European Conference on Computational Fluid Dynamics
ECCOMAS CFD 2006
P. Wesseling, E. Oñate and J. Périaux (Eds)

A PROGRESSIVE EULER-LAGRANGE FORMULATION FOR AEROELASTICITY MODEL OF QUASI-AXISYMMETRICAL AIRSHIP

Philippe Destuynder* **and Françoise Santi***

*Chaire de Calcul Scientifique, Conservatoire National des Arts et Métiers
292, rue saint Martin, 75141 Paris, France.
e-mail: destuynd@cnam.fr, santi@cnam.fr
web page: http://www.cnam/maths

Key words: Fluid Dynamics, Aeroelasticity, Stability, Airship Technology

Abstract. *An aeroelastic modelling of an axisymmetrical airship is discussed in this paper. The main difficulty is due to the fact that instabilities are not axisymmetrical. Furthermore an Euler-Lagrange kinematical coupling is necessary in order to represent the fluid-structure interaction. Our strategy is based on some arbitrary change of domain (for the air flow) which are governed by the movement of the structure. Some numerical results illustrate the method.*

1 INTRODUCTION

Let us consider an airship which is assumed to be axisymmetrical and which is set into an air flow which is not necessarily axisymmetrical. Nevertheless, it is assumed that the perturbation with respect to the main axis of symmetry of the airship is small. Furthermore the movement of the airship eigenmodes of the structure (including rigid body motion) are also assumed to be small enough in order to permit a Taylor expansion of the non-symmetrical solution using axisymmetrical models.

• The first case that we consider corresponds to small angle of attack without flexibility of the airship. Therefore one can use a Fourier decomposition with respect to the cylindro-polar angle of the 3-D solution assuming that there is no instabilities (cf. figure 1). But this is possible if one neglect the coupling between the harmonics which appears in the non-linear convection terms. From a linearization around the axisymmetrical steady state, this decoupling is straightforward. Then, one can derive a first order approximation of the aerodynamical coefficients. Three of them are meaningful: the drag coefficient C_D, he lift coefficient C_L and the pitching coefficient C_m.

• The second case is more complicated and concerns flexible movements of the airship.

The basic point of our method consists in updating the geometry for the Navier-Stokes solver using an arbitrary Euler-Lagrange updating that we call *progessive updating.* The idea is to use a continuous prolongation of the movement of the airship inside the fluid. But it is restricted to a neighbourhood of the structure. Then a Fourier decomposition with respect to the cylindro-polar angle is used again (see figure 1).

2 HOW TO TAKE INTO ACCOUNT THE PITCHING ANGLE

2.1 Steady state approximation

First of all let us define some notations used in the following. The open set occupied by the fluid when the flow is axisymmetrical is denoted by $\Omega(0)$. It corresponds to a rigid airship. The boundary of the airship at rest is $S(0)$. The remaining part of the boundary of $\Omega(0)$ is split into two component. One denoted Γ_0 is defined by:

$$\Gamma_0 = \{x \in R^3,\ \mathbf{e}(\alpha).\nu(x) \leq 0\} \tag{1}$$

where $\nu(x)$ is the unit outwards normal to the boundary of $\Omega(0)$ at point x. Therefore, Γ_0 is the intake of the flow. The flow velocity on Γ_0 is given by:

$$\mathbf{u} = U\mathbf{e}(\alpha),\ \text{on } \Gamma_0 \text{ and } \mathbf{u} = 0 \text{ on } S(0), \tag{2}$$

where $\mathbf{e}(\alpha)$ is the direction of the flow at infinity and U its magnitude ($U > 0$).
The complementary of Γ_0 and $S(0)$ is the output of the airflow and it is denoted by Γ_1. The boundary condition corresponds to a free edge (Neuman condition), but better results are obtained when one is using the characteristics method. This last condition enables to stabilize the flow near the output. The flow velocities (say $\mathbf{u}$) and the pressure field p are solution of the following system:

$$\begin{cases} (\mathbf{u}, p) \in V \times L^2(\Omega(0)) \text{ such that:} \\ \forall \mathbf{v} \in V,\ \int_{\Omega(0)} \varrho[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \otimes \nabla \mathbf{u}].\mathbf{v} - p\ div(\mathbf{v}) + 2\mu\gamma(\mathbf{u}) : \gamma(\mathbf{v}) = 0, \\ \forall q \in L^2(\Omega(0)),\ -\int_{\Omega(0)} q div(\mathbf{u}) = 0. \end{cases} \tag{3}$$

The following notations have been used:

$$\mathbf{u} = u_i \mathbf{e}_i,\ [\mathbf{u} \otimes \nabla \mathbf{u}]_j = \sum_{k=1,3} u_k \partial_k u_j,\ [\gamma(\mathbf{u})]_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \tag{4}$$

Furthermore μ is the viscisity and ϱ is the mass density. Finally V is the functional space:

$$V = \{\mathbf{v} = v_i \mathbf{e}_i,\ v_i \in H^1(\Omega(0));\ v_i = 0 \text{ on } \Gamma_0 \cup S(0)\} \tag{5}$$

and it is equiped with the norm induced by $H^1(\Omega(0))$. Existence and uniqueness are an open problem, the only known results are in 2D [7]. Concerning the numerical scheme,

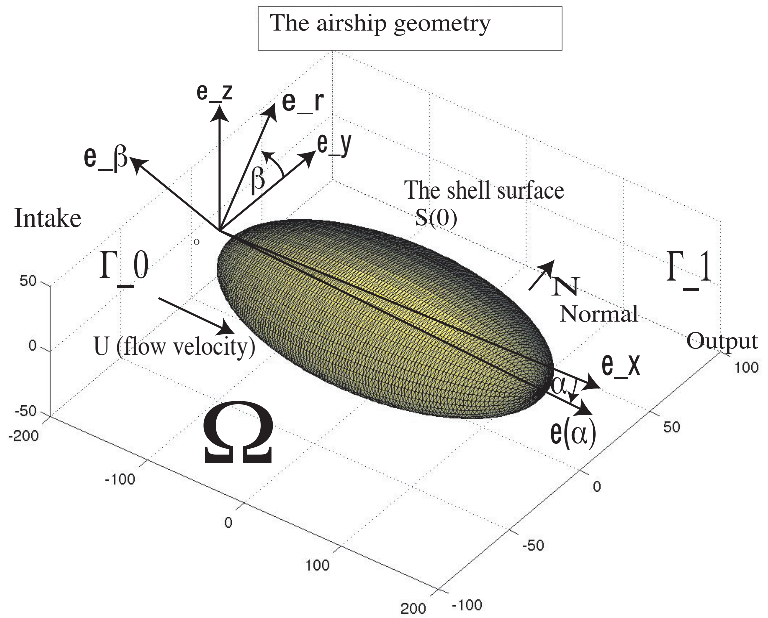

Figure 1: The airship and the notations

the mixed formulation is well adapted [5]. This is the method that has been used. When the flow direction vector **e** is parallel to the axis of symmetry of the airship, there is at least one axisymmetrical solution. It can be computed using cylindro-polar coordinates (r, β, x) (see figure **??**). The velocity field **u** is expressed in the frame: $(\mathbf{e}_r, \mathbf{e}_\beta, \mathbf{e}_x)$. Let us set:

$$\mathbf{u} = u_r\mathbf{e}_r + u_\beta\mathbf{e}_\beta + u_x\mathbf{e}_x.$$

When the flow is axisymmetrical all the derivatives with respect to β are zero and one has: $u_\beta = 0$. In this case the solution is denoted $(\mathbf{u}^0, p^0)$ and is solution of the following system:

$$\left\{\begin{array}{l} \forall \mathbf{v} \in V_0,\ \varrho \displaystyle\int_{\omega(0)} \frac{\partial \mathbf{u}^0}{\partial t}.\mathbf{v} + [\mathbf{u}^0 \otimes \nabla \mathbf{u}^0].\mathbf{v} \\ \qquad\qquad\qquad - \displaystyle\int_{\omega(0)} p^0\, div(\mathbf{v}) + 2\mu \int_{\omega(0)} \gamma(\mathbf{u}^0) : \gamma(\mathbf{v}) = 0, \\ \forall q \in L^2(\omega(0)),\ - \displaystyle\int_{\omega(0)} q div(\mathbf{u}^0) = 0, \end{array}\right. \tag{6}$$

with the boundary conditions:

$$\mathbf{u}^0 = Ue_x \text{ sur } \gamma(0),\ \text{et } \mathbf{u}^0 = 0 \text{ sur } s(0). \tag{7}$$

The following notations have been used (figure **??**):

$$\Omega(0) = \{(r, \beta, x) | (r, x) \in \omega(0),\ \beta \in [0, 2\pi[\},\ \Gamma_0 = \gamma_0 \times [0, 2\pi[,\ S(0) = s(0) \times [0, 2\pi[.$$

and:

$$V_0 = \{\mathbf{v} = (v_r, v_x) \in [H^1(\omega(0))]^2,\ \mathbf{v} = 0 \text{ sur } \gamma_0 \cup s(0)\}. \tag{8}$$

Let us come back to the general case in which the pitching angle α is not zero. Nevertheless $\mathbf{e}(\alpha)$ is assumed to be close to $\mathbf{e}_x$. In fact the only non axisymmetrical condition is the Dirichlet condition satisfied by the velocity on the boundary Γ_0. Let us set:

$$\left\{ \begin{array}{l} \mathbf{e}(\alpha) = cos(\alpha)\mathbf{e}_x + sin(\alpha)sin(\beta)\mathbf{e}_r + sin(\alpha)cos(\beta)\mathbf{e}_\beta, \\ \mathbf{u} = \mathbf{u}^0 + \mathbf{u}^\alpha, \quad p = p^0 + p^\alpha, \end{array} \right. \tag{9}$$

Let us now formulate the linearized model with respect to α, $(\mathbf{u}^\alpha, p^\alpha)$ is the solution of which. Using a Fourier decomposition with respect to the angle β, one observes that only the harmonic 1 is different from zero. Hence, using a complex representation for sake of brevety, one obtains:

$$(\mathbf{u}^\alpha, p^\alpha) \simeq (\mathbf{u}^1, p^1) = \sum_{\pm} (u_r^{\pm 1}\mathbf{e}_r + u_\beta^{\pm 1}\mathbf{e}_\beta + u_x^{\pm 1}\mathbf{e}_x, p^{\pm 1}) e^{\pm i\beta}, \tag{10}$$

where $(\mathbf{u}^1, p^1) \in V \times L^2(\Omega(0))$ is solution of an axisymmetrical model (hence 2D!) excepted concerning the boundary condition:

$$\left\{ \begin{array}{l} \forall \mathbf{v} \in V,\ \varrho \displaystyle\int_{\Omega(0)} \frac{\partial \mathbf{u}^1}{\partial t}.\mathbf{v} + [\mathbf{u}^0 \otimes \nabla \mathbf{u}^1 + \mathbf{u}^1 \otimes \nabla \mathbf{u}^0].\mathbf{v} \\ \qquad\qquad\qquad - \displaystyle\int_{\Omega(0)} p^1\, div(\mathbf{v}) + 2\mu \int_{\Omega(0)} \gamma(\mathbf{u}^1) : \gamma(\mathbf{v}) = 0, \\ \forall q \in L^2(\Omega(0)),\ - \displaystyle\int_{\Omega(0)} q div(\mathbf{u}^1) = 0. \end{array} \right. \tag{11}$$

Furthermore these boundary conditions satisfied by $\mathbf{u}^1$ on Γ_0 are:

$$\mathbf{u}^1 = U sin(\alpha)[sin(\beta)\mathbf{e}_r + cos(\beta)\mathbf{e}_\beta]. \tag{12}$$

Let us point out that the component (see figure **??**) $u_\beta^1 \mathbf{e}_\beta$ is not zero along the axis of symmetry, but one has: $(\mathbf{u}^\theta, \mathbf{e}_x) = 0$. In order to ensure that $\gamma(\mathbf{u}) \in L^2(\omega(0))$ it is necessary that: $u_r + iu_\beta = 0$ on this axis corresponding to $r = 0$. The solution to the previous system is proportional to $sin(\alpha)$. It can be solved with a two dimensional problem but with a coupling with u_β. It is worth to recall the explicit expression of the strain tensor expressed in the basis: $(\mathbf{e}_r, \mathbf{e}_\beta, \mathbf{e}_x)$.

$$\nabla(.) = \begin{pmatrix} \dfrac{\partial}{\partial r}(.) \\ \\ \dfrac{1}{r}\dfrac{\partial}{\partial \beta}(.) \\ \\ \dfrac{\partial}{\partial x} \end{pmatrix},$$

Let us set: $\mathbf{u} = u_r\mathbf{e}_r + u_\beta\mathbf{e}_\beta + u_x\mathbf{e}_x$,

then: $div(\mathbf{u}) = \dfrac{1}{r}[\dfrac{\partial(ru_r)}{\partial r} + \dfrac{\partial u_\beta}{\partial \beta}] + \dfrac{\partial u_x}{\partial x}$

$$\nabla\mathbf{u} = \begin{pmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{\partial u_\beta}{\partial r} & \dfrac{\partial u_x}{\partial r} \\ \\ \dfrac{1}{r}\dfrac{\partial u_r}{\partial \beta} - \dfrac{u_\beta}{r} & \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\beta}{\partial \beta} & \dfrac{1}{r}\dfrac{\partial u_x}{\partial \beta} \\ \\ \dfrac{\partial u_r}{\partial x} & \dfrac{\partial u_\beta}{\partial x} & \dfrac{\partial u_x}{\partial x} \end{pmatrix} \tag{13}$$

$$\gamma(\mathbf{u}) = \begin{pmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{2}(\dfrac{\partial u_\beta}{\partial r} + \dfrac{1}{r}\dfrac{\partial u_r}{\partial \beta} - \dfrac{u_\beta}{r}) & \dfrac{1}{2}(\dfrac{\partial u_x}{\partial r} + \dfrac{\partial u_r}{\partial x}) \\ \\ \dfrac{1}{2}(\dfrac{\partial u_\beta}{\partial r} + \dfrac{1}{r}\dfrac{\partial u_r}{\partial \beta} - \dfrac{u_\beta}{r}) & \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\beta}{\partial \beta} & \dfrac{1}{2}(\dfrac{1}{r}\dfrac{\partial u_x}{\partial \beta} + \dfrac{\partial u_\beta}{\partial x}) \\ \\ \dfrac{1}{2}(\dfrac{\partial u_x}{\partial r} + \dfrac{\partial u_r}{\partial x}) & \dfrac{1}{2}(\dfrac{1}{r}\dfrac{\partial u_x}{\partial \beta} + \dfrac{\partial u_\beta}{\partial x}) & \dfrac{\partial u_x}{\partial x} \end{pmatrix}$$

2.2 Quasi steady state

Let us consider two time scalings. The first one is connected to the frequencies f_s of the structure, and the second one to the flow velocity U and the wave length L of an eigenmode. Let us set:

$$f_r = \frac{Lf_s}{U}. \tag{14}$$

The steady state approximation can usually be applied if $f_r << 1$. Furthermore it is assumed that the magnitude of the structural displacements is small enough. Nevertheless it is necessary to take into account the changes in the flow velocity due to these movements. This leads to the concept of the aerodynamical damping [4], [3]. Three terms are meaningful in the dynamical contribution. One is the classical relative acceleration. The second one is the acceleration of the frame connected to the structure, and the third one is the gyroscopic effect. Let us introduce the relative velocity $\mathbf{v}_a$ on Γ_0:

$$\mathbf{v}_a = U\mathbf{e}(\alpha) - \dot{\mathbf{Z}}^0 - \dot{\mathbf{R}} \wedge \mathbf{ox}, \tag{15}$$

The second term is the rigid body motion of the structure: $\dot{\mathbf{Z}}^0$ is the velocity at point o and $\dot{\mathbf{R}}$ is the rotation vector. The flow model is then similar to (3), excepted the new terms due to the acceleration and those on Γ_0 which become:

$$\mathbf{u} = \mathbf{v}_a = U(cos(\alpha)\mathbf{e}_x + sin(\alpha)\mathbf{e}_z) - \dot{\mathbf{Z}}^0 - \dot{\mathbf{R}} \wedge \mathbf{ox}. \tag{16}$$

Thus: $\dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \otimes \nabla \mathbf{u}$ is replaced by: $\dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \otimes \nabla \mathbf{u} + \gamma^e + 2\dot{\mathbf{R}} \wedge \mathbf{u}$, where γ^e is given by:

$$\gamma^e = \ddot{Z}^0 + \ddot{\mathbf{R}} \wedge \mathbf{ox} + \dot{\mathbf{R}} \wedge (\dot{\mathbf{R}} \wedge \mathbf{ox}).$$

Let us consider a movement of the structure which implies a translation $d_z \mathbf{e}_z$ and a rotation: $\alpha \mathbf{e}_y$.The boundary condition becomes:

$$\begin{aligned} \mathbf{u} = (Ucos(\alpha) - \dot{\alpha} r sin(\beta))\mathbf{e}_x + (Usin(\alpha) + \dot{\alpha} x - \dot{d}_z) sin(\beta) \mathbf{e}_r \\ +(Usin(\alpha) + \dot{\alpha} x - \dot{d}_z)\mathbf{e}_\beta \end{aligned} \tag{17}$$

Furthermore, the acceleration of the frame is:

$$\gamma^e = \ddot{d}_z \mathbf{e}_z + \dot{d}_z \dot{\alpha} \mathbf{e}_x + \ddot{\alpha} \mathbf{e}_y \wedge \mathbf{ox} + \dot{\alpha}^2 \mathbf{e}_y \wedge (\mathbf{e}_y \wedge \mathbf{ox}).$$

The linearisation around the axisymmetrical solution gives:

$$\frac{\partial \mathbf{u^1}}{\partial t} + \mathbf{u^0} \otimes \nabla \mathbf{u^1} + \mathbf{u^1} \otimes \nabla \mathbf{u^0} + \gamma^{eL} + 2\dot{\alpha} \mathbf{e}_y \wedge \mathbf{u^0} \tag{18}$$

where the linearized acceleration is denoted by γ^{eL}, and is such that:

$$\gamma^{eL} = \ddot{d}_z \mathbf{e}_z + \ddot{\alpha} \mathbf{e}_y \wedge ox = \ddot{d}_z [sin(\beta)\mathbf{e}_r + cos(\beta)\mathbf{e}_\beta] + \ddot{\alpha}[cos(\beta)\mathbf{e}_r - sin(\beta)\mathbf{e}_\beta] \wedge ox. \tag{19}$$

Concerning the gyroscopic term, one gets:

$$\gamma^c = 2\dot{\alpha}[cos(\beta)\mathbf{e}_r - sin(\beta)\mathbf{e}_\beta] \wedge \mathbf{u}^0. \tag{20}$$

All the addditional terms (six) are confined on the first Fourier harmonic with respect to the angle β. The solution method requires to solve seven independent linear $2D$ models.
• The first one corresponds to the axisymmetrical flow with a flow velocity at the infinity equal to $Ucos(\alpha)\mathbf{e}_x \simeq U\mathbf{e}$. The solution is denoted $(\mathbf{u}^0, p^0)$.
• The six following ones corresponds to the first Fourier harmonic in β.
i) One is dependent on α and the solution is proportional to $U\ sin(\alpha) \simeq U\alpha$.
ii) Another one is proportional to $\dot{\alpha}$.
iii) In a similar way there is one proportional to $\dot{d}_z$.
iv) Then, one contribution is proportional to $\ddot{d}_z$,
v) and a similar one is proportional to $\ddot{\alpha}$.
vi) Finally the gyroscopic term is proportional to $\dot{\alpha}$.
Finally, let us set, using a complex representation for sake of brevity:

$$\mathbf{u} = \mathbf{u}^0 + \sum_{\pm} e^{\pm i\beta}[U\alpha \mathbf{u}^{\pm i} + \dot{\alpha}\mathbf{u}^{\pm t} + \dot{d}_z \mathbf{u}^{\pm g} + \ddot{d}_z u^{\pm ma} + \ddot{\alpha}\mathbf{u}^{\pm ia} + \dot{\alpha}\mathbf{u}^{\pm c}]. \tag{21}$$

3 COMPUTATION OF THE AERODYNAMICAL FORCES

3.1 Slow movements

Let us consider the simpliest case where the structure is stationary versus the flow direction. The unit normal to $S(0)$ is denoted by $\mathbf{N}$ and its projection onto the plane $(\mathbf{e}_r, \mathbf{e}_x)$ is ν. The mechanical stress is:

$$\mathbf{T} = p\mathbf{N} - 2\mu\gamma(\mathbf{u}).\mathbf{N} = \mathbf{T}^0 + \mathbf{T}^1,$$

with:

$$\mathbf{T}^0 = p^0\mathbf{N} - 2\mu\gamma(\mathbf{u}^0).\mathbf{N},\ \text{et}\ \mathbf{T}^1 = p^1\mathbf{N} - 2\mu\gamma(\mathbf{u}^1).\mathbf{N}.$$

Because $\mathbf{T}^1$ is proportional to $sin(\alpha)$ and thus to α after linearisation, one obtains a linear expression with respect to α. Let us now consider a movement of the rigid structure represented by:

$$\delta\dot{\mathbf{Z}}(x) = \delta\dot{\mathbf{Z}}(0) + \delta\dot{\mathbf{R}} \wedge \mathbf{ox}.$$

The virtual work of the aerodynamical forces is:

$$P(\delta\dot{\mathbf{Z}}) = \int_{S(0)} \mathbf{T}^0.\delta\dot{\mathbf{Z}}^0 + \int_{S(0)} (\mathbf{T}^1, \delta\dot{\mathbf{R}}, \mathbf{ox}).$$

But $\mathbf{T}^1$ is proportional to $sin(\alpha)$, and α is constant along the structure, therefore:

$$P(\delta\dot{\mathbf{Z}}) = P^0(\delta\dot{\mathbf{Z}}) + sin(\alpha)P^1(\delta\dot{\mathbf{Z}}).$$

This enables one to formulate a stability model for the coupled system as follows:

$$J_0\ddot{\alpha} = \xi sin(\alpha) \simeq \xi\alpha,\ \ \text{pour}\ \alpha\ \ \text{petit}\ , \tag{22}$$

therefore the stability depends on the sign of ξ. In fact the question is to set the center of rotation denoted by $\mathbf{o}$, with respect to the aerodynamical centre.

3.2 Steady aeroelasticity

From section 2.2, one can write the forces applied to the structure as follows:

$$\mathbf{T} = Ucos(\alpha)T^a + Usin(\alpha)\mathbf{T}^i + \dot{\alpha}\mathbf{T}^t + \dot{d}_z\mathbf{T}^g + \ddot{\alpha}\mathbf{T}^{ia} + \ddot{d}_z\mathbf{T}^{ma} + \dot{\alpha}\mathbf{T}^c, \tag{23}$$

where:

$\mathbf{T}^a$ is the stress vector due to the axisymmetrical flow $(cos(\alpha) \simeq 1)$
$\mathbf{T}^i$ is the stress vector due to the pitching angle $\alpha \simeq sin(\alpha)$
$\mathbf{T}^t$ is the stress vector due to the pitching velocity
$\mathbf{T}^g$ is the stress vector due to the galloping (24)
$\mathbf{T}^{ia}$ is the added inertia for the pitching
$\mathbf{T}^{ma}$ is the added mass due to the galloping
$\mathbf{T}^c$ is the stress vector due to the gyroscopic effect

After a linearization of the full system with respect to the pitching angle (α) and the galloping (d_z), one obtains:

$$M^a \begin{pmatrix} \ddot{\alpha} \\ \ddot{d}_z \end{pmatrix} + C^a \begin{pmatrix} \dot{\alpha} \\ \dot{d}_z \end{pmatrix} + K^a \begin{pmatrix} \alpha \\ d_z \end{pmatrix} = 0 \tag{25}$$

with:
- M^a is the full mass matrix,
- C^a is the damping (symmetrical part) and the gyroscopic (skew part) matrix,
- $K^a = \begin{pmatrix} k_t & 0 \\ k_g & 0 \end{pmatrix}$ is the aerodynamical stiffness matrix.

Because K^a not symmetrical, one can observe (at a critical velocity of the steady flow), a flutter instability (mode crossing). If the damping becomes negative, one says that there is a wake-flutter instability. Hence the stability analysis consists in studying the real part of λ solution to:

$$det(-\lambda^2 M^a + i\lambda C^a + K^a) = 0. \tag{26}$$

4 PROGRESSIVE EULER-LAGRANGE METHOD

The coupling equation between the fluid and the structure should be written in the deformed configuration. Because the steady state is not neglectible, additional terms due to the rotation of the normal appear in the model. The best way to write correctly this compatibility condition in our opinion, is to use a progressive Euler-Lagrange frame. First of all let us recall the formulation of the shell model for the structure.

4.1 The shell model

The classical Koiter model has been used in order to compute the eigenmodes $\mathbf{w}_n$. The corresponding frequencies are denoted by f_n.
Let us define by $\mathbf{z}$ the displacement field of the structure. If $m(.,.)$ is the inertia bilinear form, $a(.,.)$ the stiffness one and $\mathcal{W}$ the admissible displacement space, the eigenvalue problem consists in finding ($\lambda_n = (2\pi f_n)^2, \mathbf{w}_n$) such that:

$$\begin{cases} \mathbf{w}_n \in \mathcal{W},\ \lambda_n \in R^+, \text{ such that:} \\ \forall v \in \mathcal{W},\ \lambda_n m(\mathbf{w}_n, v) = a(\mathbf{w}_n, v) \end{cases} \tag{27}$$

Let us assume that the structural movement is well represented by a space of N eigenmodes denoted by $\mathcal{W}_N$:

$$\mathbf{z} = \sum_{n=1,N} \kappa_n(t)\mathbf{w}_n. \tag{28}$$

Let us denote by $\mathbf{N}$ the unit normal to the shell oriented towards the inside of the fluid. From shell theory the deformed normal becomes:

$$\mathbf{N}' = \mathbf{N} + \zeta(t), \tag{29}$$

where $\zeta(t)$ is the inplane rotation which depends on $\mathbf{z}$. Let us extend $\mathbf{z}$ inside the fluid by:

$$\left| \begin{array}{l} i)\ \theta = (\theta_i),\ i = 1,2,\ \theta_i \in W^{1,\infty}(\Omega(0)), \\ ii) \text{ the support of } \theta \text{ being included in a neighbourhood of the shell } S(0), \\ iii)\ \theta = \mathbf{z} \text{ on } S(0). \end{array} \right. \tag{30}$$

Let us now define the mapping F^θ from $\Omega(0)$ onto $\Omega(\mathbf{z})$ (deformed configuration):

$$x \in \Omega(0) \to x^\theta = x + \theta(x) \in \Omega(\mathbf{z}). \tag{31}$$

• **Change of functions**: Let φ be a function defined on $\Omega(\mathbf{z})$. We set: $\varphi^\theta(x) = \varphi o F^\theta(x)$.

• **Changes in the integrals**: $\int_{\Omega(\mathbf{z})} \varphi = \int_{\Omega(0)} \varphi^\theta det(I + D\theta)$,

where $D\theta$ is the Jacobian matrix associated to θ. Its transpose in the polar coordinate system $(\mathbf{e}_r, \mathbf{e}_\beta, \mathbf{e}_x)$ is $\nabla\theta$.

• **Changes in the derivatives**: $(\frac{\partial \varphi}{\partial x^\theta})^\theta = \frac{\partial \varphi}{\partial x} o(I + D\theta)^{-1}$

• **Divergence for a vector p**: $(div(\mathbf{p}))^\theta = \frac{1}{det(I + D\theta)} div((I + D\theta)^{-1}\mathbf{p}^\theta det(I + D\theta))$

• **Change in the convection term**: $(\mathbf{u} \otimes \nabla \mathbf{u})^\theta = \mathbf{u}^\theta \otimes (I +^t D\theta)^{-1} \nabla \mathbf{u}^\theta$

• **Changes in the strain rates**: $(\gamma(\mathbf{u}))^\theta = \gamma^\theta(\mathbf{u}^\theta) = \frac{1}{2}((I +^t D\theta)^{-1}\nabla\mathbf{u}^\theta +^t \nabla\mathbf{u}^\theta(I + D\theta^{-1}))$.

These formulae enables one to formulate an equivalent flow problem but set on $\Omega(0)$.

4.2 Progressive Euler-Lagrange formulation

Using the mapping F^θ and setting: $(\mathbf{u}^\theta, p^\theta) = (\mathbf{u}, p) o F^\theta$, we derive the following model.

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{u}^\theta, p^\theta) \in V \times L^2(\Omega(0)) \text{ such that:} \\ \\ \forall \mathbf{v} \in V,\ \varrho \int_{\Omega(0)} [\frac{\partial \mathbf{u}^\theta}{\partial t}.\mathbf{v} + \mathbf{u}^\theta \otimes (I +^t D\theta)^{-1} \nabla \mathbf{u}^\theta.\mathbf{v}] det(I + D\theta) \\ \\ - \int_{\Omega(0)} p^\theta\ div((I + D\theta)^{-1}\mathbf{u}^\theta det(I + D\theta)) \\ \\ +\mu \int_{\Omega(0)} ((I +^t D\theta)^{-1}\nabla\mathbf{u}^\theta +^t \nabla\mathbf{u}^\theta(I + D\theta^{-1})) : (I +^t D\theta)^{-1}\nabla\mathbf{v} det(I + D\theta) = 0, \\ \\ \forall q \in L^2(\Omega(0)),\ - \int_{\Omega(0)} q\ div((I + D\theta)^{-1}\mathbf{u}^\theta det(I + D\theta)) = 0. \end{array} \right. \tag{32}$$

If $\theta = 0$ the obtained model is exactly the axisymmetrical one. Let us introduce a linearization with respect to θ, which is a linear function of $\mathbf{z}$. Let us set $(\mathbf{u}^\theta, p^\theta) = (\mathbf{u}^0, p^0) + (\mathbf{u}^1, p^1) + \ldots$

anb by introducing this approximation into (32) one obtains that $(\mathbf{u}^1, p^1)$ is solution of:

$$\left\{\begin{array}{l}
\text{find } (\mathbf{u^1}, p^1) \in V \times L^2(\Omega(0)) \text{ such that:}\\
\forall \mathbf{v} \in V,\ \varrho \int_{\Omega(0)} \frac{\partial \mathbf{u}^1}{\partial t}.\mathbf{v} + [\mathbf{u}^0 \otimes \nabla \mathbf{u}^1 + \mathbf{u}^1 \otimes \nabla \mathbf{u}^0].\mathbf{v} - \int_{\Omega(0)} p^1\ div(\mathbf{u}^1)\\
+\mu \int_{\Omega(0)} \gamma(\mathbf{u}^1) : \gamma(\mathbf{v})\\
= -\varrho \int_{\Omega(0)} [(\frac{\partial \mathbf{u}^0}{\partial t} + \mathbf{u}^0 \otimes \nabla \mathbf{u}^0)\ div(\theta) - \mathbf{u}^0.{}^t D\theta.\nabla \mathbf{u}^0].\mathbf{v}\\
+ \int_{\Omega(0)} p^0 div(\mathbf{u}^0) div(\theta) - p^0 div(D\theta.\mathbf{v})\\
-\mu \int_{\Omega(0)} 2\gamma(\mathbf{u}^0) : \gamma(\mathbf{v}) div(\theta) - ({}^t D\theta.\nabla \mathbf{u}^0 +^t \nabla \mathbf{u}^0.D\theta).\nabla \mathbf{v}\\
+2\mu \int_{\Omega(0)} \gamma(\mathbf{u}^0) :^t D\theta.\nabla \mathbf{v},\\
\forall q \in L^2(\Omega(0)),\ -\int_{\Omega(0)} q\ div(\mathbf{u}^1) = \int_{\Omega(0)} q div(\mathbf{u}^0) div(\theta) - p^0 div(D\theta.\mathbf{v}).
\end{array}\right. \tag{33}$$

4.3 Fourier decomposition

In order to simplify the three dimensional flow model we make use of a Fourier decomposition in β. The only harmonics which are different from zero are those which are contained in the structural displacement $\mathbf{z}$.

4.4 Kinematical continuity between the fluid and the structure

In the deformed configuration one has:

$$\mathbf{u}(F^\theta(\mathbf{x}, t), t) = \frac{\partial \mathbf{z}}{\partial t}(\mathbf{x}, t),\ \forall \mathbf{x} \in S(0) \tag{34}$$

which is equivalent in $\Omega(0)$ to the following relation:

$$\mathbf{u}^\theta(\mathbf{x}, t) = \frac{\partial \mathbf{z}}{\partial t}(\mathbf{x}, t). \tag{35}$$

Let us consider for instance the normal component to the shell. The unit normal to the surface $S(\mathbf{z})$ is denoted by $\mathbf{N'}$ and we already point out that: $\mathbf{N'} = \mathbf{N} + \zeta(\mathbf{z})$, where ζ is the inplane rotation. Let us set: $\mathbf{u} = \mathbf{u}^0 + \mathbf{u}^1$, and thus we derive the kinematical continuity condition:

$$(\mathbf{u}^1, \mathbf{N}) + (\zeta(\mathbf{z}), \mathbf{u}^0) = (\frac{\partial \mathbf{z}}{\partial t}, \mathbf{N}). \tag{36}$$

Let us point out that the time derivative of $\mathbf{u}^1$ appears in the model, and this implies the second order time derivative of $\mathbf{z}$ which acts an an added mass term.
The Fourier decomposition of $\zeta(\mathbf{z})$ only implies the harmonics contained in $\mathbf{z}$. The unit normal $\mathbf{N}$ can be written as follows :

$$\mathbf{N} = cos(\kappa)\mathbf{e}_r + sin(\kappa)\mathbf{e}_x,$$

where κ is the angle between $\mathbf{N}$ and $\mathbf{e}_r$. Hence, all the harmonics in β are decoupled.

5 FORCES DUE TO THE FLUID

5.1 Forces due to the eigenmodes

On the surface $S(\mathbf{z})$ the stress vector is:

$$\mathbf{T} = -p\mathbf{N}' + 2\mu\gamma(\mathbf{u}).\mathbf{N}'. \tag{37}$$

Using the mapping $\mathbf{T}^\theta$, this quantity becomes at order one on $S(0)$:

$$\mathbf{T}^\theta = -p^0\mathbf{N} + 2\mu\gamma(\mathbf{u}^0) - p^1\mathbf{N} - p^0\zeta(\mathbf{z}) + 2\mu\gamma(\mathbf{u}^1) - \mu({}^tD\theta.\nabla\mathbf{u}^0 +^t \nabla\mathbf{u}^0.D\theta). \tag{38}$$

The two first terms correspond to the axisymmetrical flow. The four next ones are due to the dynamical behaviour. Let us assume that the reduced frequency is small enough in order to justify the use of the steady flow. In fact they are proportional to: $\mathbf{z}$, $\dfrac{\partial \mathbf{z}}{\partial t}$, and $\dfrac{\partial^2 \mathbf{z}}{\partial t^2}$. Let us set here again:

$$\mathbf{z} = \sum_{n=1,N} \kappa_n(t)\mathbf{w}_n.$$

This enables one to write at order one:

$$\mathbf{T}^\theta = \mathbf{T}^0 + \sum_{n=1,N} [\kappa_n(t)\mathbf{T}^{0z} + \frac{\partial \kappa_n}{\partial t}\mathbf{T}^{1z} + \frac{\partial^2 \kappa_n}{\partial t^2}\mathbf{T}^{2z}]. \tag{39}$$

Hence, in order to compute the previous term, one has to solve $3N + 1$ axisymmetrical and independent problems. For instance concerning the harmonic n one has the following expression to compute:

$$\mathcal{F}_n = \int_{S(0)} [(\mathbf{T}^0, \mathbf{w}_n) + \kappa_n(t)(\mathbf{T}^{0z}, \mathbf{w}_n) + \frac{\partial \kappa_n}{\partial t}(\mathbf{T}^{1z}, \mathbf{w}_n) + \frac{\partial^2 \kappa_n}{\partial t^2}(\mathbf{T}^{2z}, \mathbf{w}_n)]. \tag{40}$$

Let us denote by $\mathcal{F}$ the vector in R^N the component of which are $\mathcal{F}_n$. Then:

$$\mathcal{F}_n = \mathbf{F}_n^0 + \kappa_n(t)\mathbf{F}_n^{0z} + \dot{\kappa}_n(t)\mathbf{F}_n^{1z} + \ddot{\kappa}_n(t)\mathbf{F}_n^{2z} \tag{41}$$

5.2 The aeroelastic model

Let us denote by $\mathbf{Z}$ the vector in R^N the component of which being ζ_n and which are the coefficients of the eigenvectors $\mathbf{w}_n$. Then:

$$M\frac{\partial^2 \mathbf{Z}}{\partial t^2} + K\mathbf{Z} = \mathcal{F}(\mathbf{Z}, \frac{\partial \mathbf{Z}}{\partial t}, \frac{\partial^2 \mathbf{Z}}{\partial t^2}), \tag{42}$$

One should add initial conditions. Furthermore, the right hand side $\mathcal{F}$ depends on $\mathbf{Z}$ and its time derivatives. The matrices M and K are diagonal in the eigenvector basis. Let us set:

$$\mathcal{F} = \mathcal{F}^0 - K^a\mathbf{Z} - C^a\frac{\partial \mathbf{Z}}{\partial t} - M^a\frac{\partial^2 \mathbf{Z}}{\partial t^2}. \tag{43}$$

The coupled system becomes, with already mentioned notations:

$$(M + M^a)\frac{\partial^2 \mathbf{Z}}{\partial t^2} + C^a\frac{\partial \mathbf{Z}}{\partial t} + (K + K^a)\mathbf{Z} = \mathcal{F}^0 \tag{44}$$

The aeroelastic study consists in computing the eigenvalues λ with respect to U:

$$det(-\lambda^2(M + M^a) + i\lambda C^a + (K + K^a)) = 0. \tag{45}$$

An instability can occur if the imaginary part of λ is negative.

5.3 Discussion about the influence of the various terms appearing in the aeroelastic model and example

The effect of the added mass matrix is to reduce the eigenfrequencies. The even part of the matrix C is an aerodynamical damping. It can contribute to a so-called wake flutter. The odd part of C is the Corriolis effect and in most cases, stabilizes the system. The matrix $K + K^a$ is the augmented stiffness and is no more symmetrical because of the aerodynamical forces. A classical flutter instability can appear if two eigevalues are crossing each other. Let us give a simple example. It corresponds to a pitching or a galloping movement of the airship. The eigen-

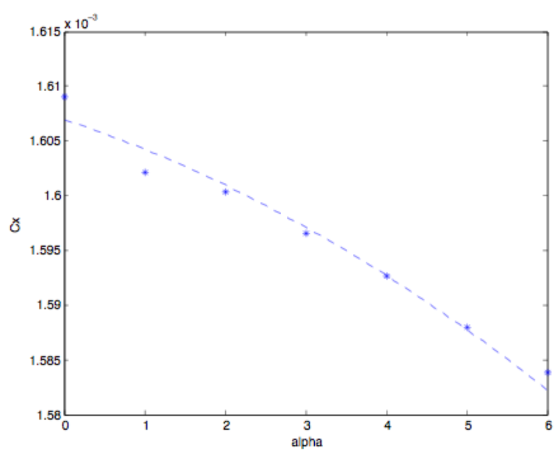

Figure 2: Drag coefficient with respect to the velocity

values have been computed for several values of the angle of attack α and taking into account the aerodynamical forces due to $\dot{d}_z$ and $\dot{\alpha}$. Furthermore, the lift and the pitching moment coefficients have been computed (see figure 3). One can see on figure 3 left, that the airship is stable -from the static point of view- versus a pitching movement ($c_m \leq 0$). The aerodynamical centre

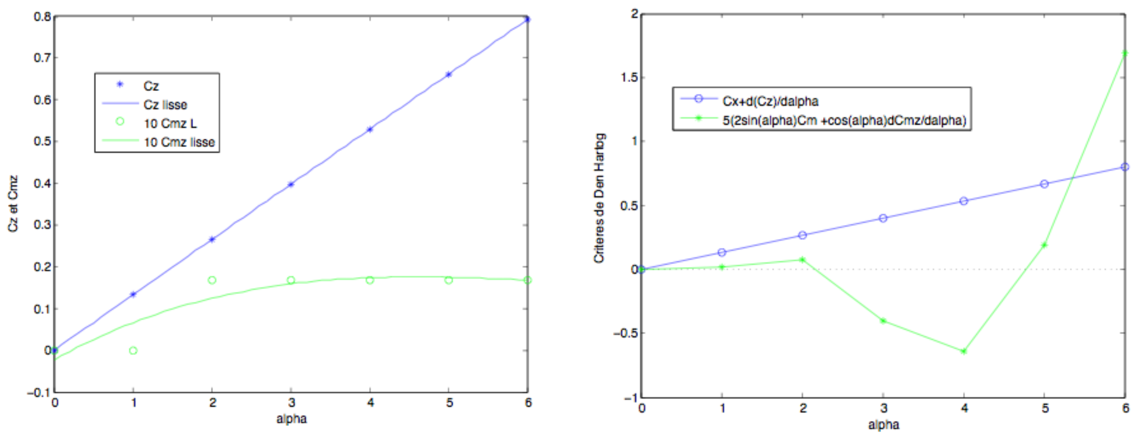

Figure 3: Left: lift and pitching moment at the center versus α; Right: imaginary part of λ for a pitching movement (green) and a galloping (blue)

is located in the front part of the airship. The drag coefficient has been plotted on figure 2. Even if it decreases, it is not meaningful because of the scale used. But concerning the aerodynamical damping it is quite zero for very small angle of attack. Then it is slightly negative for $\alpha \simeq 4$. But it becomes positive for larger value of α, (see on the right on figure 3).

6 CONCLUSION

A simplified method for studying the aeroelasic stability of an axisymmetrical body is suggested. The method enables to take account the small perturbation with respect to the axis of symmetry in an aeroelastic analysis.

Acknowlegment *This work has been carried out in collaboration with the University of Pau, France. The authors thank professor M. Amara for his support.*

REFERENCES

[1] S. Amara, Etude des vibrations propres d'un ellipsoïde plongé dans un écoulement. Mémoire d'ingénieur CNAM, Paris, (2006).

[2] Ph. Destuynder, *M*odélisation des coques minces élastiques, Masson, Paris, (1990).

[3] H. Dowell, C. Jr. Curtiss, R.H. Scanlan, F. Sisto, *A* modern course in aeroelasticity, Kluwer academic publishers, (1989).

[4] Y.C. Fung , *A*roelasticity Prentice Hall, New-York, (1968).

[5] V. Girault P.A. Raviart, *F*inite element Methods for Navier-Stokes Equations, Springer-Verlag, New York, (1986).

[6] G. A. Khoury, *A*irship Technology, Cambridge University Press, (1999).

[7] J.L. Lions, *Quelques méthodes de résolution des problèmes non linéaires*, Dunod, Paris, (1969).

[8] K. Suzuki N. Kikuchi T. Kosawada, Axisymmetric vibrations of thin shells of revolution, Bulletin of J.S.M.E., **227**, Vol. 27, May (1984).

[9] Sites consultés: 1) http://www.aat-fr.com/SiteFrancais/DGPA.htm
2) http://www.goodfellow.com/