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Quantum groups and Askey-Wilson polynomials

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"Quantum groups and Askey-Wilson polynomials"

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Abstract

In this thesis, we introduce the quantum groups $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ as Hopf algebras. We study their representations, including their similarities and differences with the classical theory. Following [10], we show that the eigenvectors of Koorwinder's twisted primitive elements of $\mathcal{U}_q(\mathfrak{su}(2))$ are dual q-Krawtchouk polynomials. We use this explicit expression to define generalised matrix elements and spherical functions in $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$. Then we use the Haar functional to show that these generalised matrix elements are Askey-Wilson polynomials with two continuous and two discrete parameters.

Next, we show a new result. Namely, two twisted primitive elements of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ generate Zhedanov's Askey-Wilson algebra AW(3). Consequently, AW(3) is embedded as a subalgebra into $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. We use this to show that overlap functions of twisted primitive elements in $\mathcal{U}_q(\mathfrak{su}(2))$ are q-Racah polynomials. With that, we derive a summation formula connecting q-Racah and dual q-Krawtchouk polynomials.

Keywords. quantum groups, $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$, $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$, spherical functions, generalised matrix elements, Hopf algebras, twisted primitive elements, representations, Haar functional, Askey-Wilson algebra, orthogonal polynomials, Askey-Wilson polynomials, q-Racah polynomials, dual q-Krawtchouk polynomials.

Preface

This thesis was written as the final assignment for the master applied mathematics at TU Delft. Before I take the time to thank everybody who helped me along the way in my studies, let me first drop some reflective thoughts. During my study years, I started to appreciate the beauty and mystery behind mathematics more and more. I always see mathematics as a world with magnificent structures and surprising stories to be unveiled by carefully and curiously walking through it. In the final years of my study, I started to realise how lucky I am there is so much knowledge collected in study books, universities, and math.stackexchange. Learning something from someone else is so much easier than figuring it out yourself. For example, at high school nowadays they teach integration, something that took mathematicians many years to figure out. It feels like a great privilege to stand on the shoulders of past mathematicians to be able to look and search further into the magical world of mathematics.

Also, I think it is a wonderful thing we live in a society where it is possible for many[1] to do a study and fore some, such as me, it even brings 'brood op de plank'. The latter is mainly due to the abundance of applications mathematics has in a wide variety of 'real life' subjects. For this, I am very grateful, for otherwise, chances would have been way smaller I could be full time working on mathematics.

Then I would like to thank my friends and family, who have been supporting me, in good and bad times, the past eight years in achieving the valuable '*ir.*' title. A special thanks to my roommates Peter, Joeri, and Lorenzo who played for 'teddy bear' when I was talking about Hopf algebras and quantum groups. Also, I want to thank my sister Regine who, after she asked if she could help make some pictures for my thesis and I replied I had none, helped proving theorem 13.3 instead. I also want to thank Jan and Dion for being on my thesis committee and taking the time to read my thesis. Then lastly, a huge thanks to my daily supervisor Wolter, without whom I would have been studying represenations of quantum groups. I always found our weekly meetings enjoyable and would like to thank you for answering my numerous questions, listening to my bad jokes, and giving valuable feedback. I've greatly appreciated your patience, flexibility (you always had time for small questions), and optimism over the past nine months and I look forward to working together for another six years.

I hope that reading this thesis is helpful (or at least a bit enjoyable) and I happily answer anyone who has questions or ideas about the content of it.

Carel Wagenaar

Delft, April 2021

[1] I wish that at some point anybody who wants to study does not have any barriers to do so.

Contents

Introduction

One of the reasons to study quantum groups and their representations is the intimate connection with orthogonal polynomials. Roughly speaking, quantum groups are not groups, but algebras closely related to Lie groups and Lie algebras. Although we will not be concerned with it here, I do want to point out that quantum groups are not purely interesting mathematical objects. They also appear naturally in a wide variety of fields, such as statistical physics, quantum field theory, and topology. We will study the quantum groups $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$, which are related to the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ and Lie group $\mathrm{SL}(2,\mathbb{C})$, and their connection with the so-called Askey-Wilson polynomials. Connections between orthogonal polynomials and representations of classical groups and algebras, such as $\mathrm{SL}(2,\mathbb{C})$ and $\mathfrak{sl}(2,\mathbb{C})$, have been known for a long time and studied extensively. For example, matrix elements of $\mathrm{SL}(2,\mathbb{C})$ can be expressed in terms of Jacobi polynomials. In 1985, Askey and Wilson defined q-analogues that generalise these Jacobi polynomials. These *Askey-Wilson* polynomials, also called q-Racah when taken discrete, are the most general explicit orthogonal polynomials in one variable that are currently known.

In this thesis, which consists of three parts, we will show two different connections between quantum groups and Askey-Wilson polynomials. Part I and II will primarily consist of known theory in the literature, part III will show a new result. In part I, we will develop a framework from which we will build the relation between quantum groups and orthogonal polynomials. This framework will lay the groundwork for part II and III, which both show a different connection between the quantum group $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and Askey-Wilson polynomials. These last two parts can be read independently from each other. In part II, we will closely follow the lecture notes [10] from Koelink. In the third part, we will show a new result: an embedding of the algebra AW(3), which is closely related to Askey-Wilson polynomials, into $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. Let us briefly discuss the content of these three parts in some more detail.

In part I, we will introduce $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$, which is described in the language of Hopf algebras. A reader who encounters these definitions for the first time might wonder what the exact intuition behind this is. I urge these readers not to worry too much about this in the beginning and just try to play around with these definitions as being a mathematical puzzle. To encourage this, we will accompany these definitions with five examples of (Hopf) algebras, to let these definitions come more alive. From these examples, the quantum group $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and the algebra generated by its linear dual, $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$, will be the quantum versions of the classical universal enveloping algebra of $\mathfrak{sl}(2,\mathbb{C})$ and the algebra of polynomials on $\mathrm{SL}(2,\mathbb{C})$ respectively. Then, we will briefly recall some representation theory and sketch the analogy between the classical and the quantum theory. Thereafter, we dive into the representation theory of the quantum group $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and its real form $\mathcal{U}_q(\mathfrak{su}(2))$. Using Koornwinder's twisted primitive elements of the quantum group $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$, we will make a connection with orthogonal polynomials. This is done by explicitly calculating the eigenvectors of self-adjoint twisted primitive elements. These eigenvectors can be expressed in terms of dual q-Krawtchouk polynomials, which are a special case of Askey-Wilson polynomials.

In the second part, we will develop a duality between the quantum groups $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$. Using this, we can define spherical elements, a non-classical equivalent of spherical functions, and generalised matrix elements. Via the Haar functional on $\mathcal{A}_q(\mathrm{SU}(2))$, we will show that these generalised matrix elements are Askey-Wilson polynomials with two continuous and two discrete parameters. This generalises the classical theory, where we only get two discrete parameters for the Jacobi polynomials.

Then, in the third part we will show our new result. Namely, the algebra generated by two twisted primitive elements is essentially Zhedanov's Askey-Wilson algebra 'AW(3)'. As the name suggests, this algebra is connected with Askey-Wilson polynomials, or in our case, their discrete version called q-Racah polynomials. As a consequence, we can show another link between $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and Askey-Wilson polynomials.

Notation. In this thesis, $\mathbb{Z}_+$ will denote the set of nonnegative integers. δ_{ij} will be the Kronecker delta function, defined by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases}$$

Furthermore, π^l and $\mathrm{d}\pi^l$ will be used for the classical $(2l+1)$-dimensional representations of $\mathrm{SL}(2,\mathbb{C})$ and its Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ respectively. The $(2l+1)$-dimensional representation in the quantum setting will be denoted by t^l. Lastly, I want to clarify the difference between the algebras $\mathfrak{sl}(2,\mathbb{C}) \leftrightarrow \mathfrak{su}(2)$, $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C})) \leftrightarrow \mathcal{U}_q(\mathfrak{su}(2))$ and $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C})) \leftrightarrow \mathcal{A}_q(\mathrm{SU}(2))$, as they look quite similar and may be confusing for the reader who does not encounter those terms often. The algebras with $\mathfrak{su}(2)/\mathrm{SU}(2)$ are called a real form of the ones with $\mathfrak{sl}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{C})$ and are 'just' the same algebra together with a $*$-structure. Without going into too much detail about what a $*$-structure is, one can think of this as defining what the adjoint operator has to be.

Part I. Introduction to Representations of Quantum Algebras

1. Algebras

In this section we will give five important examples of algebras which will later turn into Hopf algebras. Why we take these five examples will become clear later on. For now, just take these examples in as building blocks for understanding (Hopf) algebras. First, we will give a definition of an algebra. We will always work with unital[2] associative algebras over $\mathbb{C}$, .

Definition 1.1. *An associative algebra with unit, or just algebra, is a linear space A together with a bilinear mapping $m : A \times A \to A$, called multiplication, that is associative and has unit $1_A \in A$, i.e.*

$$m(m(a,b),c) = m(a,m(b,c)), \quad m(1_A,a) = m(a,1_A) = a \quad \text{for all } a,b,c \in A.$$

Define $\eta : \mathbb{C} \to A$ by

$$\eta(\lambda) = \lambda 1_A \text{ for all } \lambda \in \mathbb{C}.$$

Then a scalar $\lambda \in \mathbb{C}$ can be considered as an element of A by identifying λ with $\eta(\lambda)$.

Remark 1.2. We will often omit the $'m'$ and just write $ab := m(a,b)$.

Let us look at a few important examples.

Example (1). Let G be a finite group. Then we can look at $A = C(G)$, all functions $f : G \to \mathbb{C}$. Then A is an algebra with multiplication and unit given by

$$(fg)(h) = f(h)g(h), \quad 1_A(h) = 1, \quad f,g \in C(G) \text{ and } \ h \in G.$$

In this thesis, we will often look at associative algebras with generators $X_1, ..., X_n$, subject to relations of the form

$$p(X_1, ...X_n) = 0.$$

Here, $p(X_1, ..., X_n)$ are elements of the algebra, which are polynomials in $X_1, ..., X_n$. Examples of elements are

$$2X_1^2 X_n X_1 + 3 \cdot 1_A, \quad \text{and} \quad 19iX_2^4 X_1 + 4X_n,$$

where 1_A is the unit of the algebra. We often omit writing out the unit, i.e. $3 := 3 \cdot 1_A$. Another way of looking at this is the following. Elements of the algebra are finite linear combinations of words, where the words are formed with 'letters' $X_1, \ldots, X_n$. That is, words are finite products of $X_1, \ldots, X_n$. For example, $X_1^2 X_n$ and $X_2^3 X_1 X_n^3 X_2$ are words. Multiplication is concatenation of words, e.g. we have the product

$$m(X_1^2 X_n, X_2^3 X_1 X_n^3 X_2) = X_1^2 X_n X_2^3 X_1 X_n^3 X_2.$$

The reader who is wondering if this is well defined makes a good point. Proving that requires some abstract algebra. Since we will mainly be concerned with doing computations in such algebras, we do not have to worry about the algebraic definition behind it[3], and we will focus on the practical side.

Example (2). For our second example, we will look at $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$, the universal enveloping algebra of the Lie algebra[4] $\mathfrak{sl}(2,\mathbb{C})$. Recall that $\mathfrak{sl}(2,\mathbb{C})$ is the Lie algebra of 2×2 matrices with trace 0. It has a basis $\{H, E, F\}$ given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and multiplication, called the Lie bracket,

$$[X, Y] = XY - YX.$$

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[2] I.e. every algebra A has a unit, denoted by 1_A

[3] Officially, one has to take the quotient F/I, where F is the free algebra generated by $X_1, ..., X_n$ and I is the two-sided ideal generated by the relations. For example, if we have relations $w_1 = 0, \ldots, w_k = 0$, then we take $I = \text{span}\{aw_j b : \ a, b \in F, \ j = 1, \ldots k\}$. For more details, see for example [8].

[4] This can be done for arbitrary Lie algebras $\mathfrak{g}$. Also $\mathcal{U}(\mathfrak{g})$ can be made into a Hopf algebra in the same way as $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$. We will only look at $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ because it is the analog of the quantum group $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ which will play a crucial role in these notes.

This Lie bracket leads to relations

$$[H,E]=2E,\quad [H,F]=-2F,\quad [E,F]=H. \tag{1.1}$$

Now, $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ is the algebra generated by the basic elements H, E and F of $\mathfrak{sl}(2,\mathbb{C})$ subject to the relations (1.1) of the Lie bracket $[\cdot,\cdot]$, which are the only relations. For example, $EF \notin \mathfrak{sl}(2,\mathbb{C})$ since the matrix product of E and F does not have trace 0. Also, $E^2 = 0$ in $\mathfrak{sl}(2,\mathbb{C})$. However, $EF, E^2 \in \mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ as two abstract elements, both unequal to 0.

Example (3). Let $\mathrm{SL}(2,\mathbb{C})$ be the group of 2×2 matrices with determinant 1. The group $\mathrm{SL}(2,\mathbb{C})$ is not an algebra since adding two matrices will not be in $\mathrm{SL}(2,\mathbb{C})$ in general. However, $\mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$, the complex valued polynomials with the 4 matrix entries a, b, c and d subject to $ad-bc=1$ as variables, is an algebra. It is similar to example 1, only $\mathrm{SL}(2,\mathbb{C})$ is not a finite group and we will only look at polynomials. The product and unit are defined in the same way as example 1. For instance,

$$p_1\begin{pmatrix} a & b \\ c & d\end{pmatrix} = 2a^2+ic^3d+25, \quad \text{and} \quad p_2\begin{pmatrix} a & b \\ c & d\end{pmatrix} = 2c, \tag{1.2}$$

are elements in $\mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$. Four important elements of $\mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$ are,

$$\alpha\begin{pmatrix} a & b \\ c & d\end{pmatrix} = a, \qquad \beta\begin{pmatrix} a & b \\ c & d\end{pmatrix} = b$$
$$\gamma\begin{pmatrix} a & b \\ c & d\end{pmatrix} = c, \qquad \delta\begin{pmatrix} a & b \\ c & d\end{pmatrix} = d.$$

These are building block for any $p \in \mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$. For example, let $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$, then p_1 and p_2 given in (1.2) can also be written as

$$p_1(g) = 2\alpha(g)^2 + i\gamma(g)^3\delta(g)+25, \quad \text{and} \quad p_2(g) = 2\gamma(g).$$

Example (4). Let us now look at a crucial example, the quantised universal enveloping algebra for $\mathfrak{sl}(2,\mathbb{C})$. Let $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ be the complex unital associative algebra generated by K, E, F, K^{-1} subject to the relations

$$KK^{-1} = 1 = K^{-1}K, \quad KE = qEK, \quad KF = q^{-1}FK, \quad EF - FE = \frac{K^2-K^{-2}}{q-q^{-1}}, \tag{1.3}$$

where $q \in \mathbb{C}\backslash\{-1,0,1\}$[5]. To understand the name $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ for this algebra, let $K = \exp\left(\frac{q-1}{2}H\right)$ and therefore $K^{-1} = \exp\left(\frac{1-q}{2}H\right)$. If we now let $q \uparrow 1$, we get (formally, if H is bounded in some sense) from the second relation of (1.3) using the power series for the exponential function, that

$$\begin{aligned}
\left(\sum_{n=0}^{\infty}\frac{(\frac{q-1}{2})^n H^n}{n!}\right)E &= qE\left(\sum_{n=0}^{\infty}\frac{\left(\frac{q-1}{2}\right)^n H^n}{n!}\right)\\
\implies 2E + (q-1)HE &= 2qE + q(q-1)EH + \mathcal{O}\left((1-q)^2\right)\\
\implies 2E &= HE - EH \ (= [H,E]),
\end{aligned}$$

where we divided by $q-1$ and then let $q \uparrow 1$ in the last step. Similarly we can obtain

$$[H,F] = -2F, \ \text{and} \ [E,F] = H,$$

the exact same relations (1.1) as in (the universal enveloping of) $\mathfrak{sl}(2,\mathbb{C})$.

Example (5). This fifth example will be the quantum $\mathrm{SL}(2,\mathbb{C})$ group. Let $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ be the complex unital associative algebra generated by α, β, γ and δ subject to the relations

$$\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad ,\gamma\delta = q\delta\gamma,$$
$$\beta\gamma = \gamma\beta, \quad \alpha\delta - q\beta\gamma = 1 = \delta\alpha - q^{-1}\beta\gamma,$$

[5]Later on, we will also demand that q is not a root of unity. This is to make sure we have irreducible representations for each dimension. On the algebra level, this does not matter.

where $q \in \mathbb{C}\backslash\{0\}$. Observe that elements of $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ are polynomials in α, β, γ and δ subject to the relations above. Taking $q = 1$ gives the link between $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$, by identifying (a, b, c, d) with $(\alpha, \beta, \gamma, \delta)$. Of course, the difference here is that $a, b, c, d \in \mathbb{C}$ and $\alpha, \beta, \gamma, \delta$ are (at this point) abstract elements of an algebra. Let p be a polynomial in 4 variables, then we can interpret $p\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ as an element in $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$, but, if $q = 1$, also as an element of $\mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$, which justifies the name.

Before we end this section, we will discuss one more topic that will be important in sections 12 and 13. An element z in an algebra A is said to be a *central element*, if it commutes with every element of the algebra, i.e.

$$az = za \quad \text{for all } a \in A.$$

The unit 1_A of an algebra is an obvious example of a central element. For some algebras, such as $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ we have a special central element called a *Casimir element*. For $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$, we have a Casimir element Ω given by

$$\Omega = \frac{q^{-1}K^2 + qK^{-2}}{(q - q^{-1})^2} + EF = \frac{qK^2 + q^{-1}K^{-2}}{(q - q^{-1})^2} + FE. \tag{1.4}$$

One can verify that Ω commutes with all generators of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. Consequently, Ω is a central element of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$.

2. Hopf algebras

In this section we will add some more structure to an algebra so that it becomes a Hopf algebra. We will take two steps along the way: a coalgebra and bialgebra. Again, do not worry to much about these definitions (yet). I will include the same examples as before to make these algebras come more alive.

So without further ado, let us construct Hopf algebras. Note that if A and B are algebras, $A \otimes B$ is an algebra as well with unit $1_A \otimes 1_B$ and multiplication given by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2 \otimes b_1 b_2).$$

Or in more algebraic terms,

$$m_{A\otimes B} = m_A \otimes m_B \circ (\mathrm{id} \otimes \sigma \otimes \mathrm{id}),$$

where σ is the flip mapping $\sigma(a \otimes b) = b \otimes a$. In particular, $A \otimes A$ is an algebra.

Definition 2.1. *A coalgebra is a linear space A together with a linear mapping $\Delta : A \to A \otimes A$, called the comultiplication, and non-zero linear mapping $\varepsilon : A \to \mathbb{C}$, called the counit, such that for all $a \in A$,*

$$(\mathrm{id} \otimes \Delta) \circ \Delta(a) = (\Delta \otimes \mathrm{id}) \circ \Delta(a), \tag{2.1}$$

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta(a) = a = (\varepsilon \otimes \mathrm{id}) \circ \Delta(a). \tag{2.2}$$

Remark 2.2. The requirement for A is a linear space, not an algebra. Then one might wonder why it is still called a coalgebra. That is because an algebra is a triplet (A, m, η), where A is a linear space, m is the multiplication and η is the map called unit. A coalgebra is also a triplet (A, Δ, ε) but then the two maps Δ, the comultiplication, and ε called counit going into the opposite direction.

If an algebra is a coalgebra at the same time, we get the following.

Definition 2.3. *A bialgebra is an algebra A, such that A is also a coalgebra and the comultiplication Δ and counit ε are algebra homomorphisms from A to $A \otimes A$. That is*

$$\Delta(ab) = \Delta(a)\Delta(b) \text{ and } \varepsilon(ab) = \varepsilon(a)\varepsilon(b) \quad \text{for all } a, b \in A.$$

Remark 2.4. Since ε is non-zero, there exists an $a \in A$ such that $\varepsilon(a) = \lambda \neq 0$. Therefore,

$$\lambda = \varepsilon(a) = \varepsilon(1_A a) = \varepsilon(1_A)\lambda,$$

and $\varepsilon(1_A) = 1$.

Now one more ingredient is necessary to get a Hopf algebra, a map S called the antipode.

Definition 2.5. *A Hopf algebra is a bialgebra A together with a linear mapping $S : A \to A$, called the antipode, such that for all $a \in A$ we have*

$$m \circ (S \otimes \mathrm{id}) \circ \Delta(a) = \eta \circ \varepsilon(a) = m \circ (\mathrm{id} \otimes S) \circ \Delta(a). \tag{2.3}$$

Remark 2.6. Writing out the comultiplication we get something of the form

$$\Delta(a) = \sum_i a_i \otimes a_i'.$$

To ease notation, we will use the very convenient[6] and often used Sweedler's sigma notation,

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}, \tag{2.4}$$

$$(\mathrm{id} \otimes \Delta)\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes a_{(3)}, \tag{2.5}$$

where the latter part is well defined by (2.1).

[6]For example, relation (2.3) can now be written as $m \circ (\mathrm{id} \otimes S)\Delta(a) = \sum_{(a)} a_{(1)} S(a_{(2)})$

We also need the notion of a Hopf algebra (iso)morphism. This is what you expect it to be: a mapping that preserves the structure of both Hopf algebras.

Definition 2.7. *A linear map $\phi : A \to B$ between Hopf algebras A and B is called a Hopf algebra morphism if it is an algebra homomorphism and*

$$(\phi \otimes \phi) \circ \Delta_A = \Delta_B \circ \phi(a), \quad \phi \circ S_A = S_b \circ \phi(a), \quad \varepsilon_A = \varepsilon_B \circ \phi.$$

The map ϕ is called a Hopf algebra isomorphism if there also exists a Hopf algebra morphism $\phi^{-1} : B \to A$ such that

$$\phi^{-1} \circ \phi = \mathrm{id}_A \quad \textit{and} \quad \phi \circ \phi^{-1} = \mathrm{id}_B.$$

There are many useful relations between S, ε and Δ that can be deduced from their definitions, see e.g. [8, 10]. However, we will not do that here since it is not necessary for our purposes. We will only show the following proposition, which will also assist the reader in getting some insight in how to deal algebraically with Hopf algebras. If a bi-algebra can be made into a Hopf algebra, there is only one way to do this. That is, when an antipode S exists, it is unique. Furthermore, it is antimultiplicative and $\varepsilon \circ S = \varepsilon$.

Proposition 2.8. *Let A be a bialgebra and S an antipode making A into a Hopf Algebra. Then*

(i) S is unique.
(ii) $\varepsilon \circ S = \varepsilon$.
(iii) S is an anti-multiplicative, i.e. $S(ab) = S(b)S(a)$ for $a, b \in A$.

Proof. Let us first prove (i). Define B to be the algebra of linear maps from A to itself, in particular, $S \in B$. However, we will not use the usual[7] product of $\mathrm{End}(A)$. Let $F, G, H \in B$, define the product by

$$(F * G)(a) = m \circ (F \otimes G) \circ \Delta(a) = \sum_{(a)} F(a_{(1)})G(a_{(2)}).$$

This product is associative since

$$(F * (G * H))(a) = \sum_{(a)} F(a_{(1)})G(a_{(2)})H(a_{(3)}) = ((F * G) * H)(a).$$

The unit 1_B of this algebra B is $\eta \circ \varepsilon$. Indeed, by (2.2) we have

$$(F * (\eta \circ \varepsilon))(a) = m \circ (F \otimes \mathrm{id})(\mathrm{id} \otimes (\eta \circ \varepsilon))\Delta(a) = m \circ (F(a) \otimes 1) = F(a),$$

and similarly $((\eta \circ \varepsilon) * F)(a) = F(a)$. Now, by (2.3) and the definition of the product $*$, we have that S is a two-sided inverse of id_A, the identity map on A. That is,

$$(\mathrm{id}_A * S) = (S * \mathrm{id}_A) = (\eta \circ \varepsilon) = 1_B.$$

Since the inverse is unique in an associative algebra[8] and B was defined independent of the antipode, our S is unique as well.
For (ii), we will use (2.2) to obtain

$$a = (\varepsilon \otimes \mathrm{id})\Delta(a). \tag{2.6}$$

Applying $\mathrm{id} \otimes S$ first and then ε gives us,

$$\varepsilon(S(a)) = (\varepsilon \otimes \varepsilon)(\mathrm{id} \otimes S)\Delta(a).$$

Since ε is an algebra homomorphism, $\varepsilon \otimes \varepsilon = \varepsilon \circ m$. Together with (2.3) we get

$$\varepsilon(S(a)) = \varepsilon \circ m(\mathrm{id} \otimes S)\Delta(a) = \varepsilon \circ \eta \circ \varepsilon(a) = \varepsilon(a),$$

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[7]Normally the product of $F, G \in \mathrm{End}(A)$ is just the composition of F and G. That is, $(FG)(a) = (F \circ G)(a) = F(G(a))$ for $a \in A$.

[8]If S and S' are both an inverse of T, then $S = S(TS') = (ST)S' = S'$.

since (by remark 2.4) $\varepsilon \circ \eta$ is the identity map[9] on $\mathbb{C}$.
Lastly we need to prove (iii). Applying S on both sides of (2.6) and using (ii) gives

$$S(a) = \sum_{(a)} S(a_{(1)})\varepsilon(a_{(2)}). \tag{2.7}$$

Therefore, using the commutativity of $\mathbb{C}$ and that ε is an algebra homomorphism, we obtain

$$S(b)S(a) = \sum_{(a),(b)} S(b_{(1)})S(a_{(1)})\varepsilon(a_{(2)}b_{(2)})$$

From (2.3) and using that Δ is a homomorphism, we have for $c, d \in A$,

$$\varepsilon(cd) = \sum_{(c),(d)} c_{(1)}d_{(1)}S(c_{(2)}d_{(2)}). \tag{2.8}$$

This gives together with (2.5),

$$S(b)S(a) = \sum_{(a),(b)} S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)}S(a_{(3)}b_{(3)}).$$

Using (2.8) again and then (2.7), we have

$$S(b)S(a) = \varepsilon(a_{(1)})\varepsilon(b_{(1)})S(a_{(2)}b_{(2)}) = S(ab). \qquad \square$$

Let us look at the five examples. It is a good exercise to calculate for yourself that the comultiplication, counit and antipode given are indeed algebra morpisms and satisfy (2.1), (2.2) and (2.3).

Example (1). The algebra $C(G)$ is a Hopf algebra by defining

$$\begin{aligned}
&\Delta(f)(g,h) = f(gh), \quad f \in C(G),\ g,h \in G,\\
&\varepsilon(f) = f(e), \quad f \in C(G) \text{ and } e \text{ is the unit of } G,\\
&S(f)(g) = f(g^{-1}), \quad f \in C(G),\ g \in G.
\end{aligned}$$

Example (2). $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ can be made into a Hopf algebra by setting for $X \in \mathfrak{sl}(2,\mathbb{C})$,

$$\begin{aligned}
&\Delta(X) = X \otimes 1_U + 1_U \otimes X, \quad \Delta(1_U) = 1_U \otimes 1_U,\\
&\varepsilon(X) = 0, \quad \varepsilon(1_U) = 1, \quad S(X) = -X, \quad S(1_U) = 1_U,
\end{aligned}$$

where 1_U is the unit in $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ and $1 \in \mathbb{C}$. Then extend Δ and ε as algebra homomorphisms and S as an antihomomorphism, i.e.

$$\Delta(XY) = \Delta(X)\Delta(Y), \quad \varepsilon(XY) = \varepsilon(X)\varepsilon(Y), \quad S(XY) = S(Y)S(X).$$

Example (3). The comultiplication and counit of $\mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$ are the same as in example 1. In particular we have

$$\begin{aligned}
&\Delta(p)\left[\begin{pmatrix} a_1 & b_1 \\ c_1 & c_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right] = p\begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix},\\
&\varepsilon(p)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = p\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S(p)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = p\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
\end{aligned}$$

Notice the link here between comultiplication and matrix multiplication and between the antipode and taking the inverse, i.e. $S^2 = \mathrm{id}$.

Example (4). $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ becomes a Hopf algebra by taking

$$\begin{aligned}
&\Delta(K) = K \otimes K, \quad \Delta(E) = K \otimes E + E \otimes K^{-1},\\
&\Delta(F) = K \otimes F + F \otimes K^{-1}, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},\\
&\varepsilon(K) = \varepsilon(K^{-1}) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0.\\
&S(K) = K^{-1}, \quad S(K^{-1}) = K, \quad S(E) = -q^{-1}E, \quad S(F) = -qF.
\end{aligned} \tag{2.9}$$

[9] Since $\eta(\lambda) = \lambda 1_A$ and $\varepsilon(1_A) = 1$, $\varepsilon \circ \eta(\lambda) = \lambda$ for $\lambda \in \mathbb{C}$.

Notice here that since $q \neq 1$ we have $S^2 \neq \mathrm{id}$. It is a good exercise to check that these maps are well-defined in combination with the commutation relations (1.3) and that these maps satisfy the requirements of a Hopf algebra.

Example (5). The quantum group $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ is a Hopf algebra by defining

$$\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma, \quad \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta,$$
$$\Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma, \quad \Delta(\delta) = \gamma \otimes \beta + \delta \otimes \delta,$$
$$\varepsilon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}.$$

Notice the similarity with example 3 for the three maps, where α, β, γ and δ have to be interpreted as the variables of a polynomial. The comultiplication can here in a way be seen as matrix multiplication with itself.

Lastly in this section, we will discuss the topic of Hopf $*$-algebras.

Definition 2.9. *An algebra A together with a map $* : A \to A$ (notation $a^* := *(a)$, $a \in A$) is called a $*$-algebra if $*$ is an antilinear antimultiplicative involution. That is, for all $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$ we have*

$$(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*, \quad (ab)^* = b^* a^*, \quad (a^*)^* = a.$$

Definition 2.10. *A Hopf $*$-algebra is a Hopf algebra A, such that A is a $*$-algebra and Δ and ε are $*$-homomorphisms. That is, Δ and ε are algebra homomorphisms and*

$$\Delta(a^*) = \Delta(a)^*, \quad \varepsilon(a^*) = \varepsilon(a)^* = \overline{\varepsilon(a)}.$$

A Hopf $*$-algebra morphism $\phi : A \to B$ of two Hopf $*$-algebras A and B is a Hopf algebra morphism such that $\phi(a^*) = (\phi(a))^*$.

There is no requirement for a relation between S and $*$. However, this follows naturally. For a proof, see e.g. [10, Proposition 1.2.5].

Proposition 2.11. *Let A be a Hopf $*$-algebra, then $(S \circ *)^2 = \mathrm{id}_A$, in particular $S^{-1} = * \circ S \circ *$.*

3. Representation theory

In this section we will go over some basic representation theory and motivate the setting we will choose for the quantum case. We will cover some classical theory about $\mathrm{SL}(2,\mathbb{C})$ as well. It is important to realise that the knowledge about representations of quantum groups[10] is build upon the research already done in representation theory. In many cases, the classical theory can be extended to quantum groups in a straightforward way. However, often it will be a non-trivial extension[11], e.g. subgroups and $*$-structures are different in the quantum setting.

3.1. **Representations: general theory.** It is of great interest to look at representations of groups and algebras. If V is a vector space, then we will use $\mathrm{GL}(V)$ for the invertible linear operators on V and $\mathrm{End}(V)$ for all linear operators on V.

Definition 3.1. *Let V be a vector space.*

(i) Let G be a group. A map $\pi : G \to GL(V)$ is a representation of G on V if

$$\pi(gh) = \pi(g)\pi(h), \text{ for all } g, h \in G.$$

(ii) Let A be an algebra. A linear map $\pi : A \to End(V)$ is a representation of A on V if

$$\pi(ab) = \pi(a)\pi(b), \text{ for all } a, b \in A,$$
$$\pi(1_A) = \mathrm{id}_V,$$

where 1_A is the unit of A and id_V the identity map on V.

Remark 3.2. Unless explicitly mentioned otherwise, we will work with finite-dimensional linear spaces. Much of the theory goes on in the infinite-dimensional case. However, this requires much more technicalities from a functional analysis point of view, such as boundedness and domains. In this thesis we will primarily be concerned with the algebra behind it all.

Remark 3.3. Note that (i) and (ii) mean that π is a group and algebra homomorphism respectively, i.e. π preserves the structure of its domain and codomain. Also, it automatically[12] follows that $\pi(e_G)$ gets mapped to id_V.

Two representations $(\pi^{(1)}, V_1)$ and $(\pi^{(2)}, V_2)$ of a group G where $\dim(V_1) = \dim(V_2)$, can be equal after a change of basis from V_1 and V_2. We want to consider those as 'the same'. Therefore, $(\pi^{(1)}, V_1)$ and $(\pi^{(2)}, V_2)$ are called *equivalent* if there exists a bijective mapping $B : V_1 \to V_2$ such that

$$B\pi^{(1)}(g) = \pi^{(2)}(g)B \text{ for all } g \in G.$$

Similarly, two representations $(\pi^{(1)}, V_1)$ and $(\pi^{(2)}, V_2)$ of an algebra A are equivalent if $B\pi^{(1)}(a) = \pi^{(2)}(a)B$ for all $a \in A$. If $W \subseteq V$ and $\pi(a)W \subseteq W$ for all $a \in A$, we call W *invariant*[13] under π. Note that $W = 0$ and $W = V$ are always invariant subspaces. If those are the only invariant subspaces, we call π *irreducible*. Irreducible representation can be seen as the 'building blocks' of representations and are therefore studied the most. If the representation is on a Hilbert space H instead of an ordinary linear space, we can define *unitary* representations. A representation π is unitary if

$$\langle \pi(g)h_1, h_2 \rangle_H = \langle h_1, \pi(g^{-1})h_2 \rangle_H$$

for all $h_1, h_2 \in H$. That is, taking the inverse of an element g in the group corresponds to taking the adjoint of the operator representing g. Unitary representations are completely reducible. This follows from the fact that if $W \subset V$ is an invariant subspace, its orthogonal complement $W^\perp$ is also an invariant subspace, since

$$\langle \pi(g)w^\perp, w \rangle = \langle w^\perp, \pi(g^{-1})w \rangle = 0, \quad w^\perp \in W^\perp, \ w \in W.$$

The result then follows by induction on the dimension of V.

[10] Something worth mentioning is that there is no strict definition of a 'quantum group'. There are just several examples of them. E.g. $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ are often called quantum groups (although they are algebras and not even groups).

[11] Otherwise it would make no sense to study them

[12] $\pi(g) = \pi(e_G g) = \pi(e_G)\pi(g)$ implies $id_V = \pi(e_G)$ by multiplying with $\pi(g)^{-1}$ on the right side.

[13] This works similarly for group representations.

3.2. **Representation of SL$(2,\mathbb{C})$.** We will briefly discuss some classical well known (e.g.[12]) representation theory of SL$(2,\mathbb{C})$. Fix $l \in \frac{1}{2}\mathbb{Z}_+$ and let π^l be the $2l+1$-dimensional representation of SL$(2,\mathbb{C})$ on the vector space H_l of complex homogeneous polynomials in two variables of degree $2l$ defined by

$$(\pi^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} f)(x,y) := f(ax+cy, bx+dy), \;\; f \in H_l. \tag{3.1}$$

Take the standard basis $(\psi_n^l)_{n=-l}^l$ given by

$$\psi_n^l := \begin{bmatrix} 2l \\ l-n \end{bmatrix}^{\frac{1}{2}} x^{l-n} y^{l+n}, \tag{3.2}$$

then we define the matrix elements

$$\pi_{mn}^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left\langle \pi^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} \psi_n^l, \psi_m^l \right\rangle_{H_l}.$$

Now, matrix elements can be seen as function $\pi_{mn}^l : \mathrm{SL}(2,\mathbb{C}) \to \mathbb{C}$. Moreover, they are polynomials with variables a, b, c and d such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$. In fact, they are multiples of Jacobi polynomials (see e.g. [5]). Furthermore, π^l restricted the subgroup of unitary matrices, called SU(2), is a unitary representation.

3.3. **Approach in the quantum setting.** We want to find the quantum analogue of SL$(2,\mathbb{C})$, and corresponding q-hypergeometric orthogonal polynomials, as well. A first approach is to look at "representations" of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where $\alpha, \beta, \gamma, \delta \in \mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$. However, the questions arises how to do this. For example, do these elements have an algebra or group structure? What is the multiplication? Interpreting it like this can be done, as shown in [17], by defining compact matrix pseudogroups. Koornwinder [12] then showed that the matrix elements of this representation can be expressed as orthogonal polynomials, namely as little q-Jacobi polynomials or as q-Krawtchouk polynomials. However, there was hope that quantum groups also provided a setting for Askey-Wilson polynomials. The method above did not yield results in that direction unfortunately. Another way of looking at quantum groups was needed.

This other way is to focus on $\mathfrak{sl}(2,\mathbb{C})$, since we established a quantum version $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ of this Lie algebra. The Lie group SL$(2,\mathbb{C})$ and Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ have an intimate connection. Let $X \in \mathfrak{sl}(2,\mathbb{C})$ and $t \in \mathbb{R}$, since

$$\det(\exp(tX)) = \exp(\mathrm{tr}\,(tX)) = \exp(0) = 1,$$

the one-parameter subgroup $K = \{\exp(tX), t \in \mathbb{R}\}$ is a subalgebra of SL$(2,\mathbb{C})$. This works the other way around too. Every $Y \in \mathrm{SL}(2,\mathbb{C})$ can be written[14] as $Y = \exp(tX)$ for $t \in \mathbb{R}$. Here, X is a linear combination of the basis elements E, F and H of $\mathfrak{sl}(2,\mathbb{C})$. In this way, we can relate to the $(2l+1)$-dimensional Lie group representation π^l a Lie algebra representation $\mathrm{d}\pi^l$ by defining

$$\mathrm{d}\pi^l(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi^l(\exp(tX)), \quad X \in \mathfrak{sl}(2,\mathbb{C}),$$

This is well defined, see e.g. [6]. For the matrix elements of this Lie algebra representation, we have the identity

$$\mathrm{d}\pi_{nm}^l(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi_{nm}^l(\exp(tX)).$$

Since the matrix elements $\pi_{mn}^l \in \mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$, we can study $p \in \mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$, by looking at it as a functional on $X \in \mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$,

$$\langle X, p \rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p(\exp(tX)), \quad X \in \mathfrak{sl}(2,\mathbb{C}), \; p \in \mathrm{Pol}(\mathrm{SL}(2,\mathbb{C})). \tag{3.3}$$

[14] Or as a limit, i.e. $\lim_{t\to\infty} \exp(tX)$.

Therefore, it makes sense to look at representations of $\mathfrak{sl}(2,\mathbb{C})$ to find polynomials on the group $\mathrm{SL}(2,\mathbb{C})$. Thus, if we can establish such a duality relation in the quantum setting, we might find the quantum equivalents of π^l_{nm}. Of course, one has to be careful here. In what sense are these elements still polynomials in the quantum setting? Normally they are polynomials in a, b, c and d which come from an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$. However, we do not have the quantum version of $\mathrm{SL}(2,\mathbb{C})$ yet. The trick here is to look at the standard representation in two dimensions, i.e.

$$\pi^{\frac{1}{2}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Recall that we defined the elements $\alpha, \beta, \gamma, \delta \in \mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$ as

$$\alpha\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a, \qquad \beta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = b$$
$$\gamma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = c, \qquad \delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = d.$$

Since $\det(g) = 1$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$, we have

$$\alpha\gamma - \beta\gamma = 1.$$

We can write α, β, γ and δ in terms of matrix elements of the Lie algebra representation using the linear functionals (3.3), i.e.

$$\langle X, \alpha\rangle = \mathrm{d}\pi^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}}(X), \quad \langle X, \beta\rangle = \mathrm{d}\pi^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}(X),$$
$$\langle X, \gamma\rangle = \mathrm{d}\pi^{\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}}(X), \quad \langle X, \delta\rangle = \mathrm{d}\pi^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(X).$$

Note that using this 'duality bracket' $\langle\cdot,\cdot\rangle$, we do not need $\mathrm{SL}(2,\mathbb{C})$ or derivatives explicitly to analyse $\mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$. This idea will be used in the quantum setting, where we have access to the analogue of $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$, but not of $\mathrm{SL}(2,\mathbb{C})$. The identities above will be used to define linear functionals $\alpha, \beta, \gamma, \delta$ on $X \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. It turns out that the algebra generated by $\alpha, \beta, \gamma, \delta$ is exactly $\mathcal{A}_q(\mathrm{SU}(2))$ from example 5, this will be Theorem 7.7. To achieve all this, it is crucial we establish a duality relation between $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$. That is where the dualilty of Hopf algebras comes in, which will be the subject of section 7.

4. Representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$

In this section we will, as the title suggests, look at representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. First, we will discuss some important elements of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$: the group-like and twisted primitive elements. Secondly, we will show there are four possible inequivalent irreducible representations for each dimension '$2l+1$', $l \in \frac{1}{2}\mathbb{Z}_+$. Then we look at the three possible $*$-structures, also called real forms, on $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. We will focus on $\mathcal{U}_q(\mathfrak{su}(2))$, the one with finite-dimensional irreducible $*$-representations. Putting a $*$-structure on $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ is important since it gives us a notion of a Hilbert space and therefore orthogonality. This will relate to the orthogonality relation of the orthogonal polynomials.

4.1. **Group-like and twisted primitive elements.** The definition of group-like and twisted primitive elements might seem a bit arbitrary at first sight. In this section we will use the structure of these elements for finding representations. However, twisted primitive elements are far more important than only that, as will become clear in later sections. In order not to leave the reader completely in the dark at this point, I will briefly sketch their application. They will be important in both part II en III of this thesis. First of all, the twisted primitive elements are the key into the relation between the so called spherical elements of $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ and the Askey-Wilson polynomials. Furthermore, the subalgebra generated by two specific twisted primitive elements will be isomorphic to the Askey-Wilson algebra. This algebra has, non surprisingly, a connection with Askey-Wilson polynomials as well.

Let us now turn to the definition of twisted primitive and group-like elements. Recall that in the classical case $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$, we have

$$\Delta(1) = 1\otimes 1, \quad \Delta(X) = 1\otimes X + X\otimes 1, \quad X\in\mathfrak{sl}(2,\mathbb{C}).$$

To this end, we define special elements of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ that have a somewhat similar structure.

Definition 4.1. *$X \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ ($X \neq 0$) is called **group-like** if*

$$\Delta(X) = X\otimes X.$$

*An element $Y \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ is called a **twisted primitive element** with respect to a group-like element X if*

$$\Delta(Y) = X\otimes Y + Y\otimes S(X).$$

Remark 4.2. Observe that this captures the structure of $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$. The group-like elements correspond to $1\in\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$, the twisted primitive elements to $X\in\mathfrak{sl}(2,\mathbb{C})$.

Remark 4.3. Using the structure of the Hopf algebra, we can find

$$\varepsilon(X) = 1 \text{ if } X \text{ is group-like,} \tag{4.1}$$

$$\varepsilon(Y) = 0 \text{ if } Y \text{ is twisted primitive.} \tag{4.2}$$

Indeed, if X is group-like, (2.2) implies

$$X = (\varepsilon\otimes\mathrm{id})\Delta(X) = \varepsilon(X)X \quad\Longrightarrow\; \varepsilon(X) = 1.$$

If Y is twisted primitive, (2.2) gives

$$Y = (\mathrm{id}\otimes\varepsilon)\Delta(Y) = X\varepsilon(Y) + Y\varepsilon(S(X)) = X\varepsilon(Y) + Y,$$

since X is group-like and $\varepsilon(S(X)) = \varepsilon(X)$ by proposition 2.8(ii). Therefore, above equation implies $\varepsilon(Y) = 0$.

Some quick observations here are that K^m, $m\in\mathbb{Z}$ are group-like elements and that E, F are twisted primitive elements with respect to the group-like element K, since

$$\Delta(K^m) = \Delta(K)^m = K^m\otimes K^m,$$
$$\Delta(E) = K\otimes E + E\otimes K^{-1} = K\otimes E + E\otimes S(K),$$

and similarly for $\Delta(F)$. This is different from the classical setting $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$, where 1 is the only group-like element and all generators $X\in\mathfrak{sl}(2,\mathbb{C})$ are twisted primitive elements. The group-like and twisted primitive elements of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ have been classified completely. However, we first need an important lemma. For the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ the Poincaré-Birkhoff-Witt (PBW) theorem[15] gives a linear basis of the form $H^kF^mE^n$, $k \in \mathbb{Z}$ and $m,n \in \mathbb{Z}_+$. This can be generalised to $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$.

Lemma 4.4. *The elements $K^kF^mE^n$, where $k \in \mathbb{Z}$ and $m,n \in \mathbb{Z}_+$, form a linear basis for $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$.*

Remark 4.5. That $K^kF^mE^n$ span $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ is the easy part. The tough part is to show this elements are linearly independent. This is quite some work and the algebraic theory required goes beyond the goal of this thesis. Interesting is that the proof is not a trivial extension of the classical case.

With this lemma, we can find the group-like and twisted primitive elements.

Proposition 4.6. *In $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$,*

(i) K^m, $m \in \mathbb{Z}$ are the group-like elements;
(ii) span$\{E,F,K-K^{-1}\}$ is the linear space of twisted primitive elements with respect to K;
(iii) span$\{K^m-K^{-m}\}$, is the linear space of twisted primitive elements with respect to K^m, $m \in \mathbb{Z}\backslash\{1\}$.

Sketch of proof. This proposition can be proven by explicitly calculating $\Delta(K^m), \Delta(E^n)$ and $\Delta(F^n)$ with lemma 5.7, where $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$. Then use this to calculate $\Delta(X)$, where $X \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ is taken arbitrary and written out in the PBW basis of lemma 4.4. That is

$$\Delta(X) = \sum_{k,m,n} c_{kmn}\Delta(K^k)\Delta(F^m)\Delta(E^n).$$

Compare this outcome with the required definition of group-like and primitive elements to find conditions for k, m and n. □

4.2. **Representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$.** Let us now turn to the representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. From now on, we will assume that q is not a root of unity, i.e. $q^m \neq 1$ for $m \in \mathbb{Z}$. Then we have the following classification for the finite-dimensional representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$.

Theorem 4.7. *For each dimension $2l+1$, $l \in \frac{1}{2}\mathbb{Z}_+$, there are four inequivalent irreducible representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. There exists a basis $\{e^l_{-l}, e^l_{-l+1}, \ldots, e^l_{l-1}, e^l_l\}$ such that these representations t^l_λ are given by*

$$\begin{aligned} t^l_\lambda(K)e^l_n &= \lambda q^{-n}e^l_n, \quad t^l_\lambda(K^{-1}) = \lambda^{-1}q^n e^l_n, \quad t^l_\lambda(F)e_n = e^l_{n+1}, \\ t^l_\lambda(E)e^l_n &= \frac{q^{2l+1}\lambda^2(1-q^{-2n-2l}) + q^{-2l-1}\lambda^{-2}(1-q^{2n+2l})}{(q-q^{-1})^2}e^l_{n-1}, \end{aligned} \tag{4.3}$$

where $\lambda \in \{1,-1,i,-i\}$ and $e^l_{l+1} = 0 = e^l_{-l-1}$.

Proof. Let t be an irreducible representation of V, where $\dim(V) = 2l+1$. Also, let μ be an eigenvalue of $t(K)$ for the eigenvector $v \in V$. We will show there is a basis $\{e^l_n\}^l_{n=-l}$ for V which can be 'laddered through' by $t(E)$ and $t(F)$. That is,

$$t(E)e^l_n = c_n e^l_{n-1} \quad \text{and} \quad t(F)e^l_n = d_n e^l_{n+1}.$$

From the commutation relation $KE = qEK$, we obtain

$$t(K)t(E)v = q\ t(E)t(K)v = q\mu\ t(E)v.$$

Therefore, $t(E)^jv$ is an eigenvector of $t(K)$ with eigenvalue $q^j\mu$. Because q is not a root of unit, these eigenvalues are all distinct and $\{t(E)^jv\}_{j\geq 0}$ are linearly independent. Since the dimension of V is finite, there is an integer $j \geq 1$ such that $t(E)^jv = 0$. Let us take $e^l_{-l} = t^l(E)^{j-1}v$ and

[15]This theorem actually works for any universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$.

$e^l_{-l+k} = t(F)^k e^l_{-l}$. The intuition here is that e^l_{-l} is the lowest basis vector on the ladder and we can use $t(F)$ to climb up. We claim that

$$t(E)e^l_{-l} = 0, \quad t(K)e^l_n = \lambda q^{-n} e^l_n \quad \text{and} \quad t(F)^{2l+1}e^l_{-l} = 0, \tag{4.4}$$

where $\lambda = q^{-l+j-1}\mu$ is chosen such that $t(K)e^l_0 = \lambda e^l_0$, as we will see later. The first claim follows easily from the fact that $t(E)^j v = 0$ and the definition of e^l_{-l}. Similarly as we did for $t(E)$, we can use the commutation relation $KF = q^{-1}FK$ to prove the second claim of (4.4),

$$t(K)e^l_n = t(K)t(F)^{n+l}e^l_{-l} = q^{-n-l}t(F)^{n+l}t(K)e^l_{-l} = \mu q^{-n-l+j-1}e^l_n = \lambda q^{-n}e^l_n, \tag{4.5}$$

where we used that e^l_{-l} is an eigenvector of $t(K)$ with eigenvalue μq^{j-1}. We can also see from (4.5) that $\{e^l_n\}_{n\geq -l}$ are linearly independent, since they have different eigenvalues. Since $\dim(V) = 2l+1$, there must be a $N \leq 2l$ such that

$$e_{-l+N} \neq 0 \quad \text{and} \quad e_{-l+N+1} = 0.$$

Observe that by the last commutation relation of (1.3) and $t(E)e^l_{-l} = 0$,

$$W = \text{span}\{t(F)^k e^l_{-l}, k = 0, .., N\}$$

is an non-zero invariant subspace of V. Therefore, $W = V$ by the irreducibility of t. Thus $N = 2l$, which proves the third equation of (4.4).

Let us calculate the action of $t(E)$ on e^l_n. Using $KE = qEK$, we can see that $t(E)e^l_n$ is an eigenvector of $t(K)$ with eigenvalue $q^{-(n-1}$,

$$t(K)t(E)e^l_n = q^{-(n-1)}t(E)e^l_n.$$

Therefore, $t(E)e^l_n$ must be a multiple of e^l_{n-1}, i.e.

$$t(E)e^l_n = c_n e^l_{n-1}.$$

The commutation relation

$$EF - FE = \frac{K^2 - K^{-2}}{q - q^{-1}},$$

implies

$$c_{n+1} - c_n = \frac{\lambda^2 q^{-2n} - \lambda^{-2}q^{2n}}{q - q^{-1}} \quad \text{and} \quad c_l = \frac{\lambda^{-2}q^{2l} - \lambda^2 q^{-2l}}{q - q^{-1}}.$$

Since we need $c_{-l} = 0$, we have

$$c_n = \frac{q^{2l+1}\lambda^2(1 - q^{-2n-2l}) + q^{-2l-1}\lambda^{-2}(1 - q^{2n+2l})}{(q - q^{-1})^2}.$$

Combining this with our condition for c_l, we obtain $\lambda^4 = 1$. The four different solutions for λ all lead to a different spectrum for $t(K)$. This shows that these representations are inequivalent, proving the theorem. □

It turns out that we have the isomorphism $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C})) \cong \mathcal{U}_{q^{-1}}(\mathfrak{sl}(2,\mathbb{C}))$. Therefore, without loss of generality we can take $|q| \leq 1$ from now on. In fact, this is the only non-trivial case when $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{U}_p(\mathfrak{sl}(2,\mathbb{C}))$ are isomorphic.

Theorem 4.8. *$\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C})) \cong \mathcal{U}_p(\mathfrak{sl}(2,\mathbb{C}))$ as Hopf algebras if and only if $p = q^{-1}$ or $p = q$.*

Proof. Let $\phi : \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C})) \to \mathcal{U}_p(\mathfrak{sl}(2,\mathbb{C}))$ be a Hopf algebra isomorphism, then we need to have

$$(\phi \otimes \phi) \circ (\Delta_q(X)) = \Delta_p(\phi(X)),$$

where Δ_q and Δ_p are the comultiplication of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{U}_p(\mathfrak{sl}(2,\mathbb{C}))$ respectively. Consequently, if X is a group-like element, we must have $\phi(X) \otimes \phi(X) = \Delta_p(\phi(X))$. If Y is twisted primitive with respect to X, we get

$$\phi(X) \otimes \phi(Y) + \phi(Y) \otimes S_p(\phi(X)) = \Delta_p(\phi(Y)),$$

where S_p is the antipode of $\mathcal{U}_p(\mathfrak{sl}(2,\mathbb{C}))$. Thus ϕ maps

(i) group-like elements to group-like elements,

(ii) twisted primitive elements with respect to X to twisted primitive elements with respect to $\phi(X)$.

Since K^m, $m \in \mathbb{Z}$ are the only group-like elements, we must have $\phi(K^m) = K^{\tau(m)}$, where τ is a bijection of $\mathbb{Z}$. Because ϕ is an algebra homomorphism, τ must be linear as well. Then, the only two possibilities are $\tau(1) = 1$ and $\tau(1) = -1$. In the latter case we have $\phi(K) = \phi(K^{-1})$. Since ϕ maps twisted primitive elements with respect to K to twisted primitive elements with respect to $\phi(K) = K^{-1}$, proposition 4.6(ii), (iii) shows that ϕ maps the three-dimensional space $\text{span}\{E, F, K - K^{-1}\}$ to the one-dimensional space $\text{span}\{K^{-1} - K\}$. This contradicts ϕ being an isomorphism, thus we must have $\phi(K^m) = K^m$.
Now, the map $L : X \to KXK^{-1}$ maps the space of twisted primitive elements with respect to K onto itself. Since ϕ is an algebra homomorphism, we have

$$L(\phi(X)) = K\phi(X)K^{-1} = \phi(KXK^{-1}) = \phi(L(X)).$$

Suppose X is an eigenvector of L with eigenvalue λ. Then $L(\phi(X)) = \phi(L(X)) = \lambda\phi(X)$. Thus X and $\phi(X)$ have the same eigenvalue. Consequently, L has the same spectrum in $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ as in $\mathcal{U}_p(\mathfrak{sl}(2,\mathbb{C}))$. The three-dimensional space of twisted primitive elements in $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ has three eigenvectors with eigenvalues $\{q, q^{-1}, 1\}$ given by

$$KEK^{-1} = qE, \quad KFK^{-1} = q^{-1}F, \text{ and } K(K - K^{-1})K^{-1} = K - K^{-1}.$$

Similarly, in $\mathcal{U}_p(\mathfrak{sl}(2,\mathbb{C}))$ the space of twisted primitive elements has eigenvalues $\{p, p^{-1}, 1\}$. Therefore, $p = q$ or $p = q^{-1}$, proving the 'only if' part.
For the other direction, we will show that both cases $p = q$ and $p = q^{-1}$ have algebra isomorphisms. If $p = q$, we have an isomorphism ϕ defined on the generators by $\phi(K) = K$, $\phi(E) = \lambda E$ and $\phi(F) = \lambda^{-1}F$ for some non-zero $\lambda \in \mathbb{C}$. If $p = q^{-1}$, we can take $\phi(K) = K$, $\phi(E) = \lambda F$ and $\phi(F) = \lambda^{-1}E$. One can verify easily that these are indeed Hopf algebra isomorphisms, which proves the statement. □

4.3. **$*$-structures on $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$.** It is of interest to put a $*$-structure on our Hopf algebra $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$, since we can then come into the setting of Hilbert spaces instead of ordinary linear spaces. The tools of Hilbert spaces, e.g. inner products, self-adjointness and orthogonality, will be used to connect Hopf algebras and orthogonal polynomials. For that to happen, we want to carry over the $*$-structure of our Hopf $*$-algebra to the representation.

Definition 4.9. *A representation t of a Hopf $*$-algebra A on a Hilbert space H is called a $*$-representation if $t(X^*)$ is the adjoint operator of $t(X)$ in H for all $X \in A$. That is,*

$$\langle t(X)v, w\rangle_H = \langle v, t(X^*)w\rangle_H \quad \textit{for all } X \in A \textit{ and } v, w \in H,$$

where $\langle \cdot, \cdot \rangle_H$ is the inner product on H.

If we can make $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ into a Hopf $*$-algebra and find a $*$-representation, we can use the $*$-structure of the algebra to analyse the resulting linear operators coming from the $*$-representation.

There are three inequivalent $*$-structures of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. Two $*$-structures on a Hopf algebra are equivalent if there exists a Hopf algebra isomorphism ϕ of the Hopf algebra onto itself that intertwines the two $*$-structures. That is, $*$ and $\dagger$ are equivalent $*$ structures if

$$\phi(X^*) = \phi(X)^\dagger, \quad X \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C})).$$

Theorem 4.10. *The possible inequivalent $*$-structures, also called real forms, on $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ are*

(i) $|q| = 1$; $K^* = K$, $E^* = -E$, $F^* = -F$, named $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$,

(ii) $-1 < q < 1$; $K^* = K$, $E^* = F$, $F^* = E$, named $\mathcal{U}_q(\mathfrak{su}(2))$.

(iii) $-1 < q < 1$; $K^* = K$, $E^* = -F$, $F^* = -E$, named $\mathcal{U}_q(\mathfrak{su}(1,1))$.

Proof. One can easily verify that $*$ maps $1 \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ to itself by applying $*$ to $1X = X = X1$. If we apply $*$ to the structure relations (1.3) of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$, we obtain $(K^{-1})^* = (K^*)^{-1}$ and

$$E^*K^* = \overline{q}K^*E^*, \quad F^*K^* = \overline{q}^{-1}K^*F^*, \quad F^*E^* - E^*F^* = \frac{(K^*)^2 - (K^*)^{-2}}{\overline{q} - \overline{q}^{-1}}.$$

Therefore, we can see K^*, E^*, F^* as generators of $\mathcal{U}_{\overline{q}^{-1}}(\mathfrak{sl}(2,\mathbb{C}))$ and $*$ as an antilinear isomorphism between $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{U}_{\overline{q}^{-1}}(\mathfrak{sl}(2,\mathbb{C}))$. Following the proof of theorem 4.8, we see first of all that via the same argument we have $K^* = K$. Moreover, for the map L defined there, we must have unique eigenvectors (up to a constant). Therefore, (i) $E^* = \lambda E$ and $F^* = \lambda^{-1}F$ or (ii) $E^* = \lambda F$ and $F^* = \lambda^{-1}E$ for some nonzero $\lambda \in \mathbb{C}$. In the first case, we have

$$\lambda KE = K^*E^* = \overline{q}^{-1}E^*K^* = \overline{q}^{-1}\lambda EK,$$

thus $q = \overline{q}^{-1}$, which implies $|q| = 1$. Since $(E^*)^* = E$, we need $|\lambda| = 1$. These all lead to an equivalent[16] $*$-structure, $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. In case (ii), we can argue via similar reasoning and obtain $q = \overline{q}$, which implies $q \in \mathbb{R}$. Again, using $(E^*)^* = E$ we find $\lambda \in \mathbb{R}$. Which gives two inequivalent $*$-structures: $\mathcal{U}_q(\mathfrak{su}(2))$ where $\lambda > 0$ and $\mathcal{U}_q(\mathfrak{su}(1,1))$ where $\lambda < 0$. $\square$

Remark 4.11. This is a point where the quantum setting differs from the classical $\mathfrak{sl}(2,\mathbb{C})$, which has only two possible inequivalent $*$-structures. Since $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{su}(1,1)$, they have the same representation theory, something which is not longer true in the quantum case.

The real form $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ is of less interest for our purposes since the condition $|q| = 1$ does not correspond with the setting for Askey-Wilson polynomials, where $0 < q < 1$. The third $*$-structure, $\mathcal{U}_q(\mathfrak{su}(1,1))$, only has infinite-dimensional $*$-representations and will give rise to continuous orthogonal polynomials (e.g. [6]). In this thesis we will focus on the finite-dimensional representations of the second real form $\mathcal{U}_q(\mathfrak{su}(2))$. Since SU(2) is a compact group, $\mathcal{U}_q(\mathfrak{su}(2))$ is sometimes called the compact real form of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. The polynomials we will find depend on q^2, therefore we can only consider the case $0 < q < 1$ without much loss of generality.

Let us now obtain a $*$-representation from (4.3). Because $K^* = K$, the only way to make this into a $*$-representation of $\mathcal{U}_q(\mathfrak{su}(2))$ is to choose $\lambda = \pm 1$. We will focus on the case $\lambda = 1$; or said differently, when the spectrum of K, $\sigma(K)$, is contained in $q^{\frac{1}{2}\mathbb{Z}}$.

Theorem 4.12. *For $l \in \frac{1}{2}\mathbb{Z}_+$, the unique $(2l+1)$-dimensional $*$-representation t^l of $\mathcal{U}_q(\mathfrak{su}(2))$ such that $\sigma(K) \subset q^{\frac{1}{2}\mathbb{Z}}$, is given by*

$$t^l(K)e^l_n = q^{-n}e^l_n, \qquad t^l(K^{-1})e^l_n = q^n e^l_n,$$

$$t^l(E)e^l_n = \frac{\sqrt{(q^{-l+n-1} - q^{l-n+1})(q^{-l-n} - q^{l+n})}}{q^{-1} - q}e^l_{n-1},$$

$$t^l(F)e^l_n = \frac{\sqrt{(q^{-l+n} - q^{l-n})(q^{-l-n-1} - q^{l+n+1})}}{q^{-1} - q}e^l_{n+1},$$

where $(e^l_n)^l_{n=-l}$ is the orthonormal basis for $\mathbb{C}^{2l+1}$ and $e_{l+1} = 0 = e_{-l-1}$.

Remark 4.13. When l is fixed and the meaning is clear, we just write $e_n := e^l_n$.

Proof. Take $l \in \frac{1}{2}\mathbb{Z}_+$ and let t^l denote the $(2l+1)$-dimensional representation from (4.3) where $\lambda = 1$. Since $K^* = K$ and $q \in \mathbb{R}$ we already have $t^l(K^*) = t^l(K)^*$. We will rescale our inner product such that $t^l(E)^* = t^l(E^*)(= t^l(F))$. Taking the adjoint on both sides then automatically gives $t^l(F^*) = t^l(F)^*$ as well. We rescale by preserving orthogonality, but changing the norm of the basis vectors. That is, we define

$$d(n)e'_n = e_n, \quad d(n) \in \mathbb{R}$$

[16] If $E^* = e^{i\theta_1}E$ and $E^\dagger = e^{i\theta_2}E$, take the intertwiner generated by $\phi(K) = K$, $\phi(E) = e^{\frac{1}{2}i(\theta_1-\theta_2)}E$ and $\phi(F) = e^{-\frac{1}{2}i(\theta_1-\theta_2)}F$.

and
$$\langle e'_m, e'_n\rangle = \delta_{mn}.$$
We need to calculate for which constants $d(n)$ we have a $*$-representation. We have
$$\left\langle t^l(E)^* e_n, e_m\right\rangle = \left\langle e_n, t^l(E)e_m\right\rangle = \langle e_n, C(m)e_{m-1}\rangle = C(n+1)d(n)^2\delta_{n,m-1},$$
where
$$C(n) = \frac{q^{2l+1}(1-q^{-2n-2l}) + q^{-2l-1}(1-q^{2n+2l})}{(q-q^{-1})^2}$$
is the constant from $t^l(E)$ in (4.3). On the other hand, we have
$$\left\langle t^l(E^*)e_n, e_m\right\rangle = \left\langle t^l(F)e_n, e_m\right\rangle = d(n+1)^2\delta_{n+1,m}.$$
Therefore, we have a $*$-representation if
$$C(n+1)d(n)^2 = d(n+1)^2, \quad \text{for } n = -l, -l+1.., l-1.$$
Observe that for $0 < q < 1$ we have $C(n) > 0$ and
$$\sqrt{C(n)} = \frac{\sqrt{(q^{-l+n-1} - q^{l-n+1})(q^{-l-n} - q^{l+n})}}{q^{-1} - q},$$
$$\sqrt{C(n+1)} = \frac{\sqrt{(q^{-l+n} - q^{l-n})(q^{-l-n-1} - q^{l+n+1})}}{q^{-1} - q},$$
which are exactly the constants in the statement of this theorem for the operators $t^l(E)$ and $t^l(F)$ respectively. Now, if we take $d(n)$ such that
$$\frac{d(n+1)}{d(n)} = \sqrt{C(n+1)},$$
we have an irreducible $*$-representation of $\mathcal{U}_q(\mathfrak{su}(2))$. Taking $d(-l) = 1$ and using the recursive formula above, gives
$$d(n) = \prod_{k=-l+1}^{n} \sqrt{C(k)}.$$
Therefore,
$$t^l(E)e'_n = \frac{d(n-1)}{d(n)}C(n)e'_{n-1} = \sqrt{C(n)}e'_{n-1},$$
and
$$t^l(F)e'_n = \frac{d(n+1)}{d(n)}e'_{n+1} = \sqrt{C(n+1)}e'_{n+1},$$
as desired. □

In an irreducible representation, the Casimir element always acts as a scalar. Let us calculate the value of that scalar for the Casimir Ω defined in 1.4. We have
$$t^l(\Omega)e_n = \frac{q^{-2n-1} + q^{2n+1}}{(q-q^{-1})^2}e_n + \frac{(q^{-l+n} - q^{l-n})(q^{-l-n-1} - q^{l+n+1})}{(q-q^{-1})^2}e_n = \frac{q^{-2l-1} + q^{2l+1}}{(q-q^{-1})^2}e_n. \quad (4.6)$$
Observe that the value of the Casimir depends on the dimension of the representation.

5. Orthogonal polynomials

One of the reasons to study representations of quantum groups is because of the intimate connection with orthogonal polynomials. Using these representations, one can prove important properties of these polynomials such as symmetry and orthogonality relations. That is why we go over some basic theory of orthogonal polynomials that is necessary for our purposes. No in-depth theory of orthogonal polynomials is required. Also, we will look at orthogonal polynomials on the real line only.

In this section, we will introduce orthogonal polynomials and look at important classical and non-classical examples. Before starting, I want to emphasise the crucial role of Askey-Wilson polynomials. These are the most general orthogonal polynomials in one variable we explicitly know. They are non-classical polynomials, but all classical polynomials, such as Jacobi polynomials, are special cases of them by a transformation of parameters and then taking the limit of $q \to 1$. That is why these Askey-Wilson polynomials are of great interest and looking for them in representation theory has been of high importance.

5.1. **Orthogonal polynomials: general theory.**

Let us now start with the definitions and notation required for orthogonal polynomials. Denote by p_n a polynomial of degree n with real coefficients. That is

$$p_n(x) = \sum_{k=0}^{n} a_k x^n,\ a_k \in \mathbb{R},\ a_n \neq 0.$$

Roughly said, polymonials $p_n, p_m : \mathbb{R} \to \mathbb{R}$ are orthogonal with respect to a certain measure μ on $\mathbb{R}$ if

$$\int_{\mathbb{R}} p_n(x) p_m(x)\ \mathrm{d}\mu(x) = 0.$$

when $n \neq m$. For example, the Legendre polynomials are orthogonal with respect to the Lebesgue measure on $[-1,1]$. Often, the measure μ can be written as $w(x)\ \mathrm{d}x$, where $w(x)$ is a weight function and $\mathrm{d}x$ the Lebesgue measure on some interval of the real line. E.g, let p_n, p_m be Chebyshev polynomials, then

$$\int_{-1}^{1} p_n(x) p_m(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}} = h_n \delta_{nm},$$

where $\mathrm{d}x$ is the Lebesgue measure on $[-1,1]$ and h_n is a constant depending on n. Let us now formally introduce this, where we remind the reader that technicalities involving measures and Hilbert spaces will not play a role in this thesis. However, we still use this definition, because we do want to be mathematically correct.

Definition 5.1. *Let μ be a nonnegative Borel measure on $\mathbb{R}$ such that* $\mathrm{supp}(\mu)$ *contains at least a countably infinite number of points and all monomials are integrable, i.e.*

$$\int_{\mathbb{R}} x^n \mathrm{d}\mu(x) < \infty \text{ for all } n \in \mathbb{Z}_+.$$

This measure induces an inner product for polynomials $p_n, p_m : \mathbb{R} \to \mathbb{R}$ defined by

$$\langle p_n, p_m \rangle = \int_{\mathbb{R}} p_n(x) p_m(x)\, \mathrm{d}\mu(x).$$

*Polynomials $\{p_n\}_{n \in \mathbb{Z}_+}$, are called **orthogonal** if*

$$\langle p_n, p_m \rangle = h_n \delta_{nm}$$

*for constants h_n. If for all $n \in \mathbb{Z}_+$ we have $h_n = 1$, the polynomials are called **orthonormal**.*

Remark 5.2. Orthogonal polynomials can be made orthonormal easily. Let $\{P_n\}_{n \in \mathbb{Z}_+}$ be polynomials that satisfy the orthogonality relation

$$\int_{\mathbb{R}} P_n(x) P_m(x)\, \mathrm{d}\mu(x) = \delta_{mn} h_n.$$

Then $\{p_n\}_{n\in\mathbb{Z}_+}$, defined by

$$p_n(x) = \sqrt{\frac{h_0}{h_n}}\frac{P_n(x)}{P_0},$$

are orthonormal w.r.t the normalised measure

$$\mathrm{dm}(x) = P_0^2\frac{\mathrm{d}\mu(x)}{h_0}.$$

Often, orthogonal polynomials will be defined such that $P_0 = 1$, which simplifies above formulas.

Orthogonal polynomials are uniquely determined by the measure up to a scalar depending on the degree n. There are several ways of describing the same orthogonal polynomials. One important characterization is the three-term recurrence relation.

Theorem 5.3. *Let $\{p_n\}_{n\in\mathbb{Z}_+}$ be a set of orthogonal polynomials with respect to μ. Then there exist $A_n, B_n, C_n \in \mathbb{R}$ such that*

$$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x),$$

where we use the convention $p_{-1}(x) = 0$.

Proof. Take p_n arbitrary. Observe that $\{p_k\}_{k=0}^n$ are linearly independent and thus a linear basis for polynomials of degree $\leq n$. Therefore, we have[17]

$$\int_{\mathbb{R}} p_n(x)x^k \,\mathrm{d}\mu(x) = 0 \text{ if } k = 0, .., n-1. \tag{5.1}$$

Moreover, there exists constants b_k such that $xp_n(x) = \sum_{k=0}^{n+1} b_k p_k(x)$. Hence we have

$$\int_{\mathbb{R}} xp_n(x)p_k(x)\,\mathrm{d}\mu(x) = b_k\int_{\mathbb{R}} p_k(x)^2\,\mathrm{d}\mu(x) = b_k h_k,$$

where $h_k > 0$. By (5.1), the LHS is 0 for $k = 0, .., n-2$. Since $h_k \neq 0$, we have $b_k = 0$ for those k. We conclude that

$$xp_n(x) = b_{n+1}p_{n+1}(x) + b_n p_n(x) + b_{n-1}p_{n-1}(x).$$ □

Remark 5.4. The connection of orthogonal polynomials with representations of quantum groups, which are linear operators, often comes from this three-term recurrence relation. The following will be explained in more detail later. The idea is that the matrix representing elements of a (quantum) group will be a self-adjoint tridiagonal matrix. Then this matrix will correspond to a three-term recurrence relation acting on the linear space of polynomials. The eigenvalues will be the input for the variable of the polynomials. Moreover, the eigenvectors of a self-adjoint operator form an orthogonal basis for the linear space. This orthogonality will correspond with the orthogonality of the polynomials.

On the other hand, a three-term recurrence relation where $A_n \neq 0$ together with an initial condition $p_0(x) = a_0$ defines a set of polynomials. One can then ask if there always exists a measure such that these polynomials are orthogonal. The answer is yes and is called Favard's Theorem. Since we won't not need that theorem in this thesis and the proof is quite complex[18], we will not show it here. A sketch of this proof can be found in [10].

The above can be done as well if the measure μ has finite support. Then we will have a finite set of orthogonal polynomials. In this thesis, we will cross paths with some of these polynomials as well. A finite set of orthogonal polynomials still has a three-term recurrence relation, but the A_n will vanish at some point. The orthogonality will have the form

$$\sum_{k=0}^{N} w(x_k)p_n(x_k)p_m(x_k) = h_n\delta_{mn},$$

where $w(x_k)$ is the weight function corresponding to the measure with finite support.

[17] Actually, (5.1) is an equivalent definition for orthogonal polynomials.

[18] It uses the Spectral theorem to create a projection valued Borel measure on $\mathbb{R}$.

5.2. **Hypergeometric and q-hypergeometric functions.** Many special functions such as the exponential, Beta and Gamma function, as well as (orthogonal) polynomials can be defined in terms of *(q-)hypergeometric series*. Before introducing those, we first need some classical notation and their corresponding quantum version. Many of the q-analogues of classical formulas and functions are based on the observation that

$$\lim_{q\to 1} \frac{1-q^a}{1-q} = a,$$

using l'Hopital's rule.

Definition 5.5. *Define the* ***shifted factorial*** *by*

$$(a)_n = a(a+1)\cdots(a+n-1), \quad n \in \mathbb{Z}_+$$

Note that $(1)_n = n!$*. The product of* k *shifted factorials* $(a_i)_n$ $(i = 1, ..., k)$ *is notated by* $(a_1, \ldots, a_k)_n$*. The* ***binomial coefficient*** *is defined by*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1)_n}{(1)_k (1)_{n-k}} \left(= \begin{bmatrix} n \\ n-k \end{bmatrix} \right).$$

Let $q \in \mathbb{C}\backslash\{1\}$*, the* ***q-shifted factorial*** *is defined by*

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}).$$

Furthermore, $(a_1, .., a_k; q)_n$ *is the notation for the product of* k q*-shifted factorials* $(a_i;q)_n$ $(i = 1, ..., k)$*. The* ***q-binomial coefficient*** *is defined by*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \left(= \begin{bmatrix} n \\ n-k \end{bmatrix}_q \right)$$

We define the empty product to be 1*.*

Remark 5.6. Using l'Hopital's rule, we obtain

$$\lim_{q\to 1} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n, \tag{5.2}$$

$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}. \tag{5.3}$$

Which motivates the names of the q-extensions of the classical ones.

Similar to the classical binomial formula

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k},$$

we have the following quantum analogue.

Lemma 5.7 (q-Binomial formula)**.** *Let* x, y *be elements of an associative algebra that satisfy* $xy = qyx$*. Then for* $n \in \mathbb{Z}_+$ *we have,*

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} x^k y^{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k x^{n-k}.$$

Sketch of proof. A straightforward calculation shows that

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}.$$

Then use induction to show the required formula. □

Now we are ready to define (q-)hypergeometric series.

Definition 5.8. *For $r, s \in \mathbb{Z}_+$, and $a_1, ..., a_r, b_1, ..., b_s, x \in \mathbb{C}$, the hypergeometric series ${}_rF_s$ is defined by*

$$ {}_rF_s \left[\begin{matrix} a_1, .., a_r \\ b_1, ..., b_s \end{matrix} ; x \right] = \sum_{n=0}^{\infty} \frac{(a_1, ..., a_r)_n}{(b_1, ..., b_s)_n} \frac{x^n}{n!}. $$

The q-hypergeometric series ${}_r\phi_s$ is defined by

$$ {}_r\phi_s \left[\begin{matrix} a_1, .., a_r \\ b_1, ..., b_s \end{matrix} ; q, x \right] = \sum_{n=0}^{\infty} \frac{(a_1, ..., a_r; q)_n}{(b_1, ..., b_s; q)_n} \frac{\left((-1)^n q^{n(n-1)/2}\right)^{1+s-r} x^n}{(q; q)_n}. $$

Remark 5.9. When dealing with q-hypergeometric series, the standard assumption is $0 < q < 1$. Then the following limit of q-shifted factorials,

$$ (a; q)_\infty := \lim_{n \to \infty} (a; q)_n, $$

is well defined. Indeed, let $s_n = aq^k$. Then the infinite product

$$ \prod_{k=0}^{\infty} (1 - s_n) $$

converges if

$$ \sum_{k=0}^{\infty} |s_n| = |a| \sum_{k=0}^{\infty} q^k $$

is convergent.

Remark 5.10. As said before, (q-)hypergeometeric series generalise many well known functions. For example, we have

$$ e^x = {}_0F_0 \left[\begin{matrix} - \\ - \end{matrix} ; x \right]. $$

Remark 5.11. Of course, one should look at well-definedness and the radius of convergence of the (q-)hypergeometeric series.

(i) Observe that both $(-k)_n = 0$ and $(q^{-k}; q) = 0$ when $k \in \mathbb{Z}_+$ and $n > k$. Therefore, the hypergeometric series terminates after $n + 1$ terms if for one of the a_i we have $a_i = -n$, $n \in \mathbb{Z}_+$. Similarly, the q-hypergeometric series terminates after n terms if for one of the a_i we have $a_i = q^{-n}$.

(ii) In general, we assume when dealing with hypergeomteric series that b_i is not of the form $-k$ and $b_i \neq q^{-k}$ when dealing with q-hypergeomtric series, where $k \in \mathbb{Z}_+$. The only time when this could be well defined is when the series have terminated before the k-th term.

(iii) When we do not have a terminating series and $0 < |q| < 1$, the radius of convergence R for x can be calculated by the ratio test. We have

$$ R = \begin{cases} \infty \text{ if } r < s + 1, \\ 1 \text{ if } r = s + 1, \\ 0 \text{ if } r > s + 1. \end{cases} $$

.

Remark 5.12. When the series is well defined, the q-hypergeometric series and its classical version are (formally) linked via the limit

$$ \lim_{q \to 1} {}_r\phi_s \left[\begin{matrix} q^{a_1}, ..., q^{a_r} \\ q^{b_1}, ..., q^{b_s} \end{matrix} ; q, (q-1)^{1+s-r} x \right] = {}_rF_s \left[\begin{matrix} a_1, .., a_r \\ b_1, ..., b_s \end{matrix} ; x \right]. $$

This limit is the link between the classical polynomials and their q-analogues.

Remark 5.13. In practice, we will often be dealing with q-hypergeometric series of the form ${}_{r+1}\phi_r$. Then the expression simplifies to

$$ {}_{r+1}\phi_r \left[\begin{matrix} a_1, .., a_{r+1} \\ b_1, ..., b_r \end{matrix} ; q, x \right] = \sum_{n=0}^{\infty} \frac{(a_1, ..., a_{r+1}; q)_n}{(b_1, ..., b_r; q)_n} \frac{x^n}{(q; q)_n}. $$

There are many formulas relating hypergeometric series. They can often be generalised to a q-analogue. We will not need those in these thesis, but I want to point out the existence of these formulas to give a better overview of q-hypergeometric series. These formulas once again show the intimiate link between the classical and non-classical theory. Also, they play a crucial role in explicitly calculating and relating (q-)hypergeometric series and orthogonal polynomials. For example, we have the classical Euler's transformation formula:

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] = (1-x)^{c-a-b}\, {}_2F_1\left[\begin{matrix} c-a, c-b \\ c \end{matrix}; x\right] $$

and its q-anologue

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, x\right] = \frac{(abc^{-2}x;q)_\infty}{(z;q)_\infty}\, {}_2\phi_1\left[\begin{matrix} a^{-1}c, b^{-1}c \\ c \end{matrix}; q; \frac{abx}{c}\right]. $$

5.3. **Hypergeometric orthogonal polynomials.** In the coming two subsection we will use [9] for the definitions and properties of the polynomials. The first types of orthogonal polynomials discovered are called classical. The Jacobi polynomials are the most famous and general of this type. Other classical polynomials such as Legendre, Chebyshev and Hermite are special cases of Jacobi polynomials. Later on, research was done in hypergeometric series. Classical orthogonal polynomials can be written as hypergeometric series, but also more general ones were discovered. The Wilson polynomials are the most general hypergeometric orthogonal polynomials we know. Although these hypergeometric polynomials will not play a large role in these notes, we will still briefly discuss two important examples to give the reader an idea of them, as the theory for q-hypergeomatric orthogonal polynomials emerged from them.
As said above, the most well known classical orthogonal polynomials are the Jacobi polynomials. Orthogonal polynomials have many important characteristics. We will look at their hypergeometric definition, orthogonality relation and three-term recurrence formula.

Definition 5.14. *The* ***Jacobi*** *polynomials are defined by*

$$ P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left[\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-x}{2}\right]. $$

Orthogonality. *Let $\alpha, \beta > -1$, then the following relation holds,*

$$ \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\ \mathrm{d}x = h_n \delta_{mn}, $$

where

$$ h_n = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}, $$

and Γ is the Gamma function.

Some well known orthogonal polynomials are special cases of Jacobi polynomials. For example, the Chebyshev polynomials can be obtained by taking $\alpha = \beta = -\frac{1}{2}$ and Legendre polynomials by taking $\alpha = \beta = 0$.

The most general hypergeometric orthogonal polynomials are the Wilson polynomials, which have 4 parameters.

Definition 5.15. *The* ***Wilson*** *polynomials are defined by*

$$ W_n(x^2; a, b, c, d) = (a+b)_n (a+c)_n (a+d)_n\, {}_4F_3\left[\begin{matrix} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix}; 1\right]. $$

Orthogonality. *Let $a, b, c, d > 0$, then the following relation holds,*

$$ \frac{1}{2\pi} \int_0^\infty \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)} \right|^2 W_m(x^2; a, b, c, d) W_n(x^2; a, b, c, d)\ \mathrm{d}x $$
$$ = h_n \delta_{mn}, $$

where

$$h_n = \frac{\Gamma(n+a+b)\cdots\Gamma(n+c+d)}{\Gamma(2n+a+b+c+d)}(n+a+b+c+d-1)_n n!,$$

and

$$\Gamma(n+a+b)\cdots\Gamma(n+c+d) \\ = \Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+a+d)\Gamma(n+b+c)\Gamma(n+b+d)\Gamma(n+c+d).$$

Remark 5.16. Jacobi polynomials can be found from Wilson polynomials by taking $a = b = \frac{1}{2}(\alpha+1),\ c = \frac{1}{2}(\beta+1)+it,\ d = \frac{1}{2}(\beta+1)-it$ and $x \to t\sqrt{\frac{1}{2}(1-x)}$. Then take the limit $t \to \infty$.

5.4. **q-Hypergeometric orthogonal polynomials.** In this thesis, we will be primarily concerned with the most general non-classical polynomials, the Askey-Wilson polynomials. They generalise all non-classical as well as classical polynomials. They are the q-analogue of the Wilson polynomials given in definition 5.15.

Definition 5.17. *The* ***Askey-Wilson*** *polynomials are defined by*

$$p_n(x;a,b,c,d|q) = \frac{(ab,ac,ad;q)_n}{a^n}\,{}_4\phi_3\left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix};q;q\right],\ x = \cos(\theta).$$

Orthogonality. *Let $-1 < a,b,c,d < 1$. The Askey-Wilson polynomials satisfy the following orthogonality relation,*

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{w(x)}{\sqrt{1-x^2}}p_m(x;a,b,c,d|q)p_n(x;a,b,c,d|q)\ \mathrm{d}x = h_n\delta_{mn}, \tag{5.4}$$

where

$$w(x) = \frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},ce^{i\theta},ce^{-i\theta},de^{i\theta},de^{-i\theta};q)_\infty},$$

and

$$h_n = \frac{(abcdq^{n-1};q)_n(abcdq^{2n};q)_\infty}{(q^{n+1},abq^n,adq^n,bcq^n,bdq^n,cdq^n;q)_\infty}.$$

We define $\mathrm{d}m(x;a,b,c,d|q)$ to be the normalised Askey-Wilson measure. That is,

$$\int_{\mathbb{R}} p(x)\ \mathrm{d}m(x;a,b,c,d|q) = \frac{1}{2\pi h_0}\int_{-1}^{1}\frac{w(x)}{\sqrt{1-x^2}}p(x)\ \mathrm{d}x \tag{5.5}$$

Remark 5.18. That the Askey-Wilson polynomials are indeed polynomials in the variable $x = \cos(\theta)$ comes from the terms $e^{i\theta}$ and $e^{-i\theta}$ in the ${}_4\phi_3$-series. That is,

$$(ae^{i\theta},ae^{-i\theta};q)_n = \prod_{k=0}^{n-1}(1-ae^{i\theta}q^k)(1-ae^{-i\theta}q^k) = \prod_{k=0}^{n-1}(1-2aq^k\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)+a^2q^{2k}) \\ = \prod_{k=0}^{n-1}(1-2aq^k\cos(\theta)+a^2q^{2k}).$$

Since taking $z = e^{i\theta}$ implies $x = \frac{z+z^{-1}}{2}$, the term $e^{i\theta}$ in the definition is sometimes replaced with just 'z'.

The Askey-Wilson polynomials are also orthogonal on a finite set. In this setting, they go with the name of q-Racah polynomials. They are normally defined in a different way. However, after a transformation of parameters (which we will give as well), the q-Racah polynomials defined below are exactly the same as the Askey-Wilson ones.

Definition 5.19. *The* **q-*Racah*** *polynomials are defined by*

$$R_n(y_j;\alpha,\beta,\gamma,\delta;q) = {}_4\phi_3\left[\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-j}, \gamma\delta q^{j+1} \\ \alpha q, \beta\delta q, \gamma q \end{matrix};q;q\right],\ n = 0,1,2,...,N,$$

where

$$y_j := q^{-j} + \gamma\delta q^{j+1}$$

and

$$\alpha q = q^{-N} \text{ or } \beta\delta q = q^{-N} \text{ or } \gamma q = q^{-N}, \text{ with } N \in \mathbb{Z}_+.$$

Orthogonality. *q-Racah polynomials satisfy the following orthogonality relation,*

$$\sum_{j=0}^{N} w(y_j, \alpha, \beta, \gamma, \delta; q) R_m(y_j) R_n(y_j) = h_n \delta_{mn},$$

where

$$R_n(y_j) = R_n(y_j; \alpha, \beta, \gamma, \delta; q), \tag{5.6}$$

and

$$w(y_j, \alpha, \beta, \gamma, \delta; q) = \frac{(\alpha q, \beta\delta q, \gamma q, \gamma\delta q; q)_j (1 - \gamma\delta q^{2j+1})}{(q, \alpha^{-1}\gamma\delta q, \beta^{-1}\gamma q, \delta q; q)_j (\alpha\beta q)^j (1 - \gamma\delta q)}, \tag{5.7}$$

$$h_n = \frac{(\alpha^{-1}, \beta^{-1}\gamma, \alpha^{-1}\delta, \beta^{-1}, \gamma\delta q^2; q)_\infty}{(\alpha^{-1}\beta^{-1}q^{-1}, \alpha^{-1}\gamma\delta q, \beta^{-1}\gamma q, \delta q; q)_\infty} \frac{(1 - \alpha\beta q)(\gamma\delta q)^n (q, \alpha\beta\gamma^{-1}q, \alpha\delta^{-1}q, \beta q; q)_n}{(1 - \alpha\beta q^{2n+1})(\alpha q, \alpha\beta q, \beta\delta q, \gamma q; q)_n}. \tag{5.8}$$

Recurrence relation. *The three-term recurrence relation is given by,*

$$y_j R_n(y_j) = A_n R_{n+1}(y_j) + B_n R_n(y_j) + C_n R_{n-1}(y_j), \tag{5.9}$$

where

$$\begin{cases} A_n = \dfrac{(1 - \alpha q^{n+1})(1 - \alpha\beta q^{n+1})(1 - \beta\delta q^{n+1})(1 - \gamma q^{n+1})}{(1 - \alpha\beta q^{2n+1})(1 - \alpha\beta q^{2n+2})} \\ B_n = 1 + \gamma\delta q - (A_n + C_n) \\ C_n = \dfrac{q(1 - q^n)(1 - \beta q^n)(\gamma - \alpha\beta q^n)(\delta - \alpha q^n)}{(1 - \alpha\beta q^{2n})(1 - \alpha\beta q^{2n+1})}. \end{cases} \tag{5.10}$$

Remark 5.20. The q-Racah and Askey-Wilson polynomials are linked in the following way. Do the substitution $\alpha = abq^{-1},\ \beta = cdq^{-1},\ \gamma = adq^{-1},\ \delta = ad^{-1}$ and $q^j = a^{-1}e^{-i\theta}$. Then $y_j = 2a\cos\theta$ and

$$R_n(y_j; abq^{-1}, cdq^{-1}, adq^{-1}, ad^{-1}; q) = \frac{a^n}{(ab, ac, ad; q)_n} p_n(x; a, b, c, d|q).$$

A special case of the q-Racah polynomials are the dual q-Krawtchouk polynomials. They will appear in the next section as eigenvectors of twisted primitive elements in the quantum group $\mathcal{U}_q(\mathfrak{su}(2))$.

Definition 5.21. *The* ***dual* q-*Krawtchouk*** *polynomials are defined as*

$$R_n(y_j; c, N|q) = {}_3\phi_2 \left[\begin{matrix} q^{-n}, q^{-j}, cq^{j-N} \\ q^{-N}, 0 \end{matrix} ; q; q \right],\ n = 0, 1, 2, ..., N, \tag{5.11}$$

where

$$y_j = q^{-j} + cq^{j-N}.$$

Orthogonality. *Let $c < 0$, the orthogonality relation is given by*

$$\sum_{j=0}^{N} \frac{(cq^{-N}, q^{-N}; q)_j (1 - cq^{2j-N})}{(q, cq; q)_j (1 - cq^{-N})} c^{-j} q^{j(2N-j)} R_m(y_j) R_n(y_j) = h_n \delta_{mn},$$

where

$$R_n(y_j) = R_n(y_j; c, N|q)$$

and

$$h_n = (c^{-1}; q)_N \frac{(q; q)_n}{(q^{-N}; q)_n} (cq^{-N})^n.$$

Recurrence relation. *The dual q-Krawtchouk satisfy the following three-term recurrence relation,*

$$y_j R_n(y_j) = (1 - q^{n-N} R_{n+1}(y_j) + (q^{-N} + cq^{-N})q^n R_n(y_j) + c(1 - q^n)q^{-N} R_{n-1}(y_j). \qquad (5.12)$$

Remark 5.22. The dual q-Krawtchouk polynomials can be obtained from the q-Racah polynomials by taking $\alpha = \beta = 0$, $\gamma = q^{-N-1}$ and $\delta = c$.

6. Eigenvectors of twisted primitive elements

By proposition 4.6, every twisted primitive element X w.r.t. K can be written as

$$X = a_E E + a_F F + a_\sigma (K - K^{-1}), \tag{6.1}$$

where $a_E, a_F, a_\sigma \in \mathbb{C}$. By theorem 4.12, the matrix representing X will be tridiagonal. If chosen correctly, this will correspond to a three-term recurrence relation for orthogonal polynomials, where the input for the variable are the eigenvalues of $t^l(X)$, where t^l is the $*$-representation from theorem 4.12. A first step is to take a twisted primitive element X that is self-adjoint. Then $t^l(X)$ will have an orthogonal basis of eigenvectors, which can (and will) correspond to orthogonality of polynomials. For reasons that will become clear later, we will look at the 'almost' twisted primitive elements[19] XK and $K^{-1}X$. Note that matrices representing XK and $K^{-1}X$ are also tridiagonal. Using theorem 4.10, self-adjointness in $\mathcal{U}_q(\mathfrak{su}(2))$ requires $a_F = q^{-1}\overline{a_E}$ and $a_\sigma \in \mathbb{R}$. In particular, we look at the often used twisted primitive element

$$X_{\sigma,\theta} = q^{\frac{1}{2}} e^{i\theta} E + q^{-\frac{1}{2}} e^{-i\theta} F - \frac{q^\sigma - q^{-\sigma}}{q - q^{-1}} (K - K^{-1}), \ \sigma \in \mathbb{R}. \tag{6.2}$$

$X_{\sigma,\theta}K$ and $K^{-1}X_{\sigma,\theta}$ are self-adjoint in $\mathcal{U}_q(\mathfrak{su}(2))$ and, most important, we can explicitly calculate the spectrum and eigenvectors of these operators. The tridiagonal matrix representing $X_{\sigma,\theta}K$ will correspond to the three term recurrence relation for the dual q-Krawtchouk polynomials from definition 5.21.

Theorem 6.1. *The self-adjoint operator $t^l(X_{\sigma,\theta}K)$ has an orthonormal basis of eigenvectors for $\mathbb{C}^{2l+1}$ given by*

$$v^{l,j}(\sigma,\theta) = \sum_{n=-l}^{l} v_n^{l,j}(\sigma,\theta) e_n, \quad j \in \{-l, \dots, l\}, \tag{6.3}$$

where the coefficients $v_n^{l,j}(\sigma,\theta)$ are given by

$$c_\sigma^{l,j} (-e^{-i\theta})^{l-n} q^{\sigma(l-n)} q^{\frac{1}{2}(l-n)(l-n-1)} \left(\frac{(q^{4l};q^{-2})_{l-n}}{(q^2;q^2)_{l-n}} \right)^{\frac{1}{2}} R_{l-n}(q^{2j-2l} - q^{-2j-2l-2\sigma}; q^{-2\sigma}, 2l; q^2).$$

The $c_{\sigma,\theta}^{l,j}$ are normalizing constants and R_{l-n} are dual q-Krawtchouk polynomials from definition 5.21. The corresponding eigenvalues $\lambda_j(\sigma)$ are

$$\lambda_j(\sigma) = \frac{q^{-2j-\sigma} - q^{2j+\sigma} + q^\sigma - q^{-\sigma}}{q - q^{-1}}. \tag{6.4}$$

Proof. Let $v = \sum_{n=-l}^{l} a_n e_n^l$, then

$$t^l(X_{\sigma,\theta}K)v(\sigma) = \lambda v$$

implies, using Theorem 4.12, that

$$\begin{aligned} \lambda \sum_{n=-l}^{l} a_n e_n^l = \sum_{n=-l}^{l} q^{-n} a_n \Bigg(& e^{i\theta} q^{\frac{1}{2}} \frac{\sqrt{(q^{-l+n-1} - q^{l-n+1})(q^{-l-n} - q^{l+n})}}{q^{-1} - q} e_{n-1}^l \\ & + e^{-i\theta} q^{-\frac{1}{2}} \frac{\sqrt{(q^{-l+n} - q^{l-n})(q^{-l-n-1} - q^{l+n+1})}}{q^{-1} - q} e_{n+1}^l \\ & - \frac{q^\sigma - q^{-\sigma}}{q - q^{-1}} (q^{-n} - q^n) e_n \Bigg). \end{aligned} \tag{6.5}$$

[19] In the literature, XK and $K^{-1}X$ are often also called twisted primitive elements, which they are strictly speaking not. However, it is a bit annoying (what might be the whole reason) to call them 'almost twisted primitive elements' every time, therefore I will do the same as in the literature and call them 'twisted primitive elements' as well.

Multiplying with $q^{-1}-q$ and regrouping for e_n gives

$$
\begin{aligned}
(q^{-1}-q)\lambda \sum_{n=-l}^{l} a_n e_n^l = \sum_{n=-l}^{l} \Big(& q^{-n-\frac{1}{2}} e^{i\theta} \sqrt{(q^{-l+n}-q^{l-n})(q^{-l-n-1}-q^{l+n+1})}\, a_{n+1} \\
& + q^{-n}(q^{\sigma}-q^{-\sigma})(q^{-n}-q^{n}) a_n \\
& + q^{-n+\frac{1}{2}} e^{-i\theta} \sqrt{(q^{-l+n-1}-q^{l-n+1})(q^{-l-n}-q^{l+n})}\, a_{n-1} \Big) e_n.
\end{aligned}
$$

This gives a relation for each $n \in \{-l,\dots,l\}$, which corresponds to a three term recurrence relation for the dual q-Krawtchouk polynomials. Indeed, if for one of those relations we multiply with $q^{-2l-\sigma}$, substitute $n = l-m$, and put $a_{l-m} = C_{l-m} R_m$, we obtain

$$
\begin{aligned}
((q^{-1}-q)\lambda + q^{\sigma} - q^{-\sigma}) q^{-2l-\sigma} R_m = & e^{-i\theta} q^{-3l+m-\sigma+\frac{1}{2}} \sqrt{(q^{-m-1}-q^{m+1})(q^{-2l+m}-q^{2l-m})} \\
& \times \frac{C_{l-m-1}}{C_{l-m}} R_{m+1} + q^{2m}(q^{-4l}-q^{-4l-2\sigma}) R_m \\
& + e^{i\theta} q^{-3l+m-\sigma-\frac{1}{2}} \sqrt{(q^{-m}-q^{m})(q^{-2l+m-1}-q^{2l-m+1})} \\
& \times \frac{C_{l-m+1}}{C_{l-m}} R_{m-1}.
\end{aligned}
\tag{6.6}
$$

The three-term recurrence relation for the dual q-Krawtchouck polynomials $R_m(y) = R_m(y; c, N; q)$ is given by (5.12):

$$
y_j R_m(y_j) = (1-q^{m-N}) R_{m+1}(y_j) + (q^{-N} + c q^{-N}) q^{m} R_m(y_j) + c(1-q^{m}) q^{-N} R_{m-1}(y_j),
$$

where $y_j = q^{-j} + c q^{j-N}$, $j \in \{0,\dots,N\}$ and $c<0$. Taking $c = -q^{\sigma}$, $N = 2l$ and q^2 instead of q, we obtain

$$
y_j R_m(y_j) = (1-q^{2m-4l}) R_{m+1}(y_j) + (q^{-4l} - q^{-4l-2\sigma}) q^{2m} R_m(y_j) - (1-q^{2m}) q^{-4l-2\sigma} R_{m-1}(y_j),
$$

with $y_j = q^{-2j} - q^{2j-4l-2\sigma}$ and $j \in \{0,\dots,2l\}$. Matching this with (6.6) gives

$$
\frac{C_{l-m-1}}{C_{l-m}} = \frac{1-q^{2m-4l}}{e^{-i\theta} q^{-3l+m-\sigma+\frac{1}{2}} \sqrt{(q^{-m-1}-q^{m+1})(q^{-2l+m}-q^{2l-m})}}, \tag{6.7}
$$

$$
\frac{C_{l-m+1}}{C_{l-m}} = \frac{-(1-q^{2m}) q^{-l-m-\sigma+\frac{1}{2}}}{e^{i\theta} \sqrt{(q^{-m}-q^{m})(q^{-2l+m-1}-q^{2l-m+1})}}, \tag{6.8}
$$

$$
y_j = ((q^{-1}-q)\lambda + q^{\sigma} - q^{-\sigma}) q^{-2l-\sigma}. \tag{6.9}
$$

Working out (6.8) and replacing 'm' by '$m+1$' gives exactly (6.7), which shows consistency of both equations. Working out this recurrence relation for the C_n gives

$$
a_n = c_{\sigma}^{l,j} (-e^{-i\theta})^{l-n} q^{\sigma(l-n)} q^{\frac{1}{2}(l-n)(l-n-1)} \left(\frac{(q^{4l}; q^{-2})_{l-n}}{(q^2; q^2)_{l-n}} \right)^{\frac{1}{2}} R_{l-n}(q^{2j-2l} - q^{-2j-2l-2\sigma}; -q^{2\sigma}, 2l; q^2),
$$

where $n, j \in \{-l,\dots,l\}$ and $c_{\sigma}^{l,j}$ are constants, which do not depend on n, we can choose. We take them such that every eigenvector

$$
v^{l,j}(\sigma,\theta) = \sum_{n=-l}^{l} a_n e_n
$$

has norm one. Equation (6.9) leads to eigenvalues

$$
\lambda_j(\sigma) = \frac{q^{-2j-\sigma} - q^{\sigma+2j} + q^{\sigma} - q^{-\sigma}}{q-q^{-1}}, \quad j \in \{-l,\dots,l\}.
$$

Since $X_{\sigma,\theta}K$ is self adjoint, we know that $\{v^{l,j}_{\sigma,\theta}\}_{j=-l}^{l}$ are orthogonal. $\square$

Remark 6.2. Since we know the spectrum and eigenvectors of $X_{\sigma,\theta}K$, we know the same about the most general self adjoint twisted primitive element by rescaling. That is, we know the spectrum and eigenvalues of

$$XK = q^{\frac{1}{2}}a_E EK + q^{-\frac{1}{2}}\overline{a_E}F + a_\sigma(K^2 - I), \quad \text{for}^{20} a_E \neq 0.$$

Furthermore, our basis vectors e_n can 'absorb' the $(e^{-i\theta})^{-l-n}$ term. Therefore, the θ does not change the properties of our eigenvectors and eigenvalues. That is why in most literature about $\mathcal{U}_q(\mathfrak{su}(2))$ the twisted primitive element

$$X_\sigma := X_{\sigma,\pi/2} = iq^{\frac{1}{2}}E - iq^{-\frac{1}{2}}F - \frac{q^\sigma - q^{-\sigma}}{q - q^{-1}}(K - K^{-1})$$

is analysed. We also define

$$\begin{aligned} v^{l,j}(\sigma) &:= v^{l,j}(\sigma, \pi/2), \\ v^{l,j}_n(\sigma) &:= v^{l,j}_n(\sigma, \pi/2). \end{aligned} \tag{6.10}$$

Remark 6.3. The orthogonality of the eigenvectors $v^{l,j}(\sigma)$ given by

$$\sum_{j=-l}^{l} v^{l,j}_m(\sigma)\overline{v^{l,j}_n(\sigma)} = \delta_{mn}$$

correspond to the orthogonality relations for the dual q-Krawtchouk polynomials.

Now that we know the eigenvectors and eigenvalues of $X_\sigma K$, we can do the transformation $q \leftrightarrow q^{-1}$ to obtain the same for $K^{-1}X_\tau$.

Corollary 6.4. *The self-adjoint operator $t^l(K^{-1}X_\tau)$ has an orthonormal basis of eigenvectors for $\mathbb{C}^{2l+1}$ given by*

$$\widetilde{v}^{l,j}(\tau) = \sum_{n=-l}^{l} \widetilde{v}^{l,j}_n(\tau)e_n, \quad j \in \{-l, \ldots, l\}, \tag{6.11}$$

where the coefficients $\widetilde{v}^{l,j}_n(\tau)$ are given by

$$c^{l,j}_\tau(-i)^{l-n}q^{\tau(n-l)}q^{-\frac{1}{2}(l-n)(l-n-1)}\left(\frac{(q^{-4l};q^2)_{l-n}}{(q^{-2};q^{-2})_{l-n}}\right)^{\frac{1}{2}} R_{l-n}(q^{-2j+2l} - q^{2j+2l+2\tau}; -q^{-2\tau}, 2l; q^{-2}).$$

The $c^{l,j}_\tau$ are normalizing constants and R_{l-n} are dual q-Krawtchouk polynomials from definition 5.21. The corresponding eigenvalues $\widetilde{\lambda}_j(\tau)$ are

$$\widetilde{\lambda}_j(\tau) = -\lambda_j(\tau). \tag{6.12}$$

Proof. Let $\widetilde{v} = \sum_{n=-l}^{l} a_n e^l_n$, then

$$t^l(K^{-1}X_\tau)\widetilde{v}(\tau) = \widetilde{\lambda}v$$

implies

$$\begin{aligned} \widetilde{\lambda}\sum_{n=-l}^{l} a_n e^l_n = \sum_{n=-l}^{l} q^n a_n \Bigg(& iq^{-\frac{1}{2}}\frac{\sqrt{(q^{-l+n-1} - q^{l-n+1})(q^{-l-n} - q^{l+n})}}{q^{-1} - q}e^l_{n-1} \\ & - iq^{\frac{1}{2}}\frac{\sqrt{(q^{-l+n} - q^{l-n})(q^{-l-n-1} - q^{l+n+1})}}{q^{-1} - q}e^l_{n+1} \\ & - \frac{q^\tau - q^{-\tau}}{q - q^{-1}}(q^{-n} - q^n)e_n \Bigg). \end{aligned}$$

If we do the transformation $q \leftrightarrow q^{-1}$, we see this is just (6.5) with $\widetilde{\lambda} = -\lambda$. $\square$

[20]If $a_E = 0$, the eigenvectors are just $(e_n)^l_{n-l}$.

Remark 6.5. Since we are dealing with a terminating series, we do not have to worry about convergence of the dual q-Krawtchouk polynomials for $q > 1$.

Part II. Spherical Elements and Askey-Wilson Polynomials

7. Duality of Hopf Algebras: $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$

As shown in section 3, there is a natural duality between $\mathfrak{sl}(2,\mathbb{C})$ and $\mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$ given by

$$\langle X, p\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p(\exp(tX)), \quad X \in \mathfrak{sl}(2,\mathbb{C}),\ p \in \mathrm{Pol}(\mathrm{SL}(2,\mathbb{C})).$$

Since we have a quantum version of $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ and not of $\mathfrak{sl}(2,\mathbb{C})$, we want to extend the mapping above to $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$. Then the natural questions arises what to do with a product of elements $X_1, X_2 \in \mathfrak{sl}(2,\mathbb{C})$ and with $1 \in \mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$? The natural thing to do here is to define for $p \in \mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$ that

$$\begin{aligned} \langle X_1X_2, p\rangle &= \frac{\partial^2}{\partial t_1 \partial t_2}\Big|_{t_1,t_2=0} p(\exp(t_1X_1)\exp(t_2X_2)), \\ \langle 1, p\rangle &= p\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}. \end{aligned} \tag{7.1}$$

Now we can see the use of our Hopf algebra structure. Recall from example 1 that

$$\begin{aligned} &\Delta(p)(\exp(X_1)\otimes\exp(X_2)) = p(\exp(X_1)\exp(X_2)), \\ &\Delta(p)\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} = \varepsilon(p) \end{aligned}$$

Therefore, we can rewrite (7.1) as

$$\begin{aligned} &\langle X_1X_2, p\rangle = \langle X_1 \otimes X_2, \Delta(p)\rangle, \\ &\langle 1, p\rangle = \varepsilon(p). \end{aligned}$$

In this way, we can define the duality $\langle X, p\rangle$ between any $X \in \mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ and $p \in \mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$. Moreover, we have done this in Hopf algebra language. We will use this as definition for two Hopf algebras to be in duality.

Definition 7.1. *Two Hopf algebras U and A are in duality if there exists a bilinear mapping $\langle\cdot,\cdot\rangle : U \times A \to \mathbb{C}$ such that for all $u, v \in U$ and $a, b \in A$ we have*

$$\begin{aligned} &\langle \Delta(u), a\otimes b\rangle = \langle u, ab\rangle, \quad \langle u\otimes v, \Delta(a)\rangle = \langle uv, a\rangle, \\ &\langle u, 1\rangle = \epsilon(u), \quad \langle 1, a\rangle = \epsilon(a), \\ &\langle S(u), a\rangle = \langle u, S(a)\rangle. \end{aligned} \tag{7.2}$$

The duality is called perfect if the bilinear map $\langle\cdot,\cdot\rangle$ is doubly non-degenerate[21]*.*

Then lastly, we need to know what it means for Hopf $*$-algebras to be in duality.

Definition 7.2. *Two Hopf $*$-algebras U and A are said to be in duality if they are in duality as Hopf algebras and if $\langle u^*, a\rangle = \overline{\langle u, (S(a))^*\rangle}$.*

Using proposition 2.11 one can show that this definition is automatically symmetric, i.e.

$$\langle u, a^*\rangle = \overline{\langle (S(u))^*, a\rangle}$$

follows as well.

7.1. Duality between $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$.

Now the question arises how to find the duality between our two quantum groups. If we look at the duality between $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$, we can interpret $\mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$ as linear functionals on $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$. We will try to do something similar in the quantum setting. We will look for linear functionals on $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. This goes via representation theory. We will repeat briefly what is said in section 3. Classically, $\mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$ is generated by the four polynomials $\alpha,\beta,\gamma,\delta \in \mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$ defined by

$$\alpha\begin{pmatrix} a & b \\ c & d\end{pmatrix} = a, \quad \beta\begin{pmatrix} a & b \\ c & d\end{pmatrix} = b,$$

[21] The bilinear map $\langle\cdot,\cdot\rangle : U \otimes A \to \mathbb{C}$ is doubly non-degenerate if $\langle u, a\rangle = 0$ for all $u \in U$ implies $a = 0$ and $\langle u, a\rangle = 0$ for all $a \in A$ implies $u = 0$.

$$\gamma\begin{pmatrix}a & b\\ c & d\end{pmatrix} = c, \quad \delta\begin{pmatrix}a & b\\ c & d\end{pmatrix} = d,$$

subject to $\alpha\gamma - \beta\gamma = 1$. This last equation corresponds comes from $\det(g) = 1$ for $g = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$. These α, β, γ and δ are exactly the matrix elements of the standard representation of $\mathfrak{sl}(2,\mathbb{C})$. That is, for $g = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ we have

$$\begin{pmatrix}\alpha(g) & \beta(g)\\ \gamma(g) & \delta(g)\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix} = \pi^{\frac{1}{2}}(g).$$

Thus, in the classical duality we have

$$\begin{aligned} \langle X, \alpha\rangle &= \mathrm{d}\pi^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}}(X) \quad \langle X, \beta\rangle = \mathrm{d}\pi^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}(X),\\ \langle X, \gamma\rangle &= \mathrm{d}\pi^{\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}}(X) \quad \langle X, \delta\rangle = \mathrm{d}\pi^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(X), \end{aligned} \tag{7.3}$$

where $\mathrm{d}\pi$ is the Lie algebra representation. Therefore, we will look at representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. Just like in the classical case, $t^{\frac{1}{2}}$ is a 2×2 matrix acting on a 2-dimensional space. We look at the four matrix elements of $t^{\frac{1}{2}}$ and interpret those as analogues of the classical α, β, γ and δ. Concrete, we define for $X \in \mathcal{U}_q(\mathfrak{su}(2))$ the linear functionals α, β, γ and δ by

$$\begin{pmatrix}\alpha(X) & \beta(X)\\ \gamma(X) & \delta(X)\end{pmatrix} := t^{\frac{1}{2}}(X),$$

where $t^{\frac{1}{2}}(X)$ is taken with respect to the basis from Theorem 4.12. That is,

$$\begin{aligned} \alpha(X) &= t^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}}(X) = \left\langle t^{\frac{1}{2}}(X)e_{-\frac{1}{2}}, e_{-\frac{1}{2}}\right\rangle, \quad & \beta(X) &= t^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}(X) = \left\langle t^{\frac{1}{2}}(X)e_{\frac{1}{2}}, e_{-\frac{1}{2}}\right\rangle,\\ \gamma(X) &= t^{\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}}(X) = \left\langle t^{\frac{1}{2}}(X)e_{-\frac{1}{2}}, e_{\frac{1}{2}}\right\rangle, & \delta(X) &= t^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(X) = \left\langle t^{\frac{1}{2}}(X)e_{\frac{1}{2}}, e_{\frac{1}{2}}\right\rangle, \end{aligned} \tag{7.4}$$

where $\langle\cdot,\cdot\rangle$ is the inner product on the 2-dimensional Hilbert space with orthonormal basis $\{e_{-\frac{1}{2}}, e_{\frac{1}{2}}\}$. We want to explicitly calculate these matrix elements. Since α, β, γ and δ defined above are linear functionals, we only have to know what they do on basis elements of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ from lemma 4.4, in order to know what they do on the whole of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$.

Proposition 7.3. *Let the linear functionals α, β, γ and δ on $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ be defined as above, then we have*

$$\begin{aligned} \alpha(K^kF^mE^n) &= \delta_{m0}\delta_{n0}q^{k/2}, \quad \beta(K^kF^mE^n) = \delta_{m0}\delta_{n1}q^{k/2},\\ \gamma(K^kF^mE^n) &= \delta_{m1}\delta_{n0}q^{-k/2}, \quad \delta(K^kF^mE^n) = (\delta_{m0}\delta_{n0} + \delta_{m1}\delta_{n1})q^{-k/2}. \end{aligned} \tag{7.5}$$

Proof. Since $t^{\frac{1}{2}}$ is a representation, we find

$$\begin{pmatrix}\alpha(K^kF^mE^n) & \beta(K^kF^mE^n)\\ \gamma(K^kF^mE^n) & \delta(K^kF^mE^n)\end{pmatrix} = t^{\frac{1}{2}}(K^kF^mE^n) = t^{\frac{1}{2}}(K)^k t^{\frac{1}{2}}(F)^m t^{\frac{1}{2}}(E)^n. \tag{7.6}$$

Using that

$$t^{\frac{1}{2}}(K) = \begin{pmatrix}q^{\frac{1}{2}} & 0\\ 0 & q^{-\frac{1}{2}}\end{pmatrix}, \quad t^{\frac{1}{2}}(F) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad t^{\frac{1}{2}}(E) = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$

we find (7.5) by working out the matrix product (7.6). □

These α, β, γ and δ generate an associative algebra A, which is a subalgebra of the linear dual of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. Using the duality relations (7.2), we can define a comultiplication, counit and antipode for A, turning it into a Hopf algebra. The duality at this point is not perfect. This can be fixed by adding relations to A. Then this algebra becomes exactly[22] $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ from example 5.

[22] Actually that is how $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ is normally introduced. To improve the understanding of (Hopf) Algebras, I introduced $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ in a simpler way in this thesis.

Theorem 7.4. *Let A be the associative algebra generated by the linear functionals α, β, γ and δ on $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. Then the duality relations (7.2) can be used to turn A into a Hopf algebra. The comultiplication, counit and antipode are given by*

$$\Delta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha \otimes \alpha + \beta \otimes \gamma & \alpha \otimes \beta + \beta \otimes \delta \\ \gamma \otimes \alpha + \delta \otimes \gamma & \gamma \otimes \beta + \delta \otimes \delta \end{pmatrix}, \tag{7.7}$$

$$\epsilon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}. \tag{7.8}$$

Moreover, the same relations as in $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ hold for A, i.e.

$$\begin{aligned} &\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad ,\gamma\delta = q\delta\gamma, \\ &\beta\gamma = \gamma\beta, \quad \alpha\delta - q\beta\gamma = 1 = \delta\alpha - q^{-1}\beta\gamma \end{aligned}. \tag{7.9}$$

Proof. We will use the duality relations (7.2) to define a comultiplication, counit and antipode on A. For the comultiplication, observe that

$$\begin{aligned} \left\langle X \otimes Y, \begin{pmatrix} \Delta(\alpha) & \Delta(\beta) \\ \Delta(\gamma) & \Delta(\delta) \end{pmatrix} \right\rangle &= \begin{pmatrix} \alpha(XY) & \beta(XY) \\ \gamma(XY) & \delta(XY) \end{pmatrix} = t^{\frac{1}{2}}(XY) = t^{\frac{1}{2}}(X)t^{\frac{1}{2}}(Y) \\ &= \begin{pmatrix} \alpha(X)\alpha(Y) + \beta(X)\gamma(Y) & \alpha(X)\beta(Y) + \beta(X)\delta(Y) \\ \gamma(X)\alpha(Y) + \delta(X)\gamma(Y) & \gamma(X)\beta(Y) + \delta(X)\delta(Y) \end{pmatrix} \\ &= \left\langle X \otimes Y, \begin{pmatrix} \alpha \otimes \alpha + \beta \otimes \gamma & \alpha \otimes \beta + \beta \otimes \delta \\ \gamma \otimes \alpha + \delta \otimes \gamma & \gamma \otimes \beta + \delta \otimes \delta \end{pmatrix} \right\rangle. \end{aligned}$$

Taking $k = m = n = 0$ in (7.5) gives

$$\epsilon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha(1) & \beta(1) \\ \gamma(1) & \delta(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Via (1.3) and the fact that S is antimultiplicative we have

$$S(K^k F^m E^n) = (-q)^{m-n} E^n F^m K^{-k}.$$

Using this we can derive the action of S by

$$\left\langle K^k F^m E^n, S(\alpha) \right\rangle = (-q)^{m-n} \alpha(E^n F^m K^{-k}) = (\delta_{m0}\delta_{n0} + \delta_{m1}\delta_{n1}) q^{-\frac{k}{2}} = \left\langle K^k F^m E^n, \delta \right\rangle,$$

by a similar calculation as in the proof of proposition 7.3. In the same way one can find $S(\beta) = -q^{-1}\beta$, $S(\gamma) = -q\gamma$ and $S(\delta) = \alpha$, proving (7.7) and (7.8).
Then, let us show the relations (7.9) hold in A. Using the definition of the antipode (2.3), we find

$$\begin{aligned} & m \circ (I \otimes S)\Delta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \eta \circ \epsilon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ \Longrightarrow & \begin{pmatrix} \alpha S(\alpha) + \beta S(\gamma) & \alpha S(\beta) + \beta S(\delta) \\ \gamma S(\alpha) + \delta S(\gamma) & \gamma S(\beta) + \delta S(\delta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \Longrightarrow & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

and similarly

$$\begin{aligned} & m \circ (S \otimes I)\Delta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \eta \circ \epsilon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ \Longrightarrow & \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Which lead to the relations (7.9). $\square$

Next, we will show that the relations (7.9) are in fact the only relations for A. In particular, A is exactly $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$. We first need a lemma which gives a basis for $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$. A proof can be found in [10].

Lemma 7.5. *For the Hopf algebra $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ defined in example 5, we have a linear basis given by $\delta^L\gamma^M\beta^N$, $\alpha^L\gamma^M\beta^N$, where $L, M, N \in \mathbb{Z}_+$.*

Remark 7.6. Note that this lemma implies that any $\eta \in A$ is a linear combination of $\delta^L\gamma^M\beta^N$ and $\alpha^L\gamma^M\beta^N$, $L, M, N \in \mathbb{Z}_+$. At this point we only don't know if this can be done uniquely. That is, we don't know yet if these elements are also linearly independent in A.

Theorem 7.7. *The duality between A and $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ is perfect. In particular, A is isomorphic to $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$.*

Proof. That the duality defined here is doubly non-degenerate requires quite some work. One has to explicitly calculate the duality between the basis elements of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and the elements $\delta^L\gamma^M\beta^N$ and $\alpha^L\gamma^M\beta^N$, $L, M, N \in \mathbb{Z}_+$, in A. After quite some computations (see e.g. [10]), we obtain

$$\left\langle K^k F^m E^n, \alpha^L\beta^M\gamma^N \right\rangle = \delta_{Mm}\delta_{Nn} C^{-L,M,N}_{k,m,n}, \tag{7.10}$$

$$\left\langle K^k F^m E^n, \delta^L\beta^M\gamma^N \right\rangle = \begin{cases} q^{(m-M)^2} \begin{bmatrix} L \\ m-M \end{bmatrix}_{q^2} C^{L,M,N}_{k,m,n}, \text{ if } 0 \le m-M = n-N \le L, \\ 0 \text{ otherwise,} \end{cases} \tag{7.11}$$

where

$$C^{L,M,N}_{k,m,n} = q^{k(L+M_N)/2} q^{-L(m+n)/2} q^{-n(n-1)/2} \frac{(q^2;q^2)_n (q^2;q^2)_m}{(1-q^2)^{m+n}}.$$

Now use this to prove

$$\begin{aligned} 0 = \sum_{L,L',M,N} \Big\langle X, c_{LMN}\alpha^L\beta^M\gamma^N + c'_{L'MN}\delta^{L'}\beta^M\gamma^N \Big\rangle \quad \text{for all } X \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C})) \\ \Longrightarrow c_{LMN} = 0 = c'_{L'MN} \text{ for all } L, L', M, N, \end{aligned} \tag{7.12}$$

and

$$0 = \sum_{k,m,n} c_{kmn} \left\langle K^L F^M E^n, \eta \right\rangle \quad \text{for all } \eta \in A \quad \Longrightarrow c_{kmn} = 0 \quad \text{for all } l, m, n. \tag{7.13}$$

Then (7.12) will prove that the basis elements for $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ are also linearly independent in A. Therefore, the relations (7.9) are indeed the only relations in A. That the duality between A and $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ is doubly non-degenerate then follows from (7.13). □

Recall that the linear functionals α, β, γ and δ on $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ can also be interpreted as the four matrix elements of a quantum analogue of $\mathrm{SL}(2,\mathbb{C})$. In the classical setting (see subsection 3.2), the matrix elements π^l_{mn} are polynomials with entries in $\mathrm{SL}(2,\mathbb{C})$ and they span $\mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$. Something similar happens in the quantum setting. However since we do not have direct acces to a quantum version of $\mathrm{SL}(2,\mathbb{C})$ or its representation, we have to work again with the duality between $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$. First, we need the Clebsch-Gordan decomposition and the notion of a tensor product representation. Let s en t be representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ on V and W respectively. Then we define the *tensor product representation* $t \otimes s$ on $V \otimes W$ by

$$(s \otimes t)(X)(v \otimes w) = (s \otimes t)\Delta(X)(v \otimes w), \; v \in V, \; w \in W, \; X \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C})).$$

The Clebsch-Gordan decomposition will tell us that $t^{l_1} \otimes t^{l_2}$ is reducible and can be written as a direct sum over the irreducible representations t^l.

Lemma 7.8 (Clebsch-Gordan decomposition)**.** *Let $l_1, l_2 \in \frac{1}{2}\mathbb{Z}_+$, then*

$$t^{l_1} \otimes t^{l_2} \cong \bigoplus_{l=|l_1-l_2|}^{l_1+l_2} t^l.$$

Or equivalently, there exists a bijective linear mapping $C : \mathbb{C}^{l_1} \otimes \mathbb{C}^{l_2} \to \bigoplus_{l=|l_1-l_2|}^{l_1+l_2} \mathbb{C}^l$ *such that*

$$C \circ (t^{l_1} \otimes t^{l_2})(X) = \begin{pmatrix} t^{l_0} & 0 & \cdots & 0 \\ 0 & t^{l_0+1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & t^{l_1+l_2} \end{pmatrix} (X) \circ C,$$

where $X \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ *and* $l_0 = |l_1 - l_2|$.

Proof. Although the Clebsch-Gordan decomposition is not concerned with $*$-structures, we will use that $t^{l_1} \otimes t^{l_2}$ is a $(2l_1+1)(2l_2+1)$-dimensional unitary representation of $\mathcal{U}_q(\mathfrak{su}(2))$ on V, hence completely reducible. Since

$$(t^{l_1} \otimes t^{l_2})(K)(e_n \otimes e_m) = t^{l_1}(K)e_n \otimes t^{l_2}(K)e_m = q^{-n-m} e_n \otimes e_m,$$

the spectrum of K is contained in $q^{\frac{1}{2}\mathbb{Z}}$. Since such N-dimensional representations of $\mathcal{U}_q(\mathfrak{su}(2))$ are unique by theorem 4.12, we have

$$t^{l_1} \otimes t^{l_2} \cong \bigoplus_{l=k}^{l_1+l_2} m_l t^l,$$

for certain multiplicities m_l. Do determine those, let us count the eigenvalues for K for the eigenvectors $e_{k_1} \otimes e_{k_2}$. The eigenvalue $q^{-l_1-l_2}$ only happens once, for $e_{l_1} \otimes e_{l_2}$, which gives $m_{l_1+l_2} = 1$. The eigenvalue $q^{-l_1-l_2+1}$ appears two times, which implies $m_{l_1+l_2-1} = 1$ since one was already used for $t^{l_1+l_2}$. Continuing this way, we get $m_l = 1$ for every l. The condition for k follows from requiring equal dimensions. That is, we need

$$(2l_1+1)(2l_2+1) = \sum_{l=k}^{l_1+l_2} 2k+1 = \frac{1}{2}(2l_1+1+2l_2+2k+1)(l_1+l_2+1-k).$$

Without loss of generality, suppose that $l_1 \geq l_2$. Taking $k = |l_1 - l_2| = l_1 - l_2$, gives

$$\sum_{l=k}^{l_1+l_2} 2k+1 = (2l_1+1)(2l_2+1),$$

as desired. □

We can now derive the following.

Theorem 7.9. *Define the matrix elements* $t^l_{mn} : \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C})) \to \mathbb{C}$ *w.r.t. the basis* $\{e_n\}_{n=-l}^{l}$ *given in theorem 4.12, by*

$$t^l_{mn}(X) := (t^l(X))_{m,n} = \left\langle t^l(X)e_n, e_m \right\rangle, \ X \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C})).$$

We define $t^0_{00} = 1$ *to be the unit of* $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$*. Then* $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ *has a linear basis formed by the matrix elements* t^l_{mn}*, where* $l \in \frac{1}{2}\mathbb{Z}_+$ *and* $n, m \in \frac{1}{2}\mathbb{Z}$ *such that* $-l \leq n, m \leq l$.

Sketch of proof. We need to show that any basis element of $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ given in lemma 7.5 can be written in terms of t^l_{mn}. Since α, β, γ and δ are just $t^{1/2}_{mn}$ for $m, n \in \{-\frac{1}{2}, \frac{1}{2}\}$ and $t^{l_1}_{ij} t^{l_2}_{mn} = (t^{l_1} \otimes t^{l_2})_{im;jn}$ we can use the Clebsch-Gordan decomposition to show that any basis element $\delta^L \gamma^M \beta^N$ or $\alpha^L \gamma^M \beta^N$ can be written as a linear combination of suitable t^l_{mn}. Similarly, working the other way around shows that any t^l_{mn} can be written in terms of the generators α, β, γ and δ. To show that the t^l_{mn} are linearly independent, we use the explicit formula for a specific linear functional $h : \mathcal{A}_q(\mathrm{SL}(2,\mathbb{C})) \to \mathbb{C}$, the Haar functional, on $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ given in (10.2)[23]. Suppose $\sum_{m,n,l} c_{mnl} t^l_{mn} = 0$. Multiplying by $S(t^{l'}_{ij})$ and applying the Haar functional gives

$$0 = \sum_{m,n,l} c_{mnl} h(S(t^{l'}_{ij}) t^l_{mn}) = c_{ijl'} q^{2(l'-j)} \frac{1-q^2}{1-q^{4l'+2}}.$$

[23]The result of this Haar functional does not use that t^l_{mn} is a basis for $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$, therefore we do not need to worry about circular arguments here.

Since $q^{2(l'-j)}\frac{1-q^2}{1-q^{4l'+2}} \neq 0$, we must have $c_{ijl'} = 0$. We can do this for every i, j, l', thus we have $c_{mnl} = 0$ for all m, n and l. $\square$

Using a more classical approach, see e.g. [12, (5.5)], we can express the matrix elements t^l_{mn} as polynomials of the generators α, β, γ and δ.

$$\begin{aligned} t^l_{mn} &= \begin{bmatrix} 2l \\ l-n \end{bmatrix}^{\frac{1}{2}}_{q^{-2}} \begin{bmatrix} 2l \\ l-m \end{bmatrix}^{-\frac{1}{2}}_{q^{-2}} \sum_{i=\max(0,m-n)}^{\min(l-n,l+m} q^{(l-n-i)(n-m+2i)} q^{-i(n-m+i)} \\ &\times \begin{bmatrix} l-n \\ i \end{bmatrix}_{q^{-2}} \begin{bmatrix} l+n \\ l+m-i \end{bmatrix}_{q^{-2}} \beta^i \gamma^{n-m+i} \alpha^{l-n-i} \delta^{l+m-i}. \end{aligned} \tag{7.14}$$

Taking the limit of $q \to 1$ gives the classical matrix elements π^l_{mn}, which are Jacobi polynomials. Koornwinder [12] showed that t^l_{mn} are little q-Jacobi polynomials, a quantum analogue of the Jacobi polynomials.

For fixed l, we are often interested in

$$\mathrm{A}^l_q := \mathrm{span}\{t^l_{mn} : -l \leq m, n \leq l\}. \tag{7.15}$$

Furthermore, we can explicitly compute the coproduct of t^l_{mn}.

Proposition 7.10. *We have*

$$\Delta(t^l_{mn}) = \sum_{k=-l}^{l} t^l_{mk} \otimes t^l_{kn}. \tag{7.16}$$

Proof. For $X \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$, $t^l_{mn}(X)$ is the matrix entry (m, n) of $t^l(X)$, i.e. the matrix entry in the m-th row and n-th column. If we use

$$t^l(XY) = t^l(X)t^l(Y), \quad X, Y \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C})),$$

and the duality relations (7.2), we find

$$\left\langle X \otimes Y, \Delta(t^l_{mn}) \right\rangle = \left\langle XY, t^l_{mn} \right\rangle = \left\langle t^l(XY)e_n, e_m \right\rangle = \left\langle t^l(X)t^l(Y)e_n, e_m \right\rangle.$$

Therefore, $\left\langle X \otimes Y, \Delta(t^l_{mn} \right\rangle$ is the matrix entry (m, n) of the the matrix $t^l(X)t^l(Y)$, which is by matrix multiplication equal to the standard inner product of the m-th row of $t^l(X)$ and n-th column of $t^l(Y)$. That is,

$$\left\langle X \otimes Y, \Delta(t^l_{mn} \right\rangle = \sum_{k=-l}^{l} (t^l(X))_{m,k}(t^l(Y))_{k,n} = \sum_{k=-l}^{l} \left\langle X, t^l_{mk} \right\rangle \left\langle Y, t^l_{kn} \right\rangle = \sum_{k=-l}^{l} \left\langle X \otimes Y, t^l_{mk} \otimes t^l_{kn} \right\rangle.$$

Hence,

$$\Delta(t^l_{mn}) = \sum_{k=-l}^{l} t^l_{mk} \otimes t^l_{kn}. \qquad \square$$

Remark 7.11. Notice that we can interpret the coproduct $\Delta(t^l_{mn})$ here as the opposite of matrix multiplication. This can be used to define a so-called matrix corepresentation of $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$. Via this way Koorwinder [12] showed that t^l_{mn} are little-q-Jacobi polynomials in α, β, γ and δ.

7.2. $\mathcal{A}_q(\mathrm{SU}(2))$**:** $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ **as a Hopf $*$-algebra.** Via definition 7.2, we can carry over the $*$ structure of $\mathcal{U}_q(\mathfrak{su}(2))$ to $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$. We will do this for the generators $\alpha, \beta, \gamma, \delta$ of $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$, then we extend $*$ as an anti-homomorphism[24] to the whole of $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$. This again requires the explicit expressions for the duality between basis elements of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ we used in the proof of theorem 7.7. We can then find the following.

Theorem 7.12. *There exist three inequivalent $*$-structures on $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ corresponding to the $*$-structures of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. They are given by*

[24] That is, $(\alpha\beta)^* = \beta^*\alpha^*$.

(i) $|q| = 1$, $\alpha^* = \alpha$, $\beta^* = q^{-1}\beta$, $\gamma^* = q\gamma$, $\delta^* = \delta$, *named* $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{R}))$,
(ii) $-1 < q < 1$, $\alpha^* = \delta$, $\beta^* = -q\gamma$, $\gamma^* = -q^{-1}\beta$, $\delta^* = \alpha$, *named* $\mathcal{A}_q(SU(2))$,
(iii) $-1 < q < 1$, $\alpha^* = \delta$, $\beta^* = q\gamma$, $\gamma^* = q^{-1}\beta$, $\delta^* = \alpha$, *named* $\mathcal{A}_q(\mathrm{SU}(1,1))$.

Proof. By definition 7.2, a real form of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ can be used to define a $*$-structure on $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$. We will show how to do this for $\mathcal{A}_q(\mathrm{SU}(2))$ and $\alpha^* = \delta$; other generators and real forms of $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ can be done similarly. We will prove that $\alpha^* = \delta$ on the PBW basis elements of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ given in lemma 4.4. Via definition 7.2, the action of S on $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ given in (2.9), and the anti-multiplicativity of S and $*$, we get

$$\begin{aligned}\left\langle \alpha^*, K^k F^m E^n \right\rangle &= \overline{\langle \alpha, S(K^k F^m E^n)^* \rangle} = (-q)^{m-n}\overline{\langle \alpha, ((K^{-1})^*)^k (F^*)^m (E^*)^n)\rangle} \\ &= (-q)^{m-n}\overline{\langle \alpha, K^{-k} E^m F^n \rangle}.\end{aligned}$$

Using (2.9) again, as well as the duality relation for S in (7.2), above expression is equal to

$$(-q)^{2m-2n}\overline{\langle \alpha, S(F^n E^m K^k)\rangle} = q^{2(m-n)}\overline{\langle S(\alpha), F^n E^m K^k\rangle} = q^{2(m-n)}\overline{\langle S(\alpha), K^k F^n E^m\rangle},$$

where we used the commutation relations for $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ in the last equality. Then, since $S(\alpha) = \delta$, we obtain

$$\left\langle \alpha^*, K^k F^m E^n \right\rangle = q^{2(m-n)}\overline{\langle \delta, K^k F^n E^m\rangle}.$$

Using the explicit formula for $\left\langle \delta, K^k F^n E^m \right\rangle$ given in (7.10) and (7.11), we see that above expression is non-zero only if $0 \le m = n \le 1$ and

$$\left\langle \alpha^*, K^k F^m E^n \right\rangle = \delta_{mn} q^{2m} C^{1,0,0}_{k,m,n} = \left\langle \delta, K^k F^m E^n \right\rangle,$$

where we used that $C^{L,M,N}_{k,m,n}$ is real. In the same way, one can obtain the action of $*$ on the other generators β, γ, δ. $\square$

Since we focus on $\mathcal{U}_q(\mathfrak{su}(2))$, we will look at the corresponding real form $\mathcal{A}_q(\mathrm{SU}(2))$ of $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$. We can completely classify the irreducible $*$-representations of $\mathcal{A}_q(\mathrm{SU}(2))$, including the infinite dimensional ones. Note that we can write any $\eta \in \mathcal{A}_q(\mathrm{SU}(2))$ in terms of α, α^*, γ, γ^* since $\delta^* = \alpha$ and $\beta = -q\gamma^*$. Therefore, a $*$-representation π is determined by $\pi(a)$ and $\pi(\gamma)$.

Theorem 7.13. *There are two families of irreducible inequivalent $*$-representations of $\mathcal{A}_q(\mathrm{SU}(2))$, both depending on $\theta \in [0, 2\pi)$.*

(i) *The one-dimensional representation π_θ, defined by $\pi_\theta(\alpha) = e^{i\theta}$ and $\pi_\theta(\gamma) = 0$.*
(ii) *The infinite-dimensional representation π_θ^∞ acting on the Hilbert space $l^2(\mathbb{Z}_+)$ with orthonormal basis $(e_n)_{n=0}^\infty$. This representation is defined by*

$$\pi_\theta^\infty(\alpha)e_n = \sqrt{1-q^{2n}}\,e_{n-1}, \qquad \pi_\theta^\infty(\gamma)e_n = e^{i\theta}q^n e_n, \tag{7.17}$$

with the convention $e_{-1} = 0$.

Proof. It is easy to check that the representations given are indeed $*$-representations of $\mathcal{A}_q(\mathrm{SU}(2))$. To prove that these are the only ones, suppose that π is an irreducible $*$-representation of $\mathcal{A}_q(\mathrm{SU}(2))$ on V. Let us look at $v \in \ker(\pi(\gamma))$. From the commutation relations (7.9), we obtain

$$\pi(\gamma)\pi(\alpha)v = q^{-1}\pi(\alpha)\pi(\gamma)v = 0.$$

Similarly, we find that $\pi(\beta)v, \pi(\delta)v \in \ker(\pi(\gamma))$. Therefore, $\ker(\pi(\gamma))$ is an invariant subspace. By the irreducibility of π, we have $\ker(\pi(\gamma)) = V$ or $\ker(\pi(\gamma)) = \{0\}$.

In the first case, we have $\pi(\gamma) = 0$. Since $-q\gamma^* = \beta$ and π is a $*$-representation, we must have $\pi(\beta) = -q\pi(\gamma)^* = 0$ as well. Using the commutation relations (7.9) again, we get $\pi(\alpha)\pi(\delta) = \mathrm{id} = \pi(\delta)\pi(\alpha)$. Since irreducible representations of commutative algebras are one-dimensional[25], we get $\pi(\alpha) = \lambda$ and $\pi(\delta) = \lambda^{-1}$ for $\lambda \in \mathbb{C}\backslash\{0\}$. From $\alpha^* = \delta$ we get $\lambda = e^{i\theta}$ for $\theta \in [0, 2\pi)$.

The case $\ker(\pi(\gamma)) = \{0\}$ is more complex. Using the spectral theorem for the normal operator

[25] This follows from Schur's lemma: since $\pi(\alpha)$ is an intertwiner it must be equal to $\lambda\mathrm{id}$ for a $\lambda \in \mathbb{C}$.

$\pi(\gamma)$, we can deduce[26] that its spectrum is of the form λq^n for $n \in \mathbb{Z}_+$ and $\lambda \in \mathbb{C}$. It has corresponding eigenvectors e_n. From $\alpha\gamma = q\gamma\alpha$ we get

$$q\pi(\gamma)\pi(\alpha)e_n = \pi(\alpha)\pi(\gamma)v_n = \lambda q^n \pi(\alpha)v_n.$$

Therefore, $\pi(\alpha)e_n$ is an eigenvector of $\pi(\gamma)$ corresponding to the eigenvalue λq^{n-1}, hence $\pi(\alpha)e_n = \mu_n e_{n-1}$, for some $\mu_n \in \mathbb{C}$. Similarly, we can find $\pi(\delta)e_n = \kappa_n e_{n+1}$ and $\pi(\beta)e_n = \rho_n e_n$ for some $\kappa_n, \rho_n \in \mathbb{C}$. Now suppose that $\{e_n\}_{n\in\mathbb{Z}_+}$ does not span V. Then we have an orthogonal direct sum decomposition of V, given by

$$V = W \oplus \operatorname{span}\{e_n\}_{n\in\mathbb{Z}_+},$$

for some subspace $W \subset V$. However, this contradicts the irreducibility of π since W is an invariant subspace of π. Indeed, due to the $*$-structure of $\mathcal{A}_q(\mathrm{SU}(2))$ we have for $w \in W$,

$$\langle e_n, \pi(\gamma)w\rangle_V = \langle \pi(\gamma^*)e_n, w\rangle_V = -q^{-1}\rho_n \langle e_n, w\rangle = 0,$$

hence $\pi(\gamma)w \in W$. Similarly, we can obtain that $\pi(\alpha)w, \pi(\beta)w, \pi(\delta)w \in W$. Thus $\{e_n\}_{n\in\mathbb{Z}_+}$ spans V. Using the $*$-structure of $\mathcal{A}_q(\mathrm{SU}(2))$, we get

$$\rho_n = \langle \pi(\beta)e_n, e_n\rangle_V = \langle e_n, \pi(\gamma^*)e_n\rangle_V = -q^{n+1}\overline{\lambda}.$$

In the same way we can find $\kappa_n = \overline{\mu_{n+1}}$. Then, the last commutation relation of (7.9) gives

$$|\mu_n|^2 e_n + |\lambda|^2 q^{2n} e_n = e_n,$$

for $n \geq 1$. For $n = 0$, we have

$$|\lambda|^2 e_0 = e_0.$$

This implies $\lambda = e^{i\theta}$ for some $\theta \in [0, 2\pi)$ and $|\mu_n| = \sqrt{1 - q^{2n}}$. Therefore, π is equivalent to the representation given in (7.17). □

Remark 7.14. To assist the reader in further calculations, we will also explicitly write down

$$\pi_\theta^\infty(\beta)e_n = -e^{-i\theta}q^{n+1}e_n, \quad \pi_\theta^\infty(\delta)e_n = \sqrt{1 - q^{2n+2}}e_{n+1}.$$

Later on, we will need an explicit expression for $\pi_\theta(t^l_{mn})$.

Proposition 7.15. *We have*

$$\pi_\theta(t^l_{mn}) = \delta_{mn}e^{-2in\theta}.$$

Proof. Applying π_θ to the explicit expression (7.14) gives a non-zero outcome only if there are no β and γ terms. This gives $i = 0$ and $n - m + i = 0$, which implies $n = m$ and (7.14) simplifies to

$$\alpha^{l-n}\delta^{l+n}.$$

Since $\delta = \alpha^*$, we get

$$\pi_\theta(t^l_{mn}) = \delta_{mn}\pi_\theta(\alpha)^{l-n}\pi(\alpha^*)^{l+n} = \delta_{mn}e^{(l-n)i\theta}e^{-(l+n)i\theta} = \delta_{mn}e^{-2in\theta}.$$ □

[26] The interested reader can find the proof at [11] or [15].

8. Spherical elements

8.1. **Classical motivation.** Spherical elements in the classical sense are functions $f : G \to \mathbb{C}$ on a group G that are bi-K-invariant. That is, they are left-invariant and right-invariant with respect to a subgroup K of G. That is,

$$f(kg) = f(g) = f(gk), \text{ for all } k \in K \text{ and } g \in G. \tag{8.1}$$

For example, let $G = \mathbb{C}\backslash\{0\}$ be the group where the group action is multiplication. Then the group $K = \{e^{\phi i}, \phi \in [0, 2\pi)\}$ is a subgroup of G. Multiplication with an element $e^{\phi i}$ is a rotation in the complex plane over an angle ϕ. Therefore, functions like '$f : g \to |g|^2$' that only depend on the absolute value of an element $g \in G$ are bi-K-invariant. Since our group G is commutative, a left-invariant function is automatically right-invariant, this need not be the case for non-commutative groups such as $\mathrm{SL}(2, \mathbb{C})$.

Matrix elements π^l_{mn} from a representation of a group G are functions on the group. That is, $\pi_{mn} : G \to \mathbb{C}$. Let us show an example of spherical elements in the SU(2) case. Let us take the subgroup K defined by

$$K = \left\{ k_t = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix},\ t \in \mathbb{R} \right\}.$$

We take the unitary representation π^l from (3.1) and the matrix elements

$$\pi^l_{mn}(g) = \left\langle \pi^l(g)\psi^l_n, \psi^l_m \right\rangle,$$

where $g \in \mathrm{SU}(2)$ and ψ^l_n is the basis defined in (3.2). Since

$$\pi^l(k_t)\psi^l_0(x, y) = \begin{bmatrix} 2l \\ l-n \end{bmatrix}^{\frac{1}{2}} (e^{it}x)^l(e^{-it}y)^l = \psi^l_0(x, y) \text{ for all } t \in \mathbb{R},$$

π^l_{00} is a spherical element. Indeed, it is bi-K-invariant since for $t, s \in \mathbb{R}$ we have

$$\pi^l_{00}(k_t g k_s) = \left\langle \pi^l(k_t g k_s)\psi^l_0, \psi^l_0 \right\rangle = \left\langle \pi^l(g)\pi^l(k_s)\psi^l_0, \pi^l(k_{-t})\psi^l_0 \right\rangle = \left\langle \pi^l(g)\psi^l_0,)\psi^l_0 \right\rangle,$$

where we used that $\pi^l(k_t)^* = \pi^l(k_{-t})$ because π^l is unitary and $k_t^{-1} = k_{-t}$. One can show (e.g. [16]) that π^l_{00} are Legendre polynomials of degree l. It is of high significance that in the SU(2) setting all one-parameter subgroups are conjugate, so that the choice of subgroup does not matter. Moreover, the Legendre polynomials have no parameters we can choose. This will be different in the quantum setting, where the choice of subgroup does matter. This will lead to two extra parameters. Consequently, we will find more general orthogonal polynomials.

One problem in the quantum setting is that we do not have access to the analogue of a group, only to the polynomials on the group. Therefore, we need another way of defining spherical elements. We will use the duality between $\mathfrak{sl}(2, \mathbb{C})$ and $\mathrm{Pol}(\mathrm{SL}(2, \mathbb{C}))$. Observe that for $X \in \mathfrak{sl}(2, \mathbb{C})$, the group $K = \exp(tX)$ is a subgroup of $\mathrm{SL}(2, \mathbb{C})$. For example, the subgroup

$$K = \left\{ k_t = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix},\ t \in \mathbb{R} \right\}$$

from before is generated by

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}).$$

Using (8.1), we get that a function $f \in \mathrm{Pol}(\mathrm{SL}(2, \mathbb{C}))$ on the group $\mathrm{SL}(2, \mathbb{C})$, is bi-invariant w.r.t. $K = \exp(tX)$ if for all $g \in \mathrm{SL}(2, \mathbb{C})$ we have

$$f(g\exp(tX)) = f(g) = f(\exp(tX)g) \iff \frac{\mathrm{d}}{\mathrm{d}t} f(g\exp(tX)) = 0 = \frac{\mathrm{d}}{\mathrm{d}t} f(\exp(tX)g).$$

Here, " $\implies$ " follows by using the definition for the derivative. To see " $\impliedby$ ", use that $\frac{\mathrm{d}}{\mathrm{d}t} f(g\exp(tX)) = 0$ implies $f(g(\exp(tX)) = C$, for some constant $C \in \mathbb{C}$. Then take $t = 0$.

We will show that we only need the derivative at $t = 0$ to be 0. If we use $\exp((t+s)X) = \exp(tX)\exp(sX)$ and then do the substitution $u = t - s$, we get

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=s} f(g\exp(tX)) \\ &= \frac{\mathrm{d}}{\mathrm{d}u}\Big|_{u=0} f(g\exp(sX)\exp(uX)). \end{aligned}$$

Since $g\exp(sX) \in \mathrm{SL}(2,\mathbb{C})$, we have that a function is left-invariant if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=s} f(g\exp(tX)) = 0 \text{ for all } g \in G \text{ and } s \in \mathbb{R} \iff \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(g\exp(tX)) = 0 \text{ for all } g \in G,$$

and similarly for the right-invariance. Recall that $\Delta(f)(g,h) = f(gh)$ for $g, h \in G$, thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(g\exp(tX)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Delta(f)(g, \exp(tX)).$$

Therefore, using duality from (3.3), we can write bi-K-invariance in the language of Hopf algebra as

$$(\mathrm{id}_A \otimes \langle X, \cdot\rangle) \circ \Delta(f)(g) = 0 = (\langle X, \cdot\rangle \otimes 1_A) \circ \Delta(f)(g), \tag{8.2}$$

where id_A is the identity on $A = \mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$. This can be seen as an *action* of X on f that is 0. Recall that a left action of an algebra A on another algebra B is a linear mapping $A \times B \to B$, $(a,b) \to a.b$, such that for all $a, a' \in A$ and $b \in B$ we have

$$a.(a'.b) = (aa').b \quad \text{and} \quad 1_A.b = b.$$

Similarly, we can define $b.a$ as the right action of a on b. Thus via (8.2), the language of Hopf algebras can be used to define a left and right action of $X \in \mathfrak{sl}(2,\mathbb{C})$ on $f \in \mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$. We write

$$X.f = (\mathrm{id}_A \otimes \langle X, \cdot\rangle) \circ \Delta(f) \quad \text{and} \quad f.X = (\langle X, \cdot\rangle \otimes \mathrm{id}_A) \circ \Delta(f). \tag{8.3}$$

We can extend this to $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ by defining for the left action

$$(XY).f = X.(Y.f),$$

and similarly for the right action

$$f.(XY) = (f.X).Y.$$

In summary, f is a bi-invariant function for $K = \exp(tX)$ if and only if $X.f = 0 = f.X$.

8.2. **Actions of Hopf Algebras in Duality and Spherical elements.** In the previous subsection we showed that we can define spherical elements via the language of Hopf algebras: $X.f = 0 = f.X$. We will do this for general Hopf algebras in duality and show this is well defined. Let A and U be Hopf algebras in duality. First we show how the action from U on A is done. Then we can define what spherical elements are. We will use (8.3) to define a left and right action from U on A.

Proposition 8.1. *Let A and U be Hopf algebras in duality, then we can define a left and right action from $u \in U$ on $a \in A$ by*

$$u.a = (\mathrm{id}_A \otimes \langle u, \cdot\rangle) \circ \Delta(a), \quad a.u = (\langle u, \cdot\rangle \otimes \mathrm{id}_A) \circ \Delta(a).$$

Proof. Linearity is clear. We will show that $v.(u.a) = (vu).u$ for $v, u \in U$ and $a \in A$. Indeed, (2.1) shows that

$$\begin{aligned} v.(u.a) &= (\mathrm{id}_A \otimes \langle v, \cdot\rangle \otimes \mathrm{id}_A) \circ (\Delta \otimes \langle u, \cdot\rangle) \circ \Delta(a) \\ &= (\mathrm{id}_A \otimes \langle v, \cdot\rangle \otimes \langle u, \cdot\rangle) \circ (\Delta \otimes \mathrm{id}_A) \circ \Delta(a) \\ &= (\mathrm{id}_A \otimes \langle v \otimes u, \cdot\rangle) \circ (\mathrm{id}_A \otimes \Delta) \circ \Delta(a) \\ &= (\mathrm{id}_A \otimes \langle vu, \cdot\rangle) \circ \Delta(a) = (vu).a, \end{aligned}$$

where we used that by the first line of (7.2) we have $\langle v \otimes u, \cdot\rangle \circ \Delta = \langle uv, \cdot\rangle$. Similarly, we have for the right action that $(a.u).v = a.(uv)$. $\square$

Remark 8.2. The definition of on action might not seem very intuitive at first sight. However, do not get confused by the abstract notation. There are several ways to look at this action. Often, the context of the element $u.a \in A$ will be in the duality with $v \in U$. Writing this out gives

$$\langle v, u.a \rangle = \sum_{(a)} \left\langle v, a_{(1)} \right\rangle \left\langle u, a_{(2)} \right\rangle = \langle vu, a \rangle \,. \tag{8.4}$$

and similarly

$$\langle v, a.u \rangle = \langle uv, a \rangle \,. \tag{8.5}$$

We can also look at $u.a$ just as an element in A. Then we take the coproduct of a and apply the right side of the tensor '$a_{(2)}$' to u. Thus we just have an element in A given by

$$u.a = \sum_{(a)} \left\langle u, a_{(2)} \right\rangle a_{(1)}.$$

Now we are ready to define spherical elements.

Definition 8.3. *Let A and U be Hopf algebras in duality. Then $a \in A$ is called a spherical element with respect to $u \in U$ if $u.a = 0 = a.u$, where U acts on A.*

9. Generalised matrix elements

In the previous section we defined spherical elements using that $X \in \mathfrak{sl}(2,\mathbb{C})$ generates a subgroup of $\mathrm{SL}(2,\mathbb{C})$. Recall that the twisted primitive elements in $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ are the analogues of $X \in \mathfrak{sl}(2,\mathbb{C})$ in the classical universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$. Therefore, we will find the spherical elements in $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ with respect to the twisted primitive elements. We can do this explicitly for the ones that are self-adjoint: X_σ and X_τ. We will do this via the explicit eigenvalues and eigenvectors of $X_\sigma K$ from theorem 6.1 and remark 6.2.

9.1. **Generalised matrix elements and (τ,σ)-spherical elements.** We want to find spherical elements in $\eta \in \mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ with respect to the twisted primitive elements in $X \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$, i.e.

$$X.\eta = 0 = \eta.X$$

conform definition 8.3. Here, X acts on the left, respectively right, on η. This poses the question how to approach this. First of all, observe that all constants in $\mathcal{A}_q(\mathrm{SU}(2))$ are spherical elements with respect to the twisted primitive elements $X \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. Indeed, for $1 \in \mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ and $Y \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ we have by (4.2) and (8.4),

$$\langle Y, X.1\rangle = \langle YX, 1\rangle = \epsilon(Y)\epsilon(X) = 0,$$

since X is twisted primitive. In the same way we find $1.X = 0$. To find other spherical elements, let us look at the twisted primitive element $K - K^{-1} \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ as an example. We can now see the usefulness of (8.4) again. We obtain

$$\begin{aligned}
\left\langle X, (K-K^{-1}).t^l_{ij}\right\rangle &= \left\langle X(K-K^{-1}), t^l_{ij}\right\rangle \\
&= \left\langle t^l(X(K-K^{-1}))e_j, e_i\right\rangle \\
&= \left\langle t^l(X)t^l(K-K^{-1})e_j, e_i\right\rangle \\
&= (q^{-j}-q^j)\left\langle X, t^l_{ij}\right\rangle,
\end{aligned}$$

where we used theorem 4.12 in the last step. Therefore,

$$(K-K^{-1}).t_{ij} = (q^{-j}-q^j)t_{ij}.$$

Similarly, by using (8.5) we get

$$t_{ij}.(K-K^{-1}) = (q^{-i}-q^i)t_{ij}.$$

Thus t_{ij} is a spherical element with respect to $K - K^{-1}$ if $i = 0 = j$. Since the matrix elements t^l_{ij} form a linear basis for $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$, all spherical elements with respect to $K-K^{-1}$ are t^l_{00}, $l \in \mathbb{Z}_+$, where $t^0_{00} = 1 \in \mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$. We want to find more general spherical elements. For what follows after, we will need the $*$-structure of $\mathcal{U}_q(\mathfrak{su}(2))$ and $\mathcal{A}_q(\mathrm{SU}(2))$. Let us focus on the often used self-adjoint twisted primitive element

$$X_\sigma = X_{\sigma,\pi/2} = iq^{\frac{1}{2}}E - iq^{-\frac{1}{2}}F - \frac{q^\sigma - q^{-\sigma}}{q-q^{-1}}(K-K^{-1}), \quad \sigma \in \mathbb{R},$$

from remark 6.2. Then we define the following.

Definition 9.1. *$\eta \in \mathcal{A}_q(\mathrm{SU}(2))$ is a (τ,σ)-spherical element if*

$$X_\sigma.\eta = 0 \quad \textit{and} \quad \eta.X_\tau = 0.$$

We found the spherical elements with respect to $K - K^{-1}$ by using its eigenvectors and eigenvalues. In particular, we needed its null space. Because t_{ij} are eigenvectors of $K - K^{-1}$, but not of E and F we define generalised matrix elements by using the eigenvectors of $X_\sigma K$ we found in theorem 6.1.

Definition 9.2. *Define the generalised matrix elements $b^l_{ij}(\tau,\sigma) \in \mathcal{A}_q(\mathrm{SU}(2))$ by*

$$b^l_{ij}(\tau,\sigma) = K.a^l_{ij}(\tau,\sigma),$$

where

$$a^l_{ij}(\tau,\sigma)(X) = \left\langle t^l(X)v^{l,i}(\tau), v^{l,j}(\sigma)\right\rangle,$$

and

$$v^{l,j}(\sigma) := v^l_{l,j}(\sigma, \pi/2)$$

are the eigenvectors from theorem 6.1.

The reader might wonder why we take $b^l_{ij}(\tau,\sigma)$ instead of just $a^l_{ij}(\tau,\sigma)$. This is because $a^l_{ij}(\tau,\sigma)$ will give spherical elements with respect to $X_\sigma K(\tau,\sigma)$, and $b^l_{ij}(\tau,\sigma)$ with respect to X_σ. We have the following proposition.

Proposition 9.3. *Let $b^l_{ij}(\tau,\sigma)$ as defined above.*

(i) We have

$$X_\sigma.b^l_{ij}(\tau,\sigma) = \lambda_j(\sigma)K^{-1}.b^l_{ij}(\tau,\sigma) \text{ and } b^l_{ij}(\tau,\sigma).X_\tau = \lambda_i(\tau)b^l_{ij}(\tau,\sigma).K^{-1},$$

where $\lambda_j(\sigma)$ are the eigenvalues of $X_\sigma K$ from theorem 6.1. In particular, b^l_{00} are (τ,σ)-spherical elements if $l \in \mathbb{Z}_+$.

(ii) Fix $l \in \frac{1}{2}\mathbb{Z}_+$. Then $b^l_{ij}(\tau,\sigma) \in A^l_q$, where A^l_q is defined in (7.15)*. Explicitly,*

$$b^l_{ij}(\tau,\sigma) = \sum_{n,m=-l}^{l} v^{l,j}_n(\sigma)\overline{v^{l,i}_m(\tau)}q^{-n}t^l_{mn}.$$

Proof. (i). This follows from (8.4),

$$\begin{aligned}\left\langle Y, X_\sigma.b^l_{ij}(\tau,\sigma)\right\rangle &= \left\langle YX_\sigma K, a^l_{ij}(\tau,\sigma)\right\rangle = \left\langle t^l(Y)t^l(X_\sigma K)v^{l,j}(\sigma), v^{l,i}(\tau)\right\rangle \\ &= \lambda_j(\sigma)\left\langle t^l(Y), a^l_{ij}(\tau,\sigma)\right\rangle = \lambda_j(\sigma)\left\langle Y, K^{-1}.b^l_{ij}(\tau,\sigma)\right\rangle.\end{aligned}$$

In the same way we obtain $b^l_{ij}(\tau,\sigma).X_\tau = \lambda_i(\tau)b^l_{ij}(\tau,\sigma).K^{-1}$ with (8.5).
(ii) Using theorem 6.1 for the explicit expressions of $v^{l,j}(\sigma)$ and $v^{l,i}(\tau)$, we get

$$\begin{aligned}\left\langle X, K.a^l_{ij}(\tau,\sigma)\right\rangle = \left\langle t^l(XK)v^{l,j}(\sigma), v^{l,i}(\tau)\right\rangle &= \sum_{n,m=-l}^{l}\left\langle t^l(X)t^l(K)v^{l,j}_n(\sigma)e_n, v^{l,i}_m(\tau)e_m\right\rangle \\ &= \sum_{n,m=-l}^{l} v^{l,j}_n(\sigma)\overline{v^{l,i}_m(\tau)}q^{-n}\left\langle t^l(X)e_n, e_m\right\rangle.\end{aligned}$$

□

9.2. **The $*$-subalgebra of (τ,σ)-spherical elements.** In this subsection we will show that the (τ,σ)-spherical elements form a $*$-subalgebra of $\mathcal{A}_q(\mathrm{SU}(2))$. Then we will proof that the (τ,σ)-spherical elements $b^l_{00}(\tau,\sigma)$, $l \in \mathbb{Z}_+$ we found are in fact a linear basis for this subalgebra. Lastly, we will show that this subalgebra is generated[27] by one single (τ,σ)-spherical element called $\rho_{\tau,\sigma}$. This is the non-constant part of $b^1_{00}(\tau,\sigma)$. We can then interpret this algebra of (τ,σ)-spherical elements as polynomials in the 'variable' $\rho_{\tau,\sigma}$. In section 11 we will show that the generalised matrix element $b^l_{ij}(\tau,\sigma)$ can be seen as an Askey-Wilson polynomial of degree l in the variable $\rho_{\tau,\sigma}$ with two continuous parameters, σ and τ, and two discrete ones, i and j.

Let us first show that the algebra generated by the (τ,σ)-spherical elements is a $*$-subalgebra, i.e. it is a subset of $\mathcal{A}_q(\mathrm{SU}(2))$ and it is a $*$-algebra on its own. This means that we have to show that it is closed under multiplication and taking the $*$-operation. We have the following proposition. This is a more general statement then we need right now, since we only use the case $\lambda = 0 = \mu$ in this section. However, in section 11 we will use the full proposition.

Proposition 9.4. *Let $\eta \in \mathcal{A}_q(\mathrm{SU}(2))$ be a (τ,σ)-spherical element.*

(i) η^* is a (τ,σ)-spherical element.

(ii) If $\xi \in \mathcal{A}_q(\mathrm{SU}(2))$ satisfies

$$X_\sigma.\xi = \lambda K^{-1}.\xi \quad \text{and} \quad \xi.X_\tau = \mu\xi.K^{-1}, \tag{9.1}$$

for $\lambda,\mu \in \mathbb{C}$, then $\xi\eta$ satisfies (9.1) *for the same λ and μ. Furthermore, if $\lambda,\mu \in \mathbb{R}$ then $\xi^*\xi$ is a (τ,σ)-spherical element as well.*

[27] This means that all elements of this subsalgebra are of the form $\sum_{n=0}^{l} a_n(\rho_{\tau,\sigma})^n$ for some $l \in \mathbb{Z}_+$ and $a_n \in \mathbb{C}$.

Proof. (i) We will first show that for arbitrary Hopf $*$-algebras A and U in duality we have $u.a^* = (S(X)^*, a)^*$. Since the comultiplication is a $*$-homomorphism, we have

$$u.a^* = \sum_{(a)} \left\langle u, a^*_{(2)} \right\rangle a^*_{(1)}.$$

Using definition 7.2 and the anti-linearity of $*$, this is equal to

$$\sum_{(a)} \overline{\langle S(u)^*, a_{(2)} \rangle} a^*_{(1)} = (S(u)^*, a)^*.$$

By a straightforward calculation, $S(X_\sigma)^* = -X_\sigma$. Therefore,

$$X_\sigma.\eta^* = -(X_\sigma.\eta)^* = 0.$$

Similarly one can obtain $\eta^*.X_\tau = 0$.
(ii) Using (7.2), we find

$$\langle v, u.(ab) \rangle = \sum_{(a),(b),(v),(u)} \langle v_{(1)}, a_{(1)} \rangle \langle v_{(2)}, b_{(1)} \rangle \langle u_{(1)}, a_{(2)} \rangle \langle u_{(2)}, b_{(2)} \rangle = \left\langle v, \sum_{(u)} \left(u_{(1)}.a\right) \left(u_{(2)}.b\right) \right\rangle.$$

Thus

$$u.ab = \sum_{(u)} \left(u_{(1)}.a\right) \left(u_{(2)}.b\right).$$

Since X_σ is twisted primitive w.r.t K, this gives

$$X_\sigma.(\xi\eta) = (K.\xi)(X_\sigma.\eta) + (X_\sigma.\xi)(K^{-1}.\eta) = \lambda(K^{-1}.\xi)(K^{-1}.\eta) = \lambda K^{-1}.(\xi\eta),$$

where we used in the last step that K^{-1} is group-like. In the same way we can prove

$$(\xi\eta).X_\tau = \mu(\xi\eta).K^{-1}.$$

Now, let $\lambda, \mu \in \mathbb{R}$. Then

$$\begin{aligned} X_\sigma.(\xi^*\xi) &= (K.\xi^*)(X_\sigma.\xi) + (X_\sigma.\xi^*)(K^{-1}.\xi) = \lambda(K^{-1}.\xi)^*(K^{-1}.\xi) - (X_\sigma.\xi)^*(K^{-1}.\xi) \\ &= (\lambda - \overline{\lambda})(K^{-1}.\xi)^*(K^{-1}.\xi), \end{aligned}$$

where we used just as in (i) that $u.a^* = (S(u)^*.a)^*$. Above expression is 0 for real λ. Similarly we obtain $\xi^*\xi.X_\tau = 0$. □

Corollary 9.5. *The (τ, σ)-spherical elements form a $*$-subalgebra of $\mathcal{A}_q(\mathrm{SL}(2, \mathbb{C}))$.*

Proof. Take $\lambda = 0 = \mu$ in the previous proposition. □

Now we will show that a linear basis for this subalgebra is given by the generalised matrix elements $b^l_{00}(\tau, \sigma)$ with $l \in \mathbb{Z}_+$. We will again show something more general which we will need later on.

Proposition 9.6. *Fix $l \in \frac{1}{2}\mathbb{Z}_+$ and let $\xi \in A^l_q \backslash \{0\}$ satisfy (9.1). Then $\lambda = \lambda_j(\sigma)$ and $\mu = \lambda_i(\tau)$ and $\xi = cb^l_{ij}(\tau, \sigma)$ for some $c \in \mathbb{C}$, where $\lambda_j(\sigma)$ is from theorem 6.1. In particular, the space of (τ, σ)-spherical elements in A^l_q is empty when $l \notin \mathbb{Z}_+$, and spanned by $b^l_{00}(\tau, \sigma)$ if $l \in \mathbb{Z}_+$.*

Proof. We will first prove that

$$(X_\sigma - \lambda K^{-1}).\xi = 0 \quad \text{and} \quad \xi.(X_\tau - \mu K^{-1}) = 0,$$

implies that λ is an eigenvalue of $t^l(X_\sigma K)$ and μ of $t^l(X_\tau K)$. Since t^l_{mn} form a linear basis for A^l_q, we have

$$\xi = \sum_{n,m=-l}^{l} c_{m,n} t^l_{mn}, \quad c_{m,n} \in \mathbb{C}.$$

Suppose that $X.\xi = 0$. Then, via propositions 7.10 and 8.1, we get

$$0 = \sum_{k=-l}^{l} \sum_{n,m=-l}^{l} c_{m,n} \left\langle t^l(X)e_n, e_k \right\rangle t^l_{mk}.$$

Since the t^l_{mk} are linearly independent, we need for all m, k,

$$0 = \sum_{n-l}^{l} c_{m,n} \left\langle t^l(X) e_n, e_k \right\rangle .$$

Therefore, we have for all m,

$$0 = t^l(X) \left(\sum_{n-l}^{l} c_{m,n} e_n \right) .$$

Or in other words, $v = \sum_{n-l}^{l} c_{m,n} e_n$ is in the null-space of $t^l(X)$ for all m. In the same way, we can derive

$$\xi.X = 0 \implies t^l(X^*) \left(\sum_{m-l}^{l} \overline{c_{m,n}} e_m \right) = 0 \text{ for all } n.$$

Therefore, (9.1) implies

$$t^l \left(X_\sigma - \lambda K^{-1} \right) \left(\sum_{n-l}^{l} c_{m,n} e_n \right) = 0 \text{ for all } m \implies t^l \left(X_\sigma K - \lambda \right) \left(\sum_{n-l}^{l} c_{m,n} q^n e_n \right) = 0 \text{ for all } m,$$

$$t^l \left(X_\tau - \mu K^{-1} \right)\right)^* \left(\sum_{n-l}^{l} \overline{c_{m,n}} e_m \right) = 0 \text{ for all } n \implies \quad t^l \left(X_\tau K - \mu \right) \left(\sum_{n-l}^{l} \overline{c_{m,n}} e_m \right) = 0 \text{ for all } n,$$

where we used that $K^* = K$ and $(X_\tau K)^* = X_\tau K$. From theorem 6.1 we obtain $\lambda = \lambda_j(\sigma)$ and $\mu = \lambda_i(\tau)$. Moreover, from the fact that every $\lambda_j(\sigma)$ has multiplicity one, we get

$$\sum_{n-l}^{l} c_{m,n} q^n e_n = c_m v^{l,j}(\sigma) \quad \text{and} \quad \sum_{n-l}^{l} \overline{c_{m,n}} e_m = c_n v^{l,i}(\tau),$$

for constants $c_n, c_m \in \mathbb{C}$. Combining this with the expression for $v^{l,j}(\sigma)$, we obtain

$$c_{m,n} = c \; v^{l,j}_n(\sigma) \overline{v^{l,i}_m(\tau)} q^{-n},$$

for some constant $c \in \mathbb{C}$. Using proposition 9.3(ii), we get

$$\sum_{n-l}^{l} c_{m,n} t^l_{mn} = c \; b^l_{ij}(\tau, \sigma).$$

Then, since $\lambda_j(\sigma) = 0$ only happens if l is a full integer, we get that there are no (τ, σ) spherical elements in $\mathcal{A}_q(\mathrm{SL}(2, \mathbb{C}))$ if $l \notin \mathbb{Z}_+$, and that the space of (τ, σ)-spherical elements is one-dimensional and spanned by $b^l_{00}(\tau, \sigma)$ if $l \in \mathbb{Z}_+$. □

Let us now take a closer look at the spherical element $b^1_{00}(\tau, \sigma)$. Using (6.3) and proposition 9.3 we can explicitly calculate this element in terms of the 9 matrix elements $\{t^1_{mn}\}$, where $m, n \in \{-1, 0, 1\}$. Using (7.14) we can express b^1_{00} in terms of the generators α, β, γ and δ of $\mathcal{A}_q(\mathrm{SU}(2))$. That is, $b^1_{00}(\tau, \sigma)$ can be seen as a polynomial in the four 'variables' α, β, γ and δ. Explicitly, we have that $b^1_{00}(\tau, \sigma)$ is a constant multiple of

$$2\rho_{\tau,\sigma} + \frac{(q^{-\sigma} - q^{\sigma})(q^{-\tau} - q^{\tau})}{q + q^{-1}},$$

where

$$\begin{aligned} \rho_{\tau,\sigma} =& \frac{1}{2}(\alpha^2 + \delta^2 + q\gamma^2 + q^{-1}\beta^2 + i(q^{-\sigma} - q^{\sigma})(q\delta\gamma + \beta\alpha) - i(q^{-\tau} - q^{\tau})(\delta\beta + q\gamma\alpha) \\ &+ (q^{-\sigma} - q^{\sigma})(q^{-\tau} - q^{\tau})\beta\gamma. \end{aligned} \tag{9.2}$$

Since constants are spherical elements, $\rho_{\tau,\sigma}$ is a spherical element as well. The following theorem shows that all spherical elements in $\mathcal{A}_q(\mathrm{SL}(2, \mathbb{C}))$ are polynomials in $\rho_{\tau,\sigma}$.

Theorem 9.7. *The $*$-subalgebra of (τ, σ)-spherical elements in $\mathcal{A}_q(\mathrm{SU}(2))$ is generated by the self-adjoint element $\rho_{\tau,\sigma}$.*

Proof. Using the $*$-structure given in theorem 7.12(ii) together with the explicit expression for $\rho_{\tau,\sigma}$ in (9.2) one can easily compute that $\rho^*_{\tau,\sigma} = \rho_{\tau,\sigma}$. Next, observe that $\rho^l_{\tau,\sigma} = (\rho_{\tau,\sigma})^l$ is a spherical element by proposition 9.4. We will show that every $b^l_{00}(\tau,\sigma)$ is a polynomial of degree l in $\rho_{\tau,\sigma}$. Since by proposition 9.6 the elements $\{b^l_{00}(\tau,\sigma)\}_{l\in\mathbb{Z}_+}$ form a basis for the spherical elements, this would prove the statement.

The Clebsch-Gordan decomposition from lemma 7.8 gives that $\rho^l_{\tau,\sigma} \in \bigoplus_{k=0}^l \mathrm{A}^k_q$. Moreover, if we apply the one-dimensional $*$-representation $\pi_{\theta/2}$ given in theorem 7.13(i) on $\rho_{\tau,\sigma}$, (9.2) gives us

$$\pi_{\theta/2}(\rho_{\tau,\sigma}) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \cos\theta,$$

since all the terms with β and γ vanish. Therefore,

$$\pi_{\theta/2}(\rho^k_{\tau,\sigma}) = \pi_{\theta/2}(\rho_{\tau,\sigma})^k = (\cos\theta)^k.$$

Since $\{\cos\theta\}^k)^l_{k=0}$ are linearly independent monomials[28], $\{\rho^k_{\tau,\sigma}\}^l_{k=0}$ will be linearly independent in $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$. Therefore, the space with basis $\{b^k_{00}\}^l_{k=0}$ is equal to the space with basis $\{\rho^k_{\tau,\sigma}\}^l_{k=0}$. It now follows by induction that $b^l_{00}(\tau,\sigma)$ is a polynomial of degree l in $\rho^l_{\tau,\sigma}$. $\square$

[28] That is, $\sum_{k=0}^l c_k(\cos\theta)^k = 0$ for $\theta \in [0,\pi]$ implies $c_k = 0$ for all k.

10. The Haar functional

Until now, we have obtained orthogonality via the inner product on a Hilbert space H. We looked at representations on H of elements in $\mathcal{U}_q(\mathfrak{su}(2))$ that are self-adjoint. We then used that self-adjoint operators have a basis of orthogonal eigenvectors with respect to the inner produt of H. However, this does not yield orthogonality relations for the matrix elements of a representation, e.g. t^l_{mn} or $b^l_{ij}(\tau,\sigma)$. These matrix elements are elements of $\mathcal{A}_q(\mathrm{SU}(2))$, which are polynomials in the generators α,β,γ and δ of $\mathcal{A}_q(\mathrm{SU}(2))$. Thus we need an 'inner product' on $\mathcal{A}_q(\mathrm{SU}(2))$. For this we will use the *Haar functional*. The orthogonality arising from this functional are called *Schur's orthogonality relations*. In this section we will introduce this functional, briefly sketch its usefulness in the classical theory and then explicitly calculate what the Haar functional does on the quantum group $\mathcal{A}_q(\mathrm{SU}(2))$, specifically on the subalgebra of (τ,σ)-spherical elements. Recall that this subalgebra can be interpreted as polynomials in $\rho_{\tau,\sigma}$. The main result of this section, theorem 10.10, will be the key between the Askey-Wilson polynomials and the polynomials in $\rho_{\tau,\sigma}$.

10.1. **Introduction Haar functional.** One important theorem in harmonic analysis is the existence of a left-invariant measure μ, unique up to a constant, on the open sets of certain[29] locally compact topological groups G. One can think of groups like $(\mathbb{R}^n,+)$ and $\mathrm{GL}(V)$, where V is a finite-dimensional linear space. This is called the left Haar measure and satisfies

$$\mu(gS)=\mu(S),$$

for all open sets S of G and all $g\in G$. Here, $gS=\{gs : s\in S\}$.

Example 10.1. Take $G=(\mathbb{R},+)$; the group of real numbers with $+$ as operation. Then the left Haar measure is just the Lebesgue measure on $\mathbb{R}$. For example, take $g=3$ and $S=(0,4)$, then $gS=(3,7)$ and

$$\mu((0,4))=4=\mu((3,7)).$$

From the Haar measure, we can define a Haar integral on measurable functions f on G by

$$\int_G f(x)\,\mathrm{d}\mu(x)$$

Because of the left-invariance, we have

$$\int_G f(gx)\,\mathrm{d}\mu(x)=\int_G f(x)\,\mathrm{d}\mu(x)$$

for all $g\in G$. Taking again $G=(\mathbb{R},+)$, we indeed have

$$\int_{\mathbb{R}} f(g+x)\,\mathrm{d}\mu(x)=\int_{\mathbb{R}} f(g)\,\mathrm{d}\mu(x).$$

Similarly, there exists a right Haar measure as well. In general they do not coincide, but on certain groups they do. In commutative groups they trivially coincide. For example, the Lebesgue measure is left and right invariant on $(\mathbb{R},+)$. However, commutative groups are certainly not the only situation where this happens. For example, the left Haar measure on $\mathrm{SU}(2)$ is also right-invariant.

10.2. **Classical motivation.** The following is some well known classical theory, see for example [6]. For[30] $G=\mathrm{SU}(2)$, the Haar measure, denoted by $\mathrm{d}\mu(g)$, exists and is unique op to a constant. It satisfies

$$\int_G p(hg)\,\mathrm{d}\mu(g)=\int_G p(g)\,\mathrm{d}\mu(g)=\int_G p(gh)\,\mathrm{d}\mu(g),$$

[29] Officially, we have to look at the Borel algebra of a locally compact Hausdorff topological group. This will not play a role in this thesis.

[30] Or more general, for any compact Lie group G.

For any polynomial p on G and all $h \in G$. For $p_1, p_2 \in \mathrm{Pol}\,(G)$, we can then define an inner product $\langle \cdot, \cdot \rangle_G$ by

$$\langle p_1, p_2 \rangle_G = \int_G p_1(g)\overline{p_2(g)}\,\mathrm{d}\mu(g).$$

Via Schur's lemma, one can then show Schur's orthogonality relations. Namely, matrix elements of irreducible representations are orthogonal w.r.t. this inner product.

Theorem 10.2. *Let $\pi^{(1)}$ and $\pi^{(2)}$ be irreducible inequivalent unitary representations of $G = SU(2)$ on finite-dimensional complex Hilbert spaces H_1 and H_2 respectively. Let $(e_n^{(1)})$, $n = 1, .., \dim(H_1)$, and $(e_m^{(2)})$, $m = 1, ..., \dim(H_2)$, denote orthonormal bases for H_1 and H_2. Define the matrix elements $\pi_{mn}^{(1)}$ and $\pi_{mn}^{(2)}$ by*

$$\pi_{mn}^{(k)}(g) = \left\langle \pi^{(k)}(g)e_n^{(k)}, e_m^{(k)} \right\rangle,\ k = 1, 2 \text{ and } g \in G.$$

Let $k, k' \in \{1, 2\}$, then the following orthogonality relations hold,

$$\left\langle \pi_{ij}^{(k)}, \pi_{mn}^{(k')} \right\rangle_G = \frac{1}{\dim(H_k)}\delta_{k,k'}\delta_{im}\delta_{jn}.$$

For example, let π^l be the $(2l+1)$-dimensional representation from (3.1). The matrix elements π_{mn}^l w.r.t the basis given there are the product of Jacobi polynomials of degree $l-n$ and complex exponentials in m, i.e. $P_{l-n}(x)e^{mix}$, where P_k is a Jacobi polynomial of degree k and x is the variable. Schur's orthogonality relations for these matrix elements correspond to the orthogonality of Jacobi polynomials and complex exponentials[31].

10.3. **The Haar functional on $\mathcal{A}_q(\mathrm{SU}(2))$.** Our next step is to put the idea of the Haar measure into the language of Hopf algebras. Let us take a look at at the Hopf algebra $\mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$. Let $\mathrm{d}\mu(g)$ be the Haar measure on $\mathrm{SL}(2,\mathbb{C})$. Then we have for $p \in \mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$,

$$\int_G p(g'g)\,\mathrm{d}\mu(g) = \int_G p(g)\,\mathrm{d}\mu(g),\ \forall g' \in G.$$

If we define $h(p) = \int_G p(g)\,\mathrm{d}\mu(g)$, then the above equation can be written as

$$(\mathrm{id} \otimes h)\Delta(p)(g') = h(p).$$

This follows from $\Delta(p)(g'g) = p(g'g)$. Similarly, right invariance can be written as

$$(h \otimes \mathrm{id})\Delta(p)(g') = h(p).$$

We can use this to define a Haar functional on a Hopf algebra.

Definition 10.3. *Let A be a Hopf algebra and assume that a linear functional $h : A \to \mathbb{C}$ exist that satisfies*

$$(\mathrm{id} \otimes h) \circ \Delta = \eta \circ h = (h \otimes \mathrm{id}) \circ \Delta. \tag{10.1}$$

Then h is a Haar functional, also called an invariant functional, on A. If it exists, we normalise it by $h(1_A) = 1$.

Remark 10.4. If only the first or second equation holds in (10.1), we say that h is a left, respectively right, invariant functional.

Remark 10.5. If the Haar functional exists, it is unique up to a constant. The interest reader can find a proof, using C^*-algebras, in Woronowicz's influential article [17]. Dijkhuizen and Koorwinder [4], later showed that C^*-algebra theory is not necessary for proving the uniqueness of the Haar functional. However, proving this fundamental result will not be necessary for this thesis, since we won't need uniqueness.

We explicitly know the Haar functional on $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$.

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[31]Orthogonality for complex exponentials is given by $\int_0^{2\pi} e^{imx}e^{-inx}\mathrm{d}x = 2\pi\delta_{mn}$.

Theorem 10.6. *Define $h : \mathcal{A}_q(\mathrm{SL}(2,\mathbb{C})) \to \mathbb{C}$ by $h(1) = 1$ and $h(t^l_{mn}) = 0$ for $(l,m,n) \neq (0,0,0)$. Then h is the Haar functional on $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$. Moreover, we have*

$$h(S(t^l_{mn})t^k_{ij}) = \delta_{lk}\delta_{mj}\delta_{ni}q^{2(l-n)}\frac{1-q^2}{1-q^{4l+2}}. \tag{10.2}$$

Proof. Let us first show this is well-defined. If $t^0_{00} := 1 \in \mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ could be written as a linear combination of t^l_{mn}, where $(l,m,n) \neq (0,0,0)$, we would have a problem. Let us check this cannot happen. Suppose that

$$t^0_{00} = \sum_{m,n,l} c_{mnl} t^l_{mn}.$$

Via the duality (7.2) we have $\langle 1, X\rangle = \varepsilon(X)$. Therefore, we obtain

$$\varepsilon(X) = \sum_{m,n,l} c_{mnl}\left\langle t^l(X)e^l_n, e^l_m\right\rangle, \quad \forall X \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C})).$$

In particular, this holds for all K^j, $j \in \mathbb{Z}$. This gives

$$1 = \varepsilon(K^j) = \sum_{m,n,l} c_{mnl}\left\langle t^l(K^j)e^l_n, e^l_m\right\rangle = \sum_{m,n,l} c_{mnl}q^{-nj}\left\langle e^l_n, e^l_m\right\rangle \quad \forall j \in \mathbb{Z}.$$

Since q is not a root of unity, $\left\langle e^l_n, e^l_m\right\rangle = \delta_{mn}$ and this holds for all $j \in \mathbb{Z}$, we must have $c_{nnl} = 0$ for all $n, l : n \neq 0$. Repeating this process for K^jE^i and K^jF^k, $i,k = 0,..,2l$, we can show that $c_{mnl} = 0$ for all m,n,l: $(m,n) \neq (0,0)$. Therefore,

$$\varepsilon(X) = \sum_l c_{00l}\left\langle t^l(X)e^l_0, e^l_0\right\rangle.$$

However, if $c_{00l} \neq 0$, then $\mathrm{span}\{e^l_0\}$ is an invariant subspace for t^l, contradicting the irreducibility of t^l. Indeed, suppose there exists a $Y \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$, such that $t^l(Y)e^L_0$ has a nonzero component in e^l_k, $k > 0$. We apply E^kK^jY for $j \in \mathbb{Z}$ and similarly as before we get $c_{00l} = 0$. Thus h is well-defined.
Next, we can use proposition 7.10 to show that

$$(\mathrm{id} \otimes h) \circ \Delta(t^l_{mn}) = \sum_{k=-l}^{l} t^l_{mk}h(t^l_{kn}) = \delta_{l0}t^0_{00} = \eta \circ h(t^l_{mn}).$$

Therefore, h is a left invariant functional. Similarly, we obtain that h is right invariant. Therefore, h is the Haar functional on $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$.
Next, we will prove (10.2). Using that h is right invariant, the Hopf algebra relations (2.3), one can show[32],

$$\sum_{j=-k}^{k} h\left(S(t^l_{mn})t^k_{ij}\right)t^k_{jp} = \sum_{j=-l}^{l} t^l_{mj}h\left(S(t^l_{jn})t^k_{ip}\right). \tag{10.3}$$

Define the matrix

$$T^{(n,i)}_{mj} = h\left(S(t^l_{mn})t^k_{ij}\right),$$

then (10.3) implies

$$\left(T^{(n,i)}t^k(X)\right)_{m,p} = \left(t^l(X)T^{(n,i)}\right)_{m,p} \quad \text{for all } X \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C})).$$

Invoking Schur's Lemma now gives $T^{(n,i)} = 0$ or $T^{(n,i)}$ is bijective. If $T^{(n,i)}$ is bijective, t^l and t^k are equivalent, which can only happen if $l = k$. Therefore,

$$h\left(S(t^l_{mn})t^k_{ij}\right) = T^{(n,i)}_{mj} = 0 \quad \text{if } l \neq k.$$

In the case $l = k$, (a corollary of) Schur's lemma tells us that

$$T^{(n,i)} = c^{(n,i)}\mathrm{id},$$

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[32] see e.g. [10]

for some constants $c^{(n,i)} \in \mathbb{C}$. Which implies $T^{(n,i)}_{mj} = \delta_{mj} c^{(n,i)}$. Next, for a representation t on V, we define the *contragradient* representation t^c by

$$\langle t^c(X)e_m, e_n\rangle = \langle t(S(X))e_n, e_m\rangle \,.$$

Note that we switched the m and n in the inner product. The reader should verify that this is indeed a representation. Equivalently, we can rewrite above expression to $t^c_{nn} = S(t_{mn})$. Looking at t^l, we see that the spectrum of K w.r.t the contragredient representation is also contained in $q^{\frac{1}{2}\mathbb{Z}}$. Therefore, the contragredient representation of t^l is equivalent to t^l. Hence $S(t^l_{mn}) \in \mathrm{A}^l_q$ and $S(t^0_{00}) = t^0_{00}$. This implies $h(S(\xi)) = h(\xi)$, since any $\xi \in \mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ can be written as a linear combination of t^l_{mn}, where at least[33] the factor in front of the t^0_{00} is unique. Consequently,

$$T^{(n,i)}_{mj} = h\left(S\left(S(t^l_{mn})t^k_{ij}\right)\right) = h\left(S(t^k_{ij})S^2(t^l_{mn})\right). \tag{10.4}$$

Now, S^2 are the matrix coefficients of the double contragredient representation $(t^l)^{cc}$, and is therefore equivalent to t^l. Thus there exists a bijective linear mapping G such that

$$(t^l)^{cc}(X) = Gt^l(X)G^{-1}, \quad \text{for all } X \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C})). \tag{10.5}$$

This implies

$$S^2(t^l_{mn}) = \sum_{p,r=-l}^{l} G_{mp} t^l_{pr} (G^{-1})_{rn}.$$

Substituting this into (10.4) gives,

$$\begin{aligned}\delta_{kl}\delta_{mj}c^{(n,i)} = T^{(n,i)}_{mj} &= \sum_{p,r=-l}^{l} G_{mp} h(S(t^k_{ij})t^l_{pr})(G^{-1})_{rn} = \sum_{p,r=-l}^{l} \delta_{kl}\delta_{ir} G_{mp} c^{(j,p)} (G^{-1})_{rn} \\ &= \delta_{kl}(G^{-1})_{in} \sum_{p=-l}^{l} G_{mp} c^{(j,p)}.\end{aligned}$$

This means that $c^{(n,i)} = c(G^{-1})_{in}$ for some constant $c \in \mathbb{C}$. By the relation for the antipode (2.3) and proposition 7.10, we get

$$\sum_{m=-l}^{l} S(t^l_{km} t^l_{mr}) = \varepsilon(t^l_{kr}) = \delta_{kr}.$$

Applying h on both sides gives

$$1 = \sum_{m=-l}^{l} h\left(S(t^l_{km}t^l_{mk})\right) = \sum_{m=-l}^{l} T^{(m,m)}_{kk} = \sum_{m=-l}^{l} c^{(m,m)}.$$

Therefore,

$$c \sum_{m=-l}^{l} (G^{-1})_{mm} = 1 \quad \Longrightarrow \quad c^{-1} = \mathrm{tr}\left(G^{-1}\right).$$

Combining everything so far, we have

$$h\left(S(t^l_{mn})t^k_{ij}\right) = \delta_{kl}\delta_{mj} \frac{(G^{-1})_{in}}{\mathrm{tr}\,(G^{-1})}. \tag{10.6}$$

Let us explicitly find such a bijective mapping G for which (10.5) holds. Since

$$S^2(K) = K, \quad S^2(E) = q^{-2}E, \quad S^2(F) = q^2 F,$$

we see that G defined by $Ge_n = q^{2n}e_n$ is such a mapping. Indeed, we have

$$\begin{aligned}Gt^l(E)G^{-1}e_n &= q^{-2n}Gt^l(E)e_n = q^{-2n+2(n-1)}t^l(E)e_n = q^{-2}t^l(E)e_n \\ &= t^l(S^2(E))e_n = (t^l)^{cc}(E)e_n,\end{aligned}$$

[33] The others are unique as well, but at this point we do not know yet whether the t^l_{mn} are linearly independent for $l > 0$.

and similarly for the other generators of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. Finally, calculating (10.6) gives the desired expression,

$$h\left(S(t^l_{mn})t^k_{ij}\right) = \delta_{kl}\delta_{mj}\delta_{in}q^{-2n}\left(\sum_{p=-l}^{l} q^{-2p}\right)^{-1} = \delta_{kl}\delta_{mj}\delta_{in}\frac{1-q^2}{q^{-2l}-q^{2l+2}}$$
$$= \delta_{lk}\delta_{mj}\delta_{ni}q^{2(l-n)}\frac{1-q^2}{1-q^{4l+2}}. \qquad \square$$

We will deduce Schur's orthogonality relations from this, i.e. orthogonality of the matrix elements t^l_{mn} w.r.t the Haar functional.

Proposition 10.7. *Let h be the Haar functional on $\mathcal{A}_q(\mathrm{SU}(2))$. Then we have*

$$h((t^l_{mn})^*t^k_{ij}) = \delta_{lk}\delta_{mi}\delta_{nj}q^{2(l-n)}\frac{1-q^2}{1-q^{4l+2}}. \tag{10.7}$$

In particular, $h : \mathcal{A}_q(\mathrm{SU}(2)) \to \mathbb{C}$ is a positive functional, i.e. we have $h(\xi^*\xi) > 0$ for $\xi \in \mathcal{A}_q(\mathrm{SU}(2))\backslash\{0\}$.

Proof. Using theorem 10.6 we have to show that $S(t^l_{nm}) = (t^l_{mn})^*$. Note that the order of m and n is changed. To avoid confusion let $\langle\cdot,\cdot\rangle_H$ denote the inner product on the Hilbert space induced by the orthonormal basis $(e_n)^l_{n-l}$ in theorem 4.12 and $\langle\cdot,\cdot\rangle$ denote the duality between the Hopf $*$-algebras $\mathcal{U}_q(\mathfrak{su}(2))$ and $\mathcal{A}_q(\mathrm{SU}(2))$. Using definition 7.2 and the fact that t^l is a $*$-representation of $\mathcal{U}_q(\mathfrak{su}(2))$, we obtain for all $X \in \mathcal{U}_q(\mathfrak{su}(2))$,

$$\left\langle (t^l_{mn})^*, X\right\rangle = \overline{\langle t^l_{mn}, S(X)^*\rangle} = \left\langle e_m, t^l\left(S(X)^*\right)e_n\right\rangle_H = \left\langle t^l\left(S(X)\right)e_m, e_n\right\rangle_H = \left\langle t^l_{nm}, S(X)\right\rangle$$
$$= \left\langle S(t^l_{mn}), X\right\rangle,$$

where we used (7.2) in the last step. Next, let us show that h is a positive functional. Since the matrix elements t^l_{mn} form a basis for $\mathcal{A}_q(\mathrm{SU}(2))$ and all t^l_{mn} satisfy Schur's orthogonality relation (10.7), we only have to show that

$$h((t^l_{mn})^*t^l_{mn}) > 0.$$

This easily follows from (10.7) and the fact that $q \in \mathbb{R}$. $\square$

We will need another explicit expression for the Haar functional. This equation, involving the trace, will be used to calculate the Haar functional on the subalgebra of (τ,σ)-spherical elements. We first need the following lemma, which tells us what the Haar functional does on the basis elements of $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ given in theorem 7.7.

Lemma 10.8. *We have $h(\delta^k\gamma^m\beta^n) = 0 = h(\alpha^k\gamma^m\beta^n)$ unless $k = 0$ and $m = n$. In that case, we have*

$$h(\gamma^n\beta^n) = (-q)^n\frac{1-q^2}{1-q^{2n+2}}.$$

Proof. Take $a \in \mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$ and $X \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. Using that h is right-invariant in combination with proposition (8.1), we get

$$h(X.a) = (h \otimes \langle X,\cdot\rangle)\circ\Delta(a) = \langle X,\cdot\rangle\circ(h\otimes\mathrm{id})\circ\Delta(a) = \langle X, \eta(h(a))\rangle = h(a)\langle X, 1\rangle$$
$$= h(a)\varepsilon(X). \tag{10.8}$$

Similarly, we get $h(a.X) = h(a)\varepsilon(X)$. Using (8.4), we get for $X, Y \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and $a, b \in \mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$,

$$\langle Y, X.(ab)\rangle = \langle YX, ab\rangle = \langle\Delta(Y)\Delta(X), a\otimes b\rangle = \sum_{(X)}\left\langle Y, (X_{(1)}.a)(X_{(2)}.b)\right\rangle,$$

hence

$$X.(ab) = \sum_{(X)}(X_{(1)}.a)(X_{(2)}.b). \tag{10.9}$$

Therefore, if X is group-like, we get

$$X.(ab) = (X.a)(X.b).$$

In the same way, we get that the right action is an algebra homomorphism,

$$(ab).X = (a.X)(b.X).$$

Let $a = \delta^k\gamma^m\beta^n$, then we obtain

$$X.(\delta^k\gamma^m\beta^n) = (X.\delta)^k(X.\gamma)^m(X.\beta)^n \quad \text{and} \quad (\delta^k\gamma^m\beta^n).X = (\delta.X)^k(\gamma.X)^m(\beta.X)^n. \tag{10.10}$$

Let us calculate $K.a$ and $a.K$ for the generators β, γ and δ. Using (7.7) and (7.4), we get

$$\langle Y, K.\beta\rangle = \langle YK, \beta\rangle = \langle Y \otimes K, \delta(\beta)\rangle = \langle Y, \alpha\rangle \langle K, \beta\rangle + \langle Y, \beta\rangle \langle K, \delta\rangle = q^{-1/2} \langle Y, \beta\rangle,$$

thus $K.\beta = q^{-1/2}\beta$. Continuing this way, we obtain

$$K.\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} q^{1/2}\alpha & q^{-1/2}\beta \\ q^{1/2}\gamma & q^{-1/2}\delta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.K = \begin{pmatrix} q^{1/2}\alpha & q^{1/2}\beta \\ q^{-1/2}\gamma & q^{-1/2}\delta \end{pmatrix}. \tag{10.11}$$

Combining this with (10.8), (10.10) and $\varepsilon(K) = 1$, we get

$$\begin{aligned} h(\delta^k\gamma^m\beta^n) &= \varepsilon(K)h(\delta^k\gamma^m\beta^n) = h(K.(\delta^k\gamma^m\beta^n)) = h\left((K.\delta)^k(K.\gamma)^m(K.\beta)^n\right) \\ &= q^{(-k+m-n)/2}h(\delta^k\gamma^m\beta^n), \end{aligned}$$

and similarly with $.K$,

$$h(\delta^k\gamma^m\beta^n) = \varepsilon(K)h(\delta^k\gamma^m\beta^n) = h\left((\delta^k\gamma^m\beta^n).X\right) = q^{(-k-m+n)/2}h(\delta^k\gamma^m\beta^n).$$

Therefore, $h(\delta^k\gamma^m\beta^n) \neq 0$ implies

$$-k+m-n = 0 \quad \text{and} \quad -k-m+n = 0.$$

Adding these equations gives $k = 0$, subtracting them gives $m = n$. Thus $h(\delta^k\gamma^m\beta^n)$ can be non-zero only if $k = 0$ and $m = n$. Similarly, we can find that the exact same holds for $h(\alpha^k\gamma^m\beta^n)$.

It remain to calculate $h(\gamma^n\beta^n)$. Similar to (10.9), we have for the right action

$$(ab).X = \sum_{(X)}(a.X_{(1)})(b.X_{(2)}).$$

Using this for $X = F$, $a = \delta\gamma(-q^{-1}\beta\gamma)^{n-1}$ and $b = -q^1\beta\gamma$, we get,

$$\delta\gamma(-q^{-1}\beta\gamma)^n.F = \left(\delta\gamma(-q^{-1}\beta\gamma)^{n-1}.K\right)\left(-q^{-1}\beta\gamma.F\right) + \left(\delta\gamma(-q^{-1}\beta\gamma)^{n-1}.F\right)\left(-q^{-1}\beta\gamma.K^{-1}\right).$$

Similar to (10.11), we can derive

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.F = \begin{pmatrix} 0 & 0 \\ \alpha & \beta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.K^{-1} = \begin{pmatrix} q^{-1/2}\alpha & q^{-1/2}\beta \\ q^{1/2}\gamma & q^{1/2}\delta \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \delta\gamma(-q^{-1}\beta\gamma)^n.F &= \left(q^{-1}\delta\gamma(-q^{-1}\beta\gamma)^{n-1}\right)\left(-q^{-1/2}\beta\alpha\right) + \left(\delta\gamma(-q^{-1}\beta\gamma)^{n-1}.F\right)\left(-q^{-1}\beta\gamma\right) \\ &= q^{-1/2-2n}\left((-q^{-1}\beta\gamma)^n\right)\left(1 + -q^{-1}\beta\gamma\right) + \left(\delta\gamma(-q^{-1}\beta\gamma)^{n-1}.F\right)\left(-q^{-1}\beta\gamma\right), \end{aligned}$$

where we used the commutation relations (7.9) for $\mathcal{A}_q(\mathrm{SL}(2,\mathbb{C}))$. Together with the intital condition

$$\delta\gamma(-q^{-1}\beta\gamma)^0.F = \delta\gamma.F = q^{-\frac{1}{2}}(1 + (q^{-1}+q)\beta\gamma),$$

this recurrence relation gives

$$\delta\gamma(-q^{-1}\beta\gamma)^n.F = \frac{q^{-1/2-2n}}{1-q^2}\left((1-q^{2n+2})(-q^{-1}\beta\gamma)^n - (1-q^{2n+4})(-q^{-1}\beta\gamma)^{n+1}\right).$$

Applying h and using (10.8), gives

$$h\left(\frac{q^{-1/2-2n}}{1-q^2}\left((1-q^{2n+2})(-q^{-1}\beta\gamma)^n - (1-q^{2n+4})(-q^{-1}\beta\gamma)^{n+1}\right)\right) = h(\delta\gamma(-q^{-1}\beta\gamma)^n)\varepsilon(F) = 0.$$

Solving this two-term recurrence relation, together with the initial condition $h(1) = 1$, gives

$$h(\gamma^n\beta^n) = (-q)^n \frac{1-q^2}{1-q^{2n+2}}. \qquad \square$$

Using the lemma above, we can prove an explicit expression of the Haar functional in terms of the infinite-dimensional representation π_θ^∞ from theorem 7.13. This will be crucial, since π_θ^∞ will act as a tridiagonal operator on $\rho_{\tau,\sigma}$, which will correspond to a three-term recurrence relation of orthogonal polynomials.

Theorem 10.9. *Let D be the diagonal operator on $\ell^2(\mathbb{Z}_+)$ defined on the orthonormal basis $(e_n)_{n=0}^\infty$ by $De_n = q^{2n}e_n$. Then for $a \in \mathcal{A}_q(\mathrm{SU}(2))$ we have*

$$h(a) = \frac{(1-q^2)}{2\pi}\int_0^{2\pi} tr\left(D\pi_\theta^\infty(a)\right) \mathrm{d}\theta, \tag{10.12}$$

where π_θ^∞ is the infinite-dimensional $*$-representation of $\mathcal{A}_q(\mathrm{SU}(2))$ given in theorem 7.13(ii) and tr is the trace operator.

Proof. Let us first check that the trace and integral is well-defined, i.e. we will prove that $D\pi_\theta^\infty(a)$ is a trace class operator and is uniformly bounded in θ. We will show that $\pi_\theta^\infty(a)D$ is the product of two Hilbert-Schmidt operators, hence trace class. By the cyclic property of the trace we then get the same for $D\pi_\theta^\infty(a)$.
Observe that $D^{1/2}$, given by $D^{1/2}e_n = q^n e_n$, is Hilbert-Schmidt, since

$$||D^{1/2}||_{HS}^2 \sum_{n\in\mathbb{Z}_+}\left\langle D^{1/2}e_n, D^{1/2}e_n\right\rangle = \sum_{n\in\mathbb{Z}_+} q^{2n} = \frac{1}{1-q^2}$$

is finite. Moreover, π_θ^∞ on $\alpha,\beta,\gamma,\delta$ is bounded. Therefore, the operator $\pi_\theta^\infty(a)$ is bounded for every $a \in \mathcal{A}_q(\mathrm{SU}(2))$ by some constant $C(a)$, independent of θ. Thus,

$$||\pi_\theta^\infty(a)D^{1/2}||_{HS}^2 = \sum_{n\in\mathbb{Z}_+}\left\langle \pi_\theta^\infty(a)D^{1/2}e_n, \pi_\theta^\infty(a)D^{1/2}e_n\right\rangle \le \frac{C(a)^2}{1-q^2},$$

hence $\pi_\theta^\infty(a)D^{1/2}$ is Hilbert-Schmidt as well. The trace of the product of two Hilbert-Schmidt operators is well defined. Therefore, by the cyclic property, we have

$$|\mathrm{tr}\,(D\pi_\theta^\infty(a))| = |\mathrm{tr}\,(\pi_\theta^\infty(a)D)| = \sum_{n\in\mathbb{Z}_+} |\langle \pi_\theta^\infty(a)De_n, e_n\rangle| \le \frac{C(a)}{1-q^2}.$$

This also shows that $\mathrm{tr}\,(D\pi_\theta^\infty(a))$ is integrable.

Let us now prove (10.12). Observe that by linearity, it is sufficient to prove it for the basis elements $\delta^k\gamma^m\beta^n$ and $\alpha^k\gamma^m\beta^n$, $k,m,n \in \mathbb{Z}_+$. We will prove this for $\alpha^k\gamma^b\beta^n$, the other case is similar. We check that the RHS of (10.12) coincides with the expression for $h(\alpha^k\gamma^m\beta^n)$ given in lemma 10.8. We have

$$\pi_\theta^\infty(\alpha^k\gamma^m\beta^n)e_j = e^{i\theta(m-n)}q^{(m+n)j}(-q)^n\pi_\theta^\infty(\alpha^k)e_j.$$

Therefore,

$$\mathrm{tr}\left(D\pi_\theta^\infty(\alpha^k\gamma^m\beta^n)\right) = e^{i\theta(m-n)}(-q)^n\sum_{j\in\mathbb{Z}_+} q^{(m+n)j}\left\langle D\pi_\theta(\alpha^k)e_j, e_j\right\rangle. \tag{10.13}$$

Since

$$\left\langle D\pi_\theta(\alpha^k)e_j, e_j\right\rangle = C\left\langle e_{j-k}, e_j\right\rangle = C\delta_{k0}$$

for some $C \in \mathbb{C}$, (10.13) is 0 if $k > 0$. If $k = 0$, we get

$$\begin{aligned}\mathrm{tr}\,(D\pi_\theta^\infty(\gamma^m\beta^n)) &= e^{i\theta(m-n)}(-q)^n\sum_{j\in\mathbb{Z}_+} q^{(m+n)j}\left\langle De_j, e_j\right\rangle = e^{i\theta(m-n)}(-q)^n\sum_{j\in\mathbb{Z}_+} q^{(m+n+2)j}.\\ &= \frac{(-q)^n}{1-q^{m+n+2}}e^{i\theta(m-n)}.\end{aligned}$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i\theta(m-n)}\,\mathrm{d}\theta = \delta_{mn},$$

we have

$$\frac{(1-q^2)}{2\pi}\int_0^{2\pi} \mathrm{tr}\left(D\pi_\theta^\infty(\alpha^k\gamma^m\beta^n)\right)\mathrm{d}\theta = \delta_{k0}\delta_{mn}\frac{(-q)^m(1-q^2)}{1-q^{2m+2}},$$

exactly as in lemma 10.8. □

Now we have all the ingredients to prove the key theorem that links the quantum group $\mathcal{A}_q(\mathrm{SU}(2))$ with the Askey-Wilson polynomials.

Theorem 10.10. *The Haar functional on the subalgebra of (τ,σ)-spherical elements, which is generated by $\rho_{\tau,\sigma}$, is given by*

$$h(p(\rho_{\tau,\sigma})) = \int_{\mathbb{R}} p(x)\,\mathrm{dm}(x;a,b,c,d|q^2), \tag{10.14}$$

where p is any polynomial, $dm(x;a,b,c,d|q^2)$ is the normalised measure corresponding to the Askey-Wilson polynomials given in (5.5) *and $a=-q^{\sigma+\tau+1}$, $b=-q^{-\sigma-\tau+1}$, $c=q^{\sigma-\tau+1}$ and $d=q^{-\sigma+\tau+1}$.*

Proof. The proof has quite some computational complexity. Therefore, we will restrict ourselves of proving a simpler version of the theorem. Namely for the subalgebra generated by the self-adjoint element $\alpha+\alpha^*$. Then we have for any polynomial p that

$$h\left(p\left(\frac{\alpha+\alpha^*}{2}\right)\right) = \frac{2}{\pi}\int_{-1}^{1} p(x)\sqrt{1-x^2}\,\mathrm{d}x. \tag{10.15}$$

All steps in the proof are the same in this case, but the computations done require much less work. We will proceed in three steps. First, we will show that $\pi_\theta^\infty(\alpha+\alpha^*)$ is a self-adjoint operator which is tridiagonal w.r.t the standard basis $\{e_n\}_{n=0}^\infty$ of $\ell^2(\mathbb{Z}_+)$. This will then correspond with a three-term recurrence relation for some orthonormal polynomials $\{p_n(x)\}_{n=0}^\infty$. Secondly, we make a unitary mapping $\Lambda:\ell^2(\mathbb{Z}_+)\to L^2(\mu)$ that sends e_n to the n-th degree orthonormal polynomial p_n. Here, $L^2(\mu)$ is the weighted L^2 space corresponding to the orthogonality measure μ of $\{p_n(x)\}_{n=0}^\infty$. Then lastly, we will use that Λ is unitary in combination with theorem 10.9 to link the Haar functional on $p\left(\frac{\alpha+\alpha^*}{2}\right)$ with the integral coming from the inner product of $L^2(\mu)$.

From the fact that $\alpha+\alpha^*$ is self-adjoint in $\mathcal{A}_q(\mathrm{SU}(2))$ and π_θ is a $*$-representation, we know that $\pi_\theta^\infty(\alpha+\alpha^*)$ is self-adjoint. Moreover, using the explicit expression for π_θ^∞ in theorem 7.13(ii), we obtain

$$2\pi_\theta^\infty\left(\frac{\alpha+\alpha^*}{2}\right)e_n = \sqrt{1-q^{2n}}e_{n-1}+\sqrt{1-q^{2n+2}}e_{n+1}. \tag{10.16}$$

The three-term recurrence relation for the continuous q-Hermite polynomials $H_n(x|q)$ is given by

$$2xH_n(x|q) = H_{n+1}(x|q)+(1-q^n)H_{n-1}(x|q),\quad H_{-1}(x|q)=0,\quad H_0(x|q)=1,\quad x=\cos\phi. \tag{10.17}$$

They satisfy the orthogonality relation

$$\int_{-1}^{1} H_n(x|q)H_m(x|q)w(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \delta_{mn}h_n, \tag{10.18}$$

where

$$w(x|q) = (e^{2i\phi},e^{-2i\phi};q)_\infty,\quad x=\cos\phi,$$
$$h_n = \frac{2\pi(q;q)_n}{(q;q)_\infty}.$$

The continuous q-Hermite polynomials are the special case[34] $a = b = c = d = 0$ of Askey-Wilson polynomials. Since a self-adjoint operator has orthonormal eigenvectors, we normalize the polynomials H_n and the measure $w(x)\ \mathrm{d}x$, i.e.

$$p_n(x|q) = \sqrt{\frac{h_0}{h_n}} H_n(x|q^2) = \frac{H_n(x|q^2)}{\sqrt{(q:q)_n}},$$
$$\mathrm{dm}(x|q) = w(x|q) h_0^{-1} \frac{\mathrm{d}x}{\sqrt{1-x^2}} = w(x|q) \frac{(q;q)_\infty}{2\pi} \frac{\mathrm{d}x}{\sqrt{1-x^2}}.$$

Substituting $p_n(x|q^2)$ into (10.17) and multiplying by $\sqrt{\frac{h_0}{h_n}}$, gives

$$2x p_n(x|q^2) = \sqrt{\frac{h_{n+1}}{h_n}} p_{n+1}(x|q^2) + (1-q^{2n}) \sqrt{\frac{h_{n-1}}{h_n}} p_{n-1}(x|q^2).$$

Since

$$\sqrt{\frac{h_{n+1}}{h_n}} = \sqrt{1-q^{2n+2}} \quad \text{and} \quad \sqrt{\frac{h_{n-1}}{h_n}} = \frac{1}{\sqrt{1-q^{2n}}},$$

we obtain

$$2x p_n(x|q^2) = \sqrt{1-q^{2n+2}} p_{n+1}(x|q^2) + \sqrt{1-q^{2n}} p_{n-1}(x|q^2).$$

Comparing this with (10.16), gives that $\pi_\theta\left(\frac{\alpha+\alpha^*}{2}\right)$ works as multiplication operator on $p_n(x|q^2)$. That is, if we define $\Lambda : \ell^2(\mathbb{Z}_+) \to L^2\left(\mathrm{dm}(x|q^2)\right)$ by $\Lambda e_n = p_n(x|q^2)$, where

$$\langle f, g \rangle_{L^2(\mathrm{dm}(x|q^2))} = \int_{-1}^{1} f(x) g(x)\ \mathrm{dm}(x|q^2),$$

we have

$$\left[\Lambda \circ \pi_\theta\left(\left(\frac{\alpha+\alpha^*}{2}\right)^k\right) e_n\right](x) = x^k \cdot \left[\Lambda e_n\right](x).$$

Therefore, for any polynomial p,

$$\left[\Lambda \circ \pi_\theta\left(p\left(\frac{\alpha+\alpha^*}{2}\right)\right) e_n\right](x) = p(x) \cdot \left[\Lambda e_n\right](x).$$

Moreover, using the orthonormality of $\{e_n\}_{n=0}^\infty$ in $\ell^2(\mathbb{Z}_+)$ and $\{p_n\}_{n=0}^\infty$ in $L^2\left(\mathrm{dm}(x|q^2)\right)$, we can see that Λ is unitary:

$$\langle e_n, e_m \rangle_{\ell^2(\mathbb{Z}_+)} = \delta_{mn} = \langle p_n, p_m \rangle_{L^2(\mathrm{dm}(x|q^2))} = \langle \Lambda e_n, \Lambda e_m \rangle_{L^2(\mathrm{dm}(x|q^2))}.$$

We want to use theorem 10.9, so let us calculate the trace. We get

$$\begin{aligned}
\operatorname{tr}\left(D\pi_\theta\left(p\left(\frac{\alpha+\alpha^*}{2}\right)\right)\right) &= \sum_{n\in\mathbb{Z}_+} q^{2n} \left\langle \pi_\theta\left(p\left(\frac{\alpha+\alpha^*}{2}\right)\right) e_n, e_n \right\rangle_{\ell^2(\mathbb{Z}_+)} \\
&= \sum_{n\in\mathbb{Z}_+} q^{2n} \left\langle \Lambda\pi_\theta\left(p\left(\frac{\alpha+\alpha^*}{2}\right)\right) e_n, \Lambda e_n \right\rangle_{L^2(\mathrm{dm}(x|q^2))} \\
&= \sum_{n\in\mathbb{Z}_+} q^{2n} \langle p(x)\Lambda e_n, \Lambda e_n \rangle_{L^2(\mathrm{dm}(x|q^2))} \\
&= \frac{(q;q)_\infty}{2\pi} \sum_{n\in\mathbb{Z}_+} q^{2n} \int_{-1}^{1} p(x) \left(p_n(x|q^2)\right)^2 w(x|q^2) \frac{\mathrm{d}x}{\sqrt{1-x^2}}.
\end{aligned}$$

Observe that

$$\sum_{n=0}^{N} q^{2n} p(x) \left(p_n(x|q^2)\right)^2 w(x|q^2) \le C \frac{\mathrm{d}x}{\sqrt{1-x^2}},$$

[34] One has to be careful with directly taking the limit in the definition 5.17 of Askey-Wilson polynomials due to the appearance of a^n in the denominator. The weight function is more insightful here, since we can just take $a = b = c = d = 0$ there.

for some constant C independent of N. Since the latter is integrable, we can interchange the summation and integral using dominated convergence. We obtain,

$$\operatorname{tr}\left(D\pi_\theta\left(p\left(\frac{\alpha+\alpha^*}{2}\right)\right)\right)=\frac{(q;q)_\infty}{2\pi}\int_{-1}^{1}p(x)P_{q^2}(x,x|q^2)2w(x|q^2)\frac{\mathrm{d}x}{\sqrt{1-x^2}}, \tag{10.19}$$

where $P_t(x,y|q)$ is the Poisson kernel for the continuous q-Hermite polynomials given by

$$P_t(\cos\phi,\cos\psi|q)=\sum_{n\in\mathbb{Z}_+}t^n\,(p_n(\cos\phi|q)(p_n(\cos\psi|q))=\frac{(t^2;q)_\infty}{(te^{i\phi+i\psi},te^{i\phi-i\psi},te^{-i\phi+i\psi},te^{-i\phi-i\psi},;q)_\infty}.$$

The second identity is known as Mehler's formula for Hermite polynomials (see e.g. [1]). After a nice calculation, where many terms cancel, we obtain

$$w(x|q^2)P_{q^2}(x,x|q^2)=\frac{4(1-x^2)}{(1-q^2)(q^2;q^2)_\infty}.$$

Therefore,

$$\operatorname{tr}\left(D\pi_\theta\left(p\left(\frac{\alpha+\alpha^*}{2}\right)\right)\right)=\frac{(q;q)_\infty}{2\pi}\int_{-1}^{1}p(x)\frac{4(1-x^2)}{(1-q^2)(q^2;q^2)_\infty}\frac{\mathrm{d}x}{\sqrt{1-x^2}},$$

which implies

$$\operatorname{tr}\left(D\pi_\theta\left(p\left(\frac{\alpha+\alpha^*}{2}\right)\right)\right)=\frac{2}{\pi(1-q^2)}\int_{-1}^{1}p(x)\sqrt{1-x^2}\,\mathrm{d}x.$$

Note that the RHS is independent of θ. Therefore, the result (10.15) now easily follows from theorem 10.9 after a simple integration. $\square$

11. Generalised matrix elements and Askey-Wilson polynomials

In this section we will put all pieces of the puzzle together. We link the matrix elements of the quantum group $\mathcal{A}_q(\mathrm{SU}(2))$ with the Askey-Wilson polynomials. We will establish this by combining Schur's orthogonality relations for the generalised matrix elements $b^l_{ij}(\tau,\sigma)$ in (10.2) with the measure for the Askey-Wilson polynomials obtained in theorem 10.10.

11.1. **Generalised matrix elements and orthogonal polynomials.** First, we will show that for fixed i and j the generalised matrix elements $b^l_{ij}, l \geq |i|,|j|$ can be written as a product of a polynomial in $\rho_{\tau,\sigma}$ and a lower degree generalised matrix element. These polynomials will be orthogonal w.r.t. a certain measure. In the next subsection we will prove that this measure corresponds to the one for Askey-Wilson polynomials by explicitly computing it. The conditions for i,j and l might seem a bit technical at first sight. The idea behind this is that i,j and l differ from each other by full integers and that l is larger than $|i|$ and $|j|$. So either

(i) $i,j,l \in \mathbb{Z}$ and $l \geq |i|,|j|$, or
(ii) $i,j,l \in \frac{1}{2}+\mathbb{Z}$ and $l \geq |i|,|j|$.

Theorem 11.1. *Take $i,j \in \frac{1}{2}\mathbb{Z}$ such that $i-j \in \mathbb{Z}$ and let $m = \max(|i|,|j|)$. Then there exists a system of orthogonal polynomials $(p_k)_{k=0}^{\infty}$ such that for all $l \geq m$ and $l-m \in \mathbb{Z}_+$ we have*

$$b^l_{ij}(\tau,\sigma) = b^m_{ij}(\tau,\sigma)p_{l-m}(\rho_{\tau,\sigma}). \tag{11.1}$$

The measure for these orthogonal polynomials is given by $w_m(x)\mathrm{d}m(x;a,b,c,d|q^2)$ on $\mathbb{R}$, where $\mathrm{d}m(x;a,b,c,d|q^2)$ is the normalised measure corresponding to the Askey-Wilson polynomials with the same parameters as in theorem 10.10 and

$$w_m(\cos\theta) = \left|\pi_{\theta/2}\left(b^m_{ij}(\tau,\sigma)\right)\right|^2.$$

Here $\pi_{\theta/2}$ is the one dimensional $*$-representation of $\mathcal{A}_q(\mathrm{SU}(2))$ given in 7.13(i).

Proof. Let us first show polynomials p_{l-m} exist such that (11.1) holds. We will start with investigating the RHS of (11.1) for an arbitrary polynomial of degree $l-m$. That is, we take a polynomial s_{l-m} of degree $l-m$ and consider

$$b^m_{ij}(\tau,\sigma)s_{l-m}(\rho_{\tau,\sigma}).$$

By theorem 7.9, $\mathcal{A}_q(\mathrm{SU}(2))$ has a linear basis formed by the matrix elements t^l_{mn}. Therefore, there must be $b^k \in \mathrm{A}^k_q$ such that

$$b^m_{ij}(\tau,\sigma)s_{l-m}(\rho_{\tau,\sigma}) = \sum_{k=0}^{N} b^k.$$

We will show that these b_k are multiples of our generalised matrix elements b^k_{ij}, where $m \leq k \leq l$. Since

$$b^m_{ij}(\tau,\sigma) \in \mathrm{A}^m_q \quad \text{and} \quad s_{l-m}(\rho_{\tau,\sigma}) \in \mathrm{A}^{l-m}_q,$$

the Clebsch-Gordan decomposition (lemma 7.8) tells us that we have the upper bound $N = m+(l-m) = l$ and lower bound $|m-(l-m)| = |2m-l|$. Thus,

$$b^m_{ij}(\tau,\sigma)s_{l-m}(\rho_{\tau,\sigma}) = \sum_{k=|2m-l|}^{l} b^k.$$

Since $s_{l-m}(\rho_{\tau,\sigma})$ is a (τ,σ)-spherical element, proposition 9.4 tell us that $b^m_{ij}(\tau,\sigma)s_{l-m}(\rho_{\tau,\sigma})$ satisfies (9.1) with $\lambda = \lambda_j(\sigma)$ and $\mu = \mu_i(\tau)$. Therefore,

$$\sum_{k=|2m-l|}^{l} X_\sigma.b^k = \sum_{k=|2m-l|}^{l} \lambda_j(\sigma)D.b_k \quad \text{and} \quad \sum_{k=|2m-l|}^{l} b^k.X_\tau = \sum_{k=|2m-l|}^{l} \mu_i(\tau)b_k.D. \tag{11.2}$$

Observe that for $X \in \mathcal{U}_q(\mathfrak{su}(2))$, we still have $X.b^k \in \mathrm{A}_q^k$ and $b^k.X \in \mathrm{A}_q^k$. Therefore, $X.b^{k_1}$ and $X.b^{k_2}$ are linearly independent for $k_1 \neq k_2$. Consequently, we conclude from (11.2) that for each $k = |2m-l|, ..., l-1, l$, we must have

$$X_\sigma.b^k = \lambda_j(\sigma)D.b_k \quad \text{and} \quad b^k.X_\tau = \mu_i(\tau)b_k.D.$$

Now, proposition 9.6 tells us that $b^k = 0$ for $k < \max(|i|,|j|) = m$ and $b^k = c_k b_{ij}^k(\tau,\sigma)$ for $k \geq m$ and some constants $c_k \in \mathbb{C}$. Thus,

$$b_{ij}^m(\tau,\sigma)s_{l-m}(\rho_{\tau,\sigma}) = \sum_{k=m}^{l} c_k b_{ij}^k(\tau,\sigma).$$

Since s_{l-m} is of the form,

$$s_{l-m}(\rho_{\tau,\sigma}) = \sum_{k=0}^{l-m} a_k \rho_{\tau,\sigma}^k,$$

we can see the linear mapping $s_{l-m}(\rho_{\tau,\sigma}) \to b_{ij}^m(\tau,\sigma)s_{l-m}(\rho_{\tau,\sigma})$ as a function $f : \mathbb{C}^{l-m+1} \to \mathbb{C}^{l-m+1}$ given by $(a_0,..,a_{l-m}) \to (c_m,..,c_l)$. If we can show that f is injective, we know there exist a_k such that $c_k = \delta_{kl}$. Consequently, polynomials p_{l-m} exist such that (11.1) holds. So let us show that f is injective. Suppose that $f(s_{l-m}) = f(s'_{l-m})$, i.e.

$$b_{ij}^m(\tau,\sigma)s_{l-m}(\rho_{\tau,\sigma}) = b_{ij}^m(\tau,\sigma)s'_{l-m}(\rho_{\tau,\sigma}).$$

Applying the one-dimensional $*$-representation π_θ gives

$$\pi_\theta(b_{ij}^m(\tau,\sigma))\left[\pi_\theta(s_{l-m}(\rho_{\tau,\sigma})) - \pi_\theta(s'_{l-m}(\rho_{\tau,\sigma}))\right] = 0$$

Using the explicit expression for $b_{ij}^l(\tau,\sigma)$ in proposition 9.3 in combination with proposition 7.15 gives that there exist infinitely many $\theta \in (0,\pi)$ such that $\pi_\theta(b_{ij}^m(\tau,\sigma)) \neq 0$. Hence, for these θ we must have

$$\pi_\theta(s_{l-m}(\rho_{\tau,\sigma})) = \pi_\theta(s'_{l-m}(\rho_{\tau,\sigma})).$$

As we saw in the proof of theorem 9.7, we have $\pi_\theta(\rho_{\tau,\sigma}) = \cos(2\theta)$. Thus, we get

$$s_{l-m}(\cos(2\theta)) - s'_{l-m}(\cos(2\theta)) = 0.$$

The only polynomial of degree $\leq l-m$ with infinitely many roots is 0, hence $s_{l-m} = s'_{l-m}$ and f is injective. Therefore, we can find polynomials $\{p_{l-m}\}_{l\geq m}$ such that (11.1) holds.

Next, we will use the Haar functional to show that these polynomials are orthogonal. Let $l, l' \geq m$ such that $l-m, l'-m \in \mathbb{Z}_+$. Since $b_{ij}^l(\tau,\sigma) \in \mathrm{A}_q^l$ and $b_{ij}^{l'}(\tau,\sigma) \in \mathrm{A}_q^{l'}$, Schur's orthogonality relations (10.7) give

$$h\left((b_{ij}^l(\tau,\sigma))^* b_{ij}^{l'}(\tau,\sigma) \in \mathrm{A}_q^l\right) = \delta_{l'l}h_l, \quad h_l > 0.$$

Substituting (11.1) and using that $\rho_{\tau,\sigma}$ is self-adjoint, we obtain

$$h(\overline{p}_{l-m}(\rho_{\tau,\sigma})(b_{ij}^m(\tau,\sigma))^* b_{ij}^m(\tau,\sigma)p_{l'-m}(\rho_{\tau,\sigma})) = \delta_{l'l}h_l. \tag{11.3}$$

By proposition 9.4, $(b_{ij}^m(\tau,\sigma))^* b_{ij}^m$ is a (τ,σ)-spherical element. From theorem 9.7 and the Clebsch-Gordan composition, we get

$$(b_{ij}^m(\tau,\sigma))^* b_{ij}^m(\tau,\sigma) = w_m(\rho_{\tau,\sigma}),$$

where w_m is a polynomial of degree $\leq 2m$. Substituting this into (11.3), gives

$$h(w_m(\rho_{\tau,\sigma})p_{l'-m}(\rho_{\tau,\sigma})\overline{p}_{l-m}(\rho_{\tau,\sigma})) = \delta_{l'l}h_l$$

Finally, since $w_m(\rho_{\tau,\sigma})p_{l'-m}(\rho_{\tau,\sigma})\overline{p}_{l-m}(\rho_{\tau,\sigma})$ is a polynomial in $\rho_{\tau,\sigma}$, we can use theorem 10.10 to obtain

$$\int_{\mathbb{R}} w_m(x)p_{l'-m}(x)\overline{p}_{l-m}(x)\,\mathrm{dm}(x;a,b,c,d|q^2) = \delta_{l'l}h_l,$$

for the parameters given in the theorem. This shows that the polynomials $(p_k)_{k=0}^\infty$ are orthogonal with respect to the measure $w_m(x)\ \mathrm{dm}(x;a,b,c,d|q^2)$. Applying $\pi_{\theta/2}$ to $w_m(\rho_{\tau,\sigma})$ gives

$$w_m(\cos\theta) = |\pi_{\theta/2}(b_{ij}^m(\tau,\sigma))|^2,$$

proving the theorem. □

11.2. **Generalised matrix elements and Askey-Wilson polynomials.** In this subsection, fix $i, j \in \frac{1}{2}\mathbb{Z}$ such that $i-j \in \mathbb{Z}$. Take again $m = \max(|i|, |j|)$. This means there are four possibilities, namely $m \in \{\pm i, \pm j\}$. We will treat the case $m = i$. The other cases are done similarly or can be obtained using symmetry relations for the matrix elements in $\mathcal{A}_q(\mathrm{SU}(2))$. First, we need a proposition which shows that the measure $w_m(x)\ \mathrm{dm}(x; a, b, c, d|q^2)$ is also a measure for Askey-Wilson polynomials, but with some integer shifts in the parameters a and d.

Lemma 11.2. *We have*

$$|\pi_{\theta/2}\left(b^i_{ij}(\tau, \sigma)\right)|^2\ \mathrm{dm}(x; a, b, c, d|q^2) = C^{\tau,\sigma}_{i,j}\ \mathrm{dm}(x; aq^{2i+2j}, b, c, dq^{2i-2j}|q^2),$$

where $C^{\tau,\sigma}_{ij}$ is a constant independent of x.

Sketch of proof. The proof is quite technical and we will not show it here. The idea is to write $\pi_{\theta/2}\left(b^i_{ij}(\tau, \sigma)\right)$ in terms of q-shifted factorials including $z = e^{I\theta}$, where I is the imaginary unit. Then derive the formula

$$(az, a/z; q)_r\ \mathrm{dm}(x; a, b, c, d|q^2) = \frac{(ab, ac, ad; q)_r}{(abcd; q)_r}\ \mathrm{dm}(x; aq^r, b, c, d|q^2),$$

to incorporate the $e^{I\theta}$ terms into the measure of the Askey-Wilson polynomials. See e.g. [10] for a proof. □

Let us now prove the key theorem of part II of this thesis, where we interpret generalised matrix elements from $\mathcal{A}_q(\mathrm{SU}(2))$ as Askey-Wilson polynomials with two continuous and two discrete parameters in the variable $\rho_{\tau,\sigma}$.

Theorem 11.3. *Define the constant*

$$d^l_{ij}(\tau, \sigma) = \frac{c^{l,j}_\sigma \overline{c^{l,i}_\tau}}{c^{i,j}_\sigma \overline{c^{i,i}_\tau}} \frac{q^{i-l}}{(q^{4l}; q^{-2})_{l-i}}, \tag{11.4}$$

and the polynomial[35]

$$p^{(\alpha,\beta)}_n(x; s, t|q) := p_n(x; q^{\frac{1}{2}}t/s, q^{\frac{1}{2}+\alpha}s/t, -q^{\frac{1}{2}}/(st), -stq^{\frac{1}{2}+\beta}|q),$$

where $c^{l,j}(\sigma)$ is the constant from theorem 6.1 and $p_n(x, a, b, c, d|q)$ is the Askey-Wilson polynomial from definition 5.17. Then we have

$$b^l_{ij}(\tau, \sigma) = d^l_{ij}(\tau, \sigma) b^i_{ij}(\tau, \sigma) p^{(i-j,i+j)}_{l-i}(\rho_{\tau,\sigma}; q^\tau, q^\sigma|q^2).$$

Proof. From theorem 11.1 and lemma 11.2, we immediately get

$$b^l_{ij}(\tau, \sigma) = d^l_{ij}(\tau, \sigma) b^i_{ij}(\tau, \sigma) p^{(i-j,i+j)}_{l-m}(\rho_{\tau,\sigma}; q^\sigma, q^\tau|q^2),$$

for some (at this point unknown) constants $d^l_{ij}(\tau, \sigma)$. We will show these are given by (11.4). For this proof, let us use I for the imaginary unit to avoid confusion with our $i \in \frac{1}{2}\mathbb{Z}_+$. Apply $\pi_{\theta/2}$ on both sides of the equation above. Using the explicit expression for $b^l_{ij}(\tau, \sigma)$ in proposition 9.3 and $\pi_{\theta/2}(t^l_{mn}) = \delta_{mn} e^{-I\theta}$ (proposition 7.15), we obtain

$$\sum_{n=-l}^{l} v^{l,j}_n(\sigma)\overline{v^{l,i}_n(\tau)} q^{-n} e^{-In\theta} = d^l_{ij}(\tau, \sigma)\left(\sum_{n=-i}^{i} v^{i,j}_n(\sigma)\overline{v^{i,i}_n(\tau)} q^{-n} e^{-In\theta}\right) \times \pi_{\theta/2}\left(p^{(i-j,i+j)}_{l-m}(\rho_{\tau,\sigma}; q^\sigma, q^\tau|q^2)\right).$$

[35]In this way, Askey-Wilson polynomials $p^{(\alpha,\beta)}_n(x; s, t|q)$ can be seen as the q-analogue of the Jacobi polynomials of definition 5.14. If $s = t = 1$, they are called continuous q-Jacobi polynomials.

Let us compare the terms with $e^{-Il\theta}$. For the LHS, we have $v_l^{l,j}(\sigma)\overline{v_l^{l,i}(\tau)}q^{-l}$. For the RHS, we must look at the coefficient belonging to the term $e^{-I(i-l)\theta}$ in $\pi_{\theta/2}\left(p_{l-i}^{(i-j,i+j)}(\rho_{\tau,\sigma};q^\sigma,q^\tau|q^2)\right)$. One can calculate, using definition 5.17, that this is $(q^{4l};q^{-2})_{l-i}$. Therefore, we have

$$v_l^{l,j}(\sigma)\overline{v_l^{l,i}(\tau)}q^{-l} = d_{ij}^l(\tau,\sigma)v_i^{l,j}(\sigma)\overline{v_i^{l,i}(\tau)}q^{-i}(q^{4l};q^{-2})_{l-i},$$

which implies

$$d_{ij}^l(\tau,\sigma) = \frac{v_l^{l,j}(\sigma)\overline{v_l^{l,i}(\tau)}q^{i-l}}{v_i^{l,j}(\sigma)\overline{v_i^{l,i}(\tau)}(q^{4l};q^{-2})_{l-i}}.$$

Using theorem 6.1, we see that

$$v_k^{k,j}(\sigma) = c_\sigma^{k,j},$$

which proves (11.4). □

The above theorem was first proven in 1993 by Koornwinder [13] in the special case $i=j=0$, i.e. in the case of (τ,σ)-spherical elements. The method Koornwinder used is notably different. His approach uses the Casimir and q-difference equation for Askey-Wilson polynomials instead of the method used here. Where we used that the Haar functional on the subalgebra of (τ,σ)-spherical elements is equal to an integral involving the normalised Askey-Wilson measure. The latter (theorem 10.10) actually is a consequence of Koornwinder's result.

Part III. Twisted Primitive Elements and Askey-Wilson Polynomials

12. The Askey-Wilson algebra AW(3)

This section will be similar to the Zhedanov's article [18], where the Askey-Wilson algebra AW(3) was first introduced and its connection with q-Racah polynomials was described. These discrete Askey-Wilson polynomials will appear as overlap functions between the eigenvectors of finite dimensinoal operators representing the two generators K_0 and K_1 of AW(3). We will work out the first part of [18] in more detail. In this way, we set the stage for section 13, where the primary result of this thesis is proven: the embedding of AW(3) into $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$.

12.1. **The algebra** AW(3)**.**

For convenience, let us introduce the following notation,

$$\sinh_q(x) := q^x - q^{-x} \quad \text{and} \quad \cosh_q(x) := q^x + q^{-x}.$$

This relates to

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh(x) = \frac{e^x + e^{-x}}{2},$$

which is used in [18]. We omit the constant '$\frac{1}{2}$' to avoid extra factors later on. Note that in part I and II of this thesis, we often encountered $\sinh_q$ and $\cosh_q$ already, e.g.

$$q + q^{-1} = \cosh_q(1), \quad \frac{q^{-2j-\sigma} - q^{2j+\sigma} + q^\sigma - q^{-\sigma}}{q - q^{-1}} = \frac{-\sinh_q(2j+\sigma) + \sinh_q(\sigma)}{\sinh_q(1)}.$$

We only start to introduce this notation here since we want to stick to the notation used in the corresponding literature. Let $AW(3)$ be the Askey-Wilson algebra defined by Zhedanov in [18]. This is the algebra generated by three generators K_0, K_1 and K_2 subject to the relations

$$\begin{aligned}
[K_0, K_1]_q &= K_2, \\
[K_1, K_2]_q &= BK_1 + C_0K_0 + D_0, \\
[K_2, K_0]_q &= BK_0 + C_1K_1 + D_1,
\end{aligned} \tag{12.1}$$

where B, C_0, C_1, D_0, D_1 are central elements of the algebra and $[\cdot,\cdot]_q$ is the so-called q-commutator defined by

$$[X, Y]_q = qXY - q^{-1}YX.$$

By substituting the first equation of (12.1) into the second and third, $AW(3)$ can equivalently be described as the algebra generated by two generators K_0 and K_1 subject to the relations

$$\cosh_q(2)K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 = BK_1 + C_0K_0 + D_0, \tag{12.2}$$

$$\cosh_q(2)K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 = BK_0 + C_1K_1 + D_1. \tag{12.3}$$

In this section, we will take $B, C_0, C_1, D_0, D_1 \in \mathbb{R}$. Moreover, AW(3) has a Casimir operator Q, which commutes with every $X \in$ AW(3), given by

$$\begin{aligned}
Q =& (q^{-1} - q^3)K_0K_1K_2 + q^2K_2^2 + B(K_0K_1 + K_1K_0) + C_0q^2K_0^2 + C_1q^{-2}K_1^2 \\
&+ D_0(1+q^2)K_0 + D_1(1+q^{-2})K_1.
\end{aligned} \tag{12.4}$$

We will denote by $Q_0 \in \mathbb{C}$ the value of Q in a representation of AW(3).

12.2. **Representations of** AW(3)**.**

It turns out that, under mild conditions, AW(3) has two families of inequivalent representations. In these representations, the generators (K_0, K_1) will have a special property. We define a finite-dimensional matrix A *tridiagonal* if A_{ij}, its entry in the i-th row and j-th column, is 0 when $|i-j| > 1$. It turns out that we can make a 'ladder' representation of AW(3) such that K_0 is diagonal and K_1 is tridiagonal with respect to the same basis and the other way around as well. We first need a lemma that shows that a certain quadratic equation, arising naturally in AW(3), generates a sequence $(\lambda_n(p))_{n=0}^\infty$ in $\mathbb{C}$ such that $(\lambda_n, \lambda_{n+1})$ are solutions of this equation. The idea of the proof is taken from [14].

Lemma 12.1. *Let $C_1 \in \mathbb{R}$. Define the quadratic function $p : \mathbb{C}^2 \to \mathbb{C}$ by*
$$p(\lambda, \mu) = \lambda^2 + \mu^2 - \cosh_q(2)\lambda\mu + C_1. \tag{12.5}$$
Then the equation
$$p(\lambda_n, \lambda_{n+1}) = 0,$$
generates two families of real sequences, $(\lambda_n(p))_{n=0}^\infty$ and $(-\lambda_n(p))_{n=0}^\infty$, depending on $p \in \mathbb{R}\backslash\mathbb{Z}$, that are non-degenerate[36]. These are given by
$$\lambda_n(p) = \begin{cases} \sqrt{C_1}\dfrac{\cosh_q(2n+p+1)}{\sinh_q(2)} & \text{if } C_1 > 0, \\ q^{\pm(2n+p+1)} & \text{if } C_1 = 0, \\ \sqrt{-C_1}\dfrac{\sinh_q(2n+p+1)}{\sinh_q(2)} & \text{if } C_1 < 0, \end{cases} \tag{12.6}$$
where $n \in \mathbb{Z}$.

Proof. Fix a value λ_n. Observe that the equation $p(\lambda, \lambda_n) = 0$ has, by its quadratic nature, two 'neighbouring' solutions $\lambda_{n-1}, \lambda_{n+1}$. Let us assume that λ_{n-1}, λ_n and λ_{n+1} are all distinct, since otherwise the sequence could have cycles. Then, λ_{n+1} and λ_{n-1} are the two zeroes of $p(\lambda, \lambda_n)$. Therefore,
$$p(\lambda, \lambda_n) = (\lambda - \lambda_{n+1})(\lambda - \lambda_{n-1}) = \lambda^2 - (\lambda_{n+1} + \lambda_{n-1})\lambda + \lambda_{n+1}\lambda_{n-1}.$$
Combining this with (12.5) gives,
$$\cosh_q(2)\lambda_n = \lambda_{n+1} + \lambda_{n-1} \quad \text{and} \quad \lambda_n^2 + C_1 = \lambda_{n+1}\lambda_{n-1} \tag{12.7}$$
The first is a linear recurrence relation. Solving this using $\lambda_n = r^n$ gives
$$0 = -\cosh_q(2)r^n + r^{n+1} + r^{n-1} = r^{n-1}(r^2 - \cosh_q(2) + 1),$$
which leads to $r = q^2$ or $r = q^{-2}$. Therefore, we have solutions of the form $\lambda_n = aq^{2n} + bq^{-2n}$ for $a, b \in \mathbb{C}$. The second part of (12.7) leads to conditions on a and b. Indeed, substituting our expression for λ_n gives
$$(aq^{2n} + bq^{-2n})^2 + C_1 = (aq^{2n+2} + bq^{-2n-2})(aq^{2n-2} + bq^{-2n+2}).$$
Working this out, we obtain
$$ab = \frac{C_1}{\sinh_q(2)^2}.$$
We split three cases for C_1. They all have two sequences of solutions, λ_n or $-\lambda_n$, and depend on the starting parameter $p \in \mathbb{R}$. If $C_1 > 0$, we have
$$\lambda_n = \frac{\sqrt{C_1}}{\sinh_q(2)}\cosh_q(2n + p + 1).$$
Taking $C_1 < 0$, gives
$$\lambda_n = \frac{\sqrt{-C_1}}{\sinh_q(2)}\sinh_q(2n + p + 1).$$
Then finally, $C_1 = 0$ leads to
$$\lambda_n = q^{\pm(2n+p+1)}.$$
The degeneracy $\lambda_m = \lambda_n$ can only happen if $2n + p + 1 = -2m - p - 1$ for certain $n, m \in \mathbb{Z}_+$. This implies $p = 1 - (n + m)$, which leads to the condition on p. □

Let t be a representation of AW(3). Mathematicians like short notations. Therefore, in the literature the 't' is often omitted when the representation is clear from the context. That is, for $X \in$ AW(3), we just write 'X' for the operator $t(X)$.
The following theorem shows that a representation of AW(3) has a ladder structure. That is, once we have a single eigenvector ψ of K_0, we can create a whole sequence of eigenvectors of K_0 from ψ. Moreover, the operator K_1 will be tridiagonal with respect to these eigenvectors.

[36] That is, $\lambda_n \neq \lambda_m$ if $m \neq n$.

Theorem 12.2. *Let $B, C_0, C_1, D_0, D_1 \in \mathbb{R}$. If there exists a $N+1$-dimensional representation of* AW(3) *on V such that K_0 has a non-zero eigenvalue λ of the form specified below, then it is irreducible and there exists a linear basis $\{\psi_0, \dots, \psi_N\}$ of V for which K_0 is diagonal and K_1 is tridiagonal. By rescaling the basis, this can be written in the form*

$$K_0\psi_n = k\lambda_n\psi_n, \tag{12.8}$$

$$K_1\psi_n = a_n\psi_{n-1} + b_n\psi_n + a_{n+1}\psi_{n+1}, \tag{12.9}$$

where $k \in \{-1, 1\}$ and with the convention $\psi_{-1} = 0 = \psi_{N+1}$. Here

$$\lambda_n = \begin{cases} \sqrt{C_1}\dfrac{\cosh_q(2n+p+1)}{\sinh_q(2)} & \text{if } C_1 > 0, \\ q^{\pm(2n+p+1)} & \text{if } C_1 = 0, \\ \sqrt{-C_1}\dfrac{\sinh_q(2n+p+1)}{\sinh_q(2)} & \text{if } C_1 < 0, \end{cases} \tag{12.10}$$

where $n = 0, ..., N$ and $p \in \mathbb{C}\backslash\{0, -1, .., -2N\}$ is the spectral parameter. We need $\lambda = k\lambda_n$ for certain k, p, n. Moreover, the matrix elements a_n and b_n satisfy,

$$f_nf_{n-1}a_n^2 = \frac{(B\lambda_n + D_1)(B\lambda_{n-1} + D_1)}{g_n^2} + C_0\lambda_n\lambda_{n-1} + D_0(\lambda_n + \lambda_{n-1}) - Q_0, \tag{12.11}$$

$$b_n = \frac{B\lambda_n + D_1}{g_ng_{n+1}}, \tag{12.12}$$

where $g_n = \lambda_n - \lambda_{n-1}$, $f_n = \lambda_{n+1} - \lambda_{n-1}$ and Q_0 is the value corresponding to the representation of the Casimir element Q given in (12.4).

Proof. Let us first show existence. Suppose that ψ_p is an eigenvector of K_0 with eigenvalue λ_p,

$$K_0\psi_p = \lambda_p\psi_p \tag{12.13}$$

We will show the ladder property of AW(3). That is, there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that

$$\psi_s = (\alpha K_0 + \beta K_1 + \gamma K_2)\,\psi_p \tag{12.14}$$

is also an eigenvector of K_0 with a different eigenvalue λ_s. Now, $K_0\psi_s = \lambda_s\psi_s$ implies

$$\left(\alpha K_0^2 + \beta K_0K_1 + \gamma K_0K_2\right)\psi_p = \lambda_s\left(\alpha K_0 + \beta K_1 + \gamma K_2\right)\psi_p.$$

and using (12.13), this becomes

$$\left(\alpha\lambda_p^2 + \beta K_0K_1 + \gamma K_0K_2\right)\psi_p = \lambda_s\left(\alpha\lambda_p + \beta K_1 + \gamma K_2\right)\psi_p. \tag{12.15}$$

We want to use (12.13) again to remove the rest of K_0 from this equation. In order to do that, we will use the algebra relations (12.1) to get K_0 to the right side. Using the first and third equation of (12.1), we have

$$K_0K_1 = q^{-2}K_1K_0 + q^{-1}K_2 \quad \text{and} \quad K_0K_2 = q^2K_2K_0 - qBK_0 - qC_1K_1 - qD_1.$$

Substituting this into (12.15) and then using (12.13) gives,

$$\begin{aligned}&\left(\alpha\lambda_p^2 + \beta\left(q^{-2}\lambda_pK_1 + q^{-1}K_2\right) + \gamma\left(q^2\lambda_pK_2 - q\lambda_pB - qC_1K_1 - qD_1\right)\right)\psi_p \\ &\quad = \lambda_s\left(\alpha\lambda_p + \beta K_1 + \gamma K_2\right)\psi_p.\end{aligned}$$

Comparing the coefficients of ψ_p, $K_1\psi_p$ and $K_2\psi_p$ gives three linear equations in α, β and γ,

$$0 = \alpha(\lambda_p^2 - \lambda_s\lambda_p) - q\gamma(\lambda_pB + D_1), \tag{12.16}$$

$$0 = \beta(q^{-2}\lambda_p - \lambda_s) - q\gamma C_1, \tag{12.17}$$

$$0 = q^{-1}\beta + \gamma(q^2\lambda_p - \lambda_s). \tag{12.18}$$

Multiplying (12.16) with q gives $\beta = -q\gamma(q^2\lambda_p - \lambda_s)$. Substitute this into (12.17) to obtain,

$$-q\gamma(C_1 + (q^{-2}\lambda_p - \lambda_s)(q^2\lambda_p - \lambda_s)) = 0$$

If we take $\gamma = 0$, (12.18) immediately gives $\beta = 0$. Since we want $\lambda_p \neq \lambda_s$, (12.16) gives $\alpha = 0$ as well, resulting into $\psi_s = 0$. Therefore, we need

$$0 = C_1 + (q^{-2}\lambda_p - \lambda_s)(q^2\lambda_p - \lambda_s) = \lambda_p^2 + \lambda_s^2 - (q^2 + q^{-2})\lambda_p\lambda_s + C_1. \tag{12.19}$$

Lemma 12.1 in combination with the dimension $N+1$ then gives us that there are two possibilities for the spectrum of K_0. Namely, one of the sequences $(\lambda_n)_{n=0}^N$ or $(-\lambda_n)_{n=0}^N$, where

$$\lambda_n = \begin{cases} \sqrt{C_1}\dfrac{\cosh_q(2n+p+1)}{\sinh_q(2)} & \text{if } C_1 > 0 \\ q^{\pm(2n+p+1)} & \text{if } C_1 = 0 \\ \sqrt{-C_1}\dfrac{\sinh_q(2n+p+1)}{\sinh_q(2)} & \text{if } C_1 < 0 \end{cases} \tag{12.20}$$

with $p \in \mathbb{R}$. We will focus on the positive branch $(\lambda_n)_{n=0}^N$, the other one can easily be obtained by the transformation $K_0 \leftrightarrow -K_0$. We now have a basis given by $(\psi_n)_{n=0}^N$. Moreover, every ψ_n has two 'neighbours': ψ_{n-1} and ψ_{n+1}, except for the end points ψ_0 and ψ_N. Both neighbouring eigenvectors can be obtained from ψ_n, i.e.

$$\psi_{n+1} = (\alpha_1 K_0 + \beta_1 K_1 + \gamma_1 K_2)\,\psi_n \quad \text{and} \quad \psi_{n-1} = (\alpha_{-1} K_0 + \beta_{-1} K_1 + \gamma_{-1} K_2)\,\psi_n,$$

for suitable constants $\alpha_j, \beta_j, \gamma_j \in \mathbb{C}$. Using both equations above, we can eliminate $K_2\psi_n$ from the equation and write $K_1\psi_p$ in terms of ψ_{n-1}, ψ_n and ψ_{n+1}. Therefore, K_1 is tridiagonal w.r.t the $(\psi_n)_{n=1}^N$ basis[37], which proves existence of the representation. Then we can immediately see that it is irreducible. Let V be non-zero invariant subspace of span$\{\psi_0, ..., \psi_N\}$. Take $v \in V$, then there is a smallest integer n such that ψ_n has a non-zero component in v. In this way, define the function $k : V \to \{0, ..., N\}$. We claim that by the ladder property, there is a $X \in \mathrm{AW}(3)$ such that $k(Xv) = n+1$. Indeed, using this property one can find $Y \in \mathrm{AW}(3)$ such that $Y\psi_n = \psi_{n+1}$. By the tridiagonality of K_0, we then have $k(Yv) \geq n$. The situation $k(Yv) = n$ can be solved by taking a suitable combination $X = Y + cK_0^j$, where $c \in \mathbb{C}$ and $j \in \{1, 2\}$. By repeating this process, we can find $\psi_N \in V$. By laddering down to ψ_0 again one obtains $V = \{\psi_0, ..., \psi_N\}$ which proves the irreducibility.

Let us now explicitly calculate the matrix elements of the tridiagonal matrix K_1. By rescaling the eigenvectors, we can write this in the form,

$$K_1\psi_n = a_n\psi_{n-1} + b_n\psi_n + a_{n+1}\psi_{n+1}, \tag{12.21}$$

for $n = 0, ..., N$ and $\psi_{-1} = 0 = \psi_{N+1}$. Let us calculate the matrix elements a_n and b_n. We start with the easiest, b_n. Substituting (12.21) into the AW(3) relation (12.3) and equating coefficients of the diagonal terms ψ_n, we get

$$(\cosh_q(2) - 2)\lambda_n^2 b_n = B\lambda_n + C_1 b_n + D_1.$$

Solving this gives

$$b_n = \frac{B\lambda_n - D_1}{(\cosh_q(2) - 2)\lambda_n^2 - C_1}.$$

From (12.7), we obtain

$$\cosh_q(2)\lambda_n^2 - \lambda_n^2 - (\lambda_n^2 + C_1 = (\lambda_{n+1} + \lambda_{n-1})\lambda_n - \lambda_n^2 - \lambda_{n+1}\lambda_{n-1} = g_n g_{n+1},$$

where $g_n = \lambda_n - \lambda_{n-1}$, which proves (12.12). For a_n, substitute (12.21) into the other AW(3) relation (12.2) and again equate the diagonal terms. We get

$$f_{n+1}a_{n+1}^2 - f_{n-1}a_n^2 = -\sinh_q(1)^2\lambda_n b_n^2 + Bb_n + C_0\lambda_n + D_0, \tag{12.22}$$

[37]Similarly we have that K_2 is tridiagonal w.r.t the same basis. However, we won't need that here since we will focus on K_0 and K_1 and $K_2 = [K_0, K_1]_q$.

where $f_n = \lambda_{n+1} - \lambda_{n-1}$. Another condition for a_n can be obtained by substituting (12.21) into (12.4). This gives

$$\begin{aligned} f_{n+1}f_n a_{n+1}^2 + f_n f_{n-1} a_n^2 = &- 2Q_0 - 2(\sinh_q(1)\lambda_n b_n)^2 - \cosh_q(2)\left(C_0\lambda_n^2 + C_1 b_n^2\right) \\ &+ 4B\lambda_n b_n + \cosh_q(1)^2\left(D_0\lambda_n + D_1 b_n\right), \end{aligned} \tag{12.23}$$

where $Q_0 \in \mathbb{C}$ is the constant belonging to the Casimir Q of AW(3). Combining (12.22) and (12.23) gives an explicit expression for a_n,

$$f_n f_{n-1} a_n^2 = \frac{(B\lambda_n + D_1)(B\lambda_{n-1} + D_1)}{g_n^2} + C_0\lambda_n\lambda_{n-1} + D_0(\lambda_n + \lambda_{n-1}) - Q_0. \tag{12.24}$$

Similarly, solving this for a_{n+1} gives the same expression, only with '$n+1$' instead of 'n'. This shows consistency of the two equations (12.22) and (12.23). □

The condition for the eigenvalue λ in the theorem is only to make sure that the sequence λ_n are all distinct. It is a very mild condition since only the cases where $p = 0, -1, ..., -2N$ are excluded. When the eigenvalues of K_0 are all distinct, we say that the spectrum of K_0 is *non-degenerate*. At this point it is not clear whether a finite-dimensional representation of AW(3) always exists, since we need a_n to vanishes at some point for that. The condition (12.11) shows that this happens only if

$$\frac{(B\lambda_n + D_1)(B\lambda_{n-1} + D_1)}{g_n^2} + C_0\lambda_n\lambda_{n-1} + D_0(\lambda_n + \lambda_{n-1}) - Q_0 = 0.$$

Therefore, will analyse a_n further to see when it becomes zero. We can write a_n in a compact form using a fourth degree polynomial, of which the roots will be of major interest.

Proposition 12.3. *We can rewrite* (12.11) *into*

$$a_n^2 = \frac{\mathscr{P}(\Lambda_n)}{g_n^2 f_n f_{n-1}}, \tag{12.25}$$

where $\mathscr{P}$ is a polynomial of degree ≤ 4 in the variable $\Lambda_n = \lambda_n + \lambda_{n-1}$. It is given by

$$\begin{aligned} \mathscr{P}(\Lambda_n) = \; & C_0\frac{\sinh_q(1)^2}{\cosh_q(1)^4}\Lambda_n^4 + D_0\frac{\sinh_q(1)^2}{\cosh_q(1)^2}\Lambda_n^3 + \left(\frac{B^2 - \sinh_q(1)^2 Q_0}{\cosh_q(1)^2} + \frac{\left(\sinh_q(1)^2 - 4\right)C_0C_1}{\cosh_q(1)^4}\right)\Lambda_n^2 \\ & + \left(BD_1 - \frac{4C_1D_0}{\cosh_q(1)^2}\right)\Lambda_n + D_1^2 + C_1\frac{B^2 + 4Q_0}{\cosh_q(1)^2} - \frac{4C_0C_1^2}{\cosh_q(1)^4}. \end{aligned} \tag{12.26}$$

$\mathscr{P}$ is called the characteristic polynomial of AW(3)*.*

Proof. To see this, let us first multiply both sides of (12.24) by $f_n f_{n-1}$. Then we need to show that the numerator of

$$a_n^2 = \frac{B^2\lambda_n\lambda_{n-1} + BD_1\Lambda_n + D_1^2 + g_n^2(C_0\lambda_n\lambda_{n-1} + D_0\Lambda_n - Q_0)}{g_n^2 f_n f_{n-1}}, \tag{12.27}$$

is polynomial in Λ_n of degree ≤ 4. It suffices to show that $\lambda_n\lambda_{n-1}$ and g_n^2 are polynomials in Λ_n of degree two. We have

$$\Lambda_n^2 = \lambda_n^2 + \lambda_{n-1}^2 + 2\lambda_n\lambda_{n-1}.$$

Combining this with (12.19) gives

$$\lambda_n\lambda_{n-1} = \frac{\Lambda_n^2 + C_1}{\cosh_q(1)^2} \quad \text{and} \quad g_n^2 = \frac{\sinh_q(1)^2\Lambda_n^2 - 4C_1}{\cosh_q(1)^2}.$$

substituting this into the numerator of (12.27) gives the explicit expression for $\mathscr{P}$. □

Remark 12.4. The name characteristic polynomial of AW(3) makes sense since $\mathscr{P}$ is independent of the representation. Moreover, the four zeroes of $\mathscr{P}$ will be very convenient for analysis. For example, the structure constants of AW(3) can be expressed in terms of the p_k, as we will see later.

For a finite-dimensional representation, we need a_n to vanish at some point. This happens when $\mathscr{P}(\Lambda_n) = 0$. Therefore, we can formulate the condition for the existence of a finite-dimensional representation, in terms of a condition on the zeroes of $\mathscr{P}$. From now on, take $C_0, C_1 < 0$, which will correspond to the setting of subsection 13.2. Other cases can be obtained similarly.

We can rescale our generators, i.e. $K_0' = \lambda K_0$ and $K_1' = \mu K_1$ with $\lambda, \mu \in \mathbb{R}$, without loss of generality . Then K_0' and K_1' satisfy the AW(3) relations (12.2) and (12.3). The rescaling leads to the transformation of structure constants

$$(B, C_0, C_1, D_0, D_1) \leftrightarrow (\lambda\mu B, \mu^2 C_0, \lambda^2 C_1, \lambda\mu^2 D_0, \lambda^2\mu D_1).$$

Therefore, we can assign any value we want to C_0 and C_1, as long as we don't change their sign. We rescale K_0 and K_1 such that the spectrum of K_0 is given by

$$\lambda_n = \mathrm{sinh}_q(2n + p + 1).$$

For this to happen we take

$$C_0 = C_1 = -\mathrm{sinh}_q(2)^2.$$

Furthermore, by rescaling we can also consider only the positive branch, i.e. $k = 1$ in theorem 12.2. Observe that we now have,

$$\Lambda_n = \lambda_n + \lambda_{n-1} = \mathrm{sinh}_q(2n + p + 1) + \mathrm{sinh}_q(2n + p - 1) = \mathrm{cosh}_q(1)\mathrm{sinh}_q(2n + p).$$

We will rewrite the four roots of $\mathscr{P}$ into a similar form. Note that the function $\mathrm{sinh}_q : A \to \mathbb{C}$ is bijective, where $A = \{z \in \mathbb{C} : \frac{1}{2}\pi < \mathrm{Im}(\ln(q)z) \leq \ \frac{1}{2}\pi\}$. Therefore, we can always write the four roots in the form of Λ_n. That is, they can be uniquely written as

$$\mathrm{cosh}_q(1)\mathrm{sinh}_q(p_k), \quad p_k \in A,\ k = 0, 1, 2, 3.$$

Note that every polynomial $P_n(x)$ of degree n, can be written as a product using its n roots $\{x_k\}_{k=0}^{n-1}$. That is,

$$P_n(x) = c\prod_{k=0}^{n-1}(x - x_k),$$

where $c \in \mathbb{C}$ is the constant in front of x^n. We will rewrite our polynomial $\mathscr{P}$, and therefore a_n^2 as well, into this more convenient form.

Proposition 12.5. *We have*

$$\mathscr{P}(\Lambda_n) = -\mathrm{sinh}_q(2)^2\mathrm{sinh}_q(1)^2\prod_{k=0}^{3}\left(\mathrm{sinh}_q(2n + p) - \mathrm{sinh}_q(p_k)\right). \tag{12.28}$$

Therefore, the expression (12.25) *can be written as*

$$a_n^2 = \frac{-\prod_{k=0}^{3}\left(\mathrm{sinh}_q(2n + p) - \mathrm{sinh}_q(p_k)\right)}{\mathrm{cosh}_q(2n + p)^2\mathrm{cosh}_q(2n + p + 1)\mathrm{cosh}_q(2n + p - 1)}. \tag{12.29}$$

Proof. We can write $\mathscr{P}$ into a product of its roots. That is,

$$\mathscr{P}(\Lambda_n) = A\prod_{k=0}^{3}\left(\Lambda_n - \mathrm{cosh}_q(1)\mathrm{sinh}_q(p_k)\right).$$

Comparing this with (12.26) gives

$$A = \frac{C_0\mathrm{sinh}_q(1)^2}{\mathrm{cosh}_q(1)^4} = -\frac{\mathrm{sinh}_q(2)^2\mathrm{sinh}_q(1)^2}{\mathrm{cosh}_q(1)^4}.$$

Since $\Lambda_n = \mathrm{cosh}_q(1)\mathrm{sinh}_q(2n + p)$, we get

$$\mathscr{P}(\Lambda_n) = -\mathrm{sinh}_q(2)^2\mathrm{sinh}_q(1)^2\prod_{k=0}^{3}\left(\mathrm{sinh}_q(2n + p) - \mathrm{sinh}_q(p_k)\right).$$

Let us now work out the denominator of (12.25). Using the formula

$$\sinh_q(a) - \sinh_q(b) = \sinh_q\left(\frac{a+b}{2}\right)\cosh_q\left(\frac{a-b}{2}\right),$$

we obtain

$$\begin{aligned} f_n &= \lambda_{n+1} - \lambda_{n-1} = \sinh_q(2n+p+3) - \sinh_q(2n+p-1) \\ &= \sinh_q(2)\cosh_q(2n+p+1), \end{aligned}$$

and

$$\begin{aligned} g_n &= \lambda_n - \lambda_{n-1} = \sinh_q(2n+p+1) - \sinh_q(2n+p-1) \\ &= \sinh_q(1)\cosh_q(2n+p). \end{aligned}$$

Substituting this into (12.25) proves the proposition. □

Corollary 12.6. AW(3) *has an irreducible* $(N+1)$*-dimensional representation with non-degenerate spectrum of* K_0 *if and only two of the parameters* $\{p_k\}_{k=0}^3$*, say* p_0 *and* p_1 *satisfy the 'quantisation condition'*

$$p_1 - p_0 = 2(N+1).$$

In this case, $p = p_0$ *and the representation is unique up to the sign of the spectrum of* K_0*. Moreover, the spectrum of* K_0 *determines the representation up to equivalence.*

Proof. From theorem 12.2, we know that $a_0 = 0 = a_{N+1}$ is a necessary and sufficient condition for AW(3) to have a representation of dimension $N+1$. Using (12.29), this implies for two roots of $\mathscr{P}$, say $\sinh_q(p_0)$ and $\sinh_q(p_1)$, to be equal to $\sinh_q(p)$ and $\sinh_q(2(N+1)+p)$ respectively. Consequently, we have $p_0 = p$ and $p_1 = 2(N+1)+p$. This implies $p_1 - p_0 = 2(N+1)$ and $p = p_0$. Therefore, the spectrum of K_0 is now fixed up to a sign, i.e.

$$\lambda_n = k\sinh_q(2n+p_0+1).$$

To prove that the spectrum of K_0 determines the representation up to equivalence, let us assume we have an irreducible $(N+1)$-dimensional representation of AW(3). Then by theorem 12.2, the matrix elements of K_0 and K_1 are fixed up to equivalence[38] by (12.11) and (12.12). □

Remark 12.7. Writing the roots of $\mathscr{P}$ in the form $\cosh_q(1)\sinh_q(2n+p_k)$ is convenient for analysis. Most importantly, because the parameters of the q-Racah polynomials will depend on simple linear combinations of the p_k. Moreover, note the symmetry in the p_k: interchanging any two parameters or change of sign simultaneously of an even number of the p_k does not change AW(3). This corresponds with some of the remarkable symmetry properties of the Askey-Wilson polynomials.

The structure constants B, D_0 and D_1 as well as the value of the Casimir Q can be expressed in terms of the roots of the characteristic polynomial $\mathscr{P}$.

Proposition 12.8. *We have*

$$B = \frac{\sinh_q(1)}{\cosh_q(1)}\sinh_q(2)\left(\sinh_q\left(\frac{p_0+p_1}{2}\right)\sinh_q\left(\frac{p_2+p_3}{2}\right) + \cosh_q\left(\frac{p_0-p_1}{2}\right)\cosh_q\left(\frac{p_2-p_3}{2}\right)\right),$$

$$D_0 = \frac{\sinh_q(2)^2}{\cosh_q(1)}\sum_{k=0}^{3}\sinh_q(p_k),$$

$$D_1 = -\sinh_q(1)\sinh_q(2)\left(\sinh_q\left(\frac{p_0+p_1}{2}\right)\cosh_q\left(\frac{p_2-p_3}{2}\right) + \cosh_q\left(\frac{p_0-p_1}{2}\right)\sinh_q\left(\frac{p_2-p_3}{2}\right)\right),$$

$$Q_0 = \sinh_q(2)^2\left(\prod_{k=0}^{3}\sinh_q\left(\frac{1}{2}p_k\right) + \sinh_q(1)^2 - \frac{B\sinh_q(1)^2 + D_1^2}{\sinh_q(1)^2\sinh_q(2)^2}\right).$$

[38] The equation (12.11) gives two possibilities for a_n. However, these lead to equivalent representations which can be obtained from the case where all $a_n > 0$ by doing the basis transformation $\psi_n \to k_n\psi_n$, where $k_n/k_{n-1} = \mathrm{sign}(a_n)$.

Proof. Equating the coefficients in front of Λ_n^k in (12.26) with (12.28) for $k = 0, 1, 2, 3$ gives four equations for B, D_0, D_1 and Q. Working out the term (12.28) corresponding to Λ_n^3, we obtain

$$-\sinh_q(2)^2\sinh_q(1)^2\left(\sum_{k=0}^{3}\sinh_q(p_k)\right)\sinh_q(2n+p)^3 = -\frac{\sinh_q(2)^2\sinh_q(1)^2}{\cosh_q(1)^3}\left(-\sum_{k=0}^{3}\sinh_q(p_k)\right)\Lambda_n^3.$$

Comparing this with (12.28) gives

$$D_0\frac{\sinh_q(1)^2}{\cosh_q(1)^2} = \frac{\sinh_q(2)^2\sinh_q(1)^2}{\cosh_q(1)^3}\left(\sum_{k=0}^{3}\sinh_q(p_k)\right),$$

which leads to the desired expression for D_0. Looking at the coefficients of Λ_n^k for $k = 0, 1, 2$ and solving for B, D_1 and Q gives the other formulas. □

Because of the symmetry between K_0 and K_1, we can interchange the roles of these operators and find the following theorem, which shows the remarkable duality property of AW(3).

Theorem 12.9. *If there exists a representation of* AW(3) *on* V *with* $\dim(V) = N + 1$*, then there exists a dual basis* $\{\phi_0, \ldots, \phi_N\}$ *of* V *for which* K_1 *is diagonal and* K_0 *tridiagonal. Explicitly we have*

$$K_1\phi_m = \mu_m\phi_m \tag{12.30}$$

$$K_0\phi_m = c_m\phi_{m-1} + d_m\phi_m + c_{m+1}\phi_{m+1}, \tag{12.31}$$

for $m = 0, ..., N$*. The matrix elements of* K_0 *and* K_1 *w.r.t the basis* $\{\phi_0, \ldots, \phi_N\}$ *are determined by* $\mu_m = \tilde{\lambda}_m$*,* $c_m = \tilde{a}_m$ *and* $d_m = \tilde{b}_m$*, where* $(\tilde{\lambda}_m, \tilde{a}_m, \tilde{b}_m)$ *can be obtained from* (λ_m, a_m, b_m) *after interchanging* $C_0 \leftrightarrow C_1$ *and* $D_0 \leftrightarrow D_1$*. Moreover, in the case* $C_0 = C_1 = -\sinh_q(2)^2$ *we have*

$$c_m^2 = \frac{-\prod_{k=0}^{3}\left(\sinh_q(2m+s_0) - \sinh_q(s_0)\right)}{\cosh_q(2m+s_0)^2\cosh_q(2m+s_0+1)\cosh_q(2m+s_0-1)}, \tag{12.32}$$

where $\sinh_q(s_k)$ *are the roots of the characteristic polynomial* $\mathscr{P}$ *of* AW(3) *where* $D_0 \leftrightarrow D_1$ *are interchanged. The* p_k *and* s_k *are linked via*

$$s_0 = \frac{1}{2}\Sigma_p - p_1, \quad s_1 = \frac{1}{2}\Sigma_p - p_2, \quad s_2 = \frac{1}{2}\Sigma_p - p_3, \quad s_3 = \frac{1}{2}\Sigma_p - p_2, \quad \Sigma_p = \sum_{k=0}^{3} p_k. \tag{12.33}$$

Proof. Take $\tilde{K_0} = K_1$ and $\tilde{K_1} = K_0$. Let $\widehat{\mathrm{AW}(3)}$ denote AW(3) with structure constants

$$(\tilde{B}, \tilde{C_0}, \tilde{C_1}, \tilde{D_0}, \tilde{D_1}) = (B, C_1, C_0, D_1, D_0).$$

That is, we interchanged $C_0 \leftrightarrow C_1$ and $D_0 \leftrightarrow D_1$. Then $\tilde{K_0}$ and $\tilde{K_1}$ satisfy the relations (12.2) and (12.3) for $\widehat{\mathrm{AW}(3)}$. Therefore, we can follow the exact same steps we did before to obtain expressions for μ_m, c_m and d_m. Using proposition 12.8 we can find four equations that relate p_k and s_k by equating

$$\tilde{B} = B, \quad \tilde{D_0} = D_1, \quad \tilde{D_1} = D_0, \quad \text{and} \quad \tilde{Q}_0 = Q_0,$$

where $\tilde{Q}$ is the Casimir operator of $\widehat{\mathrm{AW}(3)}$. Solving these equations gives the desired formulas relating s_k and p_k. □

Remark 12.10. The sharp reader might wonder if we have the same representation in both cases. Once we fix the representation corresponding to the positive branch of eigenvalues for K_0, the operator representing K_1 is fixed as well. Then the spectrum of K_1 is of the form $\pm\mu_n$. If it might happen we have the negative branch for K_1, we can just look at $-K_1$, which also satisfies the AW(3) relations. Therefore, we can proceed without loss of generality.

12.3. **Overlap functions of K_0 and K_1.** We want K_0 and K_1 to be self-adjoint, since their eigenvectors will then form an orthonormal basis. This requires $a_n, b_n, \lambda_n \in \mathbb{R}$. Now, we can now look at the unitary matrix for the change of basis between the eigenvectors $\{\psi_n\}_{n=0}^N$ of K_0 and $\{\phi_m\}_{m=0}^N$ of K_1. We have

$$\phi_m = \sum_{n=0}^{N} \langle \phi_m, \psi_n \rangle \psi_n.$$

We define overlap functions $P_n(\mu_m)$ by separating $\psi_0(m) := \langle \phi_m, \psi_0 \rangle$ from the inner product, i.e.

$$P_n(\mu_m) = \frac{\langle \phi_m, \psi_n \rangle}{\langle \phi_m, \psi_0 \rangle} := \frac{\langle \phi_m, \psi_n \rangle}{\psi_0(m)}.$$

We have to check if this is well defined. If there exists a $0 \leq m \leq N$ such that $\psi_0(m) = 0$, we would have

$$0 = \psi_0(m) = \mu_m^{-1} \langle K_1 \phi_m, \psi_0 \rangle = \mu_m^{-1} \langle \phi_m, K_1 \psi_0 \rangle = \mu_m^{-1} \left(\langle \phi_m, b_0 \psi_0 \rangle + \langle \phi_m, a_1 \psi_1 \rangle \right).$$

This implies

$$\langle \phi_m, \psi_1 \rangle = 0.$$

Repeating this, we obtain $\langle \phi_m, \psi_n \rangle = 0$ for all $0 \leq n \leq N$. A contradiction since $\{\psi_0, .., \psi_N\}$ forms a basis.
Note that at this point, we don't know what kind of functions $P_n(\mu_m)$ are. We only know what they do at the points μ_m. Now we make the crucial observation that we can derive a three-term recurrence relation for $P_n(\mu_m)$. On the one hand, (12.30) tells us that the operator K_1 has eigenvectors $\{\phi_m\}_{m=0}^N$. On the other hand, K_1 acts as a tridiagonal operator on $\{\phi_n\}_{n=0}^N$ by (12.9). Since K_1 is self adjoint we have,

$$\mu_m \langle \phi_m, \psi_n \rangle = \langle K_1 \phi_m, \psi_n \rangle = \langle \phi_m, K_1 \psi_n \rangle = a_n \langle \phi_m, \psi_{n-1} \rangle + b_n \langle \phi_m, \psi_n \rangle + a_{n+1} \langle \phi_m, \psi_{n+1} \rangle,$$

where we used that $a_n, b_n \in \mathbb{R}$. Therefore,

$$\mu_m P_n(\mu_m) = a_n P_{n-1}(\mu_m) + b_n P_n(\mu_m) + a_{n+1} P_{n+1}(\mu_m). \tag{12.34}$$

Now it becomes clear why we separated $\psi_0(m)$. Since we now have the initial condition

$$P_0(\mu_m) = 1,$$

for a three-term recurrence relation. Together with the other initial condition $P_{-1}(\mu_m) = 0$, (12.34) generates polynomials $(P_n)_{n=0}^N$ in μ_m. We will show that these are q-Racah polynomials of the most general form[39]. We will do this by comparing the three-term recurrence relations for the overlap functions and q-Racah polynomials. The parameters for the q-Racah polynomials depend on p_0, p_1, p_2, p_3

Theorem 12.11. *Let $P_n(\mu_m)$ be the overlap functions defined by*

$$P_n(\mu_m) = \frac{\langle \phi_m, \psi_n \rangle}{\psi_0(m)},$$

where $\psi_0(m) = \langle \phi_m, \psi_0 \rangle$. Then

$$P_n(\mu_m) = \sqrt{\frac{h_0}{h_n}} R_n(y_m; \alpha, \beta, \gamma, \delta; q^2),$$

where $R_n(\mu_m; \alpha, \beta, \gamma, \delta; q)$ are q-Racah polynomials from definition 5.19 and h_n are normalising constants given in (5.8). *The parameters are given by*

$$\alpha = -q^{p_0+p_2}, \ \beta = q^{p_0-p_2}, \ \gamma = q^{p_0-p_1}, \ \delta = -q^{p_2+p_3}, \ m = 0, 1, ..., N. \tag{12.35}$$

[39] I.e., we have as much freedom as possible in choosing the parameters.

Proof. We have to check that the three-term recurrence relation for the overlap functions (12.34) are the same as the one for q-Racah polynomials (5.9). Note that the first only depends on $(a_n)_{n=0}^N$ and $(b_n)_{n=0}^N$, while the latter has three coefficients for each n: A_n, B_n and C_n. This is because the q-Racah polynomials in definition 5.19 are not normalised. Orthonormal polynomials p_n always have a three-term recurrence relation of the form

$$xp_n(x) = \tilde{a}_{n+1}p_{n+1}(x) + \tilde{b}_n p_n(x) + \tilde{a}_n p_{n-1}(x).$$

Therefore, we normalise (5.9) by substituting the normalised q-Racah polynomials $p_n(y_m; \alpha, \beta, \gamma, \delta; q^2)$ given by

$$p_n(y_m; \alpha, \beta, \gamma, \delta; q^2) = \sqrt{\frac{h_0}{h_n}} R_n(y_m; \alpha, \beta, \gamma, \delta; q^2).$$

If we also divide the recurrence relation by $\sqrt{h_n}$, we get

$$y_m p_n(y_m) = \sqrt{\frac{h_{n+1}}{h_n}} A_n p_{n+1}(y_m) + B_n p_n(y_m) + \sqrt{\frac{h_{n-1}}{h_n}} C_n p_{n-1}(y_m). \tag{12.36}$$

Let us first find the right eigenvalues. Filling in the parameters from (12.35), gives

$$y_m = q^{-2m} + \gamma\delta q^{2m+2} = q^{-2m} - q^{p_0 - p_1 + p_2 + p_3 + 2m + 2} = q^{-2m} - q^{2\Sigma_p - 2p_1 + 2m + 2}.$$

By the link (12.33) between p_k and s_k, this becomes

$$y_m = q^{-2m} - q^{2s_0 + 2m + 2} = -q^{s_0+1} \sinh_q(2m + s_0 + 1) = -\sqrt{-\gamma\delta q^2}\mu_m.$$

Therefore, dividing (12.36) by $-\sqrt{-\gamma\delta q^2}$ and equating this with (12.34) gives

$$-\sqrt{\frac{h_{n+1}}{-\gamma\delta q^2 h_n}} A_n = a_{n+1}, \quad -\frac{B_n}{\sqrt{-\gamma\delta q^2}} = b_n, \quad -\sqrt{\frac{h_{n-1}}{-\gamma\delta q^2 h_n}} C_n = a_n. \tag{12.37}$$

We have to check if these equations hold. We will not show the full computations here. The idea is quite simple: square the first and the third term of (12.37) and fill in the explicit expressions for $a_n^2, b_n, h_n, A_n, B_n$ and C_n using (12.12), (12.29), (5.8) and (5.10) respectively. After a precise calculation one can verify (12.37), proving the theorem. $\square$

Remark 12.12. The orthogonality of the q-Racah polynomials correspond to the orthogonality of the eigenvectors ψ_n of K_0 and ϕ_m of K_1. Indeed, since $\{\phi_m\}_{m=0}^N$ is an orthonormal basis, we have

$$\psi_n = \sum_{m=0}^N \langle \psi_n, \phi_m \rangle \phi_m.$$

Therefore, by the orthogonality of ψ_n,

$$\delta_{kn} = \langle \psi_k, \psi_n \rangle = \sum_{m=0}^N \langle \psi_k, \phi_m \rangle \langle \phi_m, \psi_n \rangle = \sum_{m=0}^N w(m) P_k(\mu_m) P_n(\mu_m),$$

where $w(m)$ is the weight function given by

$$w(m) = |\langle \phi_m, \psi_0 \rangle|^2. \tag{12.38}$$

Similarly, we can get

$$\delta_{km} = \tilde{w}(m) \sum_{n=0}^N P_n(\mu_k) P_n(\mu_m),$$

where

$$\tilde{w}(m) = \psi_0(k)\psi_0(m).$$

13. Twisted primitive elements and the Askey-Wilson algebra

In this section we will show that the algebra AW(3) is embedded as a subalgebra in $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. Namely, the subalgebra generated by the twisted primitive elements[40] $X_aK + a_\sigma I$ and $K^{-1}X_b - b_\tau I$ and the Casimir Ω, where

$$X_a = q^{\frac{1}{2}}a_E E + q^{-\frac{1}{2}}a_F F + a_\sigma(K - K^{-1}), \quad X_b = q^{\frac{1}{2}}b_E E + q^{-\frac{1}{2}}b_F F + b_\sigma(K - K^{-1}).$$

After writing the article [5], the author had the idea that there might be such an embedding possible. Earlier, embeddings of AW(3) in algebras related to $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ had been done. For example, [2] did an embedding with 3 parameters and [7] in the threefold tensor product of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. In this section, we will show that AW(3) is directly embedded as a subalgebra in $\mathcal{U}_q\left(\mathfrak{sl}(2,\mathbb{C}),\Omega_0\right)$ with all parameters of AW(3) available. The algebra $\mathcal{U}_q\left(\mathfrak{sl}(2,\mathbb{C}),\Omega_0\right)$ is $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ where a value $\Omega_0 \in \mathbb{C}$ for the Casimir Ω is fixed. Formally, this is the algebra $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ with the extra relation $\Omega = \Omega_0 \cdot 1$. Note that this makes sense, since in an irreducible representation t of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$, the Casimir operator acts as a constant times the identity,

$$t(\Omega) = c \cdot \mathrm{id}, \quad c \in \mathbb{C}.$$

We will use this embedding in combination with the results of section 12 to show that the overlap functions between twisted primitive elements of $\mathcal{U}_q(\mathfrak{su}(2))$ in a finite-dimensional representation are q-Racah polynomials in their most general form. This result has been shown for infinite-dimensional representations of $\mathcal{U}_q(\mathfrak{su}(1,1))$ in [5] by a direct computation in $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$.

13.1. **Embedding** AW(3) **in** $\mathcal{U}_q\left(\mathfrak{sl}(2,\mathbb{C}),\Omega_0\right)$.

Recall from (1.4), that the Casimir Ω of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ is given by

$$\Omega = \frac{q^{-1}K^2 + qK^{-2}}{(q-q^{-1})^2} + EF = \frac{qK^2 + q^{-1}K^{-2}}{(q-q^{-1})^2} + FE,$$

is a central element. We have the following main result of this section.

Theorem 13.1. *Let K_0 and K_1 to be the twisted primitive elements given by,*

$$\begin{aligned} K_0 &= q^{\frac{1}{2}}a_E EK + q^{-\frac{1}{2}}a_F FK + a_\sigma K^2 && (13.1)\\ &= q^{-\frac{1}{2}}a_E KE + q^{\frac{1}{2}}a_F KF + a_\sigma K^2, \\ K_1 &= q^{\frac{1}{2}}b_E K^{-1}E + q^{-\frac{1}{2}}b_F K^{-1}F - b_\tau K^{-2} && (13.2)\\ &= q^{-\frac{1}{2}}b_E EK^{-1} + q^{\frac{1}{2}}b_F FK^{-1} - b_\tau K^{-2}, \end{aligned}$$

where $a_E, a_F, a_\sigma, b_E, b_F, b_\tau \in \mathbb{C}$. Then K_0 and K_1 satisfy the AW(3) relations (12.2) *and* (12.3) *with*

$$\begin{aligned} B &= \sinh_q(1)^2\left((a_Eb_F + a_Fb_E)\Omega - a_\sigma b_\tau\right), \\ C_0 &= -\cosh_q(1)^2 b_Eb_F, \\ C_1 &= -\cosh_q(1)^2 a_Ea_F, && (13.3)\\ D_0 &= \cosh_q(1)\left(\sinh_q(1)^2 b_Eb_Fa_\sigma\Omega + (a_Eb_F + a_Fb_E)b_\tau\right), \\ D_1 &= -\cosh_q(1)\left(\sinh_q(1)^2 a_Ea_Fb_\tau\Omega + (a_Eb_F + a_Fb_E)a_\sigma\right). \end{aligned}$$

Note that B, D_0, D_1 are in general no complex numbers, but central elements.

Remark 13.2. Observe that taking

$$a_E = b_E = e^{i\theta}, \quad a_F = b_E = e^{-i\theta}, \quad a_\sigma = -\frac{\sinh_q(\sigma)}{\sinh_q(1)} \quad \text{and} \quad b_\tau = -\frac{\sinh_q(\tau)}{\sinh_q(1)},$$

gives $K_0 = X_{\sigma,\theta}K - a_\sigma$ and $K_1 = K^{-1}X_{\tau,\theta} + b_\tau$, where $X_{\sigma,\theta}$ is defined by (6.2).

[40] We have changed the twisted primitive elements by a constant. Because otherwise the calculations do not give the desired answer when taken a_σ and b_τ nonzero. As a consequence, the eigenvalues of the operators are shifted. However, it does not change the eigenvectors.

Proof. The proof requires a long and precise calculation. For the readability of this thesis, the full proof can be found in appendix A. The idea of the proof is quite simple. Fill in the twisted primitive elements K_0 and K_1 in the AW(3) relation (12.2) and then do a precise calculation, where we use the relations (1.3) in $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and that Ω is central. The hard part is to be precise and recognise how to deal with the many terms that arise in the expression. The computation is done for the first relation (12.2). Then, we can exploit the symmetry between K_0 and K_1 by doing the transformation $K \leftrightarrow K^{-1}$, $q \leftrightarrow q^{-1}$ and $a_\sigma \leftrightarrow -b_\tau$ and obtain (12.3). □

It is interesting and quite tricky what the above theorem implies. It cannot be seen as a direct embedding of AW(3), with complex parameters, as a subalgebra into $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ because of the appearance of the Casimir Ω. Although Ω is a central element, it is of course no element of $\mathbb{C}$. However, in an irreducible representation, Ω always acts as a scalar times the identity. To solve this problem, we define the algebra $\mathcal{U}_q\left(\mathfrak{sl}(2,\mathbb{C}),\Omega_0\right)$, where we add an extra relation to $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$:

$$\Omega = \Omega_0 \cdot 1, \quad \Omega_0 \in \mathbb{C}, \quad 1 \in \mathcal{U}_q\left(\mathfrak{sl}(2,\mathbb{C}),\Omega_0\right).$$

By theorem 4.3, a $(2l+1)$-dimensional irreducible representation of $\mathcal{U}_q\left(\mathfrak{sl}(2,\mathbb{C}),\Omega_0\right)$ exists if and only if

$$\Omega_0 \cdot \mathrm{id} = t^l_\lambda(\Omega), \quad \text{where } \lambda^4 = 1.$$

The natural question arises whether every instance of AW(3) can be obtained from $K_0, K_1 \in \mathcal{U}_q\left(\mathfrak{sl}(2,\mathbb{C}),\Omega_0\right)$. That is, given a set of structure constants

$$(B, C_0, C_1, D_0, D_1) \in \mathbb{C}^5$$

for AW(3), can we always find

$$(a_E, a_F, a_\sigma, b_E, b_F, b_\tau, \Omega_0) \in \mathbb{C}^7,$$

such that AW(3) is isomorphic to the subalgebra of $\mathcal{U}_q\left(\mathfrak{sl}(2,\mathbb{C}),\Omega_0\right)$ generated by (K_0, K_1) given in (13.1) and (13.2). We show that the answer is, almost always, yes. The only exception is when $B = C_0 = C_1 = 0$, $D_0 D_1 = 0$ but either D_0 or D_1 is non-zero. We first define the function $\zeta : \mathbb{C}^7 \to \mathbb{C}^5$ which sends $(a_E, a_F, a_\sigma, b_E, b_F, b_\tau, \Omega_0)$ to the values for (B, C_0, C_1, D_0, D_1) given in (13.3). That is

$$\zeta(a_E, a_F, a_\sigma, b_E, b_F, b_\tau, \Omega_0) = \begin{pmatrix} \sinh_q(1)^2\left((a_E b_F + a_F b_E)\Omega_0 - a_\sigma b_\tau\right) \\ -\cosh_q(1)^2 b_E b_F \\ -\cosh_q(1)^2 a_E a_F \\ \cosh_q(1)\left(\sinh_q(1)^2 b_E b_F a_\sigma \Omega_0 + (a_E b_F + a_F b_E) b_\tau\right) \\ -\cosh_q(1)\left(\sinh_q(1)^2 a_E a_F b_\tau \Omega_0 + (a_E b_F + a_F b_E) a_\sigma\right) \end{pmatrix}.$$

Theorem 13.3. *Let $z = (B, C_0, C_1, D_0, D_1) \in \mathbb{C}^5$ be structure constants of* AW(3) *such that D_0 or D_1 is not the only non-zero constant. Then there exists a $w \in \mathbb{C}^7$ that satisfies $\zeta(w) = z$. In particular, for every instance of* AW(3)*, there exist $K_0, K_1 \in \mathcal{U}_q\left(\mathfrak{sl}(2,\mathbb{C}),\Omega_0\right)$ such that* AW(3) *is isomorphic to the subalgebra of $\mathcal{U}_q\left(\mathfrak{sl}(2,\mathbb{C}),\Omega_0\right)$ generated by K_0 and K_1.*

Proof. Take structure constants $z = (B, C_0, C_1, D_0, D_1) \in \mathbb{C}^5$ for AW(3) arbitrary. If we can find a $w \in \mathbb{C}^7$ such that $\zeta(w) = z$, we can take $(a_E, a_F, a_\sigma, b_E, b_F, b_\tau, \Omega_0) = w$ in theorem 13.1 to find $K_0, K_1 \in \mathcal{U}_q\left(\mathfrak{sl}(2,\mathbb{C}),\Omega_0\right)$ that satisfy the AW(3) relations (12.2) and (12.3) for the given structure constants. This induces an isomorphism between AW(3) and the subalgebra of $\mathcal{U}_q\left(\mathfrak{sl}(2,\mathbb{C}),\Omega_0\right)$ generated by K_0, K_1. So let us show we can always find such a $w \in \mathbb{C}^7$, or in other words, that ζ is surjective. Note that there has to be a certain redundancy in ζ since we go from 7 degrees of freedom to 5. One of the redundancies comes from Ω_0, the other one from a_E, a_F, b_E and b_F, which appear only in three different ways: '$a_E a_F$', '$b_E b_F$' and '$a_E b_F + a_F b_E$'. Let us fix $\Omega_0 \neq 0$ and take

$$b_E = -\frac{C_0}{\cosh_q(1)^2 b_F} \quad \text{and} \quad a_F = -\frac{C_1}{\cosh_q(1)^2 a_E}, \tag{13.4}$$

and

$$x = a_E b_F + a_F b_E = a_E b_F + \frac{C_0 C_1}{\cosh_q(1)^4 a_E b_F}. \tag{13.5}$$

For arbitrary C_0, C_1, x, we can find a_E, a_F, b_E, b_F that satisfy (13.4) and (13.5). Therefore, we have to solve the following three equations

$$\begin{aligned} B &= \sinh_q(1)^2 \left(x\Omega_0 - a_\sigma b_\tau\right) \\ D_0 &= \cosh_q(1)\left(\frac{-\sinh_q(1)^2}{\cosh_q(1)^2} C_0 a_\sigma \Omega_0 + x b_\tau\right) \\ D_1 &= -\cosh_q(1)\left(\frac{-\sinh_q(1)^2}{\cosh_q(1)^2} C_1 b_\tau \Omega_0 + x a_\sigma\right), \end{aligned}$$

where we have three variables x, a_σ, b_τ. Note the symmetry in how these variables appear. Let us assume that $C_0, C_1 \neq 0$. Then we can rewrite above equations to the following form,

$$x_1 - a_1 x_2 x_3 = b_1, \tag{13.6}$$
$$x_2 - a_2 x_1 x_3 = b_2, \tag{13.7}$$
$$x_3 - a_3 x_1 x_2 = b_3, \tag{13.8}$$

with three variables x_1, x_2, x_3 and where $a_1, a_2, a_3 \neq 0$, since $C_0, C_1, \Omega_0 \neq 0$. We will show the above system always has a solution. We can eliminate x_1 by substituting (13.6) into (13.7). We obtain

$$\begin{aligned} x_2 - a_2(b_1 + a_1 x_2 x_3)x_3 &= b_2, \\ x_3 - a_3(b_1 + a_1 x_2 x_3)x_2 &= b_3. \end{aligned}$$

Which we can rewrite to the system

$$x_2 - c_1 x_2 (x_3)^2 = d_1, \tag{13.9}$$
$$x_3 - c_2 (x_2)^2 x_3 = d_2, \tag{13.10}$$

where $c_1, c_2 \neq 0$, since a_1, a_2, a_3 are non-zero. The first equation implies

$$x_2 = \frac{d_1}{1 - c_1 (x_3)^2}. \tag{13.11}$$

If $x_3 \neq \pm (c_1)^{-1/2}$, we can substitute this into the second. We then get

$$\frac{(x_3 - d_2)(1 - c_1(x_3)^2)^2 - c_2 (d_1)^2 x_3}{(1 - c_1(x_3)^2)^2} = 0. \tag{13.12}$$

Since $c_1 \neq 0$, the numerator is a polynomial of degree 5 in x_3. It always has five (in general complex) roots, since $\mathbb{C}$ is algebraically closed. If it has a root which is not of the form $x_3 = \pm(c_1)^{-1/2}$, we can use (13.11) to find x_2 and (13.6) to find x_1 and we are done. If it has a solution of the form $x_3 = \pm(c_1)^{-1/2}$, it follows from (13.12) that

$$\pm c_2 (d_1)^2 (c_1)^{-1/2} = 0.$$

Since $c_1, c_2 \neq 0$, this implies

$$d_1 = 0.$$

However, in that case the system (13.9) and (13.10) can be solved by taking $x_2 = 0$ and $x_3 = d_2$.

If $C_0 = 0$, $C_1 = 0$ or both, the system 13.9,(13.10) becomes

$$x_1 - a_1 x_2 x_3 = b_1, \qquad (1 - \delta_{C_0 0})x_2 - a_2 x_1 x_3 = b_2,$$
$$(1 - \delta_{C_1 0})x_3 - a_3 x_1 x_2 = b_3.$$

This can be solved similarly as before, the only two exceptions being $C_0 = C_1 = b_1 = b_2 = 0,\ b_3 \neq 0$ and $C_0 = C_1 = b_1 = b_3 = 0,\ b_2 \neq 0$. If we are in one of those cases, say the first, we obtain

$$x_1 - a_1 x_2 x_3 = 0, \tag{13.13}$$
$$- a_2 x_1 x_3 = 0, \tag{13.14}$$
$$- a_3 x_1 x_2 = b_3. \tag{13.15}$$

The second equation (13.14) demands x_1 or x_3 to be zero, while the third one (13.15) requires x_1 and x_2 to be non-zero. Therefore, we need $x_3 = 0$. However, then (13.13) tells us that $x_1 = 0$ has to hold, which violates (13.15). This corresponds to the case $B = C_0 = C_1 = D_0 = 0$ and $D_1 \neq 0$. $\square$

13.2. **q-Racah polynomials as overlap functions of twisted primitive elements.** Up to this point we know that AW(3) is embedded in $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and that there exists a representation of AW(3) such that the overlap functions of K_0 and K_1 are q-Racah polynomials. In general, a representation of a subalgebra need not necessarily be extendable to the whole algebra[41]. Since a representation of an algebra is always a representation of its subalgebras, we suspect that the representation of AW(3) is the same as one of the four representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ in theorem 4.3. We will show[42] that this is indeed the case for the $*$-representation of $\mathcal{U}_q(\mathfrak{su}(2))$. Then we can apply theorem 12.11 to show that the overlap functions of twisted primitive elements are q-Racah polynomials.

We will approach as follows. From corollary 12.6, we know that if a representation of AW(3) exists, it is unique if the spectrum of K_0 is fixed[43]and non-degenerate. Therefore, if we can show this for our twisted primitive elements in $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$, we know we must have the same representation. We will focus on the generators K_0 and K_1 that are self-adjoint. From the expression for K_0 and K_1 given in theorem 13.1 this requires

$$a_F = \overline{a_E} \quad \text{and} \quad b_F = \overline{b_E}.$$

This leads to

$$C_0 = -\text{cosh}_q(1)^2|a_E|^2 \quad \text{and} \quad C_1 - = \text{cosh}_q(1)^2|b_E|^2,$$

and thus $C_0, C_1 < 0$. Similarly as in the previous section, we will reduce the spectrum of K_0 to the simplest case. That is, we take

$$C_0 = C_1 = -\text{sinh}_q(2)^2.$$

Since $\text{sinh}_q(1)\text{cosh}_q(1) = \text{sinh}_q(2)$, this is satisfied if

$$a_E = -e^{i\theta}\text{sinh}_q(1) \quad \text{and} \quad b_E = e^{i\phi}\text{sinh}_q(1).$$

The resulting parameters will depend on the difference '$\theta - \phi$'. Therefore, we can take $\phi = 0$ without loss of generality. Recall that we know the eigenvalues of $X_\sigma K$ and $K^{-1}X_\tau$. We will write our generators K_0 and K_1 as a linear combination of those,

$$\widehat{K_0} = \text{sinh}_q(1)\left(\frac{\text{sinh}_q(\sigma)}{\text{sinh}_q(1)} - X_{\sigma,\theta}K\right) = -q^{\frac{1}{2}}e^{i\theta}\text{sinh}_q(1)EK - q^{-\frac{1}{2}}e^{-i\theta}\text{sinh}_q(1)FK + \text{sinh}_q(\sigma)K^2,$$

$$\widehat{K_1} = \text{sinh}_q(1)\left(K^{-1}X_\tau + \frac{\text{sinh}_q(\tau)}{\text{sinh}_q(1)}\right) = q^{-\frac{1}{2}}\text{sinh}_q(1)K^{-1}E + q^{\frac{1}{2}}\text{sinh}_q(1)K^{-1}F + \text{sinh}_q(\tau)K^{-2}.$$

We can now easily calculate the eigenvalues and eigenvectors of $\widehat{K_0}$ and $\widehat{K_1}$ using theorem 6.1 and corollary 6.4. It follows the spectrum of K_0 is non-degenerate. Therefore, the representation of $\mathcal{U}_q(\mathfrak{su}(2))$ corresponds to the one for AW(3) that leads to q-Racah polynomials. We will use theorem 13.3 to find an isomorphism between AW(3) and the subalgebra of $\mathcal{U}_q\left(\mathfrak{sl}(2,\mathbb{C}),\Omega_0\right)$ generated by $\widehat{K_0}, \widehat{K_1}$. Note that t^l from theorem 4.12 is a representation of $\mathcal{U}_q\left(\mathfrak{sl}(2,\mathbb{C}),\Omega_0\right)$ if and only if we have

$$\Omega_0 = t^l(\Omega) = \frac{\text{cosh}_q(2l+1)}{\text{sinh}_q(1)^2},$$

[41]For example, the subalgebra of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ generated by $\{K, K^{-1}\}$ has a one-dimensional representation t given by $t(K) = 2$ and $t(K^{-1}) = \frac{1}{2}$ which cannot be extended to $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$. Indeed, from $KE = qEK$ and $q \neq 1$ it follows that $E = 0$. Therefore, the relation $EF - FE = \frac{K^2-K^{-2}}{q-q^{-1}}$ can never be satisfied.

[42]Other representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ also correspond to representations of AW(3). However, we will focus on the case when K_0 and K_1 are self-adjoint. The cases $\lambda = 1, -1$ in theorem 4.3 are equivalent. The representations for $\lambda = i, -i$ are equivalent as well and can be obtained from the case $\lambda = 1, -1$ by doing the substitution of structure constants in AW(3).

[43]That is, we know we have the positive or negative branch of eigenvalues for K_0

where we used (4.6).

Corollary 13.4. *Fix*

$$\Omega_0 = \frac{\cosh_q(2l+1)}{\sinh_q(1)^2},$$

and let AW(3) *be the algebra with constants given by*

$$(B, C_0, C_1, D_0, D_1) = \zeta(a_E, a_F, a_\sigma, b_E, b_F, b_\tau, \Omega_0),$$

where $a_E, a_F, a_\sigma, b_E, b_F, b_\tau$ *come from* $\widehat{K_0}, \widehat{K_1}$*, i.e.*

$$a_E = -q^{\frac{1}{2}} q^{i\theta} \sinh_q(1), \quad a_F = -q^{-\frac{1}{2}} q^{-i\theta} \sinh_q(1), \quad a_\sigma = \sinh_q(\sigma),$$
$$b_E = q^{-\frac{1}{2}} \sinh_q(1), \quad b_F = q^{\frac{1}{2}} \sinh_q(1), \quad a_\tau = -\sinh_q(\tau).$$

Denote by A be the subalgebra of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}), \Omega_0)$ generated by $\widehat{K_0}, \widehat{K_1}$. Let $\Phi : \mathrm{AW}(3) \to A$ be the isomorphism from theorem 13.3 and let π be the irreducible $(2l+1)$-dimensional representation of AW(3) *given in corollary 12.6 corresponding to the positive branch of eigenvalues for K_0. Then t^l, the $(2l+1)$-dimensional $*$-representation of $\mathcal{U}_q(\mathfrak{su}(2))$, is a representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}), \Omega_0)$ and for $X \in \mathrm{AW}(3)$, we have*

$$t^l(\Phi(X)) = \pi(X).$$

Proof. First observe that the $*$-representation of $\mathcal{U}_q(\mathfrak{su}(2))$ is automatically a representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}), \Omega_0)$, since Ω_0 is equal to the constant belonging to the operator $t^l(\Omega)$. Using the isomorphism given in theorem 13.3, t^l is also a representation of AW(3). From corollary 12.6 we know that the spectrum of K_0, if non-degenerate, uniquely determines[44] the finite-dimensional representation. Since

$$\widehat{K_0} = \sinh_q(1)\left(\frac{\sinh_q(\sigma)}{\sinh_q(1)} - X_{\sigma,\theta}K\right),$$

we can use the eigenvalues for $X_{\sigma,\theta}K$ from theorem 6.1 and find the spectrum $\{\lambda_j\}_{j=-l}^{l}$ of $\widehat{K_0}$, we get

$$\lambda_j = \sinh_q(2j+\sigma), \quad j = -l, ..., l.$$

By the injectivity of $\sinh_q$ on the real line, the spectrum of $\widehat{K_0}$ is non-degenerate. Moreover, observe that the spectrum of $\widehat{K_0}$ and $\widehat{K_1}$ in $\mathcal{U}_q(\mathfrak{su}(2))$ indeed correspond to the one given by (12.10). By doing the substitution $n = j + l$ in λ_j above, we get

$$\lambda_n = \sinh_q(2n + \sigma - 2l), \quad n = 0, ..., 2l, \tag{13.16}$$

as spectrum for $\widehat{K_0}$. Similarly, we can find that the eigenvalues μ_m of $\widehat{K_0}$ are given by

$$\mu_m = \sinh_q(2n + \tau - 2l).$$

This shows that we have the positive branch (i.e. $k = 1$ in theorem 12.2) of eigenvalues for both generators. □

$\widehat{K_0}$ and $\widehat{K_1}$ have the same eigenvectors as $X_{\sigma,\theta+\pi}K$ and $K^{-1}X\tau$ respectively. That is,

$$\widehat{K_0}\lambda_n = \lambda_n v^{l,n}(\sigma, \theta - \pi) \quad \text{and} \quad \widehat{K_0}\lambda_n = \mu_m \tilde{v}^{l,n}(\tau),$$

where $v^{l,n}(\sigma, \theta - \pi)$ and $\tilde{v}^{l,n}(\tau)$ are given by theorem 6.1 and corollary 6.4 after the substitution $n = j + l$. Let us show that the overlap functions $P_n(\mu_m)$ of our twisted primitive elements are q-Racah polynomials. They are defined similarly as in theorem 12.11,

$$P_n(\mu_m) = \frac{\left\langle \tilde{v}^{l,m}(\tau), v^{l,n}(\sigma,\theta)\right\rangle}{v_0(m)},$$

where $v_0(m) = \left\langle \tilde{v}^{l,m}(\tau), v^{l,0}(\sigma,\theta)\right\rangle$ is separated to get the initial condition $P_0(\mu_m) = 1$. Recall from theorem 12.11 that the parameters of these polynomials are determined by $\{p_k\}_{k=0}^3$, where

[44] If the spectrum K_0 is non-degenerate, it is unique op to a sign. Therefore, if the spectrum of K_0 is fixed, the representation is unique.

$\sinh_q(p_k)$ are the roots of the characteristic polynomial $\mathscr{P}$. We can get the parameters p_0, p_1 from the spectrum of K_0 and the quantisation condition,

$$p_1 - p_0 = 2(N+1).$$

From proposition 12.8 we can determine p_2 and p_3. In order to make calculation simpler, we replace θ by ${}^q\log(e)\theta$, which leads to the replacing $e^{i\theta}$ by $q^{i\theta}$.

Theorem 13.5. *The overlap functions $P_n(\mu_m)$ of the twisted primitive elements*

$$\widehat{K_0} = -q^{\frac{1}{2}}q^{i\theta}\sinh_q(1)EK - q^{-\frac{1}{2}}q^{-i\theta}\sinh_q(1)FK + \sinh_q(\sigma)K^2,$$
$$\widehat{K_1} = q^{-\frac{1}{2}}\sinh_q(1)K^{-1}E + q^{\frac{1}{2}}\sinh_q(1)K^{-1}F + \sinh_q(\tau)K^{-2},$$

are normalised q-Racah polynomials

$$\sqrt{\frac{h_0}{h_n}}R_n(y_m;\alpha,\beta,\gamma,\delta;q^2), \quad m,n = 0,...,N.$$

Here, $R_n(y_m;\alpha,\beta,\gamma,\delta;q^2)$ and h_n are from definition 5.19 and (5.8) *respectively. The parameters are given by*

$$\alpha = -q^{\sigma-2l-1+i\theta+\tau},\ \beta = q^{\sigma-2l-1-i\theta-\tau},\ \gamma = q^{-4l-2},\ \delta = -q^{2\tau}. \tag{13.17}$$

Moreover, the normalised weight function of the q-Racah polynomials is linked with $v_0(m)$ via the formula

$$|v_0(m)|^2 = \frac{w(y_m,\alpha,\beta,\gamma,\delta;q^2)}{h_0}, \tag{13.18}$$

where $w(y_m,\alpha,\beta,\gamma,\delta;q^2)$ is given in (5.7).

Proof. From corollary 13.4 we know that the $*$-representation of $\mathcal{U}_q(\mathfrak{su}(2))$ corresponds to the representation of AW(3) used in theorem 12.11. The parameters $(\alpha,\beta,\gamma,\delta)$ of the q-Racah polynomials are expressed in p_0, p_1, p_2 and p_3. We claim that

$$p_0 = \sigma - 2l - 1, \quad p_1 = \sigma + 2l + 1, \tag{13.19}$$
$$p_2 = \tau + i\theta, \quad p_3 = \tau - i\theta. \tag{13.20}$$

First of all, using (13.16) and the second part of corollary 12.6, we can read of the spectral parameter $p_0 = \sigma - 2l - 1$. Then, by the quantisation condition of the same corollary, we have

$$p_1 = 2(2l+1) + p_0 = \sigma + 2l + 1, \tag{13.21}$$

proving (13.19). For p_2 and p_3, we will use that we have two expressions for B and D_0. In theorem 13.1, B and D_0 are expressed in terms of $(a_E, a_F, a_\sigma, b_E, b_F, b_\tau, \Omega_0)$ we used for $\widehat{K_0}$ and $\widehat{K_1}$ and $\mathcal{U}_q\left(\mathfrak{sl}(2,\mathbb{C}),\Omega_0\right)$. On the other hand, proposition 12.8 gives B and D_0 as a function of the p_k. We know that $(a_E, a_F, a_\sigma, b_E, b_F, b_\tau, \Omega_0)$ is equal to

$$\left(-q^{\frac{1}{2}}q^{i\theta}\sinh_q(1), -q^{-\frac{1}{2}}q^{-i\theta}\sinh_q(1), \sinh_q(\sigma), q^{-\frac{1}{2}}\sinh_q(1), q^{\frac{1}{2}}\sinh_q(1), -\sinh_q(\tau), \frac{\cosh_q(2l+1)}{\sinh_q(1)^2}\right).$$

Therefore, we have for D_0,

$$\begin{aligned}\frac{\sinh_q(2)^2}{\cosh_q(1)}\sum_{k=0}^{3}\sinh_q(p_k) &= \cosh_q(1)\left(\sinh_q(1)^2 b_E b_F a_\sigma \Omega_0 + (a_E b_F + a_F b_E)b_\tau\right)\\ &= \cosh_q(1)\left(\sinh_q(1)^2\sinh_q(\sigma)\cosh_q(2l+1) + \sinh_q(1)^2\cosh_q(i\theta)\sinh_q(\tau)\right)\\ &= \frac{\sinh_q(2)^2}{\cosh_q(1)}\left(\sinh_q(\sigma)\cosh_q(2l+1) + \cosh_q(i\theta)\sinh_q(\tau)\right).\end{aligned}$$

Since $\sinh_q(a)\cosh_q(b) = \sinh_q(a+b) + \sinh_q(a-b)$ and this becomes

$$\sum_{k=0}^{3}\sinh_q(p_k) = \sinh_q(\sigma+2l+1) + \sinh_q(\sigma-2l-1) + \sinh_q(\tau+i\theta) + \sinh_q(\tau-i\theta).$$

Using our expressions for p_0 and p_1, the first two terms on both sides cancel. Therefore, we obtain

$$\sinh_q(p_2) + \sinh_q(p_3) = \sinh_q(\tau + i\theta) + \sinh_q(\tau - i\theta). \tag{13.22}$$

Let us derive another equation for p_2 and p_3, using the two expressions for B. We have

$$\begin{aligned}
&\frac{\sinh_q(1)}{\cosh_q(1)}\sinh_q(2)\left(\sinh_q\left(\frac{p_0+p_1}{2}\right)\sinh_q\left(\frac{p_2+p_3}{2}\right) + \cosh_q\left(\frac{p_0-p_1}{2}\right)\cosh_q\left(\frac{p_2-p_3}{2}\right)\right)\\
&\qquad = \sinh_q(1)^2\left((a_E b_F + a_F b_E)\Omega_0 - a_\sigma b_\tau\right)\\
&\qquad = \sinh_q(1)^2\left(-\cosh_q(i\theta)\cosh_q(2l+1) + \sinh_q(\sigma)\sinh_q(\tau)\right)\\
&\qquad = \frac{\sinh_q(1)}{\cosh_q(1)}\sinh_q(2)\left(-\cosh_q(i\theta)\cosh_q(2l+1) + \sinh_q(\sigma)\sinh_q(\tau)\right).
\end{aligned}$$

Using that $p_0 + p_1 = 2\sigma$ and $p_0 - p_1 = -2(2l+1)$, we obtain

$$\begin{aligned}
\sinh_q(\sigma)\sinh_q\left(\frac{p_2+p_3}{2}\right) + \cosh_q(-2l-1)\cosh_q\left(\frac{p_2-p_3}{2}\right)&\\
= \sinh_q(\sigma)\sinh_q(\tau) + \cosh_q(i\theta)\cosh_q(-2l+1).&
\end{aligned} \tag{13.23}$$

Together with (13.22), this proves our claim (13.20) for p_2 and p_3. We can now derive the parameters for the q-Racah polynomials using (12.35). The formula for the weight function follows from (12.38). □

Since we know our overlap functions explicitly, we can derive the following, quite difficult, summation formula for dual q-Krawtchouk and q-Racah polynomials.

Corollary 13.6. *We have*

$$\begin{aligned}
\sum_{k=0}^{N} C(N,k,\sigma,\tau,\theta,m)(-q^{i\theta})^k R_k(\tilde{y}_n;\alpha\beta\gamma^{-1};\, N;q)R_k(\tilde{y}_m;\delta^{-1};N;q^{-1})&\\
= \sqrt{w(y_m,\alpha,\beta,\gamma,\delta;q)}R_n(y_m;\alpha,\beta,\gamma,\delta;q),&
\end{aligned}$$

where $R_n(y_j;q^c,N;q)$ are dual q-Krawtchouk polynomials from definition 5.21, $R_n(y_j;\alpha,\beta,\gamma,\delta;q)$ are q-Racah polynomials, $w(y_m,\alpha,\beta,\gamma,\delta;q)$ its weight function given by (5.7), $\alpha\beta,\delta < 0$ and $\gamma q = q^{-N}$. The constant can be explicitly computed.

Proof. From theorem 13.5 we get

$$\frac{\left\langle \tilde{v}^{l,m}(\tau), v^{l,n}(\sigma,\theta)\right\rangle}{\left\langle \tilde{v}^{l,m}(\tau), v^{l,0}(\sigma,\theta)\right\rangle} = \sqrt{\frac{h_0}{h_n}}R_n(y_m;-q^{\sigma-2l-1+i\theta+\tau},q^{\sigma-2l-1-i\theta-\tau},q^{-4l-2},-q^{2\tau};q^2).$$

Using the explicit formulas for $v^{l,n}(\sigma,\theta)$ and $\tilde{v}^{l,m}(\tau)$ in theorem 6.1 and corollary 6.4 we get a summation formula close to the form we want. From (13.18), we obtain

$$|\left\langle \tilde{v}^{l,m}(\tau), v^{l,0}(\sigma,\theta)\right\rangle|^2 = w(y_m,\alpha,\beta,\gamma,\delta;q).$$

Thus we get

$$\left\langle \tilde{v}^{l,m}(\tau), v^{l,0}(\sigma,\theta)\right\rangle = \eta_m\sqrt{w(y_m,\alpha,\beta,\gamma,\delta;q)},$$

where η_m is a phase factor which can be explicitly calculated for every m. Putting this η_m into the constant $C(N,k,\sigma,\tau,\theta,m)$ gives the desired formula. □

Summary and concluding remarks

This thesis provides a small insight into the remarkable connection between quantum groups and orthogonal polynomials. The first two parts showed known results from the literature, while in the last part, we proved a new and interesting embedding of the Askey-Wilson algebra AW(3) into $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$.

In part I and II, we put some focus on the similarities and differences between the representation theory in the classical- and quantum setting. We saw that the latter is often an extension of the first and in many cases a non-trivial one. This makes researching quantum groups worthwhile. For example, we saw that choice of the subgroup for invariant functions did not matter in the classical case, but it did in the quantum setting. Some results, such as theorem 10.9 do not even have a classical equivalent, since all representations of $\mathrm{Pol}(\mathrm{SL}(2,\mathbb{C}))$ are one-dimensional.

In all three parts, we saw the crucial role of Koornwinder's twisted primitive elements. They facilitate an important link between $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and Askey-Wilson polynomials. In part II, they were used for finding the space of (τ,σ)-spherical elements, while in part III we showed they can be seen as generators of Zhedanov's Askey-Wilson algebra AW(3). Interestingly, both approaches lead to different Askey-Wilson polynomials.

Lastly, the new result proved in this thesis raises some interesting questions. For example, the embedding of AW(3) into the threefold tensor product of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ (see [7]) was linked with higher-rank Askey-Wilson algebras [3]. We suspect that our result can be extended to these higher-rank Askey-Wilson algebras as well. Moreover, we proved that q-Racah polynomials are overlap functions of twisted primitive elements in the $\mathcal{U}_q(\mathfrak{su}(2))$ setting. This corresponds to the case $C_1 < 0$ in AW(3). Self-adjointness of twisted primitive elements for $C_1 > 0$ corresponds to $\mathcal{U}_q(\mathfrak{su}(1,1))$, which has no finite-dimensional representations. It is of interest to see if for $C_1 > 0$, there exist representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ as well where the overlap functions of the twisted primitive elements are q-Racah polynomials. If this would be the case, the $*$-structure of AW(3) would not correspond to a real form of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$.

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Appendix A. Proof theorem 13.1

Let $K_0, K_1 \in \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ be given by

$$\begin{aligned}
K_0 &= q^{\frac{1}{2}} a_E EK + q^{-\frac{1}{2}} a_F FK + a_\sigma K^2,\\
&= q^{-\frac{1}{2}} a_E KE + q^{\frac{1}{2}} a_F KF + a_\sigma K^2.\\
K_1 &= q^{\frac{1}{2}} b_E K^{-1}E + q^{-\frac{1}{2}} b_F K^{-1}F - b_\tau K^{-2},\\
&= q^{-\frac{1}{2}} b_E EK^{-1} + q^{\frac{1}{2}} b_F FK^{-1} - b_\tau K^{-2},
\end{aligned}$$

where $a_E, a_F, a_\sigma, b_E, b_F, b_\tau \in \mathbb{C}$. We will show that K_0 and K_1 satisfy the AW(3) relations

$$\begin{aligned}
\cosh_q(2)K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 &= BK_1 + C_0K_0 + D_0,\\
\cosh_q(2)K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 &= BK_0 + C_1K_1 + D_1,
\end{aligned}$$

with

$$\begin{aligned}
B &= \sinh_q(1)^2\left((a_Eb_F + a_Fb_E)\Omega - a_\sigma b_\tau\right),\\
C_0 &= -\cosh_q(1)^2 b_Eb_F,\\
C_1 &= -\cosh_q(1)^2 a_Ea_F,\\
D_0 &= \cosh_q(1)\left(\sinh_q(1)^2 b_Eb_Fa_\sigma\Omega + (a_Eb_F + a_Fb_E)b_\tau\right),\\
D_1 &= -\cosh_q(1)\left(\sinh_q(1)^2 a_Ea_Fb_\tau\Omega + (a_Eb_F + a_Fb_E)a_\sigma\right).
\end{aligned}$$

First, take $a_\sigma = 0 = b_\tau$ and calculating $K_1K_0K_1$, $K_1^2K_0$ and $K_0K_1^2$ gives

$$\begin{aligned}
K_1K_0K_1 =& q^{-1\frac{1}{2}} b_E^3 K^{-1} + q^{-\frac{1}{2}} b_Ea_Eb_F E^2FK^{-1} + q^{-\frac{1}{2}} b_Ea_Fb_E EFEK^{-1}\\
&+ q^{\frac{1}{2}} b_Ea_Fb_F EF^2K^{-1} + q^{-\frac{1}{2}} b_Fa_Eb_E FE^2K^{-1} + q^{\frac{1}{2}} b_Fa_Eb_F FEFK^{-1}\\
&+ q^{\frac{1}{2}} b_Fa_Fb_E F^2EK^{-1} + q^{1\frac{1}{2}} b_Fa_Fb_F F^3K^{-1}.\\
K_1^2K_0 =& q^{-3\frac{1}{2}} b_Eb_Ea_E E^3K^{-1} + q^{-\frac{1}{2}} b_Eb_Ea_F E^2FK^{-1} + q^{-\frac{1}{2}} b_Eb_Fa_E EFEK^{-1}\\
&+ q^{2\frac{1}{2}} b_Eb_Fa_F EF^2K^{-1} + q^{-2\frac{1}{2}} b_Fb_Ea_E FE^2K^{-1} + q^{\frac{1}{2}} b_Fb_Ea_F FEFK^{-1}\\
&+ q^{\frac{1}{2}} b_Fb_Fa_E F^2EK^{-1} + q^{3\frac{1}{2}} b_Fb_Fa_F F^3K^{-1}.\\
K_0K_1^2 =& q^{\frac{1}{2}} b_Eb_Eb_E E^3K^{-1} + q^{1\frac{1}{2}} a_Eb_Eb_F E^2FK^{-1} + q^{-\frac{1}{2}} a_Eb_Fb_E EFEK^{-1}\\
&+ q^{\frac{1}{2}} a_Eb_Fb_F EF^2K^{-1} + q^{-\frac{1}{2}} a_Fb_Eb_E FE^2K^{-1} + q^{\frac{1}{2}} a_Fb_Eb_F FEFK^{-1}\\
&+ q^{-1\frac{1}{2}} a_Fb_Fb_E F^2EK^{-1} + q^{-\frac{1}{2}} a_Fb_Fb_F F^3K^{-1}.
\end{aligned}$$

This gives

$$\begin{aligned}
(q^2+q^{-2})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 =& b_EE^2FK^{-1}(q^{-2\frac{1}{2}}a_Eb_F - q^{-\frac{1}{2}}b_Ea_F)\\
&+ b_EEFE\left((q^{1\frac{1}{2}}a_Fb_E + q^{-2\frac{1}{2}}) - 2q^{-\frac{1}{2}}a_Eb_F\right)\\
&+ b_FEF^2K^{-1}(q^{-1\frac{1}{2}}b_Ea_F - q^{\frac{1}{2}}a_Eb_F)\\
&+ b_EFE^2K^{-1}(q^{1\frac{1}{2}}b_Fa_E - q^{-\frac{1}{2}}a_Fb_E)\\
&+ b_FFEFK^{-1}\left((q^{2\frac{1}{2}} + q^{-1\frac{1}{2}})a_Eb_F - 2q^{\frac{1}{2}}b_Ea_F\right)\\
&+ b_FF^2EK^{-1}(q^{2\frac{1}{2}}a_Fb_E - q^{\frac{1}{2}}b_Fa_E).
\end{aligned}$$

Note that the E^3 and F^3 exactly cancel out. We now put the terms with two E's first and two F's second. Then, if we give the constants that arise a (temporary) name, we have

$$b_E\left(c_1E^2F + c_2EFE + c_3FE^2\right)K^{-1} + b_F\left(c_4EF^2 + c_5FEF + c_6F^2E\right)K^{-1}. \tag{A.1}$$

If we now use

$$EF - FE = \frac{K^2 - K^{-2}}{q - q^{-1}},$$

we obtain

$$\begin{aligned}c_1E^2F + c_2EFE + c_3FE^2 &= (c_1+c_2+c_3)EFE + c_1E\frac{K^2-K^{-2}}{q-q^{-1}} - c_3\frac{K^2-K^{-2}}{q-q^{-1}}E \\ &= (c_1+c_2+c_3)EFE + \frac{c_1-q^2c_3}{q-q^{-1}}EK^2 + \frac{c_3-q^2c_1}{q-q^{-1}}K^{-2}E.\end{aligned}$$

Similarly we get

$$\begin{aligned}c_4EF^2 + c_5FEF + c_6F^2E &= (c_4+c_5+c_6)FEF - c_6F\frac{K^2-K^{-2}}{q-q^{-1}} + c_4\frac{K^2-K^{-2}}{q-q^{-1}}F \\ &= (c_4+c_5+c_6)FEF + \frac{q^{-2}c_4-c_6}{q-q^{-1}}FK^2 + \frac{q^{-2}c_6-c_4}{q-q^{-1}}K^{-2}F.\end{aligned}$$

Now the magic starts to happen. A calculation shows that

$$c_1+c_2+c_3 = q^{-\frac{1}{2}}(a_Eb_F+a_Fb_E)(q-q^{-1})^2 = q^{-1}(c_3-c_1q^2)(q-q^{-1})$$

and

$$c_4+c_5+c_6 = q^{\frac{1}{2}}(a_Eb_F+a_Fb_E)(q-q^{-1})^2 = q(q^{-2}c_6-c_4)(q-q^{-1}).$$

Furthermore,

$$\frac{c_1-q^2c_3}{q-q^{-1}}EK^2 = q^{\frac{1}{2}}\left(a_Fb_E - (q^2+1+q^{-2})a_Eb_F\right),$$

as well as

$$\frac{q^{-2}c_4-c_6}{q-q^{-1}} = q^{-\frac{1}{2}}\left(a_Eb_F - (q^2+1+q^{-2})a_Fb_E\right).$$

Therefore, (A.1) is equal to

$$\begin{aligned}&(a_Eb_F+a_Fb_E)(q-q^{-1})^2\left[\left(EF+\frac{qK^{-2}}{(q-q^{-1})^2}\right)b_Eq^{-\frac{1}{2}}E + \left(FE+\frac{q^{-1}K^{-2}}{(q-q^{-1})^2}\right)b_Fq^{\frac{1}{2}}F\right]K^{-1} \\ &+\left(a_Fb_E-(q^2+1+q^{-2})a_Eb_F\right)q^{\frac{1}{2}}b_EEK + \left(a_Eb_F-(q^2+1+q^{-2})a_Fb_E\right)q^{-\frac{1}{2}}b_FFK.\end{aligned}$$

If we now add and substract

$$(a_Eb_F+a_Fb_E)\left(q^{\frac{1}{2}}b_EEK + q^{-\frac{1}{2}}b_FFK\right),$$

we obtain

$$\begin{aligned}(q^2+q^{-2})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 =&(a_Eb_F+a_Fb_E)(q-q^{-1})^2\Omega K_1 \\ &-(q+q^{-1})^2b_Eb_FK_0.\end{aligned} \tag{A.2}$$

If we make this slightly more interesting and take $\sigma\in\mathbb{R}$ (and still $\tau=0$), we obtain as extra term compared to the case $\sigma=0$

$$(q^2+q^{-2})a_\sigma K_1K^2K_1 - a_\sigma K_1^2K^2 - a_\sigma K^2K_1^2. \tag{A.3}$$

We have

$$\begin{aligned}K_1K^2K_1 &= (b_Eq^{-\frac{1}{2}}EK^{-1}+b_Fq^{\frac{1}{2}}FK^{-1})K^2(b_Eq^{\frac{1}{2}}K^{-1}E+b_Fq^{-\frac{1}{2}}K^{-1}F) \\ &= b_E^2E^2 + b_Eb_F\left(q^{-1}EF+qFE\right) - b_F^2F^2, \\ K_1^2K^2 &= \left(b_E^2EK^{-2}E + b_Eb_F\left(q^{-1}EK^{-2}F+qFK^{-2}E\right) - b_F^2FK^{-2}F\right)K^2, \\ &= q^{-2}b_E^2E^2 + b_Eb_F\left(qEF+q^{-1}FE\right) - q^2b_F^2F^2, \\ K^2K_1^2 &= K^2\left(b_E^2EK^{-2}E + b_Eb_F\left(q^{-1}EK^{-2}F+qFK^{-2}E\right) - b_F^2FK^{-2}F\right) \\ &= q^2b_E^2E^2 + b_Eb_F\left(qEF+q^{-1}FE\right) - q^{-2}b_F^2F^2.\end{aligned}$$

This gives that (A.3) is equal to

$$a_\sigma b_Eb_F\left(q^{-1}(q^{-2}-q^2)EF + q(q^2-q^{-2})FE\right).$$

Note that the E^2 and F^2 terms exactly cancel each other. Using that

$$EF = FE + \frac{K^2 - K^{-2}}{q - q^{-1}}$$

we obtain

$$q^{-1}(q^2 - q^{-2})EF = q^{-1}\left((q^2 - q^{-2})FE + (q + q^{-1})(K^2 - K^{-2})\right).$$

Since

$$(q - q^{-1})^2(q + q^{-1})FE = (q - q^{-1})^2(q + q^{-1})\Omega - (q + q^{-1})\left(qK^2 - q^{-1}K^{-2}\right),$$

we get,

$$\begin{aligned}
-q^{-1}(q^2 - q^{-2})EF + q(q^2 - q^{-2})FE &= (q - q^{-1})(q^2 - q^{-2})FE - q^{-1}(q + q^{-1})(K^2 - K^{-2}) \\
&= (q - q^{-1})^2(q + q^{-1})FE - q^{-1}(q + q^{-1})(K^2 - K^{-2}) \\
&= (q - q^{-1})^2(q + q^{-1})\Omega - (q + q^{-1})^2K^2.
\end{aligned}$$

Therefore, (A.3) is equal to

$$a_\sigma b_E b_F(q - q^{-1})^2(q + q^{-1})\Omega - a_\sigma b_E b_F(q + q^{-1})^2K^2. \tag{A.4}$$

The second term in front of the $a_\sigma K^2$ is exactly the same as the constant in front of the K_0 in (A.2), just like we wanted. So (A.2) stays the same for σ nonzero, only now with a added central element,

$$\begin{aligned}
(q^2 + q^{-2})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 =&(a_Eb_F + a_Fb_E)(q - q^{-1})^2\Omega K_1 - (q + q^{-1})^2b_Eb_FK_0 \\
&+ a_\sigma b_E b_F(q - q^{-1})^2(q + q^{-1})\Omega.
\end{aligned} \tag{A.5}$$

Let us now take $\sigma = 0$ but τ nonzero. Denote by $K_1(0)$ the term K_1 where τ is taken to be 0. We now obtain the extra term, compared to (A.2),

$$\begin{aligned}
(q^2 + q^{-2})&\left(-b_\tau K_1(0)K_0K^{-2} - b_\tau K^{-2}K_0K_1(0) + b_\tau^2K^{-2}K_0K^{-2}\right) \\
&-K_0\left(b_\tau^2K^{-4} - b_\tau\left(K_1(0)K^{-2} + K^{-2}K_1(0)\right)\right) \\
&-\left(b_\tau^2K^{-4} - b_\tau\left(K_1(0)K^{-2} + K^{-2}K_1(0)\right)\right)K_0.
\end{aligned} \tag{A.6}$$

We have

$$\begin{aligned}
K_0K_1(0) &= qa_Eb_EE^2 + a_Eb_FEF + a_Fb_EFE + q^{-1}a_Fb_FF^2, \\
K_1(0)K_0 &= q^{-1}a_Eb_EE^2 + a_Fb_EEF + a_Eb_FFE + qa_Fb_FF^2.
\end{aligned}$$

Therefore, we get for the terms of (A.6),

$$\begin{aligned}
K_1(0)K_0K^{-2} &= (q^{-1}a_Eb_EE^2 + a_Fb_EEF + a_Eb_FFE + qa_Fb_FF^2)K^{-2}, \\
K^{-2}K_0K_1(0) &= (q^{-1}a_Eb_EE^2 + a_Eb_FEF + a_Fb_EFE + qa_Fb_FF^2)K^{-2}, \\
K^{-2}K_0K^{-2} &= (q^{-1\frac{1}{2}}a_EE + q^{1\frac{1}{2}}a_FF)K^{-3}, \\
K_0K^{-4} &= (q^{\frac{1}{2}}a_EE + q^{-\frac{1}{2}}a_FF)K^{-3}, \\
K_0K_1(0)K^{-2} &= (qa_Eb_EE^2 + a_Eb_FEF + a_Fb_EFE + q^{-1}a_Fb_FF^2)K^{-2}, \\
K_0K^{-2}K_1(0) &= (qa_Eb_EE^2 + q^2a_Eb_FEF + q^{-2}a_Fb_EFE + q^{-1}a_Fb_FF^2)K^{-2}, \\
K^{-4}K_0 &= (q^{-3\frac{1}{2}}a_EE + q^{3\frac{1}{2}}a_FF)K^{-3}, \\
K_1(0)K^{-2}K_0 &= (q^{-3}a_Eb_EE^2 + q^2a_Fb_EEF + q^{-2}a_Eb_FFE + q^3a_Fb_FF^2)K^{-2}, \\
K^{-2}K_1(0)K_0 &= (q^{-3}a_Eb_EE^2 + a_Fb_EEF + a_Eb_FFE + q^3a_Fb_FF^2)K^{-2}.
\end{aligned}$$

Observe that all the terms with E^2K^{-2}, EK^{-3} and FK^{-3} in (A.6) cancel each other. Therefore, only the terms with EFK^{-2} and FEK^{-2} remain,

$$\begin{aligned}
&- b_\tau(q^2 + q^{-2})(a_Fb_E + a_Eb_F)\left(EF + FE\right)K^{-2} + b_\tau(a_Fb_E + a_Eb_F)\left((1 + q^2)EF + (1 + q^{-2}FE\right) \\
&= -b_\tau(a_Fb_E + a_Eb_F)\left((q^{-2} - 1)EF + (q^2 - 1)FE\right)K^{-2}
\end{aligned}$$

Using again that

$$EF = FE + \frac{K^2 - K^{-2}}{q - q^{-1}},$$

we obtain

$$\begin{aligned}\left((q^{-2}-1)EF + (q^2-1)FE\right)K^{-2} &= \left((q-q^{-1})^2 FE + q^{-1}K^{-2} - q^{-1}K^2\right)K^{-2} \\ &= (q-q^{-1})^2\Omega K^{-2} - (q+q^{-1})I.\end{aligned}$$

Thus (A.6) is equal to

$$-b_\tau(a_F b_E + a_E b_F)(q-q^{-1})^2\Omega K^{-2} + (a_F b_E + a_E b_F)b_\tau(q+q^{-1}).$$

Again, this first term in front of K^{-2} is exactly what we need to get a similar outcome as (A.2), only again with an added constant,

$$\begin{aligned}(q^2+q^{-2})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 = (a_F b_E + a_E b_F)(q-q^{-1})^2\Omega K_1 - (q+q^{-1})^2 K_0 \\ +(a_F b_E + a_E b_F)b_\tau(q+q^{-1})I.\end{aligned} \tag{A.7}$$

Now finally, taking both σ and τ nonzero gives us the extra terms where both a_σ and b_τ appear. That is,

$$\begin{aligned}(q^2+q^{-2})\left(a_\sigma b_\tau^2 K^{-2} - 2a_\sigma b_\tau K_1(0)\right) - (2a_\sigma b_\tau^2 K^{-2} - 2a_\sigma b_\tau K_1(0) \\ - a_\sigma b_\tau K^{-2}K_1(0)K^2 - a_\sigma b_\tau K^2 K_1(0)K^{-2}).\end{aligned} \tag{A.8}$$

Since

$$K^{-2}K_1(0)K^2 = q^{-2}b_E q^{\frac{1}{2}}EK + q^2 b_F q^{-\frac{1}{2}}FK,$$

and

$$K^2K_1(0)K^{-2} = q^2 b_E q^{\frac{1}{2}}EK + q^{-2}b_F q^{-\frac{1}{2}}FK,$$

we have for the last part of (A.8) that,

$$-a_\sigma b_\tau K^{-2}K_1(0)K^2 - a_\sigma b_\tau K^2K_1(0)K^{-2} = -a_\sigma b_\tau(q^2+q^{-2})K_1(0).$$

Therefore, the expression (A.8) is equal to

$$-a_\sigma b_\tau(q-q^{-1})^2(K_1(0) - b_\tau K^{-2}) = -a_\sigma b_\tau(q-q^{-1})^2K_1.$$

Adding everything together we obtain

$$\begin{aligned}(q^2+q^{-2})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 = (q-q^{-1})^2\left((a_E b_F + a_F b_E)\Omega - a_\sigma b_\tau\right)K_1 \\ -(q+q^{-1})^2 b_E b_F K_0 + (q+q^{-1})\left(a_\sigma(q-q^{-1})^2 b_E b_F\Omega + (a_F b_E + a_E b_F)b_\tau\right),\end{aligned} \tag{A.9}$$

as desired. Now lastly, let us now do the transformation $q \leftrightarrow q^{-1}$, $K \leftrightarrow K^{-1}$ and $\sigma \leftrightarrow -\tau$ in the equation above. Note that this does not change Ω and most important, the structure relations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ remain the same. It leads to interchanging $K_0 \leftrightarrow K_1$ and $a_\sigma \leftrightarrow -b_\tau$, while the computations done remain the same