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Hedging error as generalized timing risk

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This paper introduces a methodology to disentangle the *hedging error* associated with the hedging of exotic derivatives, whose payment time is unknown at inception. We derive the mathematical representation for a one-dimensional setting: we identify and characterize the hedging error and discuss the economic intuition of *hedging error as a generalized timing risk*. We then provide its mathematical integral representation to: (i) disentangle the hedging error into a specific set of positions in barrier options, (ii) re-iterate the procedure to the second order to reduce the hedging error cost. We provide an illustrative example via a dedicated numerical study. From a theoretical point of view, this paper states the foundations for future extensions in the directions of: (i) building a general multidimensional framework, (ii) re-iterating the procedure to higher orders, (iii) investigate the bridge with advanced analytics methodologies and techniques.

Keywords: Timing risk; Barrier options; Semi-static hedge; Hedging error

1. Introduction

The uncertainty characterizing financial markets renders both pricing and hedging activities crucial from a risk management perspective (Carr *et al.* 1998, Carr and Wu 2014, Derman *et al.* 1994, Engelmann *et al.* 2006, Milne and Madan 1994, Carr and Madan 2001). In the context of trading, the payoff of many OTC exotic derivatives occurs at a random time τ when the underlying asset price crosses a certain barrier. Depending on the type of derivative, the random time at which the payment will occur is not known, even if, in some cases, the amount to be paid is known (e.g. constant payment). In Carr and Picron (1999), the authors show how a static position in plain vanilla European options could be used to hedge against this *timing risk* component in the case of *barrier options*. The authors show how to decompose the timing risk component into an integral (w.r.t. maturity) of knock-in options by considering a *calendar-spread* approach. Calendar spread refers to hedging via a portfolio composed of derivatives with different maturities. As an alternative, *strike-spread* refers to a portfolio of path-independent options with different strikes (and same maturity). The paper by Bowie and Carr (1994) introduces a *strike-spread* approach under a one-dimensional Black–Scholes setting, which provides a perfect hedge, e.g. no hedging error. It represents the first contribution studying a static hedge of single barrier options and look-back options. Other contributions have extended their results by taking into account: (i) dividend paying underlying assets (Carr and Chou 1997a), (ii) more complex distributions for the underlying asset dynamics (Chou and Goergiev 1998), (iii) multiple barrier options (Carr and Chou 1997b). As a common feature, in some of these cases, the reflection principle of Brownian motion and put-call symmetry arguments play a central role (Carr and Lee 2009). In the same direction, the work by Carr and Nadtochiy (2011) provides an elegant mathematical solution to the problem in the one-dimensional setting. Among contributions on *calendar-spread* approaches, the work by Derman *et al.* (1995) introduces the concept of static hedge via options with distinct maturities, and more recently the article by Kim and Lim (2021) nicely reduces the problem to solving an integral equation.

The present paper introduces an alternative methodology to identify the *timing risk* component associated with the hedging of exotic derivatives as defined in Carr and Picron (1999), decompose it into a specific set of positions in barrier options and then build a semi-static hedge à la Bowie and Carr (1994) for each hedging option. The proposed economic intuition behind the concept of timing risk is key to: (i) enable disentangling the hedging error into specific components, (ii) build

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a general framework where the procedure can be re-iterated to higher orders.

In order to illustrate this intuition and its implementation, the present paper considers knock-in barrier options under a one dimensional setting. We identify and characterize the (first order) hedging error and discuss the economic intuition of *hedging error as a generalized timing risk*. We then provide its mathematical integral representation to: (i) disentangle the hedging error into a specific set of positions in barrier options, (ii) re-iterate the procedure to the second order to reduce the hedging error cost. We provide an illustrative example and a dedicated numerical analysis. From a theoretical point of view, the proposed methodology represents the basis to build extensions in the direction of a general multi-dimensional framework and procedure re-iteration to higher orders.

The paper is organized as follows: section 2 defines the mathematical framework underlying the concept of timing risk, its value and its decomposition. Section 3 introduces the proposed methodology to represent the hedging error as a generalized timing risk: it describes how to identify the first-order hedging error and how to re-iterate the hedging procedure to the second order. Section 4 provides an illustrative example and a simulation study. Section 5 gives some concluding remarks and discusses future research directions.

2. The value of timing risk and its decomposition

The payoff of many exotic derivatives occurs at the first passage time of the underlying asset price to a constant barrier. Depending on the type of derivative, even if in some cases the amount to be paid could be known (e.g. constant payment), the random time at which the payment will occur is not. The concept of *timing risk* introduced in Carr and Picron (1999) refers to the uncertainty associated with the payment time linked to the derivative contract conditions. In its essence, the concept of timing risk can be mathematically represented as a random variable, whose present value can be formulated as a discounted expectation. For a generic time $t \geq 0$, definition 2.1 provides a mathematical formulation for *the value of timing risk*.

Definition 2.1 (The Value of Timing Risk) *Let $r > 0$ be the constant risk-free rate of the economy, τ a stopping time, T the finite maturity of the exotic derivative and $\{\mathcal{F}_t\}$ the filtration. Let $\mathrm{Tr}_t(\tau, T)$ denote the value of timing risk, computed as the present value at time t of one unit paid at random (stopping) time τ:*

$$\mathrm{Tr}_t(\tau, T) := 1_{\{t\leq\tau\}}\mathbb{E}[\mathrm{e}^{-r(\tau-t)}1_{\{\tau\leq T\}}|\mathcal{F}_t], \tag{1}$$

with $1_{\{\cdot\}}$ denoting the indicator function.

A timing risk decomposition is possible by means of integration by parts formula. In Carr and Picron (1999), the authors discuss the Black–Scholes case. Building on this idea, we derive analytic results for a one-dimensional setting (section 3): we identify the hedging error and discuss the economic intuition of *hedging error as a generalized timing risk*. We then provide its mathematical formalization to: (i) disentangle the hedging error into a specific set of positions in barrier options, (ii) re-iterate the procedure to reduce the hedging error. Proposition 2.2 states the timing risk decomposition result for the case of finite maturity T and discusses the limiting behavior for $T \to \infty$.

Proposition 2.2 (Timing Risk Decomposition) *Let us assume $\mathbb{P}(\tau < s|\mathcal{F}_t)$ to be continuously differentiable in variable s, and*

$$\mathbb{P}(\tau < \infty) = \lim_{t\to\infty}\mathbb{P}(\tau < t) = 1.$$

The value of timing risk $\mathrm{Tr}_t(\tau, T)$ at time t given in equation (1) has the following expression for finite maturity T:

$$\begin{aligned}\mathrm{Tr}_t(\tau, T) = 1_{\{t\leq\tau\}}\Big\{&\mathrm{e}^{-r(T-t)}\mathbb{E}[1_{\{\tau\leq T\}}|\mathcal{F}_t]\\ &+\int_t^T \mathrm{e}^{-r(s-t)}\mathbb{E}\left[1_{\{\tau\leq s\}}|\mathcal{F}_t\right] r\,\mathrm{d}s\Big\}.\end{aligned} \tag{2}$$

Letting $T \to \infty$, the value of timing risk $\mathrm{Tr}_t(\tau, T)$ tends to:

$$\lim_{T\to\infty}\mathrm{Tr}_t(\tau, T) = 1_{\{t\leq\tau\}}\int_t^\infty \mathbb{E}\left[\mathrm{e}^{-r(s-t)}1_{\{\tau\leq s\}}|\mathcal{F}_t\right] r\,\mathrm{d}s. \tag{3}$$

Proof The result in equation (2) directly follows by applying integration by parts formula to equation (1). Assumption $\mathbb{P}(\tau < \infty) = 1$ implies the result given in equation (3). ■

The value of timing risk at time t can be disentangled into the sum of distinct components: (i) the value of a European cash-or-nothing knock-in option with payoff 1 at maturity T, (ii) the integral of European cash-or-nothing knock-in options with payoff $r\,\mathrm{d}s$ at maturity s. Notice that the integral implies a continuum of options with maturities $s \in [t, T]$. Formalizing this economic intuition is the basis we consider to extend the timing risk representation to the concept of hedging error and to build its decomposition based on re-iterating the procedure.

3. Hedging error and timing risk

This section introduces the mathematical foundation to represent the *hedging error as a generalized timing risk*. The value of timing risk at generic time t in equation (2) enables to (i) formalize its mathematical representation as discounted expectation, depending on the random time τ, (ii) disentangle it into specific components allowing to identify specific risk drivers. Leveraging on this, we consider an economic agent holding a position on an exotic derivative and implementing a hedging strategy. We identify the hedging error associated with the strategy and introduce its mathematical representation as *generalized timing risk*. Subsequently, we introduce a methodology to re-iterate the procedure to the second order and reduce the hedging cost.

Let us consider the case of an agent holding a position on a specific exotic derivative, whose payment will occur at random time τ coinciding with the underlying asset price crossing a constant barrier K. From a theoretical point of view, we consider the payment at random time τ as described by the payoff function f. In general terms, this contract is associated with a barrier K triggering the payment of a specific payoff at random time τ, with τ indicating the instant at which the underlying asset price, namely X_t, crosses the barrier. If a hedging strategy h is in place, the cost associated with the strategy can be measured via the related *hedging error*. Definition 3.1 introduces the mathematical representation of the hedging error value as discounted expectation.

DEFINITION 3.1 (Hedging Error) *Let $r>0$ be the constant risk free rate of the economy, τ a stopping time, T the finite maturity of the exotic derivative and $\{\mathcal{F}_t\}$ the filtration. Let $\mathrm{He}^1_t(\tau,T)$ denote the hedging error, computed as the present value at time t of the hedging error $\mathrm{He}^1_\tau(\tau,T)$ at random (stopping) time τ:*

$$\mathrm{He}^1_t(\tau,T) := 1_{\{t\le\tau\}}\mathbb{E}[\mathrm{e}^{-r(\tau-t)}\mathrm{He}^1_\tau(\tau,T)|\mathcal{F}_t], \tag{4}$$

with $\mathrm{He}^1_\tau(\tau,T)$ given by

$$\mathrm{He}^1_\tau(\tau,T) := \mathbb{E}[\mathrm{e}^{-r(T-\tau)}p_h(X_T)1_{\{\tau\le T\}}|\mathcal{F}_\tau], \tag{5}$$

where X_T is the underlying asset price at T, and function p_h is the payoff obtained in the case $\tau\le T$ according to the hedging strategy h.

REMARK 3.2 (Hedging Error as Generalized Timing Risk) This paper introduces the economic intuition of *hedging error as a generalized timing risk*. In equation (4) it is shown that the value of the hedging error at time t is equivalent to the discounted price of a pseudo-option with payoff $p_h(X_T)$. We refer to equation (4) as the mathematical representation of the hedging error as a generalized timing risk, since the same structure of equation (1) is derived. Notice that equation (4) embeds not only the timing risk uncertainty associated with the random time τ, but also to the payment level, associated with the stochastic nature of X_t and the resulting final (uncertain) payoff $p_h(X_T)$.

Let us see how the hedging strategy can be built and how to derive the resulting first-order hedging error to fully characterize its expression in equation (4) for a specific case. We refer to 'first-order hedging error' as we will introduce 'second order' one later. As illustrative example, we consider the case of a knock-in option in a one-dimensional setting (ref. assumptions 3.3) and derive the hedging error resulting from the implementation of the hedging strategy. The results can be extended via symmetrization arguments to the case of knock-out barrier options.

ASSUMPTION 3.3 *Let us consider a time homogeneous one-dimensional diffusion and denote a general price function (natural-scaled process) with $S_t := \Psi(X_t)$, where X_t is the underlying asset price, and Ψ is the Lamperti transform* (Lamperti 1962, Karatzas and Shreve 1991). *We assume that Ψ^{-1} is a real-valued function with at most exponential growth (i.e.* $\log|f|$ *is with at most linear growth).*

The state variable process S_t satisfies the following stochastic differential equation (SDE):

$$\mathrm{d}S_t = \mathrm{d}(\Psi(X_t)) = b(S_t)\,\mathrm{d}t + \mathrm{d}W_t, \qquad S_0 = x_0, \tag{6}$$

where W_t is a one-dimensional Wiener process under the risk-neutral measure, and b is a globally Lipschitz continuous function. The risk-free rate in the economy is $r>0$.

The remainder of the paper considers the hedging error and its related quantities as directly dependent on the natural-scaled process S_t.

REMARK 3.4 Examples of underlying asset price processes X_t enabling to apply our setting are Geometric Brownian motion and the exponential of Ornstein–Uhlenbeck process. More generally, if X_t satisfies

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t,$$

where $\mu(X_t)$ and $\sigma(X_t)$ represent drift and volatility, we can retrieve the setting described in assumption 3.3 under specific conditions for the existence of the Lamperti transform (Lamperti 1962) and its inverse.

Let us consider a knock-in option written on the underlying asset price X_t and work with its natural scaled process $S_t = \Psi(X_t)$ whose SDE is given in equation (6). Let us suppose that the knock-in condition is triggered if S_t hits the barrier K before the maturity T. Let $S_0 > K$, and τ be the first hitting time of S_t crossing K, defined as

$$\tau = \inf\{t:\ S_t = K\}. \tag{7}$$

At maturity T, the option payoff is

$$f(S_T)1_{\{\tau\le T\}} \tag{8}$$

with f measurable function on the real line with at most exponential growth.

REMARK 3.5 The results derived by Menozzi *et al.* (2021) ensure the integrability condition $\mathbb{E}[|f(S_T)|]<\infty$ when f is a measurable function with at most exponential growth and drift b in equation (6) satisfies global Lipschitz continuity.

DEFINITION 3.6 (Hedging Instruments) *We define the hedging instruments with payoff functions:*

$$\begin{aligned} f_{K+}(S_T) &:= f(S_T)1_{\{S_T>K\}},\\ f_{K-}(S_T) &:= f_{K+}(2K-S_T) = f(2K-S_T)1_{\{S_T\le K\}}, \end{aligned} \tag{9}$$

associated with options having a non-zero outcome in distinct complementary cases, i.e. $1_{\{S_T>K\}}, 1_{\{S_T\le K\}}$.

3.1. The first-order hedging error

This section introduces the formal mathematical representation of the *first-order hedging error*. In order to do this, we first introduce the definition of a specific hedging strategy h^1 à la Bowie and Carr (1994) and then build the corresponding hedging error.

DEFINITION 3.7 (First-Order Hedging Strategy h^1) *We define the first-order hedging strategy h^1 based on the following steps:*

(1) *at inception, $t=0$, the agent buys path-independent options with payoff functions $f(S_T)1_{\{S_T\le K\}}$ and $f_{K-}(S_T)$;*

Table 1. First-order hedging strategy h^1.

Strategy h^1	$t=0$	τ
Buy	$f(S_T)1_{\{S_T\le K\}}$ $f_{K-}(S_T)$	$f_{K+}(S_T)$
Sell		$f_{K-}(S_T)$
Net position	$f(S_T)1_{\{S_T\le K\}}+f_{K-}(S_T)$	$f_{K+}(S_T)-f_{K-}(S_T)$

Note: The table reports the structure of the hedging strategy h^1 introduced in definition 3.7 and associated with the first-order hedging error.

Table 2. Payoff at maturity T.

Knock-in time τ, Maturity T	if $\tau<T$	if $\tau>T$
Payoff at maturity T	$f(S_T)1_{\{S_T\le K\}}+f_{K+}(S_T)=f(S_T)$	0

Note: The table provides an overview of the payoff at maturity T, as a result of the implementation of the first-order hedging strategy h^1 reported in table 1.

(2) *at the knock-in (random) time τ, the agent sells the latter option with payoff $f_{K-}(S_T)$ and buys the option with payoff $f_{K+}(S_T)$.*

Table 1 reports the structure of the hedging strategy and the net position both at inception $t=0$ and at the random hitting time τ. Based on the implementation of h^1, table 2 reports the mathematical formulation of the payoff at maturity T, depending on the knock-in triggering event being before or after the maturity:

- if $\tau<T$, the option to be hedged has been knocked-in, thus the hedging strategy h^1 provides $f(S_T)1_{\{S_T\le K\}}+f_{K+}(S_T)$ at maturity T;
- if $\tau>T$, the two options bought at inception give a null payoff at maturity T.

Case 3.1 provides the analytic expression of the payoff at maturity and the corresponding hedging instruments in the case of a put-option. Results for the case of a call option can be derived analogously by means of put-call symmetry arguments.

Case 3.1 *Put option*. Let us consider the case of a plain-vanilla put option written on X_T at maturity T. In this special case, function f in equation (8) becomes:

$$f(S_T)=\max(K'-\Psi^{-1}(S_T),0),$$

with K' strike price and Ψ natural scale function. Recalling the option barrier K, we have $0<K'<2K$; based on definition 3.1, by means of equation (9), we define the corresponding *hedging instruments* as:

$$\begin{aligned}f_{K+}(S_T)&=\max(K'-\Psi^{-1}(S_T),0)1_{\{S_T>K\}},\\ f_{K-}(S_T)&=f_{K+}(2K-\Psi^{-1}(S_T))\\ &=\max(K'-2K+\Psi^{-1}(S_T),0)1_{\{S_T\le K\}}.\end{aligned}\tag{10}$$

Note that f and f_{K+} are bounded functions, while f_{K-} is a function with at most exponential growth w.r.t. S_T.

Proposition 3.8 *Let us consider a generic payoff function f and assume it is a measurable function with at most exponential growth. Under assumption 3.3 for the underlying asset dynamics, the mathematical representation of the corresponding first-order hedging error is given as*

$$\mathrm{He}^1_\tau(\tau,T)=\mathbb{E}[\mathrm{e}^{-r(T-\tau)}p_{h^1}(S_T)1_{\{\tau\le T\}}|\mathcal{F}_\tau],\tag{11}$$

with

$$p_{h^1}(S_T)=f_{K+}(S_T)-f_{K-}(S_T),\tag{12}$$

where h^1 is the first-order hedging strategy introduced in definition 3.7, functions f_{K+}, f_{K-} are given in equation (9) and S_T is the scaled underlying asset value at T, following the SDE in equation (6).

Proof The proof of the results follows by considering: (i) definition 3.1, and (ii) the implementation of the hedging strategy reported in table 1, for the resulting error at random time τ. Computing the hedging error at time $\tau<T$ means computing the expected value of the net position at time τ for the hedging strategy h^1. The net position is given in table 2 and the hedging error at random time τ is computed as:

$$\mathrm{He}^1_\tau(\tau,T)=\mathbb{E}[\mathrm{e}^{-r(T-\tau)}(f_{K+}(S_T)-f_{K-}(S_T))1_{\{\tau\le T\}}|\mathcal{F}_\tau].$$

■

By leveraging on the economic intuition of hedging error as a generalized timing risk, theorem 3.11 provides the integral representation of the hedging error by extending the validity of the timing risk decomposition in equation (2) to the first-order hedging error. Definition 3.9 introduces the operator $\mathbf{J}_t$. Proposition 3.8, definition 3.9 and lemma 3.10 enable to prove the main result stated in theorem 3.11.

Definition 3.9 *Given $t>0, x\in\mathbf{R}$, let $\mathbf{J}_t$ be the operator defined as*

$$\begin{aligned}\mathbf{J}_t g(x):=b(x)\int_{\mathbf{R}}\partial_y p(t,x,y)(g(y)1_{\{y>K\}}\\ -g(2K-y)1_{\{y\le K\}})\,\mathrm{d}y,\end{aligned}\tag{13}$$

with g being a measurable function with at most exponential growth, b the drift in SDE equation (6), K the barrier for the knock-in condition in equation (7) and $p(t,x,y)$ the Gaussian kernel representing the transition density of a Brownian motion, given by

$$\begin{aligned}p(t,x,y)&:=\mathbb{P}(W_t\in\mathrm{d}y|W_0=x)/\mathrm{d}y\\ &=\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x-y)^2}{2t}\right).\end{aligned}\tag{14}$$

Lemma 3.10 *Let $g(\cdot)$ be a function on $\mathbf{R}$ with at most exponential growth. Suppose that, at inception, the Brownian* *motion satisfies* $W_0 > K$. *Then, for any fixed time* $t > 0$, *we have:*

$$g(W_t)1_{\{\tau^W<t\}} = u(t,W_0) - v(t,W_0) + \int_0^t (\partial_x u(t-s,W_s) - \partial_x v(t-s,W_s)1_{\{s\le\tau^W\}})\,\mathrm{d}W_s, \tag{15}$$

where

$$u(t,x) = \int_{\mathbf{R}} \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{(x-y)^2}{2t}} g(y)\,\mathrm{d}y,$$
$$v(t,x) = \int_K^\infty \frac{1}{\sqrt{2\pi t}} \left(\mathrm{e}^{-\frac{(x-y)^2}{2t}} - \mathrm{e}^{-\frac{(2K-x-y)^2}{2t}}\right) g(y)\,\mathrm{d}y,$$
$$\tau^W := \inf\{s>0, W_s = K\}.$$

Proof Observe that

$$\begin{aligned}&\mathbb{E}[g(W_t)1_{\{\tau^W<t\}}|W_0=x]\\ &= \mathbb{E}[g(W_t)|W_0=x] - \mathbb{E}[g(W_t)1_{\{\tau^W>t\}}|W_0=x]\\ &= u_0(t,x) - v_0(t,x),\end{aligned}$$

where u_0 and v_0 are the unique solutions of

$$\partial_t u - \frac{1}{2}\partial_x^2 u = 0, u(0,x) = g(x), \quad u \in C^{1,2}(\mathbf{R})$$

and

$$\partial_t v - \frac{1}{2}\partial_x^2 v = 0, v(0,x) = g(x), \lim_{x\to K} v(t,x) = 0,$$
$$v \in C^{1,2}((K,\infty)),$$

respectively. We have $u = u_0$ and $v = v_0$, the latter being the consequence of the reflection principle. Applying Itô's formula to $u(t-s,W_s)$ and $v(t-s\wedge\tau^W, W_{s\wedge\tau^W})$ and letting $s\to t$, we obtain

$$g(W_t) = u(t,W_0) + \int_0^t \partial_x u(t-s,W_s)\,\mathrm{d}W_s,$$

and

$$g(W_t)1_{\{\tau^W>t\}} = v(t,W_0) + \int_0^{t\wedge\tau^W} \partial_x v(t-s,W_s)\,\mathrm{d}W_s,$$

respectively. Thus we obtain (15). ■

We now state the main result in theorem 3.11 and its corollary 3.13 on the integral representation of the *first-order hedging error* for the case of knock-in options.

THEOREM 3.11 (First-Order Hedging Error: Integral Representation) *Under assumption* 3.3, *the first-order hedging error* $\mathrm{He}^1_t(\tau,T)$ *evaluated at time* $t\le T$ *has the following integral representation:*

$$\mathrm{He}^1_t(\tau,T) = 1_{\{t\le\tau\}} \int_t^T \mathrm{e}^{-r(T-s)}\mathbb{E}[\mathrm{e}^{-r(s-t)} \times 1_{\{\tau\le s\}}\mathbf{J}_{T-s}f(S_s)|\mathcal{F}_t]\,\mathrm{d}s, \tag{16}$$

with f measurable function with at most exponential growth and operator $\mathbf{J}_t$ *introduced in definition* 3.9.

Proof We can write

$$\mathrm{He}^1_t(\tau,T) = 1_{\{t\le\tau\}}\mathbb{E}[\mathrm{e}^{-r(\tau-t)}\mathrm{He}^1_\tau(\tau,T)|\mathcal{F}_t]$$

based on proposition 3.8. Considering equation (11), we can write:

$$\mathrm{He}^1_t(\tau,T) = 1_{\{t\le\tau\}}\mathrm{e}^{-r(T-t)}\mathbb{E}[(f_{K+}(S_T) - f_{K-}(S_T))1_{\{\tau\le T\}}|\mathcal{F}_t]$$

and from the Markov property of S_t, it is sufficient to prove the formula for $t=0$. By means of Cameron–Martin–Maruyama–Girsanov theorem (Rogers and Williams 2000), we have:

$$\begin{aligned}&\mathbb{E}[(f_{K+}-f_{K-})(S_T)1_{\{\tau\le T\}}|S_0=x]\\ &= \mathbb{E}[(f_{K+}-f_{K-})(W_T)1_{\{\tau^W\le T\}}\\ &\quad\times \mathrm{e}^{\int_0^T b(W_s)\,\mathrm{d}W_s - \frac{1}{2}\int_0^T (b(W_s))^2\mathrm{d}s}|W_0=x]\end{aligned} \tag{17}$$

for any $x > K$, where

$$\tau^W := \inf\{t>0 : W_t = K\}.$$

Note that Cameron–Martin–Maruyama–Girsanov theorem is valid for the linear growth b (see Beneš 1971 and Karatzas and Shreve 1991, Corollary 3.5.16). By lemma 3.10, we write:

$$\begin{aligned}&(f_{K+}-f_{K-})(W_T)1_{\{\tau^W\le T\}}\\ &= \mathbb{E}[(f_{K+}-f_{K-})(W_T)1_{\{\tau^W\le T\}}|W_0=x]\\ &\quad+ \int_0^T (\partial_x u(T-s,W_s) - \partial_x v(T-s,W_s)1_{\{s\le\tau^W\}})\,\mathrm{d}W_s,\\ &= u(T,W_0) - v(T,W_0) + \int_0^T (\partial_x u(T-s,W_s)\\ &\quad - \partial_x v(T-s,W_s)1_{\{s\le\tau^W\}})\,\mathrm{d}W_s\end{aligned} \tag{18}$$

where

$$u(t,x) = \int_{\mathbf{R}} \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{(x-y)^2}{2t}} (f_{K+}(y) - f_{K-}(y))\,\mathrm{d}y \tag{19}$$

and

$$v(t,x) = \int_K^\infty \frac{1}{\sqrt{2\pi t}} \left(\mathrm{e}^{-\frac{(x-y)^2}{2t}} - \mathrm{e}^{-\frac{(2K-x-y)^2}{2t}}\right) \times (f_{K+}(y) - f_{K-}(y))\,\mathrm{d}y. \tag{20}$$

Observe that the following holds:

$$\begin{aligned}v(t,x) &= \int_K^\infty \frac{1}{\sqrt{2\pi t}} \left(\mathrm{e}^{-\frac{(x-y)^2}{2t}} - \mathrm{e}^{-\frac{(2K-x-y)^2}{2t}}\right) f_{K+}(y)\,\mathrm{d}y\\ &= \int_K^\infty \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{(x-y)^2}{2t}} f(y)\,\mathrm{d}y\\ &\quad - \int_{-\infty}^K \mathrm{e}^{-\frac{(x-y)^2}{2t}} f(2K-y)\,\mathrm{d}y\\ &= \int_{\mathbf{R}} \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{(x-y)^2}{2t}} (f_{K+}(y) - f_{K-}(y))\,\mathrm{d}y\\ &= u(t,x).\end{aligned} \tag{21}$$

Therefore, we can re-write equation (18) as equivalent to:

$$(f_{K+}-f_{K-})(W_T)1_{\{\tau^W\le T\}} = \int_0^T \partial_x u(T-s,W_s)1_{\{\tau^W\le s\}}\,\mathrm{d}W_s,$$

since based on equation (21) we have:

$$\mathbb{E}[(f_{K+}-f_{K-})(W_T)1_{\{\tau^W\le T\}}|W_0=x] = u(T,x)-v(T,x)=0$$

and

$$\int_0^T (\partial_x u(T-s,W_s)-\partial_x v(T-s,W_s)1_{\{s\le\tau^W\}})\,\mathrm{d}W_s = \int_0^T \partial_x u(T-s,W_s)1_{\{\tau^W\le s\}}\,\mathrm{d}W_s.$$

Moreover, we have:

$$\mathrm{e}^{\int_0^T b(W_s)\,\mathrm{d}W_s-\frac{1}{2}\int_0^T(b(W_s))^2\mathrm{d}s} = 1+\int_0^T \mathrm{e}^{\int_0^u b(W_s)\,\mathrm{d}W_s-\frac{1}{2}\int_0^T(b(W_s))^2\mathrm{d}s}b(W_u)\,\mathrm{d}W_u,$$

from equation (17); by means of Itô's isometry, we can state

$$\begin{aligned}\mathbb{E}&[(f_{K+}-f_{K-})(S_T)1_{\{\tau\le T\}}|S_0=x]\\ &=\mathbb{E}\left[\int_0^T \partial_x u(T-s,W_s)1_{\{\tau^W\le s\}}\right.\\ &\quad\left.+\,\mathrm{e}^{\int_0^s b(W_v)\,\mathrm{d}W_v-\frac{1}{2}\int_0^s(b(W_v))^2\mathrm{d}v}b(W_s)\,\mathrm{d}s\right|W_0=x\Bigg].\end{aligned}\tag{22}$$

Once again, by considering Cameron-Martin-Maruyama-Girsanov theorem, we obtain:

$$\begin{aligned}\mathbb{E}&[(f_{K+}-f_{K-})(S_T)1_{\{\tau\le T\}}|S_0=x]\\ &=\mathbb{E}\left[\int_0^T b(S_s)\partial_x u(T-s,S_s)1_{\{\tau\le s\}}\,\mathrm{d}s\right|S_0=x\Bigg]\\ &\quad\times\text{(based on Eq. (19))}\\ &=\mathbb{E}\left[\int_0^T b(S_s)\int_{\mathbf{R}}\partial_x p(T-s,S_s,y)\right.\\ &\quad\left.+\,(f_{K+}(y)-f_{K-}(y))\,\mathrm{d}y1_{\{\tau\le s\}}\,\mathrm{d}s\right|S_0=x\Bigg]\\ &=\mathbb{E}\left[\int_0^T 1_{\{\tau\le s\}}\mathbf{J}_{T-s}f(S_s)\,\mathrm{d}s\right|S_0=x\Bigg],\end{aligned}$$

proving the result. ∎

Remark 3.12 In the proof of proposition 2.2, the standard integration by parts is used, while the key mathematical tool enabling the proof of theorem 3.11 is Itô's isometry in equation (22), which can be seen as a special case of the integration by parts in Malliavin sense (Nualart 2006).

The integral representation of the first-order hedging error in equation (16) states an important result of the present paper. The first-order hedging error admits a mathematical integral representation: it is equivalent to the integral of an infinitesimal amount $\mathrm{d}s$ of knock-in options prices with payoff $\mathbf{J}_{T-s}f(S_s)$ at each maturity s. In principle, the agent can approximate the integral representation via discretization, i.e. by holding an amount $s_{i+1}-s_i$ of knock-in options with payoff $\mathbf{J}_{T-s_i}f(S_{s_i})$ for a partition $t=s_0<s_1<\cdots<s_N=T$, as in a classical calendar spread approach (see Derman *et al.* 1995). By leveraging on this discretization, the integral representation enables a 'decomposition' of the hedging error.

Corollary 3.13 states a fundamental result implementing the intuition behind the methodology proposed in this paper for the hedging error as a timing risk. It provides the mathematical representation for the value of a knock-in option at time t and its link to the first-order hedging error. Moreover, corollary 3.13 shows how to disentangle it based on the generalized timing risk representation. The procedure can then be re-iterated to derive the second-order hedging error (subsection 3.2).

The integral representation for the value of a knock-in option at a generic time t is given in corollary 3.13 as follows.

Corollary 3.13 *Let f be a measurable function with at most exponential growth. Under assumption* 3.3, *for any time $t\le\min\{\tau,T\}$, the value at t of a knock-in option with generic payoff f is given as*

$$\begin{aligned}p_{\mathrm{ki}}(t) &:= \mathbb{E}[\mathrm{e}^{-r(T-t)}f(S_T)1_{\{\tau\le T\}}|\,\mathcal{F}_t]\\ &=\mathbb{E}[\mathrm{e}^{-r(T-t)}f(S_T)1_{\{S_T\le K\}}|\,\mathcal{F}_t]\\ &\quad+\mathbb{E}[\mathrm{e}^{-r(T-t)}f_{K+}(2K-S_T)|\,\mathcal{F}_t]\\ &\quad+\int_t^T \mathrm{e}^{-r(T-s)}\mathbb{E}[\mathrm{e}^{-r(s-t)}1_{\{\tau\le s\}}\mathbf{J}_{T-s}f(S_s)|\,\mathcal{F}_t]\,\mathrm{d}s,\end{aligned}$$

with f_{K+},f_{K-} given in equation (9).

Theorem 3.11 and corollary 3.13 represent the building blocks enabling to re-iterate the procedure and obtain higher orders hedges.

3.2. The second-order hedging error

Theorem 3.11 —equation (16)—states that the first-order hedging error $\mathrm{He}^1_t(\tau,T)$ is decomposed into the integral of a continuum of knock-in options whose payoff is $\mathbf{J}_{T-s}f$, maturing at s, with knock-in boundary K. This section introduces a methodology to re-iterate the hedging to the second order and provides a proof of the main results.

Definition 3.14 (Second-Order Hedging Strategy h^2) *We introduce the notion of second-order hedge à la* Bowie and Carr (1994) *defined as the integral (with respect to the maturity s) of a specific second-order hedging strategy $h^2(s)$ for each maturity s.*

We define the second-order hedging strategy $h^2(s)$ for maturity s based on the following steps:

1. *at inception $t=0$, the agent buys* $\mathrm{d}s$ *units of path-independent options with payoff $\mathbf{J}_{T-s}f(S_s)1_{\{S_s<K\}}$ and $(\mathbf{J}_{T-s}f)_{K-}(S_s)=(\mathbf{J}_{T-s}f)_{K+}(2K-S_s)=(\mathbf{J}_{T-s}f)(2K-S_s)1_{\{S_s<K\}}$;*
2. *at the knock-in (random) time τ, the agent sells the option with payoff $(\mathbf{J}_{T-s}f)_{K-}(S_s)=(\mathbf{J}_{T-s}f)_{K+}(2K-$*

Table 3. Second-order hedging strategy h^2.

Strategy $h^2(s)$	$t = 0$	τ
Buy	$\mathbf{J}_{T-s}f(S_s)1_{\{S_s\le K\}}\,\mathrm{d}s$ $(\mathbf{J}_{T-s}f)_{K-}(S_s)\,\mathrm{d}s$ $(0 \le s \le T)$	$(\mathbf{J}_{T-s}f)_{K_+}(S_s)\,\mathrm{d}s$ $(\tau \le s \le T)$
Sell		$(\mathbf{J}_{T-s}f)_{K-}(S_s)\,\mathrm{d}s$ $(\tau \le s \le T)$
Net position	$(\mathbf{J}_{T-s}f(S_s)1_{\{S_s\le K\}}$ $+(\mathbf{J}_{T-s}f)_{K-}(S_s))\,\mathrm{d}s$	$((\mathbf{J}_{T-s}f)_{K_+}(S_s)$ $-(\mathbf{J}_{T-s}f)_{K-}(S_s))\,\mathrm{d}s$

Note: The table reports the structure of the hedging strategy h^2 introduced in definition 3.14 and associated with the second-order hedging error.

Table 4. Payoff at maturity s.

	if $\tau \le s$	if $\tau > s$
Final payoff at maturity s	$(\mathbf{J}_{T-s}f(S_s)1_{\{S_s\le K\}}+(\mathbf{J}_{T-s}f)_{K_+}(S_s))\,\mathrm{d}s$ $= \mathbf{J}_{T-s}f(S_s)\,\mathrm{d}s$	0

Note: As a result of the implementation of hedging strategy h^2 [table 3], the table reports the payoff at maturity s.

$S_s) = (\mathbf{J}_{T-s}f)(2K - S_s)1_{\{S_s<K\}}$ *and buys* $\mathrm{d}s$ *units of the one with payoff* $(\mathbf{J}_{T-s}f)_{K_+}(S_s) = \mathbf{J}_{T-s}f(S_s)1_{\{S_s>K\}}$.

Functions f_{K+}, f_{K-} *are given in equation* (9), S_T *is the scaled underlying asset value at T, following the SDE in equation* (6), *and operator* $\mathbf{J}_t$ *is introduced in definition* 3.9.

Table 3 reports the structure of the hedging strategy and the net position both at inception $t = 0$ and at the random hitting time τ. Based on the implementation of h^2, table 4 reports the mathematical formulation of the payoff at maturity s, depending on the knock-in triggering event being before or after it:

- if $\tau \le s$, that is, if the option to be hedged has been knocked-in, the strategy leaves $(\mathbf{J}_{T-s}f(S_s)1_{\{S_s<K\}} + \mathbf{J}_{T-s}f(S_s)1_{\{S_s>K\}})\,\mathrm{d}s = \mathbf{J}_{T-s}f(S_s)\,\mathrm{d}s$;
- if $\tau > s$, the two options bought at the initial date pay nothing.

Thus, the second-order hedging strategy h^2, integrated in s, hedges the knock-in option at maturity s.

PROPOSITION 3.15 (Second-Order Hedging Error: Integral Representation) *Under assumption* 3.3, *the second-order hedging error* $\mathrm{He}^2_\tau(\tau, T)$ *evaluated at time* τ *has the following integral representation:*

$$\mathrm{He}^2_\tau(\tau,T) := \int_\tau^T \mathbb{E}[\mathrm{e}^{-r(T-\tau)}1_{\{\tau\le T\}}p_{h^2}(S_s)\,|\mathcal{F}_\tau]\,\mathrm{d}s, \tag{23}$$

with

$$p_{h^2}(S_s) := \{(\mathbf{J}_{T-s}f)_{K_+}(S_s) - (\mathbf{J}_{T-s}f)_{K-}(S_s)\}, \tag{24}$$

where h^2 *is the second-order hedging strategy in definition* 3.14, f *a measurable function with at most exponential growth and* $\mathbf{J}_t$ *the operator introduced in definition* 3.9.

The integral representation of the second-order hedging error evaluated at time $t < T$ *is given by*

$$\mathrm{He}^2_t(\tau,T) := 1_{\{t\le\tau\}}\mathbb{E}[\mathrm{e}^{-r(\tau-t)}\mathrm{He}^2_\tau(\tau,T)|\mathcal{F}_t] \tag{25}$$

which is equivalent to

$$\mathrm{He}^2_t(\tau,T) = 1_{\{t\le\tau\}}\mathbb{E}\left[\mathrm{e}^{-r(T-t)}\int_t^T \{(\mathbf{J}_{T-s}f)_{K_+}(S_s) - (\mathbf{J}_{T-s}f)_{K-}(S_s)\}1_{\{\tau\le s\}}\,\mathrm{d}s|\mathcal{F}_t\right]. \tag{26}$$

Proof We can prove intuitively and heuristically the result by considering: (i) definition 3.1 applied to the first-order hedging error, (ii) the implementation of the hedging strategy h^2 in definition 3.14, reported in table 4, and focusing on the resulting error at random time τ. Computing the hedging error at time $\tau < T$ means computing the expected value of the net position at time τ for h^2, integrating w.r.t. the maturities s. Based on the results in table 4, we can write the hedging error at random time τ as:

$$\mathrm{He}^2_\tau(\tau,T) = \int_\tau^T \mathbb{E}[\mathrm{e}^{-r(T-\tau)}1_{\{\tau<s\}}\{(\mathbf{J}_{T-s}f)_{K_+}(S_s) - (\mathbf{J}_{T-s}f)_{K-}(S_s)\}\,\mathrm{d}s|\mathcal{F}_\tau],$$

which proves the result given in equations (23)–(24). Moreover, we have

$$\begin{aligned} &1_{\{t\le\tau\}}\mathbb{E}[\mathrm{e}^{-r(\tau-t)}\mathrm{He}^2_\tau(\tau,T)1_{\{\tau<T\}}|\mathcal{F}_t] \\ &= 1_{\{t\le\tau\}}\mathbb{E}[\mathrm{e}^{-r(\tau-t)}\int_\tau^T \mathrm{e}^{-r(T-\tau)}1_{\{\tau\le s\}}1_{\{\tau<T\}}\{(\mathbf{J}_{T-s}f)_{K_+}(S_s) \\ &\quad - (\mathbf{J}_{T-s}f)_{K-}(S_s)\}\,\mathrm{d}s|\mathcal{F}_t] \\ &= 1_{\{t\le\tau\}}\mathbb{E}[\mathrm{e}^{-r(T-t)}\int_t^T 1_{\{\tau\le s\}}\{(\mathbf{J}_{T-s}f)_{K_+}(S_s) \\ &\quad - (\mathbf{J}_{T-s}f)_{K-}(S_s)\}\,\mathrm{d}s|\mathcal{F}_t] \end{aligned}$$

as desired. ■

4. First- and second-order hedging errors: an illustrative example

We discuss our theoretical results for the case of constant drift $b(x) = b$ and payoff function $f \equiv 1$. From the perspective of financial modeling, this represents the special case of a bond payoff under a Black–Scholes setting. This enables us to build an analytically tractable proof of concept for the first and second-order hedging errors. We define the *hedging error cost* both in absolute and relative terms (definition 4.5) and provide a study on the behavior of the hedging error cost reduction obtained when considering re-iterating the procedure from the first to the second order for different parameters' values.

We study the behavior of the first- and second-order hedging errors at random time τ for different values of b and τ. To capture this dependence, for the purpose of this analysis, we use the following notation:

$$\mathrm{He}^1_\tau(b,r;\tau,T) := \mathrm{He}^1_\tau(\tau,T); \quad \mathrm{He}^2_\tau(b,r;\tau,T) := \mathrm{He}^2_\tau(\tau,T), \tag{27}$$

with $\mathrm{He}^1_\tau(\tau,T), \mathrm{He}^2_\tau(\tau,T)$ defined respectively in equations (11) and (23).

We work under assumption 3.3, with SDE for the underlying log-price S_t given in equation (6) as:

$$\mathrm{d}S_t = \mathrm{d}(\Psi(X_t)) = \mathrm{d}(\log(X_t)) = b\,\mathrm{d}t + \mathrm{d}W_t, \qquad S_0 = x_0,$$

and hitting time τ defined in equation (7). In this case, we consider the scale function satisfying $S_t = \Psi(X_t) = \log(X_t)$.

Proposition 4.1 *Assume that $b(x) \equiv b$, with b being in this case a constant drift in SDE equation* (6)*. Consider the operator $\mathbf{J}_t(x), t > 0$ given in definition* 3.9*. The following equivalence holds:*

$$\mathbf{J}_t g(2K - x) = -\mathbf{J}_t g(x), \tag{28}$$

based on direct substitution and the reflection principle of Brownian motion.

Proof By applying definition 3.9 to $\mathbf{J}_t g(2K - x)$ and then direct computation, we can prove the result:

$$\begin{aligned}
&\mathbf{J}_t g(2K - x)\\
&= b\int_{\mathbf{R}} \frac{(y - 2K + x)}{t} \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{(2K-x-y)^2}{2t}}\\
&\quad \times (g(y)1_{\{y>K\}} - g(2K - y)1_{\{y\le K\}})\,\mathrm{d}y\\
&= b\int_{\mathbf{R}} \frac{(x - (2K - y))}{t} \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{(2K-y-x)^2}{2t}}\\
&\quad \times (g(y)1_{\{y>K\}} - g(2K - y)1_{\{y\le K\}})\,\mathrm{d}y\\
&= -\mathbf{J}_t g(x).
\end{aligned}$$

■

Remark 4.2 Observe that the result stated in proposition 4.1 relates to an asymmetry property of the operator $\mathbf{J_t}$. In the case of constant payoff, e.g. $f \equiv 1$, the integral expression for operator $\mathbf{J}_t$ in definition 3.9 is given as

$$\mathbf{J}_t 1(x) = -2b\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-K)^2}{2t}\right). \tag{29}$$

Proposition 4.3 *Under assumption* 3.3*, when $f \equiv 1$, the first and second-order hedging errors given in equations* (11) *and* (23) *have the following analytic expression:*

$$\begin{aligned}
\mathrm{He}^1_\tau(b,r;\tau,T) &= \mathrm{e}^{-r(T-\tau)}1_{\{\tau\le T\}}\\
&\quad \times (\Phi(-b(T-\tau)) - \Phi(b(T-\tau))),
\end{aligned} \tag{30}$$

$$\begin{aligned}
\mathrm{He}^2_\tau(b,r;\tau,T) &= -1_{\{\tau\le T\}}\mathrm{e}^{-r(T-\tau)}\frac{b}{\pi}\\
&\quad \times \int_0^{\sqrt{T-\tau}} \int_{-b\sqrt{u(\sqrt{T-\tau}-u)}}^{b\sqrt{u(\sqrt{T-\tau}-u)}} \mathrm{e}^{-\frac{u^2+s^2}{2}}\,\mathrm{d}u\,\mathrm{d}s,
\end{aligned} \tag{31}$$

with Φ cumulative distribution function of a standard Normal random variable

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \mathrm{e}^{-\frac{u^2}{2}}\,\mathrm{d}u.$$

Proof The result follows by considering Remark 4.2, proposition 4.1 and by direct analytic computation. The result for the first-order hedging $\mathrm{He}^1_\tau(b,r;\tau,T)$ error is obtained as follows:

$$\begin{aligned}
\mathrm{He}^1_\tau(b,r;\tau,T) &= \mathrm{e}^{-r(T-\tau)}\mathbb{E}[(1_{\{S_T>K\}} - 1_{\{S_T\le K\}})|\mathcal{F}_\tau]1_{\{\tau\le T\}}\\
&= \mathrm{e}^{-r(T-\tau)}1_{\{\tau\le T\}}\left(\int_{b(T-\tau)}^\infty \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{u^2}{2}}\,\mathrm{d}u\right.\\
&\quad \left. - \int_{-\infty}^{b(T-\tau)} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{u^2}{2}}\,\mathrm{d}u\right)\\
&= \mathrm{e}^{-r(T-\tau)}1_{\{\tau\le T\}}\\
&\quad \times (\Phi(-b(T-\tau)) - \Phi(b(T-\tau))).
\end{aligned}$$

The result for the second-order hedging error $\mathrm{He}^2_\tau(b,r;\tau,T)$ is obtained as follows:

$$\begin{aligned}
&\mathrm{He}^2_\tau(b,r;\tau,T)\\
&= 1_{\{\tau\le T\}}\int_\tau^T \mathrm{e}^{-r(T-\tau)}\mathbb{E}[\mathbf{J}_{T-s}f(S_s)1_{\{S_s>K\}}\\
&\quad - \mathbf{J}_{T-s}f(2K - S_s)1_{\{S_s\le K\}}|\mathcal{F}_\tau]\,\mathrm{d}s\\
&= -2b1_{\{\tau\le T\}}\int_\tau^T \mathrm{e}^{-r(T-\tau)}\left\{\int_K^\infty p(s-\tau,K,y\right.\\
&\quad - b(s-\tau))p(T-s,y,K)\,\mathrm{d}y\\
&\quad \times \left(-\int_{-\infty}^K p(s-\tau,K,y-b(s-\tau))\right.\\
&\quad \left.\left. \times p(T-s,y,K)\,\mathrm{d}y\right\}\,\mathrm{d}s\right)\\
&= -2b1_{\{\tau\le T\}}\int_\tau^T \mathrm{e}^{-r(T-\tau)}\frac{1}{\sqrt{2\pi(T-\tau)}}\mathrm{e}^{-\frac{(s-\tau)^2}{2(T-\tau)}}\\
&\quad \times \left(\Phi\left(b\sqrt{\frac{(s-\tau)(T-s)}{T-\tau}}\right)\right.\\
&\quad \left. - \Phi\left(-b\sqrt{\frac{(s-\tau)(T-s)}{T-\tau}}\right)\right)\mathrm{d}s\\
&= -2b1_{\{\tau\le T\}}\mathrm{e}^{-r(T-\tau)}\frac{1}{\sqrt{2\pi}}\int_0^{\sqrt{T-\tau}} \mathrm{e}^{-\frac{u^2}{2}}\\
&\quad \times \left(\Phi\left(b\sqrt{u(\sqrt{T-\tau}-u)}\right)\right.\\
&\quad \left. - \Phi\left(-b\sqrt{u(\sqrt{T-\tau}-u)}\right)\right)\mathrm{d}u\\
&= -1_{\{\tau\le T\}}\mathrm{e}^{-r(T-\tau)}\frac{b}{\pi}\int_0^{\sqrt{T-\tau}}\int_{-b\sqrt{u(\sqrt{T-\tau}-u)}}^{b\sqrt{u(\sqrt{T-\tau}-u)}} \mathrm{e}^{-\frac{u^2+s^2}{2}}\,\mathrm{d}u\,\mathrm{d}s.
\end{aligned}$$

■

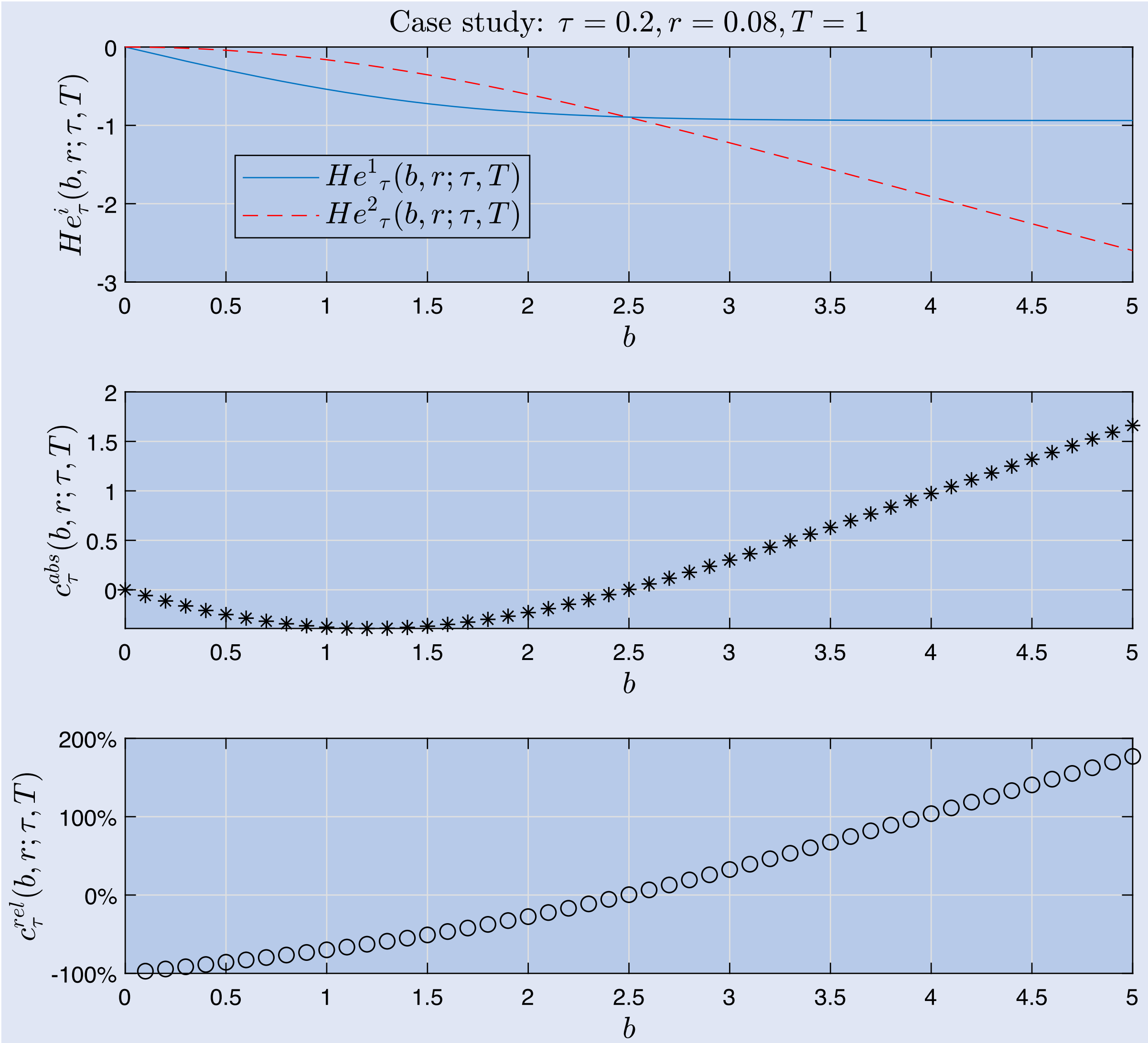

Figure 1. Hedging errors and the associated absolute and relative cost reduction. The plot at the top reports the *hedging errors* $\mathrm{He}^i_\tau(b,r;\tau,T)$, for the first and second order, namely, $i = 1, 2$ as function of the drift parameter b. The analytic expression of hedging errors is given in equation (30) for $\mathrm{He}^1_\tau(b,r;\tau,T)$ and in equation (31) for $\mathrm{He}^2_\tau(b,r;\tau,T)$. The absolute hedging cost reduction $c^{abs}_\tau(b,r;\tau,T)$ and the relative hedging cost reduction $c^{rel}_\tau(b,r;\tau,T)$ reported in the other plots (i.e. middle, bottom plot) are given in equations (33)–(34).

PROPOSITION 4.4 *The limiting behavior of the first-order hedging error given in proposition* 4.3, *equation* (30), *as function of the constant drift b is*

$$\lim_{b\to 0} \mathrm{He}^1_\tau(b,r;\tau,T) = 0,$$

$$\lim_{b\to +\infty} \mathrm{He}^1_\tau(b,r;\tau,T) = -\mathrm{e}^{-r(T-\tau)} 1_{\{\tau\le T\}}. \qquad (32)$$

Proof The results follow by direct analytic computation considering the mathematical representation of the first-order hedging error given in equation (30). ∎

Proposition 4.4 states that, for $\tau < T$, the first-order hedging error is a bounded function of the drift b, and assumes negative real values in $[-\mathrm{e}^{-r(T-\tau)}, 0]$. Starting from this result, we study how the hedging error decomposition applied to $\mathrm{He}^1_\tau(b,r;\tau,T)$ can be exploited to reduce the hedging cost, after re-iterating it to the second order.

DEFINITION 4.5 (Hedging Error Cost) *The hedging error cost could be measured in absolute and relative terms, respectively, via functions* $c^{abs}_\tau(b,r;\tau,T)$ *and* $c^{rel}_\tau(b,r;\tau,T)$ *defined as follows:*

$$c^{abs}_\tau(b,r;\tau,T) := |\mathrm{He}^2_\tau(b,r;\tau,T)| - |\mathrm{He}^1_\tau(b,r;\tau,T)|, \quad (33)$$

$$c^{rel}_\tau(b,r;\tau,T) := \frac{|\mathrm{He}^2_\tau(b,r;\tau,T)| - |\mathrm{He}^1_\tau(b,r;\tau,T)|}{|\mathrm{He}^1_\tau(b,r;\tau,T)|}, \quad (34)$$

with $\mathrm{He}^1_\tau(b,r;\tau,T), \mathrm{He}^2_\tau(b,r;\tau,T)$ *given in equations* (30)–(31).

Figures 1 and 2 report the behavior of the hedging errors (both first and second order) and the hedging cost reduction as given in definition 4.5 as function of the drift parameter b. Each plot is built based on a specific value of the random time τ, namely $\tau = 0.2, \tau = 0.8$ for the same level of risk-free interest rate ($r = 0.08$) and option maturity ($T = 1$). In this way, we isolate the effect of the timing risk component deriving from the hitting time τ. Let us denote with $\hat{b}$ the value

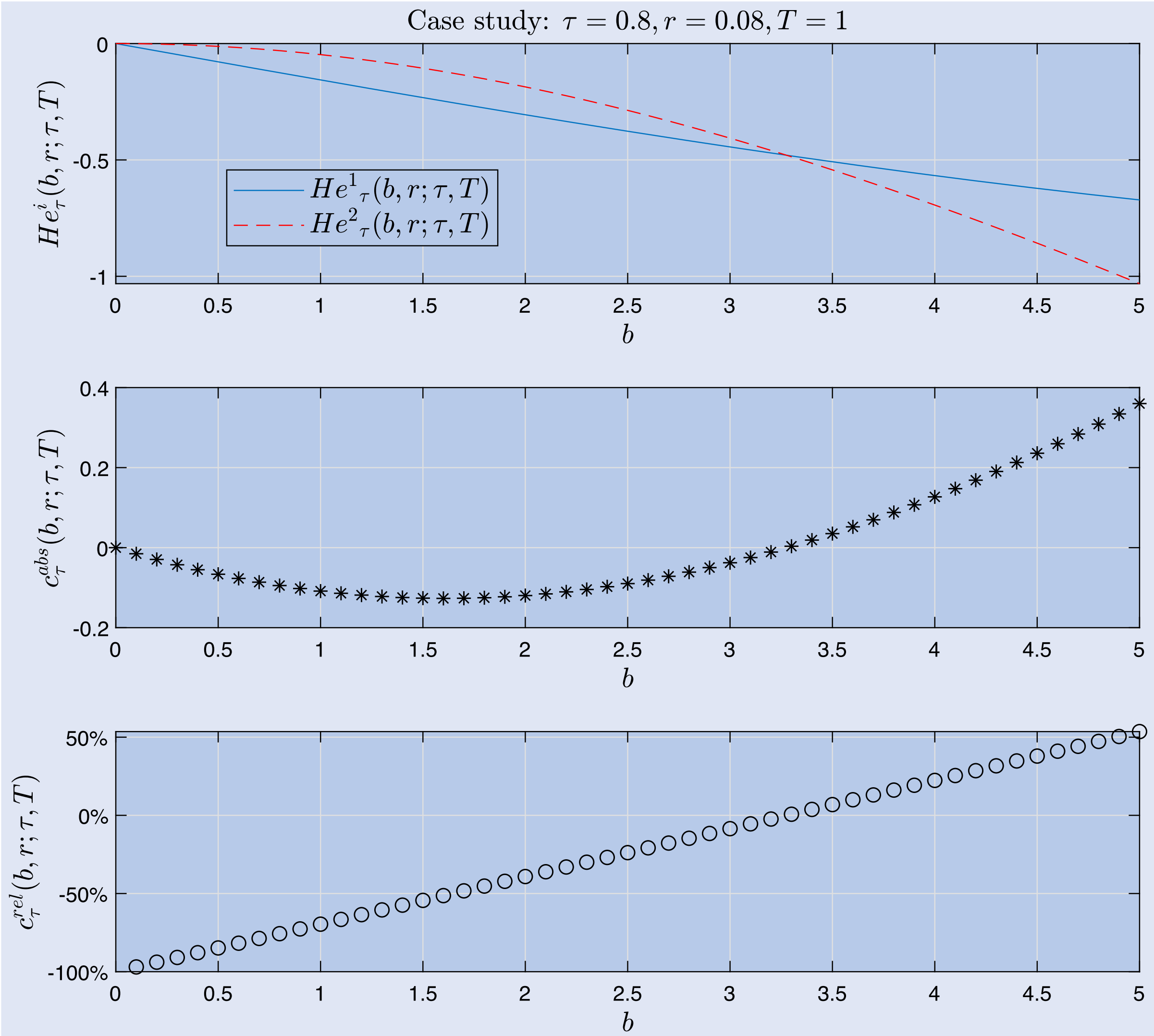

Figure 2. Hedging errors and the associated absolute and relative cost reduction. The plot at the top reports the *hedging errors* $\mathrm{He}^i_\tau(b, r; \tau, T)$, for the first and second order, namely, $i = 1, 2$ as function of the drift parameter b. The analytic expression of hedging errors is given in equation (30) for $\mathrm{He}^1_\tau(b, r; \tau, T)$ and in equation (31) for $\mathrm{He}^2_\tau(b, r; \tau, T)$. The absolute hedging cost reduction $c^{abs}_\tau(b, r; \tau, T)$ and the relative hedging cost reduction $c^{rel}_\tau(b, r; \tau, T)$ reported in the other plots (i.e. middle, bottom plot) are given in equations (33)–(34).

at which the first and second-order hedging errors coincide, namely:

$$\hat{b}: \quad \mathrm{He}^1_\tau(\hat{b}, r; \tau, T) = \mathrm{He}^2_\tau(\hat{b}, r; \tau, T),$$
$$|\mathrm{He}^2_\tau(\hat{b}, r; \tau, T)| \leq |\mathrm{He}^1_\tau(\hat{b}, r; \tau, T)|, \quad \text{for } b \in [0, \hat{b}].$$

The numerical analysis on this illustrative example shows $\hat{b}$ as an increasing function of the random hitting time τ. The drift value equalizing the first and second-order hedging error increases with τ, while the value of the errors at that point decreases in absolute value. When $b \in [0, \hat{b}]$, having a second-order semi-static hedging in place enables the holder of the position to reduce the hedging cost. This is possible by means of the integral decomposition of the first-order hedging error (theorem 3.11), which can be represented as a *generalized timing risk*. In this interval, the absolute cost reduction $c^{abs}_\tau(b, r; \tau, T)$ has a convex shape: initially increasing in absolute value as the drift increases and then reducing in absolute value until the drift $b = \hat{b}$. The relative cost measure $c^{rel}_\tau(b, r; \tau, T)$ is close to a full off-set of the hedging cost when the drift is small, and then decreases in absolute value until the drift $b = \hat{b}$. Re-iterating the hedging strategy from the first to the second order produces higher cost reductions when the hitting time is closer to inception, as in this illustrative example, the uncertainty associated with the financial position is mainly driven by the timing risk component.

5. Conclusion

This paper introduces a methodology to disentangle the *hedging error* associated with the hedging of exotic derivatives, whose payment time is unknown at inception. From a financial point of view, it introduces the financial economic intuition of *hedging error as generalized timing risk* and provides its mathematical formalization. Based on this idea, we define the hedging error and then motivate how to: (i) enable disentangling the first-order hedging error into specific components, (ii) state the mathematical derivation of the results and

(iii) re-iterate the procedure to the second order. We analyze the hedging error cost reduction obtained by re-iterating the procedure from first to second order via an illustrative example, considering both an absolute and relative cost assessment. Results show that the reduction can vary depending on both the drift of the underlying asset value dynamic and the random hitting time, which embeds the *timing risk uncertainty*. These findings represent the basis for further research extensions. From a theoretical point of view, the core mathematical results enable us to consider alternative directions: (i) building a general multidimensional framework, (ii) re-iterating the procedure to higher orders, (iii) investigating the bridge with advanced analytics methodologies and techniques, as in recent literature within the field of quantitative finance (Buehler *et al.* 2019, Lütkebohmert *et al.* 2022).

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