# Full-Step Interior-Point Methods for Symmetric Optimization

Guoyong Gu

# Propositions

accompanying the thesis

# Full-Step Interior-Point Methods for Symmetric Optimization

Guoyong Gu

1. For since the fabric of the universe is most perfect and the work of a most wise creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear. — Leonhard Euler
2. The analysis of feasible interior-point methods provides a sound basis for the analysis of full-step infeasible interior-point methods.
3. The 'adaptive strategy' improves the practical efficiency of full-step infeasible interior-point methods tremendously without changing the theoretical efficiency.
4. There is still some room to improve the implementation of 'adaptive strategy'.
5. To get an algorithm run is just the first step; it's a long way to make it robust and efficient.
6. MEX-files are not appropriate for all applications in Matlab.
7. From the recent experience at TU Delft of migrating to the centralized ICT infrastructure, it appears that centralization may conflict with local interests.
8. Practice is the sole criterion for testing truth.
9. In developing countries seeking to expand their economic activities, consideration for environmental conservation often receives a low priority.
10. To teach is to learn.

*These propositions are considered opposable and defendable and as such have been approved by the supervisor, Prof. dr. ir. C. Roos.*

# Stellingen

behorende bij het proefschrift

# Volle-stap Inwendige Punt Methoden voor Symmetrische Optimalisering

Guoyong Gu

1. Want aangezien de structuur van het universum uiterst volmaakt is en het werk van een uitermate wijze schepper, gebeurt er nergens in het universum iets waaruit niet een of andere regel van maximum of minimum blijkt. — Leonhard Euler
2. De analyse van toelaatbare inwendige punt methoden vormt een solide basis voor de analyse van niet-toelaatbare inwendige punt methoden.
3. De 'adaptieve strategie' verbetert de practische efficiëntie van volle-stap inwendige punt methoden zonder dat de theoretische efficiëntie verandert.
4. Er is nog ruimte om de implementatie van de 'adaptieve strategie' te verbeteren.
5. Het werkend krijgen van een algoritme is niet meer dan een eerste stap; het is een lange weg om het robuust en efficiënt te maken.
6. MEX-files zijn niet geschikt voor alle toepassingen in Matlab.
7. Uit de recente ervaring aan de TU Delft van migratie naar een gecentraliseerde ICT-infrastructuur, blijkt dat centralisatie ten koste kan gaan van locale belangen.
8. De praktijk is het enige criterium voor het testen van waarheid.
9. In ontwikkelingslanden die streven naar uitbreiding van hun economische activiteiten krijgt de aandacht voor milieubehoud vaak een lage prioriteit.
10. Onderwijzen is leren.

*Deze stellingen worden opponeerbaar en verdedigbaar geacht en zijn als zodanig goedgekeurd door de promotor, Prof. dr. ir. C. Roos.*

# Full-Step Interior-Point Methods for Symmetric Optimization

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PROEFSCHRIFT

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus Prof. dr. ir. J.T. Fokkema,
voorzitter van het College voor Promoties,
in het openbaar te verdedigen op
vrijdag 18 september 2009 om 12.30 uur
door

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Bachelor of Science in Mathematics,
Nanjing University, China,
geboren te Jiangsu, China.

Dit proefschrift is goedgekeurd door de promotor:

Prof. dr. ir. C. Roos

Samenstelling promotiecommissie:

| | |
|---|---|
| Rector Magnificus | voorzitter |
| Prof. dr. ir. C. Roos | Technische Universiteit Delft, promotor |
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| Prof. dr. ir. H.J. Sips | Technische Universiteit Delft, reservelid |

Dit proefschrift kwam tot stand met financieel support van

NWO
Nederlandse Organisatie voor Wetenschappelijk Onderzoek

en onder auspiciën van

Thomas Stieltjes Institute
for Mathematics

ISBN 978-90-9024574-4

Typeset by the author with the LaTeX Documentation System.

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Dedicated to my wife

*Shuoyang Wang*

# Acknowledgements

This thesis marks the end of a long journey in pursuing my PhD degree on optimization. After all these years, I've got quite a list of people who contributed in some way to this thesis, for which I would like to express thanks.

Foremost, I would like to thank Prof. Kees Roos, my supervisor and promotor, for generously accepting me as a student in the Optimization Group at TUDelft and subsequently scientific support, guidance and encouragements during the numerous talks we had. I have never heard him raise his voice, nor saw him in a bad temper, or not being pleased to see me. His suggestions were always as "If I were you, I would ...", "I suggest ...", etc. Indeed, I have never been forced to do anything.

My deep gratitude also extends to Prof. Bingsheng He from Nanjing University, China. He had been my mentor of graduate study for the first two years, leading me into the wonderful world of optimization, and later advised, encouraged and recommended me to study interior-point methods in the Optimization Group at TUDelft.

Thanks are also due to all colleagues and guests of our Optimization Group, secretaries and computer support group of our department, and the secretaries of CICAT. Especially, I would like to thank our guest Prof. Goran Lesaja, who read my thesis twice very carefully and gave many valuable suggestions. My special thanks also go to Veronique van der Varst from CICAT, for her professional management and kindly help.

My research was financially supported by the Netherlands Organization for Scientific Research (NWO) under project number 613.000.441, which is kindly acknowledged.

Moving towards more personal acknowledgements. I am tempted to individually thank all of my friends who, from my childhood until graduate school, have joined me in the discovery of what is life about and how to make the best of it. However, because the list might be too long and by fear of leaving someone out, I will simply say thank you very much to you all.

I would like to manifest my deep gratitude to my parents and parents in law, for their relentless support and unconditional love. I'm also grateful to the relatives of my family, for their continuous support and help to my family.

Last but not the least, I would like to thank my wife Shuoyang Wang. She has been, always, my pillar, my joy and my guiding light. To her I dedicate this thesis.

*Guoyong Gu* *Delft, the Netherlands*
*September, 2009*

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# Chapter 1

# Introduction

> *For since the fabric of the universe is most perfect and the work of a most wise creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear.*
>
> — Leonhard Euler, 1744

In mathematics, optimization refers to the study of problems in which one seeks to minimize or maximize a real function by systematically choosing the values of real or integer variables from within an allowed set.

Optimization is a branch of applied mathematics that is widely and increasingly used in science, engineering, economics, management, industry, and other areas. As a flourishing research activity, it has led to theoretical and computational advances, new technologies and new methods in developing optimal designs of different systems, efficiency and robustness in minimizing the costs of operations in a process, etc.

## 1.1 Milestones in linear optimization

Linear Optimization (LO), as the name implies, is concerned with the minimization or maximization of a linear function subject to linear equality and (or) linear inequality constraints on its variables.[1]

[1] Historically, the field of optimization was founded by George Dantzig, who invented the Simplex method, a systematic method to solve so-called Linear Programming problems. Nowadays, the word "programming" usually refers to the activity of writing computer programs, and as a consequence its use instead of the more natural word "optimization" gives rise to confusion. Following others, like Padberg [82], we prefer to use the name Linear Optimization in this thesis. It may be noted that in the nonlinear branches of the field of Mathematical Programming (like Combinatorial Optimization, Discrete Optimization, Semidefinite Optimization, etc) this terminology has already become generally accepted.

### 1.1.1 The simplex method

The feasible region of an LO problem in standard form (cf. Section 2.1) is a polyhedron. If the LO problem is solvable, then there must exist an optimal vertex of the polyhedron. Unfortunately, the number of the vertices associated with an LO problem can be exponential in its dimensions. Except for small size problems, this number is so large that it prevents examining all possible vertices for searching an optimal vertex ([114, Chapter 1]).

The simplex method, invented in 1947 by George Dantzig [15], is a procedure for examining candidate vertices for optimality in an intelligent fashion. It constructs a sequence of adjacent vertices with improving values of the objective function. Thus, the method travels along edges of the polyhedron until it reaches an optimal vertex.

Complexity theory attempts to describe how difficult it is for an algorithm to find a solution to a problem. An important aspect of the theory is to categorize computational problems and algorithms into complexity classes. The complexity theory framework focusses mainly on decision problems. Such problems have only two possible solutions or answers, "yes" or "no". The complexity class P is the class of problems that can be solved in polynomial time, and the complexity class NP consists of problems which given a certificate, a "yes" answer can be verified in polynomial time. The most important open question of complexity theory is whether the complexity class P is the same as the complexity class NP, or whether it is a strict subset as is generally believed [31, 83]. As this led to the development of the theory of NP-completeness and many deep extensions, people wanted to know whether LO is solvable in polynomial time or not.

In 1972, Klee and Minty [52] gave an example of an LO problem in which the polyhedron is a distortion of an $n$-dimensional cube. They showed that the simplex method using Dantzig's classic most-negative-reduced-cost pivot rule (frequently referred to as the "Dantzig rule") visits all $2^n$ vertices before arriving at the optimal vertex. This means that the worst-case complexity of the method is exponential. Since then exponential examples have been found for almost every deterministic pivot rule, including the best-neighbor rule which solves LO problems over the original Klee-Minty examples in just one pivot, and the steepest-edge rule which is the basis for the fast simplex codes of today. It is still an open question if there is a pivot rule with polynomial time, or even sub-exponential worst-case complexity.

Despite its exponential worst-case complexity, the expected behavior of the simplex method on a random LO problem drawn from some distribution is polynomial [1, 2, 13, 98]. The simplex method turns out to be remarkably efficient in practice. On average, the number of vertices visited by the simplex method seems to be roughly linear or even logarithmic to the size of the problem. Improved in various ways, the simplex method continues to be one of the best choices for solving LO problems.

### 1.1.2 The ellipsoid method

The ellipsoid method was originally developed for convex nonlinear optimization by Yudin and Nemirovski [116] and independently, related work was done by Shor [97], but it became famous when Khachiyan [50, 51] used it to obtain a polynomial-time algorithm for LO.

The basic idea of the ellipsoid method is well-known. At each iteration an ellipsoid is given which contains all optimal solutions. By considering the center of the ellipsoid, a hyperplane is constructed so that all optimal solutions lie on one side of the hyperplane (and the center either lies strictly on the other side (a deep cut) or on the hyperplane itself (a central cut)). Then a new ellipsoid is found which contains all points in the old ellipsoid and on the correct side of the hyperplane. The new constructed ellipsoid contains the optimal solution set, and undergoes a guaranteed reduction in volume, so that the solution set is squeezed more tightly at each iteration [104].

For an LO problem with $n$ inequalities and integral data with total bit length $L$, the ellipsoid method generates an approximate solution from which an exact solution can easily be obtained in $O(n^2L)$ iterations, requiring $O(n^4L)$ arithmetic operations on numbers with $O(L)$ digits.[2] This gives a polynomial bound, and thus Khachiyan was the first to show that LO is in the class P of polynomial-time solvable problems.

Based on the expectation that a polynomial LO algorithm would be faster than the simplex method, the ellipsoid method was studied intensively by theoreticians as well as practitioners. It was a great disappointment that the number of steps required for the ellipsoid method to terminate is typically very close to its worst-case bound, so that despite a number of refinements the method is not competitive for LO [10]. Thus, after the dust eventually settled, the prevalent view among LO researchers was that Khachiyan had proved a genuinely polynomial LO algorithm, but the simplex method remained the clear winner in practice.

### 1.1.3 Interior-point methods

The above irony, the fact that an algorithm with the desirable theoretical property of polynomiality might nonetheless compare unfavorably with the (worst-case exponential) simplex method, set the stage for exciting new developments. Karmarkar's projective method [49] is an important improvement on the theoretical result of Khachiyan that an LO problem can be solved in polynomial time. It sparks great interest and activity, which lead to the whole field of Interior-Point Methods (IPMs).

The new ideas employed are very intriguing: at each iteration a projective transformation is used to bring the current iterate into the center of the feasible region, and a nonlinear potential function, invariant under such transformations,

[2] These bounds appear in Khachiyan [51]; his earlier extended abstract [50] gave a bound of $O(n^5L)$ arithmetic operations on numbers with $O(nL)$ digits.

is used to measure progress. The idea of making a projective transformation is to bring the current iterate to a point far from the constraints so that a steepest descent step (on the transformed objective function or the potential function) will give good decrease. Projective transformations are not used in IPMs nowadays. However, the key concept of making a transformation or changing the metric so that the current iterate is in some sense far from the boundary remains highly valuable [104].

The complexity bound was only slightly better than that of the ellipsoid method. For a problem with $n$ inequalities and integer data of total bit length $L$, the number of steps of the algorithm is $O(nL)$, each step requires $O(n^{2.5})$ arithmetic operations, and each arithmetic operation requires a precision of $O(L)$ bits. So totally we need $O(n^{3.5}L)$ arithmetic operations on numbers with $O(L)$ digits.

Besides having polynomial complexity, IPMs are also highly efficient in practice. Computational experience with sophisticated procedures suggests that the number of iterations grows much more slowly than the dimension grows. After an initial controversy it has been established that for very large, sparse problems, subsequent variants of Karmarkar's method often outperform the simplex method.

## 1.2 More about IPMs

Since the publication of the seminal paper of Karmarkar [49], IPMs have revolutionized virtually every area of continuous optimization. Many new algorithms were proposed and almost all of these algorithms have been shown to be efficient, at least from a theoretical point of view. In this section, we recall some possible links with earlier literature, their generalization to convex optimization, and primal-dual IPMs for a subclass of convex optimization, namely Symmetric Optimization (SO)[3].

The aim of this section is to give some background of our research. For a comprehensive treatment of IPMs, we refer to the books of Roos, Terlaky and Vial [93], Vanderbei [108], Wright [112] and Ye [114] for LO; for convex optimization, the seminal monograph of Nesterov and Nemirovski [81], the books of Ben-Tal and Nemirovski [8], Nesterov [80], and Renegar [88]; for Semidefinite Optimization (SDO), the handbook of Wolkowicz, Saigal, and Vandenberghe [110]; and for general nonlinear optimization, the survey articles of Forsgren, Gill and Wright [26] and Gould, Orban and Toint [34].

[3]Under mild assumptions, any optimization problem can be reformulated as a conic convex optimization problem (which optimizes a linear objective function subject to linear constraints and over a pointed, closed, convex cone). If in addition the associated cone is symmetric (cf. Definition 4.19), we call it an SO problem.

### 1.2.1 IPMs for LO

Shortly after the publication of Karmarkar's projective algorithm for LO, some possible links with earlier literature were found.

Gill, Murray, Saunders, Tomlin and Wright [32] noticed the close similarity between the search directions in Karmarkar's algorithm and in the logarithmic barrier approach proposed by Frisch [29] and later extensively studied by Fiacco and McCormick [25] in the context of nonlinear optimization. Indeed, the logarithmic barrier approach could also be proved to be polynomial. Renegar [87] proposed an algorithm with $O(\sqrt{n}L)$ iterations, an improvement over Karmarkar's algorithm. Renegar's scheme was a clever implementation of Huard's method of centers [45, 46].

Moreover, it turns out that the IPM implemented by Karmarkar [3] (the affine-scaling method), besides being discovered simultaneously by a number of researchers in the mid 1980s [19], had in fact been proposed in 1967 [17] and analyzed in 1974 [18] by Dikin. The polynomiality question for the (primal) affine-scaling method is unsettled. Instead we have primal-dual variants of Dikin's affine-scaling method that are polynomial [48, 70]. We refer the reader to Tsuchiya [106] for a survey.

Another very important concept in the IPM literature is the central path, which was first recognized by Sonnevend [99] and Meggido [64]. Most polynomial-time variants of IPMs use the central path as a guideline to the optimal set and some variant of Newton's method to follow the central path approximately. Particularly, Megiddo [64] related the central path to the classical barrier path in the framework of the primal-dual complementarity relationship. Kojima, Mizuno and Yoshise [55] used this framework to describe a primal-dual interior-point method that traces the central trajectory and has a worst time complexity of $O(nL)$ iterations. Monteiro and Adler [69] present a path following primal-dual algorithm that requires $O(\sqrt{n}L)$ iterations.

### 1.2.2 IPMs for convex optimization

Convex analysis, the mathematics of convex sets, functions, and optimization problems, is a well developed subfield of mathematics [90]. The first formal argument that convex optimization problems are easier to solve than general nonlinear optimization problems was made by Nemirovski and Yudin in their 1983 book [73].

> *In fact the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity.*
>
> — Rockafellar, 1993, [91]

The IPM approach to LO has a natural generalization to the wider field of convex (nonlinear) optimization. Nesterov and Nemirovski [81] are the first to

point out that IPMs can solve many convex optimization problems. Specifically, it was proved that generic convex problems[4], under some computability and boundedness assumptions, are polynomially solvable. In contrast to this, no efficient algorithms for typical generic non-convex problems are known, and there are strong reasons to believe that no such algorithms exist (e.g., programs with quadratic objective and constraints are not polynomially solvable unless P=NP [74]).

The barrier function is a smooth convex function defined in the interior of the set, tending to $+\infty$ as the boundary is approached. It turned out that the key property is that the barrier function should be self-concordant, i.e., its derivatives satisfy certain Lipschitz continuity properties.

Nesterov and Nemirovski [81] also showed that, at least in principle, any convex optimization problem could be provided with a self-concordant barrier. However, as the generated barrier could not be efficiently evaluated in general, this is purely an existence result. So the class of optimization problems to which the generalized IPMs can be efficiently applied consists of those with a computationally tractable self-concordant barrier. To contrast with the general case, Nesterov and Nemirovski listed a considerable number of important problems where computationally tractable self-concordant barriers are available, and provided a calculus for constructing such functions for more complicated sets [75]. A very significant special case is that of the positive semidefinite cone, leading to SDO. Independently, Alizadeh [4] developed an efficient IPM for SDO, with the motivation of obtaining strong bounds for combinatorial optimization problems.

The theory of self-concordant barriers is limited to convex optimization. However, this limitation has become less burdensome as more and more scientific and engineering problems have been shown to be amenable to convex optimization formulations. A number of seemingly non-convex problems arising in engineering design can be reformulated as convex optimization problems: see Ben-Tal and Nemirovski [8] and Boyd and Vandenberghe [14].

### 1.2.3 Primal-dual IPMs for SO

Primal-dual path-following IPMs trace simultaneously the primal and dual central paths. It turns out that tracing the paths together is much more advantageous than tracing only one of them. As there are a lot of symmetries in SO, IPMs may achieve maximal flexibility. Consequently, we may generalize primal-dual IPMs for LO to SO.

In their seminal papers [77, 78], Nesterov and Todd provided a theoretical foundation of efficient IPMs for convex optimization problems expressed in conic form, when the cone and its associated barrier are self-scaled. The class of self-scaled cones is defined by a set of properties owned by the associated self-concordant

[4]Theoretically, any optimization problem is equivalent to a convex problem. Make the objective function linear by adding a new variable if necessary and replace the feasible set by its convex hull. Here, we exclude these theoretical cases.

barriers. Prima-dual IPMs achieve their full power when the underlying cones are self-scaled, which is the case in linear, second-order cone, and semidefinite optimization.

Güler [38] observed that self-scaled cones are precisely symmetric cones, which have been much studied and even characterized. See, for example, the comprehensive book of Faraut and Koranyi [20], where they stated that symmetric cones coincide with cones of squares in Euclidean Jordan algebras. Hence, we may use the well developed theory of Euclidean Jordan algebras to deal with SO.

Faybusovich is the first who has exploited the advantages given by the Jordan algebraic setting in the study of SO. He started his in-depth research by a study of non-degeneracy conditions for SO in [23]. Subsequently, he analyzed various interior-point strategies for SO in [21, 22, 24], where Jordan algebras played a crucial role. The ideas of Faybusovich have been followed by many optimizers. For instance, Sturm presented the theoretical basis of his SeDuMi software in terms of Jordan algebras [101]. Later, Schmieta and Alizadeh [94, 95], Hauser and Lim [42] and Rangarajan [86] used this setting in further studies of IPMs for SO.

### 1.2.4 Feasible or infeasible IPMs

IPMs are iterative algorithms. To initialize the algorithm a starting point is needed. According to the choice of the starting point, one may distinguish between feasible IPMs and infeasible IPMs (IIPMs).

Feasible IPMs start with a strictly feasible point[5] and maintain strict feasibility during the solution process. An elegant and theoretically sound method to find a strictly feasible starting point is to use a self-dual embedding model, by introducing artificial variables. The idea of self-dual embedding for LO dates back to the 1950's and the work of Goldman and Tucker [33]. With the arrival of IPMs, the embedding idea was revived by Ye, Todd, and Mizuno [115]. Subsequent references are [16, 59, 93]. Some well-known software packages are based on this approach; for example, MOSEK[6] [6] and SeDuMi[7] [100] are based on the use of the self-dual embedding model. Also, the leading commercial LO package CPLEX[8] includes the self-dual embedding as a possible option. Despite the desirable theoretical properties, the self-dual embedding model adds two dense columns and rows in the coefficient matrix, which may need some special tricks to handle [93, Chapter 20].

Most of the existing software packages use IIPMs. IIPMs deal in a more straightforward way with the situation where no feasible starting point is available. For LO, IIPM starts with an arbitrary positive but infeasible starting point, and

[5] A strictly feasible point is also called an interior point. For LO, it is defined as a point that satisfies all constraints and, in particular, strictly satisfies all inequality constraints.

[6] MOSEK is available from http://www.mosek.com/.

[7] SeDuMi is available from http://sedumi.ie.lehigh.edu/.

[8] CPLEX is available from http://cplex.com/.

seeks feasibility and optimality simultaneously. The first primal-dual IIPMs were proposed by Lustig [60] and Tanabe [102]. Global convergence was shown by Kojima, Megiddo, and Mizuno [54], whereas Zhang [117] proved an $O(n^2L)$ iteration bound under the assumption that both primal and dual problems have feasible points. Shortly after that, Mizuno [65] proved that a modification of the Kojima-Meggido-Mizuno algorithm also has an $O(n^2L)$ iteration bound. Mizuno also constructed an IIPM that has $O(nL)$ iteration bound by using the idea of the interior-point predictor-corrector algorithm proposed by Mizuno, Todd and Ye [66]. Another $O(nL)$ iteration bound predictor-corrector IIPM for LO was proposed by Potra [85]. Other relevant references are [9, 11, 27, 56, 67, 79, 76, 84, 96, 105, 111]. A detailed discussion and analysis of IIPMs can be found in the book by Wright [112] and, with less detail, in the books by Ye [114] and Vanderbei [108]. The performance of existing IIPMs highly depends on the choice of the starting point, which makes these methods less robust than the methods that use the self-dual embedding technique.

## 1.3 Motivation and outline

This thesis mainly deals with the generalization to SO of a full-Newton step IIPM for LO that was proposed by Roos [92] in 2006. The analysis of our algorithm highly depends on the use of Euclidean Jordan algebras.

### 1.3.1 Motivation

In this subsection, we introduce the main idea underlying the full-step IIPMs presented in this thesis.

In IIPMs, the iterates are not feasible, and apart from reaching optimality one needs to strive for feasibility. This is reflected by the choice of the search direction for classical IIPMs. In LO, normally, after a full step the new iterate satisfies the feasibility constraints, except possibly the nonnegativity constraints. In fact, in general, the new iterate will have negative components, and, to keep the iterate positive, one is forced to take a damped step, for which the step size usually is found by performing a line search.

Instead of reaching feasibility in one step, the last two decades have made it very clear that to get a theoretically more efficient method one should be less greedy and work with a search direction that moves the iterates only slowly in the direction of feasibility and optimality. This is because the best complexity results hold for methods that are much less greedy and that use full-Newton steps. The reason is that only then one can take full advantage of the efficiency of Newton's method, which is the workhorse in all IPMs. Similarly, striving to reach feasibility in one step might be too greedy and may deteriorate the overall behavior of a method. One should better exercise a little patience and move slower in the direction of feasibility. Therefore, in our approach the search directions are

designed in such a way that a full-Newton step reduces the sizes of the residual vectors with the same speed as the duality gap. The outcome of the analysis and the subsequent numerical test in this thesis show that this is a sound strategy.

### 1.3.2 Outline

Based on the aforementioned motivation, Roos [92] designed the first primal-dual IIPM with full-Newton steps for LO problems. The complexity of his algorithm coincides with the best iteration bound for IIPM algorithms, namely,

$$O\left(n \log \frac{(x^0)^T s^0, \left\|b - Ax^0\right\|, \left\|c - A^T y^0 - s^0\right\|}{\varepsilon}\right).^9$$

Here we consider LO problems in standard form (see Section 2.1), $x^0 > 0$, $y^0$ and $s^0 > 0$ denote the starting points, and $b - Ax^0$ and $c - A^T y^0 - s^0$ are the initial primal and dual residue vectors, respectively, whereas $\varepsilon$ is an upper bound for the duality gap and the norms of residual vectors upon termination of the algorithm.

This thesis can basically be divided into two parts. The first part is based on [35], where we give an improved variant of the full-Newton step IIPM for LO. Then in the second part, which is based on [37], we generalize the improved IIPM for LO to SO, using the well developed properties of Jordan algebras.

More in detail, the first part consists of two chapters. In Chapter 2, we present some results of feasible IPMs for LO, as these will be used to analyze the centering steps of our IIPM. The improved version of the full-Newton step IIPM for LO is given in Chapter 3. As in Roos's original IIPM, every main step of our improved full-Newton step IIPM consists of one feasibility step and several centering steps. We use a more natural feasibility step, which targets at the $\mu^+$-center of the new perturbed problems. For the centering steps, we apply a sharper quadratic convergence result, which results in a slightly wider neighborhood for the feasibility steps.

The second part of the thesis starts from Chapter 4, where we recall the main properties of Jordan algebras, as well as their connection to symmetric cones. In addition, we derive more properties which are needed in our generalization. Based on the results of Jordan algebras, we give a full Nesterov-Todd step (NT-step) feasible IPM for SO in Chapter 5. As for the linear case, this is needed in analyzing the centering steps of the IIPM for SO. Our main result follows in Chapter 6, where we generalize the improved full-Newton step (which can also be viewed as NT-step) IIPM for LO to full NT-step IIPM for SO. Moreover, to improve the practical efficiency, we propose a more aggressive adaptive updating strategy, and numerical tests are given in Chapter 7.

Finally, Chapter 8 offers some concluding remarks and topics for further research, and as a supplement, in the appendix, we give a counter example to the

[9] If the norm is not further specified, $\|\cdot\|$ will always refer to the Euclidean norm or 2-norm.

conjecture in [92]. Though our simplified analysis of IIPM for LO does not depend directly on the conjecture anymore, it does indicate that one of our main inequalities for the analysis of the IIPM for LO is tight in the order.

# Chapter 2

# A Feasible IPM for LO

In preparation for dealing with our IIPM for LO, in this chapter we briefly recall the classical way to obtain a polynomial-time primal-dual path-following feasible IPM for LO. We refer to [93, 114] for more details.

## 2.1 LO problem in standard form

We consider the LO problem in the standard form

$$\min \left\{ c^T x : \ Ax = b, \ x \geq 0 \right\}, \tag{P}$$

where $A \in \mathbf{R}^{m \times n}$, $c, x \in \mathbf{R}^n$, and $b \in \mathbf{R}^m$. Without loss of generality we assume that $A$ has full row rank $m$.

The problem (P) has the following dual problem:

$$\max \left\{ b^T y : \ A^T y + s = c, \ s \geq 0 \right\}, \tag{D}$$

where $s \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$. We call (D) the standard dual problem.

The feasible regions of (P) and (D) are denoted by $\mathcal{P}$ and $\mathcal{D}$, respectively:

$$\begin{aligned} \mathcal{P} &:= \{x : Ax = b, \ x \geq 0\}, \\ \mathcal{D} &:= \left\{(y, s) : A^T y + s = c, \ s \geq 0\right\}. \end{aligned}$$

If $\mathcal{P}$ is empty we call (P) infeasible, otherwise feasible. Assume (P) is feasible, then if the objective value $c^T x$ is unbounded below on $\mathcal{P}$, (P) is called unbounded, otherwise bounded. We use similar terminology for the dual problem (D).

Since we assumed that $A$ has full (row) rank $m$, we have one-to-one correspondence between $y$ and $s$ in the pairs $(y, s) \in \mathcal{D}$. In order to facilitate the discussion we feel free to refer to any pair $(y, s) \in \mathcal{D}$ either by $y \in \mathcal{D}$ or $s \in \mathcal{D}$. The (relative)

interiors of $\mathcal{P}$ and $\mathcal{D}$ are denoted by $\operatorname{int}\mathcal{P}$ and $\operatorname{int}\mathcal{D}$:

$$\operatorname{int}\mathcal{P} := \{x : Ax = b,\ x > 0\},$$
$$\operatorname{int}\mathcal{D} := \left\{(y, s) : A^T y + s = c,\ s > 0\right\}.$$

In addition, we say that (P) and (D) satisfy the Interior-Point Condition (IPC) if both $\operatorname{int}\mathcal{P}$ and $\operatorname{int}\mathcal{D}$ are nonempty.

## 2.2 Duality results

In this section, we recall the well known duality results for LO problem in standard form.

**Proposition 2.1** (Weak duality, cf. [93, Proposition II.1]). *Let $x$ and $s$ be feasible for (P) and (D), respectively. Then $c^T x - b^T y = x^T s \geq 0$. Consequently, $c^T x$ is an upper bound for the optimal value of (D), if it exists, and $b^T y$ is a lower bound for the optimal value of (P), if it exists. Moreover, if the duality gap $x^T s$ is zero then $x$ is an optimal solution of (P) and $(y, s)$ is an optimal solution of (D).*

A direct consequence of Proposition 2.1 is that if one of the problems (P) and (D) is unbounded, then the other problem is infeasible. The classical duality results for the primal and dual problems in standard form boil down to the following two results.

**Theorem 2.2** (Strong Duality, cf. [93, Theorem II.2]). *If (P) and (D) are feasible then both problems have optimal solutions. Then, if $x \in \mathcal{P}$ and $(y, s) \in \mathcal{D}$, these are optimal solutions if and only if $x^T s = 0$. Otherwise neither of the two problems has optimal solutions: either both (P) and (D) are infeasible or one of the two problems is infeasible and the other one is unbounded.*

**Theorem 2.3** (Goldman-Tucker Theorem, cf. [93, Proposition II.3]). *If (P) and (D) are feasible then there exists a strictly complementary pair of optimal solutions, that is an optimal solution pair $(x^*, s^*)$ satisfying $x^* + s^* > 0$.*

## 2.3 The central path

The strong duality theorem (Theorem 2.2) indicates that finding an optimal solution of (P) and (D) is equivalent to solving the following system:

$$\begin{aligned} Ax &= b, & x &\geq 0, \\ A^T y + s &= c, & s &\geq 0, \\ xs &= 0.^1 \end{aligned}$$

In these so-called optimality conditions the first two constraints represent primal and dual feasibility, whereas the last equation is the so-called complementarity condition. The nonnegativity constraints in the feasibility conditions make the problem already nontrivial: only iterative methods can find solutions of linear systems involving inequality constraints. The complementarity condition is nonlinear, which makes it extra hard to solve this system.

To solve the above system, the simplest approach is to apply Newton's method directly, using a step length less than one if it is necessary to maintain positivity of $x$ and $s$. However, as $x$ and $s$ move too sharply toward the boundary, the step length needs to be set to a small value. In path-following IPMs, a less greedy approach is used to satisfy the complementarity condition. We replace the complementarity condition by the so-called centering condition $xs = \mu e$, where $\mu$ may be any positive number. This yields the system

$$\begin{aligned} Ax &= b, \qquad x \geq 0, \\ A^T y + s &= c, \qquad s \geq 0, \\ xs &= \mu e. \end{aligned} \tag{2.1}$$

Surprisingly enough, if this system has a solution for some $\mu > 0$, then a solution exists for every $\mu > 0$, and this solution is unique. This happens if and only if (P) and (D) satisfy the IPC (cf. [93, Chapter 5]).

If the IPC is satisfied, then the solution of (2.1) is denoted by $(x(\mu), y(\mu), s(\mu))$ and called the $\mu$-center of (P) and (D). The set of all $\mu$-centers forms a path, which is called the central path. As $\mu$ goes to zero, $x(\mu)$ and $(y(\mu), s(\mu))$ converge to optimal solutions of (P) and (D), respectively. Of course, system (2.1) is still hard to solve, but by applying Newton's method one can easily find approximate solutions.

## 2.4 The Newton step

We proceed by describing Newton's method for solving (2.1), with $\mu$ fixed. Given any $x$ and $(y, s)$, we want to find displacements $\Delta x$, $\Delta y$, and $\Delta s$ such that

$$\begin{aligned} A(x + \Delta x) &= b, \\ A^T(y + \Delta y) + s + \Delta s &= c, \\ (x + \Delta x)(s + \Delta s) &= \mu e. \end{aligned}$$

Neglecting the quadratic term $\Delta x \Delta s$ in the left-hand side of the third equation, we obtain the following linear system of equations in the search directions $\Delta x$,

[1] For LO, we denote by 0 and $e$ (used later) the zero and all-one vector, respectively, of appropriate size. Moreover, if $x$, $s \in \mathbf{R}^n$, then $xs$ denotes the componentwise (or Hadamard) product of the vectors $x$ and $s$.

$\Delta y$, and $\Delta s$:

$$\begin{aligned} A\Delta x &= b - Ax, \\ A^T\Delta y + \Delta s &= c - A^T y - s, \\ s\Delta x + x\Delta s &= \mu e - xs. \end{aligned} \tag{2.2}$$

Since $A$ has full row rank, and the vectors $x$ and $s$ are positive, one may easily verify that the coefficient matrix in the linear system (2.2) is nonsingular. Hence this system uniquely defines the search directions $\Delta x$, $\Delta y$, and $\Delta s$. These search directions are used in all existing primal-dual (feasible and infeasible) IPMs and are equivalent to Newton's method for solving the equations in system (2.1).

If $x$ is primal feasible and $(y, s)$ dual feasible, then it follows that $b - Ax = 0$ and $c - A^T y - s = 0$, whence the above system reduces to

$$\begin{aligned} A\Delta x &= 0, \\ A^T\Delta y + \Delta s &= 0, \\ s\Delta x + x\Delta s &= \mu e - xs, \end{aligned} \tag{2.3}$$

which gives the usual search directions for primal-dual feasible IPMs.

We use full-Newton steps, thus, the new iterate is given by

$$\begin{aligned} x^+ &= x + \Delta x, \\ y^+ &= y + \Delta y, \\ s^+ &= s + \Delta s. \end{aligned}$$

## 2.5 A full-Newton step feasible IPM algorithm

In the analysis of the algorithm, we need a quantity that measures proximity of the feasible triple $(x, y, s)$ to the current $\mu$-center $(x(\mu), y(\mu), s(\mu))$. Following [93], this quantity is defined as follows:

$$\delta(x, s; \mu) := \delta(v) := \frac{1}{2}\left\|v - v^{-1}\right\|, \quad \text{where } v := \sqrt{\frac{xs}{\mu}}.^2 \tag{2.4}$$

A graphical illustration of one iteration of the algorithm is given in Figure 2.1. This figure depicts the $xs$-space projected onto its first two coordinates. In the $xs$-space the cental path of the primal-dual pair of problems is a straight line consisting of $\mu e$, for $\mu > 0$. The small neighborhoods of $\mu$ and $\mu^+$-centers are illustrated by shaded regions around the $\mu$ and $\mu^+$-centers, respectively. At every iteration, we start from a point in the small neighborhood of the $\mu$-center, and by one full-Newton step (illustrated by dashed curve with arrow) we come to the small neighborhood of the $\mu^+$-center.

[2]The short-hand notation in the definition of $v$ means that $v$ is the vector obtained by taking square roots of the elements of the vector $xs/\mu$.

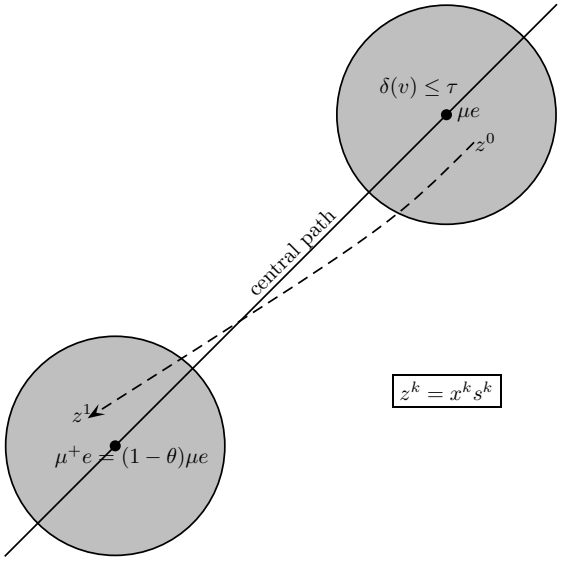


Figure 2.1: One iteration of the full-Newton step feasible IPM.

Assume that a primal feasible solution $x^0 > 0$ and a dual feasible pair $(y^0, s^0)$ with $s^0 > 0$ are given that are "close to" $x(\mu)$ and $(y(\mu), s(\mu))$, respectively, for some $\mu = \mu^0$. The primal-dual feasible IPM algorithm is given in Algorithm 2.1.

## 2.6 The analysis of the Newton step

An important observation on system (2.3) is that $\Delta x$ lies in the null space of $A$, whereas $\Delta s$ belongs to the row space of $A$. This implies that $\Delta x$ and $\Delta s$ are orthogonal, i.e., $\Delta x^T \Delta s = 0$. As a consequence we have the important property that after a full-Newton step the duality gap assumes the same value as at the $\mu$-centers, namely $n\mu$.

**Lemma 2.4** (cf. [93, Lemma II.47])**.** *After a primal-dual Newton step, one has* $(x^+)^T s^+ = n\mu$.

It is crucial for us to know the effect on $\delta(x, s; \mu)$ of a full-Newton step targeting at the $\mu$-center of $(P)$ and $(D)$. For that purpose [93, Theorem II.50] was used in [92]. This theorem states that if $\delta := \delta(x, s; \mu) \leq 1$, then the primal-dual Newton step is feasible, i.e., $x^+$ and $s^+$ are nonnegative, and, moreover, if $\delta < 1$, then $x^+$ and $s^+$ are positive and

$$\delta(x^+, s^+; \mu) \leq \frac{\delta^2}{\sqrt{2(1-\delta^2)}}.$$

**Algorithm 2.1** A full-Newton step feasible IPM for LO.

**Input:**
accuracy parameter $\varepsilon > 0$;
barrier update parameter $\theta$, $0 < \theta < 1$;
strictly feasible triple $(x^0, y^0, s^0)$
with $(x^0)^T s^0 = n\mu^0$ and $\delta(x^0, s^0; \mu^0) \leq 1/2$.
**Begin**
$x := x^0$; $y := y^0$; $s := s^0$; $\mu := \mu^0$;
**while** $x^T s \geq \varepsilon$
$\mu$-update:
$\mu := (1 - \theta)\mu$;
Newton step:
$(x, y, s) := (x, y, s) + (\Delta x, \Delta y, \Delta s)$.
**endwhile**
**end**

This result implies that the Newton process is locally quadratically convergent, and has been crucial in the analysis in [92]. Here, we use a tighter upper bound for $\delta(x^+, s^+; \mu)$, which provides a slightly wider neighborhood for the feasibility step of our IIPM. As the previous lemma, we recall it without proof.

**Theorem 2.5** ([93, Theorem II.52]). *If* $\delta := \delta(x, s; \mu) < 1$ *then*

$$\delta(x^+, s^+; \mu) \leq \frac{\delta^2}{\sqrt{2(1 - \delta^4)}}.$$

As a result, the following corollary follows trivially.

**Corollary 2.6.** *If* $\delta := \delta(x, s; \mu) \leq \frac{1}{\sqrt[4]{2}}$, *then* $\delta(x^+, s^+; \mu) \leq \delta^2$.

## 2.7 Iteration bound

The iteration bound depends on a lemma that quantifies the effect of the proximity measure on an update of the barrier parameter to $\mu^+ = (1 - \theta)\mu$.

**Lemma 2.7** (cf. [93, Lemma II.54]). *Let* $(x, s)$ *be a positive primal-dual pair and* $\mu > 0$ *such that* $x^T s = n\mu$. *Moreover, let* $\delta := \delta(x, s; \mu)$ *and let* $\mu^+ = (1 - \theta)\mu$. *Then*

$$\delta(x, s; \mu^+)^2 = (1 - \theta)\delta^2 + \frac{\theta^2 n}{4(1 - \theta)}.$$

For Algorithm 2.1, at the start of each iteration we have $x^T s = n\mu$ and $\delta(x, s; \mu) \leq 1/2$. After the barrier parameter is updated to $\mu^+ = (1 - \theta)\mu$, with

$\theta = 1/\sqrt{2n}$, Lemma 2.7 yields that $\delta(x, s; \mu^+) \leq 1/\sqrt{2}$ (cf. the proof of [93, Theorem II.53]). Then after the primal-dual Newton step to the $\mu^+$-center we have, by Corollary 2.6, $\delta(x^+, s^+, \mu^+) \leq 1/2$. Also, due to Lemma 2.4, we have $(x^+)^T s^+ = n\mu^+ = (1-\theta)n\mu$.

As the duality gap is reduced by the factor $(1-\theta)$, the total number of iterations now easily follows from the following lemma.

**Lemma 2.8** (cf. [93, Lemma I.36]). *After at most*

$$\left\lceil \frac{1}{\theta} \log \frac{n\mu^0}{\varepsilon} \right\rceil$$

*iterations we have $n\mu \leq \varepsilon$.*

The above results are summarized in the next theorem which requires no further proof.

**Theorem 2.9** (cf. [93, Theorem II.53]). *If $\theta = 1/\sqrt{2n}$, then the algorithm requires at most*

$$\left\lceil \sqrt{2n} \log \frac{n\mu^0}{\varepsilon} \right\rceil$$

*iterations. The output is a primal-dual pair $(x, s)$ such that $x^T s \leq \varepsilon$.*

# Chapter 3

# An Infeasible IPM for LO

In the case of an infeasible IPM for LO we call a triple $(x, y, s)$ an $\varepsilon$-optimal solution of (P) and (D) if the 2-norms of the residual vectors $b-Ax$ and $c-A^Ty-s$ do not exceed $\varepsilon$, and also the duality gap satisfies $x^Ts \leq \varepsilon$.

In this chapter, we present several improvements of the full-Newton step infeasible IPM for LO introduced by Roos [92]. Each main step of the method consists of a feasibility step and several centering steps. We use a more natural feasibility step, which targets at the $\mu^+$-center of the new perturbed problems. As for the centering steps, we apply a sharper quadratic convergence result, which results in a slightly wider neighborhood for the feasibility steps.

## 3.1 The perturbed problems

We assume that (P) and (D) have an optimal solution $(x^*, y^*, s^*)$, which implies that the duality gap vanishes, i.e., $(x^*)^Ts^* = 0$. As has become usual for infeasible IPMs, we start the algorithm with a triple $(x^0, y^0, s^0)$ and $\mu^0 > 0$ such that

$$x^0 = \zeta e, \quad y^0 = 0, \quad s^0 = \zeta e, \quad \mu^0 = \zeta^2, \tag{3.1}$$

where $\zeta$ is a (positive) number such that

$$x^* + s^* \leq \zeta e. \tag{3.2}$$

The IIPM algorithm present in this chapter will generate an $\varepsilon$-optimal solution of (P) and (D), or establish that there do not exist optimal solutions satisfying (3.2). The initial values of the primal and dual residual vectors are denoted as $r_p^0$ and $r_d^0$, respectively. So we have

$$\begin{aligned} r_p^0 &= b - Ax^0, \\ r_d^0 &= c - A^Ty^0 - s^0. \end{aligned}$$

In general, we have $r_p^0 \neq 0$ and $r_d^0 \neq 0$. In other words, the initial iterate is not feasible. The iterates generated by the algorithm will (in general) be infeasible for (P) and (D) as well, but they will be feasible for perturbed versions of (P) and (D) that we introduce in the sequel.

For any $\nu$ with $0 < \nu \leq 1$ we consider the perturbed problem ($P_\nu$), defined by

$$\min \left\{ \left(c - \nu r_d^0\right)^T x : Ax = b - \nu r_p^0,\ x \geq 0 \right\}, \tag{$P_\nu$}$$

and its dual problem ($D_\nu$), which is given by

$$\max \left\{ \left(b - \nu r_p^0\right)^T y : A^T y + s = c - \nu r_d^0,\ s \geq 0 \right\}. \tag{$D_\nu$}$$

Note that if $\nu = 1$, then $x = x^0$ yields a strictly feasible solution of ($P_\nu$) and $(y, s) = (y^0, s^0)$ a strictly feasible solution of ($D_\nu$). We conclude that if $\nu = 1$, then ($P_\nu$) and ($D_\nu$) satisfy the IPC.

**Theorem 3.1** ([114, Theorem 5.13])**.** *The original problems, (P) and (D), are feasible if and only if for each $\nu$ satisfying $0 < \nu \leq 1$ the perturbed problems ($P_\nu$) and ($D_\nu$) satisfy the IPC.*

*Proof.* Suppose that (P) and (D) are feasible. Let $\bar{x}$ be a feasible solution of (P) and $(\bar{y}, \bar{s})$ a feasible solution of (D). Then $A\bar{x} = b$ and $A^T\bar{y} + \bar{s} = c$, with $\bar{x} \geq 0$ and $\bar{s} \geq 0$. Now let $0 < \nu \leq 1$, and consider

$$\begin{aligned} x &= (1-\nu)\bar{x} + \nu x^0, \\ y &= (1-\nu)\bar{y} + \nu y^0, \\ s &= (1-\nu)\bar{s} + \nu s^0. \end{aligned}$$

One has

$$Ax = A((1-\nu)\bar{x} + \nu x^0) = (1-\nu)A\bar{x} + \nu Ax^0 = (1-\nu)b + \nu Ax^0 = b - \nu r_p^0,$$

showing that $x$ is feasible for ($P_\nu$). Similarly, we have $(y, s)$ is feasible for ($D_\nu$). Since $\nu > 0$, $x$ and $s$ are positive, thus proving that ($P_\nu$) and ($D_\nu$) satisfy the IPC.

To prove the inverse implication, suppose ($P_\nu$) and ($D_\nu$) satisfy the IPC for each $\nu$ satisfying $0 < \nu \leq 1$. Obviously, then ($P_\nu$) and ($D_\nu$) are feasible for these values of $\nu$. Letting $\nu$ go to zero it follows that (P) and (D) are feasible. □

It should be mentioned that this kind of perturbed problems have been studied first in [67], and later also in [28].

In the sections to follow we assume that (P) and (D) are feasible. Only in Section 3.6 will we discuss how our algorithm can be used to detect infeasibility or unboundedness of (P) and (D). It may be worth noting that if $x^0$ and $(y^0, s^0)$ are feasible for (P) and (D), then ($P_\nu$) $\equiv$ (P) and ($D_\nu$) $\equiv$ (D) for each $\nu \in (0, 1]$.

## 3.2 The central path of the perturbed problems

Let (P) and (D) be feasible and $0 < \nu \leq 1$. Then Theorem 3.1 implies that the perturbed problems ($P_\nu$) and ($D_\nu$) satisfy the IPC, and hence their central paths exist. This means that the system

$$Ax = b - \nu r_p^0, \quad x \geq 0, \tag{3.3}$$

$$A^T y + s = c - \nu r_d^0, \quad s \geq 0, \tag{3.4}$$

$$xs = \mu e$$

has a unique solution for every $\mu > 0$. This unique solution is denoted by $(x(\mu,\nu), y(\mu,\nu), s(\mu,\nu))$, and is the $\mu$-center of the perturbed problems ($P_\nu$) and ($D_\nu$). In what follows the parameters $\mu$ and $\nu$ always satisfy the relation $\mu = \nu\mu^0$. Thus we may denote the $\mu$-centers of the perturbed problems ($P_\nu$) and ($D_\nu$) simply as $(x(\nu), y(\nu), s(\nu))$.

Note that since $x^0 s^0 = \mu^0 e$, $x^0$ is the $\mu^0$-center of the perturbed problem ($P_1$) and $(y^0, s^0)$ the $\mu^0$-center of ($D_1$). In other words, $(x(1), y(1), s(1)) = \big(x^0, y^0, s^0\big)$.

## 3.3 A full-Newton step infeasible IPM algorithm

We just established that if $\nu = 1$ and $\mu = \mu^0$, then $x = x^0$ is the $\mu$-center of the perturbed problem ($P_\nu$) and $(y, s) = (y^0, s^0)$ the $\mu$-center of ($D_\nu$). This is our initial iterate.

We measure proximity to the $\mu$-center of the perturbed problems by the quantity $\delta(x, s; \mu)$ as defined in (2.4). Thus, initially we have $\delta(x, s; \mu) = 0$. In what follows we assume that at the start of each iteration, just before the feasibility step, $\delta(x, s; \mu)$ is smaller than or equal to a (small) threshold value $\tau > 0$. So this is certainly true at the start of the first iteration.

Now we describe one (main) iteration of our algorithm. Suppose that for some $\mu \in (0, \mu^0]$ we have $(x, y, s)$ satisfying the feasibility conditions (3.3) and (3.4) with $\nu = \mu/\mu^0$, and such that $x^T s = n\mu$ and $\delta(x, s; \mu) \leq \tau$. We reduce $\mu$ to $\mu^+ = (1-\theta)\mu$, and accordingly $\nu$ to $\nu^+ = (1-\theta)\nu = \mu^+/\mu^0$ with $\theta \in (0, 1)$. Next, we find a new iterate $(x^+, y^+, s^+)$ that satisfies (3.3) and (3.4) with $\nu$ replaced by $\nu^+$, and such that $(x^+)^T s^+ = n\mu^+$ and $\delta(x^+, s^+; \mu^+) \leq \tau$.

To be more precise, this is achieved as follows. Each main iteration consists of a feasibility step and a few centering steps. The feasibility step serves to get an iterate $(x^f, y^f, s^f)$ that is strictly feasible for ($P_{\nu^+}$) and ($D_{\nu^+}$) and is close to their $\mu^+$-center $(x(\nu^+), y(\nu^+), s(\nu^+))$. In fact, the feasibility step is designed in such a way that $\delta(x^f, s^f; \mu^+) \leq 1/\sqrt[4]{2}$, i.e., $(x^f, y^f, s^f)$ lies in the quadratic convergence neighborhood with respect to the $\mu^+$-center of ($P_{\nu^+}$) and ($D_{\nu^+}$). We then can easily get an iterate $(x^+, y^+, s^+)$ that is strictly feasible for ($P_{\nu^+}$) and ($D_{\nu^+}$) and such that $(x^+)^T s^+ = n\mu^+$ and $\delta(x^+, s^+; \mu^+) \leq \tau$, just by performing a few centering steps starting from $(x^f, y^f, s^f)$ and targeting at the $\mu^+$-center of ($P_{\nu^+}$) and ($D_{\nu^+}$).

A graphical illustration of one main iteration of the algorithm is given in Figure 3.1. Again, the figure is drawn in the $xs$-space projected onto its first two coordinates. Hence, the central paths of the perturbed problems are straight lines. We start with $z^0$, which is feasible and close to the $\mu$-center of the perturbed problem pair ($P_\nu$) and ($D_\nu$). After the feasibility step, the iterate is feasible for the new perturbed problem pair ($P_{\nu^+}$) and ($D_{\nu^+}$). In addition, this iterate lies in the quadratic convergence neighborhood with respect to the $\mu^+$-center of the new perturbed problem pair (the region shaded with light gray). Then we perform several centering steps, which bring our iterate close enough to the $\mu^+$-center of the new perturbed problem pair.

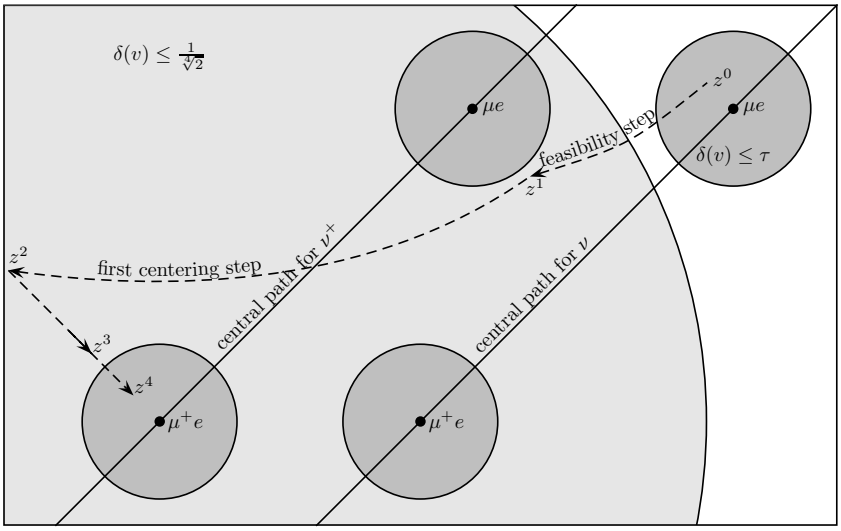


Figure 3.1: One main iteration of the full-Newton step infeasible IPM.

A formal description of the algorithm is given in Algorithm 3.1. Recall that after each iteration the residuals and the duality gap are reduced by the factor $(1-\theta)$. The algorithm stops if the norms of the residuals and the duality gap are less than the accuracy parameter $\varepsilon$.

## 3.4 Analysis of the feasibility step

In this section, we define and analyze the feasibility step. This is the most difficult part of the analysis. In essence we follow the same chain of arguments as in [92], but at several places the analysis is simpler, more precise, and also more elegant.

**Algorithm 3.1** A full-Newton step infeasible IPM for LO.

**Input:**
accuracy parameter $\varepsilon > 0$;
update parameter $\theta$, $0 < \theta < 1$;
threshold parameter $\tau > 0$;
initialization parameter $\zeta > 0$.
**Begin**
$x := \zeta e$; $y := 0$; $s := \zeta e$; $\mu := \mu^0 = \zeta^2$; $\nu := 1$;
**while** $\max(x^T s, \|b - Ax\|, \|c - A^T y - s\|) \geq \varepsilon$
feasibility step:
$(x, y, s) := (x, y, s) + (\Delta^f x, \Delta^f y, \Delta^f s)$;
update of $\mu$ and $\nu$:
$\mu := (1 - \theta)\mu$; $\nu := (1 - \theta)\nu$;
centering steps:
**while** $\delta(x, s; \mu) \geq \tau$
$(x, y, s) := (x, y, s) + (\Delta x, \Delta y, \Delta s)$.
**endwhile**
**endwhile**
**end**

### 3.4.1 Definition

In this subsection, we describe the feasibility step in detail. The analysis will follow in subsequent subsections. Suppose we have strictly feasible iterate $(x, y, s)$ for $(P_\nu)$ and $(D_\nu)$. This means that $(x, y, s)$ satisfies (3.3) and (3.4), with $\nu = \mu/\mu^0$. We need displacements $\Delta^f x$, $\Delta^f y$, and $\Delta^f s$ such that

$$\begin{aligned} x^f &= x + \Delta^f x, \\ y^f &= y + \Delta^f y, \\ s^f &= s + \Delta^f s, \end{aligned}$$

are feasible for $(P_{\nu^+})$ and $(D_{\nu^+})$. One may easily verify that $(x^f, y^f, s^f)$ satisfies (3.3) and (3.4), with $\nu$ replaced by $\nu^+ = (1 - \theta)\nu$, only if the first two equations in the following system are satisfied.

$$A\Delta^f x = \theta\nu r_p^0, \tag{3.5}$$

$$A^T \Delta^f y + \Delta^f s = \theta\nu r_d^0, \tag{3.6}$$

$$s\Delta^f x + x\Delta^f s = (1 - \theta)\mu e - xs. \tag{3.7}$$

The third equation is inspired by the third equation in system (2.3) that we used to define search directions for the feasible IPM, except that we are now targeting at the $\mu^+$-center of $(P_{\nu^+})$ and $(D_{\nu^+})$.

Indeed, (3.7) is the linearization of $x^f s^f = (1-\theta)\mu e$. While in [92], the linearization of $x^f s^f = \mu e$ is used (targeting at the $\mu$-center), and in [62], the linearization of $x^f s^f = xs$ (targeting at the old $xs$). As our aim is to calculate a feasible solution to the problem pair $(\mathrm{P}_{\nu^+})$ and $(\mathrm{D}_{\nu^+})$, which should also lie in the quadratic convergence neighborhood with respect to its $\mu^+$-center, the direction used here (directly targeting at the $\mu^+$-center of $(\mathrm{P}_{\nu^+})$ and $(\mathrm{D}_{\nu^+})$) is more natural and intuitively better.

We conclude that after the feasibility step the iterate satisfies the affine equations (3.3) and (3.4), with $\nu = \nu^+$. The hard part in the analysis will be to guarantee that $x^f$ and $s^f$ are positive and satisfy $\delta(x^f, s^f; \mu^+) \leq 1/\sqrt[4]{2}$.

### 3.4.2 Feasibility

As we have established in Subsection 3.4.1, the feasibility step generates new iterate $(x^f, y^f, s^f)$ that satisfies the feasibility conditions for $(\mathrm{P}_{\nu^+})$ and $(\mathrm{D}_{\nu^+})$, except possibly the nonnegativity constraints. This is handled in this subsection.

Define

$$d_x^f := \frac{v\Delta^f x}{x}, \quad d_s^f := \frac{v\Delta^f s}{s}, \tag{3.8}$$

where $v$ is defined in (2.4). We have, using (3.7) and (3.8),

$$\begin{aligned} x^f s^f &= xs + (s\Delta^f x + x\Delta^f s) + \Delta^f x \Delta^f s \\ &= (1-\theta)\mu e + \Delta^f x \Delta^f s \\ &= \mu\left[(1-\theta)e + d_x^f d_s^f\right], \end{aligned} \tag{3.9}$$

where the last equality follows from $xs = \mu v^2$.

**Lemma 3.2** (cf. [93, Lemma II.46]). *The iterate $(x^f, y^f, s^f)$ is feasible if and only if $(1-\theta)e + d_x^f d_s^f \geq 0$ and strictly feasible if and only if $(1-\theta)e + d_x^f d_s^f > 0$.*

*Proof.* The "only if" part of both statements in the lemma follows immediately from (3.9). For the proof of the converse implication we introduce a step length $\alpha \in [0,1]$, and define

$$\begin{aligned} x^\alpha &= x + \alpha\Delta^f x, \\ s^\alpha &= s + \alpha\Delta^f s. \end{aligned}$$

We then have $x^0 = x$, $x^1 = x^f$ and similarly $s^0 = s$, $s^1 = s^f$. Hence, we have $x^0 s^0 = xs > 0$. The proof uses a continuity argument, namely that $x^1$ and $s^1$ are nonnegative if $x^\alpha s^\alpha$ is positive for all $\alpha$ in the open interval $(0,1)$. We write

$$x^\alpha s^\alpha = (x + \alpha\Delta^f x)(s + \alpha\Delta^f s) = xs + \alpha(s\Delta^f x + x\Delta^f s) + \alpha^2 \Delta^f x \Delta^f s.$$

Using (2.4), (3.7) and (3.8) we obtain

$$\begin{aligned} x^\alpha s^\alpha &= xs + \alpha[(1-\theta)\mu e - xs] + \alpha^2 \Delta^f x \Delta^f s \\ &= \mu[(1-\alpha)v^2 + \alpha(1-\theta)e + \alpha^2 d_x^f d_s^f]. \end{aligned}$$

Suppose $(1-\theta)e + d_x^f d_s^f \geq 0$, i.e., $d_x^f d_s^f \geq -(1-\theta)e$. Substitution gives

$$\begin{aligned} x^\alpha s^\alpha &\geq \mu\left[(1-\alpha)v^2 + \alpha(1-\theta)e - \alpha^2(1-\theta)e\right] \\ &= \mu(1-\alpha)\left[v^2 + \alpha(1-\theta)e\right]. \end{aligned}$$

Since $v^2$ and $e$ are positive it follows that $x^\alpha s^\alpha > 0$ for $0 \leq \alpha < 1$. Hence, none of the entries of $x^\alpha$ and $s^\alpha$ vanish for $0 \leq \alpha < 1$. Since $x^0$ and $s^0$ are positive, this implies that $x^\alpha > 0$ and $s^\alpha > 0$ for $0 \leq \alpha < 1$. Therefore, by continuity, the vectors $x^1$ and $s^1$ cannot have negative entries. This completes the proof of the first statement in the lemma.

Assuming $(1-\theta)e + d_x^f d_s^f > 0$, we derive in the same way

$$x^\alpha s^\alpha > \mu(1-\alpha)\left[v^2 + \alpha(1-\theta)e\right].$$

This implies that $x^1 s^1 > 0$. Hence, by continuity, $x^1$ and $s^1$ must be positive, proving the second statement in the lemma. □

### 3.4.3 Proximity

A crucial element in the analysis is to show that after the feasibility step we have

$$\delta(x^f, s^f; \mu^+) \leq 1/\sqrt[4]{2},$$

i.e., the iterate $(x^f, y^f, s^f)$ is within the neighborhood where the Newton process targeting at the $\mu^+$-center of $(\mathrm{P}_{\nu^+})$ and $(\mathrm{D}_{\nu^+})$ is quadratically convergent.

We proceed by deriving an upper bound for $\delta(x^f, s^f; \mu^+)$. According to definition (2.4) one has

$$\delta(x^f, s^f; \mu^+) = \frac{1}{2}\left\| v^f - \frac{e}{v^f} \right\|, \quad \text{where } v^f = \sqrt{\frac{x^f s^f}{\mu^+}}.$$

In the sequel we denote $\delta(x^f, s^f; \mu^+)$ also shortly by $\delta(v^f)$, and we have the following result.

**Lemma 3.3.** *If* $\|d_x^f d_s^f\|_\infty < 1-\theta$, *then*

$$4\delta(v^f)^2 \leq \frac{\left\| \frac{d_x^f d_s^f}{1-\theta} \right\|^2}{1 - \left\| \frac{d_x^f d_s^f}{1-\theta} \right\|_\infty}.$$

*Proof.* After division of both sides in (3.9) by $\mu^+$ we get

$$(v^f)^2 = \frac{\mu\left[(1-\theta)e + d_x^f d_s^f\right]}{\mu^+} = e + \frac{d_x^f d_s^f}{1-\theta}.$$

Hence, we have

$$\begin{aligned} 4\delta(v^f)^2 &= \sum_{i=1}^{n}\left((v_i^f)^2+(v_i^f)^{-2}-2\right) = \sum_{i=1}^{n}\left(1+\frac{d_{xi}^f d_{si}^f}{1-\theta}+\frac{1}{1+\frac{d_{xi}^f d_{si}^f}{1-\theta}}-2\right) \\ &= \sum_{i=1}^{n}\frac{\left(\frac{d_{xi}^f d_{si}^f}{1-\theta}\right)^2}{1+\frac{d_{xi}^f d_{si}^f}{1-\theta}} \le \sum_{i=1}^{n}\frac{\left(\frac{d_{xi}^f d_{si}^f}{1-\theta}\right)^2}{1-\left|\frac{d_{xi}^f d_{si}^f}{1-\theta}\right|} \le \sum_{i=1}^{n}\frac{\left(\frac{d_{xi}^f d_{si}^f}{1-\theta}\right)^2}{1-\left\|\frac{d_x^f d_s^f}{1-\theta}\right\|_\infty} \\ &= \frac{\left\|\frac{d_x^f d_s^f}{1-\theta}\right\|^2}{1-\left\|\frac{d_x^f d_s^f}{1-\theta}\right\|_\infty}, \end{aligned}$$

where the inequalities are due to $\left\|d_x^f d_s^f\right\|_\infty < 1-\theta$. This proves the lemma. □

As we may easily verify that

$$\left\|d_x^f d_s^f\right\|_\infty \le \left\|d_x^f d_s^f\right\| \le \left\|d_x^f\right\|\left\|d_s^f\right\| \le \frac{1}{2}\left(\left\|d_x^f\right\|^2+\left\|d_s^f\right\|^2\right), \tag{3.10}$$

substitution into the inequality of Lemma 3.3 yields that

$$4\delta(v^f)^2 \le \frac{\left\|\frac{d_x^f d_s^f}{1-\theta}\right\|^2}{1-\left\|\frac{d_x^f d_s^f}{1-\theta}\right\|_\infty} \le \frac{\frac{1}{4}\left(\frac{\left\|d_x^f\right\|^2+\left\|d_s^f\right\|^2}{1-\theta}\right)^2}{1-\frac{1}{2}\frac{\left\|d_x^f\right\|^2+\left\|d_s^f\right\|^2}{1-\theta}}. \tag{3.11}$$

Thus, we have derived an upper bound for $\delta(v^f)$, but in terms of $\left\|d_x^f\right\|^2+\left\|d_s^f\right\|^2$. To proceed, we need an upper bound for $\left\|d_x^f\right\|^2+\left\|d_s^f\right\|^2$.

### 3.4.4 Upper bound for $\left\|d_x^f\right\|^2+\left\|d_s^f\right\|^2$

Obtaining an upper bound for $\left\|d_x^f\right\|^2+\left\|d_s^f\right\|^2$ is the subject of this subsection. In the sequel this will enable us to find a default value for the update parameter $\theta$, such that after the feasibility step the iterate $(x^f, y^f, s^f)$ lies in the quadratic convergence neighborhood with respect to the $\mu^+$-center of the perturbed problems $(P_{\nu^+})$ and $(D_{\nu^+})$, namely $\delta(x^f, s^f; \mu^+) \le 1/\sqrt[4]{2}$.

One may easily check that the system (3.5)–(3.7), which defines the search directions $\Delta^f x$, $\Delta^f y$, and $\Delta^f s$, can be expressed in terms of the scaled search directions $d_x^f$ and $d_s^f$ as follows:

$$\bar{A}d_x^f = \theta\nu r_p^0, \tag{3.12}$$

$$\bar{A}^T\frac{\Delta^f y}{\mu}+d_s^f = \theta\nu v s^{-1} r_d^0, \tag{3.13}$$

$$d_x^f + d_s^f = (1-\theta)v^{-1} - v, \tag{3.14}$$

where

$$\bar{A} = AV^{-1}X, \quad V = \mathrm{diag}(v), \quad X = \mathrm{diag}(x).$$

From the above definition of $\bar{A}$ we deduce that $\bar{A} = \sqrt{\mu}\, AD$, where

$$D = \mathrm{diag}\left(\frac{xv^{-1}}{\sqrt{\mu}}\right) = \mathrm{diag}\left(\sqrt{\frac{x}{s}}\right) = \mathrm{diag}\left(\sqrt{\mu}\, vs^{-1}\right).$$

For the moment, let us define

$$\bar{r}_p := \theta\nu r_p^0, \quad \bar{r}_d := \theta\nu r_d^0, \quad \bar{r} := (1-\theta)v^{-1} - v. \tag{3.15}$$

With $\xi := -\frac{\Delta^f y}{\mu}$ we then have (by eliminating $d_s^f$ from (3.12)–(3.14))

$$\sqrt{\mu}\, AD d_x^f = \bar{r}_p, \tag{3.16}$$

$$\sqrt{\mu}\, DA^T\xi + d_x^f = \bar{r} - \frac{1}{\sqrt{\mu}} D\bar{r}_d. \tag{3.17}$$

By multiplying both sides of (3.17) from the left with $\sqrt{\mu}\, AD$ and using (3.16) it follows that

$$\mu AD^2A^T\xi + \bar{r}_p = \sqrt{\mu}\, AD\left(\bar{r} - \frac{1}{\sqrt{\mu}} D\bar{r}_d\right).$$

Therefore,

$$\xi = \frac{1}{\mu}(AD^2A^T)^{-1}\left[\sqrt{\mu}\, AD\left(\bar{r} - \frac{1}{\sqrt{\mu}} D\bar{r}_d\right) - \bar{r}_p\right].$$

Substitution into (3.17) gives

$$\begin{aligned} d_x^f &= \bar{r} - \frac{1}{\sqrt{\mu}} D\bar{r}_d - \frac{1}{\sqrt{\mu}}\, DA^T(AD^2A^T)^{-1}\left[\sqrt{\mu}\, AD\left(\bar{r} - \frac{1}{\sqrt{\mu}} D\bar{r}_d\right) - \bar{r}_p\right] \\ &= \left[I - DA^T(AD^2A^T)^{-1}AD\right]\left(\bar{r} - \frac{1}{\sqrt{\mu}} D\bar{r}_d\right) + \frac{1}{\sqrt{\mu}}\, DA^T(AD^2A^T)^{-1}\bar{r}_p. \end{aligned}$$

To simplify notation we denote

$$\bar{P} = DA^T(AD^2A^T)^{-1}AD.$$

Note that $\bar{P}$ is (the matrix of) the orthogonal projection to the row space of the matrix $AD$. We now may write

$$d_x^f = [I - \bar{P}]\left(\bar{r} - \frac{1}{\sqrt{\mu}} D\bar{r}_d\right) + \frac{1}{\sqrt{\mu}}\, DA^T(AD^2A^T)^{-1}\bar{r}_p.$$

Let $(\bar{x}, \bar{y}, \bar{s})$ be such that $A\bar{x} = b$ and $A^T\bar{y} + \bar{s} = c$. Then we may write

$$\begin{aligned} \bar{r}_p &= \theta\nu r_p^0 = \theta\nu(b - Ax^0) = \theta\nu A(\bar{x} - x^0), \\ \bar{r}_d &= \theta\nu r_d^0 = \theta\nu(c - A^Ty^0 - s^0) = \theta\nu\left(A^T(\bar{y} - y^0) + \bar{s} - s^0\right). \end{aligned}$$

Thus we obtain

$$d_x^f = [I - \bar{P}] \left( \bar{r} - \frac{\theta\nu}{\sqrt{\mu}} D \left( A^T(\bar{y} - y^0) + \bar{s} - s^0 \right) \right) + \frac{\theta\nu}{\sqrt{\mu}} \bar{P} D^{-1}(\bar{x} - x^0).$$

Since $I - \bar{P}$ is the orthogonal projection to the null space of $AD$, we have

$$[I - \bar{P}] D A^T (\bar{y} - y^0) = 0,$$

and the expression for $d_x^f$ reduces to

$$d_x^f = [I - \bar{P}] \left( \bar{r} - \frac{\theta\nu}{\sqrt{\mu}} D \left( \bar{s} - s^0 \right) \right) + \frac{\theta\nu}{\sqrt{\mu}} \bar{P} D^{-1}(\bar{x} - x^0).$$

To proceed we further simplify the notation by defining

$$u^x = \frac{\theta\nu}{\sqrt{\mu}} D^{-1}(\bar{x} - x^0), \quad u^s = \frac{\theta\nu}{\sqrt{\mu}} D(\bar{s} - s^0). \tag{3.18}$$

Then we may write

$$d_x^f = [I - \bar{P}](\bar{r} - u^s) + \bar{P} u^x.$$

For $d_s^f$ we obtain, by using (3.14) and the definition of $\bar{r}$ in (3.15),

$$d_s^f = \bar{r} - d_x^f = \bar{r} - [I - \bar{P}]\bar{r} + [I - \bar{P}]u^s - \bar{P}u^x = [I - \bar{P}]u^s + \bar{P}(\bar{r} - u^x).$$

We denote $[I - \bar{P}]\bar{r} = \bar{r}_1$ and $\bar{P}\bar{r} = \bar{r}_2$, and use similar notations for the projections of $u^x$ and $u^s$. Then from the above expressions for $d_x^f$ and $d_s^f$ we derive that

$$\begin{aligned} d_x^f &= \bar{r}_1 - u_1^s + u_2^x, \\ d_s^f &= u_1^s + \bar{r}_2 - u_2^x. \end{aligned}$$

Therefore, using orthogonality of the vectors with different subscripts, we may write

$$\begin{aligned} &\left\| d_x^f \right\|^2 + \left\| d_s^f \right\|^2 \\ &= \|\bar{r}_1 - u_1^s\|^2 + \|u_2^x\|^2 + \|u_1^s\|^2 + \|\bar{r}_2 - u_2^x\|^2 \\ &= \|\bar{r}_1\|^2 + \|u_1^s\|^2 - 2\bar{r}_1^T u_1^s + \|u_2^x\|^2 + \|u_1^s\|^2 + \|\bar{r}_2\|^2 + \|u_2^x\|^2 - 2\bar{r}_2^T u_2^x \\ &= \|\bar{r}\|^2 + 2\|u_2^x\|^2 + 2\|u_1^s\|^2 - 2\bar{r}_1^T u_1^s - 2\bar{r}_2^T u_2^x. \end{aligned}$$

Further by the Cauchy-Schwartz inequality and the properties of orthogonal projection, we obtain

$$\begin{aligned} \left\| d_x^f \right\|^2 + \left\| d_s^f \right\|^2 &\le \|\bar{r}\|^2 + 2\|u_2^x\|^2 + 2\|u_1^s\|^2 + 2\|\bar{r}_1\|\,\|u_1^s\| + 2\|\bar{r}_2\|\,\|u_2^x\| \\ &\le \|\bar{r}\|^2 + 2\|u_2^x\|^2 + 2\|u_1^s\|^2 + \|\bar{r}_1\|^2 + \|u_1^s\|^2 + \|\bar{r}_2\|^2 + \|u_2^x\|^2 \\ &\le 2\|\bar{r}\|^2 + 3\left( \|u^x\|^2 + \|u^s\|^2 \right). \end{aligned} \tag{3.19}$$

Since $v$ and $v^{-1} - v$ are orthogonal and $\|v\|^2 = n$, we have from (3.15) and the definition of $\delta$

$$\begin{aligned}\|\bar{r}\|^2 &= \left\|(1-\theta)v^{-1} - v\right\|^2 = \left\|(1-\theta)(v^{-1}-v) - \theta v\right\|^2 \\ &= (1-\theta)^2 \left\|v^{-1} - v\right\|^2 + \theta^2 \|v\|^2 = 4(1-\theta)^2\delta^2 + \theta^2 n. \end{aligned} \tag{3.20}$$

Due to (3.18) we have

$$\|u^x\|^2 + \|u^s\|^2 = \frac{\theta^2\nu^2}{\mu}\left(\left\|D^{-1}(\bar{x}-x^0)\right\|^2 + \left\|D(\bar{s}-s^0)\right\|^2\right). \tag{3.21}$$

Let $(x^*, y^*, s^*)$ be an optimal solution satisfying (3.2). It follows that $Ax^* = b$ and $A^T y^* + s^* = c$. Therefore we may take $\bar{x} = x^*$, $\bar{y} = y^*$ and $\bar{s} = s^*$. Since $x^*$ is feasible for (P) we have $x^* \geq 0$. Similarly we derive $s^* \geq 0$. Hence we have $0 \leq x^* \leq x^* + s^* \leq \zeta e$, or equivalently $0 \leq \bar{x} \leq \zeta e$. In the same way we derive that $0 \leq \bar{s} \leq \zeta e$. Moreover, we derive that

$$0 \leq x^0 - \bar{x} \leq \zeta e, \quad 0 \leq s^0 - \bar{s} \leq \zeta e.$$

Thus, it follows that

$$\begin{aligned}&\left\|D^{-1}(\bar{x}-x^0)\right\|^2 + \left\|D(\bar{s}-s^0)\right\|^2 \\ &\leq \zeta^2\left(\left\|D^{-1}e\right\|^2 + \|De\|^2\right) = \zeta^2 e^T\left(\frac{s}{x} + \frac{x}{s}\right) \\ &= \zeta^2 e^T\left(\frac{x^2+s^2}{xs}\right) \leq \frac{\zeta^2 e^T\left(x^2+s^2\right)}{\min_i |x_i s_i|} \leq \frac{\zeta^2\left[e^T(x+s)\right]^2}{\mu \min_i v_i^2}. \end{aligned} \tag{3.22}$$

Summarizing, while using (3.19), (3.20), (3.21), (3.22) and $\mu = \nu\zeta^2$, we obtain

$$\left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2 \leq 8(1-\theta)^2\delta^2 + 2\theta^2 n + \frac{3\theta^2\left[e^T(x+s)\right]^2}{\zeta^2 \min_i v_i^2}. \tag{3.23}$$

To continue, we need an upper bound for $e^T(x+s)$ and a lower bound for $\min_i v_i$, which we derive in the next subsection.

### 3.4.5 Bounds for $e^T(x+s)$ and $v_i$

Recall that $x$ is feasible for $(\mathrm{P}_\nu)$ and $(y,s)$ for $(\mathrm{D}_\nu)$ with $x^T s = n\mu$ and, moreover, $\delta(x,s;\mu) \leq \tau$, i.e., the iterate is close to the $\mu$-center of $(\mathrm{P}_\nu)$ and $(\mathrm{D}_\nu)$. Based on this information, we present the following two lemmas to estimate an upper bound for $e^T(x+s)$ and a lower bound for $\min_i v_i$.

**Lemma 3.4.** *Let $x$ and $(y,s)$ be feasible for the perturbed problems $(\mathrm{P}_\nu)$ and $(\mathrm{D}_\nu)$, respectively, with $x^T s = n\mu$, $\zeta$ as defined in (3.2), and $(x^0, y^0, s^0)$ as in (3.1). We then have*

$$e^T(x+s) \leq 2n\zeta.$$

*Proof.* Let $(x^*, y^*, s^*)$ be an optimal solution satisfying (3.2). Then from the feasibility conditions (3.3) and (3.4) of the perturbed problems $(P_\nu)$ and $(D_\nu)$, it is easily seen that

$$\begin{aligned} A\left[x - \nu x^0 - (1-\nu)x^*\right] &= 0, \\ A^T\left[y - \nu y^0 - (1-\nu)y^*\right] + \left[s - \nu s^0 - (1-\nu)s^*\right] &= 0. \end{aligned}$$

This implies that $[x - \nu x^0 - (1-\nu)x^*]$ and $[s - \nu s^0 - (1-\nu)s^*]$ belong to the null space and row space of $A$, respectively. Thus,

$$\left[x - \nu x^0 - (1-\nu)x^*\right]^T \left[s - \nu s^0 - (1-\nu)s^*\right] = 0.$$

By expanding the above equality and using the fact that $(x^*)^T s^* = 0$, we obtain

$$\begin{aligned} \nu\left(\left(x^0\right)^T s + \left(s^0\right)^T x\right) = x^T s + \nu^2 \left(x^0\right)^T s^0 - (1-\nu)\left(x^T s^* + s^T x^*\right) \\ + \nu(1-\nu)\left(\left(x^0\right)^T s^* + \left(s^0\right)^T x^*\right). \quad (3.24) \end{aligned}$$

Since $(x^0, y^0, s^0)$ is as defined in (3.1), we have

$$\begin{aligned} \left(x^0\right)^T s + \left(s^0\right)^T x &= \zeta e^T (x+s), \\ \left(x^0\right)^T s^0 &= n\zeta^2, \\ \left(x^0\right)^T s^* + \left(s^0\right)^T x^* &= \zeta e^T (x^* + s^*). \end{aligned}$$

Due to (3.2) we have $e^T(x^* + s^*) \le n\zeta$. Furthermore, $x^T s = n\mu = \nu\zeta^2 n$, and $x^T s^* + s^T x^* \ge 0$. Substitution of these relations into (3.24) gives

$$\nu\zeta e^T(x+s) \le \nu\zeta^2 n + n\nu^2\zeta^2 + \nu(1-\nu)n\zeta^2 = 2n\nu\zeta^2.$$

This implies the lemma. □

**Lemma 3.5** (cf. [93, Theorem II.62]). *Let $\rho(\delta) = \delta + \sqrt{1+\delta^2}$. Then*

$$\frac{1}{\rho(\delta)} \le v_i \le \rho(\delta), \quad 1 \le i \le n.$$

Substituting the results of the above two lemmas into (3.23), we obtain

$$\begin{aligned} \left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2 &\le 8(1-\theta)^2\delta^2 + 2\theta^2 n + 12\theta^2 n^2 \rho(\delta)^2 \\ &\le 8\delta^2 + 2\theta^2 n + 12\theta^2 n^2 \rho(\delta)^2. \quad (3.25) \end{aligned}$$

**Remark 3.6.** In [92], the bound for $e^T(x/s + s/x)$ is derived as follows:

$$e^T\left(\frac{x}{s} + \frac{s}{x}\right) \le 2e^T\left(\frac{x(\nu)^2}{\mu} + \frac{s(\nu)^2}{\mu}\right) = \frac{4n\zeta^2}{\mu}\kappa(\zeta,\nu)^2 \le \frac{4n\zeta^2}{\mu}\bar{\kappa}(\zeta)^2.^1$$

It was proved that $\bar{\kappa}(\zeta) \leq \sqrt{2n}$ and conjectured, based on extensive numerical tests, that $\bar{\kappa}(\zeta) \leq 1$. As the complexity bound depends linearly on the parameter $\bar{\kappa}(\zeta)$, this would yield an $O(\sqrt{n}\log(n/\varepsilon))$ iteration bound full-Newton step infeasible IPM for LO. Unfortunately, the conjecture turns out to be not true. A counter example is given in Appendix A. During the construction of the counter example, we have a byproduct that $e^T(x(\nu)^2 + s(\nu)^2)$ is indeed in the order of $n^2\zeta^2$. Since $x(\nu)$ and $s(\nu)$ are special cases of our $x$ and $s$, we may conclude that our bound for $e^T(x^2 + s^2)$ derived here, namely,

$$e^T(x^2 + s^2) \leq \left[e^T(x + s)\right]^2 \leq 4n^2\zeta^2,^2$$

is tight in the order.

## 3.5 Choosing the update parameter $\boldsymbol{\theta}$

We want to choose $\theta$, with $0 < \theta < 1$, as large as possible, and such that $(x^f, y^f, s^f)$ lies in the quadratic convergence neighborhood with respect to the $\mu^+$-center of the perturbed problems $(P_{\nu^+})$ and $(D_{\nu^+})$, i.e., $\delta(x^f, s^f; \mu^+) \leq 1/\sqrt[4]{2}$. Using (3.11), we derive this is the case when

$$\frac{\frac{1}{4}\left(\frac{\left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2}{1-\theta}\right)^2}{1 - \frac{1}{2}\frac{\left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2}{1-\theta}} \leq 2\sqrt{2}.$$

Considering $\frac{\left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2}{1-\theta}$ as a single term, and by some elementary calculations, we obtain that the above inequality is equivalent to

$$\frac{\left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2}{1-\theta} \leq 2\sqrt{2}\left(\sqrt{1+\sqrt{2}} - 1\right) \approx 1.566. \tag{3.26}$$

Note that by (3.10), this implies

$$\left\|d_x^f d_s^f\right\|_\infty \leq \frac{1}{2}\left(\left\|d_x^f\right\|^2 + \left\|d_s^f\right\|^2\right) \leq \sqrt{2}\left(\sqrt{1+\sqrt{2}} - 1\right)(1-\theta) \approx 0.783(1-\theta),$$

which, by Lemma 3.2, guarantees that the iterate $(x^f, y^f, s^f)$ is strictly feasible. The above inequality also implies that the condition for Lemma 3.3 holds.

Due to (3.25), the inequality (3.26) holds if

$$8\delta^2 + 2\theta^2 n + 12\theta^2 n^2 \rho(\delta)^2 \leq 2\sqrt{2}\left(\sqrt{1+\sqrt{2}} - 1\right)(1-\theta).$$

[1] The definitions of $\kappa(\zeta, \nu)$ and $\bar{\kappa}(\zeta)$ can be found in Section A.1.

[2] The inequalities follow from (3.22) and Lemma 3.4.

For the threshold parameter, we choose $\tau = 1/16$.[3] As already indicated, we have $\delta \leq \tau$. Obviously, the left-hand side of the above inequality is increasing in $\delta$, due to the definition $\rho(\delta) = \delta + \sqrt{1+\delta^2}$. Using this fact one may easily verify that if

$$\theta = \frac{1}{4n}, \tag{3.27}$$

then the above inequality is satisfied, i.e., with $\theta$ as chosen above, the iterate $(x^f, y^f, s^f)$ is strictly feasible and satisfies $\delta(x^f, s^f; \mu^+) \leq 1/\sqrt[4]{2}$.

## 3.6 Iteration bound

In the previous sections we have found that if at the start of a main iteration the iterate satisfies $\delta(x, s; \mu) \leq \tau$, with $\tau = 1/16$, then after the feasibility step, with $\theta$ as defined in (3.27), the iterate satisfies $\delta(x^f, s^f; \mu^+) \leq 1/\sqrt[4]{2}$.

After the feasibility step we perform a few centering steps in order to get the iterate $(x^+, y^+, s^+)$ which satisfies $(x^+)^T s^+ = n\mu^+$ and $\delta(x^+, s^+; \mu^+) \leq \tau$. By using Corollary 2.6, the required number of centering steps can easily be obtained. Indeed, assuming $\delta(x^f, s^f; \mu^+) \leq 1/\sqrt[4]{2}$, after $k$ centering steps we will have iterate $(x^+, y^+, s^+)$ that is still feasible for $(\mathrm{P}_{\nu^+})$ and $(\mathrm{D}_{\nu^+})$ and satisfies

$$\delta(x^+, s^+; \mu^+) \leq \left(\frac{1}{\sqrt[4]{2}}\right)^{2^k}.$$

From this one easily deduces that $\delta(x^+, s^+; \mu^+) \leq \tau$ will hold after at most

$$2 + \left\lceil \log_2 \left( \log_2 \frac{1}{\tau} \right) \right\rceil \tag{3.28}$$

centering steps.

With $\tau = 1/16$, we have that, after the feasibility step, at most

$$2 + \left\lceil \log_2 \left( \log_2 \frac{1}{\tau} \right) \right\rceil = 4$$

centering steps then suffice to get the iterate $(x^+, y^+, s^+)$ satisfy $\delta(x^+, s^+; \mu^+) \leq \tau$ again. Thus, each main iteration consists of at most 5 inner iterations (one feasibility step and at most 4 centering steps), in each of which we need to compute a search direction (for either a feasibility step or a centering step).

It has become a custom to measure the complexity of an IPM by the required number of inner iterations. In each main iteration both the duality gap and the norms of the residual vectors are reduced by the factor $(1 - \theta)$. Hence,

[3] To make full use of the centering steps (choosing $\tau$ such that $\log_2 \left(\log_2 \frac{1}{\tau}\right)$ in (3.28) is an integer), $\tau$ may be chosen as $2^{(2^1)}, 2^{(2^2)}, 2^{(2^3)}, \ldots$. Among these values, we use $\tau = 1/16$.

using $\left(x^0\right)^T s^0 = n\zeta^2$, it follows from Lemma 2.8 that the total number of main iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max\left\{n\zeta^2, \left\|r_p^0\right\|, \left\|r_d^0\right\|\right\}}{\varepsilon}.$$

Taking the value of $\theta$ as in (3.27), the total number of inner iterations is bounded above by

$$20n \log \frac{\max\left\{n\zeta^2, \left\|r_p^0\right\|, \left\|r_d^0\right\|\right\}}{\varepsilon}.$$

Thus, we may state without further proof the main result of this chapter.

**Theorem 3.7.** *If (P) and (D) are feasible and $\zeta > 0$ is such that $\|x^* + s^*\|_\infty \leq \zeta$ for some optimal solutions $x^*$ of (P) and $(y^*, s^*)$ of (D), then after at most*

$$20n \log \frac{\max\left\{n\zeta^2, \left\|r_p^0\right\|, \left\|r_d^0\right\|\right\}}{\varepsilon}$$

*inner iterations the algorithm finds an $\varepsilon$-optimal solution of (P) and (D).*

Note that this bound is slightly better than that in [92, Theorem 4.8].

**Remark 3.8.** The above iteration bound is derived under the assumption that there exists some optimal solutions of (P) and (D) with $\|x^* + s^*\|_\infty \leq \zeta$. One might ask what happens if this condition is not satisfied. In that case, during the course of the algorithm it may happen that after some main steps the proximity measure $\delta$ (after the feasibility step) exceeds $1/\sqrt[4]{2}$, because otherwise there is no reason why the algorithm would not generate an $\varepsilon$-optimal solution. So if this happens it tell us that the problems (P) and (D) do not have any optimal solution that satisfies $\|x^* + s^*\|_\infty \leq \zeta$. Recall that our starting point is defined in (3.1), which depends on $\zeta$. It may happen that the value of $\zeta$ has been chosen too small. If it is the case one might run the algorithm once more with a larger $\zeta$.

# Chapter 4

# Analysis of Symmetric Cones

For LO, we have recalled a full-Newton step feasible IPM and derived an improved full-Newton step infeasible IPM. In the following chapters, we generalize these methods to the more general class of SO problems. In order to do this we need to deal first with Jordan algebras.

Jordan algebras were initially created in quantum mechanics, and they turned out to have a very large spectrum of applications. Indeed, some Jordan algebras were proved more than a decade ago to be an indispensable tool in the unified study of IPMs for SO [23].

As a preparation, this chapter offers an introduction to the theory of Jordan algebras, Euclidean Jordan algebras and symmetric cones. The presentation is mainly based on Faraut and Korányi [20]. After presenting the basic properties of Jordan algebras and Euclidean Jordan algebras, we state a fundamental result, due to M. Koecher and E.B. Vinberg, namely that the cone of squares in a Euclidean Jordan algebra is a symmetric cone, and every symmetric cone is obtained in this way. Based on this, we derive some more properties of Euclidean Jordan algebras and their associated symmetric cones, as needed for the optimization techniques we present later. We restrict ourselves to finite-dimensional algebras over the real field $\mathbf{R}$. For omitted proofs and more details, we refer to the given references and also to [7, 20, 47, 53, 63, 109].

## 4.1 Jordan algebras

In this section, we introduce Jordan algebras as well as some of their basic properties.

### 4.1.1 Definition

**Definition 4.1** (Bilinear map)**.** Let $\mathcal{J}$ be a finite-dimensional vector space over $\mathbf{R}$. A map $\circ : \mathcal{J} \times \mathcal{J} \mapsto \mathcal{J}$ is called bilinear if for all $x, y, z \in \mathcal{J}$ and $\alpha, \beta \in \mathbf{R}$:

(i) $(\alpha x + \beta y) \circ z = \alpha(x \circ z) + \beta(y \circ z)$;

(ii) $x \circ (\alpha y + \beta z) = \alpha(x \circ y) + \beta(x \circ z)$.

**Definition 4.2** ($\mathbf{R}$-algebra)**.** A finite-dimensional vector space $\mathcal{J}$ over $\mathbf{R}$ is called an algebra over $\mathbf{R}$ if a bilinear map from $\mathcal{J} \times \mathcal{J}$ into $\mathcal{J}$ is defined.

**Definition 4.3** (Jordan algebra)**.** Let $\mathcal{J}$ be a finite-dimensional $\mathbf{R}$-algebra along with a bilinear map $\circ : \mathcal{J} \times \mathcal{J} \mapsto \mathcal{J}$. Then $(\mathcal{J}, \circ)$ is called a Jordan algebra if for all $x, y \in \mathcal{J}$ the following holds:

(i) $x \circ y = y \circ x$ (Commutativity);

(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, where $x^2 = x \circ x$ (Jordan's Axiom).

In the sequel, we always assume that $(\mathcal{J}, \circ)$ is a Jordan algebra, which we simply denote as $\mathcal{J}$. For an element $x \in \mathcal{J}$, let $L(x) : \mathcal{J} \mapsto \mathcal{J}$ be the linear map defined by

$$L(x)y := x \circ y, \quad \text{for all } y \in \mathcal{J}.$$

Consequently, Jordan's Axiom in Definition 4.3 means that the operators $L(x)$ and $L(x^2)$ commute.

We define $x^n$ $(n \geq 2)$ recursively by $x^n = x \circ x^{n-1}$. An algebra is said to be power associative if, for any $x$ in the algebra, $x^m \circ x^n = x^{m+n}$. This means that the subalgebra generated by $x$ is associative. Jordan algebras are not necessarily associative, but they are power associative [20, Proposition II.1.2].

### 4.1.2 Characteristic polynomial

Let $\mathcal{J}$ be a Jordan algebra over $\mathbf{R}$. An element $e \in \mathcal{J}$ is said to be an identity element if

$$e \circ x = x \circ e = x, \quad \text{for all } x \in \mathcal{J}.$$

Note that the identity element $e$ is unique: if $e_1$ and $e_2$ are identity elements of $\mathcal{J}$, then $e_1 = e_1 \circ e_2 = e_2$.

From now on, we always assume the existence of the identity element $e$. Let $\mathbf{R}[X]$ denote the algebra over $\mathbf{R}$ of polynomials in one variable with coefficients in $\mathbf{R}$. For an element $x$ in $\mathcal{J}$ we define

$$\mathbf{R}[x] := \{p(x) : p \in \mathbf{R}[X]\}.$$

Since $\mathcal{J}$ is a finite-dimensional vector space, for each $x \in \mathcal{J}$, there exists a positive integer $k$ (bounded above by the dimension of $\mathcal{J}$) such that $e, x, x^2, \ldots, x^k$

are linearly dependent. This implies the existence of a nonzero polynomial $p \in \mathbf{R}[X]$, such that $p(x) = 0$. If, in addition, this polynomial is monic (i.e., with the leading coefficient equal to 1) and of minimal degree, we call it the minimal polynomial of $x$.

The minimal polynomial of an element $x \in \mathcal{J}$ is unique. As if $p_1$ and $p_2$ are two distinct minimal polynomials of $x$, their difference $p_1 - p_2$ vanishes in $x$ as well. Since $p_1$ and $p_2$ are monic and of the same degree, the degree of $p_1 - p_2$ is smaller than that of $p_1$ (and $p_2$). This contradicts the minimality of the degree of $p_1$ (and $p_2$).

We define the degree of an element $x \in \mathcal{J}$, denoted as $\deg(x)$, as the degree of the minimal polynomial of $x$. Obviously, this number is bounded by the dimension of the vector space $\mathcal{J}$. Moreover, we define the rank of $\mathcal{J}$ as

$$r := \max \{\deg(x) : x \in \mathcal{J}\},$$

which is again bounded by the dimension of $\mathcal{J}$. An element $x \in \mathcal{J}$ is called regular if $\deg(x) = r$. An important result concerning regular elements is contained in the following proposition.

**Proposition 4.4** (cf. [20, Proposition II.2.1])**.** *The set of regular elements is open and dense in $\mathcal{J}$. There exist polynomials $a_1, a_2, \ldots, a_r$ on $\mathcal{J}$ such that the minimal polynomial of every regular element $x$ is given by*

$$f(\lambda; x) = \lambda^r - a_1(x)\lambda^{r-1} + a_2(x)\lambda^{r-2} + \cdots + (-1)^r a_r(x).$$

*The polynomials $a_1, \ldots, a_r$ are unique and $a_i$ is homogeneous of degree $i$.*

The polynomial $f(\lambda; x)$ is called the characteristic polynomial of the regular element $x$. Since the regular elements are dense in $\mathcal{J}$, by continuity we may extend the polynomials $a_i(x)$ and consequently the characteristic polynomial to all elements of $\mathcal{J}$. Note that the characteristic polynomial is a polynomial of degree $r$ in $\lambda$, where $r$ is the rank of $\mathcal{J}$. Moreover, the minimal polynomial coincides with the characteristic polynomial for regular elements, but it divides the characteristic polynomial of non-regular elements.

The coefficient $a_1(x)$ is called the trace of $x$, denoted as $\operatorname{tr}(x)$, and the coefficient $a_r(x)$ is called the determinant of $x$, denoted as $\det(x)$. In the following proposition, we recall an important property about the trace.

**Proposition 4.5** (cf. [20, Proposition II.4.3])**.** *The symmetric bilinear form $\operatorname{tr}(x \circ y)$ is associative, i.e.,*

$$\operatorname{tr}((x \circ y) \circ z) = \operatorname{tr}(x \circ (y \circ z)), \quad \textit{for all } x, y, z \in \mathcal{J}.$$

An element $x$ is said to be invertible if there exists an element $y$ in $\mathbf{R}[x]$ such that $x \circ y = e$. Since $\mathbf{R}[x]$ is associative, $y$ is unique. It is called the inverse of $x$ and denoted by $x^{-1}$. A relationship between $x^{-1}$ and the inverse of $L(x)$ is given below.

**Proposition 4.6** (cf. [20, Proposition II.2.3]). *If $L(x)$ is invertible, then $x$ is invertible and $x^{-1} = L(x)^{-1}e$.*

**Remark 4.7.** The equality $x \circ y = e$ does not by itself imply that $y$ is the inverse of $x$, and the fact that $x$ is invertible does not imply that $L(x)$ is invertible, as can be seen in the following example. Let $\mathbf{S}^2$ be the vector space of $2 \times 2$ symmetric matrices, and the binary operation $\circ$ defined by

$$X \circ Y = \frac{XY + YX}{2}.^1$$

Since the usual product of matrices is associative, $(\mathbf{S}^2, \circ)$ is a Jordan algebra, with its identity element equals to the $2 \times 2$ identity matrix. Let

$$X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & \alpha \\ \alpha & -1 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}, \quad \alpha \in \mathbf{R}.$$

Then $X$ is invertible and $X^{-1} = X$. We have $X \circ Y = e$, but $Y$ does not belong to $\mathbf{R}[X]$ for $\alpha \neq 0$. Also, $L(X)$ is not invertible since $L(X)Z = X \circ Z = 0$.

### 4.1.3 Quadratic representation

Let $\mathcal{J}$ be a finite-dimensional Jordan algebra over $\mathbf{R}$ with the identity element $e$. For $x \in \mathcal{J}$ we define

$$P(x) = 2L(x)^2 - L(x^2),$$

where $L(x)^2 = L(x)L(x)$. The map $P(\cdot)$ is called the quadratic representation of $\mathcal{J}$. Below we list several important properties of the quadratic representation.

**Proposition 4.8** (cf. [20, Proposition II.3.1]). *An element $x \in \mathcal{J}$ is invertible if and only if $P(x)$ is invertible. In this case,*

$$P(x)x^{-1} = x,$$
$$P(x)^{-1} = P(x^{-1}).$$

**Proposition 4.9** (cf. [20, Proposition II.3.3]). *One has:*

*(i) The differential of the map $x \mapsto x^{-1}$ is $-P(x)^{-1}$.*

*(ii) If $x$ and $y$ are invertible, then $P(x)y$ is invertible and*

$$(P(x)y)^{-1} = P(x^{-1})y^{-1}.$$

*(iii) For any two elements $x$ and $y$:*

$$P(P(y)x) = P(y)P(x)P(y).$$

In particular, the equation in (*iii*) of the above proposition is known as the fundamental formula (cf. [53, Chapter IV.1]).

[1] Normally, we use capitalized letters to denote matrices.

### 4.1.4 Examples

To make our presentation more concrete, we examine the following two examples of Jordan algebras.

*Example* 4.10 (The quadratic terms algebra $\mathbf{L}^{n+1}$). The quadratic terms algebra, or Jordan spin algebra, or spin factor is widely used in applications, ranging from statistics to relativistic mechanics. In optimization, we utilize this algebra to deal with second-order cone optimization. Let $\mathbf{L}^{n+1}$ be the $(n+1)$-dimensional real vector space whose elements are indexed from zero. For simplicity, we denote $(x_0; x_1; \ldots; x_n) \in \mathbf{R}^{n+1}$ as $x = (x_0; \bar{x})$ with $\bar{x} := (x_1; \ldots; x_n) \in \mathbf{R}^n$.[2] Define the product as

$$x \circ y := (x^T y; x_0\bar{y} + y_0\bar{x}).$$

It is easily verified that $(\mathbf{L}^{n+1}, \circ)$ is a Jordan algebra. The identity element is the vector $e = (1; 0; \ldots; 0)$. Every vector $x$ satisfies the quadratic equation

$$x^2 - 2x_0 x + (x_0^2 - \|\bar{x}\|^2)e = 0.$$

Thus, the rank of the quadratic terms algebra $\mathbf{L}^{n+1}$ is 2, independent of the dimension of its underlying vector space. Each element $x$ has two eigenvalues, $x_0 \pm \|\bar{x}\|$. In addition, $\mathrm{tr}(x) = 2x_0$ and $\det(x) = x_0^2 - \|\bar{x}\|^2$. Except for multiples of the identity element, every element has degree 2. From the definition of Jordan multiplication, it is seen that

$$L(x) = \begin{bmatrix} x_0 & \bar{x}^T \\ \bar{x} & x_0 I \end{bmatrix}.^3$$

Then, by definition, and after some elementary calculations, we easily get

$$P(x) = 2L(x)^2 - L(x^2) = \begin{bmatrix} x^T x & 2x_0\bar{x}^T \\ 2x_0\bar{x} & \det(x)I + 2\bar{x}\bar{x}^T \end{bmatrix}.$$

*Example* 4.11 (Jordan algebra of $\mathbf{S}^n$). Let $\mathbf{S}^n$ be the matrix space of $n \times n$ real symmetric matrices with the symmetrized multiplication defined as

$$X \circ Y = \frac{XY + YX}{2}.$$

Here, $XY$ denotes the usual matrix product. Commutativity and Jordan's Axiom are readily seen to be satisfied by this new multiplication. Hence, $(\mathbf{S}^n, \circ)$ is a Jordan algebra, and its identity element is the identity matrix $I$. Since $X \circ X = XX$, we can easily conclude that the powers of $X$ are equal wether we consider the Jordan algebra $(\mathbf{S}^n, \circ)$ or the algebra of symmetric matrices with the usual product.

[2] We follow the convention of some high level programming languages, such as MATLAB, and use ";" for adjoining vectors in a column.

[3] We use $I$ to denote the identity matrix of appropriate size.

Thus, the characteristic polynomials and consequently, eigenvalues, trace and determinant coincide with the usual ones in symmetric matrices. Note that $\deg(X)$ is the number of distinct eigenvalues of $X$ and, thus, is at most $n$ for an $n \times n$ symmetric matrix. In other words the rank of the Jordan algebra $(\mathbf{S}^n, \circ)$ is $n$. The "vec" operator concatenates the columns of a matrix (in the natural order) into a vector. Applying it, we get

$$\operatorname{vec}(X \circ Y) = \operatorname{vec}\left(\frac{XY + YX}{2}\right) = \frac{1}{2}(I \otimes X + X \otimes I)\operatorname{vec}(Y),$$

where $\otimes$ denotes the Kronecker product[4]. Thus, for $(\mathbf{S}^n, \circ)$,

$$L(X) = \frac{1}{2}(I \otimes X + X \otimes I).$$

By using the properties of the Kronecker product, it easily follows that

$$P(X) = X \otimes X.$$

By definition, we obtain

$$P(X)Y = XYX.$$

## 4.2 Euclidean Jordan algebras versus symmetric cones

Euclidean Jordan algebras (also called formally real Jordan algebras, cf. [7] or [20, Section III.1, Section VIII.4]) form a subclass of Jordan algebras. In this section, we recall their basic properties as well as their relation with symmetric cones.

### 4.2.1 Euclidean Jordan algebras

We consider a finite-dimensional Jordan algebra $\mathcal{J}$ over $\mathbf{R}$ and assume the existence of the identity element $e$. The Jordan algebra $\mathcal{J}$ is said to be Euclidean if there exists a positive definite symmetric bilinear form on $\mathcal{J}$ which is associative; in other words, there exists an inner product denoted by $\langle \cdot, \cdot \rangle$, such that

$$\langle x \circ y, z \rangle = \langle x, y \circ z \rangle, \quad \text{for all } x, y, z \in \mathcal{J}.$$

In the sequel, unless stated otherwise, we always assume that $\mathcal{J}$ is a Euclidean Jordan algebra with the identity element $e$. An element $c \in \mathcal{J}$ is said to be an idempotent if $c^2 = c$. Two idempotents $c_1$ and $c_2$ are said to be orthogonal if $c_1 \circ c_2 = 0$. Since

$$\langle c_1, c_2 \rangle = \langle c_1^2, c_2 \rangle = \langle c_1, c_1 \circ c_2 \rangle,$$

[4]See [44, 61] for properties of the Kronecker product.

orthogonal idempotents are orthogonal with respect to the inner product. Moreover, an idempotent is primitive if it is non-zero and cannot be written as the sum of two (necessarily orthogonal) non-zero idempotents. We say that $\{c_1, \ldots, c_r\}$ is a complete system of orthogonal primitive idempotents, or Jordan frame, if each $c_i$ is a primitive idempotent and

$$c_i \circ c_j = 0, \quad i \neq j,$$
$$\sum_{i=1}^{r} c_i = e.$$

Jordan frames play a crucial role in characterizing the elements of Euclidean Jordan algebras. The results are known as "spectral theorem", which are listed below. Note that in the case of real symmetric matrices, it specializes to the usual spectral theorem (see Example 4.37).

**Theorem 4.12** (Spectral theorem, first version, cf. [20, Theorem III.1.1])**.** *For $x$ in $\mathcal{J}$, there exist unique real numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$, all distinct, and a unique complete system of orthogonal idempotents $c_1, c_2, \ldots, c_k$ such that*

$$x = \sum_{i=1}^{k} \lambda_i c_i.$$

*For each $i = 1, \ldots, k$, we have $c_i \in \mathbf{R}[x]$. The numbers $\lambda_i$ are said to be the eigenvalues and $\sum_{i=1}^{k} \lambda_i c_i$ the spectral decomposition of $x$.*

Now it is possible to extend the definition of any real valued, continuous univariate function $f(\cdot)$ to elements of a Euclidean Jordan algebra, using eigenvalues:

$$f(x) := f(\lambda_1)c_1 + \cdots + f(\lambda_k)c_k.$$

Particularly, we have:

**Inverse:** $x^{-1} := \lambda_1^{-1}c_1 + \cdots + \lambda_r^{-1}c_k$, whenever all $\lambda_i \neq 0$ and undefined otherwise;

**Square root:** $x^{1/2} := \lambda_1^{1/2}c_1 + \cdots + \lambda_r^{1/2}c_k$, whenever all $\lambda_i \geq 0$ and undefined otherwise;

**Square:** $x^2 := \lambda_1^2 c_1 + \cdots + \lambda_r^2 c_k$.

For each $i = 1, \ldots, k$ we have $c_i \in \mathbf{R}[x]$. Moreover

$$x^{-1} \circ x = (\lambda_1^{-1}c_1 + \cdots + \lambda_k^{-1}c_k) \circ (\lambda_1 c_1 + \cdots + \lambda_r c_k) = e.$$

Hence, the expression of $x^{-1}$ defined here, using a spectral decomposition of $x$, coincides with the algebraic definition of inverse in Subsection 4.1.2. In the same way it follows that $x^{1/2} \circ x^{1/2} = x$ and $x^2 = x \circ x$.

**Theorem 4.13** (Spectral theorem, second version, cf. [20, Theorem III.1.2])**.** *Suppose $\mathcal{J}$ has rank $r$. Then for $x$ in $\mathcal{J}$ there exist a Jordan frame $c_1, c_2, \ldots, c_r$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that*

$$x = \sum_{i=1}^{r} \lambda_i c_i.$$

*The numbers $\lambda_i$ (with their multiplicities) are uniquely determined by $x$. Furthermore,*

$$\operatorname{tr}(x) = \sum_{i}^{r} \lambda_i, \quad \det(x) = \prod_{i=1}^{r} \lambda_i.$$

*More generally,*

$$a_k(x) = \sum_{1 \le i_1 < \cdots < i_k \le r} \lambda_{i_1} \ldots \lambda_{i_k},$$

*where $a_k (1 \le k \le r)$ is the polynomial defined in Proposition 4.4.*

It follows that a Jordan frame always contains $r$ primitive idempotents, where $r$ is the rank of $\mathcal{J}$. In fact, the above $\lambda_1, \ldots, \lambda_r$, are exactly the roots of the characteristic polynomial $f(\lambda; x)$ (see Proposition 4.4). To express their dependence on $x$, we denote them as $\lambda_1(x), \ldots, \lambda_r(x)$, or simply as a vector $\lambda(x) \in \mathbf{R}^r$. We call them the eigenvalues (or spectral values) of $x$. Moreover, we denote the largest eigenvalue of $x$ as $\lambda_{\max}(x)$, and analogously the smallest as $\lambda_{\min}(x)$. Note that since $e$ has eigenvalue 1, with multiplicity $r$, it follows that $\operatorname{tr}(e) = r$ and $\det(e) = 1$.

**Theorem 4.14** (cf. [20, Theorem III.1.5])**.** *Let $\mathcal{J}$ be a Jordan algebra over $\mathbf{R}$ with the identity element $e$. The following two properties are equivalent.*

*(i) $\mathcal{J}$ is a Euclidean Jordan algebra.*

*(ii) The symmetric bilinear form $\operatorname{tr}(x \circ y)$ is positive definite.*

The above theorem implies that if $\mathcal{J}$ is a Euclidean Jordan algebra, then $\operatorname{tr}(x \circ y)$ is an inner product. In the sequel, $\langle x, y \rangle$ will always denote this inner product, and we refer to it as the trace inner product.

The norm induced by the above inner product is named as the Frobenius norm, which is given by

$$\|x\|_F := \sqrt{\langle x, x \rangle} = \sqrt{\operatorname{tr}(x^2)}.$$

In fact, the above norm can also be obtained via eigenvalues. By Theorem 4.13, $x \in \mathcal{J}$ has a spectral decomposition $x = \sum_{i=1}^{r} \lambda_i(x) c_i$, and $x^2 = \sum_{i=1}^{r} \lambda_i^2(x) c_i$. Hence

$$\|x\|_F = \sqrt{\langle x, x \rangle} = \sqrt{\operatorname{tr}(x^2)} = \sqrt{\sum_{i=1}^{r} \lambda_i^2(x)} = \|\lambda(x)\|.$$

Moreover, the operators $L(x)$ and $P(x)$ are self-adjoint with respect to this inner product (cf. Proposition 4.5 and the definition of the quadratic representation, respectively).

### 4.2.2 Symmetric cones

In this subsection, we recall some definitions concerning symmetric cones.

**Definition 4.15** (Convex set)**.** A set $\mathcal{K}$ is convex if for any $x, y \in \mathcal{K}$ and any $\alpha$ with $0 \leq \alpha \leq 1$, we have $\alpha x + (1-\alpha)y \in \mathcal{K}$.

**Definition 4.16** (Cone)**.** A set $\mathcal{K}$ is called a cone if for every $x \in \mathcal{K}$ and $\alpha \geq 0$, we have $\alpha x \in \mathcal{K}$.

Therefore, a set $\mathcal{K}$ is called a convex cone if it is convex and a cone, which means that for any $x, y \in \mathcal{K}$ and $\alpha, \beta \geq 0$, we have $\alpha x + \beta y \in \mathcal{K}$.

**Definition 4.17** (Dual cone)**.** Let $\mathcal{K} \subseteq \mathcal{J}$ be a cone. The set

$$\mathcal{K}^* := \{y \in \mathcal{J} : \langle x, y \rangle \geq 0, \text{ for all } x \in \mathcal{K}\}$$

is called the dual cone of $\mathcal{K}$.

As the name suggests, $\mathcal{K}^*$ is a cone, and is always convex, even when the original cone is not. If cone $\mathcal{K}$ and its dual $\mathcal{K}^*$ coincide, we say that $\mathcal{K}$ is self-dual. In particular, this implies that $\mathcal{K}$ has a nonempty interior and does not contain any straight line (i.e., it is pointed).

**Definition 4.18** (Homogeneity)**.** The convex cone $\mathcal{K}$ is said to be homogeneous if for every pair $x, y \in \operatorname{int} \mathcal{K}$, there exists an invertible linear operator $g$ for which $g\mathcal{K} = \mathcal{K}$ and $gx = y$.

In fact, the above linear operator $g$ is an automorphism of the cone $\mathcal{K}$, i.e., $g \in \operatorname{Aut}(\mathcal{K})$, which is defined later in subsection 4.2.5.

**Definition 4.19** (Symmetric cone)**.** The convex cone $\mathcal{K}$ is said to be symmetric if it is self-dual and homogeneous.

In [20] the self-dual cone and consequently the symmetric cone are defined to be open. Here, we follow the definition used by the optimization community (cf. e.g. [95, Definition 1], [14, Section 2.6]). This minor difference will not affect the essence.

### 4.2.3 One-to-one correspondence

In this subsection, we recall a fundamental result which establishes the one-to-one correspondence between (cones of squares of) Euclidean Jordan algebras and symmetric cones.

Let $\mathcal{J}$ be a Euclidean Jordan algebra. We define the cone of squares $\mathcal{K}(\mathcal{J})$ of $\mathcal{J}$ as

$$\mathcal{K}(\mathcal{J}) := \left\{x^2 : x \in \mathcal{J}\right\}.$$

The set $\mathcal{K}(\mathcal{J})$ is a cone and therefore its dual $\mathcal{K}^*(\mathcal{J})$ is a closed convex cone:

$$\mathcal{K}^*(\mathcal{J}) = \left\{y \in \mathcal{J} : \langle y, x^2\rangle \geq 0, \text{ for all } x \in \mathcal{J}\right\}.$$

Since

$$\langle y, x^2\rangle = \langle y \circ x, x\rangle = \langle L(y)x, x\rangle,$$

we have

$$\mathcal{K}^*(\mathcal{J}) = \{y \in \mathcal{J} : L(y) \text{ is positive semidefinite}\}.$$

The following theorem brings together some major properties of the cone of squares in a Euclidean Jordan algebra.

**Theorem 4.20** (cf. [20, Theorem III.2.1, Proposition III.2.2])**.** *Let $\mathcal{J}$ be a Euclidean Jordan algebra, then $\mathcal{K}(\mathcal{J})$ is a symmetric cone, and is the set of elements $x$ in $\mathcal{J}$ for which $L(x)$ is positive semidefinite. Furthermore, if $x$ is invertible, then*

$$P(x)\operatorname{int}\mathcal{K}(\mathcal{J}) = \operatorname{int}\mathcal{K}(\mathcal{J}).$$

The above theorem indicates that the cone of squares in a Euclidean Jordan algebra is a symmetric cone. Conversely, given any symmetric cone in a Euclidean space, one may define a Euclidean Jordan algebra such that the given cone is its cone of squares (cf. [20, Theorem III.3.1]). Therefore, we have the following Jordan algebraic characterization of symmetric cones.

**Theorem 4.21** (cf. [20, Section III.2–5])**.** *A cone is symmetric if and only if it is the cone of squares of some Euclidean Jordan algebra.*

Because of the above one-to-one correspondence, the notions of cone of squares in a Euclidean Jordan algebra and symmetric cone are equivalent. In the sequel, $\mathcal{K}$ will always denote a symmetric cone, and $\mathcal{J}$ a Euclidean Jordan algebra for which $\mathcal{K}$ is its cone of squares.

We associate with the proper cone[5] $\mathcal{K}$ the partial order defined by

$$x \succeq_{\mathcal{K}} y \quad \Leftrightarrow \quad x - y \in \mathcal{K}.$$

We also write $y \preceq_{\mathcal{K}} x$ for $x \succeq_{\mathcal{K}} y$. Similarly, we define an associated strict partial order by

$$x \succ_{\mathcal{K}} y \quad \Leftrightarrow \quad x - y \in \operatorname{int}\mathcal{K},$$

and write $y \prec_{\mathcal{K}} x$ for $x \succ_{\mathcal{K}} y$.

The next proposition contains a result of crucial importance in the design of IPMs within the framework of Jordan algebras.

**Proposition 4.22** (cf. [23, Lemma 2.2])**.** *Let $x, s \in \mathcal{K}$. Then $\operatorname{tr}(x \circ s) \geq 0$. We have $\operatorname{tr}(x \circ s) = 0$ if and only if $x \circ s = 0$.*

[5] A cone is called a proper cone if it is pointed, closed, convex, and with nonempty interior.

### 4.2.4 Simple Jordan algebras

Another consequence of the one-to-one correspondence between Euclidean Jordan algebras and symmetric cones is the unique decomposition of every symmetric cone into a direct product of irreducible ones. As usual, we start with some definitions.

**Definition 4.23** (Ideal)**.** Let $\mathcal{J}$ be an $\mathbf{R}$-algebra. An ideal $\mathcal{I}$ of $\mathcal{J}$ is a vector subspace of $\mathcal{J}$ such that for every $x \in \mathcal{I}$ and every $y \in \mathcal{J}$ the elements $x \circ y$ and $y \circ x$ belong to $\mathcal{I}$.

**Definition 4.24** (Simple algebra)**.** An $\mathbf{R}$-algebra $\mathcal{J}$ is simple if it contains only two ideals, namely $\{0\}$ and $\mathcal{J}$ (trivial ideals).

**Proposition 4.25** (cf. [20, Proposition III.4.4])**.** *If $\mathcal{J}$ is a Euclidean Jordan algebra, then it is, in a unique way, a direct sum of simple ideals.*

The previous proposition immediately implies that any Euclidean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras.

A symmetric cone $\mathcal{K}$ in a Euclidean space $\mathcal{J}$ is said to be irreducible if there do not exist non-trivial subspaces $\mathcal{J}_1$, $\mathcal{J}_2$, and symmetric cones $\mathcal{K}_1 \subset \mathcal{J}_1$, $\mathcal{K}_2 \subset \mathcal{J}_2$, such that $\mathcal{J}$ is the direct sum of $\mathcal{J}_1$ and $\mathcal{J}_2$, and $\mathcal{K}$ is the direct sum of $\mathcal{K}_1$ and $\mathcal{K}_2$.

**Proposition 4.26** (cf. [20, Proposition III.4.5])**.** *Any symmetric cone $\mathcal{K}$ is, in a unique way, the direct product of irreducible symmetric cones.*

The following theorem states that there are only five kinds of simple Euclidean Jordan algebras and correspondingly five kinds of irreducible symmetric cones.

**Theorem 4.27** (cf. [20, Chapter V])**.** *Let $\mathcal{J}$ be a simple Euclidean Jordan algebra. Then $\mathcal{J}$ is isomorphic to one of the following algebras.*

*(i) The algebra in space $\mathbf{R}^{n+1}$ with Jordan multiplication defined as*

$$x \circ y = (x^T y; x_0 \bar{y} + y_0 \bar{x}),$$

*where $x := (x_0; \bar{x})$ and $y := (y_0; \bar{y})$ with $x_0, y_0 \in \mathbf{R}$ and $\bar{x}, \bar{y} \in \mathbf{R}^n$.*

*(ii) The algebra of real symmetric matrices with Jordan multiplication defined as*

$$X \circ Y = (XY + YX)/2.$$

*(iii) The algebra of complex Hermitian matrices with Jordan multiplication defined as in (ii).*

*(iv) The algebra of quaternion Hermitian matrices, with Jordan multiplication defined as in (ii).*

*(v) The algebra of $3 \times 3$ octonion Hermitian matrices with Jordan multiplication defined as in (ii).*

### 4.2.5 Automorphisms

Let $\mathcal{J}$ be a Euclidean Jordan algebra and $\mathcal{K}$ its cone of squares (or equivalently its associated symmetric cone). In this section, we recall definitions and some properties concerning automorphisms of $\mathcal{J}$ and $\mathcal{K}$.

We denote henceforth the set of all invertible linear maps from $\mathcal{J}$ into itself by $\mathrm{GL}(\mathcal{J})$.

**Definition 4.28** (Automorphism of $\mathcal{J}$)**.** A map $g \in \mathrm{GL}(\mathcal{J})$ is called an automorphism of $\mathcal{J}$ if for every $x$ and $y$ in $\mathcal{J}$, we have $g(x \circ y) = g(x) \circ g(y)$, or equivalently, $gL(x)g^{-1} = L(gx)$. The set of automorphisms of $\mathcal{J}$ is denoted as $\mathrm{Aut}(\mathcal{J})$.

**Definition 4.29** (Automorphism of $\mathcal{K}$)**.** A map $g \in \mathrm{GL}(\mathcal{J})$ is called an automorphism of $\mathcal{K}$ if $g\mathcal{K} = \mathcal{K}$. The set of automorphisms of $\mathcal{K}$ is denoted as $\mathrm{Aut}(\mathcal{K})$.

We say that a linear map is orthogonal if $g^* = g^{-1}$. The set of orthogonal automorphisms of $\mathcal{K}$ is denoted as $\mathrm{OAut}(\mathcal{K})$, that is

$$\mathrm{OAut}(\mathcal{K}) = \left\{ g \in \mathrm{Aut}(\mathcal{K}) : g^* = g^{-1} \right\}.$$

We would like to stress that $\mathrm{Aut}(\mathcal{J}) \neq \mathrm{Aut}(\mathcal{K})$. For example, $g = 2P(e)$ is in $\mathrm{Aut}(\mathcal{K})$, but it does not satisfy $g(x \circ y) = gx \circ gy$.

**Proposition 4.30** (cf. [20, Proposition II.4.2])**.** *The trace and the determinant are invariant under* $\mathrm{Aut}(\mathcal{J})$*.*

The next proposition establishes a connection between the automorphism group of a Euclidean Jordan algebra and the orthogonal automorphism group of its associated symmetric cone.

**Proposition 4.31** (cf. [109, Theorem 2.8.4])**.** *We have*

$$\mathrm{Aut}(\mathcal{J}) = \mathrm{OAut}(\mathcal{K}).$$

**Proposition 4.32** (cf. [20, Proposition IV.2.5])**.** *Let $\mathcal{J}$ be a simple Euclidean Jordan algebra. If $\{c_1, \ldots, c_r\}$ and $\{d_1, \ldots, d_r\}$ are two Jordan frames, then there exists an automorphism $g$ in* $\mathrm{Aut}(\mathcal{J})$ *such that*

$$gc_i = d_i, \quad \textit{for all } 1 \leq i \leq r.$$

### 4.2.6 The Peirce decomposition

In this subsection we recall the Peirce decomposition, first with respect to a single idempotent, and then with respect to a complete system of primitive idempotents, or Jordan frame. Much of this could be done more generally, but we restrict ourselves to the case of Euclidean Jordan algebras.

Notice that for an idempotent $c$, since $c^2 = c$, one can show that [20, Proposition III.1.3]

$$2L(c)^3 - 3L(c)^2 + L(c) = 0.$$

Therefore, eigenvalues of $L(c)$ are 0, 1/2 and 1. Furthermore, the eigenspace corresponding to each eigenvalue of $L(c)$ is the set of $x$ such that $L(c)x = ix$ or equivalently $c \circ x = ix$, for $i = 0, 1/2, 1$. Therefore,

**Theorem 4.33** (Peirce decomposition, type I, cf. [20, Section IV.1])**.** *Let $\mathcal{J}$ be a Euclidean Jordan algebra and $c$ an idempotent. Then $\mathcal{J}$, as a vector space, can be decomposed as*

$$\mathcal{J} = \mathcal{J}_0(c) \oplus \mathcal{J}_{1/2}(c) \oplus \mathcal{J}_1(c),$$

*where*

$$\mathcal{J}_i(c) = \{x : c \circ x = ix\}.$$

*The subspaces $\mathcal{J}_0(c)$ and $\mathcal{J}_1(c)$ are subalgebras of $\mathcal{J}$. They are orthogonal in the sense that*

$$\mathcal{J}_0(c) \circ \mathcal{J}_1(c) = \{0\}.$$

*Furthermore,*

$$(\mathcal{J}_0(c) \oplus \mathcal{J}_1(c)) \circ \mathcal{J}_{1/2}(c) \subseteq \mathcal{J}_{1/2}(c),$$
$$\mathcal{J}_{1/2}(c) \circ \mathcal{J}_{1/2}(c) \subseteq \mathcal{J}_0(c) \oplus \mathcal{J}_1(c).$$

With respect to a Jordan frame, one can give a finer decomposition:

**Theorem 4.34** (Peirce decomposition, type II, cf. [20, Theorem IV.2.1])**.** *Let $\mathcal{J}$ be a Euclidean Jordan algebra and $\{c_1, \ldots, c_r\}$ a Jordan frame. Then*

*(i) The space $\mathcal{J}$ decomposes in the following orthogonal direct sum:*

$$\mathcal{J} = \bigoplus_{i \le j} \mathcal{J}_{ij},$$

*where*

$$\mathcal{J}_{ii} = \mathcal{J}_i(c_i) = \mathbf{R}c_i,$$
$$\mathcal{J}_{ij} = \mathcal{J}_{1/2}(c_i) \cap \mathcal{J}_{1/2}(c_j), \quad i \neq j.$$

*(ii) If we denote by $P_{ij}$ the orthogonal projection onto $\mathcal{J}_{ij}$, then*

$$P_{ii} = P(c_i),$$
$$P_{ij} = 4L(c_i)L(c_j).$$

*(iii) Furthermore,*

$$\mathcal{J}_{ij} \circ \mathcal{J}_{ij} \subseteq \mathcal{J}_{ii} + \mathcal{J}_{jj},$$
$$\mathcal{J}_{ij} \circ \mathcal{J}_{jk} \subseteq \mathcal{J}_{ik}, \quad \text{if } i \neq k,$$
$$\mathcal{J}_{ij} \circ \mathcal{J}_{kl} \subseteq \{0\}, \quad \text{if } \{i, j\} \cap \{k, l\} = \emptyset.$$

As a consequence, we have

**Corollary 4.35** (cf. [95, Lemma 12])**.** *Lex $x \in \mathcal{J}$ and its spectral decomposition $x = \sum_{i=1}^{r} \lambda_i c_i$. Then the following statements hold.*

*(i) The matrices, $L(x)$ and $P(x)$ commute and thus share a common system of eigenvectors; in fact the $c_i$ are among their common eigenvectors.*

*(ii) The eigenvalues of $L(x)$ have the form*

$$\frac{\lambda_i + \lambda_j}{2}, \quad 1 \leq i \leq j \leq r.$$

*(iii) The eigenvalues of $P(x)$ have the form*

$$\lambda_i \lambda_j, \quad 1 \leq i \leq j \leq r.$$

### 4.2.7 Examples continued

Given some associative inner product, the two Jordan algebra examples in Subsection 4.1.4 turn out to be Euclidean Jordan algebras. In this subsection, we further investigate their cones of squares and spectral decompositions.

*Example* 4.36 (The quadratic terms algebra $\mathbf{L}^{n+1}$). Consider the Jordan algebra $(\mathbf{L}^{n+1}, \circ)$ defined in Example 4.10. We have that $(\mathbf{L}^{n+1}, \circ)$ is Euclidean Jordan algebra with the associative inner product defined as

$$\langle x, y \rangle := \operatorname{tr}(x \circ y) = 2x^T y.$$

It is straightforward to show that the cone of squares of $\mathbf{L}^{n+1}$ is

$$\mathbf{L}_+^{n+1} = \left\{ x \in \mathbf{R}^{n+1} : x_0 \geq \|\bar{x}\| \right\},$$

which is known as the second-order cone, the Lorentz cone[6], the quadratic cone, or the ice-cream cone. Moreover, any $x \in \mathbf{L}^{n+1}$, with $\bar{x} \neq 0$ (the spectral decomposition is trivial for $\bar{x} = 0$), has the spectral decomposition

$$x = \lambda_1 c_1 + \lambda_2 c_2,$$

where $\lambda_1 = x_0 - \|\bar{x}\|$, $\lambda_2 = x_0 + \|\bar{x}\|$ are eigenvalues, and

$$c_1 = \frac{1}{2} \begin{bmatrix} 1 \\ -\frac{\bar{x}}{\|\bar{x}\|} \end{bmatrix}, \quad c_2 = \frac{1}{2} \begin{bmatrix} 1 \\ \frac{\bar{x}}{\|\bar{x}\|} \end{bmatrix}.$$

[6]The cone is called after the Dutch physician Hendrik Antoon Lorentz who, together with his student Pieter Zeeman received the Noble prize in 1902 for their work on the so-called "'Zeeman-effect".

*Example* 4.37 (Jordan algebra of $\mathbf{S}^n$). Let $(\mathbf{S}^n, \circ)$ be the Jordan algebra defined in Example 4.11. With the associative inner product defined as

$$\langle X, Y \rangle := \operatorname{tr}(X \circ Y) = \operatorname{tr}(XY),$$

$(\mathbf{S}^n, \circ)$ is a Euclidean Jordan algebra. A symmetric matrix is square of another symmetric matrix if and only if it is positive semidefinite. Thus the cone of squares of $\mathbf{S}^n$ is the cone of positive semidefinite matrices. Every symmetric matrix can be diagonalized by an orthogonal matrix: $X = Q\Lambda Q^T$. This relation may be written as

$$X = \lambda_1 q_1 q_1^T + \cdots + \lambda_n q_n q_n^T,$$

where the $\lambda_i$ are the eigenvalues of $X$ and $q_i$, the columns of $Q$, are their corresponding eigenvectors. Since all $q_i$ form an orthonormal set, it follows that the set of rank one matrices $q_i q_i^T$ form a Jordan frame:

$$(q_i q_i^T)^2 = q_i q_i^T, \quad (q_i q_i^T)(q_j q_j^T) = 0 \text{ for } i \neq j, \quad \sum_{i=1}^{n} q_i q_i^T = e.$$

This gives a spectral decomposition of $X$.

## 4.3 More algebraic properties

In this section we recall or derive some more properties of Euclidean Jordan algebras and their associated symmetric cones. These results play a key role in our analysis of optimization techniques for symmetric cones. Recall that we always assume $\mathcal{K}$ is a symmetric cone (or equivalently the cone of squares of some Euclidean Jordan algebra).

### 4.3.1 NT-scaling

When defining the search directions in our algorithms, we need a rescaling of the space in which the symmetric cone lives. In this subsection, we show the existence and uniqueness of the NT-scaling point $w$ corresponding to any points $x, s \in \operatorname{int} \mathcal{K}$, such that $P(w)$ takes $s$ into $x$. This was done by Nesterov and Todd for self-scaled cones in [77, 78]. Later, Faybusovich [24] derived it in the framework of Euclidean Jordan algebras.

As a preparation, we recall the following two propositions.

**Proposition 4.38.** *If $x \in \mathcal{K}$, then $x^{1/2}$ is well defined and $P(x^{1/2}) = P(x)^{1/2}$.*

*Proof.* Since $x$ is in the cone of squares, all its eigenvalues are nonnegative. Hence, $x^{1/2}$ is well defined. By the definition of the quadratic representation, we have

$$P(x^{1/2})e = \left(2L(x^{1/2})^2 - L(x)\right) e = x.$$

Therefore, by the fundamental formula (cf. Proposition 4.9),

$$P(x) = P(P(x^{1/2})e) = P(x^{1/2})P(e)P(x^{1/2}) = P(x^{1/2})^2.$$

Moreover, since $P(x^{1/2})$ is self-adjoint, $P(x)^{1/2} = P(x^{1/2})$. □

**Proposition 4.39** (cf. [58, Lemma 2.3])**.** *Given $x, s \in \operatorname{int} \mathcal{K}$ with $P(x) = P(s)$, then $x = s$.*

*Proof.* Since $P(x) = P(s)$, we have

$$x^2 = P(x)e = P(s)e = s^2.$$

It follows that

$$0 = x^2 - s^2 = (x + s) \circ (x - s) = L(x + s)(x - s).$$

Since $x, s \in \operatorname{int} \mathcal{K}$, it follows that $x + s \in \operatorname{int} \mathcal{K}$. The proposition follows from the positive definiteness of $L(x + s)$. □

**Proposition 4.40** (NT-scaling, cf. [24, Lemma 3.2])**.** *Given $x, s \in \operatorname{int} \mathcal{K}$, there exists a unique $w \in \operatorname{int} \mathcal{K}$ such that*

$$x = P(w)s.$$

*Moreover,*

$$w := P(x)^{1/2}(P(x)^{1/2}s)^{-1/2} \quad \left[= P(s)^{-1/2}(P(s)^{1/2}x)^{1/2}\right], \tag{4.1}$$

*and we call $w$ the scaling point of $x$ and $s$ (in this order).*

*Proof.* From Theorem 4.20, Proposition 4.9, and Proposition 4.38, it readily follows that

$$w = P(x)^{1/2}(P(x)^{1/2}s)^{-1/2} \in \operatorname{int} \mathcal{K}.$$

By Proposition 4.38 and the fundamental formula, we have

$$P(w) = P(x)^{1/2}P((P(x)^{1/2}s)^{-1/2})P(x)^{1/2}.$$

Hence

$$\begin{aligned} P(w)^{-1}x &= P(x)^{-1/2}P((P(x)^{1/2}s)^{1/2})P(x)^{-1/2}x \\ &= P(x)^{-1/2}P((P(x)^{1/2}s)^{1/2})e \\ &= P(x)^{-1/2}P(x)^{1/2}s = s. \end{aligned}$$

Conversely, assume $x = P(w)s$. Then by the fundamental formula,

$$P(x) = P(w)P(s)P(w).$$

Therefore,

$$\begin{aligned}\left[P(x)^{1/2}P(w^{-1})P(x)^{1/2}\right]^2 &= P(x)^{1/2}P(w)^{-1}P(x)P(w)^{-1}P(x)^{1/2}\\ &= P(x)^{1/2}P(s)P(x)^{1/2},\end{aligned}$$

or equivalently, by taking square root at both sides,

$$P(P(x)^{1/2}w^{-1}) = P((P(x)^{1/2}s)^{1/2}).$$

By Proposition 4.39,

$$P(x)^{1/2}w^{-1} = (P(x)^{1/2}s)^{1/2}.$$

Then $w = P(x)^{1/2}(P(x)^{1/2}s)^{-1/2}$ follows from part $(ii)$ of Proposition 4.9. Hence the uniqueness is proved.

For the proof of the second equality of (4.1), we have, by Proposition 4.39, that this equality holds if

$$P(x)^{1/2}P((P(x)^{1/2}s)^{-1/2})P(x)^{1/2} = P(s)^{-1/2}P((P(s)^{1/2}x)^{1/2})P(s)^{-1/2}$$

or equivalently,

$$P(s)^{1/2}P(x)^{1/2}P((P(x)^{1/2}s)^{-1/2})P(x)^{1/2}P(s)^{1/2} = P((P(s)^{1/2}x)^{1/2}).$$

Since both sides are positive definite, the above equation follows from

$$\left[P(s)^{1/2}P(x)^{1/2}P((P(x)^{1/2}s)^{-1/2})P(x)^{1/2}P(s)^{1/2}\right]^2 = \left[P((P(s)^{1/2}x)^{1/2})\right]^2,$$

which holds trivially by the fundamental formula. The proof is complete. □

**Remark 4.41.** In fact, the point $w$ is exactly the geometric mean of $x$ and $s^{-1}$ [109, Section 5.4]. Hence the above results also follows from the properties of geometric mean, which is studied in detail by Lim [58]. Moreover, the point $w$ coincides with the unique scaling point introduced by Nesterov and Todd [77, 78] for self-scaled cones.

### 4.3.2 Similarity

Recall that two matrices $X$ and $S$ are similar if they share the same set of eigenvalues; in this case, we write $X \sim S$. Analogously, we say that two elements $x$ and $s$ in $\mathcal{J}$ are similar, denoted as $x \sim s$, if $x$ and $s$ share the same set of eigenvalues. In what follows, we list some basic results regarding similarity. For more details we refer to [95, 101, 109].

**Proposition 4.42** ([95, Proposition 19])**.** *Two elements $x$ and $s$ of a Euclidean Jordan algebra are similar if and only if $L(x)$ and $L(s)$ are similar.*

**Proposition 4.43** ([95, Corollary 20]). *Let $x$ and $s$ be two elements in $\operatorname{int}\mathcal{K}$. Then $x$ and $s$ are similar if and only if $P(x)$ and $P(s)$ are similar.*

**Proposition 4.44** ([95, Proposition 21]). *Let $x$ and $s$ be two elements in $\operatorname{int}\mathcal{K}$, then $P(x)^{1/2}s$ and $P(s)^{1/2}x$ are similar.*

Following are two important generalizations. Because of their importance we include the proofs, where we often use the fact that for any two positive definite matrices $X$ and $S$ we have $XS \sim X^{1/2}SX^{1/2}$.

**Lemma 4.45** ([95, Proposition 21]). *Let $x, s, u \in \operatorname{int}\mathcal{K}$. Defining $\tilde{x} = P(u)x$ and $\tilde{s} = P(u^{-1})s$, one has*

$$P(\tilde{x}^{1/2})\tilde{s} \sim P(x^{1/2})s.$$

*Proof.* By the fundamental formula,

$$P(P(\tilde{x}^{1/2})\tilde{s}) = P(\tilde{x}^{1/2})P(\tilde{s})P(\tilde{x}^{1/2}) \sim P(\tilde{x})P(\tilde{s}).$$

Similarly $P(P(x^{1/2})s) \sim P(x)P(s)$. Since both $P(\tilde{x}^{1/2})\tilde{s}$ and $P(x^{1/2})s$ lie in $\operatorname{int}\mathcal{K}$ (cf. Theorem 4.20), by Proposition 4.43, it suffices to show that $P(\tilde{x})P(\tilde{s}) \sim P(x)P(s)$. Using the fundamental formula again, we obtain

$$\begin{aligned} P(\tilde{x})P(\tilde{s}) &= P(P(u)x)P(P(u^{-1})s) \\ &= P(u)P(x)P(u)P(u^{-1})P(s)P(u^{-1}) \\ &\sim P(x)P(s). \end{aligned}$$

Hence the proof is complete. □

**Lemma 4.46** (cf. [109, Proposition 3.2.4]). *Let $x, s \in \operatorname{int}\mathcal{K}$, and $w$ the scaling point of $x$ and $s$, then*

$$(P(x^{1/2})s)^{1/2} \sim P(w^{1/2})s.$$

*Proof.* From Theorem 4.20 and Proposition 4.38, it readily follows that

$$(P(x^{1/2})s)^{1/2} \in \operatorname{int}\mathcal{K}, \quad P(w^{1/2})s \in \operatorname{int}\mathcal{K}.$$

Then Proposition 4.43 implies that the statement is equivalent to

$$P((P(x^{1/2})s)^{1/2} \sim P(P(w^{1/2})s).$$

Since we have, by the fundamental formula,

$$P(P(w^{1/2})s) = P(w)^{1/2}P(s)P(w)^{1/2} \sim P(w)P(s).$$

the statement is also equivalent to

$$P((P(x^{1/2})s)^{1/2} \sim P(w)P(s).$$

From Proposition 4.40, we have that $w = P(x^{1/2})(P(x^{1/2})s)^{-1/2}$. Then by the fundamental formula and Proposition 4.38

$$P(w) = P(P(x^{1/2})(P(x^{1/2})s)^{-1/2}) = P(x^{1/2})P(P(x^{1/2})s)^{-1/2}P(x^{1/2}).$$

Hence, by substitution and using the fundamental formula again, we derive

$$\begin{aligned} P(w)P(s) &= P(x^{1/2})P(P(x^{1/2})s)^{-1/2}P(x^{1/2})P(s) \\ &\sim P(P(x^{1/2})s)^{-1/2}P(x^{1/2})P(s)P(x^{1/2}) \\ &= P(P(x^{1/2})s)^{1/2}. \end{aligned}$$

The lemma follows. □

### 4.3.3 Inequalities

To analyze our algorithms, we need some inequalities, which are presented in this subsection.

**Lemma 4.47.** *Let $x \in \mathcal{K}$, then*

$$\operatorname{tr}(x^2) \leq \operatorname{tr}(x)^2.$$

*Proof.* Since $x \in \mathcal{K}$, we have $\lambda(x) \geq 0$. Therefore

$$\operatorname{tr}(x^2) = \sum_{i=1}^{r} \lambda_i(x)^2 \leq \left(\sum_{i=1}^{r} \lambda_i(x)\right)^2 = \operatorname{tr}(x)^2.$$

This proves the lemma. □

**Lemma 4.48.** *Let $x \in \mathcal{J}$, then*

$$\left\|x^2\right\|_F \leq \|x\|_F^2.$$

*Proof.* Using the definition of Frobenius norm, we have

$$\left\|x^2\right\|_F^2 = \operatorname{tr}((x^2)^2) \leq \operatorname{tr}(x^2)^2 = (\|x\|_F^2)^2,$$

where the inequality follows from Lemma 4.47. Since both $\left\|x^2\right\|_F$ and $\|x\|_F$ are nonnegative, the lemma follows. □

**Lemma 4.49.** *Let $\mathcal{J}$ be a Euclidean Jordan algebra, and $x, s \in \mathcal{J}$ with $\langle x, s\rangle = 0$. Then one has*

*(i)* $-\frac{1}{4}\|x+s\|_F^2 e \preceq_{\mathcal{K}} x \circ s \preceq_{\mathcal{K}} \frac{1}{4}\|x+s\|_F^2 e$;

*(ii)* $\|x \circ s\|_F \leq \frac{1}{2\sqrt{2}}\|x+s\|_F^2$.

*Proof.* We write

$$x \circ s = \frac{1}{4}\left((x+s)^2 - (x-s)^2\right).$$

Since $(x+s)^2 \in \mathcal{K}$, we have

$$x \circ s + \frac{1}{4}(x-s)^2 \in \mathcal{K}.$$

Using

$$(x-s)^2 \preceq_{\mathcal{K}} \lambda_{\max}\left((x-s)^2\right) e \preceq_{\mathcal{K}} \|x-s\|_F^2\, e,$$

it follows that

$$x \circ s + \frac{1}{4}\|x-s\|_F^2\, e \in \mathcal{K},$$

which means that $-\frac{1}{4}\|x-s\|_F^2\, e \preceq_{\mathcal{K}} x \circ s$. In a similar way one derives that $x \circ s \preceq_{\mathcal{K}} \frac{1}{4}\|x+s\|_F^2\, e$. Since $\langle x, s\rangle = 0$, it follows that $\|x-s\|_F = \|x+s\|_F$, and hence, part $(i)$ of the lemma follows.

For the proof of part $(ii)$, we observe that

$$\begin{aligned}\|x \circ s\|_F^2 &= \left\|\frac{1}{4}((x+s)^2 - (x-s)^2)\right\|_F^2 = \frac{1}{16}\operatorname{tr}\left(((x+s)^2 - (x-s)^2)^2\right) \\ &= \frac{1}{16}\left[\operatorname{tr}\left((x+s)^4\right) + \operatorname{tr}\left((x-s)^4\right) - 2\operatorname{tr}\left((x+s)^2 \circ (x-s)^2\right)\right].\end{aligned}$$

Since $(x+s)^2$ and $(x-s)^2$ belong to $\mathcal{K}$, the trace of their product is nonnegative. Thus, we obtain

$$\|x \circ s\|_F^2 \leq \frac{1}{16}\left[\operatorname{tr}\left((x+s)^4\right) + \operatorname{tr}\left((x-s)^4\right)\right] = \frac{1}{16}\left[\left\|(x+s)^2\right\|_F^2 + \left\|(x-s)^2\right\|_F^2\right].$$

Using Lemma 4.48 and $\|x+s\|_F = \|x-s\|_F$ again, we get

$$\|x \circ s\|_F^2 \leq \frac{1}{16}\left[\|x+s\|_F^4 + \|x-s\|_F^4\right] = \frac{1}{8}\|x+s\|_F^4.$$

This implies part $(ii)$ of the lemma. Hence, the proof of the lemma is complete. □

**Proposition 4.50** (cf. [23, Theorem 5.13])**.** *Given $x \in \operatorname{int}\mathcal{K}$, we have*

$$\langle x, s\rangle > 0, \quad \textit{for all } s \in \mathcal{K} \setminus \{0\}.$$

**Lemma 4.51.** *If $x \circ s \in \operatorname{int}\mathcal{K}$, then $\det(x) \neq 0$.*

*Proof.* By Theorem 4.13, the element $x$ can be write as

$$x = \sum_{i=1}^{r} \lambda_i(x) c_i,$$

where $\{c_1, \ldots, c_r\}$ is a Jordan frame. Suppose $\det(x) = 0$, then there must exist an integer $k$ with $1 \leq k \leq r$, such that $\lambda_k(x) = 0$. Since $x \circ s \in \operatorname{int} \mathcal{K}$ and $c_k \in \mathcal{K} \setminus \{0\}$, by Proposition 4.50,

$$0 < \langle x \circ s, c_k \rangle = \langle s, x \circ c_k \rangle = 0.$$

This contradiction completes the proof. □

**Lemma 4.52** (Cauchy-Schwarz inequality)**.** *Let $x, s \in \mathcal{J}$, then*

$$\langle x, s \rangle \leq \|x\|_F \|s\|_F .$$

*Proof.* As the inequality is trivially true in the case $s = 0$, we may assume $\langle s, s \rangle$ is nonzero. Let $\alpha \in \mathbf{R}$, then

$$0 \leq \|x - \alpha s\|_F^2 = \langle x - \alpha s, x - \alpha s \rangle = \langle x, x \rangle - 2\alpha \langle x, s \rangle + \alpha^2 \langle s, s \rangle.$$

The above expression is valid for any $\alpha$, which implies that discriminant is negative, i.e.,

$$4\langle x, s \rangle^2 - 4\langle s, s \rangle \langle x, x \rangle \leq 0,$$

and from this the lemma follows. □

**Lemma 4.53.** *Let $x, s \in \mathcal{J}$, then*

$$\|x \circ s\|_F \leq \frac{1}{2} \left\| x^2 + s^2 \right\|_F .$$

*Proof.* Since

$$x^2 + s^2 + 2x \circ s = (x + s)^2 \in \mathcal{K},$$
$$x^2 + s^2 - 2x \circ s = (x - s)^2 \in \mathcal{K},$$

we have

$$\langle x^2 + s^2 + 2x \circ s, x^2 + s^2 - 2x \circ s \rangle \geq 0,$$

which is equivalent to

$$\langle x^2 + s^2, x^2 + s^2 \rangle - 4\langle x \circ s, x \circ s \rangle \geq 0,$$

or, equivalently,

$$\left\| x^2 + s^2 \right\|_F^2 - 4 \|x \circ s\|_F^2 \geq 0.$$

This implies the lemma. □

**Lemma 4.54.** *Let $x \in \mathcal{J}$ and $s \in \mathcal{K}$, then*

$$\lambda_{\min}(x) \operatorname{tr}(s) \leq \operatorname{tr}(x \circ s) \leq \lambda_{\max}(x) \operatorname{tr}(s).$$

*Proof.* For any $x \in \mathcal{J}$ we have $\lambda_{\max}(x)e - x \in \mathcal{K}$. Furthermore, since $s \in \mathcal{K}$, it follows that

$$\operatorname{tr}((\lambda_{\max}(x)e - x) \circ s) \geq 0.$$

Hence, the second inequality in the lemma follows by writing

$$\operatorname{tr}(x \circ s) \leq \operatorname{tr}(\lambda_{\max}(x)e \circ s) = \lambda_{\max}(x) \operatorname{tr}(s).$$

The proof of the first inequality goes in the similar way. □

**Lemma 4.55** (cf. [86, Lemma 2.9])**.** *Given $x \in \operatorname{int} \mathcal{K}$, we have*

$$\left\|x - x^{-1}\right\|_F \leq \frac{\left\|x^2 - e\right\|_F}{\lambda_{\min}(x)}.$$

*Proof.* The assertion follows from

$$\left\|x - x^{-1}\right\|_F = \left\|L(x^{-1})(x^2 - e)\right\|_F \leq \left\|L(x^{-1})\right\|_F \left\|(x^2 - e)\right\|_F,$$

where $\left\|L(x^{-1})\right\|_F$ is the operator norm induced by $\|\cdot\|_F$. Since the $\|\cdot\|_F$ is the norm induced by the inner product $\langle \cdot, \cdot \rangle$, the operator norm coincides with the spectral norm. Using Corollary 4.35, we have

$$\left\|L(x^{-1})\right\| = \lambda_{\max}(L(x^{-1})) = \lambda_{\max}(x^{-1}).$$

Hence, it follows that

$$\left\|x - x^{-1}\right\|_F = \lambda_{\max}(x^{-1}) \left\|(x^2 - e)\right\|_F = \frac{\left\|(x^2 - e)\right\|_F}{\lambda_{\min}(x)},$$

which proves the lemma. □

**Lemma 4.56** (cf. [95, Lemma 30])**.** *Let $x, s \in \operatorname{int} \mathcal{K}$, then*

$$\left\|P(x)^{1/2}s - e\right\|_F \leq \|x \circ s - e\|_F.$$

*Proof.* By the definition of the Frobenius norm, we have

$$\begin{aligned}\left\|P(x)^{1/2}s - e\right\|_F^2 &= \langle P(x)^{1/2}s - e, P(x)^{1/2}s - e\rangle \\ &= \langle P(x)^{1/2}s, P(x)^{1/2}s\rangle - 2\langle P(x)^{1/2}s, e\rangle + \langle e, e\rangle \\ &= \langle s, P(x)s\rangle - 2\langle x, s\rangle + \langle e, e\rangle.\end{aligned}$$

Analogously, we have

$$\|x \circ s - e\|_F^2 = \langle x \circ s - e, x \circ s - e\rangle = \langle s, L(x)^2 s\rangle - 2\langle x, s\rangle + \langle e, e\rangle.$$

Note that $L(x)^2 - P(x)$, by Corollary 4.35, has eigenvalues of the form

$$\left(\frac{\lambda_i + \lambda_j}{2}\right)^2 - \lambda_i\lambda_j = \left(\frac{\lambda_i - \lambda_j}{2}\right)^2 \geq 0.$$

Therefore $L(x)^2 \succeq P(x)$. This implies the lemma. □

Next, we recall the Lyapunov lemma for Euclidean Jordan algebras.

**Lemma 4.57** (Lyapunov lemma)**.** *Let $x \in \operatorname{int} \mathcal{K}$. Then*

$$L(x)^{-1} \operatorname{int} \mathcal{K} \subseteq \operatorname{int} \mathcal{K}, \quad L(x)^{-1}\mathcal{K} \subseteq \mathcal{K}.$$

Sturm [101, Corollary 3], along with extensions to many properties of symmetric matrices, provided a proof of the above lemma. Later, an alternative constructive proof was given by Rangarajan [86, Lemma 2.12], and a different argument given by Baes [7, Remakrk 2.8.9]. This lemma is needed for the proof of the next lemma.

**Lemma 4.58** (cf. [101, Theorem 4])**.** *Let $x, s \in \operatorname{int} \mathcal{K}$, then*

$$\lambda_{\min}(P(x)^{1/2}s) \geq \lambda_{\min}(x \circ s).$$

*Proof.* We have

$$L(x)(s - \lambda_{\min}(x \circ s)x^{-1}) = x \circ s - \lambda_{\min}(x \circ s)e \in \mathcal{K}.$$

Pre-multiplying the above equality with $L(x)^{-1}$, and using Lemma 4.57, it follows that

$$s - \lambda_{\min}(x \circ s)x^{-1} \in \mathcal{K}.$$

If we pre-multiply this inclusion with $P(x)^{1/2}$, we obtain

$$P(x)^{1/2}s - \lambda_{\min}(x \circ s)e \in \mathcal{K},$$

which shows that $\lambda_{\min}(P(x)^{1/2}s) \geq \lambda_{\min}(x \circ s)$. □

An alternative proof for the above lemma can be found in [86, Lemma 3.5], which also uses the Lyapunov lemma. In fact, analogously we have [95, Lemma 30]

$$\lambda_{\max}(P(x)^{1/2}s) \leq \lambda_{\max}(x \circ s),$$

i.e., the eigenvalues of $x \circ s$ are more disperse than that of $P(x)^{1/2}s$.

# Chapter 5

# A Feasible IPM for SO

In this chapter we present a full NT-step feasible IPM for SO and its analysis. The quadratic convergence result in Theorem 5.6 unifies the analysis known for LO (cf. [93, Theorem II.50]) and SDO (cf. [16, Lemma 7.4]). It will also be used later on when dealing with a full NT-step infeasible IPM for SO, which is the main purpose of this thesis.

## 5.1 The SO problem

During the last two decades, major developments in convex optimization were focusing on Conic Optimization (CO), primarily, on LO, Second-Order Cone Optimization (SOCO), and SDO. The conic form reveals rich structure of these problems. It allows to exploit this structure in order to solve the problems efficiently.

In this section, we introduce the SO problems in conic form. This unifies the representation for LO, SOCO and SDO. For more details, we refer to [74, 95].

Let $\mathcal{J}$ be a Euclidean Jordan algebra with rank $r$ and cone of squares (i.e., its associated symmetric cone) $\mathcal{K}$. Consider the primal-dual pair of SO problems

$$\min \{\langle c, x\rangle : Ax = b,\ x \in \mathcal{K}\} \tag{CP}$$

and

$$\max \left\{b^T y : A^T y + s = c,\ y \in \mathbf{R}^m,\ s \in \mathcal{K}\right\}. \tag{CD}$$

Here $c$ and the rows of $A$ lie in $\mathcal{J}$, and $b \in \mathbf{R}^m$. Without loss of generality we assume that the rows of $A$ are linearly independent. If $a_i$ is the $i$-th row of $A$, then $Ax = b$ means that

$$\langle a_i, x\rangle = b_i, \quad \text{for each } i = 1, \ldots, m,$$

while $A^T y + s = c$ means

$$\sum_{i=1}^{m} y_i a_i + s = c.$$

We say that $x$ is in the null space of $A$ if $x \in \mathcal{J}$ and $Ax = 0$, and $s$ in the row space of $A$ (or column space of $A^T$), if $s = A^T y$ for some $y \in \mathbf{R}^m$. Moreover, we say that $x$ and $s$ are orthogonal with respect to the trace inner product if $\mathrm{tr}(x \circ s) = 0$ (or equivalently $\langle x, s \rangle = 0$). Note that if $x$ is in the null space of $A$ and $s$ in the row space of $A$, then

$$\langle x, s \rangle = \langle x, A^T y \rangle = (Ax)^T y = 0,$$

i.e., they are orthogonal (with respect to the trace inner product).

## 5.2 Conic duality

We call (CP) feasible if there exists $x \in \mathcal{K}$ such that $Ax = b$, and strictly feasible, if in addition, $x \in \operatorname{int} \mathcal{K}$. Similarly, we call (CD) feasible if there exists $(y, s) \in \mathbf{R}^m \times \mathcal{K}$ such that $A^T y + s = c$, and strictly feasible, if in addition $s \in \operatorname{int} \mathcal{K}$.

Let $x$ and $(y, s)$ be a primal-dual feasible pair, i.e., a pair comprised of feasible solutions to (CP) and (CD). Then

$$\langle c, x \rangle - b^T y = \langle A^T y + s, x \rangle - b^T y = \langle x, s \rangle \geq 0,$$

where $\langle x, s \rangle$ is called the duality gap.

**Theorem 5.1** ([8, Theorem 2.4.1])**.** *If the primal problem (CP) is strictly feasible and below bounded, then the dual (CD) is solvable and the optimal values in the problems coincide. Similarly, if the dual (CD) is strictly feasible and above bounded, then the primal (CP) is solvable and the optimal values coincide. Moreover, if both of the problems (CP) and (CD) are strictly feasible, then both of them are solvable, and the optimal values coincide.*

## 5.3 The central path

In the sequel to the current section, we always assume that both (CP) and (CD) satisfy the IPC, i.e., both (CP) and (CD) are strictly feasible. Then it follows from Proposition 4.22 and Theorem 5.1 that finding an optimal solution of (CP) and (CD) is equivalent to solving the following system (cf. [23, 95]).

$$\begin{aligned} Ax &= b, \quad x \in \mathcal{K}, \\ A^T y + s &= c, \quad s \in \mathcal{K}, \\ x \circ s &= 0. \end{aligned} \tag{5.1}$$

The basic idea of primal-dual IPMs is to replace the third equation in (5.1), the so-called complementarity condition for (CP) and (CD), by the parameterized equation $x \circ s = \mu e$, with $\mu > 0$. Thus we consider the system

$$\begin{aligned} Ax &= b, \qquad x \in \mathcal{K}, \\ A^T y + s &= c, \qquad s \in \mathcal{K}, \\ x \circ s &= \mu e. \end{aligned} \tag{5.2}$$

For each $\mu > 0$ the system (5.2) has a unique solution $(x(\mu), y(\mu), s(\mu))$, and we call $x(\mu)$ and $(y(\mu), s(\mu))$ the $\mu$-centers of (CP) and (CD), respectively. The set of $\mu$-centers (with $\mu$ running through all positive real numbers) gives a homotopy path, which is called the central path of (CP) and (CD). If $\mu \to 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition, the limit yields optimal solutions for (CP) and (CD) [23].

## 5.4 The NT-search direction

The natural way to define a search direction is to follow Newton's approach and to linearize the third equation in (5.2). This leads to the system

$$\begin{aligned} A\Delta x &= 0, \\ A^T \Delta y + \Delta s &= 0, \\ x \circ \Delta s + s \circ \Delta x &= \mu e - x \circ s. \end{aligned} \tag{5.3}$$

Due to the fact that $x$ and $s$ do not operator commute in general, i.e.,

$$L(x)L(s) \neq L(s)L(x),$$

the system (5.3) does not always have a unique solution. In particular, for SDO, the system defines the AHO direction, which is not necessarily unique [23, 71]. It is now well known that this difficulty can be solved by applying a scaling scheme as follows [95]. Let $u \in \operatorname{int} \mathcal{K}$. Then we have

$$x \circ s = \mu e \quad \Leftrightarrow \quad P(u)x \circ P(u^{-1})s = \mu e.$$

Since $x$, $s \in \operatorname{int} \mathcal{K}$, this is an easy consequence of Proposition 4.9 $(ii)$, as becomes clear when using the fact that $x \circ s = \mu e$ holds if and only if $x = \mu s^{-1}$. Now, replacing the third equation in (5.3) by $P(u)x \circ P(u^{-1})s = \mu e$, and then applying Newton's method, we obtain the system

$$\begin{aligned} A\Delta x &= 0, \\ A^T \Delta y + \Delta s &= 0, \\ P(u)x \circ P(u^{-1})\Delta s + P(u^{-1})s \circ P(u)\Delta x &= \mu e - P(u)x \circ P(u^{-1})s. \end{aligned} \tag{5.4}$$

Some best-known choices of $u$ for SDO are listed in the Table 5.1. In fact, these scaling points are designed for, or can easily be generalized to SO. By choosing $u$ appropriately, namely, in a subclass of the Monteiro-Zhang family called the commutative class [72, 95, 103, 118], system (5.4) can be used to define search directions uniquely.

| $u$ | Reference |
|---|---|
| $I$ | Alizadeh et al. [5] |
| $X^{-\frac{1}{2}}$ | Kojima et al. [57], Monteiro [68] |
| $S^{\frac{1}{2}}$ | Helmberg et al. [43], Kojima et al. [57], Monteiro [68] |
| $W^{-1/2}$ [1] | Nesterov and Todd [77, 78] |

Table 5.1: Choices of the scaling matrix for SDO.

Here we focus on the scaling point $u = w^{-1/2}$, where $w$ is the NT-scaling point of $x$ and $s$ as defined in Subsection 4.3.1. For that case we define

$$v := \frac{P(w)^{-1/2}x}{\sqrt{\mu}} \quad \left[= \frac{P(w)^{1/2}s}{\sqrt{\mu}}\right], \tag{5.5}$$

and

$$d_x := \frac{P(w)^{-1/2}\Delta x}{\sqrt{\mu}}, \quad d_s := \frac{P(w)^{1/2}\Delta s}{\sqrt{\mu}}. \tag{5.6}$$

This enables us to rewrite the system (5.4) as follows:

$$\sqrt{\mu}AP(w)^{1/2}d_x = 0, \tag{5.7}$$

$$\left(\sqrt{\mu}AP(w)^{1/2}\right)^T \frac{\Delta y}{\mu} + d_s = 0, \tag{5.8}$$

$$d_x + d_s = v^{-1} - v. \tag{5.9}$$

It is easy to verify that the substitution of (5.5) and (5.6) into the first two equations of (5.4) yields (5.7) and (5.8). It is less obvious that the third equation in (5.4) yields (5.9). By substitution we get, after dividing both sides by $\mu$, $v \circ (d_x + d_s) = e - v^2$. This can be written as $L(v)(d_x + d_s) = e - v^2$. After multiplying both sides from the left with $L(v)^{-1}$, while using $L(v)^{-1}e = v^{-1}$ and $L(v)^{-1}v^2 = v$, we obtain (5.9). It easily follows that the above system has a unique solution. Since (5.7) requires that $d_x$ belongs to the null space of $\sqrt{\mu}AP(w)^{1/2}$, and (5.8) that $d_s$ belongs to the row space of $\sqrt{\mu}AP(w)^{1/2}$, it follows that system (5.7)–(5.9) determines $d_x$ and $d_s$ uniquely as the (mutually orthogonal with respect to the trace inner product) components of the vector $v^{-1} - v$ in these two spaces. From (5.9) and the orthogonality of $d_x$ and $d_s$ we obtain

$$\|d_x\|_F^2 + \|d_s\|_F^2 = \|d_x + d_s\|_F^2 = \left\|v^{-1} - v\right\|_F^2. \tag{5.10}$$

[1] $W = X^{\frac{1}{2}}(X^{\frac{1}{2}}SX^{\frac{1}{2}})^{-\frac{1}{2}}X^{\frac{1}{2}}$ $\left[= S^{-1/2}(S^{1/2}XS^{1/2})^{1/2}S^{-1/2}\right]$.

Therefore the displacements $d_x, d_s$ (and since $\sqrt{\mu}AP(w)^{1/2}$ has full row rank, also $\Delta y$) are zero if and only if $v^{-1} - v = 0$. In this case it easily follows that $v = e$, and this implies that $x$ and $(y, s)$ coincide with the respective $\mu$-centers. To get the search directions $\Delta x$ and $\Delta s$ in the original space we simply transform the scaled search directions back to the $x$ and $s$-space by using (5.6):

$$\begin{aligned} \Delta x &= \sqrt{\mu}P(w)^{1/2}d_x, \\ \Delta s &= \sqrt{\mu}P(w)^{-1/2}d_s. \end{aligned} \tag{5.11}$$

Then the new iterate is obtained by taking a full NT-step, as follows:

$$\begin{aligned} x^+ &= x + \Delta x, \\ y^+ &= y + \Delta y, \\ z^+ &= z + \Delta z. \end{aligned} \tag{5.12}$$

## 5.5 Proximity measure

For the design and analysis of the algorithms, we need to measure the distance of the iterate $(x, y, s)$ to the current $\mu$-center $(x(\mu), y(\mu), s(\mu))$. The aim of this section is to present such a measure and to show how it depends on the eigenvalues of the vector $v$.

The proximity measure that we are going to use is defined as follows:

$$\delta(x, s; \mu) \equiv \delta(v) := \frac{1}{2}\left\|v - v^{-1}\right\|_F, \tag{5.13}$$

where $v$ is defined in (5.5). Note that this proximity measure is a natural generalization from the linear case (cf. (2.4)). It follows that

$$4\delta^2(v) = \left\|v - v^{-1}\right\|_F^2 = \operatorname{tr}(v^2) + \operatorname{tr}(v^{-2}) - 2\operatorname{tr}(e), \tag{5.14}$$

which expresses $\delta^2(v)$ in the eigenvalues of $v^2$ and its inverse.

## 5.6 A full NT-step feasible IPM algorithm

The full NT-step feasible IPM algorithm is given in Algorithm 5.1. We show below (cf. Lemma 5.4) that after a full NT-step the duality gap $\langle x, s\rangle$ gets its target value $\mu \operatorname{tr}(e)$. The algorithm stops when the duality gap that equals $\mu \operatorname{tr}(e)$ is less than $\varepsilon$.

## 5.7 The analysis of the NT-step

### 5.7.1 Feasibility

Our aim is to find a condition that guarantees feasibility of the iterate after a full NT-step. As before, let $x, s \in \operatorname{int} \mathcal{K}$, $\mu > 0$ and let $w$ be the scaling point of $x$ and

**Algorithm 5.1** A full NT-step feasible IPM for SO.

**Input:**
accuracy parameter $\varepsilon > 0$;
barrier update parameter $\theta$, $0 < \theta < 1$;
strictly feasible triple $(x^0, y^0, s^0)$
such that $\operatorname{tr}(x^0 \circ s^0) = \mu^0 \operatorname{tr}(e)$ and $\delta(x^0, s^0; \mu^0) \leq 1/2$.
**Begin**
$x := x^0$; $y := y^0$; $s := s^0$; $\mu := \mu^0$;
**while** $\operatorname{tr}(x \circ s) \geq \varepsilon$
$\mu$-update:
$\mu := (1 - \theta)\mu$;
NT-step:
$(x, y, s) := (x, y, s) + (\Delta x, \Delta y, \Delta s)$.
**endwhile**
**end**

$s$. Using (5.5), (5.6) and (5.12), we obtain

$$\begin{aligned} x^+ &= x + \Delta x = \sqrt{\mu} P(w)^{1/2}(v + d_x), \\ s^+ &= s + \Delta s = \sqrt{\mu} P(w)^{-1/2}(v + d_s). \end{aligned} \tag{5.15}$$

Since $P(w)^{1/2}$ and its inverse $P(w)^{-1/2}$ are automorphisms of $\operatorname{int} \mathcal{K}$ (cf. Theorem 4.20), $x^+$ and $s^+$ will belong to $\operatorname{int} \mathcal{K}$ if and only if $v + d_x$ and $v + d_s$ belong to $\operatorname{int} \mathcal{K}$. For the proof of our main result in this subsection, which is Lemma 5.3, we need the following lemma.

**Lemma 5.2.** *If $\delta(v) \leq 1$ then $e + d_x \circ d_s \in \mathcal{K}$. Moreover, if $\delta(v) < 1$ then $e + d_x \circ d_s \in \operatorname{int} \mathcal{K}$.*

*Proof.* Since $d_x$ and $d_s$ are orthogonal with respect to the trace inner product, Lemma 4.49 implies that the absolute values of the eigenvalues of $d_x \circ d_s$ do not exceed $\frac{1}{4} \|d_x + d_s\|_F^2$. In addition, it follows from (5.10) and (5.13) that

$$\|d_x + d_s\|_F^2 = \left\|v - v^{-1}\right\|_F^2 = 4\delta(v)^2.$$

Hence, the absolute values of the eigenvalues of $d_x \circ d_s$ do not exceed $\delta(v)^2$. This implies that $1 - \delta(v)^2$ is a lower bound for the eigenvalues of $e + d_x \circ d_s$. Hence, if $\delta(v) \leq 1$, then $e + d_x \circ d_s \in \mathcal{K}$ and if $\delta(v) < 1$, then $e + d_x \circ d_s \in \operatorname{int} \mathcal{K}$. This proves the lemma. □

**Lemma 5.3.** *The full NT-step is feasible if $\delta(v) \leq 1$ and strictly feasible if $\delta(v) < 1$.*

*Proof.* We introduce a step length $\alpha$ with $0 \le \alpha \le 1$, and define

$$v_x^\alpha = v + \alpha d_x,$$
$$v_s^\alpha = v + \alpha d_s.$$

We then have $v_x^0 = v$, $v_x^1 = v + d_x$, and similarly $v_s^0 = v$, $v_s^1 = v + d_s$. It follows from (5.9) that

$$\begin{aligned} v_x^\alpha \circ v_s^\alpha &= (v + \alpha d_x) \circ (v + \alpha d_s) = v^2 + \alpha v \circ (d_x + d_s) + \alpha^2 d_x \circ d_s \\ &= v^2 + \alpha v \circ (v^{-1} - v) + \alpha^2 d_x \circ d_s = (1-\alpha)v^2 + \alpha e + \alpha^2 d_x \circ d_s. \end{aligned}$$

Since $\delta(v) \le 1$, Lemma 5.2 implies that $d_x \circ d_s \succeq_{\mathcal{K}} -e$. Substitution gives

$$v_x^\alpha \circ v_s^\alpha \succeq_{\mathcal{K}} (1-\alpha)v^2 + \alpha e - \alpha^2 e = (1-\alpha)(v^2 + \alpha e).$$

If $0 \le \alpha < 1$, the last vector belongs to $\operatorname{int} \mathcal{K}$, i.e., we have

$$v_x^\alpha \circ v_s^\alpha \succ_{\mathcal{K}} 0, \quad \text{for } \alpha \in [0,1).$$

By Lemma 4.51, $\det(v_x^\alpha)$ and $\det(v_s^\alpha)$ do not vanish for $\alpha \in [0,1)$. Since

$$\det(v_x^0) = \det(v_s^0) = \det(v) > 0,$$

by continuity, $\det(v_x^\alpha)$ and $\det(v_s^\alpha)$ stay positive for all $\alpha \in [0,1)$. Moreover, by Theorem 4.13, this implies that all the eigenvalues of $v_x^\alpha$ and $v_s^\alpha$ stay positive for all $\alpha \in [0,1)$. Again by continuity, we obtain that all the eigenvalues of $v_x^1$ and $v_s^1$ are nonnegative. This proves that if $\delta(v) \le 1$ then $v + d_x \in \mathcal{K}$ and $v + d_s \in \mathcal{K}$.

For $\delta(v) < 1$, we have by Lemma 5.2, $d_x \circ d_s \succ_{\mathcal{K}} -e$ and similar arguments imply that $\det(v_x^\alpha)$ and $\det(v_s^\alpha)$ do not vanish for $\alpha \in [0,1]$, whence $v + d_x \in \operatorname{int} \mathcal{K}$ and $v + d_s \in \operatorname{int} \mathcal{K}$. This proves the lemma. □

**Lemma 5.4.** *Let $x, s \in \operatorname{int} \mathcal{K}$ and $\mu > 0$, then $\langle x^+, s^+ \rangle = \mu \operatorname{tr}(e)$.*

*Proof.* Due to (5.15) we may write

$$\langle x^+, s^+ \rangle = \langle \sqrt{\mu} P(w)^{1/2}(v + d_x), \sqrt{\mu} P(w)^{-1/2}(v + d_s) \rangle = \mu \langle v + d_x, v + d_s \rangle.$$

Using (5.9) we obtain

$$\begin{aligned} \langle v + d_x, v + d_s \rangle &= \langle v, v \rangle + \langle v, d_x + d_s \rangle + \langle d_x, d_s \rangle \\ &= \langle v, v \rangle + \langle v, v^{-1} - v \rangle + \langle d_x, d_s \rangle \\ &= \operatorname{tr}(e) + \langle d_x, d_s \rangle. \end{aligned}$$

Since $d_x$ and $d_s$ are orthogonal with respect to the trace inner product, the lemma follows. □

### 5.7.2 Quadratic convergence

In this subsection we prove quadratic convergence to the target point $(x(\mu), s(\mu))$ when taking full NT-steps. According to (5.5), the $v$-vector after the step is given by:

$$v^+ := \frac{P(w^+)^{-1/2}x^+}{\sqrt{\mu}} \quad \left[= \frac{P(w^+)^{1/2}s^+}{\sqrt{\mu}}\right], \tag{5.16}$$

where $w^+$ is the scaling point of $x^+$ and $s^+$.

**Lemma 5.5** ([109, Proposition 5.9.3])**.** *One has*

$$v^+ \sim \left(P(v+d_x)^{1/2}(v+d_s)\right)^{1/2}.$$

*Proof.* It readily follows from (5.16) and Lemma 4.46 that

$$\sqrt{\mu}v^+ = P(w^+)^{1/2}s^+ \sim \left(P(x^+)^{1/2}s^+\right)^{1/2}.$$

Due to (5.15) and Lemma 4.45, we may write

$$\begin{aligned} P(x^+)^{1/2}s^+ &= \mu P\left(P(w)^{1/2}(v+d_x)\right)^{1/2} P(w)^{-1/2}(v+d_s) \\ &\sim \mu P(v+d_x)^{1/2}(v+d_s). \end{aligned}$$

From this the lemma follows. □

**Theorem 5.6.** *If $\delta := \delta(v) < 1$, then the full NT-step is strictly feasible and*

$$\delta(v^+) \leq \frac{\delta^2}{\sqrt{2(1-\delta^2)}}.$$

*Proof.* Since $\delta := \delta(v) < 1$, from Lemma 5.3 and its proof, it follows that $v + d_x$, $v + d_s$, and $(v + d_x) \circ (v + d_s)$ belong to the $\operatorname{int} \mathcal{K}$. Let us, for the moment, denote

$$\begin{aligned} u &:= P(v+d_x)^{1/2}(v+d_s), \\ \bar{u} &:= (v+d_x) \circ (v+d_s). \end{aligned}$$

Then, it follows from Lemma 5.5 that $v^+ \sim u^{1/2}$. Therefore

$$2\delta(v^+) = \left\|v^+ - (v^+)^{-1}\right\|_F = \left\|u^{1/2} - u^{-1/2}\right\|_F.$$

By applying Lemma 4.55, we obtain

$$2\delta(v^+) = \left\|u^{1/2} - u^{-1/2}\right\|_F \leq \frac{\|u-e\|_F}{\lambda_{\min}(u^{1/2})} = \frac{\|u-e\|_F}{\lambda_{\min}(u)^{1/2}}.$$

In addition, we derive from Lemma 4.56 and Lemma 4.58 that

$$2\delta(v^+) \leq \frac{\|u-e\|_F}{\lambda_{\min}(u)^{1/2}} \leq \frac{\|\bar{u}-e\|_F}{\lambda_{\min}(\bar{u})^{1/2}}.$$

It follows from (5.9) that

$$\begin{aligned}\bar{u} &= (v+d_x)\circ(v+d_s) = v^2 + v\circ(d_x+d_s) + d_x\circ d_s \\ &= v^2 + v\circ(v^{-1}-v) + d_x\circ d_s = e + d_x\circ d_s,\end{aligned}$$

and the substitution gives

$$2\delta(v^+) \leq \frac{\|d_x\circ d_s\|_F}{\lambda_{\min}(e+d_x\circ d_s)^{1/2}} = \frac{\|d_x\circ d_s\|_F}{[1+\lambda_{\min}(d_x\circ d_s)]^{1/2}}.$$

Now we apply Lemma 4.49. Part $(i)$ of this lemma implies that $\delta^2$ is a upper bound for $\|\lambda(d_s\circ d_s)\|_\infty$, as we already established in the proof of Lemma 5.2. Also using part $(ii)$ of Lemma 4.49 we may now write

$$2\delta(v^+) \leq \frac{\|d_x\circ d_s\|_F}{[1+\lambda_{\min}(d_x\circ d_s)]^{1/2}} \leq \frac{\frac{1}{2\sqrt{2}}\|d_x+d_s\|_F^2}{\sqrt{1-\delta^2}} = \frac{\sqrt{2}\delta^2}{\sqrt{1-\delta^2}},$$

which implies the lemma. □

As a result, the following corollary readily follows.

**Corollary 5.7.** *If $\delta(v) \leq 1/\sqrt{2}$, then the full NT-step is strictly feasible and $\delta(v^+) \leq \delta(v)^2$.*

## 5.8 Updating the barrier parameter $\mu$

In this section we establish a simple relation between the values of our proximity measure just before and after a $\mu$-update.

**Lemma 5.8.** *Let $x, s \in \operatorname{int}\mathcal{K}$, $\operatorname{tr}(x\circ s) = \mu\operatorname{tr}(e)$, and $\delta := \delta(x,s;\mu)$. If $\mu^+ = (1-\theta)\mu$ for some $0<\theta<1$, then*

$$\delta(x,s;\mu^+)^2 = \frac{\theta^2\operatorname{tr}(e)}{4(1-\theta)} + (1-\theta)\delta^2.$$

*Proof.* When updating $\mu$ to $\mu^+$, the vector $v$ is divided by the factor $\sqrt{1-\theta}$. Hence we may write

$$4\delta(x,s;\mu^+)^2 = \left\|\frac{v}{\sqrt{1-\theta}} - \sqrt{1-\theta}v^{-1}\right\|_F^2 = \left\|\frac{\theta v}{\sqrt{1-\theta}} + \sqrt{1-\theta}(v-v^{-1})\right\|_F^2.$$

We show below that the vectors $v$ and $v - v^{-1}$ are orthogonal with respect to the trace inner product. From (5.5), it readily follows that $\mu \operatorname{tr}(v^2) = \operatorname{tr}(x \circ s)$, which implies $\operatorname{tr}(v^2) = \operatorname{tr}(e)$. Hence, we have

$$\operatorname{tr}(v \circ (v - v^{-1})) = \operatorname{tr}(v^2 - e) = \operatorname{tr}(v^2) - \operatorname{tr}(e) = 0.$$

Therefore, we may proceed as follows:

$$4\delta(x, s; \mu^+)^2 = \frac{\theta^2}{1-\theta} \|v\|_F^2 + (1-\theta) \left\|v - v^{-1}\right\|_F^2 = \frac{\theta^2}{1-\theta} \operatorname{tr}(e) + 4(1-\theta)\delta^2.$$

This implies the lemma. □

## 5.9 Iteration bound

We conclude this chapter with an iteration bound for the Algorithm 5.1. The arguments here generalize those in Section 2.7 for LO.

At the start of each iteration we have $\operatorname{tr}(x \circ s) = \mu r$ ($r = \operatorname{tr}(e)$ is the rank of the associated Euclidean Jordan algebra) and $\delta(x, s; \mu) \leq 1/2$. After the barrier parameter is updated to $\mu^+ = (1-\theta)\mu$, with $\theta = 1/\sqrt{2r}$, Lemma 5.8 yields that $\delta(x, s; \mu^+) \leq 1/\sqrt{2}$ (cf. the proof of [93, Theorem II.53]). Then, after the full NT-step to the $\mu^+$-center we have, by Corollary 5.7, $\delta(x^+, s^+, \mu^+) \leq 1/2$. Moreover, using Lemma 5.4, we have that the target value of the duality gap is attained, i.e., $\operatorname{tr}(x^+ \circ s^+) = \mu^+ r = (1-\theta)\mu r$.

Since for each iteration the duality gap is reduced by the factor $(1-\theta)$, the total number of iterations now easily follows from Lemma 2.8 (we replace $n$ by $r = \operatorname{tr}(e)$).

**Theorem 5.9.** *If $\theta = 1/\sqrt{2r}$, where $r = \operatorname{tr}(e)$ is the rank of the associated Euclidean Jordan algebra, then the number of iterations of Algorithm 5.1 does not exceed*

$$\left\lceil \sqrt{2r} \log \frac{\mu^0 r}{\varepsilon} \right\rceil.$$

# Chapter 6

# An Infeasible IPM for SO

A triple $(x, y, s)$ is called an $\varepsilon$-optimal solution of (CP) and (CD) if the norms of the residual vectors $b - Ax$ and $c - A^T y - s$ do not exceed $\varepsilon$, and also the duality gap satisfies $\mathrm{tr}(x \circ s) \leq \varepsilon$. In this chapter we present an infeasible-start algorithm that generates an $\varepsilon$-optimal solution of (CP) and (CD), if it exists, or establish that no such solution exists.

## 6.1 The perturbed problems

We assume (CP) and (CD) have an optimal solution $(x^*, y^*, s^*)$ with vanishing duality gap, i.e., $\mathrm{tr}(x^* \circ s^*) = 0$. As it is common for infeasible IPMs we start the algorithm with a triple $(x^0, y^0, s^0)$ and $\mu^0 > 0$ such that

$$x^0 = \zeta e, \quad y^0 = 0, \quad s^0 = \zeta e, \quad \mu^0 = \zeta^2, \tag{6.1}$$

where $\zeta$ is a (positive) number such that

$$x^* + s^* \preceq_{\mathcal{K}} \zeta e. \tag{6.2}$$

The algorithm presented in this chapter will generate an $\varepsilon$-optimal solution of (CP) and (CD), or establish that there do not exist optimal solutions with vanishing duality gap satisfying (6.2). The initial values of the primal and dual residual vectors are denoted as $r_p^0$ and $r_d^0$, respectively. So we have

$$\begin{aligned} r_p^0 &= b - Ax^0, \\ r_d^0 &= c - A^T y^0 - s^0. \end{aligned}$$

In general, we have $r_p^0 \neq 0$ and $r_d^0 \neq 0$, i.e., the initial iterate is not feasible. The iterates generated by the algorithm will (in general) be infeasible for (CP) and (CD) as well, but they will be feasible for perturbed versions of (CP) and (CD) that we introduce in the sequel.

For any $\nu$ with $0 \leq \nu \leq 1$ we consider the perturbed problem (CP$_\nu$), defined by

$$\min\left\{(c - \nu r_d^0)^T x : b - Ax = \nu r_p^0,\ x \in \mathcal{K}\right\}, \tag{CP$_\nu$}$$

and its dual problem (CD$_\nu$), given by

$$\max\left\{(b - \nu r_p^0)^T y : c - A^T y - s = \nu r_d^0,\ s \in \mathcal{K}\right\}. \tag{CD$_\nu$}$$

Note that these problems are defined in such a way that if $(x, y, s)$ is feasible for (CP$_\nu$) and (CD$_\nu$) then the residual vectors for the given triple $(x, y, s)$ with respect to the original problems (CP) and (CD) are $\nu r_p^0$ and $\nu r_d^0$, respectively.

If $\nu = 1$ then $x = x^0$ yields a strictly feasible solution of (CP$_\nu$), and $(y, s) = (y^0, s^0)$ a strictly feasible solution of (CD$_\nu$). This means that if $\nu = 1$, then (CP$_\nu$) and (CD$_\nu$) satisfy the IPC.

**Lemma 6.1** (cf. [114, Theorem 5.13])**.** *Let (CP) and (CD) be feasible and* $0 < \nu \leq 1$*. Then, the perturbed problems (CP$_\nu$) and (CD$_\nu$) satisfy the IPC.*

*Proof.* Let $\bar{x}$ be a feasible solution of (CP) and $(\bar{y}, \bar{s})$ a feasible solution of (CD), i.e., $A\bar{x} = b$, $\bar{x} \in \mathcal{K}$ and $A^T\bar{y} + \bar{s} = c$, $\bar{s} \in \mathcal{K}$. Consider

$$\begin{aligned} x &= (1-\nu)\bar{x} + \nu x^0, \\ y &= (1-\nu)\bar{y} + \nu y^0, \\ s &= (1-\nu)\bar{s} + \nu s^0. \end{aligned}$$

Since $x$ is the sum of the vectors $(1-\nu)\bar{x} \in \mathcal{K}$ and $\nu x^0 \in \operatorname{int}\mathcal{K}$, we have $x \in \operatorname{int}\mathcal{K}$. Moreover

$$b - Ax = b - A[(1-\nu)\bar{x} + \nu x^0] = b - (1-\nu)b - \nu A x^0 = \nu(b - Ax^0) = \nu r_p^0,$$

showing that $x$ is strictly feasible for (CP$_\nu$). In precisely the same way one shows that $(y, s)$ is strictly feasible for (CD$_\nu$). Thus we have shown that (CP$_\nu$) and (CD$_\nu$) satisfy the IPC. □

## 6.2 The central path of the perturbed problems

Let (CP) and (CD) be feasible and $0 < \nu \leq 1$. Then Lemma 6.1 implies that the problems (CP$_\nu$) and (CD$_\nu$) satisfy the IPC, and therefore their central paths exist. This means that for every $\mu > 0$ the system

$$b - Ax = \nu r_p^0, \quad x \in \mathcal{K}, \tag{6.3}$$

$$c - A^T y - s = \nu r_d^0, \quad s \in \mathcal{K}, \tag{6.4}$$

$$x \circ s = \mu e \tag{6.5}$$

has a unique solution. This solution is denoted as $(x(\mu,\nu), y(\mu,\nu), s(\mu,\nu))$. These are the $\mu$-centers of the perturbed problems $(\mathrm{CP}_\nu)$ and $(\mathrm{CD}_\nu)$. In the sequel the parameters $\mu$ and $\nu$ will always be in a one-to-one correspondence, according to

$$\mu = \nu\mu^0 = \nu\zeta^2.$$

Therefore, we feel free to omit one parameter and denote $(x(\mu,\nu), y(\mu,\nu), s(\mu,\nu))$ simply as $(x(\nu), y(\nu), s(\nu))$.

Due to the choice of the initial iterate, according to (6.1), we have $x^0 \circ s^0 = \mu^0 e$. Hence $x^0$ is the $\mu^0$-center of the perturbed problem $(\mathrm{CP}_1)$ and $(y^0, s^0)$ the $\mu^0$-center of the perturbed problem $(\mathrm{CD}_1)$. In other words, $(x(1), y(1), s(1)) = (x^0, y^0, s^0)$.

## 6.3 A full NT-step infeasible IPM algorithm

We just established that if $\nu = 1$ and $\mu = \mu^0$, then $x = x^0$ and $(y, s) = (y^0, s^0)$ are the $\mu$-centers of $(\mathrm{CP}_\nu)$ and $(\mathrm{CD}_\nu)$, respectively. This is our initial iterate.

We measure proximity to the $\mu$-center of the perturbed problems by the quantity $\delta(x, s; \mu)$ as defined in (5.13). So, initially we have $\delta(x, s; \mu) = 0$. In the sequel we assume that at the start of each iteration, just before the feasibility step, $\delta(x, s; \mu)$ is smaller than or equal to a (small) threshold value $\tau > 0$. This condition is certainly satisfied at the start of the first iteration. Since we then have $\delta(x, s; \mu) = 0$. Also at the start we have $\mathrm{tr}(x \circ s) = \mu^0 \,\mathrm{tr}(e)$.

Now we describe one (main) iteration of our algorithm. Suppose we have $x$, and $(y, s)$ satisfying the feasibility conditions (6.3) and (6.4) for $\nu = \mu/\mu^0$, and such that $\mathrm{tr}(x \circ s) = \mu\,\mathrm{tr}(e)$ and $\delta(x, s; \mu) \leq \tau$. We reduce $\nu$ to $\nu^+ = (1-\theta)\nu$, and accordingly $\mu$ to $\mu^+ = \nu^+\mu^0 = (1-\theta)\mu$ with $\theta \in (0, 1)$. Then, we find a new iterate $(x^+, y^+, s^+)$ that satisfies (6.3) and (6.4), with $\nu$ replaced by $\nu^+$, and such that $\mathrm{tr}(x^+ \circ s^+) = \mu^+ \,\mathrm{tr}(e)$ and $\delta(x^+, s^+; \mu^+) \leq \tau$.

More in detail, every (main) iteration consists of a feasibility step and a few centering steps. The feasibility step serves to get an iterate $(x^f, y^f, s^f)$ that is strictly feasible for $(\mathrm{CP}_{\nu^+})$ and $(\mathrm{CD}_{\nu^+})$, and such that $\delta(x^f, s^f; \mu^+) \leq 1/\sqrt{2}$. In other words, $(x^f, y^f, s^f)$ belongs to the quadratic convergence neighborhood with respect to the $\mu^+$-center of $(\mathrm{CP}_{\nu^+})$ and $(\mathrm{CD}_{\nu^+})$. Hence, because the NT-step is quadratically convergent in that region, a few centering steps, starting from $(x^f, y^f, s^f)$ and targeting at the $\mu^+$-center of $(\mathrm{CP}_{\nu^+})$ and $(\mathrm{CD}_{\nu^+})$ will generate an iterate $(x^+, y^+, s^+)$ that is strictly feasible for $(\mathrm{CP}_{\nu^+})$ and $(\mathrm{CD}_{\nu^+})$ and satisfies $\mathrm{tr}(x^+ \circ s^+) = \mu^+ \,\mathrm{tr}(e)$ and $\delta(x^+, s^+; \mu^+) \leq \tau$.

A formal description of the algorithm is given in Algorithm 6.1. Recall that after each iteration the residuals and the duality gap are reduced by the factor $(1-\theta)$. The algorithm stops if the norms of the residuals and the duality gap are less than the accuracy parameter $\varepsilon$.

**Algorithm 6.1** A full NT-step infeasible IPM for SO.

**Input:**
accuracy parameter $\varepsilon > 0$;
update parameter $\theta$, $0 < \theta < 1$;
threshold parameter $\tau > 0$;
initialization parameter $\zeta > 0$.
**Begin**
$x := \zeta e$; $y := 0$; $s := \zeta e$; $\mu := \mu^0 = \zeta^2$; $\nu := 1$;
**while** $\max\left(\mathrm{tr}(x \circ s), \|b - Ax\|_F, \left\|c - A^T y - s\right\|_F\right) \geq \varepsilon$
feasibility step:
$(x, y, s) := (x, y, s) + (\Delta^f x, \Delta^f y, \Delta^f s)$;
update of $\mu$ and $\nu$:
$\mu := (1 - \theta)\mu$; $\nu := (1 - \theta)\nu$;
centering steps:
**while** $\delta(x, s; \mu) \geq \tau$
$(x, y, s) := (x, y, s) + (\Delta x, \Delta y, \Delta s)$.
**endwhile**
**endwhile**
**end**

## 6.4 Analysis of the feasibility step

In this section, we define and analyze the feasibility step. This is the most difficult part of the analysis. In essence we follow the same chain of arguments as in Section 3.4.

### 6.4.1 Definition

We describe the feasibility step in detail. The analysis will follow in subsequent subsections. Suppose we have a strictly feasible iterate $(x, y, s)$ for $(\mathrm{CP}_\nu)$ and $(\mathrm{CD}_\nu)$. This means that $(x, y, s)$ satisfies the feasibility conditions (6.3) and (6.4) with $\nu = \mu/\zeta^2$. We need displacements $\Delta^f x$, $\Delta^f y$ and $\Delta^f s$ such that

$$\begin{aligned} x^f &:= x + \Delta^f x, \\ y^f &:= y + \Delta^f y, \\ s^f &:= s + \Delta^f s, \end{aligned} \tag{6.6}$$

are feasible for $(\mathrm{CP}_{\nu^+})$ and $(\mathrm{CD}_{\nu^+})$. One may easily verify that $(x^f, y^f, s^f)$ satisfies (6.3) and (6.4), with $\nu$ replaced by $\nu^+$ and $\mu$ by $\mu^+ = \nu^+ \mu^0 = (1 - \theta)\mu$, only

if the first two equations in the following system are satisfied.

$$\begin{aligned} A\Delta^f x &= \theta\nu r_p^0, \\ A^T\Delta^f y + \Delta^f s &= \theta\nu r_d^0, \\ P(u)x \circ P(u^{-1})\Delta^f s + P(u^{-1})s \circ P(u)\Delta^f x &= (1-\theta)\mu e - P(u)x \circ P(u^{-1})s. \end{aligned}$$

The third equation is inspired by the third equation in the system (5.4) that we used to define search directions for the feasible case, except that we target at the $\mu^+$-centers of $(\mathrm{CP}_{\nu^+})$ and $(\mathrm{CD}_{\nu^+})$. As in the feasible case, we use the NT-scaling scheme to guarantee that the above system has a unique solution. So we take $u = w^{-1/2}$, where $w$ is the NT-scaling point of $x$ and $s$ as given in (4.1). Then the third equation becomes

$$\begin{aligned} P(w)^{-1/2}x \circ P(w)^{1/2}\Delta^f s + P(w)^{1/2}s \circ P(w)^{-1/2}\Delta^f x = \\ (1-\theta)\mu e - P(w)^{-1/2}x \circ P(w)^{1/2}s. \end{aligned} \quad (6.7)$$

Due to this choice of $u$ the coefficient matrix of the resulting system is exactly the same as in the feasible case, and hence it defines the feasibility step uniquely.

By its definition, after the feasibility step the iterate satisfies the affine equations in (6.3) and (6.4), with $\nu$ replaced by $\nu^+$. The hard part in the analysis will be to guarantee that $x^f, s^f \in \operatorname{int}\mathcal{K}$ and to guarantee that the new iterate satisfies $\delta(x^f, s^f; \mu^+) \le 1/\sqrt{2}$.

Let $(x, y, s)$ denote the iterate at the start of an iteration with $\operatorname{tr}(x \circ s) = \mu \operatorname{tr}(e)$ and $\delta(x, s; \mu) \le \tau$. Recall that at the start of the first iteration this is certainly true, because $\operatorname{tr}(x^0 \circ s^0) = \mu^0 \operatorname{tr}(e)$ and $\delta(x^0, s^0; \mu^0) = 0$.

We scale the search directions, just as we did in the feasible case (cf. (5.6)), by defining

$$d_x^f := \frac{P(w)^{-1/2}\Delta^f x}{\sqrt{\mu}}, \quad d_s^f := \frac{P(w)^{1/2}\Delta^f s}{\sqrt{\mu}} \quad (6.8)$$

with $w$ denoting the scaling point of $x$ and $s$, as defined in Proposition 4.40. With the vector $v$ as defined in (5.5), the equation (6.7) can be restated as

$$\mu v \circ (d_x^f + d_s^f) = (1-\theta)\mu e - \mu v^2.$$

By multiplying both sides of this equation from left with $\mu^{-1}L(v)^{-1}$ this equation becomes

$$d_x^f + d_s^f = (1-\theta)v^{-1} - v.$$

Thus, we arrive at the following system for the scaled search directions in the feasibility step:

$$\begin{aligned} \sqrt{\mu}AP(w)^{1/2}d_x^f &= \theta\nu r_p^0, \\ \left(\sqrt{\mu}AP(w)^{1/2}\right)^T \frac{\Delta^f y}{\mu} + d_s^f &= \frac{1}{\sqrt{\mu}}\theta\nu P(w)^{1/2} r_d^0, \\ d_x^f + d_s^f &= (1-\theta)v^{-1} - v. \end{aligned} \quad (6.9)$$

To get the search directions $\Delta^f x$ and $\Delta^f s$ in the original $x$ and $s$-space we use (6.8), which gives

$$\begin{aligned}\Delta^f x &= \sqrt{\mu} P(w)^{1/2} d_x^f,\\ \Delta^f s &= \sqrt{\mu} P(w)^{-1/2} d_s^f.\end{aligned}$$

The new iterate is obtained by taking a full step, as given by (6.6). Hence, we have

$$\begin{aligned}x^f &= x + \Delta^f x = \sqrt{\mu} P(w)^{1/2}(v + d_x^f),\\ s^f &= s + \Delta^f s = \sqrt{\mu} P(w)^{-1/2}(v + d_s^f).\end{aligned} \tag{6.10}$$

From the third equation in (6.9) we derive that

$$(v + d_x^f) \circ (v + d_s^f) = v^2 + v \circ [(1-\theta)v^{-1} - v] + d_x^f \circ d_s^f = (1-\theta)e + d_x^f \circ d_s^f. \tag{6.11}$$

As we mentioned before, the analysis of the algorithm as presented below is much more difficult than in the feasible case. The main reason for this is that the scaled search directions $d_x^f$ and $d_s^f$ are not (necessarily) orthogonal (with respect to the trace inner product).

### 6.4.2 Feasibility

Using the same arguments as in Subsection 5.7 it follows from (6.10) that $x^f$ and $s^f$ are strictly feasible if and only if $v + d_x^f$ and $v + d_s^f$ belong to $\operatorname{int} \mathcal{K}$. Using this we have the following lemma.

**Lemma 6.2.** *The iterate $(x^f, y^f, s^f)$ is feasible if*

$$(1-\theta)e + d_x^f \circ d_s^f \in \mathcal{K},$$

*and strictly feasible if*

$$(1-\theta)e + d_x^f \circ d_s^f \in \operatorname{int} \mathcal{K}.$$

*Proof.* Just as in the proof of Lemma 5.3 we introduce a step length $\alpha$ with $0 \le \alpha \le 1$, and define

$$\begin{aligned}v_x^\alpha &= v + \alpha d_x^f,\\ v_s^\alpha &= v + \alpha d_s^f.\end{aligned}$$

We then have $v_x^0 = v$, $v_x^1 = v + d_x^f$, and similarly $v_s^0 = v$, $v_s^1 = v + d_s^f$. From the third equation in (6.9), i.e., $d_x^f + d_s^f = (1-\theta)v^{-1} + v$, it follows that

$$\begin{aligned}v_x^\alpha \circ v_s^\alpha &= (v + \alpha d_x^f) \circ (v + \alpha d_s^f) = v^2 + \alpha v \circ (d_x^f + d_s^f) + \alpha^2 d_x^f \circ d_s^f\\ &= v^2 + \alpha v \circ [(1-\theta)v^{-1} - v] + \alpha^2 d_x^f \circ d_s^f\\ &= (1-\alpha)v^2 + \alpha(1-\theta)e + \alpha^2 d_x^f \circ d_s^f.\end{aligned}$$

If $(1-\theta)e + d_x^f \circ d_s^f \in \mathcal{K}$, we have $d_x^f \circ d_s^f \succeq_{\mathcal{K}} -(1-\theta)e$. Substituting this into the above equality gives

$$v_x^\alpha \circ v_s^\alpha \succeq_{\mathcal{K}} (1-\alpha)v^2 + \alpha(1-\theta)e - \alpha^2(1-\theta)e = (1-\alpha)(v^2 + \alpha(1-\theta)e).$$

Since $v^2 \in \operatorname{int}\mathcal{K}$, we have $v^2 + \alpha(1-\theta)e \in \operatorname{int}\mathcal{K}$. Hence,

$$v_x^\alpha \circ v_s^\alpha \succeq_{\mathcal{K}} (1-\alpha)(v^2 + \alpha(1-\theta)e) \succ_{\mathcal{K}} 0, \quad \text{for } \alpha \in [0,1).$$

By Lemma 4.51, it follows that $\det(v_x^\alpha)$ and $\det(v_s^\alpha)$ do not vanish for $\alpha \in [0,1)$. Since

$$\det(v_x^0) = \det(v_s^0) = \det(v) > 0,$$

by continuity, $\det(v_x^\alpha)$ and $\det(v_s^\alpha)$ stay positive for all $\alpha \in [0,1)$. Moreover, by Theorem 4.13, this implies that all the eigenvalues of $v_x^\alpha$ and $v_s^\alpha$ stay positive for all $\alpha \in [0,1)$. Again by continuity, we obtain that all the eigenvalues of $v_x^1$ and $v_s^1$ are nonnegative. This proves that if $(1-\theta)e + d_x^f \circ d_s^f \in \mathcal{K}$, then $v + d_x^f \in \mathcal{K}$ and $v + d_s^f \in \mathcal{K}$, i.e., the iterate $(x^f, y^f, s^f)$ is feasible.

If $(1-\theta)e + d_x^f \circ d_s^f \in \operatorname{int}\mathcal{K}$, or equivalently $d_x^f \circ d_s^f \succ_{\mathcal{K}} -(1-\theta)e$, similar arguments imply that $\det(v_x^\alpha)$ and $\det(v_s^\alpha)$ do not vanish for $\alpha \in [0,1]$, whence $v + d_x^f \in \operatorname{int}\mathcal{K}$ and $v + d_s^f \in \operatorname{int}\mathcal{K}$. This proves the lemma. □

It is clear from the above lemma that the feasibility of the iterate $(x^f, y^f, s^f)$ highly depends on the eigenvalues of the vector $d_x^f \circ d_s^f$.

### 6.4.3 Proximity

We proceed by deriving an upper bound for $\delta(x^f, s^f; \mu^+)$. Let $w^f$ be the scaling point of $x^f$ and $s^f$. Denoting the $v$-vector after the feasibility step with respect to the $\mu^+$-center as $v^f$, we have, according to (5.5),

$$v^f := \frac{P(w^f)^{-1/2}x^f}{\sqrt{\mu(1-\theta)}} \quad \left[= \frac{P(w^f)^{1/2}s^f}{\sqrt{\mu(1-\theta)}}\right]. \tag{6.12}$$

**Lemma 6.3.** *One has*

$$\sqrt{1-\theta}\, v^f \sim \left[P(v + d_x^f)^{1/2}(v + d_s^f)\right]^{1/2}.$$

*Proof.* It follows from (6.12) and Lemma 4.46 that

$$\sqrt{\mu(1-\theta)}\, v^f = P(w^f)^{1/2}s^f \sim (P(x^f)^{1/2}s^f)^{1/2}.$$

Due to (6.10) and Lemma 4.45, we may write

$$\begin{aligned} P(x^f)^{1/2}s^f &= \mu P\left(P(w)^{1/2}(v + d_x^f)\right)^{1/2} P(w)^{-1/2}(v + d_s^f) \\ &\sim \mu P(v + d_x^f)^{1/2}(v + d_s^f). \end{aligned}$$

Thus, we obtain

$$\sqrt{\mu(1-\theta)}\, v^f \sim \sqrt{\mu}\left[P(v+d_x^f)^{1/2}(v+d_s^f)\right]^{1/2},$$

and the statement of the lemma follows. □

The above lemma implies that

$$(v^f)^2 \sim P\left(\frac{v+d_x^f}{\sqrt{1-\theta}}\right)^{1/2}\left(\frac{v+d_s^f}{\sqrt{1-\theta}}\right). \tag{6.13}$$

In the sequel we denote $\delta(x^f, s^f; \mu^+)$ shortly as $\delta(v^f)$.

**Lemma 6.4.** *If* $\left\|\lambda(d_x^f \circ d_s^f)\right\|_\infty < 1-\theta$, *then*

$$4\delta(v^f)^2 \le \frac{\left\|\frac{d_x^f \circ d_s^f}{1-\theta}\right\|_F^2}{1-\left\|\frac{\lambda(d_x^f \circ d_s^f)}{1-\theta}\right\|_\infty}.$$

*Proof.* Since $\left\|\lambda(d_x^f \circ d_s^f)\right\|_\infty < 1-\theta$, from Lemma 6.2 and its proof, it follows that $v+d_x^f$, $v+d_s^f$, and $(v+d_x^f)\circ(v+d_s^f)$ belong to the $\operatorname{int}\mathcal{K}$. Let us, for the moment, denote

$$u := P\left(\frac{v+d_x^f}{\sqrt{1-\theta}}\right)^{1/2}\left(\frac{v+d_s^f}{\sqrt{1-\theta}}\right),$$
$$\bar{u} := \left(\frac{v+d_x^f}{\sqrt{1-\theta}}\right)\circ\left(\frac{v+d_s^f}{\sqrt{1-\theta}}\right).$$

Then, since $v^f \sim u^{1/2}$ (cf. (6.13)), we have

$$2\delta(v^f) = \left\|v^f - (v^f)^{-1}\right\|_F = \left\|u^{1/2} - u^{-1/2}\right\|_F.$$

By applying Lemma 4.55, we obtain

$$2\delta(v^f) = \left\|u^{1/2} - u^{-1/2}\right\|_F \le \frac{\|u-e\|_F}{\lambda_{\min}(u^{1/2})} = \frac{\|u-e\|_F}{\lambda_{\min}(u)^{1/2}}.$$

Now, from Lemma 4.56 and Lemma 4.58 we derive

$$2\delta(v^f) \le \frac{\|u-e\|_F}{\lambda_{\min}(u)^{1/2}} \le \frac{\|\bar{u}-e\|_F}{\lambda_{\min}(\bar{u})^{1/2}}.$$

From the third equation in (6.9) it follows that

$$\begin{aligned}(1-\theta)\bar{u} &= (v+d_x^f)\circ(v+d_s^f) = v^2 + v\circ(d_x^f+d_s^f) + d_x^f\circ d_s^f \\ &= v^2 + v\circ((1-\theta)v^{-1} - v) + d_x^f\circ d_s^f = (1-\theta)e + d_x^f\circ d_s^f,\end{aligned}$$

and substitution into the above inequality gives

$$2\delta(v^f) \leq \frac{\left\| \frac{d_x^f \circ d_s^f}{1-\theta} \right\|_F}{\lambda_{\min}\left(e + \frac{d_x^f \circ d_s^f}{1-\theta}\right)^{1/2}} \leq \frac{\left\| \frac{d_x^f \circ d_s^f}{1-\theta} \right\|_F}{\left(1 - \left\| \frac{\lambda(d_x^f \circ d_s^f)}{1-\theta} \right\|_\infty\right)^{1/2}},$$

which proves the lemma. □

From the definition of the Frobenius norm, Lemma 4.53, Lemma 4.48, we have

$$\begin{aligned} \left\|\lambda(d_x^f \circ d_s^f)\right\|_\infty &\leq \left\|d_x^f \circ d_s^f\right\|_F \leq \frac{1}{2}\left\|(d_x^f)^2 + (d_s^f)^2\right\|_F \\ &\leq \frac{1}{2}\left(\left\|(d_x^f)^2\right\|_F + \left\|(d_s^f)^2\right\|_F\right) \leq \frac{1}{2}\left(\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2\right). \end{aligned} \tag{6.14}$$

Substitution of the above inequality into the inequality of Lemma 6.4 yields that

$$4\delta(v^f)^2 \leq \frac{\left\| \frac{d_x^f \circ d_s^f}{1-\theta} \right\|_F^2}{1 - \left\| \frac{\lambda(d_x^f \circ d_s^f)}{1-\theta} \right\|_\infty} \leq \frac{\frac{1}{4}\left( \frac{\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2}{1-\theta} \right)^2}{1 - \frac{1}{2}\frac{\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2}{1-\theta}}. \tag{6.15}$$

We have derived an upper bound for $\delta(v^f)$, in terms of $\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2$. As our ultimate goal is to choose $\theta$, $0 < \theta < 1$, as large as possible, such that $\delta(v^f) \leq 1/\sqrt{2}$, we need an upper bound for $\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2$.

### 6.4.4 Upper bound for $\left\|\boldsymbol{d_x^f}\right\|_{\boldsymbol{F}}^{\boldsymbol{2}} + \left\|\boldsymbol{d_s^f}\right\|_{\boldsymbol{F}}^{\boldsymbol{2}}$

Obtaining an upper bound for $\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2$ is the goal of this subsection. In the sequel this will enable us to find a default value for the update parameter $\theta$.

For the moment, let us define

$$\bar{r}_p := \theta\nu r_p^0, \quad \bar{r}_d := \theta\nu r_d^0, \quad \bar{r} := (1-\theta)v^{-1} - v. \tag{6.16}$$

With $\xi := -\frac{\Delta^f y}{\mu}$, the system (6.9) (by eliminating $d_s^f$) reduces to

$$\sqrt{\mu}AP(w)^{1/2}d_x^f = \bar{r}_p, \tag{6.17}$$

$$\left(\sqrt{\mu}AP(w)^{1/2}\right)^T \xi + d_x^f = \bar{r} - \frac{1}{\sqrt{\mu}}P(w)^{1/2}\bar{r}_d. \tag{6.18}$$

Multiplying both sides of (6.18) from the left with $\sqrt{\mu}\,AP(w)^{1/2}$ and using (6.17) it follows that

$$\mu AP(w)A^T\xi + \bar{r}_p = \sqrt{\mu}\,AP(w)^{1/2}\left(\bar{r} - \frac{1}{\sqrt{\mu}}P(w)^{1/2}\bar{r}_d\right).$$

Therefore,

$$\xi = \frac{1}{\mu}(AP(w)A^T)^{-1}\left[\sqrt{\mu}\,AP(w)^{1/2}\left(\bar{r} - \frac{1}{\sqrt{\mu}}P(w)^{1/2}\bar{r}_d\right) - \bar{r}_p\right]. \tag{6.19}$$

Substitution into (6.18) gives

$$\begin{aligned} d_x^f &= \bar{r} - \frac{1}{\sqrt{\mu}}P(w)^{1/2}\bar{r}_d \\ &\quad - \frac{1}{\sqrt{\mu}}\,P(w)^{1/2}A^T(AP(w)A^T)^{-1}\left[\sqrt{\mu}\,AP(w)^{1/2}\left(\bar{r} - \frac{1}{\sqrt{\mu}}P(w)^{1/2}\bar{r}_d\right) - \bar{r}_p\right] \\ &= \left[I - P(w)^{1/2}A^T(AP(w)A^T)^{-1}AP(w)^{1/2}\right]\left(\bar{r} - \frac{1}{\sqrt{\mu}}P(w)^{1/2}\bar{r}_d\right) \\ &\quad + \frac{1}{\sqrt{\mu}}\,P(w)^{1/2}A^T(AP(w)A^T)^{-1}\bar{r}_p. \end{aligned}$$

To simplify the above expression we denote

$$\bar{P} = P(w)^{1/2}A^T(AP(w)A^T)^{-1}AP(w)^{1/2}.$$

Note that $\bar{P}$ is (the matrix of) the orthogonal projection (with respect to the trace inner product) to the row space of the matrix $AP(w)^{1/2}$. We now may write

$$d_x^f = [I - \bar{P}]\left(\bar{r} - \frac{1}{\sqrt{\mu}}P(w)^{1/2}\bar{r}_d\right) + \frac{1}{\sqrt{\mu}}\,P(w)^{1/2}A^T(AP(w)A^T)^{-1}\bar{r}_p.$$

Let $(\bar{x}, \bar{y}, \bar{s})$ be such that $A\bar{x} = b$ and $A^T\bar{y} + \bar{s} = c$. Then we may write

$$\begin{aligned} \bar{r}_p &= \theta\nu r_p^0 = \theta\nu(b - Ax^0) = \theta\nu A(\bar{x} - x^0), \\ \bar{r}_d &= \theta\nu r_d^0 = \theta\nu(c - A^Ty^0 - s^0) = \theta\nu\left(A^T(\bar{y} - y^0) + \bar{s} - s^0\right). \end{aligned}$$

Substituting the expressions of $\bar{r}_p$ and $\bar{r}_d$ into the expression for $d_x^f$, we obtain

$$d_x^f = [I - \bar{P}]\left(\bar{r} - \frac{\theta\nu}{\sqrt{\mu}}P(w)^{1/2}\left(A^T(\bar{y} - y^0) + \bar{s} - s^0\right)\right) + \frac{\theta\nu}{\sqrt{\mu}}\,\bar{P}P(w)^{-1/2}(\bar{x} - x^0).$$

Since $I - \bar{P}$ is the orthogonal projection to the null space of $AP(w)^{1/2}$ we have

$$[I - \bar{P}]P(w)^{1/2}A^T(\bar{y} - y^0) = 0.$$

Hence, it follows that

$$d_x^f = [I - \bar{P}]\left(\bar{r} - \frac{\theta\nu}{\sqrt{\mu}}P(w)^{1/2}\left(\bar{s} - s^0\right)\right) + \frac{\theta\nu}{\sqrt{\mu}}\,\bar{P}P(w)^{-1/2}(\bar{x} - x^0).$$

To proceed we further simplify the above expression by defining

$$u^x = \frac{\theta\nu}{\sqrt{\mu}} P(w)^{-1/2}(\bar{x} - x^0), \quad u^s = \frac{\theta\nu}{\sqrt{\mu}} P(w)^{1/2}(\bar{s} - s^0). \tag{6.20}$$

Then we may write

$$d_x^f = [I - \bar{P}](\bar{r} - u^s) + \bar{P}u^x.$$

For $d_s^f$ we obtain, by using the third equation in (6.9) and the definition (6.16) of $\bar{r}$,

$$d_s^f = \bar{r} - d_x^f = \bar{r} - [I - \bar{P}]\bar{r} + [I - \bar{P}]u^s - \bar{P}u^x = [I - \bar{P}]u^s + \bar{P}(\bar{r} - u^x).$$

We denote $[I-\bar{P}]\bar{r} = \bar{r}_1$ and $\bar{P}\bar{r} = \bar{r}_2$, and use similar notations for the projections of $u^x$ and $u^s$. Then from the above expressions for $d_x^f$ and $d_s^f$ we derive that

$$d_x^f = \bar{r}_1 - u_1^s + u_2^x, \quad d_s^f = u_1^s + \bar{r}_2 - u_2^x.$$

Therefore, using the orthogonality (with respect to the trace inner product) of vectors with different subscripts, we may write

$$\begin{aligned}
&\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2 \\
&= \|\bar{r}_1 - u_1^s\|_F^2 + \|u_2^x\|_F^2 + \|u_1^s\|_F^2 + \|\bar{r}_2 - u_2^x\|_F^2 \\
&= \|\bar{r}_1\|_F^2 + \|u_1^s\|_F^2 - 2\langle \bar{r}_1, u_1^s\rangle + \|u_2^x\|_F^2 + \|u_1^s\|_F^2 + \|\bar{r}_2\|_F^2 + \|u_2^x\|_F^2 - 2\langle \bar{r}_2, u_2^x\rangle \\
&= \|\bar{r}\|_F^2 + 2\|u_2^x\|_F^2 + 2\|u_1^s\|_F^2 - 2\langle \bar{r}_1, u_1^s\rangle - 2\langle \bar{r}_2, u_2^x\rangle.
\end{aligned}$$

Using the Cauchy-Schwartz inequality (cf. Lemma 4.52), and the properties of orthogonal projection, we further obtain

$$\begin{aligned}
\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2 &\le \|\bar{r}\|_F^2 + 2\|u_2^x\|_F^2 + 2\|u_1^s\|_F^2 + 2\|\bar{r}_1\|_F \|u_1^s\|_F + 2\|\bar{r}_2\|_F \|u_2^x\|_F \\
&\le \|\bar{r}\|_F^2 + 2\|u_2^x\|_F^2 + 2\|u_1^s\|_F^2 + \|\bar{r}_1\|_F^2 + \|u_1^s\|_F^2 + \|\bar{r}_2\|_F^2 + \|u_2^x\|_F^2 \\
&\le 2\|\bar{r}\|_F^2 + 3\left(\|u^x\|_F^2 + \|u^s\|_F^2\right).
\end{aligned} \tag{6.21}$$

Since $v$ and $v^{-1} - v$ are orthogonal (with respect to the trace inner product) and $\|v\|_F^2 = \mathrm{tr}(e)$, we have

$$\begin{aligned}
\|\bar{r}\|_F^2 &= \left\|(1-\theta)v^{-1} - v\right\|_F^2 = \left\|(1-\theta)(v^{-1} - v) - \theta v\right\|_F^2 \\
&= (1-\theta)^2\left\|v^{-1} - v\right\|_F^2 + \theta^2\|v\|_F^2 = 4(1-\theta)^2\delta(v)^2 + \theta^2 r,
\end{aligned} \tag{6.22}$$

where $r = \mathrm{tr}(e)$ is the rank of the associated Euclidean Jordan algebra. On the other hand, due to (6.20) we have

$$\|u^x\|_F^2 + \|u^s\|_F^2 = \frac{\theta^2\nu^2}{\mu}\left(\left\|P(w)^{-1/2}(\bar{x} - x^0)\right\|_F^2 + \left\|P(w)^{1/2}(\bar{s} - s^0)\right\|_F^2\right). \tag{6.23}$$

Let $(x^*, y^*, s^*)$ be the optimal solution satisfying (6.2). It follows that $Ax^* = b$ and $A^T y^* + s^* = c$. Therefore, we may choose $\bar{x} = x^*$, $\bar{y} = y^*$ and $\bar{s} = s^*$. Since $x^*$ is feasible for (CP) we have $x^* \succeq_{\mathcal{K}} 0$. Also $s^* \succeq_{\mathcal{K}} 0$. Hence we have $0 \preceq_{\mathcal{K}} x^* \preceq_{\mathcal{K}} x^* + s^* \preceq_{\mathcal{K}} \zeta e$, or equivalently $0 \preceq_{\mathcal{K}} \bar{x} \preceq_{\mathcal{K}} \zeta e$. In a similar way we derive that $0 \preceq_{\mathcal{K}} \bar{s} \preceq_{\mathcal{K}} \zeta e$. Therefore, it follows that

$$0 \preceq_{\mathcal{K}} x^0 - \bar{x} \preceq_{\mathcal{K}} \zeta e, \quad 0 \preceq_{\mathcal{K}} s^0 - \bar{s} \preceq_{\mathcal{K}} \zeta e.$$

We first consider the term $\left\|P(w)^{-1/2}(\bar{x} - x^0)\right\|_F^2$. Using that $P(w)^{1/2}$ is self-adjoint with respect to the inner product and $P(w)e = w^2$, we have

$$\begin{aligned}
\left\|P(w)^{-1/2}(\bar{x} - x^0)\right\|_F^2 &= \left\|P(w)^{-1/2}(x^0 - \bar{x})\right\|_F^2 = \langle P(w)^{-1}(x^0 - \bar{x}), x^0 - \bar{x}\rangle \\
&= \langle P(w)^{-1}(x^0 - \bar{x}), \zeta e\rangle - \langle P(w)^{-1}(x^0 - \bar{x}), \zeta e - (x^0 - \bar{x})\rangle \\
&\leq \langle P(w)^{-1}(x^0 - \bar{x}), \zeta e\rangle = \zeta\langle P(w)^{-1}e, x^0 - \bar{x}\rangle \\
&= \zeta\langle P(w)^{-1}e, \zeta e\rangle - \zeta\langle P(w)^{-1}e, \zeta e - (x^0 - \bar{x})\rangle \\
&\leq \zeta^2 \operatorname{tr}(w^{-2}).
\end{aligned}$$

Similarly, it follows that

$$\left\|P(w)^{1/2}(\bar{s} - s^0)\right\|_F^2 \leq \zeta^2 \operatorname{tr}(w^2).$$

Substitution of the last two inequalities into (6.23) gives

$$\|u^x\|_F^2 + \|u^s\|_F^2 \leq \frac{\theta^2\nu^2\zeta^2}{\mu} \operatorname{tr}(w^2 + w^{-2}).$$

By using $\mu = \nu\mu^0 = \nu\zeta^2$ we derive

$$\|u^x\|_F^2 + \|u^s\|_F^2 \leq \theta^2\nu \operatorname{tr}(w^2 + w^{-2}). \tag{6.24}$$

**Lemma 6.5.** *One has*

$$\operatorname{tr}(w^2 + w^{-2}) \leq \frac{\operatorname{tr}(x+s)^2}{\mu\lambda_{\min}(v)^2}.$$

*Proof.* For the moment, let $u := (P(x^{1/2})s)^{-1/2}$. Then, by Proposition 4.40, $w = P(x^{1/2})u$. Using that $P(x^{1/2})$ is self-adjoint, and also Lemma 4.54, we obtain

$$\operatorname{tr}(w^2) = \langle P(x^{1/2})u, P(x^{1/2})u\rangle = \langle u, P(x)u\rangle \leq \lambda_{\max}(u) \operatorname{tr}(P(x)u).$$

Using the same arguments as above and the fact that $P(x)e = x^2$ we may write

$$\operatorname{tr}(P(x)u) = \operatorname{tr}(P(x)u \circ e) = \langle P(x)u, e\rangle = \langle u, P(x)e\rangle = \langle u, x^2\rangle \leq \lambda_{\max}(u) \operatorname{tr}(x^2).$$

Combining the above inequalities we obtain

$$\operatorname{tr}(w^2) \leq \lambda_{\max}(P(x^{-1/2})s^{-1}) \operatorname{tr}(x^2).$$

Due to $P(s^{1/2})x \sim P(x^{1/2})s \sim (P(w)^{1/2}s)^2 \sim (P(w^{-1/2})x)^2 = \mu v^2$, we have

$$\lambda_{\max}(P(x^{-1/2})s^{-1}) \operatorname{tr}(x^2) = \frac{\operatorname{tr}(x^2)}{\lambda_{\min}(P(x^{1/2})s)} = \frac{\operatorname{tr}(x^2)}{\mu\lambda_{\min}(v)^2}.$$

Thus, we obtain

$$\operatorname{tr}(w^2) \leq \frac{\operatorname{tr}(x^2)}{\mu\lambda_{\min}(v)^2}.$$

Recalling that $w^{-1}$ is the scaling point of $s$ and $x$, it follows from the above inequality, by interchanging the role of $x$ and $s$, that

$$\operatorname{tr}(w^{-2}) \leq \frac{\operatorname{tr}(s^2)}{\mu\lambda_{\min}(v)^2}.$$

By adding the last two inequalities we obtain

$$\operatorname{tr}(w^2 + w^{-2}) \leq \frac{\operatorname{tr}(x^2) + \operatorname{tr}(s^2)}{\mu\lambda_{\min}(v)^2}.$$

Since $x, s \in \mathcal{K}$, we have $\operatorname{tr}(x \circ s) \geq 0$. Together with the fact that $\operatorname{tr}(u^2) \leq \operatorname{tr}(u)^2$ for each $u \in \mathcal{K}$ (cf. Lemma 4.47), we obtain

$$\operatorname{tr}(x^2) + \operatorname{tr}(s^2) \leq \operatorname{tr}(x^2) + \operatorname{tr}(s^2) + 2\operatorname{tr}(x \circ s) = \operatorname{tr}\left((x+s)^2\right) \leq \operatorname{tr}(x+s)^2.$$

Substitution yields

$$\operatorname{tr}(w^2 + w^{-2}) \leq \frac{\operatorname{tr}(x+s)^2}{\mu\lambda_{\min}(v)^2},$$

which completes the proof. □

Substituting the result of the above lemma into inequality (6.24), we obtain

$$\|u^x\|_F^2 + \|u^s\|_F^2 \leq \theta^2\nu\frac{\operatorname{tr}(x+s)^2}{\mu\lambda_{\min}(v)^2}. \tag{6.25}$$

Concluding this subsection, we have, by substituting (6.22) and (6.25) into (6.21), and using the fact that $\mu = \nu\mu^0 = \nu\zeta^2$,

$$\begin{aligned}\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2 &\leq 2\left[4(1-\theta)^2\delta(v)^2 + \theta^2 r\right] + 3\theta^2\nu\frac{\operatorname{tr}(x+s)^2}{\mu\lambda_{\min}(v)^2} \\ &= 2\left[4(1-\theta)^2\delta(v)^2 + \theta^2 r\right] + 3\theta^2\frac{\operatorname{tr}(x+s)^2}{\zeta^2\lambda_{\min}(v)^2},\end{aligned} \tag{6.26}$$

where $r$ is the rank of the associated Euclidean Jordan algebra. To continue, we need an upper bound for $\operatorname{tr}(x+s)$, and a lower bound for $\lambda_{\min}(v)$, which we will derive in the next subsection.

### 6.4.5 Bounds for $\mathbf{tr}(x+s)$ and $\boldsymbol{\lambda}_{\min}(v)$

**Lemma 6.6.** *Let $x$ and $(y,s)$ be feasible for the perturbed problems ($CP_\nu$) and ($CD_\nu$), respectively, and $\operatorname{tr}(x\circ s)=\mu\operatorname{tr}(e)$. With $(x^0,y^0,s^0)$ as defined in (6.1) and $\zeta$ as in (6.2), we then have*

$$\operatorname{tr}(x+s)\le 2\zeta r,$$

*where $r=\operatorname{tr}(e)$ is the rank of the associated Euclidean Jordan algebra.*

*Proof.* Let $(x^*,y^*,s^*)$ be the optimal solution satisfying (6.2). Then from the feasibility conditions of the perturbed problems ($\mathrm{CP}_\nu$) and ($\mathrm{CD}_\nu$), it is easily seen that

$$\begin{aligned} A\left[x-\nu x^0-(1-\nu)x^*\right]&=0,\\ A^T\left[y-\nu y^0-(1-\nu)y^*\right]+\left[s-\nu s^0-(1-\nu)s^*\right]&=0.\end{aligned}$$

This implies that

$$\operatorname{tr}\left(\left(x-\nu x^0-(1-\nu)x^*\right)\circ\left(s-\nu s^0-(1-\nu)s^*\right)\right)=0.$$

By expanding the above equality and using the fact that $\operatorname{tr}(x^*\circ s^*)=0$, we obtain

$$\begin{aligned}\nu\left(\operatorname{tr}(x^0\circ s)+\operatorname{tr}(s^0\circ x)\right)&=\operatorname{tr}(x\circ s)+\nu^2\operatorname{tr}(x^0\circ s^0)-(1-\nu)\operatorname{tr}(x\circ s^*+s\circ x^*)\\&\quad+\nu(1-\nu)\operatorname{tr}(x^0\circ s^*+s^0\circ x^*).\end{aligned}\tag{6.27}$$

Since $(x^0,y^0,s^0)$ is as defined in (6.1), we have

$$\begin{aligned}\operatorname{tr}(x^0\circ s)+\operatorname{tr}(s^0\circ x)&=\zeta\operatorname{tr}(x+s),\\ \operatorname{tr}(x^0\circ s^0)&=\zeta^2\operatorname{tr}(e),\\ \operatorname{tr}(x^0\circ s^*)+\operatorname{tr}(s^0\circ x^*)&=\zeta\operatorname{tr}(x^*+s^*).\end{aligned}$$

Due to (6.2) we have $\operatorname{tr}(x^*+s^*)\le\zeta\operatorname{tr}(e)$. Furthermore, $\operatorname{tr}(x\circ s)=\mu\operatorname{tr}(e)=\nu\zeta^2\operatorname{tr}(e)$. Finally, $\operatorname{tr}(x\circ s^*+s\circ x^*)\ge 0$. Substitution of the above equations into (6.27) gives

$$\nu\zeta\operatorname{tr}(x+s)\le\nu\zeta^2\operatorname{tr}(e)+\nu^2\zeta^2\operatorname{tr}(e)+\nu(1-\nu)\zeta^2\operatorname{tr}(e)=2\nu\zeta^2\operatorname{tr}(e),$$

which implies the lemma. □

**Lemma 6.7** (cf. [93, Lemma II.62])**.** *If $\delta:=\delta(v)$ is defined by (5.13), then*

$$\frac{1}{\rho(\delta)}\le\lambda_{\min}(v)\le\lambda_{\max}(v)\le\rho(\delta),$$

*where*

$$\rho(\delta):=\delta+\sqrt{1+\delta^2}.$$

*Proof.* By the definition of $\delta(v)$ (cf. (5.13)), we have

$$2\delta = \left\|v - v^{-1}\right\|_F = \sqrt{\sum_{i=1}^{r} \left(\lambda_i(v) - 1/\lambda_i(v)\right)^2}.$$

Since $\lambda_i(v) > 0$, we derive

$$-2\delta\lambda_i(v) \leq 1 - \lambda_i(v)^2 \leq 2\delta\lambda_i(v),$$

which implies

$$\lambda_i(v)^2 - 2\delta\lambda_i(v) - 1 \leq 0 \leq \lambda_i(v)^2 + 2\delta\lambda_i(v) - 1.$$

Rewriting this as

$$(\lambda_i(v) - \delta)^2 - 1 - \delta^2 \leq 0 \leq (\lambda_i(v) + \delta)^2 - 1 - \delta^2,$$

we obtain

$$(\lambda_i(v) - \delta)^2 \leq 1 + \delta^2 \leq (\lambda_i(v) + \delta)^2,$$

which implies

$$\lambda_i(v) - \delta \leq |\lambda_i(v) - \delta| \leq \sqrt{1 + \delta^2} \leq \lambda_i(v) + \delta.$$

Thus, we obtain the bounds for $\lambda_i(v)$

$$-\delta + \sqrt{1 + \delta^2} \leq \lambda_i(v) \leq \delta + \sqrt{1 + \delta^2} = \rho(\delta).$$

The lower bound above can be written as

$$-\delta + \sqrt{1 + \delta^2} = \frac{1}{\delta + \sqrt{1 + \delta^2}} = \frac{1}{\rho(\delta)},$$

which proves the lemma. □

By substituting the results of Lemma 6.6 and Lemma 6.7 into (6.26), with $\delta := \delta(v)$, we derive an upper bound for $\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2$ as follows.

$$\begin{aligned}\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2 &\leq 2\left[4(1-\theta)^2\delta^2 + \theta^2 r\right] + 3\theta^2 \frac{(2\zeta r)^2 \rho(\delta)^2}{\zeta^2} \\ &= 2\left[4(1-\theta)^2\delta^2 + \theta^2 r\right] + 12\theta^2 r^2 \rho(\delta)^2, \end{aligned} \tag{6.28}$$

where $r = \mathrm{tr}(e)$ is the rank of the associated Euclidean Jordan algebra.

## 6.5 Choosing the update parameter $\theta$

We want to choose $\theta$, $0 < \theta < 1$, as large as possible, and such that $(x^f, y^f, s^f)$ lies in the quadratic convergence neighborhood with respect to the $\mu^+$-center of the perturbed problems $(\mathrm{CP}_{\nu^+})$ and $(\mathrm{CD}_{\nu^+})$, i.e., $\delta(v^f) \leq 1/\sqrt{2}$. By (6.15), we derive that this is the case when

$$\frac{\frac{1}{4}\left(\frac{\|d_x^f\|_F^2+\|d_s^f\|_F^2}{1-\theta}\right)^2}{1-\frac{1}{2}\frac{\|d_x^f\|_F^2+\|d_s^f\|_F^2}{1-\theta}} \leq 2.$$

Considering $\frac{\|d_x^f\|_F^2+\|d_s^f\|_F^2}{1-\theta}$ as a single term, and performing some elementary calculations, we obtain that

$$\frac{\|d_x^f\|_F^2 + \|d_s^f\|_F^2}{1-\theta} \leq 2\sqrt{3} - 2 \approx 1.4641. \tag{6.29}$$

By (6.28), the above inequality holds if

$$2\left[4(1-\theta)^2\delta^2 + \theta^2 r\right] + 12\theta^2 r^2 \rho(\delta)^2 \leq (2\sqrt{3}-2)(1-\theta).$$

Choosing $\tau = 1/16$, one may easily verify that if

$$\theta = \frac{1}{4r}, \tag{6.30}$$

then the above inequality is satisfied. Moreover, using (6.14) and (6.29) we have

$$\left\|\lambda(d_x^f \circ d_s^f)\right\|_\infty \leq \frac{1}{2}(\|d_x^f\|_F^2 + \|d_s^f\|_F^2) \leq (\sqrt{3}-1)(1-\theta) < 1-\theta,$$

which, by Lemma 6.2, means that $(x^f, y^f, s^f)$ are strictly feasible. Thus, we have found a desired $\theta$.

## 6.6 Iteration bound

In the previous sections we have found that if at the start of an iteration the iterate satisfies $\delta(x, s; \mu) \leq \tau$, with $\tau = 1/16$, then after the feasibility step, with $\theta$ as defined in (6.30), the iterate is strictly feasible and satisfies $\delta(x^f, s^f; \mu^+) \leq 1/\sqrt{2}$, i.e., $(x^f, y^f, s^f)$ lies in the quadratic convergence neighborhood with respect to the $\mu^+$-center of the perturbed problems $(\mathrm{CP}_{\nu^+})$ and $(\mathrm{CD}_{\nu^+})$.

After the feasibility step we perform a few centering steps in order to get iterate $(x^+, y^+, s^+)$ which satisfies ${x^+}^T s^+ = n\mu^+$ and $\delta(x^+, s^+; \mu^+) \leq \tau$. From Chapter 5 we precisely know how to analyze these steps. By Corollary 5.7, after

$k$ centering steps we will have the iterate $(x^+, y^+, s^+)$ that is still feasible for $(\mathrm{CP}_{\nu^+})$ and $(\mathrm{CD}_{\nu^+})$ and such that

$$\delta(x^+, s^+; \mu^+) \leq \left(\frac{1}{\sqrt{2}}\right)^{2^k}.$$

From this one easily deduces that $\delta(x^+, s^+; \mu^+) \leq \tau$ will hold after at most

$$\left\lceil \log_2 \left( \log_2 \frac{1}{\tau^2} \right) \right\rceil = 1 + \left\lceil \log_2 \left( \log_2 \frac{1}{\tau} \right) \right\rceil \tag{6.31}$$

centering steps.

According to (6.31), and since $\tau = 1/16$, at most three centering steps then suffice to get iterate $(x^+, y^+, s^+)$ that satisfies $\delta(x^+, s^+; \mu^+) \leq \tau$ again. So each main iteration consists of at most four so-called inner iterations, in each of which we need to compute a search direction (for either a feasibility step or a centering step).

It has become a custom to measure the complexity of an IPM by the required number of inner iterations. In each main iteration both the duality gap and the norms of the residual vectors are reduced by the factor $(1 - \theta)$. Hence, using $\mathrm{tr}(x^0 \circ s^0) = r\zeta^2$, the total number of main iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max\left\{ r\zeta^2, \left\|r_p^0\right\|_F, \left\|r_d^0\right\|_F \right\}}{\varepsilon}.$$

Due to (6.30) and the fact that we need at most four inner iterations per main iteration, the total number of inner iterations is bounded above by

$$16r \log \frac{\max\left\{ r\zeta^2, \left\|r_p^0\right\|_F, \left\|r_d^0\right\|_F \right\}}{\varepsilon}.$$

Thus, we may state without further proof the main result of this chapter.

**Theorem 6.8.** *If (CP) has an optimal solution $x^*$ and (CD) has an optimal solution $(y^*, s^*)$, which satisfy $\mathrm{tr}(x^* \circ s^*) = 0$ and $x^* + s^* \preceq_{\mathcal{K}} \zeta e$ for some $\zeta > 0$, then after at most*

$$16r \log \frac{\max\left\{ r\zeta^2, \left\|r_p^0\right\|_F, \left\|r_d^0\right\|_F \right\}}{\varepsilon}$$

*inner iterations the Algorithm 6.1 finds an $\varepsilon$-optimal solution of (CP) and (CD). Here $r = \mathrm{tr}(e)$ is the rank of the associated Euclidean Jordan algebra.*

Note that this bound is slightly better than that in [92, Theorem 4.8], where the coefficient is $16\sqrt{2}$.

**Remark 6.9.** As for LO, the iteration bound in Theorem 6.8 is derived under the assumption that there exists an optimal solution pair $(x^*, y^*, s^*)$ of (CP) and (CD) with vanishing duality gap and satisfying $x^* + s^* \preceq_{\mathcal{K}} \zeta e$. During the course of the algorithm, if at some main iteration, the proximity measure $\delta$ after the feasibility step exceeds $1/\sqrt{2}$, then it tells us that the above assumption does not hold. It may happen that the value of $\zeta$ has been chosen too small. In this case one might run the algorithm once more with a larger value of $\zeta$.

## 6.7 Adaptive updating strategy

Our full NT-step infeasible IPM algorithm (Algorithm 6.1) admits the best known iteration bound, but from a practical perspective a severe shortcoming is its worst-case-oriented nature: it will always perform according to its worst-case theoretical complexity bound. An obvious way to reduce the number of iterations is to make larger updates of the barrier parameter while keeping the iterates in the region of quadratic convergence. This is called the adaptive updating strategy, which we discuss in this section.

Observe that the iteration bound in Theorem 6.8 was obtained by requiring that after each feasibility step the proximity measure satisfies $\delta(v^f) \leq 1/\sqrt{2}$. In order to make clear how this observation can be used to improve the performance of the algorithm without losing the iteration bound, let us briefly recall the idea behind the proof of this theorem. At the start of a main iteration we are given $(x, y, s)$ satisfying the feasibility conditions (6.3) and (6.4) for $\nu = \mu/\mu^0$, and such that $\operatorname{tr}(x \circ s) = \mu \operatorname{tr}(e)$ and $\delta(x, s; \mu) \leq \tau$. We reduce $\nu$ to $\nu^+ = (1-\theta)\nu$, and accordingly $\mu$ to $\mu^+ = (1-\theta)\mu$ with $\theta \in (0, 1)$. Then we make a feasibility step to the $\mu^+$-center of the perturbed problem pair $(\mathrm{CP}_{\nu^+})$ and $(\mathrm{CD}_{\nu^+})$. The feasibility step yields $(x^f, y^f, s^f)$, which are strictly feasible for $(\mathrm{CP}_{\nu^+})$ and $(\mathrm{CD}_{\nu^+})$, and such that $\delta(x^f, s^f; \mu^+) \leq 1/\sqrt{2}$. In other words, $(x^f, y^f, s^f)$ belongs to the quadratic convergence neighborhood with respect to the $\mu^+$-center of $(\mathrm{CP}_{\nu^+})$ and $(\mathrm{CD}_{\nu^+})$. Since $\tau = 1/16$, at most three centering steps then suffice to get the iterate $(x^+, y^+, s^+)$ that is feasible for $(\mathrm{CP}_{\nu^+})$ and $(\mathrm{CD}_{\nu^+})$ and that satisfies $\operatorname{tr}(x^+ \circ s^+) = \mu^+ \operatorname{tr}(e)$ and $\delta(x^+, s^+; \mu^+) \leq \tau$.

Our estimates in the proof of Theorem 6.8 were such that it has become clear that the default value $\theta = 1/(4r)$ guarantees that $\delta(x^f, s^f; \mu^+) \leq 1/\sqrt{2}$ will hold. But actually the new proximity may be much smaller than $1/\sqrt{2}$.

This opens a way to speed up the algorithm without degrading the iteration bound. For if we take $\theta$ larger than the default value $1/(4r)$, thus, enforcing a deeper update of the barrier parameter in such a way that $\delta(x^f, s^f; \mu^+) \leq 1/\sqrt{2}$ still holds, then the analysis in the proof of the theorem remains valid but the number of iterations decreases. The question arises of how deep the update might be.

To answer the above question, we need to separate the system (6.9) into fol-

lowing two systems,

$$\begin{aligned}\sqrt{\mu}AP(w)^{1/2}d_x^{ff} &= \nu r_p^0,\\ \left(\sqrt{\mu}AP(w)^{1/2}\right)^T\frac{\Delta^{ff}y}{\mu}+d_s^{ff} &= \frac{1}{\sqrt{\mu}}\nu P(w)^{1/2}r_d^0,\\ d_x^{ff}+d_s^{ff} &= -v^{-1},\end{aligned}$$

and

$$\begin{aligned}\sqrt{\mu}AP(w)^{1/2}d_x^{fc} &= 0,\\ \left(\sqrt{\mu}AP(w)^{1/2}\right)^T\frac{\Delta^{fc}y}{\mu}+d_s^{fc} &= 0,\\ d_x^{fc}+d_s^{fc} &= v^{-1}-v.\end{aligned}$$

Obviously, each system has a unique solution, and

$$(d_x^f,\Delta^f y,d_s^f)=\theta(d_x^{ff},\Delta^{ff}y,d_s^{ff})+(d_x^{fc},\Delta^{fc}y,d_s^{fc}). \tag{6.32}$$

**Remark 6.10.** The above two systems share the same coefficient matrix. Hence, solving them both just adds slightly more computational work. In other words, the adaptive strategy will not burden every main iteration too much (assume the same number of centering iterations).

On the other hand, from the definition of the Frobenius norm, we have

$$\left\|\lambda(d_x^f\circ d_s^f)\right\|_\infty \le \left\|d_x^f\circ d_s^f\right\|_F.$$

Hence, provided that $\left\|d_x^f\circ d_s^f\right\|_F<1-\theta$, the result of Lemma 6.4 can be relaxed to

$$4\delta(v^f)^2 \le \frac{\left\|\frac{d_x^f\circ d_s^f}{1-\theta}\right\|_F^2}{1-\left\|\frac{d_x^f\circ d_s^f}{1-\theta}\right\|_F}.$$

This implies that if

$$\left\|\frac{d_x^f\circ d_s^f}{1-\theta}\right\|_F<1 \quad\text{and}\quad \frac{\left\|\frac{d_x^f\circ d_s^f}{1-\theta}\right\|_F^2}{1-\left\|\frac{d_x^f\circ d_s^f}{1-\theta}\right\|_F}\le 2,$$

then $\delta(x^f,s^f;\mu^+)\le 1/\sqrt{2}$ will hold, i.e., after the feasibility step, the iterate $(x^f,y^f,s^f)$ lies in the quadratic convergence neighborhood with respect to the $\mu^+$-center of the perturbed problems ($\mathrm{CP}_{\nu^+}$) and ($\mathrm{CD}_{\nu^+}$). Considering $\left\|\frac{d_x^f\circ d_s^f}{1-\theta}\right\|_F$ as a single term, we derive that if

$$\left\|\frac{d_x^f\circ d_s^f}{1-\theta}\right\|_F\le\sqrt{3}-1,$$

then $\delta(x^f, s^f; \mu^+) \leq 1/\sqrt{2}$. Substituting (6.32) into the above inequality, we obtain

$$\left\| \frac{(\theta d_x^{ff} + d_x^{fc}) \circ (\theta d_s^{ff} + d_s^{fc})}{1-\theta} \right\|_F \leq \sqrt{3} - 1. \tag{6.33}$$

The above inequality is equivalent to the following quartic polynomial inequality in $\theta$

$$\begin{aligned}
&\theta^4 \left\| d_x^{ff} \circ d_s^{ff} \right\|_F^2 \\
&+\theta^3 \left[ 2\langle d_x^{ff} \circ d_s^{ff}, d_x^{fc} \circ d_s^{ff} + d_x^{ff} \circ d_s^{fc} \rangle \right] \\
&+\theta^2 \left[ 2\langle d_x^{ff} \circ d_s^{ff}, d_x^{fc} \circ d_s^{fc} \rangle + \left\| d_x^{fc} \circ d_s^{ff} + d_x^{ff} \circ d_s^{fc} \right\|_F^2 - (\sqrt{3}-1)^2 \right] \\
&+\theta \left[ 2\langle d_x^{fc} \circ d_s^{ff} + d_x^{ff} \circ d_s^{fc}, d_x^{fc} \circ d_s^{fc} \rangle + 2(\sqrt{3}-1)^2 \right] \\
&+ \left[ \left\| d_x^{fc} \circ d_s^{fc} \right\|_F^2 - (\sqrt{3}-1)^2 \right] \leq 0.
\end{aligned} \tag{6.34}$$

Our goal is to choose $\theta$, $0 < \theta < 1$, as large as possible, such that (6.34) holds for all values in $[0, \theta]$. From the analysis in Section 6.4 we know beforehand that under the conditions of Theorem 6.8 the value of $\theta$ is at least $1/(4r)$, i.e., if $\theta \in [0, 1/(4r)]$, then (6.34) always holds. On the other side, if (6.33) holds for all $0 < \theta < 1$, then we may easily verify that $\theta = 1$ will yield an optimal solution to (CP) and (CD). Otherwise, we have, as a result of equivalence, that (6.34) will not hold. This, by continuity, implies that there exist at least one $\theta$ such that $\theta \in [1/(4r), 1)$ and (6.34) holds with equality. Using the properties of polynomials, we need to choose the smallest real root between zero and one as our update parameter $\theta$. Remember that under the conditions of Theorem 6.8, the resulting $\theta$ must larger than or equal to $1/(4r)$.

# Chapter 7

# Numerical Experiments

In this chapter we present numerical tests of our full NT-step IIPM for SO. All our numerical experiments are carried out on a workstation with Intel® Core™ 2 Duo CPU at 3GHz and 4GB of physical memory. The workstation runs MATLAB version 7.7.0.471 (R2008b) on Windows XP Professional operating system.

## 7.1 Some details about the implementation

We implement our full NT-step IIPM for SO with both short (with $\theta = 1/(4r)$) and adaptive updates. The input arguments closely follow these used in SeDuMi[1], namely,

$A$, $b$, $c$: SO problem data as described in Section 5.1;
$K$: a structure variable defines the structure of the conic constraints:
  $K.l$ is the number of nonnegative components,
  $K.q$ lists the dimensions of second-order cone constraints,
  $K.s$ lists the dimensions of positive semidefinite constraints;
$\varepsilon$: accuracy parameter (optional);
$\zeta$: initialization parameter $\zeta > 0$ (optional).

### 7.1.1 Initialization parameter

Our algorithms can start with an infeasible starting point. However, the performance of these algorithms is quite sensitive to the choice of the starting point. Theoretically, we need a starting point that satisfies (6.1) and (6.2). It is desirable to choose the initialization parameter $\zeta$ such that the starting point at least has the same order of magnitude as an optimal solution. If $\zeta$ is not known, the

[1]SeDuMi is available from http://sedumi.ie.lehigh.edu/.

following $\zeta$ is used: [2]

$$\zeta = \max\left\{\xi^l, \eta^l, \xi_i^q, \eta_i^q, \xi_i^s, \eta_i^s\right\},$$

where

$$\begin{aligned}
\xi^l &= \max\left\{10, \sqrt{K.l}, \sqrt{K.l}\max_{1\le k\le m}\frac{1+|b_k|}{1+\|A^l(k,:)\|}\right\};\\
\eta^l &= \max\left\{10, \sqrt{K.l}, \max\left\{\left\|c^l\right\|, \left\|A^l(1,:)\right\|, \ldots, \left\|A^l(m,:)\right\|\right\}\right\};\\
\xi_i^q &= \max\left\{10, \sqrt{K.q(i)}, \sqrt{K.q(i)}\max_{1\le k\le m}\frac{1+|b_k|}{1+\|A^{q_i}(k,:)\|}\right\};\\
\eta_i^q &= \max\left\{10, \sqrt{K.q(i)}, \max\left\{\left\|c^{q_i}\right\|, \left\|A^{q_i}(1,:)\right\|, \ldots, \left\|A^{q_i}(m,:)\right\|\right\}\right\};\\
\xi_i^s &= \max\left\{10, \sqrt{K.s(i)}, K.s(i)\max_{1\le k\le m}\frac{1+|b_k|}{1+\|A^{s_i}(k,:)\|}\right\};\\
\eta_i^s &= \max\left\{10, \sqrt{K.s(i)}, \max\left\{\left\|c^{s_i}\right\|, \left\|A^{s_i}(1,:)\right\|, \ldots, \left\|A^{s_i}(m,:)\right\|\right\}\right\}.
\end{aligned}$$

Here $A^l$ is an $m \times K.l$ matrix block of $A$ corresponding to the nonnegative block of $x$, and similarly we use notations $A^{q_i}$, $A^{s_i}$ and $c^l$, $c^{q_i}$, $c^{s_i}$.

### 7.1.2 Stopping criteria

If the accuracy parameter $\varepsilon$ is not given, we use

$$\begin{aligned}
\varepsilon &= 10^{\left\lceil \lg\left(\max\left(\mathrm{tr}(x\circ s), \|b-Ax\|_F, \left\|c-A^Ty-s\right\|_F\right)/10^{16}\right)\right\rceil}\\
&= 10^{\left\lceil \lg\left(\max\left(\mathrm{tr}(x\circ s), \|b-Ax\|_F, \left\|c-A^Ty-s\right\|_F\right)\right)\right\rceil - 16}
\end{aligned}$$

as our default accuracy parameter. So we reduce the initial duality gap and the resuduals by a factor of approximately $10^{16}$. The algorithm is stopped when any of the following cases occur.

(i) Solutions with the desired accuracy have been obtained, i.e.,

$$\max\left(\mathrm{tr}(x\circ s), \|b-Ax\|_F, \left\|c-A^Ty-s\right\|_F\right) \le \varepsilon.$$

[2] Our choice of $\zeta$ is based on the choice of the initial iterate for SDPT3 [107, Section 3]. In fact, with the starting point

$$x^0 = \zeta_x e, \quad y^0 = 0, \quad s^0 = \zeta_s e, \quad \mu^0 = \zeta_x\zeta_s,$$

where $\zeta_x$ and $\zeta_s$ are (positive) numbers such that

$$x^* \preceq_{\mathcal{K}} \zeta_x e, \quad s^* \preceq_{\mathcal{K}} \zeta_s e,$$

we may still prove that the inequality (6.28) holds and subsequently, the value of the updater parameter $\theta$ and Theorem 6.8 stay the same. Here, the initialization parameters may be chosen as

$$\zeta_x = \max\left\{\xi^l, \xi_i^q, \xi_i^s\right\}, \quad \zeta_s = \max\left\{\eta^l, \eta_i^q, \eta_i^s\right\}.$$

(ii) There is no optimal solution with vanishing duality gap that satisfies (6.1) and (6.2) namely, after the feasibility step, we have

$$\begin{cases} \delta > 1/\sqrt{2}, & \text{for short updates;} \\ \theta < 1/(4r), & \text{for adaptive updates.} \end{cases}$$

(iii) Numerical problems are encountered, such as the iterate violates the cone constraints, or after three centering steps, the proximity measure $\delta(v)$ is still larger than $\tau = 1/16$, etc.

### 7.1.3 Scaling point for semidefinite cones

The main purpose of this subsection is to show that the NT-scaling point (cf. (4.1)) for semidefinite cones can be computed using two Cholesky factorizations and one Singular Value Decomposition (SVD). The presentation here mainly follows [103].

Let the Cholesky factorizations of the positive definite matrices $X$ and $S$ be

$$X = LL^T, \quad S = RR^T,$$

and let $UDV^T = R^TL$ be the SVD of $R^TL$. Define $Q := L^{-1}X^{1/2}$, then

$$QQ^T = L^{-1}X^{1/2}X^{1/2}L^{-T} = L^{-1}XL^{-T} = L^{-1}LL^TL^{-T} = I,$$

i.e., $Q$ is an orthogonal matrix. It is easily seen that

$$X^{1/2}SX^{1/2} = Q^T(L^TR)(R^TL)Q = (Q^TV)D^2(V^TQ).$$

Since $Q^TV$ is orthogonal, we have

$$(X^{1/2}SX^{1/2})^{-1/2} = (Q^TV)D^{-1}(V^TQ).$$

Then the NT-scaling point $W$ can be computed easily:

$$W = LVD^{-1}V^TL^T = GG^T,$$

where

$$G := LVD^{-1/2}.$$

In the implementation, we use the scaling matrices $G$ and $G^{-1} = D^{1/2}V^TL^{-1}$ in the computation of the feasibility directions or centering directions. For example, $X$ and $S$ are scaled to the same diagonal matrix $D$,

$$G^TSG = G^{-1}XG^{-T} = D.$$

**Remark 7.1.** In the implementation, we do not need to keep the orthogonal matrix $U$; $V$ and $D$ are all we need. In fact, $VD^2V^T$ is the eigenvalue decomposition of $L^TRR^TL = L^TSL$, so we could use this instead of the SVD. But the condition number of this matrix is the square of that of $R^TL$, which may lead to a less stable algorithm.

## 7.2 Short updates versus adaptive updates

In this section we start with a straightforward implementation of the full NT-step IIPM for SO with standard short updates, namely, Algorithm 6.1 with $\theta = 1/(4r)$. After that we make some modifications of the algorithm as described in Section 6.7 that increase the practical efficiency of the algorithm without destroying the theoretical iteration bound.

We solve the problem truss1[3] (from SDPLIB [12]) using both the short updating algorithm and the adaptive updating one. For both algorithms, the initialization parameter $\zeta$ is computed as described in Subsection 7.1.1 (for truss1, $\zeta = 10$), and the accuracy parameter $\varepsilon$ is set to $10^{-9}$. For each iteration we list the iteration number (It.), the update parameter $\theta$, proximity measure after the feasibility step $\delta(v^f)$, proximity measure after the centering steps $\delta(v)$, parameter $\nu$, the duality gap $\mathrm{tr}(x \circ s)$, the norm of primal infeasibility $\|r_p\|_F$ ($r_p = b - Ax$), and the norm of dual infeasibility $\|r_d\|_F$ ($r_d = c - A^T y - s$).

For the algorithm with short updates, we need 1437 main iterations to reach our accuracy. Since there are too many iterations, in Table 7.1, after the first iteration we show only iterations whose number is a multiple of 100, and the last iteration. Note that the update parameter $\theta$ is constant ($\theta = 1/(4r)$) in this example.

| It. | $\theta$ | $\delta(v^f)$ | $\delta(v)$ | $\nu$ | $\mathrm{tr}(x \circ s)$ | $\|r_p\|_F$ | $\|r_d\|_F$ |
|---|---|---|---|---|---|---|---|
| 0 | – | – | 0 | 1.0000+00 | 1.3000+03 | 7.7363+01 | 3.5791+01 |
| 1 | 0.019231 | 2.150−04 | 1.736−08 | 9.8077−01 | 1.2750+03 | 7.5875+01 | 3.5103+01 |
| 100 | 0.019231 | 8.767−05 | 2.305−09 | 1.4344−01 | 1.8648+02 | 1.1097+01 | 5.1340+00 |
| 200 | 0.019231 | 9.911−05 | 2.075−09 | 2.0576−02 | 2.6749+01 | 1.5918+00 | 7.3645−01 |
| 300 | 0.019231 | 4.783−05 | 1.982−10 | 2.9515−03 | 3.8370+00 | 2.2834−01 | 1.0564−01 |
| 400 | 0.019231 | 7.716−06 | 8.635−13 | 4.2338−04 | 5.5040−01 | 3.2754−02 | 1.5153−02 |
| 500 | 0.019231 | 1.121−06 | 2.783−13 | 6.0732−05 | 7.8951−02 | 4.6984−03 | 2.1737−03 |
| 600 | 0.019231 | 1.610−07 | 1.883−12 | 8.7116−06 | 1.1325−02 | 6.7396−04 | 3.1180−04 |
| 700 | 0.019231 | 2.311−08 | 8.135−12 | 1.2496−06 | 1.6245−03 | 9.6675−05 | 4.4726−05 |
| 800 | 0.019231 | 3.304−09 | 7.412−11 | 1.7925−07 | 2.3303−04 | 1.3867−05 | 6.4156−06 |
| 900 | 0.019231 | 1.038−09 | 6.778−10 | 2.5713−08 | 3.3427−05 | 1.9892−06 | 9.2029−07 |
| 1000 | 0.019231 | 2.561−09 | 7.263−09 | 3.6884−09 | 4.7949−06 | 2.8534−07 | 1.3201−07 |
| 1100 | 0.019231 | 6.945−08 | 1.886−08 | 5.2907−10 | 6.8780−07 | 4.0931−08 | 1.8936−08 |
| 1200 | 0.019231 | 2.668−07 | 2.274−07 | 7.5893−11 | 9.8660−08 | 5.8713−09 | 2.7163−09 |
| 1300 | 0.019231 | 2.683−06 | 1.802−06 | 1.0886−11 | 1.4152−08 | 8.4220−10 | 3.8963−10 |
| 1400 | 0.019231 | 1.385−05 | 2.198−05 | 1.5616−12 | 2.0301−09 | 1.2080−10 | 5.5892−11 |
| 1437 | 0.019231 | 3.725−05 | 2.612−05 | 7.6127−13 | 9.8965−10 | 5.8892−11 | 2.7245−11 |

Table 7.1: Full NT-step IIPM for truss1 with short updates.

[3]We choose this problem only because it is the problem in SDPLIB with the smallest rank $r$, which in turn requires least number of main iterations when using the short updating algorithm.

For the algorithm with adaptive updates, we list all the iterations. From Table 7.2, we see that the adaptive updating strategy reduced the number of main iterations from 1437 to 10. Since for both algorithms the work in every main iteration is almost the same, this is really a huge reduction.

| It. | $\theta$ | $\delta(v^f)$ | $\delta(v)$ | $\nu$ | $\mathrm{tr}(x \circ s)$ | $\|r_p\|_F$ | $\|r_d\|_F$ |
|---|---|---|---|---|---|---|---|
| 0 | – | – | 0 | 1.0000+00 | 1.3000+03 | 7.7363+01 | 3.5791+01 |
| 1 | 0.542133 | 0.581227 | 0.002693 | 4.5787−01 | 5.9523+02 | 3.5422+01 | 1.6388+01 |
| 2 | 0.562885 | 0.528824 | 0.003163 | 2.0014−01 | 2.6018+02 | 1.5483+01 | 7.1632+00 |
| 3 | 0.695392 | 0.329394 | 0.029860 | 6.0964−02 | 7.9253+01 | 4.7164+00 | 2.1820+00 |
| 4 | 0.664871 | 0.322266 | 0.015685 | 2.0431−02 | 2.6560+01 | 1.5806+00 | 7.3124−01 |
| 5 | 0.670462 | 0.324459 | 0.014783 | 6.7327−03 | 8.7526+00 | 5.2086−01 | 2.4097−01 |
| 6 | 0.703068 | 0.351603 | 0.008555 | 1.9992−03 | 2.5989+00 | 1.5466−01 | 7.1552−02 |
| 7 | 0.832277 | 0.361225 | 0.001646 | 3.3531−04 | 4.3590−01 | 2.5940−02 | 1.2001−02 |
| 8 | 0.959861 | 0.360385 | 0.000190 | 1.3459−05 | 1.7497−02 | 1.0412−03 | 4.8171−04 |
| 9 | 0.998235 | 0.359214 | 0.000044 | 2.3749−08 | 3.0874−05 | 1.8373−06 | 8.5001−07 |
| 10 | 0.999976 | 0.046595 | 0.000068 | 5.7971−13 | 7.5362−10 | 4.4167−11 | 2.0753−11 |

Table 7.2: Full NT-step IIPM for truss1 with adaptive updates.

## 7.3 SDPLIB problems

In this section we describe the numerical results of our full NT-step IIPM for SO with adaptive updates, on problems from SDPLIB collection of Borchers [12]. To make our algorithm more efficient, we make some efforts to assemble the Shur complement matrix $AP(w)A^T$ (cf. (6.19)). Following [30], we exploit the sparsity of the part of the coefficient matrix corresponding to the semidefinite blocks. In addition, we use Mex subroutines compiled from programs written in C.

Numerical results are given in Table 7.3.[4] By default, we use initialization parameter $\zeta$ and accuracy parameter $\varepsilon$ estimated by the algorithm. We adjust the accuracy parameter $\varepsilon$ only when the algorithm fails to reach it. For each problem, we list the initialization parameter $\zeta$, the accuracy parameter $\varepsilon$, the number of main iterations required, the primal objective value, and the dual objective value.

Generally, our algorithm is slower than the latest version of SDPT3 (SDPT3-4.0-beta)[5]. We think the main reason is that our algorithm uses adaptive updating strategy, which means that the iterates always lie in the quadratic convergence

[4] As for the latest version of SDPT3 (SDPT3-4.0-beta), we cannot solve the largest two maxG problems (maxG55 and maxG60) from SDPLIB. Since in the algorithm we store $x$, $s$, as well as the search directions, this requires a lot of memory. Moreover, Matlab duplicates the data when calling a subroutine, and this makes the situation even worse. On the other hand, our algorithm is a primal-dual method storing the primal iterate $x$, it cannot exploit common sparsity in $c$ and the constraint matrix as well as dual methods or nonlinear-programming based methods.

[5] SDPT3 is available from http://www.math.nus.edu.sg/~mattohkc/sdpt3.html.

| Problem | $\zeta$ | $\varepsilon$ | it. | primal obj. | dual obj. |
|---|---|---|---|---|---|
| arch0 | 1.960080+004 | 1.0−004 | 70 | -5.664629−001 | -5.665339−001 |
| arch2 | 1.960077+004 | 1.0−004 | 66 | -6.714678−001 | -6.715268−001 |
| arch4 | 1.960077+004 | 1.0−004 | 70 | -9.725819−001 | -9.726377−001 |
| arch8 | 1.960090+004 | 1.0−004 | 69 | -7.056927+000 | -7.057000+000 |
| control01 | 2.517596+004 | 1.0−006 | 33 | -1.778463+001 | -1.778463+001 |
| control02 | 4.954101+004 | 1.0−005 | 35 | -8.300066+000 | -8.300003+000 |
| control03 | 6.120488+004 | 1.0−004 | 43 | -1.363336+001 | -1.363328+001 |
| control04 | 8.716397+004 | 1.0−004 | 46 | -1.979449+001 | -1.979425+001 |
| control05 | 1.052365+005 | 1.0−004 | 51 | -1.688373+001 | -1.688362+001 |
| control06 | 1.484766+005 | 1.0−003 | 62 | -3.730680+001 | -3.730456+001 |
| control07 | 1.588696+005 | 1.0−003 | 63 | -2.062617+001 | -2.062517+001 |
| control08 | 1.866582+005 | 1.0−003 | 62 | -2.028768+001 | -2.028649+001 |
| control09 | 2.074063+005 | 1.0−003 | 61 | -1.467638+001 | -1.467557+001 |
| control10 | 2.221099+005 | 1.0−003 | 87 | -3.853708+001 | -3.853316+001 |
| control11 | 2.727531+005 | 1.0−002 | 77 | -3.197998+001 | -3.195949+001 |
| equalG11 | 8.010000+002 | 1.0−004 | 48 | -6.291552+002 | -6.291553+002 |
| equalG51 | 1.009909+003 | 1.0−004 | 64 | -4.005601+003 | -4.005601+003 |
| gpp100 | 1.000000+002 | 1.0−005 | 47 | 4.494354+001 | 4.494354+001 |
| gpp124-1 | 1.240000+002 | 1.0−005 | 49 | 7.343071+000 | 7.343069+000 |
| gpp124-2 | 1.240000+002 | 1.0−005 | 50 | 4.686228+001 | 4.686229+001 |
| gpp124-3 | 1.240000+002 | 1.0−005 | 51 | 1.530141+002 | 1.530141+002 |
| gpp124-4 | 1.240000+002 | 1.0−005 | 88 | 4.189876+002 | 4.189876+002 |
| gpp250-1 | 2.500000+002 | 1.0−004 | 45 | 1.544498+001 | 1.544491+001 |
| gpp250-2 | 2.500000+002 | 1.0−004 | 45 | 8.186901+001 | 8.186894+001 |
| gpp250-3 | 2.500000+002 | 1.0−004 | 50 | 3.035393+002 | 3.035393+002 |
| gpp250-4 | 2.500000+002 | 1.0−004 | 56 | 7.473283+002 | 7.473282+002 |
| gpp500-1 | 5.000000+002 | 1.0−004 | 57 | 2.532062+001 | 2.532053+001 |
| gpp500-2 | 5.000000+002 | 1.0−004 | 54 | 1.560605+002 | 1.560604+002 |
| gpp500-3 | 5.000000+002 | 1.0−004 | 50 | 5.130177+002 | 5.130176+002 |
| gpp500-4 | 5.000000+002 | 1.0−004 | 50 | 1.567019+003 | 1.567019+003 |
| hinf01 | 1.200000+001 | 1.0−005 | 28 | -2.034139+000 | -2.033369+000 |
| hinf02 | 1.200000+001 | 1.0−004 | 29 | -1.097437+001 | -1.096869+001 |
| hinf03 | 1.389576+001 | 1.0−003 | 42 | -5.709390+001 | -5.700073+001 |
| hinf04 | 4.596982+001 | 1.0−004 | 34 | -2.747976+002 | -2.747807+002 |
| hinf05 | 6.392141+001 | 1.0−003 | 44 | -3.642457+002 | -3.632350+002 |
| hinf06 | 2.942494+001 | 1.0−003 | 80 | -4.500583+002 | -4.494233+002 |
| hinf07 | 8.155415+001 | 1.0−003 | 56 | -3.909963+002 | -3.908684+002 |
| hinf08 | 4.707387+001 | 1.0−003 | 39 | -1.169842+002 | -1.163869+002 |
| hinf09 | 7.130835+001 | 1.0−002 | 89 | -2.363351+002 | -2.355179+002 |
| hinf10 | 9.653183+001 | 1.0−002 | 91 | -1.101387+002 | -1.094250+002 |
| hinf11 | 1.127458+002 | 1.0−002 | 79 | -6.676545+001 | -6.631436+001 |

| problem | $\zeta$ | $\varepsilon$ | it. | primal obj. | dual obj. |
|---|---|---|---|---|---|
| hinf12 | 4.410098+001 | 1.0−001 | 73 | -3.358319+001 | -2.911997+001 |
| hinf13 | 2.800000+001 | 1.0−001 | 74 | -4.925325+001 | -4.609890+001 |
| hinf14 | 3.200000+001 | 1.0−003 | 50 | -1.301100+001 | -1.300101+001 |
| hinf15 | 3.600000+001 | 1.0−001 | 84 | -2.855134+001 | -2.545825+001 |
| infd1 | 2.076438e+003 | 1.0−007 | 18 | $\theta < 1/(4r)$ | |
| infd2 | 3.993056e+002 | 1.0−009 | 17 | $\theta < 1/(4r)$ | |
| infp1 | 9.394030e+001 | 1.0−010 | 7 | $\theta < 1/(4r)$ | |
| infp2 | 7.082406e+001 | 1.0−010 | 7 | $\theta < 1/(4r)$ | |
| maxG11 | 8.000000+002 | 1.0−007 | 49 | -6.291648+002 | -6.291648+002 |
| maxG32 | 2.000000+003 | 1.0−006 | 61 | -1.567640+003 | -1.567640+003 |
| maxG51 | 1.000000+003 | 1.0−007 | 49 | -4.006256+003 | -4.006256+003 |
| mcp100 | 1.000000+002 | 1.0−010 | 25 | -2.261574+002 | -2.261574+002 |
| mcp124-1 | 1.240000+002 | 1.0−009 | 28 | -1.419905+002 | -1.419905+002 |
| mcp124-2 | 1.240000+002 | 1.0−009 | 27 | -2.698802+002 | -2.698802+002 |
| mcp124-3 | 1.240000+002 | 1.0−009 | 27 | -4.677501+002 | -4.677501+002 |
| mcp124-4 | 1.240000+002 | 1.0−009 | 26 | -8.644119+002 | -8.644119+002 |
| mcp250-1 | 2.500000+002 | 1.0−008 | 34 | -3.172643+002 | -3.172643+002 |
| mcp250-2 | 2.500000+002 | 1.0−008 | 32 | -5.319301+002 | -5.319301+002 |
| mcp250-3 | 2.500000+002 | 1.0−008 | 32 | -9.811726+002 | -9.811726+002 |
| mcp250-4 | 2.500000+002 | 1.0−008 | 32 | -1.681960+003 | -1.681960+003 |
| mcp500-1 | 5.000000+002 | 1.0−007 | 41 | -5.981485+002 | -5.981485+002 |
| mcp500-2 | 5.000000+002 | 1.0−007 | 41 | -1.070057+003 | -1.070057+003 |
| mcp500-3 | 5.000000+002 | 1.0−007 | 39 | -1.847970+003 | -1.847970+003 |
| mcp500-4 | 5.000000+002 | 1.0−007 | 39 | -3.566738+003 | -3.566738+003 |
| qap05 | 5.905421+002 | 1.0−009 | 18 | 4.360000+002 | 4.360000+002 |
| qap06 | 8.839910+002 | 1.0−005 | 38 | 3.813595+002 | 3.813997+002 |
| qap07 | 9.601125+002 | 1.0−005 | 42 | 4.247678+002 | 4.247943+002 |
| qap08 | 1.539013+003 | 1.0−005 | 42 | 7.568138+002 | 7.568855+002 |
| qap09 | 2.407917+003 | 1.0−005 | 44 | 1.409864+003 | 1.409903+003 |
| qap10 | 2.411985+003 | 1.0−005 | 34 | 1.092337+003 | 1.092474+003 |
| qpG11 | 1.325483+003 | 1.0−006 | 53 | -2.448659+003 | -2.448659+003 |
| qpG51 | 1.656854+003 | 1.0−006 | 78 | -1.181800+004 | -1.181800+004 |
| ss30 | 1.332305+003 | 1.0−005 | 69 | -2.023950+001 | -2.023951+001 |
| theta1 | 5.000000+001 | 1.0−010 | 19 | -2.300000+001 | -2.300000+001 |
| theta2 | 1.000000+002 | 1.0−010 | 24 | -3.287917+001 | -3.287917+001 |
| theta3 | 1.500000+002 | 1.0−009 | 25 | -4.216698+001 | -4.216698+001 |
| theta4 | 2.000000+002 | 1.0−009 | 28 | -5.032122+001 | -5.032122+001 |
| theta5 | 2.500000+002 | 1.0−008 | 28 | -5.723231+001 | -5.723231+001 |
| theta6 | 3.000000+002 | 1.0−008 | 32 | -6.347709+001 | -6.347709+001 |
| thetaG11 | 8.010000+002 | 1.0−007 | 91 | -4.000000+002 | -4.000000+002 |
| thetaG51 | 1.001000+003 | 1.0−006 | 91 | -3.490000+002 | -3.490000+002 |

| problem | $\zeta$ | $\varepsilon$ | it. | primal obj. | dual obj. |
|---|---|---|---|---|---|
| truss1 | 1.000000+001 | 1.0−011 | 11 | 8.999996+000 | 8.999996+000 |
| truss2 | 1.199960+001 | 1.0−009 | 42 | 1.233804+002 | 1.233804+002 |
| truss3 | 1.500000+001 | 1.0−009 | 22 | 9.109996+000 | 9.109996+000 |
| truss4 | 1.000000+001 | 1.0−010 | 13 | 9.009996+000 | 9.009996+000 |
| truss5 | 2.999900+001 | 1.0−008 | 41 | 1.326357+002 | 1.326357+002 |
| truss6 | 1.000000+001 | 1.0−006 | 192 | 9.010013+002 | 9.010014+002 |
| truss7 | 1.000000+001 | 1.0−006 | 190 | 9.000014+002 | 9.000014+002 |
| truss8 | 5.699810+001 | 1.0−007 | 50 | 1.331146+002 | 1.331146+002 |

Table 7.3: Full NT-step IIPM for SO on SDPLIB problems.

neighborhood with respect to some perturbed problems. As a consequence, the improvement over optimality and feasibility in every main iteration is smaller than that for large update methods. Therefore, we need more main iterations, normally 2 to 3 times more than that of SDPT3[6]. Moreover, we use NT-direction in our algorithm, while the default direction in SDPT3 is HRVW/KSH/M direction, which is universally faster than NT-direction on problems with semidefinite blocks, especially for sparse problems with large semidefinite blocks. The reason that the NT-direction is slower is because computing the NT-scaling matrix requires a full eigenvalue decomposition. This computation can dominate the work at each inner iteration when the problem is sparse [107].

## 7.4 The counter example

In Appendix A we construct a counter example for LO showing that $\bar{\kappa}(\zeta)$ is in the order of $\sqrt{n}$, which in turn implies that the update parameter $\theta$ for our full NT-step IIPM algorithm with standard short updates is tight in the order. As a consequence, it seems that our full NT-step IIPM algorithm with adaptive updates will result in minor improvements over the algorithm with short updates.

In this section, we present typical numerical results of our full NT-step IIPM algorithm with adaptive updates on the counter example as well as a random problem for comparison. We follow exactly the same line as in Appendix A to generate our counter example problem with dimension $n = 400$ and $\nu = 1/400$. For the random problem, we generate the coefficient matrix $A$ randomly with the same dimension as in our counter example. Then we use the same optimal solutions as in our counter example to compose $b$ and $c$.

The output of our adaptive updating algorithm for the counter example is listed in Table 7.4. We see that at some iterations (iteration 3, 4) the update parameter $\theta$ is really small. The output for the random example is contained in

[6]The iteration number for SDPT3 can be found in their user's guide, but for HRVW/KSH/M direction.

| It. | $\theta$ | $\delta(v^f)$ | $\delta(v)$ | $\nu$ | $\mathrm{tr}(x \circ s)$ | $\|r_p\|_F$ | $\|r_d\|_F$ |
|---|---|---|---|---|---|---|---|
| 0 | – | – | – | 1.0000+00 | 4.0000+02 | 1.9967+01 | 1.1756+01 |
| 1 | 0.933523 | 0.682593 | 0.010870 | 6.6477−02 | 2.6591+01 | 1.3274+00 | 7.8154−01 |
| 2 | 0.234841 | 0.704583 | 0.013006 | 5.0865−02 | 2.0346+01 | 1.0156+00 | 5.9800−01 |
| 3 | 0.086448 | 0.705139 | 0.003284 | 4.6468−02 | 1.8587+01 | 9.2784−01 | 5.4630−01 |
| 4 | 0.097065 | 0.705043 | 0.022271 | 4.1958−02 | 1.6783+01 | 8.3777−01 | 4.9328−01 |
| 5 | 0.148434 | 0.704586 | 0.005878 | 3.5730−02 | 1.4292+01 | 7.1342−01 | 4.2006−01 |
| 6 | 0.252086 | 0.701992 | 0.000798 | 2.6723−02 | 1.0689+01 | 5.3358−01 | 3.1417−01 |
| 7 | 0.415739 | 0.677546 | 0.002496 | 1.5613−02 | 6.2452+00 | 3.1175−01 | 1.8356−01 |
| 8 | 0.584069 | 0.437202 | 0.007433 | 6.4940−03 | 2.5976+00 | 1.2967−01 | 7.6346−02 |
| 9 | 0.508806 | 0.359555 | 0.001494 | 3.1898−03 | 1.2759+00 | 6.3691−02 | 3.7501−02 |
| 10 | 0.418970 | 0.359502 | 0.000334 | 1.8534−03 | 7.4135−01 | 3.7006−02 | 2.1789−02 |
| 11 | 0.367215 | 0.359505 | 0.000126 | 1.1728−03 | 4.6911−01 | 2.3417−02 | 1.3788−02 |
| 12 | 0.337189 | 0.359505 | 0.000091 | 7.7734−04 | 3.1093−01 | 1.5521−02 | 9.1387−03 |
| 13 | 0.321393 | 0.359505 | 0.000121 | 5.2751−04 | 2.1100−01 | 1.0533−02 | 6.2016−03 |
| 14 | 0.316393 | 0.359505 | 0.000196 | 3.6061−04 | 1.4424−01 | 7.2002−03 | 4.2395−03 |
| 15 | 0.320866 | 0.359505 | 0.000335 | 2.4490−04 | 9.7960−02 | 4.8899−03 | 2.8792−03 |
| 16 | 0.334885 | 0.359500 | 0.000603 | 1.6289−04 | 6.5155−02 | 3.2524−03 | 1.9150−03 |
| 17 | 0.359699 | 0.359473 | 0.001157 | 1.0430−04 | 4.1719−02 | 2.0825−03 | 1.2262−03 |
| 18 | 0.397242 | 0.359294 | 0.002368 | 6.2866−05 | 2.5146−02 | 1.2552−03 | 7.3908−04 |
| 19 | 0.444636 | 0.358118 | 0.004724 | 3.4913−05 | 1.3965−02 | 6.9711−04 | 4.1046−04 |
| 20 | 0.462980 | 0.355261 | 0.004844 | 1.8749−05 | 7.4996−03 | 3.7436−04 | 2.2042−04 |
| 21 | 0.415502 | 0.358565 | 0.000273 | 1.0959−05 | 4.3835−03 | 2.1882−04 | 1.2884−04 |
| 22 | 0.371625 | 0.362127 | 0.002009 | 6.8863−06 | 2.7545−03 | 1.3750−04 | 8.0958−05 |
| 23 | 0.348489 | 0.364162 | 0.002897 | 4.4865−06 | 1.7946−03 | 8.9582−05 | 5.2745−05 |
| 24 | 0.341300 | 0.365351 | 0.003225 | 2.9552−06 | 1.1821−03 | 5.9007−05 | 3.4743−05 |
| 25 | 0.346493 | 0.365940 | 0.003276 | 1.9313−06 | 7.7251−04 | 3.8562−05 | 2.2705−05 |
| 26 | 0.362179 | 0.365947 | 0.003165 | 1.2318−06 | 4.9272−04 | 2.4595−05 | 1.4482−05 |
| 27 | 0.387680 | 0.365221 | 0.002902 | 7.5426−07 | 3.0170−04 | 1.5060−05 | 8.8674−06 |
| 28 | 0.423888 | 0.363604 | 0.002481 | 4.3454−07 | 1.7381−04 | 8.6764−06 | 5.1086−06 |
| 29 | 0.473897 | 0.361146 | 0.002011 | 2.2861−07 | 9.1444−05 | 4.5647−06 | 2.6877−06 |
| 30 | 0.542105 | 0.357468 | 0.001574 | 1.0468−07 | 4.1872−05 | 2.0901−06 | 1.2307−06 |
| 31 | 0.632748 | 0.350431 | 0.000973 | 3.8444−08 | 1.5378−05 | 7.6761−07 | 4.5196−07 |
| 32 | 0.753218 | 0.341228 | 0.001070 | 9.4872−09 | 3.7949−06 | 1.8943−07 | 1.1154−07 |
| 33 | 0.839775 | 0.310414 | 0.001883 | 1.5201−09 | 6.0804−07 | 3.0352−08 | 1.7871−08 |
| 34 | 0.789479 | 0.279784 | 0.000246 | 3.2001−10 | 1.2800−07 | 6.3897−09 | 3.7622−09 |
| 35 | 0.871525 | 0.279112 | 0.000034 | 4.1113−11 | 1.6445−08 | 8.2091−10 | 4.8335−10 |
| 36 | 0.973113 | 0.279029 | 0.000001 | 1.1054−12 | 4.4216−10 | 2.2071−11 | 1.2996−11 |

Table 7.4: Full NT-step IIPM with adaptive updates on the counter example.

Table 7.5. We notice that the total number of iterations is approximately half of that for the counter example.

| It. | $\theta$ | $\delta(v^f)$ | $\delta(v)$ | $\nu$ | $\mathrm{tr}(x \circ s)$ | $\|r_p\|_F$ | $\|r_d\|_F$ |
|---|---|---|---|---|---|---|---|
| 0 | – | – | – | 1.0000+00 | 4.0000+02 | 3.9863+03 | 1.9435+03 |
| 1 | 0.976793 | 0.360216 | 0.005074 | 2.3207−02 | 9.2827+00 | 9.2508+01 | 4.5103+01 |
| 2 | 0.744812 | 0.354630 | 0.000051 | 5.9221−03 | 2.3688+00 | 2.3607+01 | 1.1510+01 |
| 3 | 0.546422 | 0.354996 | 0.000090 | 2.6861−03 | 1.0744+00 | 1.0708+01 | 5.2205+00 |
| 4 | 0.455168 | 0.355445 | 0.000161 | 1.4635−03 | 5.8539−01 | 5.8339+00 | 2.8443+00 |
| 5 | 0.408278 | 0.355931 | 0.000247 | 8.6597−04 | 3.4639−01 | 3.4520+00 | 1.6830+00 |
| 6 | 0.385256 | 0.356390 | 0.000334 | 5.3235−04 | 2.1294−01 | 2.1221+00 | 1.0346+00 |
| 7 | 0.377418 | 0.356750 | 0.000402 | 3.3143−04 | 1.3257−01 | 1.3212+00 | 6.4415−01 |
| 8 | 0.380504 | 0.356919 | 0.000436 | 2.0532−04 | 8.2129−02 | 8.1847−01 | 3.9905−01 |
| 9 | 0.392228 | 0.356795 | 0.000434 | 1.2479−04 | 4.9915−02 | 4.9744−01 | 2.4253−01 |
| 10 | 0.411393 | 0.356242 | 0.000402 | 7.3451−05 | 2.9381−02 | 2.9280−01 | 1.4275−01 |
| 11 | 0.437797 | 0.355070 | 0.000347 | 4.1295−05 | 1.6518−02 | 1.6461−01 | 8.0257−02 |
| 12 | 0.472855 | 0.353104 | 0.000262 | 2.1768−05 | 8.7073−03 | 8.6774−02 | 4.2307−02 |
| 13 | 0.520383 | 0.350326 | 0.000144 | 1.0440−05 | 4.1762−03 | 4.1618−02 | 2.0291−02 |
| 14 | 0.585182 | 0.346064 | 0.000024 | 4.3309−06 | 1.7323−03 | 1.7264−02 | 8.4172−03 |
| 15 | 0.671503 | 0.337622 | 0.000038 | 1.4227−06 | 5.6907−04 | 5.6712−03 | 2.7650−03 |
| 16 | 0.788239 | 0.328767 | 0.000111 | 3.0127−07 | 1.2051−04 | 1.2009−03 | 5.8552−04 |
| 17 | 0.825182 | 0.289628 | 0.000113 | 5.2667−08 | 2.1067−05 | 2.0995−04 | 1.0236−04 |
| 18 | 0.797427 | 0.278804 | 0.000085 | 1.0669−08 | 4.2676−06 | 4.2529−05 | 2.0735−05 |
| 19 | 0.896075 | 0.278963 | 0.000050 | 1.1088−09 | 4.4351−07 | 4.4199−06 | 2.1549−06 |
| 20 | 0.983405 | 0.290358 | 0.000281 | 1.8400−11 | 7.3601−09 | 7.3313−08 | 3.5761−08 |
| 21 | 0.990449 | 0.366145 | 0.000009 | 1.7575−13 | 7.0300−11 | 7.5669−10 | 3.4167−10 |

Table 7.5: Full NT-step IIPM with adaptive updates on a random example.

In summary, the counter example does affect the performance of the full NT-step IIPM algorithm with adaptive updates, but not so severally to prevent the improvements (the update parameter $\theta$ for our short updating algorithm is $1/(4r) = 1/1600$, from Table 7.4 we see that $\theta$ is much larger than this value). We think the reasons are twofold. First, the $\bar{\kappa}(\zeta)$ is in the order of $\sqrt{n}$, but the coefficient is small; second, the update parameter $\theta$ in fact relies on $\kappa(\zeta, \nu)$ ($\bar{\kappa}(\zeta) = \max_{0<\nu\leq 1} \kappa(\zeta, \nu)$), which is not always in the order of $\sqrt{n}$ for all $0 < \nu \leq 1$.

# Chapter 8

# Conclusions

## 8.1 Concluding remarks

In this thesis we analyze full-step infeasible IPMs for LO and SO. Since the analysis requires a quadratic convergence result for the feasible IPM, primal-dual feasible IPMs with full steps are presented as well.

The work is motivated by [92], where Roos proposed the first full-Newton step IIPM for LO. The full-step IIPMs can also be viewed as homotopy methods, which turn out to have many nice properties. First, as the name suggests, they use full steps, so there is no need to calculate the step length. Second, the iterates always lie in the quadratic convergence neighborhood with respect to some perturbed problems, which makes the algorithm more stable. Third, during the solution process, both "feasibility" and "optimality" are improved at the same rate, which is also credited by Potra [85]. Finally, the iteration bound coincides with the currently best-known iteration bound for IIPMs.

As in Roos's original IIPM, each iteration of the algorithm consists of a step that restores the feasibility for an intermediate problem (the so-called feasibility step) and a few (usually one to four) centering steps. The thesis starts with some improvements over the original full-Newton step IIPM for LO. We use a more natural feasibility step, which targets at the $\mu^+$-center of the new perturbed problems. For the centering steps, we apply a sharper quadratic convergence result, which results in a slightly wider neighborhood for the feasibility steps (This is better when using adaptive updates, cf. Section 6.7). Also, the analysis is simplified, which benefits the later generalization to SO.

Based on extensive computational evidence (hundreds of thousands of randomly generated problems) Roos made a conjecture in [92], whose validity would reduce the iteration bound of full-Newton step IIPM for LO by $\sqrt{n}$. In the appendix, we falsify the conjecture by a counter example, which also indicates that one of our main inequalities for the analysis of the IIPM for LO is tight in the

order.

Using the properties of Jordan algebras, we generalize the improved version of full-Newton step IIPM for LO to full NT-step IIPM for SO, which includes LO, SOCO, and SDO as special cases. The order of the resulting iteration bound coincides with the bound derived for LO, which is the currently best known iteration bound for SO.

Although our devised IIPMs admit the best known iteration bound, from a practical perspective they will always perform according to their worst-case theoretical complexity bounds. As a remedy, we propose a more aggressive adaptive updating strategy.

Finally, we implement our full NT-step IIPM for SO with both standard short and adaptive updates. The significant improvement in performance of the adaptive updating strategy over the original short updating strategy is illustrated. The algorithm with adaptive updates is used to solve problems from the well known library SDPLIB [12] of test problems. The results are promising, and to some extend competing with SDPT3 [107].

## 8.2 Further research

We mention some possible directions for further research that are related to the topics discussed in this thesis.

- In our improved IIPM for LO, a stronger quadratic convergence result is used (cf. Theorem 2.5). This results in a slightly wider quadratic convergence neighborhood, which is better when using adaptive updates. Can we generalize this result to SO?

- In the thesis, the full-step IIPM for SO is based on the NT-direction. A natural generalization is to use directions in the commutative class, or more generally the directions in the MZ-family (maybe with different proximity measures). In fact, the HRVW/KSH/M direction generally outperforms the NT-direction for problems involving semidefinite cones [107]. Also, SDPA is based on the HRVW/KSH/M search direction [113].

- For the adaptive updating strategy we want to choose the update parameter $\theta$, $0 < \theta < 1$, as large as possible, such that after the feasibility step, the iterate lies in the quadratic convergence neighborhood (with respect to the new perturbed problem pair), or, equivalently, $\delta(v^f) \leq 1/\sqrt{2}$. Due to some difficulties in the analysis of SO, we can only use the loose bound derived in Lemma 6.4 to calculate $\theta$, which means that there is a gap between the $\theta$ used in our adaptive strategy and the optimal $\theta$ we may choose. A typical graph of $\delta(v^f)$, as a function of $\theta$, is as depicted in Figure 8.1.

- For LO, the feasible IPMs converge to the analytic center of the optimal set [93], while for SDO it becomes more complicated [40, 41]. For our IIPMs,

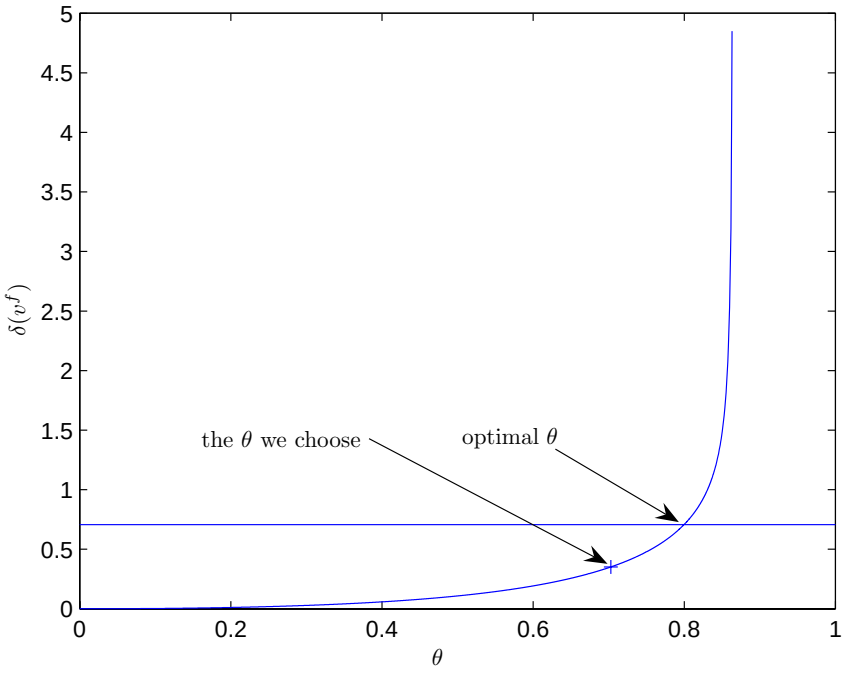


Figure 8.1: Typical behavior of $\delta(v^f)$ as a function of $\theta$.

we only know that in general the homotopy path does not converge to the analytic center of the optimal set. Thus, the convergence properties of our full-step IIPMs need further investigation.

- The SO problem is much more subtle than the LO problem. For example, it is possible that both the primal and the dual problem are feasible, but their optimal values are not equal. Also, either problem may be infeasible without there being a certificate (so-called weak infeasibility). So detecting infeasibility robustly is always a challenge for the SO solvers. For our IIPM, when it terminates without an optimal solution, we can only say that there is no optimal solution with vanishing duality gap that satisfies (6.1) and (6.2). Hence, more work is needed to detect infeasibility, indicate the infeasibility pattern, and give a certificate of infeasibility if it exists.

- In this thesis, we generalize the full-Newton step IIPM for LO to full NT-step IIPM for SO. Can we generalize the full-step IIPM to more general classes of cones. For example, the hyperbolicity cones, which are more general than symmetric cones, but have some structure that may help in developing efficient IPMs [39, 89].

# Appendix A

# Counter Example to a Conjecture

In *A full-Newton step $O(n)$ infeasible interior-point algorithm for linear optimization* [92], Roos proved that the devised full-Newton step infeasible IPM has $O(n\log(n/\varepsilon))$ worst-case iteration bound. This bound depends linearly on a parameter $\bar{\kappa}(\zeta)$, which is proved to be less than $\sqrt{2n}$. Based on extensive computational evidence (hundreds of thousands of randomly generated problems) Roos conjectured that $\bar{\kappa}(\zeta) = 1$ (Conjecture 5.1 in the above mentioned paper), which would yield an $O(\sqrt{n}\log(n/\varepsilon))$ iteration bound full-Newton step infeasible IPM. In this chapter we present an example, which is based on [36], showing that $\bar{\kappa}(\zeta)$ is in the order of $\sqrt{n}$, the same order as has been proved in Roos's original paper. In other words, the conjecture is false.

## A.1 The conjecture

We define

$$\kappa(\zeta,\nu) := \frac{\sqrt{\|x(\nu)\|^2 + \|s(\nu)\|^2}}{\zeta\sqrt{2n}}, \quad 0 < \nu \leq 1,$$

where $(x(\nu), y(\nu), s(\nu))$ is the $\mu$-center of the perturbed problems $(\mathrm{P}_\nu)$ and $(\mathrm{D}_\nu)$ with $\mu = \nu\mu^0 = \nu\zeta^2$ as defined in Subsection 3.2. In addition, we denote

$$\bar{\kappa}(\zeta) = \max_{0<\nu\leq 1} \kappa(\zeta,\nu).$$

In [92], the total number of inner iterations (cf. [92, Section 4.7]) is bounded above by

$$16\bar{\kappa}(\zeta)\sqrt{n}\log\frac{\max\left\{n\zeta^2, \|b - A\zeta e\|, \|c - \zeta e\|\right\}}{\varepsilon}.$$

There, Roos proved that

$$\bar{\kappa}(\zeta) \leq \sqrt{2n}, \tag{A.1}$$

which implies that the inner iteration bound is $O(n \log(n/\varepsilon))$ (the best iteration bound for infeasible interior-point methods). Based on extensive computational evidence (hundreds of thousands of randomly generated problems) Roos made the following conjecture.

**Conjecture A.1** (Conjecture 5.1 in [92]). *If (P) and (D) are feasible and* $\zeta \geq \|x^* + s^*\|_\infty$ *for some pair of optimal solutions* $x^*$ *and* $(y^*, s^*)$*, then* $\bar{\kappa}(\zeta) = 1$.

Clearly, the correctness of the above conjecture will reduce the iteration bound by $\sqrt{n}$.

## A.2 Counter example

Due to the choice of the optimal solution $(x^*, y^*, s^*)$, we have

$$\begin{aligned} Ax^* &= b, \quad 0 \leq x^* \leq \zeta e, \\ A^T y^* + s^* &= c, \quad 0 \leq s^* \leq \zeta e, \\ x^* s^* &= 0. \end{aligned} \tag{A.2}$$

To simplify notation in the rest of this section, we denote $x := x(\mu, \nu)$, $y := y(\mu, \nu)$ and $s := s(\mu, \nu)$. Then $x$, $y$ and $s$ are uniquely determined by the system in Subsection 3.2. Using (A.2) we get the following equivalent system

$$\begin{aligned} Ax^* - Ax &= \nu(Ax^* - A\zeta e), && x > 0, \\ A^T y^* + s^* - A^T y - s &= \nu(A^T y^* + s^* - \zeta e), && s > 0, \\ xs &= \nu\zeta^2 e. \end{aligned}$$

We rewrite this system as

$$\begin{aligned} A(x^* - x - \nu x^* + \nu\zeta e) &= 0, && x > 0, \\ A^T(y^* - y - \nu y^*) &= s - s^* + \nu s^* - \nu\zeta e, && s > 0, \\ xs &= \nu\zeta^2 e. \end{aligned} \tag{A.3}$$

Hence the maximal value that $\bar{\kappa}(\zeta)$ can attain is obtained by solving the problem

$$\max_{0<\nu\leq 1} \left\{ \frac{\sqrt{\|x\|^2 + \|s\|^2}}{\zeta\sqrt{2n}} : (A.2) \text{ and } (A.3) \right\}. \tag{A.4}$$

In this problem we maximize over all possible values of $A$, $b$, $c$, $\zeta$, $\nu$, $x^*$, $y^*$, $s^*$, $x$, $y$, and $s$ satisfying (A.2) and (A.3). Note that if (A.2) and (A.3) are satisfied,

then after replacing $x^*$, $y^*$, $s^*$, $x$, $y$, $s$, $b$, and $c$ by $x^*/\zeta$, $y^*/\zeta$, $s^*/\zeta$, $x/\zeta$, $y/\zeta$, $s/\zeta$, $b/\zeta$, and $c/\zeta$, respectively, we get a solution of (A.2) and (A.3) with $\zeta = 1$, and in that case the value of the objective function in (A.4) does not change. Hence, without loss of generality we may assume below that $\zeta = 1$.

Our aim is to construct a feasible solution for (A.2) and (A.3) such that the objective value of (A.4) is of the same order as $\sqrt{n}$, thus showing that the order of the theoretical bound for $\bar{\kappa}(\zeta)$ in (A.1) is sharp. This will be done by first constructing suitable vectors $x^*$, $y^*$, $s^*$, $x$, $y$, $s$ such that, for some fixed value of $\nu \in (0, 1)$,

$$0 \leq x^* \leq e, \quad 0 \leq s^* \leq e, \quad x^* s^* = 0, \quad x > 0, \quad s > 0, \quad xs = \nu e, \tag{A.5}$$

and such that the objective value in (A.4) is of the same order as $\sqrt{n}$. After this we will construct $A$, $b$ and $c$ such that (A.2) and (A.3) are satisfied (for $\zeta = 1$). It follows that the constructed $(x, y, s)$ is just the $\mu$-center of the perturbed problem pair $(\mathrm{P}_\nu)$ and $(\mathrm{D}_\nu)$ with $\mu = \nu\mu^0 = \nu\zeta^2 = \nu$. This will suffice to falsify Conjecture A.1.

Using that the row space of a matrix and its null space are orthogonal, we relax for the moment the first two equations in the system (A.3) to

$$(x^* - x - \nu x^* + \nu e)^T (s - s^* + \nu s^* - \nu e) = 0, \quad x > 0, \quad s > 0. \tag{A.6}$$

Since $x^*$ and $s^*$ are orthogonal, we may rewrite the above equation as follows.

$$x^T \left[ \frac{1-\nu}{\nu} s^* + e \right] + s^T \left[ \frac{1-\nu}{\nu} x^* + e \right] = (1-\nu) e^T (x^* + s^*) + n(1+\nu). \tag{A.7}$$

At this stage we choose a fixed value of $\nu \in (0, 1)$ and $x^*$ and $s^*$ such that their positive entries are small enough to have

$$\frac{1-\nu}{\nu} s^* + e \approx e, \quad \frac{1-\nu}{\nu} x^* + e \approx e, \quad (1-\nu) e^T (x^* + s^*) + n(1+\nu) \approx n(1+\nu). \tag{A.8}$$

Then it follows from (A.7) that

$$x^T e + s^T e \approx n(1+\nu).$$

Yet we choose

$$x_i = s_i = \sqrt{\nu}, \quad \text{for } i > 1, \tag{A.9}$$

leaving $x_1$ and $s_1$ free for the moment. This gives

$$x_1 + s_1 + 2(n-1)\sqrt{\nu} \approx n(1+\nu),$$

or, equivalently,

$$x_1 + s_1 \approx (n-1)\left(1 - \sqrt{\nu}\right)^2 + (1+\nu). \tag{A.10}$$

Our aim is to make $x$ and $s$ the $\mu$-centers of the perturbed problems corresponding to $\mu = \nu\mu^0 = \nu\zeta^2$, and then to compute $\kappa(\zeta, \nu)$. This holds if $xs = \mu e$. Since $\zeta = 1$, and because of (A.9), this holds if $x_1 s_1 = \nu$. We may easily check that there exists $x_1$ and $s_1$ which satisfy (A.10) and $x_1 s_1 = \nu$. Hence

$$x_1^2 + s_1^2 = (x_1 + s_1)^2 - 2x_1 s_1 \approx \left[(n-1)\left(1 - \sqrt{\nu}\right)^2 + (1+\nu)\right]^2 - 2\nu,$$

Thus we obtain

$$\|x\|^2 + \|s\|^2 \approx \left[(n-1)\left(1 - \sqrt{\nu}\right)^2 + (1+\nu)\right]^2 - 2\nu + 2(n-1)\nu.$$

Since $\zeta = 1$ this implies

$$\kappa(1, \nu) = \frac{\sqrt{\|x\|^2 + \|s\|^2}}{\sqrt{2n}} \approx \frac{\sqrt{\left[(n-1)\left(1 - \sqrt{\nu}\right)^2 + (1+\nu)\right]^2 + 2(n-2)\nu}}{\sqrt{2n}}. \tag{A.11}$$

Note that for fixed $\nu$ $(0 < \nu < 1)$ the last expression is of the same order as $\sqrt{n}$. E.g., for $\nu = 1/4$ it equals $\sqrt{(n+16)/32}$. Note that if $\nu$ is too small, then (A.8) will not be a good approximation.

Until now the vectors $x^*$, $y^*$, $s^*$, $x$, $y$, $s$ only satisfy (A.5) and (A.6). It remains to show that there exist $A$, $b$ and $c$ such that (A.2) and (A.3) are satisfied. This is easy. We take for $A$ any matrix whose row space is equal to the orthogonal complement of the linear space generated by the vector $x^* - x - \nu x^* + \nu e$. Then the vector $s^* - s - \nu s^* + \nu e$ belongs to the row space of $A$, and hence there exists a vector $y$ such that $A^T y = s^* - s - \nu s^* + \nu e$. Taking $y^* = 0$ it follows that (A.3) holds. Finally, taking $b = Ax^*$ and $c = A^T y^* + s^*$, also (A.2) holds. Thus we have shown the existence of a feasible solution of (A.4) for which the $\kappa(\zeta, \nu)$ has the order of $\sqrt{n}$, and hence $\bar{\kappa}(\zeta)$ will be at least of this order.

Just to add some numerical evidence to the above analysis we applied the above described construction for several values of $n$ and $\nu$. We took for $x^*$ and $s^*$ randomly generated nonnegative and orthogonal vectors, whose positive entries are uniformly distributed in $(0, 1/1000)$ [1]. For the computation of $x_1$ and $s_1$ we used (A.7), instead of its approximation (A.10). As a consequence $x$ and $s$ are the $\mu$-centers of the perturbed problems $(\mathrm{P}_\nu)$ and $(\mathrm{D}_\nu)$ with $\mu = \nu\zeta^2 = \nu$, and $\kappa(1, \nu)$ is well-approximated by (A.11). We choose $\nu = 1/2, 1/4, 1/16, 1/256$; the resulting values of $\kappa(\zeta, \nu)$ and $\bar{\kappa}(\zeta)$ with $\zeta = 1$ are listed in Table A.1.

For different values of $\nu$, we plot $\bar{\kappa}(1)$ according to $\sqrt{n}$ in Figure A.1. From the figure, we notice that $\bar{\kappa}(1)$ increases almost linearly with respect to $\sqrt{n}$.

[1] Here $x^* + s^* \leq 1/1000$, and we choose $\zeta = 1$ (much larger than $1/1000$). One natural question is what about the situation when $\zeta$ is tight for $x^* + s^*$. The answer is that we may slightly change our example, it still works. Indeed, upon our original example, we change $x_1^*$ to 0 and $s_1^*$ to 1, hence $\zeta = 1$ is tight. When calculate $x_1$ and $s_1$ from (A.7) and $x_1 s_1 = \nu$, we set $x_1$ the larger one and $s_1$ the smaller one accordingly. We may see from (A.7) that this will not affect our example too much, which is also indicated by the numerical test.

| | | $\nu = 1/2$ | $\nu = 1/4$ | $\nu = 1/16$ | $\nu = 1/256$ |
|---|---|---|---|---|---|
| $n = 2^2$ | $\kappa(1,\nu)$ | 0.7975 | 0.7901 | 0.9860 | 1.2829 |
| | $\bar{\kappa}(1)$ | 1.0000 | 1.0000 | 1.1344 | 1.3311 |
| $n = 3^2$ | $\kappa(1,\nu)$ | 0.8089 | 0.8834 | 1.3268 | 1.8778 |
| | $\bar{\kappa}(1)$ | 1.0299 | 1.2625 | 1.6239 | 1.9334 |
| $n = 4^2$ | $\kappa(1,\nu)$ | 0.8246 | 0.9991 | 1.6927 | 2.4944 |
| | $\bar{\kappa}(1)$ | 1.1877 | 1.5560 | 2.1203 | 2.5851 |
| $n = 5^2$ | $\kappa(1,\nu)$ | 0.8444 | 1.1307 | 2.0685 | 3.1003 |
| | $\bar{\kappa}(1)$ | 1.3430 | 1.8671 | 2.6135 | 3.2163 |
| $n = 6^2$ | $\kappa(1,\nu)$ | 0.8679 | 1.2734 | 2.4515 | 3.7157 |
| | $\bar{\kappa}(1)$ | 1.5057 | 2.1871 | 3.1220 | 3.8597 |
| $n = 7^2$ | $\kappa(1,\nu)$ | 0.8950 | 1.4236 | 2.8400 | 4.3271 |
| | $\bar{\kappa}(1)$ | 1.6770 | 2.5173 | 3.6338 | 4.4846 |
| $n = 8^2$ | $\kappa(1,\nu)$ | 0.9253 | 1.5791 | 3.2273 | 4.9449 |
| | $\bar{\kappa}(1)$ | 1.8506 | 2.8389 | 4.1382 | 5.1343 |
| $n = 9^2$ | $\kappa(1,\nu)$ | 0.9584 | 1.7389 | 3.6183 | 5.5528 |
| | $\bar{\kappa}(1)$ | 2.0335 | 3.1725 | 4.6459 | 5.7512 |

Table A.1: Typical values of $\kappa(1,\nu)$ and $\bar{\kappa}(1)$ for some values of $n$ and $\nu$.

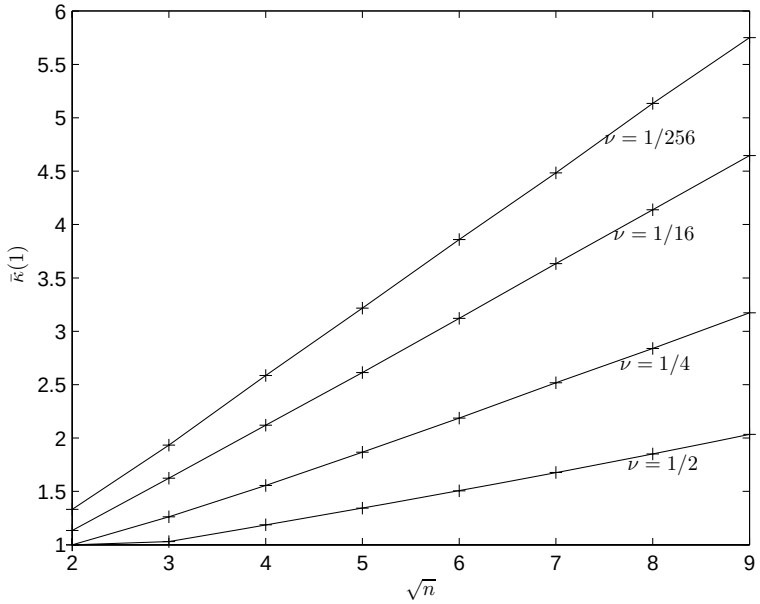


Figure A.1: Illustration of $\bar{\kappa}(1)$ vs $\sqrt{n}$, for $\nu = 1/2, 1/4, 1/16, 1/256$.

It is clear from the Table A.1 and Figure A.1 that Conjecture A.1 ([92, Conjecture 5.1]) is false.

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# Notation

## Sets

| | |
|---|---|
| $\mathbf{R}$ | the field of real numbers. |
| $\mathbf{R}_+$ | Nonnegative real number. |
| $\mathbf{R}_{++}$ | positive real numbers. |
| $\mathbf{R}^n$ | Real $n$-vectors ($n \times 1$ matrices). |
| $\mathbf{R}^{m\times n}$ | Real $m \times n$ matrices. |
| $\mathbf{L}_+^{n+1}$ | Second-order cones. |
| $\mathbf{S}^n$ | Symmetric $n \times n$ matrices. |
| $\mathbf{S}_+^n$ | Symmetric positive semidefinite $n \times n$ matrices. |
| $\mathbf{S}_{++}^n$ | Symmetric positive definite $n \times n$ matrices. |

## Vectors and matrices

| | |
|---|---|
| $0$ | Zero vector or matrix of appropriate size. |
| $I$ | Identity matrix of appropriate size. |
| $.^T$ | Transpose of a vector or matrix. |

## Jordan algebra

| | |
|---|---|
| $\circ$ | Jordan product. |
| $e$ | the identity element. |
| $\lambda(\cdot)$ | the eigenvalues. |
| $\lambda_{\min}(\cdot)$ | the smallest eigenvalue. |
| $\lambda_{\max}(\cdot)$ | the largest eigenvalue. |
| $\mathrm{tr}(\cdot)$ | trace. |
| $\det(\cdot)$ | determinant. |
| $L(\cdot)$ | Linear operator. |
| $P(\cdot)$ | Quadratic operator. |
| $\langle \cdot, \cdot \rangle$ | Inner product. |
| $\|\cdot\|$ | Euclidean norm. |
| $\|\cdot\|_F$ | Frobenius norm. |
| $\otimes$ | Kronecker product. |
| $\oplus$ | Direct sum. |

## Generalized inequalities

| | |
|---|---|
| $\preceq_{\mathcal{K}}$ | Generalized inequality induced by the proper cone $\mathcal{K}$. |
| $\prec_{\mathcal{K}}$ | Strict generalized inequality induced by the proper cone $\mathcal{K}$. |

## Abbreviations

| | |
|---|---|
| IPC | Interior-Point Condition. |
| IPM(s) | Interior-Point Method(s). |
| IIPM(s) | Infeasible Interior-Point Method(s). |
| LO | Linear Optimization. |
| SOCO | Second-Order Cone Optimization. |
| SDO | Semidefinite Optimization. |
| SO | Symmetric Optimization. |
| CO | Conic Optimization. |
| NT | Nesterov-Todd. |
| SVD | Singular Value Decomposition. |

# Index

# Summary

## Full-Step Interior-Point Methods for Symmetric Optimization

In [92] Roos proposed a full-Newton step Infeasible Interior-Point Method (IIPM) for Linear Optimization (LO). It is a primal-dual homotopy method; it differs from the classical IIPMs in that it uses only full steps. This means that no line searches are needed.

In this thesis, we first present an improved full-Newton step IIPM for LO. Then, based on the properties of Euclidean Jordan algebras, we generalize the improved full-Newton step IIPM for LO to full Nesterov-Todd step (NT-step) IIPM for Symmetric Optimization (SO). Since the analysis requires a quadratic convergence result for the feasible case, primal-dual feasible IPMs with full steps are presented as well.

Although our devised IIPMs admit the best known iteration bound, from a practical perspective they are not efficient. This is because they always perform according to their worst-case theoretical complexity bounds, which means that only tiny reductions of the so-called barrier parameter are admitted. As a remedy, we propose a more aggressive (adaptive) updating strategy.

Finally, our full NT-step IIPM for SO is implemented with both standard and adaptive updates of the barrier parameter. The significant improvement in performance of the adaptive updating strategy over the original short updating strategy is illustrated. The algorithm with adaptive updates is also used to solve problems from the well known library SDPLIB [12] of test problems. The results are promising, and to some extend competing with SDPT3 [107].

# Samenvatting

## Volle-stap Inwendige Punt Methoden voor Symmetrische Optimalisering

In [92] presenteerde en analiseerde Roos een Ontoelaatbare Inwendige Punt Methode (OIPM) voor Lineaire Optimalisering (LO). Het is een primaal-duale homotopie methode die zich onderscheidt van de klassieke methoden doordat alleen volle Newton stappen worden gebruikt. Er zijn dus geen lijnzoekmethoden nodig.

We beginnen dit proefschrift met een verbeterde volle-Newton-stap OIPM voor LO. Vervolgens, gebruikmakend van de eigenschappen van Euclidische Jordan algebras, generaliseren we de aldus verbeterde OIPM voor LO naar een volle Nesterov-Todd-stap OIPM voor Symmetrische Optimalisering (SO). Omdat de analyse van deze methode een kwadratisch convergentie resultaat gebruikt voor toelaatbare IPMn, kijken we eerst naar primaal-duale toelaatbare IPMn met volle stappen.

Ofschoon de door ons bedachte OIPMn de best bekende iteratiegrens hebben, zijn zij vanuit praktisch oogpunt inefficiënt, doordat hun praktische performance niet veel beter is dan de theoretische slechtse-geval performance. Dit komt doordat de theorie slechts kleine reducties van de zogenaamde barrière parameter toelaat. Als remedie stellen wij een agressievere (adaptieve) aanpassingstrategie voor.

Tenslotte is onze volle NT-stap OIPM voor SO geïmplementeerd met zowel standaard als adaptieve aanpassingen van de barrière parameter. De hierdoor te bereiken verbetering in performance van de adaptive strategie ten opzichte van de standaard strategie wordt aan de hand van een voorbeeld duidelijk gemaakt. Het algortime met de adaptieve strategie is ook gebruikt om problemen op te lossen uit de welbekende bibliotheek SDPLIB [12] van testproblemen. De gepresenteerde resultaten zijn veelbelovend, en tot op zekere hoogte vergelijkbaar met die van SDPT3 [107].

# Curriculum Vitae

Guoyong Gu was born on December 2, 1980, in Jiangsu, China.

Guoyong Gu finished high school in 1999 and then enrolled in the Department of Mathematics, Nanjing University. With the major of Information and Computational Science, he received his Bachelor's Degree in 2003. After that, Guoyong Gu was accepted into the graduate school with full scholarship by Nanjing University. His study was under the guidance of Prof. Bingsheng He with the specification of Computational Mathematics.

Guoyong Gu started his PhD in September 2005, at the Optimization Group, Department of Software Technology, Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology. His research was financially supported by the Netherlands Organization for Scientific Research (NWO grant 613.000.441) and under the daily supervision of Prof. Kees Roos. During his stay in Delft, he completed the course program for PhD of the Dutch Network on the Mathematics of Operations Research (LNMB) and finally got an LNMB Diploma. It might also be worth mentioning that he programmed a puzzle game, which is available via http://www.st.ewi.tudelft.nl:8080/sudoku/.