabla_A \ell_B.$$

Notice that this is the same definition as (4.12) from the derivation of the Raychaudhuri equation. We remark that this definition is quite similar to the one of the extrinsic curvature (2.9) which we saw when we showed well-posedness for the smooth spacelike Cauchy problem. The expansion may be further rewritten as

$$\theta_{AB} \equiv \frac{1}{2}\nabla_B \ell_A + \frac{1}{2}\nabla_A \ell_B = \frac{1}{2}\left(\ell^\mu \nabla_\mu g_{AB} + g_{A\mu}\nabla_B \ell^\mu + g_{\mu B}\nabla_A \ell^\mu\right) = \frac{1}{2}\pounds_\ell \check{g}_{AB}, \tag{5.4}$$

[45]Given that the ∂_r-integral curves are lightlike geodesics. This does not generally hold, we will derive a correction to the Raychaudhuri equation, expressed in terms of initial data, when the ∂_r-integral curves are nongeodesic.

where we use $\nabla g_{AB} = 0$ in the second equality. This is even more similar to the time derivative formulation of the extrinsic curvature (2.10).

The difference between the expansion (5.4) and the extrinsic curvature (2.9) is threefold. First, The expansion is defined only for its (x^A) components, whereas the extrinsic curvature is expressed for all components in the initial hypersurface. However, as argued previously, the components θ_{11} and θ_{1A} are constant (and thus vanish) since the separation in these directions is identically zero. Second, the way we defined the expansion tensor, it is actually the symmetric part of what we'd expect from the extrinsic curvature. However, since the vorticity is identically zero, we have that

$$\ell_{A;B} = \theta_{AB} + \omega_{AB} = \theta_{AB}.$$

This means that it does not make a difference wether we take the symmetric part of $\ell_{A;B}$ or just $\ell_{A;B}$. Third, the vector ℓ is tangent to the initial surface which was not the case for the analogous timelike vector ξ in the smooth spacelike Cauchy problem. It can intuitively thus not really be considered to be a time derivative in the same sense as where the orthogonal vector to a smooth spacelike Cauchy surface can be considered as such. However, mathematically, it can be considered to be the same object as the extrinsic curvature.[46] In the literature, the expansion $\theta_{\mu\nu}$ is sometimes denoted by $\chi_{\mu\nu}$ and called the *null second fundamental form on* C_O or the *null extrinsic curvature* of the initial surface C_O. We will show that, given a metric $\overline{g}$ on C_O and a specific field of connections κ, one can derive the expansion θ_{AB}. That is, the initial data $(\overline{g}_{AB}, \kappa)$ is equivalent to the initial data $(\overline{g}_{AB}, \theta_{AB})$ on the lightcone.

We shall now express the Raychaudhuri equation (5.3) purely in terms of the initial data and adapted null coordinates. In adapted null coordinates, we can express the expansion as

$$\theta_{AB} = -\overline{\Gamma}^0_{AB}\nu_0 = \frac{1}{2}\partial_r \check{g}_{AB}. \tag{5.5}$$

Where the equalities follow from the calculations

$$\theta_{AB} = \frac{1}{2}\pounds_\ell \overline{g}_{AB} = \frac{1}{2}\left(\ell^\mu \partial_\mu g_{AB} + g_{A\mu}\partial_B \ell^\mu + g_{\mu B}\partial_A \ell^\mu\right) = \frac{1}{2}\partial_r g_{AB}, \tag{5.6}$$

and

$$\overline{\Gamma}^0_{AB} = \frac{1}{2}\overline{g}^{0\mu}(\partial_A \overline{g}_{\mu B} + \partial_B \overline{g}_{A\mu} - \partial_\mu \overline{g}_{AB}) = -\frac{1}{2}\nu^0 \partial_r \overline{g}_{AB}. \tag{5.7}$$

The expression for the Lie derivative in (5.6) is derived in (Appendix: A.5). The last equality from (5.6) follows from the observation $\ell^\mu \equiv \delta^\mu_r$, and the last equality from (5.7) follows from the expression of the inverse metric (5.2). We remark that the expansion may always be expressed as $\theta_{AB} = \frac{1}{2}\partial_r g_{AB}$ if one uses a basis for which the ∂_r, and respectively ∂_A components correspond with the tangent vector and spacelike vectors in a pseudo-orthonormal basis. That is to say, we made no reference to any other properties of the (coordinates on a) lightcone in (5.4), or (5.6).

The divergence θ, and respectively shear ${\sigma_A}^B$ are defined as the trace, and traceless

[46] As far as that is possible; since ℓ is null, it cannot be normalized.

part of the expansion:

$$\theta \equiv \theta_A{}^A = \overline{g}^{AB}\theta_{AB} = \frac{1}{2}\overline{g}^{AB}\partial_r\overline{g}_{AB} = \partial_r \log\sqrt{\det \breve{g}_{\Sigma_r}} \tag{5.8}$$

and

$$\sigma_A{}^B \equiv \overline{g}^{BC}\theta_{AC} - \frac{1}{n-1}\delta_A^B\theta \tag{5.9}$$

We further define $\sigma^2 \equiv \sigma_{AB}\sigma^{AB}$. We wish to write the Raychaudhuri equation (5.3) without making reference to the parametrization of the ℓ-integral curves. In the case where $\nabla_{\partial_r}\partial_r$ is identically zero, we have that the ℓ-integral curves are geodesics. We will consider the more general case where this does not necessarily hold. In this case, recall that the $\frac{\mathrm{d}}{\mathrm{d}s}$ derivative in (5.3) is a covariant derivative. Suppose the κ connection from our initial data is given such that $\nabla_{\partial_r}\partial_r = \kappa\partial_r$. This can be done without loss of generality since the ℓ-integral curves are geodesics up to a parametrization (Friedrich and Schmid, 1987). Note that, as κ is freely prescribable initial data, it can be chosen such that it is zero everywhere. This choice may seem natural since it corresponds to ℓ-integral null curves that are geodesic.[47] However, this coordinate choice leads to unwanted logarithmic terms in conformal infinity when one is not in Minkowski spacetime (P. T. Chruściel & Paetz, 2014). It is for this reason that we wish to express the Raychaudhuri equation for arbitrary coefficient κ. To mathematically capture the effect of the connection in equation (5.3), we concentrate on the term $\frac{\mathrm{d}}{\mathrm{d}s}\theta$, which yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\theta &\equiv \nabla_{\partial_r}\theta = \nabla_{\partial_r}\partial_r \log\sqrt{\det \breve{g}_{\Sigma_r}} \\ &= \partial_r\nabla_{\partial_r}\log\sqrt{\det \breve{g}_{\Sigma_r}} + (\nabla_{\partial_r}\partial_r)\log\sqrt{\det \breve{g}_{\Sigma_r}} \\ &= \partial_r\partial_r \log\sqrt{\det \breve{g}_{\Sigma_r}} + \kappa\partial_r\log\sqrt{\det \breve{g}_{\Sigma_r}} = \partial_r\theta + \kappa\theta.\end{aligned}$$

That is to say, we see that the introduction of the connection κ leads to us picking up the extra term $\kappa\theta$ from the $\nabla_{\partial_r}\partial_r$ term after using Leibniz's rule. The value of κ at a point does not influence the other terms in the Raychaudhuri equation since the equation holds for *any* affinely-parametrized ℓ-integral curve.

Recall that the Raychaudhuri equation (5.3) holds for lightlike geodesics. In the case where the ℓ-integral curves are nongeodesic due to a nonaffine parametrization we obtain an the extra term $\kappa\theta$. In order for us to express the evolution of the expansion in terms of our coordinates we must then subtract the $\kappa\theta$ term from the left hand side of (5.3). By these considerations, we may write (5.3) as

$$\partial_r\theta - \kappa\theta = -\sigma^2 - \frac{\theta^2}{n-1}. \tag{5.10}$$

This equation serves as a constraint equation for the metric tensor $\overline{g}$ since the divergence θ and the shear σ^2 are expressed in terms of $\overline{g}$ using (5.5). A solution of equation (5.10) then defines the expansion which we've argued to be the analogue of the extrinsic curvature K from the smooth spacelike Cauchy problem.

If one considers the vacuum Einstein equation $R_{11} = 0$ one can derive the Ray-

[47]Because the ℓ integral curves then have affine parameter and thus satisfy $\ell^\nu{}_{;\mu}\ell^\mu = 0$.

chaudhuri equation and an expression for κ in terms of known quantities by assessing $\overline{R}_{11} = 0$. One finds (Choquet-Bruhat et al., 2011a; equation (6.13)):

$$\kappa = \nu^0 \partial_r \nu_0 - \frac{1}{2} \nu_0 g^{\mu\nu} \overline{\Gamma}^0_{\mu\nu} - \frac{1}{2} \theta. \tag{5.11}$$

Adapted null coordinates are singular at the vertex O and smooth everywhere else. Thus, for us to be able to prescribe smooth data, we also need that the lightcone itself is smooth.[48] It must then be the case that the map between points in the spacetime and the coordinate expression from above is a bijective one. That is, when traveling along any one of the null geodesics through $O \in \mathscr{M}$ we may not encounter other null geodesics through O. That is, the separation between two geodesics through O may not vanish for a point along the two geodesics. These possible points where a congruence of geodesics focus are called *caustics* in the literature. Since we are interested in points where geodesics that were once focused refocus, we are actually interested in conjugate points. In the section on the Raychaudhuri equation, we saw that this nonvanishing condition of the separation between lightlike geodesics through a shared point corresponds the expansion θ being nonnegative everywhere. So the Raychaudhuri equation also serves as an equation which ensures smoothness of the lightcone when the divergence θ is required to be nonnenegative. This is an additional constraint on the initial data. A good analysis of this smoothness is given in (P. T. Chruściel and Paetz, 2014).

5.3 The Raychaudhuri equation for a conformal metric

In the previous subsection we expressed the Raychaudhuri equation with respect to the metric $\overline{g}$ (equation (5.10)). As we are interested in the characteristic Cauchy problem on a lightcone in a conformally completed spacetime as well as the 'physical' spacetime, we wish to analyze the smoothness of the lightcone using the Raychaudhuri equation with respect to a conformal metric as well as the physical one. The Raychaudhuri equation is derived for null geodesics with respect to the metric $\overline{g}$. The connection coefficient κ was introduced for generality of this equation when the ∂_r-integral curves are nongeodesic. If the metric is replaced by a conformal metric, then the geodesics with respect to the conformal metric may differ from the geodesics with respect to $\overline{g}$. However, this nongeodicity is only due to the parametrization of these curves (Friedrich and Schmid, 1987), and thus due to the connection. In this section, we will analyze the Raychaudhuri equation for conformal metrics on the lightcone. In the process, we will find that we may use a conformal class $[\gamma]$ of metrics as initial data instead of a metric which satisfies the Raychaudhuri equation. Choosing a representative γ of this class will allow us to use the Raychaudhuri equation to find a conformal factor Ω for which the metric $\Omega^2\gamma$ satisfies it.

Suppose we have a metric γ on $\Sigma_r \subset C_O$ which is conformal to $\breve{g}_{\Sigma_r}$. We wish to express the parameters of the Raychaudhuri constraint equation (5.10) in terms of the metric γ. Since γ is conformal to $\breve{g}_{\Sigma_r}$, we may express the components of $\breve{g}_{\Sigma_r}$ and its inverse as:

$$\breve{g}_{AB} = \Omega^2 \gamma_{AB}, \quad \breve{g}^{AB} = \Omega^{-2} \gamma^{AB} \tag{5.12}$$

[48]Except, of course, for the vertex O.

We call Ω the *conformal factor* of the metric $\breve{g}_{\Sigma_r}$. γ will be considered as representing the class of conformal metrics to $\breve{g}_{\Sigma_r}$, which is denoted by $[\gamma]$. The trace free part of an arbitrary tensor w with components w_{AB}, $\breve{w}_{AB}$, may then be written as

$$\breve{w} \equiv w_{AB} - \frac{1}{n-1}\gamma_{AB}\gamma^{CD}w_{CD}. \tag{5.13}$$

The shear (with respect to the original metric $\breve{g}_{\Sigma_r}$) may be written in terms of the conformal metric as ${\sigma_A}^B = \frac{1}{2}\gamma^{BC}(\partial_r\gamma_{AC})\breve{}$:

$$\begin{aligned}
\gamma^{BC}(\partial_r\gamma_{AC})\breve{} &= \gamma^{BC}\partial_r\gamma_{AC} - \frac{1}{n-1}\gamma^{BC}\gamma_{AC}\gamma^{DE}\partial_r\gamma_{DE} \\
&= \gamma^{BC}\partial_r\gamma_{AC} - \frac{1}{n-1}\delta^B_A\gamma^{DE}\partial_r\gamma_{DE} \\
&= \Omega^2\breve{g}^{BC}(\Omega^{-2}\partial_r g_{AC} + g_{AC}\partial_r\Omega^{-2}) - \frac{1}{n-1}\delta^B_A\Omega^2\breve{g}^{DE}(\Omega^{-2}\partial_r g_{DE} + g_{DE}\partial_r\Omega^{-2}) \\
&= 2{\theta_A}^B + \delta^B_A\Omega^2\partial_r\Omega^{-2} - \frac{2}{n-1}\delta^B_A\theta - \delta^B_A\Omega^2\partial_r\Omega^{-2} = 2\left({\theta_A}^B - \frac{1}{n-1}\delta^B_A\theta\right) = 2{\sigma_A}^B
\end{aligned}$$

Where the "$\breve{}$"-sign denotes the trace free part of the expression in the brackets. This is a remarkable result since we previously found that the shear tensor can be expressed as ${\sigma_A}^B = \frac{1}{2}g^{BC}(\partial_r g_{AC})\breve{}$ (5.9). This shows that the shear, and thus the parameter σ^2 in the Raychaudhuri equation, is independent of the conformal factor Ω.

Expressing the divergence θ in terms of the conformal metric γ yields the following expression:

$$\begin{aligned}
\theta = \partial_r \log\sqrt{\det \breve{g}_{\Sigma_r}} = \partial_r \log\sqrt{\Omega^{2(n-1)}\det\gamma} &= \partial_r\left(\log\Omega^{(n-1)} + \log\sqrt{\det\gamma}\right) \\
&= (n-1)\partial_r\log\Omega + \partial_r\log\sqrt{\det\gamma}
\end{aligned}$$

We define the *conformal divergence* to be

$$\tau \equiv \partial_r \log\sqrt{\det\gamma},$$

that is, the divergence with respect to the conformal metric γ on Σ_r. We obtain that the divergence with respect to this conformal metric γ may be related to the divergence with respect to the original metric $\breve{g}_{\Sigma_r}$ by the expression

$$\theta = \tau + (n-1)\partial_r\log\Omega. \tag{5.14}$$

As was mentioned in the subsection containing the conformally completed Minkowski spacetime, the point sets related to null geodesics are the same for all conformal metrics on a spacetime. Recall that the Raychaudhuri constraint equation (5.10) is an equation for the null ℓ-integral curves which become geodesics when κ is identically zero. Since the connection can be specified freely, one may also consider the Raychaudhuri equation with respect to the conformal metric γ. Here we introduce the *conformal connection coefficient* ρ for the *conformal Raychaudhuri equation*

$$\partial_r\tau - \rho\tau = -\sigma^2 - \frac{\tau^2}{n-1}. \tag{5.15}$$

Notice that the shear in this equation remains unchanged from the original Raychaudhuri constraint equation since the shear depends only on the conformal class. Subtracting (5.15) from (5.10) lets us relate the conformal connection ρ, the physical connection κ, the conformal divergence τ, and the conformal factor Ω to each other.

$$\begin{aligned} 0 &= \partial_r(\theta - \tau) - \kappa\theta + \rho\tau - \frac{\tau^2 - \theta^2}{n-1} \\ &= (n-1)\partial_r\partial_r \log\Omega - \kappa(\tau + (n-1)\partial_r \log\Omega) + \rho\tau + 2\tau\partial_r \log\Omega + (n-1)(\partial_r \log\Omega)^2 \end{aligned}$$

$$\implies \partial_r(\partial_r \log\Omega) = -(\partial_r \log\Omega)^2 + \left(\kappa - \frac{2\tau}{n-1}\right)(\partial_r \log\Omega) + \frac{(\kappa - \rho)\tau}{n-1} \tag{5.16}$$

Here, we used expression (5.14) to relate θ and τ. Equation (5.16) is a first order quadratic quasilinear differential equation in $\partial_r \log\Omega$. We have a couple of observations regarding this equation. First, if we know κ, ρ, γ everywhere, and $\partial_r \log\Omega$ at some point, we may obtain a unique solution for the conformal factor on the lightcone up to the multiplication with a constant.[49] Second, if the connection coefficients κ, and ρ are equal, we obtain a Bernoulli's differential equation in $\partial_r \log\Omega$. This has the exact solution:

$$\Omega = \exp\left[C_2 + \int_1^r \frac{\exp\left(\int_1^\vartheta \kappa - \frac{2\tau}{n-1}\mathrm{d}\xi\right)}{C_1 + \int_1^\vartheta \exp\left(\int_1^\zeta \kappa - \frac{2\tau}{n-1}\mathrm{d}\xi\right)\mathrm{d}\zeta}\mathrm{d}\vartheta\right] \tag{5.17}$$

where C_1 and C_2 are integration constants which can be found from the boundary conditions. We see $C_2 = \log\Omega\ |_{r=1}$. C_1 can also be determined by considering $(\partial_r \log\Omega)\ |_{r=1}$. This yields $C_1 = (\partial_r \log\Omega)^{-1}\ |_{r=1}$. For the derivation of (5.17) we refer to (Appendix A.8). At first glance, equation (5.16) also seems to give this solution when τ is identically zero. However, we find that since the ℓ-integral null curves intersect in O, the parameter τ diverges to infinity when $r \to 0$.[50] So τ may not be taken to be identically zero.

Solution (5.17) can be used to reframe our initial value formulation. In our original formulation, the initial data consists of the triple $(\mathscr{N}, \overline{g}, \kappa)$. If we instead consider the triple $(\mathscr{N}, [\gamma], \kappa)$, we may use equation (5.16) and define $\rho \equiv \kappa$. This yields the conformal factor Ω by (5.17) which ensures that the Raychaudhuri equation holds for the metric $\breve{g}_{\Sigma_r} = \Omega^2\gamma$. That is, the Raychaudhuri equation may serve to determine the 'missing information' of the metric on the initial surface. The conformal factor is not unique since the integration constants C_1, and C_2 may be chosen freely.

In this section we saw how one can use the Raychaudhuri equation to relate initial data for conformal metrics on the Σ_r. We saw that one may use equation (5.16) to derive the conformal factor which relates the Raychaudhuri equation for both metrics. The key takeaway of this section is that one may use a conformal class of metrics as initial data on the lightcone. In this case, the Raychaudhuri equation

[49] Multiplication of Ω with a constant gives the addition of a constant after taking the logarithm. This thus vanishes in the derivatives $\partial_r \log\Omega$.

[50] This holds since the conformal divergence diverges exactly when the divergence with respect to geodesics diverges. This geodesic divergence can be expressed as $(\det M)^{-1}\frac{\mathrm{d}}{\mathrm{d}s}(\det M)$, wherein M is the separation matrix we saw in (4.24) (see section 4.3).

serves not as a constraint equation but as a differential equation from which the conformal factor can be determined. We found that the Raychaudhuri equation serves as a differential equation from which one may determine the conformal factor Ω from the derivative of its logarithm in the ∂_r direction. The (x^A) dependence of the conformal factor can be chosen arbitrarily. This corresponds with the freedom to choose coordinates on the Σ_r hypersurface.

6 Conclusion & discussion

For the smooth spacelike Cauchy problem, a form of initial data is the triple (Σ, h, K) where Σ is the Cauchy surface, and h and respectively K are the metric and extrinsic curvature on Σ. In this case, the physical constraints (2.26), and (2.21) ensure that the vacuum equations are satisfied everywhere in the Cauchy development. Moreover, through implementation of harmonic coordinates (2.30), one can write the vacuum field equations as a well-posed initial value formulation. The uniqueness condition of this initial value problem suggests that one may uniquely associate spacetimes with their initial data on a smooth spacelike Cauchy surface. This is a welcome result since generating initial data is generally easier than finding solutions to the vacuum equations. For example, a simple way to generate initial data for this form of the Cauchy problem can be found in (Lichnerowicz, 1944), (York, 1971).

We introduced conformal metrics which share the causal structure of spacetime. We derived the Raychaudhuri equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\theta = -R_{\alpha\beta}k^{\alpha}k^{\beta} + \omega^2 - \sigma^2 - \frac{\theta^2}{n-1}$$

in arbitrary dimension. The Raychaudhuri equation is an ordinary differential equation along null geodesics. It captures the evolution of the separation between infinitesimally neighboring geodesics. We've shown that the vorticity ω^2 is identically zero for infinitesimally neighboring null geodesics that share a point. This result and the null convergence condition imply that the right hand side of the Raychaudhuri equation is always nonpositive. From this, we've shown that conjugate points in finite parameter distance coincide precisely with negativity of the divergence θ at some point.

We analyzed initial data for the characteristic Cauchy problem on a lightcone. For this to be physically significant, we expanded our definition of a lightcone to include point sets generated by null geodesics through a point outside of Minkowski spacetime. Our only condition on the lightcone itself, which serves as the Cauchy surface in the Cauchy development, is that it is smooth everywhere, except for the vertex. This implies that the lightcone does not attain caustics. We've given an argument as to why this is equivalent to the nonoccurrence of conjugate points to the vertex of the lightcone along the geodesics that generate it.

We've shown how to express the Raychaudhuri equation in terms of the initial data triple $(\mathscr{N}, \overline{g}, \kappa)$. In this triple, $\mathscr{N}$ denotes the lightcone, $\overline{g}$ is the metric tensor on $\mathscr{N}$, and κ is a scalar field which serves as a connection coefficient on $\mathscr{N}$. The Raychaudhuri equation can be expressed in terms of initial data because the separation between lightlike geodesics of infinitesimally neighboring geodesics is shown to have no time component, i.e., it lies completely in the lightcone $\mathscr{N}$. Since the Raychaudhuri equation holds for arbitrary congruences of lightlike geodesics, it is given as a constraint equation on the initial data. This constraint can be analyzed further by demanding that the divergence θ is everywhere nonnegative. That is, by demanding that the divergence is nowhere negative one ensures smoothness of the lightcone $\mathscr{N}$ since no conjugate points, and thus no caustics, occur in this case. A good analysis of this constraint on the initial data is done by (P. T. Chruściel and Paetz, 2014).

Alternatively, we discussed initial data of the form $(\mathscr{N}, [\gamma], \kappa)$ which is the same as the previous form, with the exception that we use the conformal class of metrics $[\gamma]$ instead of the metric $\overline{g}$ on the lightcone. In this case, we take a representative γ of the conformal class $[\gamma]$. The Raychaudhuri equation for this representative then serves, not as a constraint equation, but as an equation from which one can find the conformal factor Ω such that the Raychaudhuri equation, with parameters defined in terms of the metric $\Omega^2\gamma$, holds. That is, the Raychaudhuri equation is used to 'complete' the initial data. In this case, we found that the difference between the Raychaudhuri equation for a metric for which it holds and for the representative of the conformal class is a Bernoulli's differential equation for which a two parameter family of exact solutions is known.

To show how one might use the Raychaudhuri equation to use a conformal class of metric tensors as initial data we made use of the property that all null geodesics which make up the lightcone share a point at the vertex of the cone. This was done in order to get rid of the vorticity term in the Raychaudhuri equation. This argument does not hold in the case where one studies the characteristic Cauchy problem on two intersecting null surfaces since the family of lightlike geodesics tangent to intersecting null surfaces do not all intersect in one point. It is for this reason that one may not generally conclude that a conformal class of metrics may be used as initial data for the characteristic Cauchy on two intersecting null surfaces from the considerations in this thesis. However, a conformal class of metrics can be shown to be taken as valid initial data for the characteristic Cauchy problem on two intersecting null surfaces as can be seen in (Rendall, 1990).

The goal of this thesis was to introduce the characteristic Cauchy problem on a lightcone, and to show how its initial data relates to the initial data for the smooth spacelike Cauchy problem for which well-posedness is known. We showed how different forms of initial data relate to one another. Specifically, the contributions of this thesis can be found in the comparison between the data (Σ, h, K) for the smooth spacelike Cauchy problem and the initial data $(\mathscr{N}, \overline{g}, \kappa)$ for the characteristic Cauchy problem on a lightcone, and in the exact expression of the conformal factor when one uses the initial data triple $(\mathscr{N}, [\gamma], \kappa)$ instead of $(\mathscr{N}, \overline{g}, \kappa)$.

In the literature one may find even more forms that initial data for the characteristic Cauchy problem may take. For example, in (P. T. Chruściel and Paetz, 2014), the use of certain components of the Weyl tensor as initial data is discussed. Some more topics which could be interesting to get a fuller understanding of the considerations in this thesis are the ways in which the Weyl tensor influences the divergence, vorticity, and shear tensors in the Raychaudhuri equation. In the Raychaudhuri equation, the Weyl tensor is not seen directly but it does cause distortions in the spacetime which may be analyzed. A starting point for this could be (Penrose, 1967). Finally, to obtain a fuller understanding of the characteristic Cauchy problem, one could also look at papers concentrating on the form of the problem where the initial surface consists of two intersecting characteristic surfaces. One may want to study (Rendall, 1990) in which a conformal class of metrics as initial data for this case is discussed.

A Appendix

A.1 Set identities for the chronological future

In this subsection we derive the set equalities

$$\mathscr{I}^+(\mathscr{I}^+(S)) = \mathscr{I}^+(S), \tag{A.1}$$

and

$$\mathscr{I}^+(\overline{S}) = \mathscr{I}^+(S) \tag{A.2}$$

for the chronological future of the set S, $\mathscr{I}^+(S)$. In (A.2) we have that $\overline{S}$ is the closure of S. Recall that we defined the chronological future as

$$\mathscr{I}^+(p) = \left\{ q \in \mathscr{M} \;\middle|\; \begin{array}{l} \text{there exists a future directed timelike curve} \\ \gamma(t) \text{ such that } \gamma(0) = p \text{ and } \gamma(1) = q \end{array} \right\},$$

and

$$\mathscr{I}^+(S) = \bigcup_{p \in S} \mathscr{I}^+(p).$$

We begin by considering equation (A.1). To prove the $\subseteq$-direction, we consider a point $q \in \mathscr{I}^+(\mathscr{I}^+(S))$ for an arbitrary set $S \in \mathscr{M}$. Since $q \in \mathscr{I}^+(\mathscr{I}^+(S))$, there exists a point $r \in \mathscr{I}^+(S)$ such that there is a timelike curve γ_2 for which $\gamma_2(0) = r$, and $\gamma_2(1) = q$. As $r \in \mathscr{I}^+(S)$, there exists a $p \in S$ such that there exists a timelike curve γ_1 for which $\gamma_1(0) = p$, and $\gamma_1(1) = r$. We now define the curve γ by

$$\gamma(t) = \begin{cases} \gamma_1(2t) & , \quad t \in [0, 1/2] \\ \gamma_2(2t-1) & , \quad t \in [1/2, 1] \end{cases}$$

We then have that γ is timelike since γ_1, and γ_2 are timelike. Moreover, $\gamma(0) = p$, and $\gamma(1) = q$. Thus, $q \in \mathscr{I}^+(p) \subseteq \mathscr{I}^+(S)$.

The $\supseteq$-direction is easier to show. Let us consider $q \in \mathscr{I}^+(S)$. We then have a $p \in S$ such that there exists a timlike curve γ for which $\gamma(0) = p$, and $\gamma(1) = q$. If we consider the point $r = \gamma(1/2)$, we see that $r \in \mathscr{I}^+(p) \subseteq \mathscr{I}^+(S)$ and $q \in \mathscr{I}^+(r) \subseteq \mathscr{I}^+(\mathscr{I}^+(S))$. □

We now shift our attention to (A.2). We begin by proving that $\mathscr{I}^+(\overline{S}) \subseteq \mathscr{I}^+(S)$.

Suppose we have a point $q \in \mathscr{I}^+(\overline{S})$. We may then take a point $p \in \overline{S}$ such that there is a timelike curve γ for which $\gamma(0) = p$, and $\gamma(1) = q$. Since $p \in \overline{S}$ we have that $p \in S$ or $p \in \overline{S} \setminus S$. In the case where $p \in S$, we trivially have that $q \in \mathscr{I}^+(S)$. We therefore concentrate on the case $p \in \overline{S} \setminus S$. In this case, p is an element of the boundary of S. This means that every neighborhood U of p intersects the set S. Since γ is timelike, we may find a neighborhood V of p such that there is a timelike curve λ_v for every point $v \in V$ which connects v and q.[51] Specifically, we may take $r \in V \cap S$ since p is in the boundary of S. We then have that there exists a λ_r which is timelike such that $r \in S$, $\lambda_r(0) = r$, and $\lambda_r(1) = q$

[51] This follows from theorem 8.1.2 in (Wald, 1984). An equivalent formulation to the claim associated with this footnote is given in (Wald, 1984) right after the set expression of the chronological future (8.1.1) on page 190.

which proves this set inclusion.

For the inclusion $\mathscr{I}^+(S) \subseteq \mathscr{I}^+(\overline{S})$ we simply use that $S \in \overline{S}$ which implies that $\mathscr{I}^+(S) \subseteq \mathscr{I}^+(\overline{S})$ by the definition of the chronological future of a set. $\square$

A.2 The contracted Bianchi identity

In this subsection, we derive the contracted Bianchi identity of the Riemann tensor. In components, the Riemann curvature tensor may be expressed as

$$R^{\lambda}{}_{\alpha\nu\mu} = \partial_\mu \Gamma^{\lambda}_{\alpha\nu} - \partial_\nu \Gamma^{\lambda}_{\alpha\mu} + \Gamma^{\lambda}_{\sigma\mu}\Gamma^{\sigma}_{\alpha\nu} - \Gamma^{\lambda}_{\sigma\nu}\Gamma^{\sigma}_{\alpha\mu}.$$

Two symmetries of the Riemann curvature tensor in components are

$$R^{\gamma}{}_{\alpha(\nu\mu)} = 0,$$

and

$$R_{(\gamma\alpha)\nu\mu} = 0.$$

These symmetries will be used extensively in the upcoming derivation. Note that for any point in the considered manifold $\mathscr{M}$, we may choose a basis of the tangent space $T_p\mathscr{M}$ such that all Christoffel symbols vanish at p. Note, however that the derivatives of the Christoffel symbols are generally not zero. Given such a basis, we can then express the Riemann tensor in components as:

$$R^{\lambda}{}_{\alpha\nu\mu} = \partial_\mu \Gamma^{\lambda}_{\alpha\nu} - \partial_\nu \Gamma^{\lambda}_{\alpha\mu}$$

From this, we consider the covariant derivatives

$$\begin{aligned}
R^{\lambda}{}_{\alpha\nu\mu;\sigma} &= \underline{\partial_\sigma \partial_\mu \Gamma^{\lambda}_{\alpha\nu}} - \partial_\sigma \partial_\nu \Gamma^{\lambda}_{\alpha\mu} \\
R^{\lambda}{}_{\alpha\mu\sigma;\nu} &= \partial_\nu \partial_\sigma \Gamma^{\lambda}_{\alpha\mu} - \partial_\nu \partial_\mu \Gamma^{\lambda}_{\alpha\sigma} \\
R^{\lambda}{}_{\alpha\sigma\nu;\mu} &= \partial_\mu \partial_\nu \Gamma^{\lambda}_{\alpha\sigma} - \underline{\partial_\mu \partial_\sigma \Gamma^{\lambda}_{\alpha\nu}}
\end{aligned}$$

If one adds these three covariant derivatives, the different kinds of underlined parts cancel with each other. One finds

$$R^{\lambda}{}_{\alpha[\nu\mu;\sigma]} = 0$$

This is known as the first Bianchi identity. Since the Riemann tensor is coordinate independent. This identity holds even if we don't use the coordinates where the Christoffel symbols vanish.

If we set $\sigma = \lambda$, we get the following expression:

$$\begin{aligned}
0 = R^{\lambda}{}_{\alpha\nu\mu;\lambda} + R^{\lambda}{}_{\alpha\mu\lambda;\nu} + R^{\lambda}{}_{\alpha\lambda\nu;\mu} &= R^{\lambda}{}_{\alpha\nu\mu;\lambda} - R_{\alpha\mu;\nu} + R_{\alpha\nu;\mu} \\
&\implies R^{\lambda\alpha}{}_{\nu\mu;\lambda} - R^{\alpha}{}_{\mu;\nu} + R^{\alpha}{}_{\nu;\mu} = 0
\end{aligned}$$

Choosing $\nu = \alpha$ as well yields

$$0 = R^{\lambda\alpha}{}_{\alpha\mu;\lambda} - R^{\alpha}{}_{\mu;\alpha} + R^{\alpha}{}_{\alpha;\mu} = -R^{\alpha\lambda}{}_{\alpha\mu;\lambda} - R^{\alpha}{}_{\mu;\alpha} + R_{;\mu} = -R^{\lambda}{}_{\mu;\lambda} - R^{\alpha}{}_{\mu;\alpha} + R_{;\mu}$$
$$\implies -2R^{\nu}{}_{\mu;\nu} + \delta^{\nu}_{\mu}R_{;\nu} = 0.$$

Note that the expression $R^{\nu}{}_{\mu;\nu} + \frac{1}{2}\delta^{\nu}_{\mu}R_{;\nu}$ is precisely the Einstein tensor with a raised first index. That is, we've derived the expression

$$G^{\mu}{}_{\nu;\mu} = 0. \tag{A.3}$$

This is known as the *contracted Bianchi identity.*

A.3 The Lie derivative of a tensor in components

The Lie derivative along a vector x of a type $(1,1)$ tensor, $\tau^{a}{}_{b}$, may be expressed in local coordinates as

$$(\pounds_x \tau)^{a}{}_{b} = x^k \nabla_k \tau^{a}{}_{b} + \tau^{a}{}_{i} \nabla_b x^i - \tau^{j}{}_{b} \nabla_j x^a \tag{A.4}$$

Where the Einstein summation convention is used. This can be expanded to the Lie derivative for an arbitrary type (p,q) tensor by taking the positive sum for every covariant component of the tensor and the negative sum for every contravariant component.

One may instead want to express the Lie derivative in terms of partial derivatives when working in coordinates. In this thesis, this is done for a type (0,2) tensor to derive one of the equalities. The Lie derivative of a type (0,2) tensor τ_{ab} may be expressed in partial derivatives by the calculation:

$$\begin{aligned}(\pounds_x \tau)_{ab} &= x^k \nabla_k \tau_{ab} + \tau_{ak} \nabla_b x^k + \tau_{kb} \nabla_a x^k \\ &= x^k(\partial_k \tau_{ab} - \Gamma^{j}_{ka}\tau_{jb} - \Gamma^{j}_{kb}\tau_{aj}) + \tau_{ak}(\partial_b x^k + \Gamma^{k}_{bj}x^j) + \tau_{kb}(\partial_a x^k + \Gamma^{k}_{ja}x^j) \\ &= x^k \partial_k \tau_{ab} + \tau_{ak}\partial_b x^k + \tau_{kb}\partial_a x^k + x^k(-\Gamma^{j}_{ka}\tau_{jb} - \Gamma^{j}_{kb}\tau_{aj} + \Gamma^{j}_{kb}\tau_{aj} + \Gamma^{j}_{ka}\tau_{jb}) \\ &= x^k \partial_k \tau_{ab} + \tau_{ak}\partial_b x^k + \tau_{kb}\partial_a x^k\end{aligned} \tag{A.5}$$

Where we used that the connection is torsion free (i.e. $\Gamma^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu}$).

A.4 Bernoulli's differential equation

The general form of the Bernoulli's differential equation is as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = q(x)y^m - p(x)y.$$

This has the exact solution

$$\left[\frac{(1-m)\int \exp\left((1-m)\int p(x)\mathrm{d}x\right) q(x)\mathrm{d}x + C_1}{\exp\left((1-m)\int p(x)\mathrm{d}x\right)}\right]^{1/(1-m)} \tag{A.6}$$

when $m \neq 1$ (Weisstein, 2026). Here, C_1 is an integration constant. Equation (5.16) is also of this form. For this equation, we have

$$y = \partial_r \log \Omega, \quad q = -1, \quad p = -\left(\kappa - \frac{2\tau}{n-1}\right), \quad m = 2.$$

Note also that the derivative is taken with respect to r. Substituting, y, q, p, and m into A.6 yields

$$\partial_r \log \Omega = \left(\frac{C_1 + \int \exp\left(\int \kappa - \frac{2\tau}{n-1} \mathrm{d}r\right)}{\exp\left(\int \kappa - \frac{2\tau}{n-1} \mathrm{d}r\right)} \right)^{-1}. \tag{A.7}$$

Introducing correct limits of the integrals, integrating and taking the exponential gives us

$$\Omega = \exp\left[C_2 + \int_1^r \frac{\exp\left(\int_1^\vartheta \kappa - \frac{2\tau}{n-1} \mathrm{d}\xi\right)}{C_1 + \int_1^\vartheta \exp\left(\int_1^\zeta \kappa - \frac{2\tau}{n-1} \mathrm{d}\xi\right) \mathrm{d}\zeta} \mathrm{d}\vartheta \right] \tag{A.8}$$

just like we wanted. Note that the lower limit of the integrals are taken to be at $r = 1$. This is an arbitrary choice, however $r = 0$ is not taken since τ diverges to infinity there and the integral could be ill behaved.

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