# Optimal Gait Switching

## Using Max-plus Linear Systems

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Master of Science Thesis

Delft University of Technology

Delft Center for Systems and Control

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Faculty of Mechanical, Maritime and Materials Engineering (3mE) · Delft University of Technology

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# Abstract

Animals change the way they walk depending on the type of terrain or how fast they move. Taking inspiration from nature, it is reasonable to think that robots should change their locomotion patters, also called "gaits", depending on their environment or the objectives of their tasks. Recently, a discrete event system (DES) model description for legged locomotion has been developed, this model is based on events, where the events describe each leg touch down and liftoff. By describing the relation between the touchdown and lift-off events of the different legs a gait is generated. This DES model can be described in the max-plus linear algebra as a linear model. Gait switches are simply done by changing system matrices, the max-plus algebra then ensures that the robot remains stable during the gait switch by forcing some of the legs to move slower over the ground. By doing this, the order in which the legs move can be changed to that of the new gait, while ensuring enough legs are on the ground to avoid falling. However, very little research has been done on the optimization of these gait switches. The goal of this thesis is to find optimal gait switches using the max-plus linear model. An optimal gait switch in this thesis is considered as a gait transition where the difference in speed of the legs on the ground during the gait switch is as small as possible.

In order to find the optimal gait switches the steady state behaviour of the gaits created with this model are analyzed. This is done by looking at the properties of the matrix that describes the gait. These properties are the eigenvalue and the eigenvector, which represent the cycle time and order in which the events happen. First an eigenvalue and eigenvector are found for the system. Then, by proving that the matrix is irreducible and that the critical graph of that matrix consists of a single strongly connected subgraph one can show that this eigenvalue and eigenvector are unique. This in turn means the steady state behaviour for these gaits is uniquely defined.

The steady state behaviour between different gaits can then be analysed by comparing the eigenvectors of the gaits. The difference between the eigenvectors can then be used as a measure of optimality. By minimizing the difference between the eigenvectors an optimal gait switch is found.

The gait switch is further optimized by manipulating the time the legs stay in the air according to the difference in the two eigenvectors. Using this method “perfect” gait switches can be obtained, where the speed of the legs on the ground during the gait switch is the same for all legs. For this to work however it is required that the legs on the ground move at the same speed for both gaits. This means the perfect gait switches are only possible if the robot is moving at the same speed in both gaits. Which in turn means the robot will not speed up when using only “perfect” gait switches.

Non-zero acceleration is achieved by manipulating the clock the robot is using for the timing. By introducing a virtual clock, that has a non-linear relation with the real clock, that determines the timing of the robot, the robot will speed up relative to the real clock, while the timing relative to the virtual clock does not change. By doing so the robot speeds up without having to change the gait parameters.

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# Chapter 1

# Introduction

## 1-1 Background

Mankind has always used tools to complete tasks they otherwise could not, or would take much more effort. These tools started very simple, but in time became more and more complex machines. Robots are an example of these complex machines. Nowadays robots are used in situations where the task is impossible to complete for humans or are too dangerous such as search and rescue operations, space exploration, bomb disposal operations, reconnaissance, and many more operations. As the complexity of these operations increases the capabilities of the robot also has to increase. For example a robot that is used for search and rescue operations has to be able to move over many different types of terrains. Each type of terrain requires the robot to move slightly different. It needs to be able to switch from one locomotion pattern to another. Locomotion patterns are often called gaits in literature.

If legs are used to move this act of moving is called legged locomotion. Legged locomotion and gaits for all sorts of animals have been researched; from the smallest insects[2] to the largest mammals and reptiles use legs to move. The number of legs these animals have can be as low as two, like humans, or up to hundreds for insects like the millipede. Besides the number of legs, many other factors determine the locomotion pattern[5], such as the size and shape of the animal, and many more factors. The gaits of horses have been extensively researched[7], but also the energetics of these gaits and their transitions[11].

These animals have inspired a lot of different models for legged locomotion for the use in robotics[9]. It has also lead to many different kind of robots, such as the human inspired biped(two legged) walking robots Leo and Flame[8] developed at the Delft University of Technology, or quadruped(four legged) robots suchs as the Sony AIBO[10] and the BigDog[16] or hexapod(six legged) robots such as the RHex[17] and its DCSC derivative [13][15] and robots with even more legs.

One of the challenges that remains in legged locomotion is how to switch from one gait to another without causing unstable behaviour and how to optimize that gait switch. Animals can do this naturally[11][5], when they adapt their motion to the surface they are walking on, or when they turn, or any other change in their locomotion. Robots need to be programmed. This can be done in several ways. One way is to continuously make small changes to the gaits in such a way that all those small changes result in the desired new gaits[4][3]. This however is a long and complicated process.

Recent developments in legged locomotion using discrete event system (DES)[13][15] have shown promising results especially with gait switches. The events of such a system are the lift-off and touch down moments of the legs. A locomotion pattern can then be described by the synchronization constraints of these events. These constraints can be written as the maximum of multiple events each with different offsets added to them. Since it only uses the max and addition operators these constraints can be written as linear max-plus algebra equations. This thesis will therefore focus on optimizing gait switches for the max-plus linear model description developed by Lopes et al.[13][15]

## 1-2 Problem statement

The main goal for this thesis is:

Determine the optimal gait switches for the max-plus linear model for a hexapod robot.

In order to reach this goal several several steps need to be taken first. In order to determine an optimal gait switch it is neccesary to know how the robot behaves when it is using those gaits. Therefore the steady state behaviour of all gaits needs to determined first. After the steady state behaviour has been determined, this needs to be used to determine the optimal gait switches, but in order to do that a method to quantitatively define the optimality of a gait needs to be defined. This method then needs to be applied to all possible gait switches in order to determine the optimal gait switches. The final part of thesis will be about new gait switching methods that can be used to further improve these gait switches.

This results in the following subgoals:

- Analyze the steady state behaviour of the gaits
- Develop a method that quantitatively defines the optimality of a gait switch and use this to determine the optimal gait switches
- Develop new methods for switching gaits

## 1-3 Contribution

The max-plus model and its control structure are based on the work of Lopes et al. [13][15]. In collaboration with Lopes et al. two more conference papers on this subject have been submitted for publication [14][12] and two journal papers are being written. My contribution to these papers is the proof of the unique eigenvalue in chapter 4 and the development of new gait switching methods which improve the gait switches, as presented in chapter 5.

## 1-4 Outline

The structure of this thesis is as follows; in chapter 2 the basics of max-plus linear algebra are explained, in chapter 3 the basics of legged locomotion and the max-plus linear model are presented, in chapter 4 the steady state behaviour of the gaits created with max-plus linear model is analyzed, in chapter 5 the steady state behaviour is used to find the optimal gait switches and determine new gait switching methods. Finally in chapter 6 the thesis will be concluded with a discussion of the research done, conclusions will be drawn and recommendation for future research will be made.

# Chapter 2

# Max-plus linear algebra

## 2-1 Introduction

In this chapter the mathematical principles of max-plus algebra will be explained. This chapter is based on the books of Bacelli et al.[1] and Heidergott et al.[6]. Starting with the most basic concepts and definitions in section 2-2. In section 2-3 Matrices and vectors for max-plus algebra will be introduced. In section 2-4 graphs in max-plus algebra will be explained. Eigenvectors and -values are explained in section 2-5. In section 2-6 the mathematical concepts for solving max-plus linear equations are presented. Section 2-7 deals with max-plus linear (MPL) systems. In section 2-8 these MPL systems are expanded to switching max-plus linear (SMPL).

## 2-2 Constants and operators

First a few constants need to be defined before the rest of the theory can be explained. These constants are $\varepsilon \stackrel{\text{def}}{=} -\infty$ and $e \stackrel{\text{def}}{=} 0$. $\varepsilon$ is called the zero-element and $e$ is called the one-element in max-plus algebra. The set $\mathbb{R}_{\max}$ is defined as $\mathbb{R} \cup \{\varepsilon\}$, where $\mathbb{R}$ is the set of real numbers.

Next the two basic operators $\oplus$ and $\otimes$ will be defined. For the elements $a, b \in \mathbb{R}_{\max}$ the $\oplus$ and $\otimes$ operators are defined as

$$a \oplus b \stackrel{\text{def}}{=} \max(a, b) \tag{2-1}$$

and

$$a \otimes b \stackrel{\text{def}}{=} a + b \tag{2-2}$$

The reason $\varepsilon$ is called the zero-element is

$$\begin{aligned} a \oplus \varepsilon &= \varepsilon \oplus a &= a \quad (2\text{-}3)\\ a \otimes \varepsilon &= \varepsilon \otimes a &= \varepsilon. \quad (2\text{-}4) \end{aligned}$$

From the following it is clear that $e$ is the one-element

$$a \otimes e = e \otimes a = a. \tag{2-5}$$

Max-Plus algebra is denoted by

$$\mathcal{R}_{\max} = (\mathbb{R}_{\max}, \oplus, \otimes, \varepsilon, e).$$

In max-plus algebra the $\otimes$ operator has priority over the $\oplus$ operator just like in conventional algebra, where $\times$ has priority over $+$. This means that for the elements $a, b, c, d \in \mathbb{R}_{\max}$

$$a \oplus b \otimes c \oplus d = a \oplus (b \otimes c) \oplus d.$$

## 2-3 Matrices and vectors

Matrices and vectors for max-plus algebra will be introduced in this section. The set of $n \times m$ matrices for max-plus algebra is denoted by $\mathbb{R}_{\max}^{n\times m}$, where $n, m \in \mathbb{N}$ and $n, m \neq 0$. The element $a_{ij}$ of a matrix $A \in \mathbb{R}_{\max}^{n\times m}$ is the element on the $i^{\text{th}}$ row and $j^{\text{th}}$ column, where $i \in \{0, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Another way to notate the same element is $[A]_{ij}$. The matrix $A$ looks like

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \tag{2-6}$$

Matrices can be added up in max-plus algebra in a similar way to that of the matrix addition in standard algebra. For the matrices $A, B \in \mathbb{R}_{\max}^{n\times m}$, the sum is denoted by $A \oplus B$ and defined as

$$\begin{aligned} [A \oplus B]_{ij} &= a_{ij} \oplus b_{ij} \\ &= \max(a_{ij}, b_{ij}), \end{aligned} \tag{2-7}$$

for $i, j \in \mathbb{N}$, $i, j \neq 0$, $i \leq n$ and $j \leq m$.

For $A \in \mathbb{R}_{\max}^{n\times m}$ and $\beta \in \mathbb{R}_{\max}$, $\beta \otimes A$ is defined as

$$[\beta \otimes A]_{ij} = \beta \otimes a_{ij} \tag{2-8}$$

for $i, j \in \mathbb{N}$, $i, j \neq 0$, $i \leq n$ and $j \leq m$.

For matrices $C \in \mathbb{R}_{\max}^{n\times l}$ and $D \in \mathbb{R}_{\max}^{l\times m}$, the matrix product $C \otimes D$ is defined as

$$\begin{aligned} [C \otimes D]_{ik} &= \bigoplus_{j=1}^{l} c_{ij} \otimes d_{jk} \\ &= \max_{j} \{c_{ij} + d_{jk}\} \end{aligned} \tag{2-9}$$

where $i, j, k \in \mathbb{N}$, $i, j, k \neq 0$, $i \leq n$, $j \leq l$ and $k \leq m$.

Next the two matrices $\mathcal{E}(n,m)$ and $E(n,m)$ are defined.

$$[\mathcal{E}(n,m)]_{ij} \stackrel{\text{def}}{=} \varepsilon \tag{2-10}$$

$E(n,m)$ is the $n \times m$ matrix defined by

$$[E(n,m)]_{ij} \stackrel{\text{def}}{=} \begin{cases} e & \text{for i = j} \\ \varepsilon & \text{otherwise} \end{cases} \tag{2-11}$$

If $n = m$ then $E(n,n)$ is called the $n \times n$ identity matrix.

The transpose of the matrix $A \in \mathbb{R}_{\max}^{n \times m}$, denoted by $A^\top$, is defined as

$$[A^\top]_{ij} = a_{ji} \tag{2-12}$$

Which is the same as the transpose in standard algebra.

The matrix power is defined as

$$A^{\otimes k} \stackrel{\text{def}}{=} \underbrace{A \otimes A \otimes \cdots \otimes A}_{k \text{ times}}, \tag{2-13}$$

where $A \in \mathbb{R}_{\max}^{n \times n}$.

If each row of a matrix $A \in \mathbb{R}_{\max}^{n \times m}$ has at least one element different from $\varepsilon$ then matrix $A$ is called regular.

A matrix $A \in \mathbb{R}_{\max}^{n \times n}$ with $a_{ij} = \varepsilon$, for $1 \leq i \leq j \leq n$, is called strictly lower triangular. If $a_{ij} = \varepsilon$, for $1 \leq i < j \leq n$ then $A$ is called lower triangular. If $A^\top$ is (strictly) lower triangular then $A$ is (strictly) upper triangular.

Vectors are defined as elements of $\mathbb{R}_{\max}^{n} \stackrel{\text{def}}{=} \mathbb{R}_{\max}^{n \times 1}$. The $i^{\text{th}}$ element of a vector $x \in \mathbb{R}_{\max}^{n}$ is denoted by $x_i$. It can also be written as $[x]_i$. The $i^{\text{th}}$ column of the identity matrix $E(n,n)$ is denoted by $e_i$ and it is called the $i^{\text{th}}$ base vector of $\mathbb{R}_{\max}^{n}$. There is also a unit vector in $\mathbb{R}_{\max}^{n}$. This vector is denoted by $\mathbf{u}$ and has all elements equal to $e$.

## 2-4 Graphs

A graph $\mathcal{G}$ is a representation of a finite set of nodes $\mathcal{N}$ and a set of arcs (or edges), $\mathcal{D} \subset \mathcal{N} \times \mathcal{N}$, which represents the connections between the nodes. When the set of arcs is ordered it means that there is a distinction between the arcs $(i,j)$ and $(j,i)$. If $i, j \in \mathcal{N}$, $(i,j) \notin \mathcal{D}$ and $(j,i) \in \mathcal{D}$ then there exists an arc from node $j$ to node $i$ but not from node $i$ to node $j$. This arc $(j,i)$ is called an incoming arc at $i$ and an outgoing arc at $j$. This means that ordered arcs have a direction, that is why a graph with an ordered set of arcs is called a directed graph. If all the arcs $(i,j) \in \mathcal{D}$ of the directed graph $\mathcal{G}$ have a weight $w(i,j)$ then $\mathcal{G}$ is called a weighted directed graph. Because the only type of graph that is of interest is the weighted directed graph the other types of

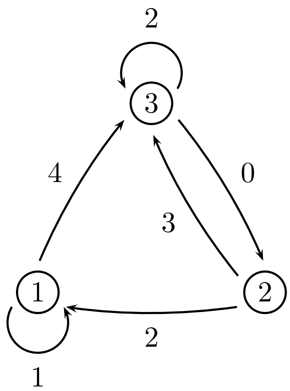


**Figure 2-1:** Graph $\mathcal{G}(A_{\text{ex}})$

graphs will not be discussed. Therefore weighted directed graphs will be referred to as graphs.

For any matrix $A \in \mathbb{R}_{\max}^{n \times n}$ a graph can be drawn, this graph is called the communication graph and is denoted by $\mathcal{G}(A)$. The set of nodes of the graph is denoted by $\mathcal{N}(A)$ and the set of arcs is denoted by $\mathcal{D}(A) \subset \mathcal{N}(A) \times \mathcal{N}(A)$. A pair $(i,j) \in \mathcal{N}(A) \times \mathcal{N}(A)$ is an arc of the graph if $a_{ji} \neq \varepsilon$. $a_{ji}$ is the weight of the arc $(i,j)$.

As an example consider the matrix

$$A_{\text{ex}} = \begin{bmatrix} 1 & 2 & \varepsilon \\ \varepsilon & \varepsilon & 0 \\ 4 & 3 & 2 \end{bmatrix}.$$

The graph $\mathcal{G}(A_{\text{ex}})$ is shown in Figure 2-1

A path from node $i$ to node $j$ is a sequence of arcs denoted by $p = ((i_k, j_k) \in \mathcal{D}(A) : k \in \{1, \ldots, m\}$ where $i = i_1$, $j_k = i_{k+1}$ for $k < m$ and $j_m = j$. The path has length $m$, denoted as $|p|_1 = m$. If $i = j$ or in other words if the end node is the same as the starting node the path is called a circuit. The circuit is called elementary if, restricted to that circuit, all the nodes of the circuit only have one ingoing and one outgoing arc. If there are several paths from $i$ to $j$ of length $m$ they can be viewed as a set denoted by $P(i,j;m)$. The weight of a path in $\mathcal{G}(A)$ is defined as the the sum of the weights of the arcs that make up the path. Formally stated, for $p = ((i_1,i_2),(i_2,i_3),\cdots,(i_m,i_{m+1})) \in P(i,j;m)$ with $i = i_1$ and $j = i_{m+1}$, the weight of $p$, denoted by $|p|_{\text{w}}$, is defined as

$$|p|_{\text{w}} = \bigotimes_{k=1}^{m} a_{i_{k+1} i_k} \tag{2-14}$$

The average weight of a path $p$ is given by $|p|_{\text{w}}/|p|_1$. When dealing with circuits the length, weight and average weight are defined similarly as paths, however the average circuit weight is called the circuit mean.

Paths in $\mathcal{G}(A)$ can be combined in order to construct a new path. For example let $p = ((i_1,i_2),(i_2,i_3))$ and $q = ((i_3,i_4),(i_4,i_5))$ be two paths in $\mathcal{G}(A)$. Then,

$$p \circ q = ((i_1,i_2),(i_2,i_3),(i_3,i_4),(i_4,i_5))$$

is a path in $\mathcal{G}(A)$ as well. The operation $\circ$ is called the concatenation of paths.

Using the powers of the matrix $A$ the maximal weight of a path of length $k$ from node $i$ to node $j$ can be found. The following definition shows how.

$$[A^{\otimes k}]_{ji} = \max\{|p|_{\mathrm{w}} : p \in P(i,j;k)\}, \tag{2-15}$$

where $A \in \mathbb{R}_{\max}^{n \times n}$, $k \geq 1$ and $[A^{\otimes k}]_{ji} = \varepsilon$ if $P(i,j;k)$ is empty.

Similarly the maximum weight of any path from node $i$ to node $j$ can be defined as

$$A^{+} \stackrel{\text{def}}{=} \bigoplus_{k=1}^{\infty} A^{\otimes k} \tag{2-16}$$

where $A \in \mathbb{R}_{\max}^{n \times n}$.

If graph $\mathcal{G}(A)$, where $A \in \mathbb{R}_{\max}^{n \times n}$, only has circuits with a circuit mean of $e$ or less, then $k$ in 2-16 can be limited to $n$.

$$A^{+} = \bigoplus_{k=1}^{\infty} A^{\otimes k} = A \oplus A^{\otimes 2} \oplus \cdots \oplus A^{\otimes n} \tag{2-17}$$

In some cases the infinity sum in 2-17 can also be limited to a value $k = m$ if $A$ is nilpotent. That is if $A^{\otimes m} = \mathcal{E}$. Such a case is discussed in section 2-4.

Next a few more concepts and definitions are given regarding graph theory. Consider a graph $\mathcal{G} = (\mathcal{N}, \mathcal{D})$, where $\mathcal{N}$ is the set of nodes and $\mathcal{D}$ the set of arcs, with nodes $i$, $j \in \mathcal{N}$. If there is a path from $i$ to $j$ then $j$ is reachable from $i$, denoted as $i\mathcal{R}j$. If for all nodes $i$, $j \in \mathcal{N}$, $j$ is reachable from $i$ then the graph $\mathcal{G}$ is called strongly connected. Consider a matrix $A \in \mathbb{R}_{\max}^{n \times n}$ and its graph $\mathcal{G}(A)$, if the graph $\mathcal{G}(A)$ is strongly connected then $A$ is irreducible. When a matrix is not irreducable it is reducible. If $i\mathcal{R}j$ and $j\mathcal{R}i$, in words if there exists a path from $i$ to $j$ and from $j$ to $i$, or if $i$ is $j$ then node $j$ communicates with node $i$ and visa versa, this is denoted as $j\mathcal{C}i$ and $i\mathcal{C}j$ respectively. When a graph $\mathcal{G} = (\mathcal{N}, \mathcal{D})$ is not strongly connected, not all nodes communicate with each other. This does not mean there are no subsets of nodes which are strongly connected. Graph $\mathcal{G}$ can be devided into these subsets of nodes. The subsets can not communicate with each other, if they do they both should belong to a larger subset containing all nodes of the two communicating subsets. These subsets will be denoted as $\mathcal{N}_1$, $\mathcal{N}_2$, $\cdots$, $\mathcal{N}_q$ and $\mathcal{N}$ can de devided into these subsets as $\mathcal{N}_1 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_q$, where $\mathcal{N}_r, r \in \{1, \ldots, q\}$. Each subset of nodes $\mathcal{N}_r$ also has a corresponding subset of arcs $\mathcal{D}_r, r \in \{1, \ldots, q\}$, only arcs that have a starting node and ending node in $\mathcal{N}_r$ are in $\mathcal{D}_r$. The combination of $\mathcal{N}_r$ and $\mathcal{D}_r$ is a subgraph of $\mathcal{G}$ denoted as $\mathcal{G}_r = (\mathcal{N}_r, \mathcal{D}_r), r \in \{1, \ldots, q\}$. The subgraph $\mathcal{G}_r = (\mathcal{N}_r, \mathcal{D}_r)$ is called a maximal strongly connected subgraph (m.s.c.s.) of graph $\mathcal{G}$ if $\mathcal{D}_r \neq \emptyset$. The set of nodes that contains node $i$ and communicates with eachother is defined as

$$[i] \stackrel{\text{def}}{=} \{j \in \mathcal{N} : i\mathcal{C}j\} \tag{2-18}$$

This means that if $i \in \mathcal{N}_r$ then $[i] = \mathcal{N}_r$, for $r \in \{1, \ldots, q\}$. One special subgraph is the graph which only has one node $i$ and that node does not communicate with any other

node. This means that $[i] = \{i\}$ and that arc set is empty. Eventhough the subset is not strongly connected it is still referred to as an m.s.c.s.
Using the previous definitions and concepts the reduced graph can be introduced.

$$\begin{aligned} &\tilde{\mathcal{G}} = (\tilde{\mathcal{N}}, \tilde{\mathcal{D}}), \text{ where} \\ &\tilde{\mathcal{N}} = \{[i_1], \cdots, [i_q]\} \\ &([i_r], [i_s]) \in \tilde{\mathcal{D}} \text{ for } r \neq s \end{aligned} \tag{2-19}$$

there also has to exist an arc $(k, l) \in \mathcal{D}$ for some $k \in [i_r]$ and $l \in [i_s]$. Consider the reducable matrix $A \in \mathbb{R}_{\max}^{n \times n}$ and its graph $\mathcal{G}(A)$. $A$ can be written in the upper block triangular form. Some relabeling of the nodes in $\mathcal{G}(A)$ might be necessary.

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & \cdots & A_{1q} \\ \mathcal{E} & A_{22} & \cdots & \cdots & A_{2q} \\ \mathcal{E} & \mathcal{E} & A_{33} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ \mathcal{E} & \mathcal{E} & \cdots & \mathcal{E} & A_{qq} \end{pmatrix}, \tag{2-20}$$

where $A_{rr}$ is the matrix represented by the subgraph $\mathcal{G}_r = (\mathcal{N}_r, \mathcal{D}_r)$ for $1 \leq r \leq q$ and where $A_{sr}$, for $1 \leq s \leq r \leq q$, is a matrix of appropriate size which has finite elements corresponding to arcs from a node in $[i_r]$ to a node in $[i_s]$ where the values corresponds to the weights of the arcs. This form of matrix $A$ is a normal form of $A$.

The concept of the cyclicity of a graph $\mathcal{G}$ is defined for the strongly connected and the not strongly connected graphs. The cyclicity of a graph $\mathcal{G}$ is denoted by $\sigma_{\mathcal{G}}$. For a strongly connected graph $\mathcal{G}$ the cyclicity $\sigma_{\mathcal{G}}$ is defined as the greatest common divisor of the lengths of all elementary circuits in $\mathcal{G}$. If $\mathcal{G}$ consists of only one node and no self-loop then $\sigma_{\mathcal{G}} \stackrel{\text{def}}{=} 1$.
For a not strongly connected graph $\mathcal{G}$ the cyclicity $\sigma_{\mathcal{G}}$ is defined as the greatest common multiple of the cyclicities of all the m.s.c.s.'s of $\mathcal{G}$.

The nodes to which node $i$ is connected can be devided into the (direct) predecessors and the (direct) successors. The direct predecessors and all predecessors are denoted by $\pi(i)$, $\pi^+(i)$ respectively. The direct successors and all successors are denoted by $\sigma(i)$, $\sigma^+(i)$ respectively. The names already imply the meaning of those sets, but the formal definitions are

$$\pi(i) \stackrel{\text{def}}{=} \{j \in \{1, \ldots, n\} : (j, i) \in \mathcal{D}\}, \tag{2-21}$$

$$\pi^+(i) \stackrel{\text{def}}{=} \{j \in \{1, \ldots, n\} : j\mathcal{R}i\}, \tag{2-22}$$

$$\sigma(i) \stackrel{\text{def}}{=} \{j \in \{1, \ldots, n\} : (i, j) \in \mathcal{D}\}, \tag{2-23}$$

$$\sigma^+(i) \stackrel{\text{def}}{=} \{j \in \{1, \ldots, n\} : i\mathcal{R}j\}, \tag{2-24}$$

further the set $\pi^*(i)= \{i\} \cup \pi^+$ and the set $\sigma^*(i)= \{i\} \cup \sigma^+$

## 2-5 Eigenvalues and -vectors

Just like in conventional algebra square matrices in max-plus algebra have eigenvalues and -vectors. The definition of eigenvalues and -vectors is quite similar

$$A \otimes v = \mu \otimes v,$$

where $A \in \mathbb{R}_{\max}^{n\times n}$, $v \in \mathbb{R}_{\max}^{n}$ and has at least one finite element and $\mu \in \mathbb{R}_{\max}$. Here $\mu$ is called an eigenvalue of $A$ and $v$ an eigenvector of $A$ associated with eigenvalue $\mu$. Square matrices may have more than one eigenvalue. The eigenvectors are never unique. If $v$ is an eigenvector then $a \otimes v$ is an eigenvector too, where $a \in \mathbb{R}$. The set of eigenvectors associated with an eigenvalue span a vector space in the max-plus sense, called the eigenspace. The eigenspace associated with eigenvalue $\mu$ is denoted by $V(A, \mu)$. If it is clear that the eigenvalue of $A$ is unique the eigenspace can be denoted as $V(A)$ and the eigenvalue as $\lambda(A)$.
Given a matrix $A \in \mathbb{R}_{\max}^{n\times n}$ with finite eigenvalue $\mu$, there exists a circuit $\gamma$ in $\mathcal{G}(A)$ such that

$$\mu = \frac{|\gamma|_{\mathrm{w}}}{|\gamma|_1}$$

This means that one of the circuit means is an eigenvalue of $A$, however it does not state from which circuit. To be able to determine which circuit mean is an eigenvalue of $A$ first some concepts need to be defined. A circuit $p$ in $\mathcal{G}(A)$ is called critical if its circuit mean is maximal. This maximal circuit mean is denoted by $\lambda$ and defined as

$$\lambda = \max_{p\in\mathcal{C}(A)} \frac{|p|_{\mathrm{w}}}{|p|_1}, \tag{2-25}$$

where $\mathcal{C}(A)$ is the set of all elementary circuits in $\mathcal{G}(A)$. If $\mathcal{C}(A) = \emptyset$ then $\lambda = -\infty$.

The critical graph of $A$, denoted by $\mathcal{G}^c(A) = (\mathcal{N}^c(A),\mathcal{D}^c(A))$, is the graph corresponding to the nodes and arcs that make up the critical circuits in $\mathcal{G}(A)$. A node $i \in \mathcal{N}^c(A)$ will sometimes be called a critical node and a subpath of a critical circuit will sometimes be called a critical path.

Now matrix $A_\lambda$ will be defined as follows

$$[A_\lambda]_{ij} = a_{ij} - \lambda, \tag{2-26}$$

where $\lambda$ is defined as in 2-25. Matrix $A_\lambda$ will sometimes be called the normalized matrix. By definition the maximum circuit mean of $\mathcal{G}^c(A_\lambda)$ is zero. Using this 2-16 can be used to find $A_\lambda^+$, this should be read as $(A_\lambda)^+$. From this it follows that

$$\forall \eta \in \mathcal{N}^c(A) : [A_\lambda^+]_{\eta\eta} = e = 0. \tag{2-27}$$

This means that every node of the critical graph is in a circuit and that every circuit of the critical graph has weight zero, since $\lambda$ is substracted from all elements. Next $A_\lambda^*$ is defined as

$$A_\lambda^* \stackrel{\text{def}}{=} E \oplus A_\lambda^+ = \bigoplus_{k\geq 0} A_\lambda^{\otimes k}, \tag{2-28}$$

where $A_\lambda^*$ should be read as $(A_\lambda)^*$. Finally let $[B]_{.k}$ denote the $k^{\text{th}}$ column of a matrix $B$.

Now if $A$ is irreducible and $v \in \mathbb{R}_{\max}^n$ is an eigenvector of $A$ associated with finite eigenvalue $\mu$ and has at least one finite element, then all elements have to differ from $\varepsilon$. It can even be stated that if $A$ is irreducible that it only has one eigenvalue. That eigenvalue is denoted by $\lambda(A)$, it is finite and equal to the maximal circuit mean of the circuits in $\mathcal{G}(A)$. In other words

$$\lambda(A) = \max_{p \in \mathcal{C}(A)} \frac{|\gamma|_{\mathrm{w}}}{|\gamma|_1}, \tag{2-29}$$

Using these concepts and definitions the following can be stated. Consider the communication graph $\mathcal{G}(A)$ of matrix $A \in \mathbb{R}_{\max}^{n \times n}$ and let it have a finite maximal circuit mean $\lambda$. Then $\lambda$ is an eigenvalue for the matrix $A$ and the column $[A_\lambda^*]_{.\eta}$ is an eigenvector of A associated with $\lambda$, for any node $\eta$ in $\mathcal{G}^c(A)$. For nodes $i, j$ belonging to $\mathcal{G}^c(A)$ there exists $a \in \mathbb{R}$ such that

$$a \otimes [A_\lambda^*]_i = [A_\lambda^*]_j \tag{2-30}$$

if and only if $i$ and $j$ belong to the same m.s.c.s. of the critical graph.

If $\mathcal{G}^c(A)$ consists of only one m.s.c.s. Then equation (2-30) will hold for any node $i, j$ that belong to the critical graph. This means that the column associated with those nodes are equal to each other, except for a max-plus multiplication, which in turn means the eigenvector is unique.

## 2-6 Solving linear equations

In this section a method for solving linear max-plus equations is discussed.

Using 2-16 and 2-28 $A^*$ can be defined for any matrix $A \in \mathbb{R}_{\max}^{n \times n}$ as follows

$$A^* \stackrel{\text{def}}{=} E \oplus A^+ = \bigoplus_{k \geq 0} A^{\otimes k} \tag{2-31}$$

The $A^*$ matrix can be used to solve the linear max-plus equation $x = A \otimes x \oplus b$, where $A \in \mathbb{R}_{\max}^{n \times n}$, $b \in \mathbb{R}_{\max}^n$ if the communication graph $\mathcal{G}(A)$ has maximum circuit mean less than or equal to $e$. The solution to the equation is $x = A^* \otimes b$. This solution is unique if the circuit weights in $\mathcal{G}(A)$ are negative.

This however only works for square matrices. For non square linear max-plus equations the following equation is considered: $A \otimes x = b$. This does not always have a solution. But there is always a greatest solution to the max-plus inequality $A \otimes x \leq b$, where $A \in \mathbb{R}_{\max}^{m \times n}$ and $b \in \mathbb{R}_{\max}^m$. This solution is called the principal solution and is denoted by $x^*(A, b)$ and is defined as

$$[x^*(A, b)]_j = \min\{b_i - a_{ij} : 1 \leq i \leq m\}, \tag{2-32}$$

for $j \in \{1, \ldots, n\}$.

## 2-7 Max-plus linear systems

In this section max-plus linear (MPL) systems will be discussed. MPL systems are, as the name suggest systems that are linear in max-plus algebra. They are discrete event systems where the events are synchronized, there are no concurrent events and there are no choices to be made that determine the order of events.
These systems are described by

$$\begin{aligned} x(k) &= A(k) \otimes x(k-1) \oplus B(k) \otimes u(k), \\ y(k) &= C(k) \otimes x(k), \end{aligned} \tag{2-33}$$

where $A \in \mathbb{R}_{\max}^{n \times n}$, $B \in \mathbb{R}_{\max}^{n \times m}$ and $C \in \mathbb{R}_{\max}^{p \times n}$, where $m$ is the number of inputs, $n$ is the number of states and $p$ is the number of outputs. Here $k$ is the event counter, the matrices $A$, $B$ and $C$ often indicate the sum of maximizations of transport times, or internal process times, etc. $x(k+1)$ indicates when the internal events happen for the $(k+1)^{\text{th}}$ time, $u(k)$ indicates when the inputs become available for the $k^{\text{th}}$ time and $y(k)$ indicates when the output events happen for the $(k)^{\text{th}}$ time.

In these systems the state-vector is often the output. This means that $y(k) = x(k)$ therefore it is often left out of the equation and the rest is rewritten as

$$x(k) = A(k) \otimes x(k-1) \oplus B(k) \otimes u(k). \tag{2-34}$$

This form is called the explicit model as it only depends on the known values $x(k-1)$, $u(k)$. The implicit form has the following format

$$x(k) = A_0(k) \otimes x(k) \oplus A_1(k) \otimes x(k-1) \oplus B_0(k) \otimes u(k). \tag{2-35}$$

If the matrices $A_0(k)$, $A_1(k)$, and $B_0(k)$ are carfefully constructed such that $A_0^*$ exists, which means $A_0$ is either nilpotent or has average circuit mean of $e$ or less, then $A_0^* \otimes A_1 = A$, $A_0^* B_0(k) = B(k)$ and 2-35 can be rewritten into 2-34.

## 2-8 Switching max-plus linear systems

This section introduces an extension to the MPL-systems: The SMPL-systems. The switching applies to the mode of operation of the discrete event system (DES). The mode is denoted by $l(k) \in \{1, \cdots, n_m\}$ for step $k$. Switching the mode of operation means the system matrices are replaced by others. An SMPL system is described by

$$\begin{aligned} x(k) &= A^{\ell(k)} \otimes x(k-1) \oplus B^{\ell(k)} \otimes u(k), \\ y(k) &= C^{\ell(k)} \otimes x(k), \end{aligned} \tag{2-36}$$

If the output is equal to the state-vector this can be rewritten into the explicit model

$$x(k) = A^{\ell(k)} \otimes x(k-1) \oplus B^{\ell(k)} \otimes u(k), \tag{2-37}$$

where $A^{\ell(k)}$ and $B^{\ell(k)}$ are the system matrices for mode $\ell(k)$. The switching makes it possible for the structure of the system to change, which means the order of events can change. The switching of modes is controlled by a switching mechanism. This mechanism is represented by the switching variable $z(k)$ which may depend on the previous state-vector $x(k-1)$, the previous mode $\ell(k-1)$, the input variable $u(k)$ and a control variable $v(k)$:

$$z(k) = \phi(x(k-1), \ell(k-1), u(k), v(k)) \in \mathbb{R}_{\max}^{n_z}, \tag{2-38}$$

where $\mathbb{R}_{\max}^{n_z}$ is partitioned in $n_m$ subsets $\mathcal{Z}^{(i)}$, $i = 1, \cdots, n_m$. The set to which $z(k)$ belongs determines what mode the system will be in. In other words, if $z(k) \in \mathcal{Z}^{(i)}$, then $\ell(k) = i$.

The implicit model is described by

$$x(k) = A_0^{\ell(k)}(k) \otimes x(k) \oplus A_1^{\ell(k)}(k) \otimes x(k-1) \oplus B_0^{\ell(k)}(k) \otimes u(k). \tag{2-39}$$

For this model it also applies that it can be rewritten into the explicit model of 2-37 if the matrices $A_0^{\ell(k)}(k)$, $A_1^{\ell(k)}(k)$ and $B_0^{\ell(k)}$ are properly constructed. Again $A_0^{*,\ell(k)}(k)$ for each mode $\ell(k)$ has to exist such that $A_0^{*,\ell(k)}(k) \otimes A_1^{\ell(k)}(k) = A^{\ell(k)}$, $A_0^{*,\ell(k)}(k) \otimes B_0^{\ell(k)} = B^{\ell(k)}$.

An important property of SMPL-systems is the maximum growth rate[18] denoted by $\lambda$. Consider an SMPL-system with matrices $A_\beta^{l(k)}$, with $[A_\beta^{l(k)}]_{ij} = [A^{l(k)}]_{ij} - \beta$. The maximum growth rate $\lambda$ of the SMPL system is the smallest $\beta$ for which there exist a max-plus diagonal matrix $P = \text{diag}(p_1, \ldots, p_n)$ with finit diagonal elements $p_i$, such that

$$[P \otimes A_\beta^{l(k)} \otimes P^{\otimes -1}]_{ij} \leq 0, \forall i, j, l, k \tag{2-40}$$

Here the inverse of $P$, denoted by $P^{\otimes -1}$ is equal to $P^{\otimes -1} = \text{diag}(p_1, \ldots, p_n)$. The maximum growth rate $\lambda$ is finite. The upper bound of $\lambda$ is equal to $\max[A^{l(k)}] \forall i, j, l, k$. Finally the maximum growth rate for a max-plus linear system is equivalent to the largest eigenvalue of the matrix $A$.

The idea of SMPL systems in the event-driven domain is parallel to that of piece-wise-affine (PWA)-systems in the time-driven domain. Just like with PWA-system analysis, where the properties of the linear subsystems are analyzed to derive the properties of the PWA-system, the properties of the MPL-subsystems can be used in the analysis of the properties of the SMPL-system.

Chapter 3

# Legged locomotion using max-plus linear systems

## 3-1 Introduction

This chapter deals with the basics of legged locomotion as discussed in section 3-2, where both continuous time and discrete event system (DES) based legged locomotion is explained. In section 3-3 the control structure that is used for the DES legged locomotion is described. Of which the max-plus gait scheduler is discussed in more detail in section 3-4 and in the final section 3-5 the supervisory control is explained.

## 3-2 Basics of legged locomotion

### 3-2-1 Continuous time legged locomotion

Before getting into the details of a complete legged locomotion model, a single leg is considered. The end effector of the leg is its foot. The foot moves in a periodic motion which consists of two phases; an aerial phase, called flight or swing and a ground phase, called stance. During swing the foot moves through the air back to the start of the trajectory. During stance the foot is used to exert force on the floor which then causes the robot to move. This periodic motion can be mapped onto a circle, where the phase of the circle represents the position of the foot as shown in Figure 3-1.

In most cases of legged locomotion more than one leg is considered. In order to achieve locomotion the legs need to be synchronized, which means the foot trajectories need to be synchronized. This can be done using a central pattern generator (CPG) such as a network of phase oscillators as discussed in chapter 2.3 of Holmes et al.[9] and used in Zhang et al.[19]. In the simplest model one oscillator is linked to each leg

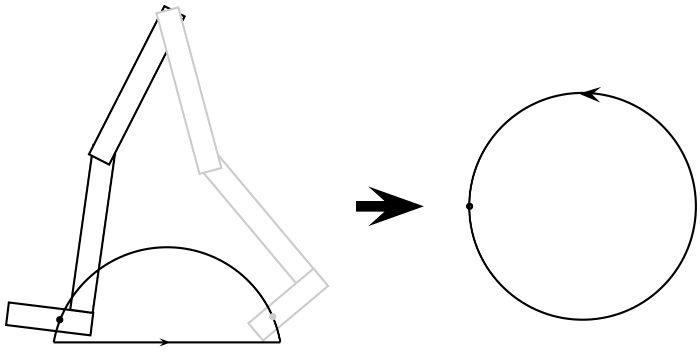

**Figure 3-1:** Periodic motion of a foot during locomotion and its abstract phase representation.

to generate periodic foot trajectories of constant angular velocity. By comparing the phase of the oscillators and using a controller to force a phase difference between them a locomotion pattern or gait is created. A gait is a cyclic motion pattern that produces the locomotion.

Consider the mapped circular motions for legs $i$, for $i \in \{1, \ldots, n\}$, that are traversed with a constant velocity $\omega_i$ . This is denoted by

$$\dot{\phi_i} = \omega_i, \text{ for } i \in \{1, \ldots, n\},$$

where $n$ is the number of legs.

These legs can then be coupled by a function $f_i(\phi_1, \ldots, \phi n)$. This results in the following notation for $\phi_i$

$$\dot{\phi_i} = \omega_i + f_i(\phi_1, \ldots, \phi n), \text{ for } i \in \{1, \ldots, n\}, \tag{3-1}$$

The function $f_i$ works as feedback, returning the phase difference to its desired value. The trajectory that the legs now follow is called a limit cycle and the function $f_i$ is used to make the limit cycle stable.

As an example, consider two legs that should have a 180° phase difference and have the same velocity ($\omega_1 = \omega_2$). This could then be forced by the following equations:

$$\dot{\phi_1} = \omega_1 + w * sin(\phi_1 - \phi_2) \tag{3-2}$$
$$\dot{\phi_2} = \omega_1 + w * sin(\phi_2 - \phi_1) \tag{3-3}$$

where $w \in \mathbb{R}^+$ is a weight that has to be determined based on the highest acceptable deviation from the set phase difference.

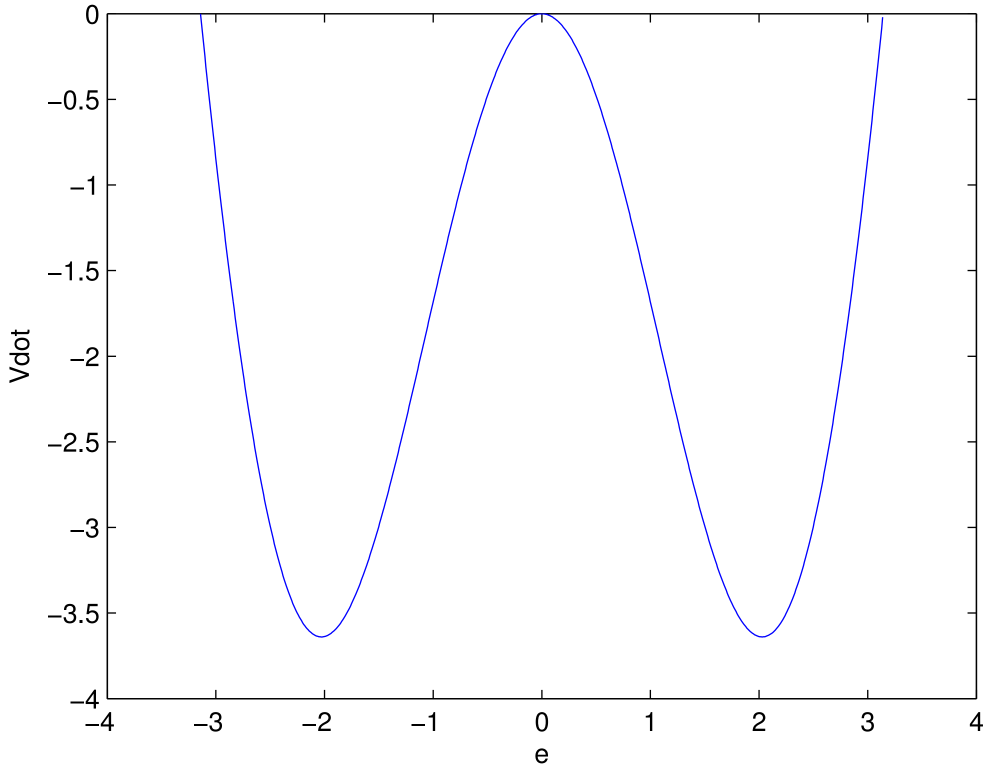


**Figure 3-2:** $\dot{V}$ for the different values of the error e.

The error term e is then defined by

$$\mathrm{e} = \phi_1 - \phi_2 + \pi$$

This error term is stable according to Lyapunov's stability criterion. This can be checked by taking

$$\begin{aligned} V &= \mathrm{e}^2 > 0 \ \forall x \neq 0, V = 0 \text{ for } x = 0 \\ \dot{V} &= 2\mathrm{e}\dot{\mathrm{e}} = -2\mathrm{e} \times \sin(\mathrm{e}) \leq 0 \text{ for } -\pi \leq \mathrm{e} \leq \pi \end{aligned}$$

$\dot{V}$ is shown in Figure 3-2. The reason for only evaluating e over the given interval is because the motion is cyclic and has period $2\pi$.

### 3-2-2 Discrete event system based legged locomotion

The periodic motion of the foot can also be described as a DES. A DES is based on events. In the case of legged locomotion these events happen periodically. Such a period is called a cycle and $k$ denotes what cycle the robot is currently in. The events that happen during the periodic motion of the foot are the touchdown event and lift-off

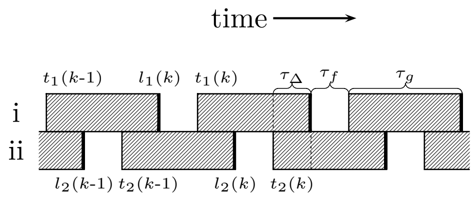


**Figure 3-3:** Time evolution of a bipedal gait. The hatched rectangles represent the leg stance and the solid thick vertical lines represent the lift off events.

event. The touchdown event is the moment the foot touches the ground for the first time in the cycle. The touchdown event in cycle $k$ is denoted by $t(k)$. The lift-off event is the moment the foot lifts off the ground. The lift-off event in cycle $k$ is denoted by $l(k)$. Now that the events have been defined the relation between these events can be defined.

Consider a single leg $i$, $i \in \{1, \ldots, n\}$, where $n$ is the number of legs of the robot. For the leg $i$ the time instance it touches down is equal to the time instance it lifted off the ground for the last time plus the time it stayed in flight, denoted by $\tau_f$.

$$t_i(k) = l_i(k) + \tau_f, \tag{3-4}$$

The time instance leg $i$ lifts off is equal to the time it touched down the last time plus the time it stayed in stance denoted by $\tau_g$. This can then be written as:

$$l_i(k) = t_i(k-1) + \tau_g, \tag{3-5}$$

$t(k-1)$ has been chosen as the touch down time of the previous cycle, that way the equations 3-4 and 3-5 can be used iteratively.

As an example consider a two legged robot with legs 1 and 2, with touchdown events $t_1$ and $t_2$ and lift-off events $l_1$ and $l_2$ respectively. These legs need to be synchronized as shown in Figure 3-3.

The synchronization of the legs is achieved by enforcing constraints on the firing order of the events of the two legs. In order to get the legs synchronized as shown in Figure 3-3, leg 1 can only lift off $\tau_\Delta$ seconds after leg 2 touched down and $\tau_g$ seconds after leg 1 has touched down the last time. For leg 2 the relationship is as follows: leg 2 can only lift off $\tau_\Delta$ seconds after leg 1 touched down and $\tau_g$ seconds after leg 2 has touched down. This can be written as:

$$\begin{aligned}
t_1(k) &= l_1(k) + \tau_f \\
t_2(k) &= l_2(k) + \tau_f \\
l_1(k) &= \max(t_1(k-1) + \tau_g, t_2(k-1) + \tau_\Delta) \\
l_2(k) &= \max(t_2(k-1) + \tau_g, t_1(k) + \tau_\Delta)
\end{aligned}$$

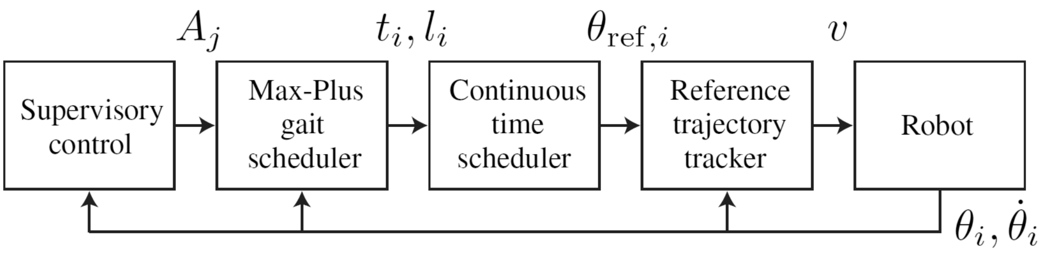


**Figure 3-4:** Block diagram of control structure for a legged robot.

This can be rewritten in max-plus algebra as

$$\begin{array}{rcl} t_1(k) & = & l_1(k) \otimes \tau_f \\ t_2(k) & = & l_2(k) \otimes \tau_f \\ l_1(k) & = & t_1(k-1) \otimes \tau_g \oplus t_2(k-1) \otimes \tau_\Delta \\ l_2(k) & = & t_2(k-1) \otimes \tau_g \oplus t_1(k) \otimes \tau_\Delta \end{array}, \tag{3-6}$$

which is linear in max-plus algebra. This DES only determines the synchronization between the legs, in order to get the robot moving a complete control structure is needed, which is discussed in the next section.

## 3-3 Control Structure

The control structure discussed in this section is shown in Figure 3-4.

- At the top of the control structure stands the supervisory control. This part of the control structure determines when to switch gaits and what gait it should switch to.
- Just below that is the max-plus gait scheduler. The max-plus gait scheduler creates the sequence of events based on the gait chosen in the supervisory control.
- The sequence of events is then used to create a continuous-time reference trajectory in the continuous time scheduler. It does this by taking a collection $p$ of the state vector $x(k) \in \mathbb{R}^{2n}_{\max}$ and the time instant $\tau \in \mathbb{R}^+$, derived from the internal clock of the robot and maps it into a piecewise linear trajectory. Symbolically this is represented by:

  $$\theta_{\mathrm{ref}} : \mathbb{R}^+ \times (\mathbb{R}^{2n}_{\max})^p \to (S^1)^n$$

  The actual piecewise linear equation is defined by:

  $$\theta_{\mathrm{ref},i}(\tau) := \begin{cases} \dfrac{\theta_l(t_i(k_{2i-1}) - \tau) + (\theta_t + 2\pi)(\tau - l_i(k_{2i}))}{t_i(k_{2i-1}) - l_i(k_{2i})} & \text{if } \tau \in [l_i(k_{2i}), t_i(k_{2i-1})] \\ \dfrac{\theta_t(l_i(k_{2i}+1) - \tau) + \theta_l(\tau - t_i(k_{2i-1}))}{l_i(k_{2i}+1) - t_i(k_{2i-1})} & \text{if } \tau \in [t_i(k_{2i-1}), l_i(k_{2i}+1)] \end{cases} \tag{3-7}$$

| | |
|---|---|
| $x(k)$ | Full state vector of the touch down and lift off events. |
| $t_i(k)$ | Touch down time for leg $i$ at iteration $k$. |
| $l_i(k)$ | Lift off time for leg $i$ at iteration $k$. |
| $i$ | Index for legs. |
| $\theta_t$ | Leg touchdown angle. |
| $\theta_l$ | Leg lift off angle. |
| $\tau$ | Current time instant. |
| $\tau_f$ | Time leg spends in flight (swing). |
| $\tau_g$ | Time leg spends on the ground (stance). |
| $\tau_\Delta$ | Double stance time |
| $k_j$ | Index function for each state vector element. |

**Table 3-1:** State variables and gait parameters

This is for $\theta_t<\theta_l$, which have been defined in 3-1. The function $\theta_{\text{ref}}$ take a $p$-collection of events since it is not necessary for the intervals $[l_i(k_{2i}), t_i(k_{2i-1})]$ and $[t_i(k_{2i-1}), l_i(k_{2i}+1)]$ to overlap for all legs. The event indices $\{k_j\} \in \mathbb{N}^8$ are chosen for each leg such that the time $\tau$ lies in the proper interval

- The last part of the control structure is the robot itself, where a PD controller is used to force the legs to follow the reference trajectory.

Since this thesis is about switching gaits using switching max-plus linear (SMPL) systems, only the max-plus gait scheduler and the supervisory control will be discussed in more detail.

## 3-4 Max-plus gait scheduler

The symbols that are used in this subsection and the following one are defined in Table 3-1.

Consider the relations written down in equation 3-4 and 3-5 and let the state vector for an $n$-legged robot be defined by

$$x(k) = [\underbrace{t_1(k) \cdots t_n(k)}_{t(k)}\, \underbrace{l_1(k) \cdots l_n(k)}_{l(k)}]^\top$$

The system equations can then be written as an implicit model as described by equation 2-39 and repeated here for convenience

$$x(k) = A_0^{\ell(k)}(k) \otimes x(k) \oplus A_1^{\ell(k)}(k) \otimes x(k-1) \oplus B_0^{\ell(k)}(k) \otimes u(k), \tag{3-8}$$

the term $B_0^{\ell(k)}(k) \otimes u(k)$ can be left out since there is no input $u(k)$.

The system equations then take the form of

$$\begin{bmatrix} t(k) \\ l(k) \end{bmatrix} = \left[\begin{array}{c|c} \mathcal{E} & \tau_f \otimes E \\ \hline \mathcal{E} & \mathcal{E} \end{array}\right] \otimes \begin{bmatrix} t(k) \\ l(k) \end{bmatrix} \oplus \left[\begin{array}{c|c} E & \mathcal{E} \\ \hline \tau_g \otimes E & E \end{array}\right] \otimes \begin{bmatrix} t(k-1) \\ l(k-1) \end{bmatrix} \tag{3-9}$$

Two $E$ matrices were added to the $A_1$ matrix which adds the requirement that the current touchdown event does not happen before the previous touchdown event and the current lift-off event does not happen before the previous lift-off event. Next the leg synchronization as described in equation 3-6 are added to this equation by adding the matrices $P$ and $Q$, which will be explained after this. The resulting implicit model is then defined as:

$$\begin{bmatrix} t(k) \\ l(k) \end{bmatrix} = \left[\begin{array}{c|c} \mathcal{E} & \tau_f \otimes E \\ \hline P & \mathcal{E} \end{array}\right] \otimes \begin{bmatrix} t(k) \\ l(k) \end{bmatrix} \oplus \left[\begin{array}{c|c} E & \mathcal{E} \\ \hline \tau_g \otimes E \oplus Q & E \end{array}\right] \otimes \begin{bmatrix} t(k-1) \\ l(k-1) \end{bmatrix} \tag{3-10}$$

This model can be rewritten as explicit model by using the solution of the the max-plus linear equation $x(k) = B \otimes x(k) \oplus c$, which is $x = B^* \otimes c$. Substitute $B$ for $A_0$ and $c$ for $A_1 \otimes x(k-1)$, this results in

$$\begin{aligned} x(k) &= A_0^* \otimes A_1 \otimes x(k-1) \\ x(k) &= A \otimes x(k-1), \end{aligned} \tag{3-11}$$

where $A = A_0^* \otimes A_1$ is called the system matrix, $A_0^*$ has to exist in order to be able to solve this. $A_0^*$ exists if $P$ is nilpotent. The proof of this can be found in Lopes et all.[15].

Next the process of building the matrices $P$ and $Q$ will be explained. Before the process of building these matrices can be explained a new way of notating the order of events has to be introduced.

Consider an $n$-legged system and let $L_1, \ldots, L_m$ be sets of integers such that

$$\bigcup_{p=1}^{m} L_p = \{1, \ldots, n\}$$
$$\forall i \neq j, l_i \cap l_j = \emptyset.$$

$L_p$ is considered to contain the indices of a set of legs that recirculates simultaneously. Define $r_p = \#L_p$. A gait $\mathbf{G}$ is defined as an ordering relation of groups of legs:

$$\mathbf{G} = L_1 \prec L_2 \prec \cdots \prec L_m. \tag{3-12}$$

Equation 3-12 states that the set of legs $L_i$ will lift off and touch down simultaneously, they precede the lift off of the set of legs $L_i + 1$. The set of legs $L_i + 1$ lift off once all legs in the set $L_i$ have touched down and have been on the ground for at least $\tau_\Delta$.

As an example consider a hexapod robot. A hexapod can have three different types of gaits, assuming the number of legs in each set is the same, these are the tripod gait; which is a gait of two sets of three legs, the quadruped gait; which consists of three sets of two legs, and the quintuped gait; which consists of six sets of one leg. The names of these gaits are derived from the minimum number of legs that are on the ground. Since only one set of legs can be in flight at the same time, the minimum number of legs for a tripod gait is three, for a quadruped gait four and for a quintuped gait five. Examples of these gaits are:

$$\begin{aligned} \mathbf{G}_{\text{tripod}} &= \{1,4,5\} \prec \{2,3,6\} \\ \mathbf{G}_{\text{quadruped}} &= \{1,4\} \prec \{5,2\} \prec \{3,6\} \\ \mathbf{G}_{\text{quintuped}} &= 1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6 \end{aligned}$$

Using this notation a systematic way of generating matrices $P$ and $Q$ in equation 3-10 is possible. Consider the gait $\mathbf{G}$ as defined in equation 3-12 then $\forall j \in \{1, \dots, m-1\}$, $\forall p \in L_j + 1$ and $\forall q \in L_j$

$$[P]_{p,q} = \tau_\Delta. \tag{3-13}$$

and $\forall p \in L_1$ and $\forall q \in L_m$

$$[Q]_{p,q} = \tau_\Delta, \tag{3-14}$$

where all other entries of $P$ and $Q$ are $\varepsilon$.

By placing $\tau_\Delta$ at these positions in $P$ the lift-off of the legs of set $L_i, \forall i \in \{2, \dots, m\}$ are forced to happen $\tau_\Delta$ after the legs of set $L_{i-1}$ touch down. This results in an overlap of the stance of the two sets of $\tau_\Delta$. The position of $\tau_\Delta$ in $Q$ enforces that the lift-off of the legs of set $L_1$ in the current cycle happens $\tau_\Delta$ after the legs of set $L_m$ touch down in the previous cycle. This ensures that there is an overlap of $\tau_\Delta$ of the stance of the two sets. This way the synchronization of the full cycle is enforced.

As an example consider the gait $\mathbf{G} = \{1, 3\} \prec \{2, 4\}$. The matrices $P$ and $Q$ are

$$P = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \tau_\Delta & \varepsilon & \tau_\Delta & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \tau_\Delta & \varepsilon & \tau_\Delta & \varepsilon \end{bmatrix}$$

$$Q = \begin{bmatrix} \varepsilon & \tau_\Delta & \varepsilon & \tau_\Delta \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \tau_\Delta & \varepsilon & \tau_\Delta \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix}$$

## 3-5 Supervisory Control

In Lopes et al.[15] there is no autonomous supervisory control yet. The gaits to switch to are selected by hand at times chosen by the person controlling the robot. Some results on gait switching are presented in the papers of Lopes et al.[13][15] though; Using the notation presented in section 3-4 an ordered pairs such as $\{1\} \prec \{2\} \prec \{3\} \prec \{4\} \prec \{5\} \prec \{6\}$ and $\{4\} \prec \{5\} \prec \{6\} \prec \{1\} \prec \{2\} \prec \{3\}$ result in different synchronization matrices $P$ and $Q$ but are equal up to an 'event shift' in the state variables. For quintuped gaits for a hexapod this means there are $5! = 120$ different gaits and a total of $6! = 720$ different gait parameterizations. Switching from 'structurally' different gait classes, for example from a tripod to quadruped gait, different transitions will occur based on the different gaits chosen from these classes. This can result in different stance times for each of the legs while switching gaits, this can cause problems such as unwanted turning of the robot and slipping of the legs. It is therefore important to find the optimal gait to switch to. Where optimal refers to minimizing the stance velocity variation of the legs that are on the ground at the same time. The 'optimal' gait to transition to can be found by looking for the gait which steady state behaviour 'resembles' the most with that of the current gait. This is done by looking at gaits that

**Figure 3-5:** Gait switch from $\{1,4,5\},\{2,3,6\}$ to $\{1,4\},\{5,2\},\{3,6\}$. The gray blocks represent the stance and the white blocks represent the flight of the legs.

are in 'structurally' different gait classes but where the ordering in the legs is as similar as possible. Such as a gait switch from $\{1,4,5\} \prec \{2,3,6\}$ to $\prec \{1,4\} \prec \{5,2\} \prec \{3,6\}$ which is shown in Figure 3-5. There is no mathematical proof these switches are indeed the optimal transitions, but extensive simulation results corroborate this hypothesis.

# Chapter 4

# Steady state behaviour

## 4-1 Introduction

In section 4-2 of this chapter the structure of the system matrices, that are made with the method discussed in section 3-4, is found using similarity transformations. This section is based on the work of Lopes et al. [14]. In section 4-3 the structure of the matrix is used to proof that the eigenvector and eigenvalue are unique under certain conditions. In the final section of this chapter, section 4-4, conclusions are drawn about the steady state behaviour of this class of matrices.

## 4-2 Matrix structure

Before the general structure of the matrices, discussed in section 3-4, is analyzed, the function needs to be defined. $\flat$ transforms a gait into a vector of integers:

$$\flat : \{[L_1]_1, \ldots, [L_1]_{i_1}\} \prec \ldots \prec \{[L_m]_1, \ldots, [L_m]_{i_m}\} \mapsto [[L_1]_1, \ldots, [L_1]_{i_1}, [l_m]_1, \ldots, [L_m]_{i_m}]^\top,$$

it is assumed here that the ordering of equation 3-12 is represented as a set of sets. As an example consider the gait $\mathbf{G} = \{1, 4, 5\} \prec \{2, 3, 6\}$, then $\flat(\mathbf{G}) = [1, 4, 5, 2, 3, 6]^\top$.

A gait $\bar{\mathbf{G}}$ is called a normal gait if the elements of $\flat(\bar{\mathbf{G}})$ are sorted in such a way that they are strictly increasing from $[\flat(\bar{\mathbf{G}})]_1$ to $[\flat(\bar{\mathbf{G}})]_n$, where $n$ is the number of legs of the robot (which is also the number of elements of $\flat(\bar{\mathbf{G}})$).

For any gait $\mathbf{G}$ a similarity matrix $C$ can be defined that transforms the system matrix $A$ of $\mathbf{G}$ into the system matrix $\bar{A}$ of the normal gait $\bar{\mathbf{G}}$. This similarity matrix has the following structure:

$$C = \begin{bmatrix} \bar{C} & \mathcal{E} \\ \mathcal{E} & \bar{C} \end{bmatrix}, \tag{4-1}$$

where $\forall i, j \in 1, \ldots, n$:

$$[\bar{C}]_{i,j} = \begin{cases} e & \text{if } [\flat(\mathbf{G})]_i = j \\ \varepsilon & \text{otherwise} \end{cases} \tag{4-2}$$

The structure of $C$ is such that

$$C \otimes C^\top = C^\top \otimes C = E$$

The similarity transformation can be seen as redefining the numbering of the legs in order to get a highly structured matrix $\bar{A}$, which makes analysis of the matrix much easier. $\bar{A}$ can be obtained using the following similarity transformation

$$\bar{A} = C \otimes A \otimes C^\top. \tag{4-3}$$

Before the structure of this matrix can be discussed the structure 'normal' counterparts of $P$ and $Q$ need analysed. These 'normal' counterparts can be obtained via the following similarity transformation:

$$\bar{P} = \bar{C} \otimes P \otimes \bar{C}^\top \tag{4-4}$$

$$\bar{Q} = \bar{C} \otimes Q \otimes \bar{C}^\top \tag{4-5}$$

$\bar{P}$ and $\bar{Q}$ have the following structure:

$$\bar{P} = \begin{bmatrix} \mathcal{E} & & \cdots & & \mathcal{E} \\ \tau_\Delta \otimes \mathbb{1}_{r_2 \times r_1} & \mathcal{E} & & & \\ \mathcal{E} & \tau_\Delta \otimes \mathbb{1}_{r_3 \times r_2} & \mathcal{E} & & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathcal{E} & \cdots & \mathcal{E} & \tau_\Delta \otimes \mathbb{1}_{r_m \times r_{m-1}} & \mathcal{E} \end{bmatrix}, \tag{4-6}$$

$$\bar{Q} = \begin{bmatrix} \mathcal{E}_{r_1 \times (n-r_m)} & \tau_\Delta \otimes \mathbb{1}_{r_1 \times r_m} \\ \mathcal{E}_{(n-r_1) \times (n-r_m)} & \mathcal{E}_{(n-r_1) \times (r_m)} \end{bmatrix}, \tag{4-7}$$

where $[\mathbb{1}]_{ij} = e$, $\forall i \in \{1, \ldots, n\}$, $\forall j \in \{1, \ldots, m\}$, where $\mathbb{1}_{n \times m} \in \mathbb{R}_{\max}^{n \times m}$ and where $m$ is the total number of sets of legs.

Replacing $P$ and $Q$ for $\bar{P}$ and $\bar{Q}$ in equation 3-10 gives the matrices $\bar{A}_0$ and $\bar{A}_1$. A closed form solution can be found for $\bar{A}_0{}^*$ by observation of the structure of $\bar{A}_0$:

$$\bar{A}_0{}^* = \bigoplus_{k=0}^{2 \times m - 1} \bar{A}_0{}^{\otimes k}$$

the sum is limited to $(2 \times m - 1)$ elements due to the structure of the matrix $\bar{A}_0$ and its powers, which have the following structure:

$$\bar{A}_0{}^{\otimes k} = \begin{cases} \begin{bmatrix} \mathcal{E} & \tau_f^{\otimes \frac{k+1}{2}} \otimes \bar{P}^{\otimes \frac{k-1}{2}} \\ \tau_f^{\otimes \frac{k-1}{2}} \otimes \bar{P}^{\otimes \frac{k+1}{2}} & \mathcal{E} \end{bmatrix} & \text{if } k \text{ is odd} \\ \begin{bmatrix} \tau_f^{\otimes \frac{k}{2}} \otimes \bar{P}^{\otimes \frac{k}{2}} & \mathcal{E} \\ \mathcal{E} & \tau_f^{\otimes \frac{k}{2}} \otimes \bar{P}^{\otimes \frac{k}{2}} \end{bmatrix} & \text{if } k \text{ is even} \end{cases}$$

Because $\bar{P}$ has a subdiagonal block structure, $\bar{P}^{\otimes l} = \mathcal{E}, \forall l \geq m$, which means that $\bar{A_0}^{\otimes k} = \mathcal{E}, \forall \geq (2 \times m)$. The resulting matrix $\bar{A_0}^*$ has the following structure

$$\bar{A_0}^* = \begin{bmatrix} W & \tau_f \otimes W \\ \bar{W} & W \end{bmatrix}$$

where

$$\begin{aligned} W &= (\tau_f \otimes \bar{P})^* && (4\text{-}8) \\ &= \bigoplus_{k=0}^{m-1} (\tau_f \otimes \bar{P})^{\otimes k}, && (4\text{-}9) \end{aligned}$$

this sum of powers is limited to the power of $(m-1)$ again because of the structure of $\bar{P}$.

$$W = \begin{bmatrix} E_{r_1} & & & \cdots & & \mathcal{E} \\ \tau_\delta \otimes \mathbb{1}_{r_2 \times r_1} & E_{r_2} & & & & \\ \tau_\delta^{\otimes 2} \otimes \mathbb{1}_{r_3 \times r_1} & \tau_\delta \otimes \mathbb{1}_{r_3 \times r_2} & E_{r_3} & & & \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ & & & & & \\ \tau_\delta^{\otimes (m-1)} \otimes \mathbb{1}_{r_m \times r_1} & \cdots & & \tau_\delta^{\otimes 2} \otimes \mathbb{1}_{r_m \times r_{m-2}} & \tau_\delta \otimes \mathbb{1}_{r_m \times r_{m-1}} & E_{r_m} \end{bmatrix}, \quad (4\text{-}10)$$

where $\tau_\delta = \tau_f \otimes \tau_\Delta$ and

$$\bar{W} = \tau_\Delta \otimes \begin{bmatrix} \mathcal{E} & & & \cdots & & \mathcal{E} \\ \mathbb{1}_{r_2 \times r_1} & \mathcal{E} & & & & \\ \tau_\delta \otimes \mathbb{1}_{r_3 \times r_1} & \mathbb{1}_{r_3 \times r_2} & \mathcal{E} & & & \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ & & & & & \\ \tau_\delta^{\otimes (m-2)} \otimes \mathbb{1}_{r_m \times r_1} & \cdots & & \tau_\delta \otimes \mathbb{1}_{r_m \times r_{m-2}} & \mathbb{1}_{r_m \times r_{m-1}} & \mathcal{E} \end{bmatrix}, \quad (4\text{-}11)$$

Using $\bar{A_0}^*$ and $\bar{A}_1$ an expression for $\bar{A}$ can be found:

$$\begin{aligned} \bar{A} &= \bar{A_0}^* \otimes \bar{A}_1 \\ &= \begin{bmatrix} W & \tau_f \otimes W \\ \bar{W} & W \end{bmatrix} \otimes \begin{bmatrix} E & \mathcal{E} \\ \tau_g \otimes E \oplus \bar{Q} & E \end{bmatrix} \\ &= \begin{bmatrix} W \oplus \tau_f \otimes \tau_g \otimes W \oplus \tau_f \otimes W \otimes \bar{Q} & \tau_f \otimes W \\ \bar{W} \oplus \tau_g \otimes W \oplus W \otimes \bar{Q} & W \end{bmatrix} && (4\text{-}12) \end{aligned}$$

Now define

$$\begin{aligned} V &= W \otimes \bar{Q} && (4\text{-}13) \\ &= \begin{bmatrix} \mathcal{E}_{r_1 \times (n-r_m)} & \tau_\Delta \otimes \mathbb{1}_{r_1 \times r_m} \\ \mathcal{E}_{r_2 \times (n-r_m)} & \tau_\Delta \otimes \tau_\delta \otimes \mathbb{1}_{r_2 \times r_m} \\ \vdots & \vdots \\ \mathcal{E}_{r_m \times (n-r_m)} & \tau_\Delta \otimes \tau_\delta^{\otimes (m-1)} \otimes \mathbb{1}_{r_m \times r_m} \end{bmatrix} && (4\text{-}14) \end{aligned}$$

and since $\mu \otimes W \geq W$ for any $\mu \geq 0$ equation 4-12 can be simplified to

$$\bar{A} = \begin{bmatrix} \tau_f \otimes (\tau_g \otimes W \oplus V) & \tau_f \otimes W \\ \tau_g \otimes W \oplus V & W \end{bmatrix}. \tag{4-15}$$

By inspecting the structure of $\bar{A}$ it is clear that rows and columns $n - r_m$ to $n$ are non-$\varepsilon$ and thus any node can reach any other node via the nodes associated to these rows and columns. This means $\bar{A}$ is irreducible and thus only has one unique eigenvalue.

As an example of this transformation consider the gait $\mathbf{G} = \{1,3\} \prec \{2,4\}$, with $\tau_\delta = \tau_\Delta \otimes \tau_f$ and $\tau_\gamma = \tau_g \otimes \tau_f$, the matrices $P, Q$ and $A$ are as follows:

$$P = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \tau_\Delta & \varepsilon & \tau_\Delta & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \tau_\Delta & \varepsilon & \tau_\Delta & \varepsilon \end{bmatrix}$$

$$Q = \begin{bmatrix} \varepsilon & \tau_\Delta & \varepsilon & \tau_\Delta \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \tau_\Delta & \varepsilon & \tau_\Delta \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix}$$

$$A = \left[\begin{array}{cccc|cccc} \tau_\gamma & \tau_\delta & \varepsilon & \tau_\delta & \tau_f & \varepsilon & \varepsilon & \varepsilon \\ \tau_\gamma \otimes \tau_\delta & \tau_\gamma \oplus \tau_\delta^{\otimes 2} & \tau_\gamma \otimes \tau_\delta & \tau_\delta^{\otimes 2} & \tau_f \otimes \tau_\delta & \tau_f & \tau_f \otimes \tau_\delta & \varepsilon \\ \varepsilon & \tau_\delta & \tau_\gamma & \tau_\delta & \varepsilon & \varepsilon & \tau_f & \varepsilon \\ \tau_\gamma \otimes \tau_\delta & \tau_\delta^{\otimes 2} & \tau_\gamma \otimes \tau_\delta & \tau_\gamma \oplus \tau_\delta^{\otimes 2} & \tau_f \otimes \tau_\delta & \varepsilon & \tau_f \otimes \tau_\delta & \tau_f \\ \hline \tau_g & \tau_\Delta & \varepsilon & \tau_\Delta & e & \varepsilon & \varepsilon & \varepsilon \\ \tau_g \otimes \tau_\delta & \tau_g \oplus \tau_\Delta \otimes \tau_\delta & \tau_g \otimes \tau_\delta & \tau_\Delta \otimes \tau_\delta & \tau_\delta & e & \tau_\delta & \varepsilon \\ \varepsilon & \tau_\Delta & \tau_g & \tau_\Delta & \varepsilon & \varepsilon & e & \varepsilon \\ \tau_g \otimes \tau_\delta & \tau_\Delta \otimes \tau_\delta & \tau_g \otimes \tau_\delta & \tau_\gamma \oplus \tau_\Delta \otimes \tau_\delta & \tau_\delta & \varepsilon & \tau_\delta & e \end{array}\right]$$

The matrix $\bar{C}$ that is obtained is:

$$\bar{C} = \begin{bmatrix} e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e \end{bmatrix}$$

The matrix $C$ is then:

$$C = \begin{bmatrix} e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \end{bmatrix}$$

The normal matrices $\bar{P}$ and $\bar{Q}$ are then:

$$\begin{aligned}\bar{P} &= \begin{bmatrix} e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e \end{bmatrix} \otimes \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \tau_\Delta & \varepsilon & \tau_\Delta & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \tau_\Delta & \varepsilon & \tau_\Delta & \varepsilon \end{bmatrix} \otimes \begin{bmatrix} e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e \end{bmatrix}^\top \\ &= \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \tau_\Delta & \tau_\Delta & \varepsilon & \varepsilon \\ \tau_\Delta & \tau_\Delta & \varepsilon & \varepsilon \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\bar{Q} &= \begin{bmatrix} e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e \end{bmatrix} \otimes \begin{bmatrix} \varepsilon & \tau_\Delta & \varepsilon & \tau_\Delta \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \tau_\Delta & \varepsilon & \tau_\Delta \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} \otimes \begin{bmatrix} e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e \end{bmatrix}^\top \\ &= \begin{bmatrix} \varepsilon & \varepsilon & \tau_\Delta & \tau_\Delta \\ \varepsilon & \varepsilon & \tau_\Delta & \tau_\Delta \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix}\end{aligned}$$

Inserting $\bar{P}$ and $\bar{Q}$ into equation 3-10 results in the following normal matrices $\bar{A}_0$ and $\bar{A}_1$:

$$\bar{A}_0 = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \tau_f & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \tau_f & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \tau_f & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \tau_f \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \tau_\Delta & \tau_\Delta & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \tau_\Delta & \tau_\Delta & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix}$$

$$\bar{A}_1 = \begin{bmatrix} e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \tau_g & \varepsilon & \tau_\Delta & \tau_\Delta & e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \tau_g & \tau_\Delta & \tau_\Delta & \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \tau_g & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \tau_g & \varepsilon & \varepsilon & \varepsilon & e \end{bmatrix}$$

From this $\bar{A}_0^*$ can be found using the powers of $\bar{A}_0$:

$$\bar{A}_0^{\otimes 0} = E,$$

$$\bar{A}_0^{\otimes 1} = \bar{A}_0,$$

$$\bar{A}_0^{\otimes 2} = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \tau_\delta & \tau_\delta & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \tau_\delta & \tau_\delta & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \tau_\delta & \tau_\delta & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \tau_\delta & \tau_\delta & \varepsilon & \varepsilon \end{bmatrix}$$

$$\bar{A}_0^{\otimes 3} = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \tau_f \otimes \tau_\delta & \tau_f \otimes \tau_\delta & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \tau_f \otimes \tau_\delta & \tau_f \otimes \tau_\delta & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix}$$

and

$$\bar{A}_0^{\otimes 4} = \mathcal{E}.$$

$\bar{A}_0^*$ is then

$$\begin{aligned} \bar{A}_0^* &= E \oplus \bar{A}_0 \oplus \bar{A}_0^{\otimes 2} \oplus \bar{A}_0^{\otimes 3} \\ &= \begin{bmatrix} e & \varepsilon & \varepsilon & \varepsilon & \tau_f & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \tau_f & \varepsilon & \varepsilon \\ \tau_\delta & \tau_\delta & e & \varepsilon & \tau_f \otimes \tau_\delta & \tau_f \otimes \tau_\delta & \tau_f & \varepsilon \\ \tau_\delta & \tau_\delta & \varepsilon & e & \tau_f \otimes \tau_\delta & \tau_f \otimes \tau_\delta & \varepsilon & \tau_f \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\ \tau_\Delta & \tau_\Delta & \varepsilon & \varepsilon & \tau_\delta & \tau_\delta & e & \varepsilon \\ \tau_\Delta & \tau_\Delta & \varepsilon & \varepsilon & \tau_\delta & \tau_\delta & \varepsilon & e \end{bmatrix} \end{aligned}$$

With $\bar{A}_0^*$ the system matrix $\bar{A}$ can be found:

$$\begin{aligned}
\bar{A} &= \bar{A}_0^* \otimes \bar{A}_1 \\
&= \begin{bmatrix}
e & \varepsilon & \varepsilon & \varepsilon & \tau_f & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \tau_f & \varepsilon & \varepsilon \\
\tau_\delta & \tau_\delta & e & \varepsilon & \tau_f \otimes \tau_\delta & \tau_f \otimes \tau_\delta & \tau_f & \varepsilon \\
\tau_\delta & \tau_\delta & \varepsilon & e & \tau_f \otimes \tau_\delta & \tau_f \otimes \tau_\delta & \varepsilon & \tau_f \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\
\tau_\Delta & \tau_\Delta & \varepsilon & \varepsilon & \tau_\delta & \tau_\delta & e & \varepsilon \\
\tau_\Delta & \tau_\Delta & \varepsilon & \varepsilon & \tau_\delta & \tau_\delta & \varepsilon & e
\end{bmatrix} \otimes \dots \\
&\quad \begin{bmatrix}
e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\tau_g & \varepsilon & \tau_\Delta & \tau_\Delta & e & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \tau_g & \tau_\Delta & \tau_\Delta & \varepsilon & e & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \tau_g & \varepsilon & \varepsilon & \varepsilon & e & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \tau_g & \varepsilon & \varepsilon & \varepsilon & e
\end{bmatrix} \\
&= C \otimes A \otimes C^\top
\end{aligned}$$

Which results in:

$$\bar{A} = \left[\begin{array}{cccc|cccc}
\tau_\gamma & \varepsilon & \tau_\delta & \tau_\delta & \tau_f & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \tau_\gamma & \tau_\delta & \tau_\delta & \varepsilon & \tau_f & \varepsilon & \varepsilon \\
\tau_\gamma \otimes \tau_\delta & \tau_\gamma \otimes \tau_\delta & \tau_\gamma \oplus \tau_\delta^{\otimes 2} & \tau_\delta^{\otimes 2} & \tau_f \otimes \tau_\delta & \tau_f \otimes \tau_\delta & \tau_f & \varepsilon \\
\tau_\gamma \otimes \tau_\delta & \tau_\gamma \otimes \tau_\delta & \tau_\delta^{\otimes 2} & \tau_\gamma \oplus \tau_\delta^{\otimes 2} & \tau_f \otimes \tau_\delta & \tau_f \otimes \tau_\delta & \varepsilon & \tau_f \\
\hline
\tau_g & \varepsilon & \tau_\Delta & \tau_\Delta & e & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \tau_g & \tau_\Delta & \tau_\Delta & \varepsilon & e & \varepsilon & \varepsilon \\
\tau_g \otimes \tau_\delta & \tau_g \otimes \tau_\delta & \tau_g \oplus \tau_\Delta \otimes \tau_\delta & \tau_\Delta \otimes \tau_\delta & \tau_\delta & \tau_\delta & e & \varepsilon \\
\tau_g \otimes \tau_\delta & \tau_g \otimes \tau_\delta & \tau_\Delta \otimes \tau_\delta & \tau_g \oplus \tau_\Delta \otimes \tau_\delta & \tau_\delta & \tau_\delta & \varepsilon & e
\end{array}\right]$$

which as expected has the structure of matrix $\bar{A}$ of equation 4-15.

## 4-3 Unique steady state behaviour

### 4-3-1 Existance

Consider the following assumptions, which, in practice, are always satisfied;
A1 $\tau_f, \tau_g > 0$
A2 $\tau_\gamma \leq \tau_\delta^{\otimes m}$

*theorem 1*: If assumption A1 is satisfied then the matrix $A$ defined by equations 3-10, 3-11 has a unique eigenvalue $\lambda = \tau_\delta^{\otimes m} \oplus \tau_\gamma$, where $\tau_\delta = \tau_\Delta \otimes \tau_f$ and $\tau_\gamma = \tau_g \otimes \tau_f$ and an eigenvector $v$ defined by

$$\begin{aligned} \forall j \in \{1, \ldots, m\}, \forall q \in L_j : \quad [v]_q &= \tau_f \otimes \tau_\delta^{\otimes j-1} \\ [v]_{q+n} &= \tau_\delta^{\otimes j-1} \end{aligned}$$

and let $[\bar{v}]_q = [v]_{q+n}$ for all $j$ and $q \in L_j$. If assumption A2 is also satisfied, then this eigenvector is unique.

Proving that $\lambda$ is an eigenvalue of $A$ and $v$ is one of its associated eigenvectors can be done by replacing $x(k-1)$ with $v$ and $x(k)$ by $\lambda \otimes v$ in equation 3-10 and show that it holds.

$$\begin{aligned} \lambda \otimes \begin{bmatrix} \tau_f \otimes \bar{v} \\ \bar{v} \end{bmatrix} &= \lambda \otimes \left[\begin{array}{c|c} \mathcal{E} & \tau_f \otimes E \\ \hline P & \mathcal{E} \end{array}\right] \otimes v \oplus \left[\begin{array}{c|c} E & \mathcal{E} \\ \hline \tau_g \otimes E \oplus Q & E \end{array}\right] \otimes v \\ &= \left[\begin{array}{c|c} E & \lambda \otimes \tau_f \otimes E \\ \hline \lambda \otimes P \oplus \tau_g \otimes E \oplus Q & E \end{array}\right] \otimes \begin{bmatrix} \tau_f \otimes \bar{V} \\ \bar{V} \end{bmatrix} \end{aligned}$$

This results in the following two expressions:

$$\lambda \otimes \tau_f \otimes \bar{v} = \tau_f \otimes \bar{v} \oplus \lambda \otimes \tau_f \otimes \bar{v} \tag{4-16}$$
$$\lambda \otimes \bar{v} = \tau_f \otimes (\lambda \otimes P \oplus \tau_g \otimes E \oplus Q) \otimes \bar{v} \oplus \bar{v} \tag{4-17}$$

Eventhough $\tau_\Delta$ can be negative $\lambda > 0$, because $\tau_\gamma > 0$, therefore equation 4-16 always holds. This means only equation 4-17 needs to be checked if it always holds. Equation 4-17 can be simplified, due to $\tau_\gamma > 0$, to:

$$\lambda \otimes \bar{v} = \tau_\gamma \otimes \bar{v} \oplus \tau_f \otimes (\lambda \otimes P \oplus Q) \otimes \bar{v} \tag{4-18}$$

Let $\tau_\Delta \otimes P_0 = P$ and $\tau_\Delta \otimes Q_0 = Q$, this means all entries of $P$ and $Q$ are either $e$ or $\varepsilon$. Since $\lambda = \tau_\delta^{\otimes m} \oplus \tau_\gamma$ and $\tau_\delta = \tau_f \otimes \tau_\Delta$ equation 4-18 can be rewritten as:

$$(\tau_\delta^{\otimes m} \oplus \tau_\gamma) \otimes \bar{v} = \tau_\gamma \otimes \bar{v} \oplus \tau_\delta \otimes ((\tau_\delta^{\otimes m} \oplus \tau_\gamma) \otimes P_0 \oplus Q_0) \otimes \bar{v} \tag{4-19}$$

The proof that equation 4-19 holds is split up into two parts.

- First part of the proof consists of showing that equation 4-19 holds for the elements $[(\tau_\delta^{\otimes m} \oplus \tau_\gamma) \otimes \bar{v}]_p$, $\forall j \in \{1, \ldots, m-1\}$ and $\forall p \in L_{j+1}$. Since all elements $[Q_0]_{p,\cdot}$

are $\varepsilon$ according to equation 3-14, because $p \notin L_1$, and $[\bar{v}]_p = \tau_\delta^{\otimes j}$, $\forall p \in L_{j+1}$:

$$\begin{aligned}
[(\tau_\delta^{\otimes m} \oplus \tau_\gamma) \otimes \bar{v}]_p &= \tau_\gamma \otimes \tau_\delta^{\otimes j} \oplus \tau_\delta \otimes [(\tau_\delta^{\otimes m} \oplus \tau_\gamma) \otimes P_0]_{p,.} \otimes \bar{v} \Leftrightarrow \\
(\tau_\delta^{\otimes m} \oplus \tau_\gamma) \otimes \tau_\delta^{\otimes j} &= (\tau_\gamma) \otimes \tau_\delta^{\otimes j} \oplus \tau_\delta \otimes \bigoplus_{q \in L_j} (\tau_\delta^{\otimes m} \oplus \tau_\gamma) \otimes [P_0]_{p,q} \otimes \bar{v}_q \Leftrightarrow \\
(\tau_\delta^{\otimes m} \oplus \tau_\gamma) \otimes \tau_\delta^{\otimes j} &= (\tau_\gamma) \otimes \tau_\delta^{\otimes j} \oplus \tau_\delta \otimes (\tau_\delta^{\otimes m} \oplus \tau_\gamma) \otimes \tau_\delta^{\otimes j-1} \Leftrightarrow \\
(\tau_\delta^{\otimes m} \oplus \tau_\gamma) \otimes \tau_\delta^{\otimes j} &= (\tau_\delta^{\otimes m} \oplus \tau_\gamma) \otimes \tau_\delta^{\otimes j}
\end{aligned}$$

This shows that equation 4-19 holds for the elements $[(\tau_\delta^{\otimes m} \oplus \tau_\gamma) \otimes \bar{v}]_p$.

- The second part of the proof consists of showing that equation 4-19 holds for the elements $[(\tau_\delta^{\otimes m} \oplus \tau_\gamma) \otimes \bar{v}]_p$, $p \in L_1$. Since all elements of $[P_0]_{p,.}$ are $\varepsilon$, according to equation 3-13, and $[\bar{v}]_p = e$ when $p \in L_1$:

$$\begin{aligned}
[(\tau_\delta^{\otimes m} \oplus \tau_\gamma) \otimes \bar{v}]_p &= (\tau_\gamma) \otimes [\bar{v}]_p \oplus \tau_\delta \otimes [Q_0]_{p,.} \otimes \bar{v} \Leftrightarrow \\
\tau_\delta^{\otimes m} \oplus \tau_\gamma &= \tau_\gamma \oplus \tau_\delta \bigoplus_{q \in L_m} [Q_0]_{p,q} \otimes [\bar{v}]_q \Leftrightarrow \\
\tau_\delta^{\otimes m} \oplus \tau_\gamma &= \tau_\gamma \oplus \tau_\delta \otimes \tau_\delta^{\otimes m-1} \Leftrightarrow \\
\tau_\delta^{\otimes m} \oplus \tau_\gamma &= \tau_\delta^{\otimes m} \oplus \tau_\gamma
\end{aligned}$$

This shows that equation 4-19 also holds for the elements $[(\tau_\delta^{\otimes m} \oplus \tau_\gamma) \otimes \bar{v}]_p$, $p \in L_1$

Combining these two parts shows that equation 4-19 holds.

### 4-3-2 Uniqueness

In the previous subsection it was shown that the unique eigenvalue of this class of matrices is equal to $\lambda = \tau_\gamma \oplus \tau_\delta^{\otimes m}$, where $m$ is the total number of sets of legs, and that there was at least one eigenvector $v$ associated to this eigenvalue. In this section it will be shown that if $\tau_\gamma \leq \tau_\delta^{\otimes m}$ this eigenvector is unique up to a max-plus multiplication with a scalar. That means that if $\lambda$ is an eigenvector then $a \otimes \lambda$, for $a \in \mathbb{R}$ is also an eigenvector. This can be proven with the theory explained at the end of section 2-5 which is repeated here for convenience:

Consider the communication graph $\mathcal{G}(A)$ of matrix $A \in \mathbb{R}_{\max}^{n \times n}$ and let it have a finite maximal circuit mean $\lambda$. Then $\lambda$ is an eigenvalue for the matrix $A$ and the column $[A_\lambda^*]_{.\eta}$ is an eigenvector of A associated with $\lambda$, for any node $\eta$ in $\mathcal{G}^c(A)$. For nodes $i, j$ belonging to $\mathcal{G}^c(A)$ there exists $a \in \mathbb{R}$ such that

$$a \otimes [A_\lambda^*]_i = [A_\lambda^*]_j \tag{4-20}$$

if and only if $i$ and $j$ belong to the same maximal strongly connected subgraph (m.s.c.s.) of the critical graph.

If $\mathcal{G}^c(A)$ consists of only one m.s.c.s., then equation (4-20) will hold for any node $i, j$ that belong to the critical graph. This means that the column associated with those nodes are equal to each other, except for a max-plus multiplication, which in turn means the eigenvector is unique.

This means that in order to proof that the eigenvector is unique it is sufficient to show that the critical graph $\mathcal{G}^c(A)$ consists of a single strongly connected subgraph. The critical graph consists of all circuits with maximal average weight: $\lambda$. This means there are three cases that need to be considered:

- $\tau_\gamma = \tau_\delta^{\otimes m} = \lambda$
- $\tau_\delta^{\otimes m} < \tau_\gamma = \lambda$
- $\tau_\gamma < \tau_\delta^{\otimes m} = \lambda$

In order to show that the critical graph $\mathcal{G}^c(A)$ consists of a single strongly connected subgraph the 'normal' matrix $\bar{A}$ is used and the graphs will be reduced using the node reductions shown in Figure 4-1. Since all paths, of length one, between two sets of nodes have the same weight, circuits that follow the same path through the sets of touchdown/lift-off nodes, but using different nodes of those sets, have the same weight. Thus only one of those circuits needs to be looked at.

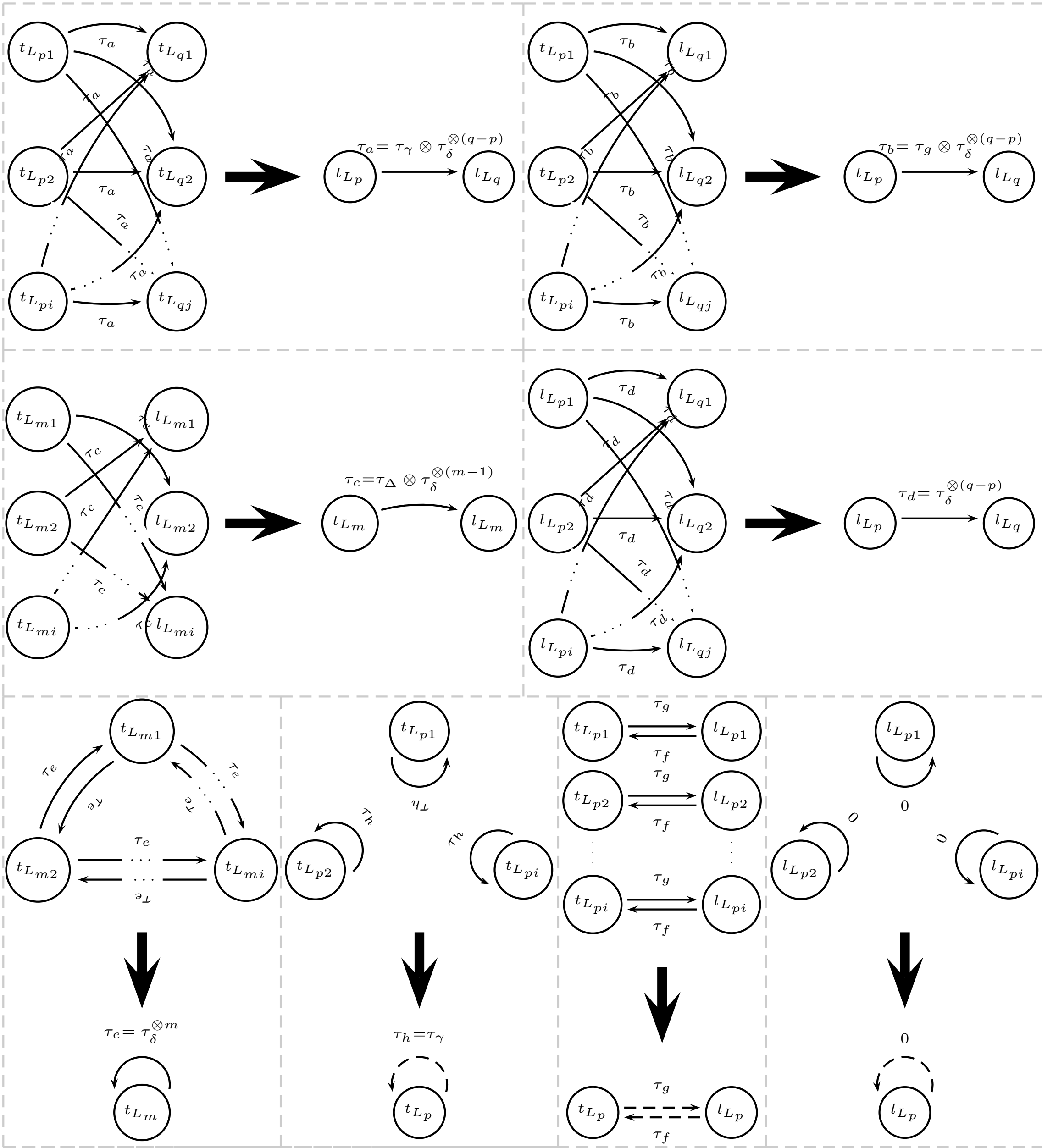


**Figure 4-1:** Node reductions in the graph of the system matrix $\bar{A}$. Each subfigure represents the groups of nodes with their equivalent weigh arcs, and the resulting simplified lumped nodes. Top left, lumped paths from all nodes of the sets $t_{L_p}$ to all nodes of the set $t_{L_q}$. Top right, lumped paths from all nodes of the sets $t_{L_p}$ to all nodes of the set $l_{L_q}$, the same figure applies for the paths from $l_{L_p}$ to $t_{L_q}$ with $\tau_a = \tau_\Delta \otimes \tau_\delta^{\otimes(q-p)}$. Middle left, lumped paths from all nodes of the sets $t_{L_m}$ to all nodes of the set $l_{L_m}$, excluding the paths between $t_{L_{mi}}$ and $l_{L_{mi}}$, $\forall i \in L_m$. Middle right, lumped paths from all nodes of the sets $l_{L_p}$ to all nodes of the set $l_{L_q}$. Bottom left, lumped paths for all nodes of the set $t_{L_m}$, excluding self loops. Bottom left center, lumped nodes for all of the self loops of the sets $t_{L_p}$, for the set $t_{L_p}$ the weight of the self loops is equal to $\lambda$. Bottom right center, lumped nodes for all circuits between touchdown and lift-off nodes of the same leg. Bottom right, lumped nodes for all self loops of the sets $l_{L_p}$

- In the first case, where $\tau_\delta^{\otimes m} = \tau_\gamma = \lambda$, the circuits of the graph of $\bar{A}$ can be split up in different groups:
  - The self loops of the touchdown and lift-off nodes; of which the self loops of the touchdown nodes have a weight of $\lambda$ and the self loops of the lift-off nodes have average circuit weight of 0. Since these self loops are of length one, the weight of the circuit is equal to the average weight which means the self loops of the touchdown nodes will be in the critical graph.
  - The circuits between the touchdown and lift-off nodes of the same leg; which have average weight $\frac{\tau_\gamma}{2}$.
  - The paths $p$, of length one, between the nodes of set $L_m$ have a weight of

$$|p|_{\mathrm{w}} = \tau_\delta^{\otimes m},$$

    so a circuit that consists of only these paths has an average weight of

$$\frac{|p|_{\mathrm{w}}}{|p|_1} = \frac{\tau_\delta^{\otimes(m\otimes r)}}{r} = \tau_\delta^{\otimes m} = \lambda,$$

    where $r$ is the number of paths of length one. Because the average weight of the circuits is equal to $\lambda$ these circuits are also in the critical graph.
  - The circuits that have not yet been considered all have to go through at least one of the nodes of set $m$. This is because there are no paths from nodes of set $L_i$ to $L_{i-1}$, $\forall i \in \{2, \dots, m-1\}$. The only paths from nodes of set $L_j$ to any of the previous sets $L_{j-k}$, $k \in \{1, \dots, j-1\}$ is for the set of nodes $t_{L_j}$ for $j = m$. This means the circuit can be split up into two paths: The path from a node of set $L_i$ to $t_{L_m}$ and the path back from $t_{L_m}$ to $L_i$. First only circuits of which all nodes are touchdown nodes will be considered. The path $q$ from a node of the set $t_{L_m}$ to $t_{L_i}$ has a weight $|q|_w$ of

$$|q|_{\mathrm{w}} = \tau_\delta^{\otimes i}.$$

    To find the weight of the path $r$ from $t_{L_i}$ to $t_{L_m}$ first take a look at the weight of the path between the sets $t_{L_p}$ and $t_{L_q}$ as shown in Figure 4-1, which is

$$\tau_\gamma \otimes \tau_\delta^{\otimes(p-q)},$$

    If you take the path from node $t_{L_i}$ to $t_{L_j}$ and from there to $t_{L_k}$ the total weight of this path of length two is

$$\tau_\gamma \otimes \tau_\delta^{\otimes(i-j)} \otimes \tau_\gamma \otimes \tau_\delta^{\otimes(j-k)} = \tau_\gamma^2 \otimes \tau_\delta^{\otimes(i-k)},$$

    which shows that it does not matter what value $j$ is, only that the length of the path matters and the starting and ending node. By expanding this to a path of $r \in \{1, \dots, m-1-i\}$ length, the weight of the path $s$ from $t_{L_i}$ to $t_{L_m}$ is:

$$|s|_{\mathrm{w}} = \tau_\gamma^{\otimes r} \otimes \tau_\delta^{\otimes(m-i)},$$

where $r$ is the length of the path.

By combining the two paths $q$ and $s$ a circle starting at node $t_{L_i}$ is made with the following weight:

$$|s \circ q|_{\mathrm{w}} = \tau_\gamma^{\otimes r} \otimes \tau_\delta^{\otimes m}$$

The average weight is then

$$\frac{|s \circ q|_{\mathrm{w}}}{|s \circ q|_1} = \frac{\tau_\gamma^{\otimes r} \otimes \tau_\delta^{\otimes m}}{r+1} = \frac{\tau_\gamma^{\otimes (r+1)}}{r+1} = \tau_\gamma = \lambda$$

The combination of these circuits that belong to the critical graph include all paths between all touchdown nodes. If you write

$$\bar{A} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \tag{4-21}$$

where $A_1 = \tau_f \otimes (\tau_g \otimes W \oplus V)$, $A_2 = \tau_f \otimes W$, $A_3 = \tau_g \otimes W \oplus V$ and $A_4 = W$. Then all elements of $A_1$ belong to the critical graph.

Now define $A > B$ as:

$$[A]_{ij} > [B]_{ij}, \forall [B]_{ij} \neq \varepsilon, i, j \in \{1, \ldots, n\}, \tag{4-22}$$

where $A, B \in \mathbb{R}_{\max}^{n \times n}$.

Using this definition the following holds true: $A_1 > A_3 > A_2 > A_4$. This means that replacing one of the touchdown nodes for a lift-off node of the same set of legs, in one of the circuits discussed before, results in the average weight of the new circuit being less than that of the original circuit, therefore only the circuits between touchdown nodes are in the critical graph. For this case the critical graph $\mathcal{G}_c(\bar{A})$ is equal to the graph $\mathcal{G}(A_1)$ as shown in the middle graph of Figure 4-2. In this case the graph consists of a single strongly connected subgraph, which means the eigenvector is unique for this case.

- In the second case, where $\tau_\delta^{\otimes m} < \tau_\gamma = \lambda$, it is clear that only circuits which consist of path with weights that are any combination of $\tau_g$ and $\tau_f$, but not $\tau_\delta$ should be considered. This limits the possible circuits to the self loops of the touchdown nodes and the circuits between touchdown and lift-off nodes of the same set. But the average weight of the circuits between touchdown and lift-off is $\tau_\gamma/2$. Thus the only circuits that are in the critical graph are the self loops of the touchdown nodes. Which is shown in the top of Figure 4-2. In this case the critical graph does not consist of a single strongly connected subgraph, but of $n$ subgraphs and thus there are $n$ max-plus linearly independant eigenvectors.
- In the third case, where $\tau_\gamma < \tau_\delta^{\otimes m} = \lambda$, the same applies as for the second case, but this time for weights that are any linear combination of $\tau_\delta$. The only paths that satisfy this condition are the paths of the set $t_{L_m}$. All single length paths of this set have a weight of $\tau_\delta^{\otimes m}$, thus any circuit consists of solely these paths will have an average weight of $\tau_\delta^{\otimes m} = \lambda$ and are in the critical graph. These are the

only circuits in the critical graph. This is shown in the bottom graph of Figure 4-2. For this case the critical graph also consists of a single strongly connected subgraph, which means the eigenvector is also unique for this case.

In practice the second case never happens because the designer of the gaits choses the values for $\tau_f, \tau_g$ and $\tau_\Delta$ in such a way that $\tau_\gamma = \tau_\delta{}^{\otimes m}$. The reason for this is that there are no advantages when chosing them differently. There is no difference in steady state behaviour between case one and three and in case two the steady state behaviour is not uniquely defined, which makes it very challenging to find optimal gait switches.

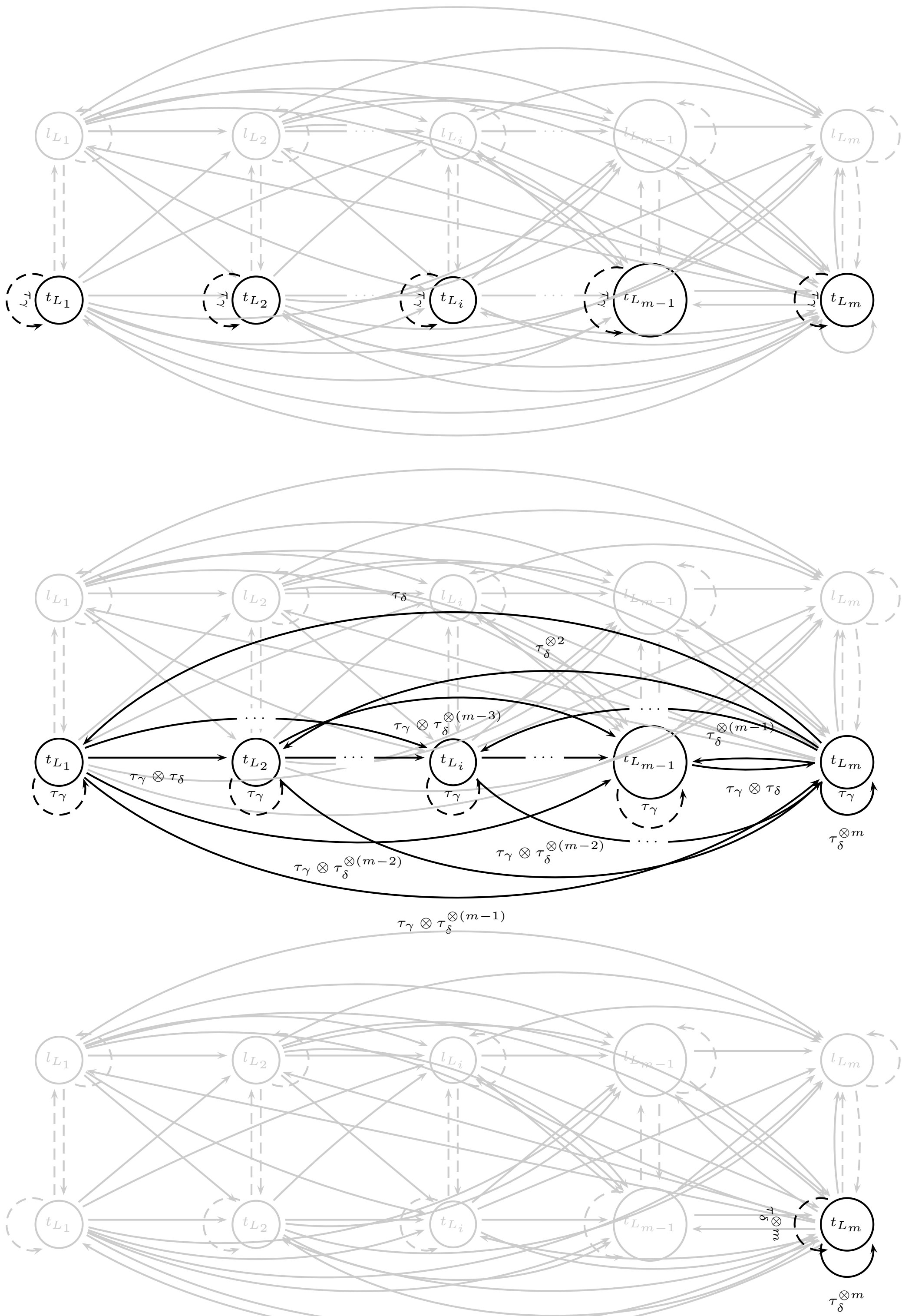


**Figure 4-2:** Critical graphs of the system matrix $\bar{A}$ for top: $\tau_\delta^{\otimes m} < \tau_\gamma = \lambda$; center: $\tau_\gamma = \tau_\delta^{\otimes m} = \lambda$; and bottom: $\tau_\gamma < \tau_\delta^{\otimes m} = \lambda$.

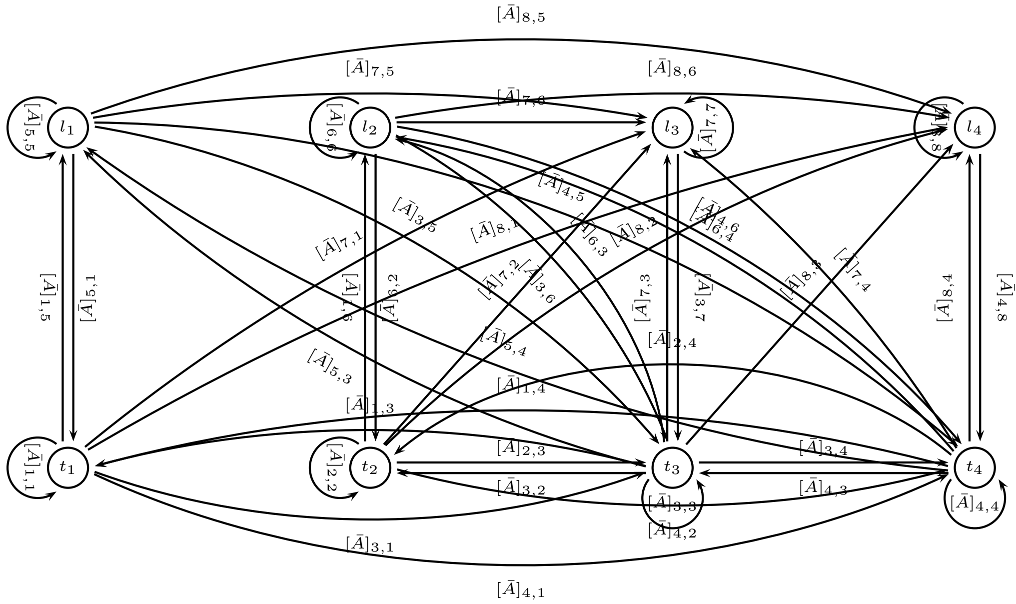


**Figure 4-3:** Graph of the matrix $\bar{A}$.

As an example consider the normal matrix $\bar{A}$ of gait $\mathbf{G} = \{1,3\} \prec \{2,4\}$ that was found in the previous section, repeated here for convenience:

$$\bar{A} = \left[\begin{array}{cccc|cccc} \tau_\gamma & \varepsilon & \tau_\delta & \tau_\delta & \tau_f & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \tau_\gamma & \tau_\delta & \tau_\delta & \varepsilon & \tau_f & \varepsilon & \varepsilon \\ \tau_\gamma \otimes \tau_\delta & \tau_\gamma \otimes \tau_\delta & \tau_\gamma \oplus \tau_\delta^{\otimes 2} & \tau_\delta^{\otimes 2} & \tau_f \otimes \tau_\delta & \tau_f \otimes \tau_\delta & \tau_f & \varepsilon \\ \tau_\gamma \otimes \tau_\delta & \tau_\gamma \otimes \tau_\delta & \tau_\delta^{\otimes 2} & \tau_\gamma \oplus \tau_\delta^{\otimes 2} & \tau_f \otimes \tau_\delta & \tau_f \otimes \tau_\delta & \varepsilon & \tau_f \\ \hline \tau_g & \varepsilon & \tau_\Delta & \tau_\Delta & e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \tau_g & \tau_\Delta & \tau_\Delta & \varepsilon & e & \varepsilon & \varepsilon \\ \tau_g \otimes \tau_\delta & \tau_g \otimes \tau_\delta & \tau_g \oplus \tau_\Delta \otimes \tau_\delta & \tau_\Delta \otimes \tau_\delta & \tau_\delta & \tau_\delta & e & \varepsilon \\ \tau_g \otimes \tau_\delta & \tau_g \otimes \tau_\delta & \tau_\Delta \otimes \tau_\delta & \tau_g \oplus \tau_\Delta \otimes \tau_\delta & \tau_\delta & \tau_\delta & \varepsilon & e \end{array}\right]$$

The graphs $\mathcal{G}(\bar{A})$ is shown in Figure 4-3.

This graph can be reduced, using the reductions shown in Figure 4-1, to the graph in Figure 4-4.

In order to proof the uniqueness of the eigenvector first consider the case $\tau_\gamma = \tau_\delta^{\otimes 2} = \lambda$. The eigenvalue is equal to the maximum average circuit weight, which means any circuit with average circuit weight $\lambda$ is in the critical graph.

- The self loops of the touchdown nodes are either $\tau_\gamma = \lambda$ or $\tau_\gamma \oplus \tau_\delta^{\otimes 2} = \lambda$. This means these self-loops are in the critical graph.
- The self loops of the lift off nodes are $e$, which means they are not in the critical graph.
- The circuits between the touchdown and lift-off nodes of the same leg; which have average weight $\frac{\tau_\gamma}{2}$, which means these circuits are not in the critical graph.

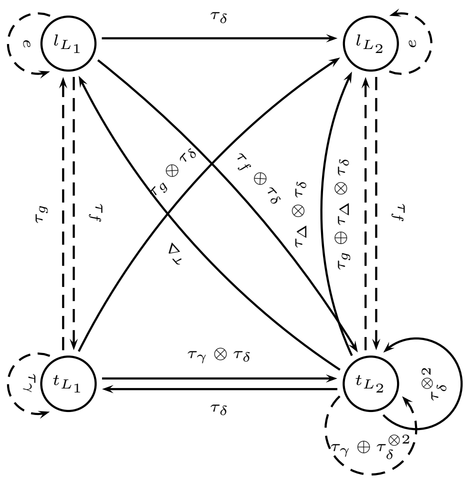


**Figure 4-4:** Reduced graph of the matrix $\bar{A}$, where $t_1, t_2 \in t_{L_1}$, $t_3, t_4 \in t_{L_2}$, $l_1, l_2 \in l_{L_1}$ and $l_3, l_4 \in l_{L_2}$.

- Next consider the circuits made with the nodes of set $t_{L_2}$, (not including the self loops), these circuits consists of paths of length one and of weight $\tau_\delta^{\otimes 2}$. Which means the average circuit weight is also $\tau_\delta^{\otimes 2} = \lambda$. This means these circuits are also in the critical graph

- The only circuits that have not been considered yet have in common that at least one node is of the set $t_{L_2}$ as can be seen from the graph in Figure 4-4, where all the paths that have not been considered yet lead to $t_{L_2}$ and only from $t_{L_2}$ back to the other sets.

  - Considered the circuits from $t_{L_1}$ to $t_{L_2}$ and back, these have a total weight of $\tau_\gamma \otimes \tau_\delta^{\otimes 2} = 2 \times \lambda$ and a length of two, which means the average weight is $\lambda$. Which in turn means these circuits are in the critical graph as well.
  - Consider the circuits from $l_{L_1}$ to $t_{L_2}$ and back, these have a total weight of $\tau_\delta^{\otimes 2} = \lambda$ and a length of two, which means the average weight is $\frac{\lambda}{2}$, therefore these circuits are not in the critical graph.
  - Next consider the circuits of length three from $t_{L_1}$ to $l_{L_1}$ to $t_{L_2}$ and back to $t_{L_1}$, these circuits have a total weight of $\tau_\gamma \otimes \tau_\delta^{\otimes 2} = 2 \times \lambda$, which means the average circuit weight is $\frac{2\times\lambda}{3}$ which means these are not in the critical graph either.
  - Next consider the circuits of length three from $t_{L_1}$ to $l_{L_2}$ to $t_{L_2}$ and back to $t_{L_1}$, these circuits have a total weight of $\tau_\gamma \otimes \tau_\delta^{\otimes 2} = 2 \times \lambda$, which means the average circuit weight is $\frac{2\times\lambda}{3}$ which means these are not in the critical graph either.
  - Finally consider the circuits of length four, from $t_{L_1}$ to $l_{L_1}$ to $l_{L_2}$ to $t_{L_2}$ and back to $t_{L_1}$. These circuits have a total weight of $\tau_\gamma \otimes \tau_\delta^{\otimes 2} = 2 \times \lambda$, which

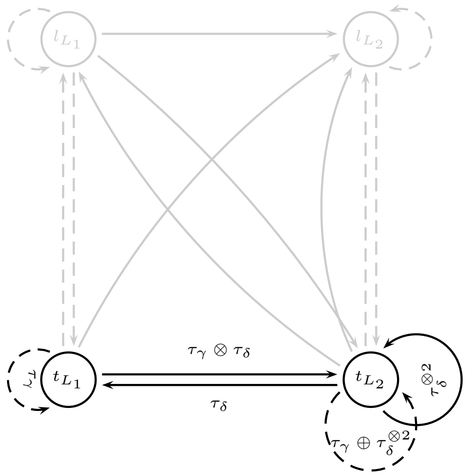


**Figure 4-5:** Critical graph of the matrix $\bar{A}$, for the case $\tau_\gamma = \tau_\delta^{\otimes 2} = \lambda$. $t_1, t_2 \in t_{L_1}$, $t_3, t_4 \in t_{L_2}$, $l_1, l_2 \in l_{L_1}$ and $l_3, l_4 \in l_{L_2}$.

means the average circuit weight is $\frac{\lambda}{2}$ which means these are not in the critical graph either.

The critical graph $\mathcal{G}_c(\bar{A})$ for this case is shown in Figure 4-5. It is clear that this critical graph consists of a single strongly connected subgraph. This means the eigenvector is unique for this case.

In the second case, where $\tau_\delta^{\otimes 2} < \tau_\gamma = \lambda$, it is clear that the circuits with their weights being a combination of $\tau_\delta$ and $\tau_\gamma$, which were in the critical graph, are now no longer in the critical graph since $\tau_\delta^{\otimes 2} < \tau_\gamma$. Therefore in this case only the circuits that had average weight $\lambda$ in the previous case and had paths with only weights being a combination of $\tau_f$ and $\tau_g$ are now in the critical graph. The only circuits that satisfy this condition are the self loops of the touchdown nodes. The critical graph for this case is shown in Figure 4-6. In this case the critical graph consists of four strongly connected subgraphs, which means there is more than one eigenvector.

In the third case, where $\tau_\gamma < \tau_\delta^{\otimes 2} = \lambda$, only the circuits of the first case that have average weight $\lambda$ and consist of paths with weights consisting of $\tau_f$ and $\tau_\Delta$ are in the critical graph. The circuits that satisfy these conditions are the circuits made with the paths between the touchdown nodes of set $t_{L_2}$. The critical graph for this case is shown in Figure 4-7. In this case the critical graph consists of a single strongly connected subgraph as well, which in turn means the eigenvector is unique.

To show what this means for the steady state behaviour consider the gait $\mathbf{G} = \{1, 3\} \prec \{2, 4\}$ with gait parameters as shown in Table 4-1 The steady state behaviour of the

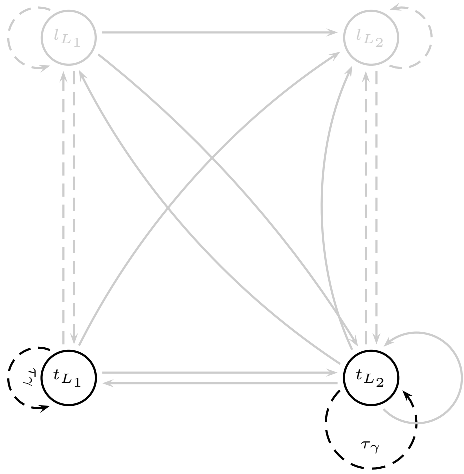


**Figure 4-6:** Critical graph of the matrix $\bar{A}$, for the case $\tau_\delta^{\otimes 2} < \tau_\gamma = \lambda$. $t_1, t_2 \in t_{L_1}$, $t_3, t_4 \in t_{L_2}$, $l_1, l_2 \in l_{L_1}$ and $l_3, l_4 \in l_{L_2}$.

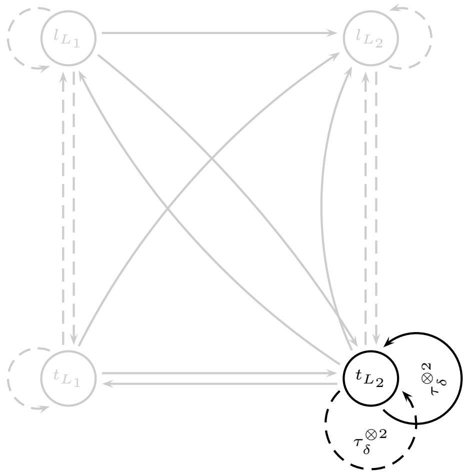


**Figure 4-7:** Critical graph of the matrix $\bar{A}$, for the case $\tau_\delta^{\otimes 2} < \tau_\gamma = \lambda$. $t_1, t_2 \in t_{L_1}$, $t_3, t_4 \in t_{L_2}$, $l_1, l_2 \in l_{L_1}$ and $l_3, l_4 \in l_{L_2}$.

**Table 4-1:** Gait parameters for gait $\mathbf{G} = \{1,3\} \prec \{2,4\}$

| Case | $\tau_f$ | $\tau_\Delta$ | $\tau_g$ | $\lambda$ |
|---|---|---|---|---|
| $\tau_\gamma = \tau_\delta^{\otimes 2}$ | 0.6 | 0.2 | 1.0 | 1.6 |
| $\tau_\delta^{\otimes 2} < \tau_\gamma$ | 0.6 | 0.0 | 1.0 | 1.6 |
| $\tau_\gamma < \tau_\delta^{\otimes 2}$ | 0.6 | 0.2 | 0.5 | 1.6 |

first and last case are the same with cycle time: $\lambda = 1.6$ and eigenvector

$$v_1 = \begin{bmatrix} 0.6 \\ 1.4 \\ 0.6 \\ 1.4 \\ 0.0 \\ 0.8 \\ 0.0 \\ 0.8 \end{bmatrix}$$

The steady state behaviour of the second case is not uniquely defined; the cycle time is unique with $\lambda = 1.6$, but the eigenvector is not unique. The structure of one of the eigenvectors can be found using *theorem 1*. This eigenvector is

$$v_2 = \begin{bmatrix} 0.6 \\ 1.2 \\ 0.6 \\ 1.2 \\ 0.0 \\ 0.6 \\ 0.0 \\ 0.6 \end{bmatrix}$$

Now the robot will be initialized at $v_1$ and $v_2$ for all three cases. The behaviour of the robot for the first case is shown in Figure 4-8, the top figure shows the behaviour when initialised at $v_1$ and the bottom at $v_2$. When initialized with $v_1$ the robot is already in its steady state behaviour, so this behaviour is just repeated every cycle. When initialized with $v_2$ the robot first forces the legs of $L_2$ to stay on the ground longer, that way the lift-off events happen at the same time as they would when the robot was initialized with $v_1$, after that the robot is in its steady state behaviour and repeats it every cycle. The reason for its unique steady state behaviour can be derived from the requirements the max-plus system is based on: the flight time has to be equal to $\tau_f$, the double stance time has to be at least $\tau_\Delta$ and the ground time at least $\tau_g$. As can be seen from the figure, these conditions are exactly satisfied if $\tau_\gamma = \tau_\delta^{\otimes 2}$

The behaviour of the robot for the second case is shown in Figure 4-9, the top figure shows the behaviour when initialized with $v_1$ and the bottom with $v_2$. It is clear from the values of $\tau_f$, $\tau_\Delta$ and $\tau_g$, and the derived values of $\tau_\delta$ and $\tau_\gamma$ that both the behaviour initialized by $v_1$ and $v_2$ satisfy the conditions put up by the max-plus model. It is clear that because $\tau_\gamma > \tau_\delta^{\otimes 2}$ there is room in the synchronization conditions for a small change in the touchdown and lift-off events of some of the legs.

For the third case the behaviour is shown in Figure 4-10. The top figure shows the behaviour when initialised with $v_1$ and the bottom with $v_2$. When looking at the values of $\tau_f$, $\tau_\Delta$ and $\tau_g$, and the derived values of $\tau_\delta$ and $\tau_\gamma$ it is clear that $\tau_g$ is too small to be able to satisfy the synchronization conditions put up by the max-plus model. The max-plus model solves this by increasing the stance time to the minimum required time, which is the same as $\tau_g$ of the first case. Therefore this system has the same steady state behaviour of the system in the first case. The response to the initialisation with $v_2$ is slightly different than that of the system in the first case; the stance time is already larger than $\tau_g$, therefore the stance time can be decreased to reach steady state faster, this is done for legs 1 and 3 in the first cycle. After that the system is in the same steady state behaviour as when it was initialized with $v_1$.

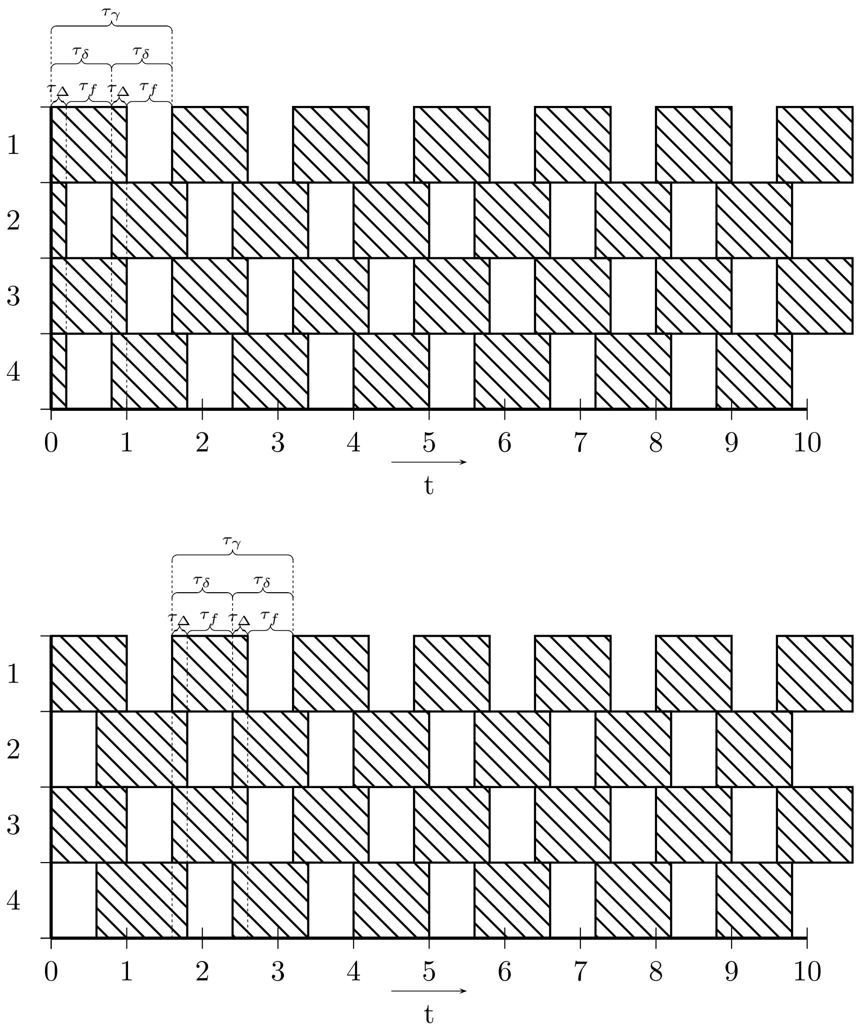


**Figure 4-8:** Steady state behaviour of the robot using $\mathbf{G} = \{1,3\} \prec \{2,4\}$ and $\tau_\gamma = \tau_\delta^{\otimes 2}$, top: when initialized with $v_1$, bottom: when initialized with $v_2$. The dashed boxes represent the stance time and the empty space in between represents the flight time.

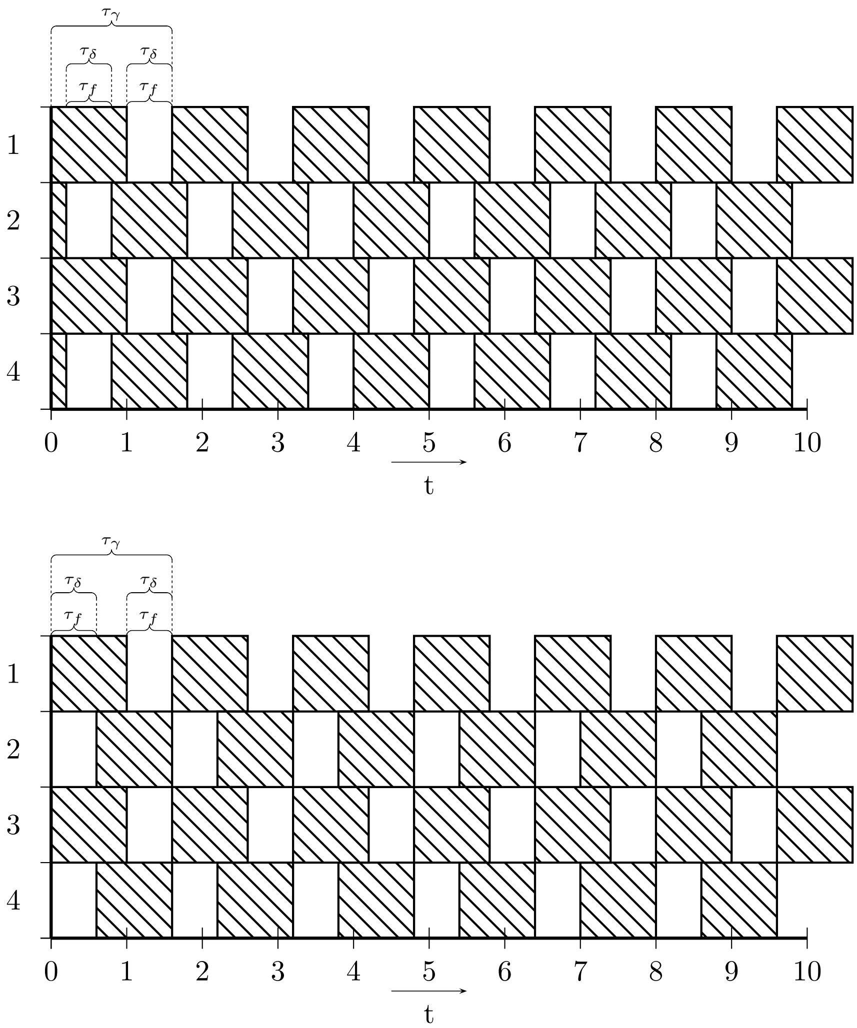


**Figure 4-9:** Steady state behaviour of the robot using $\mathbf{G} = \{1,3\} \prec \{2,4\}$ and $\tau_\gamma > \tau_\delta^{\otimes 2}$, top: when initialized with $v_1$, bottom: when initialized with $v_2$. The dashed boxes represent the stance time and the empty space in between represents the flight time.

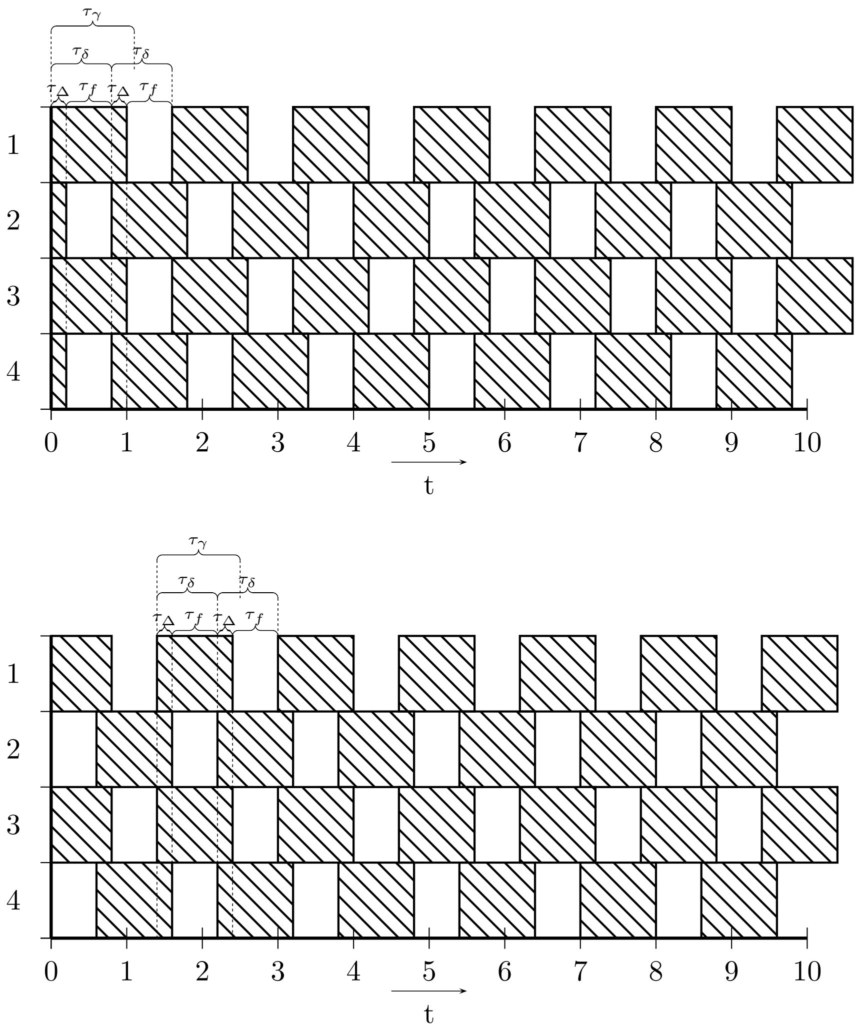


**Figure 4-10:** Steady state behaviour of the robot using $\mathbf{G} = \{1,3\} \prec \{2,4\}$ and $\tau_\gamma < \tau_\delta^{\otimes 2}$, top: when initialized with $v_1$, bottom: when initialized with $v_2$. The dashed boxes represent the stance time and the empty space in between represents the flight time.

## 4-4 Conclusion

In this chapter it has been proven that any gait, made with the method discussed in section 3-4, has a unique eigenvalue $\lambda = \tau_\delta^{\otimes m} \oplus \tau_\gamma$. In practice the values for $\tau_f, \tau_g$ and $\tau_\Delta$ are chosen in such a way that $\lambda = \tau_\gamma = \tau_\delta{}^{\otimes m}$. In this case the critical graph of the gait consists of a single strongly connected subgraph, which means the eigenvector is also uniquely defined. The eigenvector has the following structure:

$$\begin{aligned} \forall j \in \{1, \ldots, m\}, \forall q \in L_j : \quad [v]_q &= \tau_f \otimes \tau_\delta^{\otimes j-1} \\ [v]_{q+n} &= \tau_\delta^{\otimes j-1} \end{aligned}$$

The steady state behaviour of the robot is now mathematically and uniquely defined. This opens up the possibility of defining and implementing optimal gait transitions which is the subject of the next chapter.

# Chapter 5

# Gait switching

## 5-1 Introduction

This chapter uses the knowledge gained about the steady state behaviour of the gaits as discussed in the previous chapter to find a mathematical method to determine the optimal gait to switch to. In the second section this mathematical method is explained. In section 5-3 this method is applied to a hexapod robot. In section 5-4 a model is discussed that has a constant stance time. In section 5-5 a model is discussed that has different, but fixed values for the flight time for each of the legs and how these values for the flight time can be found. In section 5-6 a method is discussed that enables the robot to accelerate or decelerate without changing the gait parameters. Finally in section 5-7 a conclusion is drawn about the use of chosen mathematical method and the different models that were used.

## 5-2 Optimal Gait switching

Consider a gait $\mathbf{G}_1$, with flight-time $\tau_{f_1}$, 'double stance'-time $\tau_{\Delta_1}$ and ground-time $\tau_g$, and where the legs are divided into $m_1$ sets:

$$\mathbf{G}_1 = L_1 \prec L_2 \prec \cdots \prec L_{m_1}, \tag{5-1}$$

where each set of legs can consist of multiple legs and the total number of legs is equal to $n$. Assume for this gait that

$$\tau_{\delta_1}^{\otimes m_1} = \tau_{\gamma_1},$$

where $\tau_{\delta_1} = \tau_{f_1} \otimes \tau_{\Delta_1}$ and $\tau_{\gamma_1} = \tau_{f_1} \otimes \tau_g$, then the eigenvector of this gait has the following structure:

$$\begin{aligned} \forall j \in \{1, \ldots, m_1\}, \forall q \in L_j : \quad [v]_q &= \tau_{f_1} \otimes \tau_{\delta_1}^{\otimes j-1} \\ [v]_{q+n} &= \tau_{\delta_1}^{\otimes j-1} \end{aligned}$$

And consider another gait $\mathbf{G}_2$ with flight-time $\tau_{f_2}$, 'double stance'-time $\tau_{\Delta_2}$ and ground-time $\tau_g$, and where the legs are divided into $m_2$ sets:

$$\mathbf{G}_2 = L_1 \prec L_2 \prec \cdots \prec L_{m_2}, \tag{5-2}$$

Assume for this gait that

$$\tau_{\delta_2}^{\otimes m_2} = \tau_{\gamma_2},$$

where $\tau_{\delta_2} = \tau_{f_2} \otimes \tau_{\Delta_2}$ and $\tau_{\gamma_2} = \tau_{f_2} \otimes \tau_g$, then the eigenvector of this gait has the following structure:

$$\begin{aligned} \forall j \in \{1, \ldots, m_2\}, \forall q \in L_j : \quad [v]_q &= \tau_{f_2} \otimes \tau_{\delta_2}^{\otimes j-1} \\ [v]_{q+n} &= \tau_{\delta_2}^{\otimes j-1} \end{aligned}$$

A gait switch is considered optimal if the variation of the velocities at which the legs move while on the ground is a small as possible. This means that the variation in the time spent on the ground should be as small as possible. This can be achieved by switching from a gait $\mathbf{G}_1$ to a gait $\mathbf{G}_2$ where the difference in the timing of the touchdown (and lift-off) events is as small as possible, this difference can be found by subtracting the eigenvector of $\mathbf{G}_2$ from the eigenvector of $\mathbf{G}_1$. This leads to the following definition of an optimal gait switch: An optimal gait switch from $\mathbf{G}_1$ to a gait $\mathbf{G}_2$ minimizes the value of $\tau_{\text{diff}}$ as defined in equation 5-3.

$$\tau_{\text{diff}} = (t_{v_{\mathbf{G}_2}} - t_{v_{\mathbf{G}_1}}) - \min(t_{v_{\mathbf{G}_2}} - t_{v_{\mathbf{G}_1}}), \tag{5-3}$$

where $t_{v_{\mathbf{G}_2}}$ are the touchdown times of eigenvector $v_{\mathbf{G}_2}$, and $t_{v_{\mathbf{G}_1}}$ are the touchdown times of eigenvector $v_{\mathbf{G}_1}$. Since the touchdown and lift-off events of a single eigenvector are equal up to a timeshift of $\tau_f$ only the touchdown or lift-off events need to be considered.

In order to find the optimal gait switch it is neccesary to look into the effects of such a gait switch on the events of the legs. When switching from $\mathbf{G}_1$ to a gait $\mathbf{G}_2$ legs will move from a set $L_i$ in gait $\mathbf{G}_1$ to a set $L_j$ in gait $\mathbf{G}_2$. This results in a change of the timing of the events for that leg:

The touchdown event of a leg in the set $L_i$ in gait $\mathbf{G}_1$ happens at

$$t_{L_i} = \tau_{f_1} \otimes \tau_{\delta_1}^{\otimes(i-1)} \tag{5-4}$$

and the touchdown moment of a leg in $L_j$ in gait $\mathbf{G}_2$ happens at

$$t_{L_j} = \tau_{f_2} \otimes \tau_{\delta_2}^{\otimes(j-1)}. \tag{5-5}$$

Moving a leg from $L_i$ to $L_j$ results in a change of the touchdown (and lift-off) event of:

$$t_{L_j} - t_{L_i} = \tau_{f_2} \otimes \tau_{\delta_2}^{\otimes(j-1)} - \tau_{f_1} \otimes \tau_{\delta_1}^{\otimes(i-1)}, \tag{5-6}$$

In order to find the optimal gait transition, a combination of these set changes from $L_i$ in gait $\mathbf{G}_1$ to a set $L_j$ in gait $\mathbf{G}_2$, that minimizes the maximum of $\tau_{\text{diff}}$ needs to be found. The maximum of $\tau_{\text{diff}}$ can be minimized by minimizing the maximum absolute deviation of these set changes from the average change of these set changes.

By adding up all the possible changes: for $i \in \{1, \ldots, m_1\}$ and $j \in \{1, \ldots, m_2\}$ and dividing them by the total number of possible changes: $m_1 \times m_2$, the average change in the timing of the events of the legs is computed. This average is:

$$\frac{\tau_{f_2}^{\otimes(m_1 \times m_2)} \otimes \tau_{\delta_2}^{\otimes \frac{(m_2-1)\times m_1 \times m_2}{2}} - (\tau_{f_1}^{\otimes(m_1 \times m_2)} \otimes \tau_{\delta_1}^{\otimes \frac{(m_1-1)\times m_1 \times m_2}{2}})}{m_1 \times m_2}, \tag{5-7}$$

which can be simplified to:

$$\tau_{f_2} \otimes \tau_{\delta_2}^{\otimes \frac{m_2-1}{2}} - (\tau_{f_1} \otimes \tau_{\delta_1}^{\otimes \frac{m_1-1}{2}}). \tag{5-8}$$

Subtracting equation (5-8) from equation (5-6) results in the deviation from this average which is

$$\tau_{\delta_1}^{\otimes \frac{m_1+1-2\times i}{2}} - \tau_{\delta_2}^{\otimes \frac{m_2+1-2\times j}{2}} \tag{5-9}$$

This means that a gait switch for which the subset of all the possible changes, has the lowest maximum value for equation 5-9, is an optimal gait switch.

In the next section these equations are used to find the optimal gait switches for a hexapod robot.

## 5-3 Optimal gait switches of a hexapod

### 5-3-1 Gaits and their parameters

A hexapod has three gaits that are commonly used, assuming the number of legs are equally divided over the sets:

$$\begin{aligned} \mathbf{G}_1 &= L_1 \prec L_2 && (5\text{-}10)\\ \mathbf{G}_2 &= L_1 \prec L_2 \prec L_3 && (5\text{-}11)\\ \mathbf{G}_3 &= L_1 \prec L_2 \prec L_3 \prec L_4 \prec L_5 \prec L_6 && (5\text{-}12) \end{aligned}$$

with the parameters as shown in Table 5-1.

**Table 5-1:** Gait parameters

| Gait | $\tau_f$ | $\tau_\Delta$ | $\tau_g$ |
|---|---|---|---|
| $\mathbf{G}_1$ | $\tau_{f_1}$ | $\tau_{\Delta_1}$ | $\tau_g$ |
| $\mathbf{G}_2$ | $\tau_{f_2}$ | $\tau_{\Delta_2}$ | $\tau_g$ |
| $\mathbf{G}_3$ | $\tau_{f_3}$ | $\tau_{\Delta_3}$ | $\tau_g$ |

$\mathbf{G}_1$ is called the tripod gait, $\mathbf{G}_2$ is called the quadruped gait and $\mathbf{G}_3$ is called the quintuped gait. All possible gait switches between these gaits and the optimal gait switches will be determined next.

### 5-3-2 Tripod to quadruped gait switch

Consider the switch from $\mathbf{G}_1$ to $\mathbf{G}_2$, the possible changes in the order of the legs are:

$$\begin{aligned}
\{L_1\} \to \{L_1\} &= \tau_{f_2} - \tau_{f_1} \\
\{L_1\} \to \{L_2\} &= (\tau_{\delta_2} + \tau_{f_2}) - \tau_{f_1} \\
\{L_1\} \to \{L_3\} &= (2 \times \tau_{\delta 2} + \tau_{f_2}) - \tau_{f_1} \\
\{L_2\} \to \{L_1\} &= \tau_{f_2} - (\tau_{\delta_1} + \tau_{f_1}) \\
\{L_2\} \to \{L_2\} &= (\tau_{\delta_2} + \tau_{f_2}) - (\tau_{\delta_1} + \tau_{f_1}) \\
\{L_2\} \to \{L_3\} &= (2 \times \tau_{\delta 2} + \tau_{f_2}) - (\tau_{\delta_1} + \tau_{f_1})
\end{aligned}$$

The average change is then:

$$(\tau_{\delta_2} + \tau_{f_2}) - (\frac{\tau_{\delta_1}}{2} + \tau_{f_1})$$

The deviation from this average is:

$$\begin{aligned}
\{L_1\} \to \{L_1\} &= \frac{\tau_{\delta_1}}{2} - \tau_{\delta_2} \\
\{L_1\} \to \{L_2\} &= \frac{\tau_{\delta_1}}{2} \\
\{L_1\} \to \{L_3\} &= \frac{\tau_{\delta_1}}{2} + \tau_{\delta_2} \\
\{L_2\} \to \{L_1\} &= -(\frac{\tau_{\delta_1}}{2} + \tau_{\delta_2}) \\
\{L_2\} \to \{L_2\} &= -\frac{\tau_{\delta_1}}{2} \\
\{L_2\} \to \{L_3\} &= -\frac{\tau_{\delta_1}}{2} + \tau_{\delta_2}
\end{aligned}$$

The two switches with the largest absolute deviation are: $\{L_1\} \to \{L_3\}$ and $\{L_2\} \to \{L_1\}$ with an absolute deviation of $|\frac{\tau_{\delta_1}}{2} + \tau_{\delta_2}|$. If you leave these out the maximum deviation is less and the gait switch is still possible. Thus the optimal gait switch from $\mathbf{G}_1$ to $\mathbf{G}_2$ consists of all combination of gaits that only use the following changes in the order of the legs

$$\begin{aligned}
&\{L_1\} \to \{L_1\} \\
&\{L_1\} \to \{L_2\} \\
&\{L_2\} \to \{L_2\} \\
&\{L_2\} \to \{L_3\}
\end{aligned}$$

As an example of a gait switch from $\mathbf{G}_1$ to $\mathbf{G}_2$, the parameters of table 5-2 are used and for gait $\mathbf{G}_1$ the following gait is used

$$\mathbf{G}_1 = \{1, 4, 5\} \prec \{2, 3, 6\},$$

which is the only tripod gait possible, not counting its time shifted alternative:

$$\{2, 3, 6\} \prec \{1, 4, 5\}.$$

**Table 5-2:** Gait parameters for $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{G}_3$

| Gait | $\tau_f$ | $\tau_\Delta$ | $\tau_g$ |
|---|---|---|---|
| $\mathbf{G}_1$ | 1 | 0 | 1 |
| $\mathbf{G}_2$ | 0.5 | 0 | 1 |
| $\mathbf{G}_3$ | 0.2 | 0 | 1 |

Possible optimal gait switches from $\{1,4,5\} \prec \{2,3,6\}$, to a gait with the structure of $\mathbf{G}_2$, are:

$$\begin{gathered}
\{1,4\} \prec \{5,2\} \prec \{3,6\} \\
\{1,4\} \prec \{5,3\} \prec \{2,6\} \\
\{1,4\} \prec \{5,6\} \prec \{2,3\} \\
\{1,5\} \prec \{4,2\} \prec \{3,6\} \\
\{1,5\} \prec \{4,3\} \prec \{2,6\} \\
\{1,5\} \prec \{4,6\} \prec \{2,3\} \\
\{4,5\} \prec \{1,2\} \prec \{3,6\} \\
\{4,5\} \prec \{1,3\} \prec \{2,6\} \\
\{4,5\} \prec \{1,6\} \prec \{2,3\}
\end{gathered}$$

By adding the restriction that only legs on opposite sides (left and right) can be in the same set and those legs cannot be both at the front, middle or back of the robot either, this will result in the least stable gaits being removed from this list, the list of optimal gaits to switch to reduces to:

$$\begin{aligned}
\mathbf{G}_{21} &= \{1,4\} \prec \{5,2\} \prec \{3,6\} \\
\mathbf{G}_{22} &= \{4,5\} \prec \{1,6\} \prec \{2,3\}
\end{aligned}$$

The values of $\tau_{\text{diff}}$ for these two gaits are shown below:

$$\tau_{diff_{1\rightarrow 21}} = \begin{bmatrix} 0.5 \\ 0.0 \\ 0.5 \\ 0.5 \\ 1.0 \\ 0.5 \end{bmatrix}, \tau_{diff_{1\rightarrow 22}} = \begin{bmatrix} 1.0 \\ 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \\ 0.0 \end{bmatrix},$$

which both have a maximum value of 1.0. The value for $\tau_{\text{diff}}$ for a non-optimal gait switch to, for example

$$\mathbf{G}_{23} = \{1,4\} \prec \{3,6\} \prec \{2,5\},$$

**Figure 5-1:** Gait switches from $\mathbf{G}_1$ to $\mathbf{G}_2$. Top, optimal gait switch from $\{1,4,5\} \prec \{2,3,6\}$ to $\{1,4\} \prec \{5,2\} \prec \{3,6\}$. Middle, optimal gait switch from $\{1,4,5\} \prec \{2,3,6\}$ to $\{4,5\} \prec \{1,6\} \prec \{2,3\}$. Bottom, non-optimal gait switch from $\{1,4,5\} \prec \{2,3,6\}$ to $\{1,4\} \prec \{3,6\} \prec \{2,5\}$. The gray blocks represent the stance and the white blocks represent the flight of the legs.

is:

$$\tau_{diff_{1\rightarrow 23}} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.0 \\ 0.5 \\ 1.5 \\ 0.0 \end{bmatrix},$$

which has a maximum value of 1.5. These three gait switches are shown in Figure 5-1, where you can see that a larger value for $[\tau_{\text{diff}}]_i$ results in a longer stance time for leg $i$ during the gait switch which happens in the fourth and fifth cycle.

### 5-3-3 Quadruped to quintuped gait switch

Consider the switch from $\mathbf{G}_2$ to $\mathbf{G}_3$, the possible changes in the order of the legs are:

$$\begin{aligned}
\{L_1\} \rightarrow \{L_1\} &= \tau_{f_3} - \tau_{f_2} \\
\{L_1\} \rightarrow \{L_2\} &= (\tau_{\delta_3} + \tau_{f_3}) - \tau_{f_2} \\
\{L_1\} \rightarrow \{L_3\} &= (2 \times \tau_{\delta_3} + \tau_{f_3}) - \tau_{f_2} \\
\{L_1\} \rightarrow \{L_4\} &= (3 \times \tau_{\delta_3} + \tau_{f_3}) - \tau_{f_2} \\
\{L_1\} \rightarrow \{L_5\} &= (4 \times \tau_{\delta_3} + \tau_{f_3}) - \tau_{f_2} \\
\{L_1\} \rightarrow \{L_6\} &= (5 \times \tau_{\delta_3} + \tau_{f_3}) - \tau_{f_2} \\
\{L_2\} \rightarrow \{L_1\} &= \tau_{f_3} - (\tau_{\delta_2 + \tau_{f_2}}) \\
\{L_2\} \rightarrow \{L_2\} &= (\tau_{\delta_3} + \tau_{f_3}) - (\tau_{\delta_2} + \tau_{f_2}) \\
\{L_2\} \rightarrow \{L_3\} &= (2 \times \tau_{\delta_3} + \tau_{f_3}) - (\tau_{\delta_2} + \tau_{f_2}) \\
\{L_2\} \rightarrow \{L_4\} &= (3 \times \tau_{\delta_3} + \tau_{f_3}) - (\tau_{\delta_2} + \tau_{f_2}) \\
\{L_2\} \rightarrow \{L_5\} &= (4 \times \tau_{\delta_3} + \tau_{f_3}) - (\tau_{\delta_2} + \tau_{f_2}) \\
\{L_2\} \rightarrow \{L_6\} &= (5 \times \tau_{\delta_3} + \tau_{f_3}) - (\tau_{\delta_2} + \tau_{f_2}) \\
\{L_3\} \rightarrow \{L_1\} &= \tau_{f_3} - (2 \times \tau_{\delta 2 + \tau_{f_2}}) \\
\{L_3\} \rightarrow \{L_2\} &= (\tau_{\delta_3} + \tau_{f_3}) - (\tau_{f_2} + 2 \times \tau_{\delta 2}) \\
\{L_3\} \rightarrow \{L_3\} &= (2 \times \tau_{\delta_3} + \tau_{f_3}) - (2 \times \tau_{\delta 2} + \tau_{f_2}) \\
\{L_3\} \rightarrow \{L_4\} &= (3 \times \tau_{\delta_3} + \tau_{f_3}) - (2 \times \tau_{\delta 2 + \tau_{f_2}}) \\
\{L_3\} \rightarrow \{L_5\} &= (4 \times \tau_{\delta_3} + \tau_{f_3}) - (2 \times \tau_{\delta 2} + \tau_{f_2}) \\
\{L_3\} \rightarrow \{L_6\} &= (5 \times \tau_{\delta_3} + \tau_{f_3}) - (2 \times \tau_{\delta 2} + \tau_{f_2})
\end{aligned}$$

The average change is then:

$$(2.5 \times \tau_{\delta_3} + \tau_{f_3}) - (\tau_{\delta_2} + \tau_{f_2})$$

The deviation from this average is:

$$
\begin{aligned}
\{L_1\} \rightarrow \{L_1\} &= \tau_{\delta_2} - 2.5 \times \tau_{\delta_3}\\
\{L_1\} \rightarrow \{L_2\} &= \tau_{\delta_2} - 1.5 \times \tau_{\delta_3}\\
\{L_1\} \rightarrow \{L_3\} &= \tau_{\delta_2} - 0.5 \times \tau_{\delta_3}\\
\{L_1\} \rightarrow \{L_4\} &= \tau_{\delta_2} + 0.5 \times \tau_{\delta_3}\\
\{L_1\} \rightarrow \{L_5\} &= \tau_{\delta_2} + 1.5 \times \tau_{\delta_3}\\
\{L_1\} \rightarrow \{L_6\} &= \tau_{\delta_2} + 2.5 \times \tau_{\delta_3}\\
\{L_2\} \rightarrow \{L_1\} &= -2.5 \times \tau_{\delta_3}\\
\{L_2\} \rightarrow \{L_2\} &= -1.5 \times \tau_{\delta_3}\\
\{L_2\} \rightarrow \{L_3\} &= -0.5 \times \tau_{\delta_3}\\
\{L_2\} \rightarrow \{L_4\} &= 0.5 \times \tau_{\delta_3}\\
\{L_2\} \rightarrow \{L_5\} &= 1.5 \times \tau_{\delta_3}\\
\{L_2\} \rightarrow \{L_6\} &= 2.5 \times \tau_{\delta_3}\\
\{L_3\} \rightarrow \{L_1\} &= -(\tau_{\delta_2} + 2.5 \times \tau_{\delta_3})\\
\{L_3\} \rightarrow \{L_2\} &= -(\tau_{\delta_2} + 1.5 \times \tau_{\delta_3})\\
\{L_3\} \rightarrow \{L_3\} &= -(\tau_{\delta_2} + 0.5 \times \tau_{\delta_3})\\
\{L_3\} \rightarrow \{L_4\} &= -\tau_{\delta_2} + 0.5 \times \tau_{\delta_3}\\
\{L_3\} \rightarrow \{L_5\} &= -\tau_{\delta_2} + 1.5 \times \tau_{\delta_3}\\
\{L_3\} \rightarrow \{L_6\} &= -\tau_{\delta_2} + 2.5 \times \tau_{\delta_3}
\end{aligned}
$$

In order to get an equal number of legs in all of the six sets, at least six changes in the order of the legs are needed, which can be devided into two changes per set of legs of $\mathbf{G}_2$. The combination of those six changes, that has the lowest maximum absolute deviation, is:

$$
\begin{aligned}
\{L_1\} &\rightarrow \{L_1\}\\
\{L_1\} &\rightarrow \{L_2\}\\
\{L_2\} &\rightarrow \{L_3\}\\
\{L_2\} &\rightarrow \{L_4\}\\
\{L_3\} &\rightarrow \{L_5\}\\
\{L_3\} &\rightarrow \{L_6\}
\end{aligned}
$$

As an example of a gait switch from $\mathbf{G}_2$ to $\mathbf{G}_3$, the parameters of table 5-2 are used and for gait $\mathbf{G}_2$ the following gait is used:

$$\mathbf{G}_2 = \{1, 4\} \prec \{5, 2\} \prec \{3, 6\}.$$

The gaits with the structure of $\mathbf{G}_3$ that are optimal to switch to are:

$$\begin{gathered}
1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6 \\
1 \prec 4 \prec 5 \prec 2 \prec 6 \prec 3 \\
1 \prec 4 \prec 2 \prec 5 \prec 3 \prec 6 \\
1 \prec 4 \prec 2 \prec 5 \prec 6 \prec 3 \\
4 \prec 1 \prec 5 \prec 2 \prec 3 \prec 6 \\
4 \prec 1 \prec 5 \prec 2 \prec 6 \prec 3 \\
4 \prec 1 \prec 2 \prec 5 \prec 3 \prec 6 \\
4 \prec 1 \prec 2 \prec 5 \prec 6 \prec 3
\end{gathered}$$

The values of $\tau_{\text{diff}}$ will be calculated for

$$\begin{aligned}
\mathbf{G}_{31} &= 1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6 \\
\mathbf{G}_{32} &= 4 \prec 1 \prec 2 \prec 5 \prec 6 \prec 3
\end{aligned}$$

The values for $\tau_{diff}$ are:

$$\tau_{diff_{2\rightarrow 31}} = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.0 \\ 0.4 \\ 0.1 \\ 0.2 \end{bmatrix}, \tau_{diff_{2\rightarrow 32}} = \begin{bmatrix} 0.4 \\ 0.1 \\ 0.2 \\ 0.2 \\ 0.3 \\ 0.0 \end{bmatrix},$$

which have a maximum value of 0.4. Now consider the non-optimal switch to

$$\mathbf{G}_{33} = 1 \prec 3 \prec 5 \prec 2 \prec 4 \prec 6$$

The values for $\tau_{\text{diff}}$ for this switch is

$$\tau_{diff_{2\rightarrow 33}} = \begin{bmatrix} 0.8 \\ 0.9 \\ 0.0 \\ 1.6 \\ 0.7 \\ 0.8 \end{bmatrix},$$

which has a maximum value of 1.6.

These three gait switches are shown in Figure 5-2, where you can see that a larger value for $[\tau_{\text{diff}}]_i$ results in a longer stance time for leg $i$ during the switch which happens in the fourth and fifth cycle.

**Figure 5-2:** Gait switches from $\mathbf{G}_2$ to $\mathbf{G}_3$. Top, optimal gait switch from $\{1,4\} \prec \{5,2\} \prec \{3,6\}$ to $1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6$. Middle, optimal gait switch from $\{1,4\} \prec \{5,2\} \prec \{3,6\}$ to $4 \prec 1 \prec 2 \prec 5 \prec 6 \prec 3$. Bottom, non-optimal gait switch from $\{1,4\} \prec \{5,2\} \prec \{3,6\}$ to $1 \prec 3 \prec 5 \prec 2 \prec 4 \prec 6$. The gray blocks represent the stance and the white blocks represent the flight of the legs.

### 5-3-4 Quintuped to quadruped gait switch

Consider the switch from $\mathbf{G}_3$ to $\mathbf{G}_2$, the possible changes in the order of the legs are:

$$
\begin{aligned}
\{L_1\} \rightarrow \{L_1\} &= \tau_{f_2} - \tau_{f_3} \\
\{L_1\} \rightarrow \{L_2\} &= (\tau_{\delta_2} + \tau_{f_2}) - \tau_{f_3} \\
\{L_1\} \rightarrow \{L_3\} &= (2 \times \tau_{\delta_2} + \tau_{f_2}) - \tau_{f_3} \\
\{L_2\} \rightarrow \{L_1\} &= \tau_{f_2} - (\tau_{\delta 3 + \tau_{f_3}}) \\
\{L_2\} \rightarrow \{L_2\} &= (\tau_{\delta_2} + \tau_{f_2}) - (\tau_{\delta_3} + \tau_{f_3}) \\
\{L_2\} \rightarrow \{L_3\} &= (2 \times \tau_{\delta_2} + \tau_{f_2}) - (\tau_{\delta_3} + \tau_{f_3}) \\
\{L_3\} \rightarrow \{L_1\} &= \tau_{f_2} - (2 \times \tau_{\delta 3 + \tau_{f_3}}) \\
\{L_3\} \rightarrow \{L_2\} &= (\tau_{\delta_2} + \tau_{f_2}) - (\tau_{f_3} + 2 \times \tau_{\delta 3}) \\
\{L_3\} \rightarrow \{L_3\} &= (2 \times \tau_{\delta_2} + \tau_{f_2}) - (2 \times \tau_{\delta 3} + \tau_{f_3}) \\
\{L_4\} \rightarrow \{L_1\} &= \tau_{f_2} - (3 \times \tau_{\delta_3} + \tau_{f_3}) \\
\{L_4\} \rightarrow \{L_2\} &= (\tau_{\delta_2} + \tau_{f_2}) - (3 \times \tau_{\delta_3} + \tau_{f_3}) \\
\{L_4\} \rightarrow \{L_3\} &= (2 \times \tau_{\delta 2 + \tau_{f_2}}) - (3 \times \tau_{\delta_3} + \tau_{f_3}) \\
\{L_5\} \rightarrow \{L_1\} &= \tau_{f_2} - (4 \times \tau_{\delta_3} + \tau_{f_3}) \\
\{L_5\} \rightarrow \{L_2\} &= (\tau_{\delta_2} + \tau_{f_2}) - (4 \times \tau_{\delta_3} + \tau_{f_3}) \\
\{L_5\} \rightarrow \{L_3\} &= (2 \times \tau_{\delta 2} + \tau_{f_2}) - (4 \times \tau_{\delta_3} + \tau_{f_3}) \\
\{L_6\} \rightarrow \{L_1\} &= \tau_{f_2} - (5 \times \tau_{\delta_3} + \tau_{f_3}) \\
\{L_6\} \rightarrow \{L_2\} &= (\tau_{\delta_2} + \tau_{f_2}) - (5 \times \tau_{\delta_3} + \tau_{f_3}) \\
\{L_6\} \rightarrow \{L_3\} &= (2 \times \tau_{\delta 2} + \tau_{f_2}) - (5 \times \tau_{\delta_3} + \tau_{f_3})
\end{aligned}
$$

The average change is then:

$$(\tau_{\delta_2} + \tau_{f_2}) - (2.5 \times \tau_{\delta_3} + \tau_{f_3})$$

The deviation from this average is:

$$\begin{aligned}
\{L_1\} \rightarrow \{L_1\} &= 2.5 \times \tau_{\delta_3} - \tau_{\delta_2} \\
\{L_1\} \rightarrow \{L_2\} &= 2.5 \times \tau_{\delta_3} \\
\{L_1\} \rightarrow \{L_3\} &= 2.5 \times \tau_{\delta_3} + \tau_{\delta_2} \\
\{L_2\} \rightarrow \{L_1\} &= 1.5 \times \tau_{\delta_3} - \tau_{\delta_2} \\
\{L_2\} \rightarrow \{L_2\} &= 1.5 \times \tau_{\delta_3} \\
\{L_2\} \rightarrow \{L_3\} &= 1.5 \times \tau_{\delta_3} + \tau_{\delta_2} \\
\{L_3\} \rightarrow \{L_1\} &= 0.5 \times \tau_{\delta_3}) - \tau_{\delta_2} \\
\{L_3\} \rightarrow \{L_2\} &= 0.5 \times \tau_{\delta_3}) \\
\{L_3\} \rightarrow \{L_3\} &= 0.5 \times \tau_{\delta_3}) + \tau_{\delta_2} \\
\{L_4\} \rightarrow \{L_1\} &= -(0.5 \times \tau_{\delta_3} + \tau_{\delta_2}) \\
\{L_4\} \rightarrow \{L_2\} &= -0.5 \times \tau_{\delta_3} \\
\{L_4\} \rightarrow \{L_3\} &= -0.5 \times \tau_{\delta_3} + \tau_{\delta_2} \\
\{L_5\} \rightarrow \{L_1\} &= -(1.5 \times \tau_{\delta_3} + \tau_{\delta_2}) \\
\{L_5\} \rightarrow \{L_2\} &= -1.5 \times \tau_{\delta_3} \\
\{L_5\} \rightarrow \{L_3\} &= -1.5 \times \tau_{\delta_3} + \tau_{\delta_2} \\
\{L_6\} \rightarrow \{L_1\} &= -(2.5 \times \tau_{\delta_3} + \tau_{\delta_2}) \\
\{L_6\} \rightarrow \{L_2\} &= -2.5 \times \tau_{\delta_3} \\
\{L_6\} \rightarrow \{L_3\} &= -2.5 \times \tau_{\delta_3} + \tau_{\delta_2}
\end{aligned}$$

In order to get an equal number of legs in all of the six sets, at least six changes in the order of the legs are needed, which can be devided into two changes per set of legs of $\mathbf{G}_2$. The combination of those six changes, that has the lowest maximum absolute deviation, is:

$$\begin{gathered}
\{L_1\} \rightarrow \{L_1\} \\
\{L_2\} \rightarrow \{L_1\} \\
\{L_3\} \rightarrow \{L_2\} \\
\{L_4\} \rightarrow \{L_2\} \\
\{L_5\} \rightarrow \{L_3\} \\
\{L_6\} \rightarrow \{L_3\}
\end{gathered}$$

As an example of a gait switch from $\mathbf{G}_3$ to $\mathbf{G}_2$, the parameters of table 5-2 are used and for gait $\mathbf{G}_3$ the following gait is used:

$$\mathbf{G}_3 = 1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6.$$

The gait with the structure of $\mathbf{G}_2$ that is optimal to switch to is:

$$\mathbf{G}_{21} = \{1,4\} \prec \{5,2\} \prec \{3,6\}$$

The values for $\tau_{diff}$ are:

$$\tau_{diff_{3\to21}} = \begin{bmatrix} 0.2 \\ 0.1 \\ 0.4 \\ 0.0 \\ 0.3 \\ 0.2 \end{bmatrix},$$

which have a maximum value of 0.4. It is interesting to note that adding $\tau_{diff_{3\to21}}$ to $\tau_{diff_{2\to31}}$ results in a vector with all values equal to 0.4. This can be explained by the fact that this represents a gait switch from $\mathbf{G}_2$ to $\mathbf{G}_3$ and back to $\mathbf{G}_2$, which means that the events should happen in the same order as they started in, which is the case since all values of $\tau_{\text{diff}}$ are 0.4. The value of 0.4 means that compared to the case where the robot stayed in $\mathbf{G}_2$, there is a timeshift of 0.4.

Now consider the non-optimal switch to

$$\mathbf{G}_{23} = \{1,4\} \prec \{3,6\} \prec \{2,5\}$$

The values for $\tau_{\text{diff}}$ for this switch is

$$\tau_{diff_{3\to23}} = \begin{bmatrix} 0.5 \\ 0.9 \\ 0.2 \\ 0.3 \\ 1.1 \\ 0.0 \end{bmatrix},$$

which has a maximum value of 1.1.

These two gait switches are shown in Figure 5-3, where you can see that a larger value for $[\tau_{\text{diff}}]_i$ results in a longer stance time for leg $i$ during the switch which happens in the fourth and fifth cycle.

### 5-3-5 Quadruped to tripod gait switch

Consider the switch from $\mathbf{G}_2$ to $\mathbf{G}_1$, the possible changes in the order of the legs are:

$$\begin{aligned}
\{L_1\} \to \{L_1\} &= \tau_{f_1} - \tau_{f_2} \\
\{L_1\} \to \{L_2\} &= (\tau_{\delta_1} + \tau_{f_1}) - \tau_{f_2} \\
\{L_2\} \to \{L_1\} &= \tau_{f_1} - (\tau_{\delta_2} + \tau_{f_2}) \\
\{L_2\} \to \{L_2\} &= (\tau_{\delta_1} + \tau_{f_1}) - (\tau_{\delta_2} + \tau_{f_2}) \\
\{L_3\} \to \{L_1\} &= \tau_{f_1} - (2 \times \tau_{\delta 2} + \tau_{f_2}) \\
\{L_3\} \to \{L_2\} &= (\tau_{\delta_1} + \tau_{f_1} - (2 \times \tau_{\delta 2} + \tau_{f_2}))
\end{aligned}$$

The average change is then:

$$(\frac{\tau_{\delta_1}}{2} + \tau_{f_1}) - (\tau_{\delta_2} + \tau_{f_2})$$

**Figure 5-3:** Gait switches from $\mathbf{G}_3$ to $\mathbf{G}_2$. Top, optimal gait switch from $1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6$ to $\{1,4\} \prec \{5,2\} \prec \{3,6\}$. Bottom, non-optimal gait switch from $1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6$ to $\{1,4\} \prec \{3,6\} \prec \{2,5\}$. The gray blocks represent the stance and the white blocks represent the flight of the legs.

The deviation from this average is:

$$\begin{aligned}
\{L_1\} \rightarrow \{L_1\} &= \tau_{\delta_2} - \frac{\tau_{\delta_1}}{2} \\
\{L_1\} \rightarrow \{L_2\} &= \tau_{\delta_2} + \frac{\tau_{\delta_1}}{2} \\
\{L_2\} \rightarrow \{L_1\} &= -\frac{\tau_{\delta_1}}{2} \\
\{L_2\} \rightarrow \{L_2\} &= \frac{\tau_{\delta_1}}{2} \\
\{L_3\} \rightarrow \{L_1\} &= -(\tau_{\delta_2} + \frac{\tau_{\delta_1}}{2}) \\
\{L_3\} \rightarrow \{L_2\} &= -\tau_{\delta_2} + \frac{\tau_{\delta_1}}{2}
\end{aligned}$$

The two switches with the largest absolute deviation are: $\{L_1\} \rightarrow \{L_2\}$ and $\{L_3\} \rightarrow \{L_1\}$ with an absolute deviation of $|\tau_{\delta_2} + \frac{\tau_{\delta_1}}{2}|$. If you leave these out the maximum deviation is less and the gait switch is still possible. Thus the optimal gait switch from $\mathbf{G}_2$ to $\mathbf{G}_1$ consists of all combination of gaits that only use the following changes in the order of the legs

$$\begin{aligned}
&\{L_1\} \rightarrow \{L_1\} \\
&\{L_2\} \rightarrow \{L_2\} \\
&\{L_2\} \rightarrow \{L_2\} \\
&\{L_3\} \rightarrow \{L_2\}
\end{aligned}$$

As an example of a gait switch from $\mathbf{G}_2$ to $\mathbf{G}_1$ consider the following gait:

$$\mathbf{G}_2 = \{1,4\} \prec \{5,2\} \prec \{3,6\},$$

Possible optimal gait switches from $\{1,4,5\} \prec \{2,3,6\}$, to a gait with the structure of $\mathbf{G}_2$, are:

$$\{1,4,5\} \prec \{2,3,6\}$$
$$\{1,4,2\} \prec \{5,3,6\}$$

Of these two possible gaits only $\mathbf{G}_{11} = \{1,4,5\} \prec \{2,3,6\}$ is a proper tripod gait. The values of $\tau_{\text{diff}}$ for the switch from $\mathbf{G}_2$ to $\mathbf{G}_1$:

$$\tau_{diff_{2\to 11}} = \begin{bmatrix} 0.5 \\ 1.0 \\ 0.5 \\ 0.5 \\ 0.0 \\ 0.5 \end{bmatrix},$$

which has a maximum value of 1.0. The value for $\tau_{\text{diff}}$ for a non-optimal gait switch to, for example

$$\mathbf{G}_{12} = \{2,3,6\} \prec \{1,4,5\},$$

is:

$$\tau_{diff_{2\to 12}} = \begin{bmatrix} 2.0 \\ 0.5 \\ 0.0 \\ 2.0 \\ 1.5 \\ 0.0 \end{bmatrix},$$

which has a maximum value of 2.0. These two gait switches are shown in Figure 5-4, where you can see that a larger value for $[\tau_{\text{diff}}]_i$ results in a longer stance time for leg $i$ during the gait switch which happens in the fourth and fifth cycle.

**Figure 5-4:** Gait switches from $\mathbf{G}_2$ to $\mathbf{G}_1$. Top, optimal gait switch from $\{1,4\} \prec \{5,2\} \prec \{3,6\}$ to $\{1,4,5\} \prec \{2,3,6\}$. Bottom, non-optimal gait switch from $\{1,4\} \prec \{5,2\} \prec \{3,6\}$ to $\{2,3,6\} \prec \{1,4,5\}$. The gray blocks represent the stance and the white blocks represent the flight of the legs.

### 5-3-6 Tripod to quintuped gait switch

Consider the switch from $\mathbf{G}_1$ to $\mathbf{G}_3$, the possible changes in the order of the legs are:

$$
\begin{array}{rcl}
\{L_1\} \rightarrow \{L_1\} & = & \tau_{f_3} - \tau_{f_1} \\
\{L_1\} \rightarrow \{L_2\} & = & (\tau_{\delta_3} + \tau_{f_3}) - \tau_{f_1} \\
\{L_1\} \rightarrow \{L_3\} & = & (2 \times \tau_{\delta_3} + \tau_{f_3}) - \tau_{f_1} \\
\{L_1\} \rightarrow \{L_4\} & = & (3 \times \tau_{\delta_3} + \tau_{f_3}) - \tau_{f_1} \\
\{L_1\} \rightarrow \{L_5\} & = & (4 \times \tau_{\delta_3} + \tau_{f_3}) - \tau_{f_1} \\
\{L_1\} \rightarrow \{L_6\} & = & (5 \times \tau_{\delta_3} + \tau_{f_3}) - \tau_{f_1} \\
\{L_2\} \rightarrow \{L_1\} & = & \tau_{f_3} - (\tau_{\delta_1 + \tau_{f_1}}) \\
\{L_2\} \rightarrow \{L_2\} & = & (\tau_{\delta_3} + \tau_{f_3}) - (\tau_{\delta_1} + \tau_{f_1}) \\
\{L_2\} \rightarrow \{L_3\} & = & (2 \times \tau_{\delta_3} + \tau_{f_3}) - (\tau_{\delta_1} + \tau_{f_1}) \\
\{L_2\} \rightarrow \{L_4\} & = & (3 \times \tau_{\delta_3} + \tau_{f_3}) - (\tau_{\delta_1} + \tau_{f_1}) \\
\{L_2\} \rightarrow \{L_5\} & = & (4 \times \tau_{\delta_3} + \tau_{f_3}) - (\tau_{\delta_1} + \tau_{f_1}) \\
\{L_2\} \rightarrow \{L_6\} & = & (5 \times \tau_{\delta_3} + \tau_{f_3}) - (\tau_{\delta_1} + \tau_{f_1})
\end{array}
$$

The average change is then:

$$(2.5 \times \tau_{\delta_3} + \tau_{f_3}) - (\frac{\tau_{\delta_1}}{2} + \tau_{f_1})$$

The deviation from this average is:

$$\begin{aligned}
\{L_1\} \rightarrow \{L_1\} &= \frac{\tau_{\delta_1}}{2} - 2.5 \times \tau_{\delta_3} \\
\{L_1\} \rightarrow \{L_2\} &= \frac{\tau_{\delta_1}}{2} - 1.5 \times \tau_{\delta_3} \\
\{L_1\} \rightarrow \{L_3\} &= \frac{\tau_{\delta_1}}{2} - 0.5 \times \tau_{\delta_3} \\
\{L_1\} \rightarrow \{L_4\} &= \frac{\tau_{\delta_1}}{2} + 0.5 \times \tau_{\delta_3} \\
\{L_1\} \rightarrow \{L_5\} &= \frac{\tau_{\delta_1}}{2} + 1.5 \times \tau_{\delta_3} \\
\{L_1\} \rightarrow \{L_6\} &= \frac{\tau_{\delta_1}}{2} + 2.5 \times \tau_{\delta_3} \\
\{L_2\} \rightarrow \{L_1\} &= -(\frac{\tau_{\delta_1}}{2} + 2.5 \times \tau_{\delta_3}) \\
\{L_2\} \rightarrow \{L_2\} &= -(\frac{\tau_{\delta_1}}{2} + 1.5 \times \tau_{\delta_3}) \\
\{L_2\} \rightarrow \{L_3\} &= -(\frac{\tau_{\delta_1}}{2} + 0.5 \times \tau_{\delta_3}) \\
\{L_2\} \rightarrow \{L_4\} &= -\frac{\tau_{\delta_1}}{2} + 0.5 \times \tau_{\delta_3} \\
\{L_2\} \rightarrow \{L_5\} &= -\frac{\tau_{\delta_1}}{2} + 1.5 \times \tau_{\delta_3} \\
\{L_2\} \rightarrow \{L_6\} &= -\frac{\tau_{\delta_1}}{2} + 2.5 \times \tau_{\delta_3}
\end{aligned}$$

In order to get an equal number of legs in all of the six sets, at least six changes in the order of the legs are needed, which can be devided into two changes per set of legs of $\mathbf{G}_2$. The combination of those six changes, that has the lowest maximum absolute deviation, is:

$$\begin{aligned}
&\{L_1\} \rightarrow \{L_1\} \\
&\{L_1\} \rightarrow \{L_2\} \\
&\{L_1\} \rightarrow \{L_3\} \\
&\{L_2\} \rightarrow \{L_4\} \\
&\{L_2\} \rightarrow \{L_5\} \\
&\{L_2\} \rightarrow \{L_6\}
\end{aligned}$$

As an example of a gait switch from $\mathbf{G}_1$ to $\mathbf{G}_3$, the parameters of table 5-2 are used and for gait $\mathbf{G}_1$ the following gait is used:

$$\mathbf{G}_1 = \{1, 4, 5\} \prec \{2, 3, 6\}.$$

The gaits with the structure of $\mathbf{G}_3$ that are optimal to switch to are any combination of the left and right column of table 5-3: The values of $\tau_{\text{diff}}$ will be calculated for

$$\begin{aligned}
\mathbf{G}_{31} &= 1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6 \\
\mathbf{G}_{34} &= 5 \prec 4 \prec 1 \prec 6 \prec 3 \prec 2
\end{aligned}$$

**Table 5-3:** Optimal quintuped gaits to switch to from $\{1,4,5\} \prec \{2,3,6\}$

| | |
|---|---|
| $1 \prec 4 \prec 5$ | $2 \prec 3 \prec 6$ |
| $1 \prec 5 \prec 4$ | $2 \prec 6 \prec 3$ |
| $4 \prec 1 \prec 5$ | $3 \prec 2 \prec 6$ |
| $4 \prec 5 \prec 1$ | $3 \prec 6 \prec 2$ |
| $5 \prec 1 \prec 4$ | $6 \prec 2 \prec 3$ |
| $5 \prec 4 \prec 1$ | $6 \prec 3 \prec 2$ |

The values for $\tau_{diff}$ are:

$$\tau_{diff_{1\to 31}} = \begin{bmatrix} 0.4 \\ 0.0 \\ 0.2 \\ 0.6 \\ 0.8 \\ 0.4 \end{bmatrix}, \tau_{diff_{1\to 34}} = \begin{bmatrix} 0.8 \\ 0.4 \\ 0.2 \\ 0.6 \\ 0.4 \\ 0.0 \end{bmatrix},$$

which have a maximum value of 0.8. Now consider the non-optimal switch to

$$\mathbf{G}_{33} = 1 \prec 3 \prec 5 \prec 2 \prec 4 \prec 6$$

The values for $\tau_{\text{diff}}$ for this switch is

$$\tau_{diff_{1\to 33}} = \begin{bmatrix} 0.8 \\ 0.4 \\ 0.0 \\ 1.6 \\ 0.2 \\ 0.8 \end{bmatrix},$$

which has a maximum value of 1.6.

These three gait switches are shown in Figure 5-5, where you can see that a larger value for $[\tau_{\text{diff}}]_i$ results in a longer stance time for leg $i$ during the switch which happens in the fourth and fifth cycle.

**Figure 5-5:** Gait switches from $\mathbf{G}_1$ to $\mathbf{G}_3$. Top, optimal gait switch from $\{1,4,5\} \prec \{2,3,6\}$ to $1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6$. Middle, optimal gait switch from $\{1,4,5\} \prec \{2,3,6\}$ to $5 \prec 4 \prec 1 \prec 6 \prec 3 \prec 2$. Bottom, non-optimal gait switch from $\{1,4,5\} \prec \{2,3,6\}$ to $1 \prec 3 \prec 5 \prec 2 \prec 4 \prec 6$. The gray blocks represent the stance and the white blocks represent the flight of the legs.

### 5-3-7 Quintuped to tripod gait switch

Consider the switch from $\mathbf{G}_3$ to $\mathbf{G}_1$, the possible changes in the order of the legs are:

$$
\begin{aligned}
\{L_1\} \rightarrow \{L_1\} &= \tau_{f_1} - \tau_{f_3} \\
\{L_1\} \rightarrow \{L_2\} &= (\tau_{\delta_1} + \tau_{f_1}) - \tau_{f_3} \\
\{L_2\} \rightarrow \{L_1\} &= \tau_{f_1} - (\tau_{\delta 3 + \tau_{f_3}}) \\
\{L_2\} \rightarrow \{L_2\} &= (\tau_{\delta_1} + \tau_{f_1}) - (\tau_{\delta_3} + \tau_{f_3}) \\
\{L_3\} \rightarrow \{L_1\} &= \tau_{f_1} - (2 \times \tau_{\delta 3 + \tau_{f_3}}) \\
\{L_3\} \rightarrow \{L_2\} &= (\tau_{\delta_1} + \tau_{f_1}) - (\tau_{f_3} + 2 \times \tau_{\delta 3}) \\
\{L_4\} \rightarrow \{L_1\} &= \tau_{f_1} - (3 \times \tau_{\delta_3} + \tau_{f_3}) \\
\{L_4\} \rightarrow \{L_2\} &= (\tau_{\delta_1} + \tau_{f_1}) - (3 \times \tau_{\delta_3} + \tau_{f_3}) \\
\{L_5\} \rightarrow \{L_1\} &= \tau_{f_1} - (4 \times \tau_{\delta_3} + \tau_{f_3}) \\
\{L_5\} \rightarrow \{L_2\} &= (\tau_{\delta_1} + \tau_{f_1}) - (4 \times \tau_{\delta_3} + \tau_{f_3}) \\
\{L_6\} \rightarrow \{L_1\} &= \tau_{f_1} - (5 \times \tau_{\delta_3} + \tau_{f_3}) \\
\{L_6\} \rightarrow \{L_2\} &= (\tau_{\delta_1} + \tau_{f_1}) - (5 \times \tau_{\delta_3} + \tau_{f_3})
\end{aligned}
$$

The average change is then:

$$(\frac{\tau_{\delta_1}}{2} + \tau_{f_1}) - (2.5 \times \tau_{\delta_3} + \tau_{f_3})$$

The deviation from this average is:

$$
\begin{aligned}
\{L_1\} \rightarrow \{L_1\} &= 2.5 \times \tau_{\delta_3} - \frac{\tau_{\delta_1}}{2} \\
\{L_1\} \rightarrow \{L_2\} &= 2.5 \times \tau_{\delta_3} + \frac{\tau_{\delta_1}}{2} \\
\{L_2\} \rightarrow \{L_1\} &= 1.5 \times \tau_{\delta_3} - \frac{\tau_{\delta_1}}{2} \\
\{L_2\} \rightarrow \{L_2\} &= 1.5 \times \tau_{\delta_3} + \frac{\tau_{\delta_1}}{2} \\
\{L_3\} \rightarrow \{L_1\} &= 0.5 \times \tau_{\delta_3}) - \frac{\tau_{\delta_1}}{2} \\
\{L_3\} \rightarrow \{L_2\} &= 0.5 \times \tau_{\delta_3}) + \frac{\tau_{\delta_1}}{2} \\
\{L_4\} \rightarrow \{L_1\} &= -(0.5 \times \tau_{\delta_3} + \frac{\tau_{\delta_1}}{2}) \\
\{L_4\} \rightarrow \{L_2\} &= -0.5 \times \tau_{\delta_3} + \frac{\tau_{\delta_1}}{2} \\
\{L_5\} \rightarrow \{L_1\} &= -(1.5 \times \tau_{\delta_3} + \frac{\tau_{\delta_1}}{2}) \\
\{L_5\} \rightarrow \{L_2\} &= -1.5 \times \tau_{\delta_3} + \frac{\tau_{\delta_1}}{2}
\end{aligned}
$$

$$\begin{aligned}\{L_6\} \rightarrow \{L_1\} &= -(2.5 \times \tau_{\delta_3} + \frac{\tau_{\delta_1}}{2})\\ \{L_6\} \rightarrow \{L_2\} &= -2.5 \times \tau_{\delta_3} + \frac{\tau_{\delta_1}}{2}\end{aligned}$$

In order to get an equal number of legs in all of the six sets, at least six changes in the order of the legs are needed, which can be devided into two changes per set of legs of $\mathbf{G}_2$. The combination of those six changes, that has the lowest maximum absolute deviation, is:

$$\begin{aligned}\{L_1\} &\rightarrow \{L_1\}\\ \{L_2\} &\rightarrow \{L_1\}\\ \{L_3\} &\rightarrow \{L_1\}\\ \{L_4\} &\rightarrow \{L_2\}\\ \{L_5\} &\rightarrow \{L_2\}\\ \{L_6\} &\rightarrow \{L_2\}\end{aligned}$$

As an example of a gait switch from $\mathbf{G}_3$ to $\mathbf{G}_1$, the parameters of table 5-2 are used and for gait $\mathbf{G}_3$ the following gait is used:

$$\mathbf{G}_3 = 1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6.$$

The gait with the structure of $\mathbf{G}_1$ that is optimal to switch to is:

$$\mathbf{G}_{11} = \{1, 4, 5\} \prec \{3, 6, 2\}$$

The values for $\tau_{diff}$ are:

$$\tau_{diff_{3\rightarrow 11}} = \begin{bmatrix}0.4\\0.8\\0.6\\0.2\\0.0\\0.4\end{bmatrix},$$

which has a maximum value of 0.8.

**Figure 5-6:** Gait switches from $\mathbf{G}_3$ to $\mathbf{G}_1$. Top, optimal gait switch from $1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6$ to $\{1,4,5\} \prec \{2,3,6\}$. Bottom, non-optimal gait switch from $1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6$ to $\{2,3,6\} \prec \{1,4,5\}$. The gray blocks represent the stance and the white blocks represent the flight of the legs.

Now consider the non-optimal switch to

$$\mathbf{G}_{12} = \{2,3,6\} \prec \{1,4,5\},$$

The values for $\tau_{\text{diff}}$ for this switch is

$$\tau_{diff_{3\to 12}} = \begin{bmatrix} 2.0 \\ 0.4 \\ 0.2 \\ 1.8 \\ 1.6 \\ 0.0 \end{bmatrix},$$

which has a maximum value of 2.0.

These two gait switches are shown in Figure 5-6, where you can see that a larger value for $[\tau_{\text{diff}}]_i$ results in a longer stance time for leg $i$ during the switch which happens in the fourth and fifth cycle.

## 5-4 'Constant stance time'-model

The current max-plus model is written in such a way that the real flight time is equal to $\tau_f$ and if there are disturbances or when switching gaits the stance times are extended

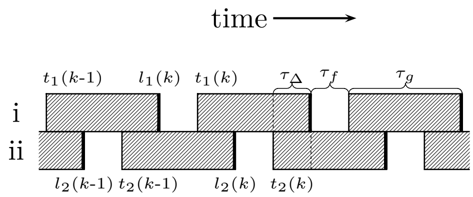


**Figure 5-7:** Time evolution of a bipedal gait. The hatched rectangles represent the leg stance and the solid thick vertical lines represent the lift off events.

to compensate for the disturbances or to switch to a new gait. This is because the synchronization constraints are put on the lift-off event. By putting the synchronization constraints on the touchdown event instead of the lift-off event the model forces the stance time to be equal to $\tau_g$ and adjusts the flight time to compensate for disturbances or to switch gaits. The model that is composed from these equations is called the 'constant stance time'-model.

The touchdown events of the original model are described as

$$t(k) = \tau_f \otimes l(k)$$

the lift-off events are described as

$$l(k) = P \otimes t(k) \oplus (\tau_g \otimes E \oplus Q) \otimes t(k-1)$$

From this description it is clear that the touchdown events only depend on the lift-off events, that the time difference is exactly $\tau_f$ and that the lift-off events depend on the touchdown events of the previous cycle with a time difference of $\tau_g$, but also on all the synchronization constraints defined by $P$ and $Q$.

These synchronization constraints can also be added to the touchdown events, then the lift-off events would only depend on the touchdown events of the previous cycle and the time difference would be exactly $\tau_g$, which means the stance time is always $\tau_g$.

Consider the example of Figure 3-3. The figure is repeated here as Figure 5-7. In order to get the legs synchronized as shown in Figure 3-3, leg 1 can only lift off $\tau_g$ seconds after leg 1 has touched down the last time. Leg 1 can only touch down $\tau_f$ seconds after leg 1 has lifted off and $\tau_\gamma$ seconds after leg 2 has touched down. For leg 2 the relationship is as follows: leg 2 can only lift off $\tau_g$ seconds after leg 2 has touched down, it can only touch down $\tau_f$ seconds after leg 2 has lifted off and $\tau_\gamma$ seconds after leg 1 has touched down. This can be written as:

$$\begin{array}{rcl} t_1(k) & = & l_1(k) \otimes \tau_f \oplus t_2(k-1) \otimes \tau_\gamma \\ t_2(k) & = & l_2(k) \otimes \tau_f \oplus t_1(k) \otimes \tau_\gamma \\ l_1(k) & = & t_1(k-1) \otimes \tau_g \\ l_2(k) & = & t_2(k-1) \otimes \tau_g \end{array} , \qquad (5\text{-}13)$$

**Figure 5-8:** Gait switch from $\{1,4\} \prec \{5,2\} \prec \{3,6\}$ to $1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6$ using the 'constant ground time'-model



**Figure 5-9:** Gait switch from $1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6$ to $\{1,4\} \prec \{5,2\} \prec \{3,6\}$ using the 'constant ground time'-model

This can be extended to any number of legs $n$. The statevector for an $n$-legged robot is defined as

$$x(k) = [\underbrace{t_1(k) \cdots t_n(k)}_{t(k)} \underbrace{l_1(k) \cdots l_n(k)}_{l(k)}]^\top$$

The system equations can then be written as an implicit model as shown in equation 5-14

$$\begin{bmatrix} t(k) \\ l(k) \end{bmatrix} = \left[\begin{array}{c|c} P_c & \tau_f \otimes E \\ \hline \mathcal{E} & \mathcal{E} \end{array}\right] \otimes \begin{bmatrix} t(k) \\ l(k) \end{bmatrix} \oplus \left[\begin{array}{c|c} Q_c \oplus E & \mathcal{E} \\ \hline \tau_g \otimes E & \mathcal{E} \end{array}\right] \otimes \begin{bmatrix} t(k-1) \\ l(k-1) \end{bmatrix}, \tag{5-14}$$

where $P_c = P \otimes \tau_f$ and $Q_c = Q \otimes \tau_f$ and where the matrices $P$ and $Q$ are defined as in equations 3-12 and 3-13.

For some switches this model works well as can be seen in Figure 5-8, where a switch from $\{1,4\} \prec \{5,2\} \prec \{3,6\}$ to $1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6$ is shown. The values for the gaits are those of Table 5-2 The switch back from $1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6$ to $\{1,4\} \prec \{5,2\} \prec \{3,6\}$ however does not work as well, as can be seen in Figure 5-9, where in the third cycle legs $2,3,5$ and $6$ are in the air at the same time. Leaving the robot standing on only legs 1 and 4.

Eventhough this model does not give the desired results, it has inspired the design of a new model which is discussed in the next section.

## 5-5 'Multiple flight time'-model

### 5-5-1 The model

As was shown in section 5-3 even the optimal gait switches have different stance times during a gait switch. The time the legs spend on the ground longer than desired is equal to $\tau_{\text{diff}}$ as defined in section 5-2. The 'multiple flight time'-model changes the flight time of each leg in order to compensate for the change of events of each leg, which are also represented by $\tau_{\text{diff}}$.

The gait switch is defined as a switch from $\mathbf{G}_1$ to $\mathbf{G}_2$. It is assumed that $\tau_g$ is the same for both gaits. Furthermore it is assumed that $\tau_{\delta_1}^{\otimes m_1} = \tau_{\gamma 1}$,where $\tau_{f\mathbf{G}_1}, \tau_{\Delta \mathbf{G}_1}$ and $\tau_g$ are the gait parameters of $\mathbf{G}_1$ and $m_1$ is the number of sets of $\mathbf{G}_1$. Finally it is also assumed that $\tau_{\delta_2}^{\otimes m_2} = \tau_{\gamma 2}$, where $\tau_{f\mathbf{G}_2}, \tau_{\Delta \mathbf{G}_2}$ and $\tau_g$ are the gait parameters of $\mathbf{G}_2$ and $m_2$ is then number of sets of $\mathbf{G}_2$.

The 'multiple flight time'-model is described in equations 5-15 and 5-16.

$$\begin{bmatrix} t(k) \\ l(k) \end{bmatrix} = \left[\begin{array}{c|c} \mathcal{E} & R \\ \hline P & \mathcal{E} \end{array}\right] \otimes \begin{bmatrix} t(k) \\ l(k) \end{bmatrix} \oplus \left[\begin{array}{c|c} E & \mathcal{E} \\ \hline \tau_g \otimes E \oplus Q & E \end{array}\right] \otimes \begin{bmatrix} t(k-1) \\ l(k-1) \end{bmatrix} \tag{5-15}$$

where

$$R = \begin{bmatrix} \tau_{f_1} & \varepsilon & \dots & \varepsilon \\ \varepsilon & \tau_{f_2} & \ddots & \varepsilon \\ \vdots & \ddots & \ddots & \vdots \\ \varepsilon & \varepsilon & \dots & \tau_{f_n} \end{bmatrix}. \tag{5-16}$$

By forcing the legs, that would normally stay on the ground longer, to stay in flight longer for the same amount of time the gait switch can be achieved without the legs having different stance times. However increasing the flight time of legs can result in an instable state of the robot as was seen in the previous section. Instead of increasing the flight time for the legs that would normally stay on the ground longer, the flight time for the other legs is decreased. This results in the same relative timing difference between the legs, without causing instability.

However this method does have its limitations. Because the flight time is decreased instead of increased the maximum relative timing difference that can be achieved in one cycle is limited to the value of $\tau_f$. That is if one or more legs stays in the air for $\tau_f$ seconds and others 0 seconds, however a flight time of zero seconds is impossible. There is a minimum time the legs have to stay in the air due to mechanical limitations which is denoted by $\tau_{f_{\min}}$. The real maximum relative timing difference is thus

$$\tau_f - \tau_{f_{\min}}.$$

This can be slightly extended by using some of the knowledge of the gaits and optimal switches. The timing between the legs in the first and last set of the gait is $\tau_g - \tau_\Delta$ for all gaits, which means that during the gait switch from $\mathbf{G}_1$ to $\mathbf{G}_2$, this will only change from $\tau_g - \tau_{\Delta_1}$ to $\tau_g - \tau_{\Delta_2}$, which means there is at least a doublestance time of

$\min(\tau_{\Delta_1}, \tau_{\Delta_2})$ between all legs during the switch. Furthermore the timing between set $L_i$ and $L_{i+1}$is $\tau_\delta$, where $i \in \{1, \ldots, m-1\}$. This means that during the gait switch it is allowed to increase $\tau_f$ to $\tau_\delta$ as long as $\tau_\Delta = 0$. The maximum relative timing difference is then:

$$\tau_\delta - \tau_{f_{\min}}.$$

In most cases however $\tau_{\text{diff}} > \tau_f + \tau_\Delta - \tau_{f_{\min}}$. This problem can be solved by letting the gait switch take two cycles. The first cycle using the model in equation 5-15 for gait $\mathbf{G}_1$ and the second cycle using the same model for gait $\mathbf{G}_2$. By doing this the maximum relative difference that can be achieved is:

$$\tau_{\delta_1} + \tau_{\delta_2} - \tau_{f_{\min}}. \tag{5-17}$$

Now if $\max(\tau_{\text{diff}}) \leq \tau_{\delta_1} + \tau_{\delta_2} - \tau_{f_{\min}}$ the gait switch can be optimized using the model of equation 5-15 in such a way that all legs have a stance time of exactly $\tau_g$ even during the gait switch.

Consider that the gait switch is started after cycle $k$ the state vectors that follows are determined as follows:

$$\begin{aligned}
x(k+1) &= A_{\mathbf{G}_1} \otimes x(k) && (5\text{-}18)\\
x(k+2) &= A_{S1} \otimes x(k+1) && (5\text{-}19)\\
x(k+3) &= A_{S2} \otimes x(k+2) && (5\text{-}20)\\
x(k+4) &= A_{\mathbf{G}_2} \otimes x(k+3) && (5\text{-}21)
\end{aligned}$$

where $A_{S1}$ is the matrix derived from equations 5-15 and 5-16 with matrix $R$= $R_{S1}$, $\tau_{\Delta_{S1}}$ and $\tau_g$ and $A_{S2}$ is the matrix derived from equations 5-15 and 5-16 with matrix $R = R_{S2}$, $\tau_{\Delta_{S2}}$ and $\tau_g$.

Matrices $R_{S1}$ and $R_{S2}$ and the values $\tau_{\Delta_{S1}}$ and $\tau_{\Delta_{S2}}$ are determined using the following method:

- First check if $\max(\tau_{\text{diff}} + 2 \times \tau_{f_{\min}}) \leq \tau_{\delta_{\mathbf{G}_1}} + \tau_{\delta_{\mathbf{G}_2}}$, if this is the case continue, if it is not the case then this method cannot perfectly optimize the gait switch.
- Then determine $\tau_{\text{diff}}$.
- Check if $\tau_{\delta_{\mathbf{G}_1}} > \tau_{\delta_{\mathbf{G}_2}}$, if this is the case then
  - Determine $R_{\mathbf{G}_1}$ using the following equation:

$$[R_{S1}]_{jj} = \min([\tau_{\text{diff}}]_j + \tau_{f_{\min}}, \tau_{\delta_{\mathbf{G}_1}}), \forall j \in \{1, \ldots, n\}$$

  - If $[R_{S1}]_{jj} - \tau_{f_{\min}} \geq \tau_{\Delta_{\mathbf{G}_1}}, \forall j \in \{1, \ldots, n\}$ then

$$\begin{aligned}
\tau_{\Delta_{S1}} &= \tau_{\Delta_{\mathbf{G}_1}}\\
R_{S1} &= R_{S1} - \tau_{\Delta_1}
\end{aligned}$$

- If this is not the case then

$$\begin{aligned} \tau_{\Delta_{S1}} &= 0 \\ R_{S1} &= R_{S1}. \end{aligned}$$

- If $[R_{S1}]_{jj} = \max([\tau_{diff}]_j + \tau_{f_{\min}}), \forall j \in \{1, \ldots, n\}$ then

$$\begin{aligned} [R_{S2}]_{jj} &= \tau_{f_{\mathbf{G}_2}}, \forall j \in \{1, \ldots, n\} \\ \tau_{\Delta_{S2}} &= \tau_{\Delta_{\mathbf{G}_2}} \end{aligned}$$

- If this is not the case then

$$\begin{aligned} [R_{S2}]_{jj} &= \tau_{\delta_{\mathbf{G}_2}} + ([\tau_{\text{diff}}]_j + \tau_{f_{\min}} - [R_{\mathbf{G}_1}]_{jj}) - \ldots \\ &\quad \max([\tau_{\text{diff}}]_j + \tau_{f_{\min}} - [R_{\mathbf{G}_1}]_{jj}), \forall j \in \{1, \ldots, n\} \end{aligned}$$

  * If $[R_{S2}]_{jj} - \tau_{f_{\min}} \geq \tau_{\Delta_{\mathbf{G}_2}}, \forall j \in \{1, \ldots, n\}$ then

$$\begin{aligned} \tau_{\Delta_{S2}} &= \tau_{\Delta_{\mathbf{G}_2}} \\ R_{S2} &= R_{S2} - \tau_{\Delta_2} \end{aligned}$$

  * If this is not the case then

$$\begin{aligned} \tau_{\Delta_{S2}} &= 0 \\ R_{S2} &= R_{S2}. \end{aligned}$$

- If this is not the case then:
  - Determine $R_{\mathbf{G}_2}$ using the following equation:

$$[R_{S2}]_{jj} = \min([\tau_{\text{diff}}]_j + \tau_{f_{\min}}, \tau_{\delta_{\mathbf{G}_2}}), \forall j \in \{1, \ldots, n\}$$

  - If $[R_{S2}]_{jj} - \tau_{f_{\min}} \geq \tau_{\Delta_{\mathbf{G}_2}}, \forall j \in \{1, \ldots, n\}$ then

$$\begin{aligned} \tau_{\Delta_{S2}} &= \tau_{\Delta_{\mathbf{G}_2}} \\ R_{S2} &= R_{S2} - \tau_{\Delta_2} \end{aligned}$$

  - If this is not the case then

$$\begin{aligned} \tau_{\Delta_{S2}} &= 0 \\ R_{S2} &= R_{S2}. \end{aligned}$$

  - If $[R_{S2}]_{jj} = \max([\tau_{diff}]_j + \tau_{f_{\min}}), \forall j \in \{1, \ldots, n\}$ then

$$\begin{aligned} [R_{S1}]_{jj} &= \tau_{f_{\mathbf{G}_1}}, \forall j \in \{1, \ldots, n\} \\ \tau_{\Delta_{S1}} &= \tau_{\Delta_{\mathbf{G}_1}} \end{aligned}$$

– If this is not the case then

$$\begin{aligned}[R_{S1}]_{jj} &= \tau_{\delta_{\mathbf{G}_1}} + ([\tau_{\text{diff}}]_j + \tau_{f_{\min}} - [R_{\mathbf{G}_2}]_{jj}) - \ldots \\ &\quad \max([\tau_{\text{diff}}]_j + \tau_{f_{\min}} - [R_{\mathbf{G}_2}]_{jj}), \forall j \in \{1, \ldots, n\}\end{aligned}$$

∗ If $[R_{S1}]_{jj} - \tau_{f_{\min}} \geq \tau_{\Delta_{\mathbf{G}_1}}, \forall j \in \{1, \ldots, n\}$ then

$$\begin{aligned}\tau_{\Delta_{S1}} &= \tau_{\Delta_{\mathbf{G}_1}} \\ R_{S1} &= R_{S1} - \tau_{\Delta_1}\end{aligned}$$

∗ If this is not the case then

$$\begin{aligned}\tau_{\Delta_{S1}} &= 0 \\ R_{S1} &= R_{S1}.\end{aligned}$$

In the next subsection this method of gait switching will be applied to several optimal gait switches, as found in section 5-3.

### 5-5-2 Application

Consider the gaits:

$$\mathbf{G}_1 = \{1, 4, 5\} \prec \{2, 3, 6\} \tag{5-22}$$
$$\mathbf{G}_2 = \{1, 4\} \prec \{5, 2\} \prec \{3, 6\} \tag{5-23}$$
$$\mathbf{G}_3 = 1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6 \tag{5-24}$$

with the gait parameters of Table 5-2.

The gaits have the following eigenvectors:

$$v_{G_1} = \begin{bmatrix}1.0\\2.0\\2.0\\1.0\\1.0\\2.0\\0.0\\1.0\\1.0\\0.0\\0.0\\1.0\end{bmatrix}, v_{G_2} = \begin{bmatrix}0.5\\1.0\\1.5\\0.5\\1.0\\1.5\\0.0\\0.5\\1.0\\0.0\\0.5\\1.0\end{bmatrix} \text{ and } v_{G_3} = \begin{bmatrix}0.2\\0.8\\1.0\\0.4\\0.6\\1.2\\0.0\\0.6\\0.8\\0.2\\0.4\\1.0\end{bmatrix}$$

There are six possible gait switches: from $\mathbf{G}_1$ to either $\mathbf{G}_2$ or $\mathbf{G}_3$, from $\mathbf{G}_2$ to either $\mathbf{G}_1$ or $\mathbf{G}_3$ and from $\mathbf{G}_3$ to either $\mathbf{G}_1$ or $\mathbf{G}_2$.

The $\tau_{\text{diff}}$ for these gait switches are:

$$\tau_{diff_{\mathbf{G}_1\to\mathbf{G}_2}} = \begin{bmatrix} 0.5\\0.0\\0.5\\0.5\\1.0\\0.5 \end{bmatrix}, \tau_{diff_{\mathbf{G}_1\to\mathbf{G}_3}} = \begin{bmatrix} 0.4\\0.0\\0.2\\0.6\\0.8\\0.4 \end{bmatrix}, \tau_{diff_{\mathbf{G}_2\to\mathbf{G}_1}} = \begin{bmatrix} 0.5\\1.0\\0.5\\0.5\\0.0\\0.5 \end{bmatrix},$$

$$\tau_{diff_{\mathbf{G}_2\to\mathbf{G}_3}} = \begin{bmatrix} 0.2\\0.3\\0.0\\0.4\\0.1\\0.2 \end{bmatrix}, \tau_{diff_{\mathbf{G}_3\to\mathbf{G}_1}} = \begin{bmatrix} 0.4\\0.8\\0.6\\0.2\\0.0\\0.4 \end{bmatrix}, \tau_{diff_{\mathbf{G}_3\to\mathbf{G}_2}} = \begin{bmatrix} 0.2\\0.1\\0.4\\0.0\\0.3\\0.2 \end{bmatrix}.$$

Using $\tau_{diff_{\mathbf{G}_1\to\mathbf{G}_2}}$ the matrices $R_{S1}$ and $R_{S2}$ and the values $\tau_{\Delta_{S1}}$ and $\tau_{\Delta_{S1}}$ are found for the gait switch from $\mathbf{G}_1$ to $\mathbf{G}_2$:

$$R_{S1} = \begin{bmatrix} 0.6 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0.1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0.6 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0.6 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1.0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.6 \end{bmatrix}, R_{S2} = \begin{bmatrix} 0.4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0.4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0.4 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0.4 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.5 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.4 \end{bmatrix}$$

and

$$\tau_{\Delta_{S1}} = 0, \tau_{\Delta_{S2}} = 0.$$

Using $\tau_{diff_{\mathbf{G}_1\to\mathbf{G}_3}}$ the matrices $R_{S1}$ and $R_{S2}$ and the values $\tau_{\Delta_{S1}}$ and $\tau_{\Delta_{S1}}$ are found for the gait switch from $\mathbf{G}_1$ to $\mathbf{G}_3$:

$$R_{S1} = \begin{bmatrix} 0.5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0.1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0.3 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0.7 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.9 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.5 \end{bmatrix}, R_{S2} = \begin{bmatrix} 0.2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0.2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0.2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0.2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.2 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.2 \end{bmatrix}$$

and

$$\tau_{\Delta_{S1}} = 0, \tau_{\Delta_{S2}} = 0.$$

Using $\tau_{diff_{\mathbf{G}_2\to\mathbf{G}_1}}$ the matrices $R_{S1}$ and $R_{S2}$ and the values $\tau_{\Delta_{S1}}$ and $\tau_{\Delta_{S1}}$ are found for the gait switch from $\mathbf{G}_2$ to $\mathbf{G}_1$:

$$R_{S1} = \begin{bmatrix} 0.4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0.5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0.4 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0.4 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.4 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.4 \end{bmatrix}, R_{S2} = \begin{bmatrix} 0.6 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1.0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0.6 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0.6 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.6 \end{bmatrix}$$

and

$$\tau_{\Delta_{S1}} = 0, \tau_{\Delta_{S2}} = 0.$$

Using $\tau_{diff_{\mathbf{G}_2 \to \mathbf{G}_3}}$ the matrices $R_{S1}$ and $R_{S2}$ and the values $\tau_{\Delta_{S1}}$ and $\tau_{\Delta_{S1}}$ are found for the gait switch from $\mathbf{G}_2$ to $\mathbf{G}_3$:

$$R_{S1} = \begin{bmatrix} 0.3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0.4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0.1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0.5 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.2 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.3 \end{bmatrix}, R_{S2} = \begin{bmatrix} 0.2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0.2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0.2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0.2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.2 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.2 \end{bmatrix}$$

and

$$\tau_{\Delta_{S1}} = 0, \tau_{\Delta_{S2}} = 0.$$

Using $\tau_{diff_{\mathbf{G}_3 \to \mathbf{G}_1}}$ the matrices $R_{S1}$ and $R_{S2}$ and the values $\tau_{\Delta_{S1}}$ and $\tau_{\Delta_{S1}}$ are found for the gait switch from $\mathbf{G}_3$ to $\mathbf{G}_1$:

$$R_{S1} = \begin{bmatrix} 0.2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0.2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0.2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0.2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.2 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.2 \end{bmatrix}, R_{S2} = \begin{bmatrix} 0.5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0.9 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0.7 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0.3 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.5 \end{bmatrix}$$

and

$$\tau_{\Delta_{S1}} = 0, \tau_{\Delta_{S2}} = 0.$$

Using $\tau_{diff_{\mathbf{G}_3 \to \mathbf{G}_2}}$ the matrices $R_{S1}$ and $R_{S2}$ and the values $\tau_{\Delta_{S1}}$ and $\tau_{\Delta_{S1}}$ are found for the gait switch from $\mathbf{G}_3$ to $\mathbf{G}_2$:

$$R_{S1} = \begin{bmatrix} 0.2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0.2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0.2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0.2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.2 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.2 \end{bmatrix}, R_{S2} = \begin{bmatrix} 0.3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0.2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0.5 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0.1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.4 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0.3 \end{bmatrix}$$

and

$$\tau_{\Delta_{S1}} = 0, \tau_{\Delta_{S2}} = 0.$$

Using these matrices and the model as described in the previous section results in the following gait switches as shown in Figure 5-10 (done in the following order: $\mathbf{G}_3 \Rightarrow \mathbf{G}_2 \Rightarrow \mathbf{G}_1 \Rightarrow \mathbf{G}_2 \Rightarrow \mathbf{G}_3 \Rightarrow \mathbf{G}_1 \Rightarrow \mathbf{G}_3$)

**Figure 5-10:** Gait switches using the multiple $\tau_f$-model with the following order for the gaits: $\mathbf{G}_3 \Rightarrow \mathbf{G}_2 \Rightarrow \mathbf{G}_1 \Rightarrow \mathbf{G}_2 \Rightarrow \mathbf{G}_3 \Rightarrow \mathbf{G}_1 \Rightarrow \mathbf{G}_3$. The gray blocks represent the stance and the white blocks represent the flight of the legs.

## 5-6 'Time' manipulation

### 5-6-1 Constant acceleration

The current max-plus model for the gaits of the robot results in a constant velocity, which is determined by the time the legs stay on the ground ($\tau_g$) and the touchdown and lift-off angles. If the touchdown and lift-off angles are fixed the only possibility to speed up or slow down the robot is by changing the value of $\tau_g$ and adjusting the values for $\tau_f$ and $\tau_\Delta$ accordingly. This however results in new gaits for which the optimal gait transitions need to be determined and the velocity would instantly change once a leg is in the next cycle, instead of gradually increasing.

The timing of the robot is based on $\tau$ which is by default equal to $t$, which is the real time. By manipulating the relation between $\tau$ and $t$ the robot can accelerate or decelerate.

In order to get a constant acceleration

$$\tau = \alpha \times (t)^2, \tag{5-25}$$

where $\alpha$ is the constant that determines the acceleration.

This can be proven by first looking at the distance $s$ the robot travels and taking the derivative with regards to time twice.

$$s(\tau) = \bar{v} \times \tau,$$

where $\bar{v}$ is a the distance one leg pushes the robot forward divided by the time the leg takes to do that: $\tau_g$.

$$s(t) = \bar{v} \times \alpha \times t^2 \tag{5-26}$$

The first derivative of $s$ is the velocity $v$:

$$v(t) = 2\bar{v} \times \alpha \times t, \tag{5-27}$$

The second derivative of $s$ is the acceleration $a$:

$$a(t) = 2\bar{v} \times \alpha. \tag{5-28}$$

There are ofcourse some limitations to the velocity and acceleration the robot can achieve. The physical limitations will be determined by the maximum acceleration and velocity that can be achieved. This will mean there will be minimum values for the flight time and the ground time. This in turn mean that different gaits will have different limitations when considering the maximum velocity. For a gait with six sets of legs the flight time is at most $0.2 \times \tau_g$. For a gait with three sets of legs the maximum flight time is already much bigger: $0.5 \times \tau_g$ and for a set of two legs it is at most equal to $\tau_g$.

When using the multiple flight time-model to achieve an optimal gait switch there are some limitations:

$$\max(\tau_{\text{diff}} + 2 \times \tau_{f_{\min}}) \leq \tau_{\delta_{\mathbf{G}_1}} + \tau_{\delta_{\mathbf{G}_2}}, \tag{5-29}$$

where $\tau_{\delta_{\mathbf{G}_1}}, \tau_{\delta_{\mathbf{G}_2}}$ are the values for $\tau_\delta$ for the starting gait $\mathbf{G}_1$ and ending gait $\mathbf{G}_2$ of the gait switch

$\tau_{f_{\min}}$ is determined by a physical limitation, which is constant relative to $t$, this means $\tau_{f_{\min}}$ has to increase at the same rate as $\frac{d\tau}{dt}$, since $\tau_{f_{\min}}$ is relative to $\tau$. That has as a result that the gait switches have to be done before the robot reaches a certain speed and equation (5-29) no longer holds. If minimum value for $\tau_f$, relative to $t$ is denoted by $\tau_{f_p}$ and the maximum value for $\tau_{f_{\min}}$ for which equation (5-29) is satisfied is denoted by $\tau_{f_q}$ then

$$\frac{d\tau}{dt} \leq \frac{\tau_{f_q}}{\tau_{f_p}},$$

by filling in the derivative this turns into

$$2\alpha \times t \leq \frac{\tau_{f_q}}{\tau_{f_p}}, \tag{5-30}$$

Assuming that $t \geq 0$ the state vectors denoted in relation to $\tau$ can be denoted in relation to $t$ by using the inverse of equation (5-25).

$$t = \sqrt{\frac{\tau}{\alpha}} \tag{5-31}$$

### 5-6-2 Application

In order to apply the previous section $\alpha$ and $\tau_{f_p}$need to be defined.

$$\begin{aligned} \alpha &= 0.1 \\ \tau_{f_p} &= 0.1 \end{aligned}$$

Consider the gaits denoted by $G_1$, $G_2$ and $G_3$, the gait switches from $G_1$ to $G_2$ and from $G_2$ to $G_3$ and their values for $\tau_{\text{diff}}$ as defined in section 5-4. By looking at the values of $\tau_{\text{diff}}$ and for the gait switch from $G_1$ to $G_2$, $\tau_{f_{\min}} \leq 0.15$ in order to satisfy equation (5-29). For the gait switch from $G_2$ to $G_3$ it is $\tau_{f_{\min}} \leq 0.25$.

**Figure 5-11:** Sequence of state vectors for a hexapod robot. Top: time relative to $\tau$, bottom: time relative to $t$. The gray blocks represent the stance and the white blocks represent the flight of the legs.

Applying equation (5-30) gives bounds for the time at which the gait switches can happen: For the gait switch from $G_1$ to $G_2$

$$t \leq 7.5,$$

$$\tau \leq \ 0.1 \times 7.5^2 = 5.625$$

this means the leg which has $\tau_f = \tau_{f_{\min}}$ has to have touched down before that time and for the gait switch from $G_1$ to $G_3$.

$$t \leq 12.5,$$

$$\tau \leq \ 0.1 \times 12.5^2 = 15.625$$

This results in the following sequence of gaits: First three cycles of $G_1$, then two cycles for the switch, then five cycles of $G_2$, then 2 cycles for the second switch, then the rest of the cycles $G_3$. This is shown in the Figure (5-11)

## 5-7 Conclusion

The goal of this chapter was to address the problem of mathematically defining an optimal gait switch and then finding it. With the use of the knowledge of the steady state behaviour it was possible to determine the measurement of optimality $\tau_{\text{diff}}$ for a gait switch from a gait $\mathbf{G}_1$ to a gait $\mathbf{G}_2$ :

$$\tau_{\text{diff}} = (t_{v_{\mathbf{G}_2}} - t_{v_{\mathbf{G}_1}}) - \min(t_{v_{\mathbf{G}_2}} - t_{v_{\mathbf{G}_1}}),$$

where $t_{v_{\mathbf{G}_2}}$ are the touchdown times of eigenvector $v_{\mathbf{G}_2}$, and $t_{v_{\mathbf{G}_1}}$ are the touchdown times of eigenvector $v_{\mathbf{G}_1}$. This measurement was then used to find optimal gait switches for a hexapod robot.

The gait switch is defined as a switch from $\mathbf{G}_1$ to $\mathbf{G}_2$. It is assumed that $\tau_g$ is the same for both gaits. Furthermore it is assumed that $\tau_{\delta_1}^{\otimes m_1} = \tau_{\gamma 1}$,where $\tau_{f\mathbf{G}_1}, \tau_{\Delta\mathbf{G}_1}$ and $\tau_g$ are the gait parameters of $\mathbf{G}_1$ and $m_1$ is the number of sets of $\mathbf{G}_1$. Finally it is also assumed that $\tau_{\delta_2}^{\otimes m_2} = \tau_{\gamma 2}$, where $\tau_{f\mathbf{G}_2}, \tau_{\Delta\mathbf{G}_2}$ and $\tau_g$ are the gait parameters of $\mathbf{G}_2$ and $m_2$ is then number of sets of $\mathbf{G}_2$.

A set of gaits for which all gait switches between the gaits in the set are optimal is:

$$\begin{aligned} \mathbf{G}_1 &= \{1,4,5\} \prec \{2,3,6\} \\ \mathbf{G}_2 &= \{1,4\} \prec \{5,2\} \prec \{3,6\} \\ \mathbf{G}_3 &= 1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6 \end{aligned}$$

An attempt was made to further improve the gait switches by rewriting the model into the form of

$$\begin{bmatrix} t(k) \\ l(k) \end{bmatrix} = \left[\begin{array}{c|c} P & \tau_f \otimes E \\ \hline \mathcal{E} & \mathcal{E} \end{array}\right] \otimes \begin{bmatrix} t(k) \\ l(k) \end{bmatrix} \oplus \left[\begin{array}{c|c} Q & \mathcal{E} \\ \hline \tau_g \otimes E & \mathcal{E} \end{array}\right] \otimes \begin{bmatrix} t(k-1) \\ l(k-1) \end{bmatrix}.$$

This however caused instable behavior for some gait switches, which lead to the abandonment of this model.

It did however inspire a different model. A model where each leg has its own flight time:

$$\begin{bmatrix} t(k) \\ l(k) \end{bmatrix} = \left[\begin{array}{c|c} \mathcal{E} & R \\ \hline P & \mathcal{E} \end{array}\right] \otimes \begin{bmatrix} t(k) \\ l(k) \end{bmatrix} \oplus \left[\begin{array}{c|c} E & \mathcal{E} \\ \hline \tau_g \otimes E \oplus Q & E \end{array}\right] \otimes \begin{bmatrix} t(k-1) \\ l(k-1) \end{bmatrix}$$

where

$$R = \begin{bmatrix} \tau_{f_1} & \varepsilon & \dots & \varepsilon \\ \varepsilon & \tau_{f_2} & \ddots & \varepsilon \\ \vdots & \ddots & \ddots & \vdots \\ \varepsilon & \varepsilon & \dots & \tau_{f_n} \end{bmatrix}.$$

$\tau_{\text{diff}}$ was used to determine the values for the diagonal elements of $R$. As long as

$$\max(\tau_{\text{diff}}) \leq \tau_{\delta_1} + \tau_{\delta_2} - \tau_{f_{\min}},$$

and $\tau_g$ is the same for both gaits this model can be used to generate gait switches that have a constant stance time during that gait switch.

The requirement that $\tau_g$ is the same for both gaits means the robot cannot change speeds without disrupting the optimal gait switches. Therefore a new method was developed to increase the speed without changing the gait parameters. This method manipulates $\tau$, which is the clock of the robot. By default $\tau = t$, where $t$ is the real time, but by rewriting that into

$$\tau = \alpha \times (t)^2,$$

the robot accelerates at a constant acceleration.

# Chapter 6

# Conclusions and recommendations

The main focus of this thesis is switching of gaits on a legged robot using a max-plus linear model for the gait generation. The research can be split up in to two parts; The analysis of the steady state behaviour of the gait generated by the model and the optimization of the gait switches using the knowledge gained from the analysis of the steady state behaviour.

## 6-1 Steady state analysis

In order to analyze the steady state behaviour of all possible gaits it was necessary to derive a general structure of the matrices that describe the gaits. This was done by a lengthy but straight forward calculation of a general matrix. The steady state behaviour is determined by two matrix properties: the eigenvalue and the eigenvector. The eigenvalue represents the cycle time, which is the time it takes for the periodic motion of the gait to repeat, the eigenvector represents the order in which the legs touchdown and lift-off and the time between them. From the structure it was possible to determine that the matrices are irreducible, which means the eigenvalue is unique. The eigenvalue $\lambda = \tau_\gamma \oplus \tau_\delta^{\otimes m}$. By analyzing the critical graph of the general matrix it was determined that if $\tau_\gamma \leq \tau_\delta^{\otimes m}$, which means $\lambda = \tau_\delta^{\otimes m}$, then the eigenvector is also unique and has the following structure:

$$\begin{aligned} \forall j \in \{1, \ldots, m\}, \forall q \in L_j : \quad [v]_q &= \tau_f \otimes \tau_\delta^{\otimes j-1} \\ [v]_{q+n} &= \tau_\delta^{\otimes j-1} \end{aligned}$$

and let $[\bar{v}]_q = [v]_{q+n}$ for all $j$ and $q \in L_j$. That means that if $\tau_\gamma \leq \tau_\delta^{\otimes m}$ the steady state behaviour is known and uniquely defined.

## 6-2 Optimization of the gait switches

With the knowledge of the steady state behaviour a quantitative measure for the optimality of a gait switch from $\mathbf{G}_1$ to a gait $\mathbf{G}_2$ could be defined. This quantative measure is the maximum of $\tau_{\text{diff}}$ which is defined as:

$$\tau_{\text{diff}} = (t_{v_{\mathbf{G}_2}} - t_{v_{\mathbf{G}_1}}) - \min(t_{v_{\mathbf{G}_2}} - t_{v_{\mathbf{G}_1}}),$$

where $t_{v_{\mathbf{G}_2}}$ are the touchdown times of eigenvector $v_{\mathbf{G}_2}$, and $t_{v_{\mathbf{G}_1}}$ are the touchdown times of eigenvector $v_{\mathbf{G}_1}$. $\tau_{\text{diff}}$ is equal to the time each leg stays on the ground longer, therefore the maximum of this represents the maximum difference between the time the legs stay on the ground.

With the use of $\tau_{\text{diff}}$ the optimal gait switches for a hexapod robot were determined. One group of gaits, consisting of a tripod, quadruped and quintuped gait, that result in optimal gait switches when switching to one of the other gaits in the group is:

$$\begin{aligned}
\mathbf{G}_{\text{tripod}} &= \{1,4,5\} \prec \{2,3,6\} \\
\mathbf{G}_{\text{quadruped}} &= \{1,4\} \prec \{5,2\} \prec \{3,6\} \\
\mathbf{G}_{\text{quintuped}} &= 1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 6
\end{aligned}$$

A different method for gait switching using a special model that is only used during these gait switches has been developed. Using these methods "Perfect" gait switches can be achieved. The method creates two new matrices $A_{S1}$ and $A_{S1}$ which have specific flight times for each leg, these flight times are chosen in such a way that they compensate for the difference in events as found by $\tau_{\text{diff}}$. The gait switch from $\mathbf{G}_1$ to $\mathbf{G}_2$ then happens as follows.

$$x(k+1) = A_{\mathbf{G}_1} \otimes x(k) \tag{6-1}$$
$$x(k+2) = A_{S1} \otimes x(k+1) \tag{6-2}$$
$$x(k+3) = A_{S2} \otimes x(k+2) \tag{6-3}$$
$$x(k+4) = A_{\mathbf{G}_2} \otimes x(k+3) \tag{6-4}$$

where the $k^{\text{th}}$ cycle is the cycle where the gait switch is initiated. Using this method gait switches can be perfected, but there are some limitations; $\max(\tau_{\text{diff}})$ has to be less or equal to $\tau_{\delta_1} + \tau_{\delta_2} - \tau_{f_{\min}}$, where $\tau_{f_{\min}}$ is the minimum flight-time that is physically possible, and $\tau_g$ has to be the same for both gaits. This means the robot cannot change its speed.

This problem can be circumvented by manipulating the clock the robot is using for the timing. By introducing a clock $\tau$, that has a quadratic relation to the real time $t$: $\tau = a \times t^2$, where $a$ determines the acceleration of the robot. The robot will speed up relative to the real time $t$, while the timing relative to the virtual clock $\tau$ is the same as it would normally be, if the clock $\tau$ was not present. By doing so the robot speeds up without having to change the gait parameters.

## 6-3 Discussion and recommendations

With the methods presented in this thesis it is possible to optimize the gait switches in such a way that all legs have the same stance time, even during the gait switch. While this does limit the gait switches to gaits with the same value for $\tau_g$ the robot can still accelerate by manipulating the clock the robot is using for the timing. While these methods have not been tested on the robot, there should not be any major problems implementing them, since these methods are based on the principles of switching max-plus linear (SMPL) systems which have already been implemented on the robot, the only difference are the system matrices. The only thing that remains is to adjust the programming to incorporate the methods and then test the programming on the robot. This was not possible during the period this research was done because the robot was not in a working condition.

The limitations of the hardware, such as the minimum flight time, the maximum acceleration and velocity, which are needed for some of the methods are not exactly known yet, but these can be determined by simple experiments on the robot or they can be determined from the analysis of a model of the robot.

There is still plenty of research left to be done, all with their own challenges:

- Gaits with aerial phases have not been considered yet, while all of these methods should work for gaits with aerial phases, there are some issues that need to be solved before aerial phases can be introduced, such as determining the minimum velocity needed for an aerial phase and what gait parameters are needed in order to get a gait with a proper aerial phase.
- Another subject of interest is the power consumption of the different gaits at different velocities; what gait is the most energetically efficient at different velocities, and use that to automatically switch gaits when speeding up or slowing down the robot.
- A subject of interest that is not that closely related to the rest is the implementation of a max-plus model for modular robots, where robots consist of modules that can connect to each other to form larger robots and where each individual module is a working robot in it self.

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# Glossary

## List of Acronyms

| | |
|---|---|
| **MPL** | max-plus linear |
| **SMPL** | switching max-plus linear |
| **PWA** | piece-wise-affine |
| **CPG** | central pattern generator |
| **DES** | discrete event system |
| **m.s.c.s.** | maximal strongly connected subgraph |

## List of Symbols

| | |
|---|---|
| $\lambda$ | Unique eigenvalue |
| $\mu$ | Eigenvalue |
| $\omega_i$ | Angular velocity |
| $\pi(i)$ | Direct predecessors of node $i$ |
| $\pi^*(i)$ | All predecessors of node $i$ plus the node itself |
| $\pi^+(i)$ | All predecessors of node $i$ |
| $\sigma(i)$ | Direct successors of node $i$ |
| $\sigma^*(i)$ | All successors of node $i$ plus the node itself |
| $\sigma^+(i)$ | All successors of node $i$ |
| $\sigma_{\mathcal{G}}$ | Cyclicity of the graph |
| $\tau$ | Time instant |
| $\tau_\delta$ | Sum of the flight and double stance time |
| $\tau_\gamma$ | Sum of the flight and stance time |

| | |
|---|---|
| $\tau_{\text{diff}}$ | Measure for the difference between two eigenvectors |
| $\tau_{f_p}$ | Minimum value for the flight time, relative to $t$ |
| $\tau_{f_q}$ | Maximum allowed value for the flight time, relative to $\tau$ |
| $\tau_{f_{\min}}$ | Minimum flight time |
| $\tau_{\Delta}$ | Double stance time |
| $\tau_f$ | Flight time |
| $\tau_g$ | Stance time |
| $\theta_{\text{ref},i}$ | Reference trajectory |
| $\theta_l$ | Lift-off angle |
| $\theta_t$ | Touchdown angle |
| $\mathcal{E}$ | Max-plus zero matrix |
| $\varepsilon$ | The zero-element of max-plus algebra |
| $\bar{\mathbf{G}}$ | Normal form of the gait |
| $\bar{A}$ | Normal form of the system matrix |
| $\bar{P}$ | Normal form of the matrix $P$ |
| $\bar{Q}$ | Normal form of the matrix $Q$ |
| $\bar{v}$ | Distance traversed by one leg per cycle |
| $\mathbb{R}$ | Set of real numbers |
| $\mathbb{R}_{\max}$ | Set of numbers used for max-plus algebra |
| $\mathbf{G}$ | Gait |
| $\mathcal{C}$ | Connected |
| $\mathcal{D}$ | Set of arcs |
| $\mathcal{D}^c$ | Arcs of the critical graph |
| $\mathcal{D}_r$ | Subset of arcs |
| $\mathcal{G}$ | Graph |
| $\mathcal{G}^c$ | Critical graph |
| $\mathcal{G}_r$ | Subgraph |
| $\mathcal{N}$ | Set of nodes |
| $\mathcal{N}^c$ | Nodes of the critical graph |
| $\mathcal{N}_r$ | Subset of nodes |
| $\mathcal{R}$ | Reachable |
| $\mathcal{R}_{\max}$ | Max-plus algebra |
| $\tilde{\mathcal{D}}$ | Set of arcs belonging to the reduced graph |
| $\tilde{\mathcal{G}}$ | Reduced graph |
| $\tilde{\mathcal{N}}$ | Set of nodes belonging to the reduced graph |
| $a$ | Acceleration of the robot |
| $A(k)$ | State matrix |
| $A_{\lambda}$ | Normalized matrix |

| | |
|---|---|
| $A_{S1}$ | System matrix for the first cycle of the gait switch using 'multiple flight time'-model |
| $A_{S2}$ | System matrix for the second cycle of the gait switch using 'multiple flight time'-model |
| $B(k)$ | Input matrix |
| $C$ | Similarity transformation matrix |
| $C(k)$ | Output matrix |
| $E$ | Max-plus identity matrix |
| $e$ | The one-element of max-plus algebra |
| $k_j$ | Index function for each state vector element |
| $l_i(k)$ | Lift-off event |
| $L_p$ | Set of legs |
| $P$ | Synchronization matrix $P$ |
| $P_c$ | Synchronization matrix $P_c$ for the 'constant stance'-time model |
| $Q$ | Synchronization matrix Q |
| $Q_c$ | Synchronization matrix $Q_c$ for the 'constant stance'-time model |
| $R$ | Matrix with multiple values for the flight time for the 'multiple flight time'-model |
| $s$ | Distance traversed |
| $t$ | Real time |
| $t_i(k)$ | Touchdown event |
| $u(k)$ | Input |
| $v$ | Eigenvector |
| $v$ | Velocity of the robot |
| $v(k)$ | Control variable |
| $x(k)$ | Statevector |
| $y(k)$ | Output |
| $z(k)$ | Switching variable |
| $_i$ | Number of the leg |
| $_p$ | Number of the set of legs |