The Thermodynamics of Economic Engineering

With Applications to Economic Growth

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With Applications to Economic Growth

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Abstract

The economic engineering group at the DCSC uses Newtonian and analytical mechanics to model economic systems but makes no use of thermodynamics. The introduction of thermodynamics to the economic engineering framework would increase the extent of analysis and interpretation of economic systems in economic engineering. In this thesis the foundations of the thermodynamics of economic engineering are developed in order to include thermodynamic modeling of economic systems in the field of economic engineering. After the development of the thermodynamics of economic engineering theory, the theory is applied to analyze economic growth, factor productivity, and the value of a business.

An axiomatic approach is taken to derive economic analogs to thermodynamic concepts. The meaning of an axiomatic approach is that these economic-thermodynamic analogs are developed in a logical order, e.g., the economic analog to temperature is not introduced before developing the economic analog to the first law. Key analogs between economics and thermodynamics are established in this thesis that include but are not limited to two fundamental economic laws, work as an expenditure, heat as an expense, temperature as a price level, entropy as a quantity referred to as human capital in the thesis. By deriving key analogs, the thesis establishes the foundational principles of thermodynamics within economic engineering.

Utilizing the theory of the thermodynamics of economic engineering results in applications to economic growth. Empirical growth accounting is modeled by the fundamental thermodynamic relationship. A relationship between linear production functions and Cobb-Douglas type production functions is shown by analyzing the productivity of an economy. Furthermore, a method to evaluate the worth of a business is created by determining the business' total potential earnings.

Additionally, the thesis shares a vision of how to include thermodynamics within a control engineering framework. A higher-dimensional energy-based approach to model dynamical systems offers a way forward to include thermodynamic energy and entropy within control formalisms. Such a framework would account for availability and heat buildup in controlled dynamical systems. One potential application is to account for the heat build up that occurs in integrated circuits.

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Preface and Acknowledgements

Sometimes the problem you end up solving was not the problem you expected to solve. When I arrived at Delft, I would have never guessed I would eventually be applying mechanics and thermodynamics to economics. For my midterm literature assignment, I applied mechanics to economic growth, but soon after I changed my path and began investigating the use of thermodynamics to model to economic systems. In order to extend economic engineering to include thermodynamics, I studied the economic engineering theory developed by my supervisor, economic literature, and literature of those applying physics to economics. This thesis is a result of putting together all my research to contribute to the field of economic engineering and economics.

I want to thank my supervisor dr.ir. M.B. Mendel for his assistance during the writing of this thesis. He has provided me with a truly unique opportunity here in the economic engineering group and has allowed me to explore past the typical program of a systems and control engineer. I also want to thank the members of my thesis committee for showing interest in my thesis and taking the time to read it. I would also like to thank my peers in the economic engineering group at the DCSC for their feedback and contributions during our meetings and discussions. Lastly, I want to thank my family for supporting me.

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Nicolas Manders

“When the going gets tough, the tough get going.”
— *Joseph P. Kennedy*

Chapter 1

Introduction

1-1 Motivation

The motivation behind this thesis is to (1), extend the field of economic engineering to include the modeling of economic growth, and (2), to put thermodynamic modeling back into control analysis. Currently, the two methods to modeling economic systems in the economic engineering group are the Newtonian method and analytical mechanic method [1]. Both of these methods describe dynamical systems. For my literature survey, I developed a Newtonian model of an economy to study macroeconomic growth based on the Solow growth model, see [2] for more on Solow growth. I concluded that this Newtonian approach to macroeconomic growth had limitations, see Section 2-4 for a more in depth explanation. Instead of trying to add source flows, negative resistances, or different forcing functions to model growth, I decided that thermodynamics—a science dedicated to analyzing changes of a system's energy—was better suited to analyze growth.

Therefore, I develop the use of thermodynamics in economic engineering to model economic systems. Chapter 3 focuses on the foundational principles of the thermodynamics of economic engineering. The chapter follows a classical axiomatic approach to thermodynamics where it derives economic analogs to thermodynamic concepts. These economic analogs to thermodynamic concepts form the foundational base of the thermodynamics of economic engineering. By following an axiomatic approach, the chapter derives economic analogs in the same logical order that the thermodynamic concepts are introduced.

Applications for the theory of Chapter 3 to economic growth are developed in Chapter 4. Chapter 3 derives the necessary theory to achieve the goal of modeling economic growth in the economic engineering framework. Topics in economic growth such as Solow growth and the Cobb-Douglas production function have background provided in Section 2-3. The thermodynamics of economic engineering theory is first applied to the empirical growth of an economy illustrating how the fundamental thermodynamic relationship models economic growth. Then the theory is applied to production functions, showing a relationship between linear production functions and production functions of a Cobb-Douglas form. This application demonstrates that the Cobb-Douglas production function takes productivities of factors as inputs. The last application of the chapter develops a method to determine the value of a business during its sale based on determining its total potential earnings and then discounting those earnings. These applications demonstrate the potential of the thermodynamics of economic engineering in future economic engineering applications.

One of the motivations behind this thesis is to understand how thermodynamic modeling can be included in controls. After developing the thermodynamics of economic engineering and some applications of the theory, I understood the way forward for both future economic interpretations and control applications is to use an energy-based method to account for thermodynamic effects of dynamic systems. Chapter 6 describes how thermodynamic modeling can be brought into controls to account for entropy generation from dissipation.

1-2 Thermodynamic Modeling of Economic Systems

Others have attempted to model economics with thermodynamics in the field called thermoeconomics, see [3, 4, 5, 6, 7, 8, 9, 10] for a selection of academic works from this field. However, the analogs proposed in the thermoeconomic field by various parties are not congruent with economic engineering. For example, in economic engineering we argue that price is analogous to momentum, that rents and wages are analogous to forces, and that cash flows are analogous to energy [1]. These economic engineering analogs are not analogs made in any of the works selected from the thermoeconomic field.

Section 2-5 discusses the background of the field of thermoeconomics and details the efforts of thermoeconomists to model economic systems as thermodynamic systems. I argue why the approach of thermoeconomists is insufficient for developing a thermodynamic-economic analogy for economic engineering.

An axiomatic analogy between thermodynamics and economics is built up in Chapter 3 analog by analog. Analogs are introduced in a specific order meant to mirror how the topic of thermodynamics itself is taught. The thermodynamic text *Principles of General Thermodynamics* by Hatsopoulos and Keenan is primarily used [11]. Laws of economics are introduced by following the introduction and development of the laws of thermodynamics. The economic analogs to concepts such as work, the first law, the second law, heat, and entropy are derived along side the order of their introduction from the thermodynamic text.

Chapter 2

Background

2-1 Introduction

This chapter presents background information from literature and pertinent topics that are used in this thesis. Section 2-2 will provide background for the Newtonian and analytical economic engineering approaches. Concepts such as surplus, utility, and consumption are summarized. Section 2-3 discusses economic growth theories and presents results from a Newtonian approach to macroeconomic growth (essentially a summary of my literature survey). Section 2-4 shares my motivation for developing thermodynamic theory to model economic growth. Section 2-5 introduces thermoeconomics and demonstrates why this field of literature is not useful for an economic engineering approach. The analogs established in thermoeconomics are not the same analogs made in economic engineering.

2-2 Economic Engineering

The overall purpose of economic engineering is to model economic systems as physical systems [1]. Analogies that exist between the dynamic behavior of mechanical and electrical systems can naturally be extended to the dynamic behavior of economic systems compared to physical systems. The primary advantage of modeling economic systems as dynamic physical systems is that control engineers can apply control theory and formalisms to economic systems.

2-2-1 Newtonian Mechanics

The starting point of economic engineering is to see that the analogs between mechanics and economics are closely related to the analogs between mechanics, electronics, pneumatics, or hydraulics. To see how different physical domains can be modeled the same way, see bond graph theory e.g. [12].

To start, a position is considered analogous to an amount of a certain asset (e.g. financial stock, capital, or inventory). These assets will each have a certain price. This price is considered analogous to momentum in a mechanical system [1].

$$q := \text{asset } [\#] \qquad\qquad p := \text{price } \left[\frac{\$}{\#}\right] \tag{2-1}$$

The amount of assets or the price can both be positive or negative. If an asset is positive then that asset is owned, and if negative, that asset is owed. Similarly, if a price is positive then it is being sold, and if a price is negative, it is being bought. The economic state of any economic agent is defined as

$$x = (\mathbf{q}\ \mathbf{p})^T \tag{2-2}$$

where $\mathbf{q}$ is a vector of assets and $\mathbf{p}$ is the corresponding vector of prices.

The time derivative of the economic state is

$$\dot{q} := \text{asset flow } \left[\frac{\#}{yr}\right] \qquad\qquad \dot{p} := \text{price change } \left[\frac{\$}{\#yr}\right] \tag{2-3}$$

The asset flow is the movement of assets measured in units over time (either in seconds, minutes, hours, days, years, etc.) One such asset flow is the quantity demanded of a consumer good. The relationship between the quantity and price of the good is governed by the law of demand: price is linearly related to the quantity demanded by the demand m, which is analogous to mass in mechanics or an inductor in mechanics [1].

The price of an asset is changed as a result of the forces acting on it (Newton's Law). Thus the movement of price is the net force acting on the system. The three types of forces that are present in an economic system are demand forces (inertial forces), inventory or holding costs (spring forces), and transaction costs (damper forces).

Following from the units of the economic state and its derivative, other physical properties can have their economic units derived. For example, energy is found to have units $\left[\frac{\$}{yr}\right]$ which is a cash flow, power is $\left[\frac{\$}{yr^2}\right]$ which is growth, and phase volume is [\$] which is the unit of action.

2-2-2 Analytical Mechanics: Hamiltonian and Lagrangian

For a text on classical analytical mechanics, see e.g. Corben [13]. In economic engineering theory the Hamiltonian and Lagrangian are related to surplus and utility. This section is split into two subsections, one for the surplus Hamiltonian and one for the utility Lagrangian. Consumption is discussed in both sections.

Hamiltonian

In classical analytical mechanics the Hamiltonian represents the total mechanical energy in a system, and in economics the Hamiltonian represents surplus [1]. Surplus is defined to be some amount of funds that can generate economic activity [1, 14].

For classical analytical mechanics there is no inclusion of damping, and for an isolated system

$$\frac{dH}{dt} = 0$$

is always true since the Hamiltonian is a conserved quantity. The economic interpretation of Hamiltonian mechanics is that surplus is the driving force of economic behavior such that Hamilton's equations

$$\frac{\partial H}{\partial q} = -\dot{p} \qquad\qquad \frac{\partial H}{\partial p} = \dot{q} \tag{2-4}$$

relate surplus to the behavior of assets and prices. It shows that price movements are inversely related to the quantity of goods and that changes of price are positively related to a quantity demanded of goods.

When a consumer consumes, they are eating away at their surplus in order to receive greater utility. Consumption and depreciation of goods is modeled as dissipation in mechanics and is an irreversible process [1, 14]. As mentioned, classical analytical mechanics does not contain endogenous damping, but it is possible to include damping in complexified analytical mechanics to model economic consumption (see Section 2-2-3).

Lagrangian

The Lagrangian is related to utility instead of surplus and is analogous to a budget constraint [1]. The Lagrangian is a function of assets and asset flows. The Hamiltonian and Lagrangian are related by a Legendre transform from prices to asset movements

$$L(q, \dot{q}) = p\dot{q} - H(q, p) \quad . \tag{2-5}$$

Economic agents attempt to maximize their utility by maximizing the quantity

$$S = \int_{t_0}^{t_1} L(q, \dot{q}) dt \tag{2-6}$$

within the time interval $[t_0, t_1]$. They do so by choosing the asset path that will result in maximum utility

$$\delta S = 0 \tag{2-7}$$

which is analogous to the stationary action principle in mechanics. This principle leads to the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = 0 \tag{2-8}$$

which gives the equations of motion for the optimal path.

The chosen path by an economic agent is the path that will include an optimal level of consumption. However, just like the Hamiltonian of classical analytical mechanics, the Lagrangian does not contain endogenous damping. See Section 2-2-3 for how damping can be included in analytical mechanics if it is first complexified.

2-2-3 Complexified Analytical Mechanics

Damping (consumption) can be included in analytical mechanics if complexified mechanics is used. The master's thesis of Coenraad Hutters [14] shows how a complex state and Hamiltonian can be used to include damping in a Hamiltonian formalism.

Complex State

The complex state is obtained by complexifying the state space. The complex state and its conjugate are

$$a = \alpha q + i\beta p \qquad \bar{a} = \alpha q - i\beta p \tag{2-9}$$

where α and β are constants that can be chosen arbitrarily. However, these constants are chosen such that a represents the square root of mechanical energy where $\alpha = \sqrt{\frac{k}{2}}$ and $\beta = \sqrt{\frac{1}{2m}}$. Then $a\bar{a}$ is equal to the sum of kinetic and potential energy.

Complex Hamiltonian and Equations of Motion

For a system with no damping, the complex Hamiltonian, $\mathcal{H}$, is equal to $a\bar{a}$. The equation of motion is then

$$\dot{a} = -i\omega_n \frac{\partial \mathcal{H}}{\partial \bar{a}} \tag{2-10}$$

where ω_n is the natural frequency and the factor of $-i$ (a -$\frac{\pi}{2}$ rotation) corresponds to a structure matrix J ($J^2 = -I$) from port-Hamiltonian theory [14, 15].

For a system with damping, the equation of motion of the complex state is

$$\dot{a} = -(\beta + i\omega_n)a + \beta\bar{a} \tag{2-11}$$

where $\beta = \frac{b}{2m}$. In order for the complex Hamilton equation to hold (2-10) the complex Hamiltonian must be equal to

$$\mathcal{H} = (1 - i\zeta)a\bar{a} - \frac{1}{2}i\zeta a^2 + \frac{1}{2}i\zeta\bar{a}^2 \quad . \tag{2-12}$$

The complex Hamiltonian can also be expressed as

$$\mathcal{H} = \mathcal{E} - i\mathcal{D} \tag{2-13}$$

where

$$\mathcal{E} = a\bar{a} \qquad\qquad \mathcal{D} = \frac{1}{2}\zeta(2a\bar{a} + a^2 - \bar{a}^2) \quad . \tag{2-14}$$

The mechanical energy in the system is equal to $\mathcal{E}$, and the dissipation function is $\mathcal{D}$.

While $\mathcal{E}$ represents the mechanical energy in the system, $\mathcal{D}$ does not represent damped energy. $\mathcal{D}$ points in the direction energy will be damped in but does not represent a total amount of damped energy. While the complex Hamiltonian is a conserved quantity, it eventually converges to zero for an unforced system. Energy itself cannot be a complex value, but the Hamiltonian is complex since it is a vector quantity.

2-3 Economic Growth Theories

This section gives background for a Newtonian approach to macroeconomics and provides motivation for using a thermodynamic approach to economic growth. For my literature assignment I used a Newtonian approach to model macroeconomic growth. I provide a brief overview of macroeconomic growth theory in Section 2-3-1. Then I summarize the results of my application of the Newtonian method from my literature survey in Section 2-3-2.

2-3-1 Solow Growth

Economic growth is of primary importance to macroeconomists [2, pp. 8]. One particular model of economic growth that has been widely used and studied over the years is the Solow growth model. Solow growth is considered a starting point for analyzing growth and used in comparison to other models [2]. The theory of the Solow model is that capital accumulation and labor growth cannot alone explain the difference in growth in countries over time [2]. The model uses a proxy measure of growth, the ratio k

$$k = \frac{K}{AL} \quad , \tag{2-15}$$

where K is the total amount of capital in an economy and AL is the total amount of effective labor. L is total labor, and A is a coefficient of effectiveness representative of a level of technology.

Capital can be described by first-order dynamics (such as spring-damper system) driven by reinvestment from the output of that economy

$$\dot{K} = -\delta K + sY(K, AL) \tag{2-16}$$

where δ is the rate of depreciation of capital and Y is a production function that relates capital and effective labor to an output. When the coefficient A enters the production function with labor as effective labor AL, it is called labor-augmenting [2, pp. 10]. The saving rate s determines the amount of output reinvested back into capital.

The production function Y relates the magnitude of stock inputs—capital, labor, and a level of technology—to a magnitude of output

$$Y(K, AL) = (AL)^{1-\alpha} K^{\alpha} \tag{2-17}$$

where α is a coefficient between zero and one. Equation (2-17) is an example of a Cobb-Douglas production function where inputs are exponentiated by their output elasticity and multiplied together to determine the output.

Labor and labor effectiveness are assumed to have exogenous growth rates, n and g

$$\dot{L} = nL \qquad\qquad \dot{A} = gA \quad . \tag{2-18}$$

The dynamics of k can then be derived

$$\dot{k} = -(\delta + n + g)k + sy(k) \tag{2-19}$$

where $y(k)$ is the intensive form of the production function [2, pp. 11]. The dynamics of k imply that an economy will save until an equilibrium is reached when capital and labor begin to grow equally. Investment per worker in an economy equals a break even investment rate at equilibrium.

2-3-2 Newtonian Approach: 2nd-Order Economic Growth Model

In this section I summarize my application of the Newtonian approach to the Solow growth model. The Newtonian approach I developed models the capital and labor systems as second-order systems. Labor is now endogenous and is driven by a fraction of a rent $sF(t)$ and a wage $W(t)$ in Equation (2-20)

$$m_K\ddot{K} + \delta_2\dot{K} + \delta_1 K = sF(t) \qquad\qquad m_L\ddot{L} + \Omega\dot{L} + aL = W(t) \tag{2-20}$$

and with k as the same definition in Equation (2-15) results in the dynamics of k

$$\ddot{k} + \frac{\delta_2}{m_K}\dot{k} + \frac{\delta_1}{m_K}k + 2\dot{k}\dot{\gamma} + k\ddot{\gamma} + k\dot{\gamma}^2 + \frac{\delta_2}{m_K}k\dot{\gamma} = s\frac{y(k)}{m_K} \quad . \tag{2-21}$$

This resulting equation captures fourth-order behavior due to the nonlinear effects

$$2\dot{k}\dot{\gamma} + k\ddot{\gamma} + k\dot{\gamma}^2 + \frac{\delta_2}{m_K}k\dot{\gamma} \quad . \tag{2-22}$$

Wages and Rents are defined as

$$W = \frac{\partial(p_0 Y)}{\partial L} \qquad\qquad F = \frac{\partial(p_0 Y)}{\partial K} \quad . \tag{2-23}$$

Here $p_0 Y$ is the price of the sole output times the output which results in a cash flow (analogous to energy). Then the way wage and rent are defined results in them being analogous to a force in Newtonian mechanics and marginal productivities of labor or capital in economics.

The limitation of this approach is the same of any Newtonian approach to a macroeconomic system. It assumes there exists measurable springs, masses, and dampers in the economy. There is only one measure of capital and only one measure of labor. It must be assumed that millions of laborers and pieces of capital all act similarly enough to be modeled by a single Newtonian systems.

2-4 Motivation for Using Thermodynamics

My motivation to use thermodynamics as a method to model economic systems was to consider economic growth as analogous to changes of a systems energy. To include growth within a Newtonian approach would require a source of energy. For instance, negative resistance could be added to a Newtonian labor or capital system to cause the systems to grow through instability. A flow source or effort source could also cause the system to grow. However, these sources would not yield economic insight or interpretation beyond the Newtonian approach.

I surmised that analyzing the interactions between economic systems that lead to economic growth could produce useful insight.

I considered how the production function could be used as a measure of energy instead of a flow of goods. In my literature survey I used the production function to create a quantity that measured energy. That quantity was p_0Y, a cash flow analogous to energy (see Section 2-3-2). By doing so, I derive wages and rents with a partial derivative with respect to this quantity that is a function of the production function. The result is that wage and rents are properties derived from a measure of energy. My thermodynamic approach to modeling economic systems came from hypothesizing that the cash flow corresponding to production was related to internal energy. Growth would be caused by interactions between economic systems that led to increases or decreases of properties such as labor or capital. So, I began to research how economics could be modeled by thermodynamics.

2-5 The Field Of Thermoeconomics

This section is divided into an overview of thermoeconomics and a short analysis of works selected from the field. The current field of thermoeconomics does not yield useful results for economic engineers.

2-5-1 Overview of Thermoeconomics

This thesis is not the first proposal of the similarities between thermodynamics and economics. There is a field of economics called thermoeconomics[1] which seeks to use thermodynamic principles in the construction of economic theory [4]. It applies thermodynamics and statistical mechanics to economics.

A selection of papers from the field are presented. These papers are not necessarily emblematic of the field, but they do cite numerous other papers which can be followed, see [3, 4, 5, 6, 7, 8, 9, 10]. Thermoeconomists are concerned with problems of the relationships between value, money, temperature, utility, entropy, and consumption.

2-5-2 Analysis of Thermoeconomics from an Economic Engineer's Perspective

The arguments proposed by thermoeconomists conflict with the arguments proposed by economic engineers. Interpretations and results from thermoeconomics do not fit in the economic engineering framework because the analogs made in thermoeconomics do not agree with analogs of the economic engineering framework. Due to these differences, ideas from thermoeconomics are not used in this thesis. The thermodynamic-economic analogs and applications established in the thesis use economic engineering as background.

A common idea in thermoeconomics is that pressure or a force is a price, see e.g. [3]. Economic engineers view price as analogous to momentum, not force. For example, the J. Bryant papers establish pressure as price as a first principle, which is not agreed with in economic engineering

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[1] Sometimes called Energy Economics, Biophysical Economics, or considered a subfield of Econophysics. Thermoeconomics is not the same as the economics of the costs involved with the transportation of energy.

[3, 4, 5]. Temperature is also considered as an economic analog to economic value in these papers by J. Bryant, which is incongruous with the physical analogy between temperature, voltage, and pressure [16].

Overall in thermoeconomics, economic value in units of money [$] is considered to be energy, see e.g. [3, 4, 5, 6, 7, 8, 9, 10]. The thermoeconomists view value as something conserved in economics, so to thermoeconomists value must be analogous to energy in physics. Economic engineers would argue that thermoeconomists are confusing price and value, since price (analogous to momentum in economic engineering) is also a conserved quantity. In economic engineering, energy is analogous to a cash flow in units of money per time [1].

Sometimes concepts, such as entropy, avoid concrete definition within thermoeconomists. For example, entropy is called "economic entropy" instead of having an economic interpretation or analogy, see e.g. [3, 4, 5]. This thesis avoid creating analogies that just add the label "economic" if possible such as by avoiding creating analogs like "economic entropy" or "economic heat".

The thermodynamic approach in this thesis differs to the approach of thermoeconomists in three key ways. One, it approaches the analogy axiomatically and builds up analogs logically following closely the thermodynamic text *Principles of General Thermodynamics* [11]. This axiomatic approach presents the analogy in a logical order. Two, the units and analogs of economic properties developed in the thermodynamic approach are consistent with those in the Newtonian and analytical mechanical approach. Three, this thesis makes the step to explicitly define the economic analogs energy, work, and heat in terms of economic concepts. This last step is a contribution to the field of thermoeconomics because it explicitly in an axiomatic way defines economic analogs to work, heat, and energy—topics that avoid concrete economic interpretations in thermoeconomics.

Chapter 3

The Thermodynamics of Economic Engineering

3-1 Introduction

This chapter builds an axiomatic analogy between classical thermodynamics and economics. The chapter develops analogs between thermodynamics and economics by following the order of the material presented by Hatsopoulos and Keenan from Part I of *Principles of General Thermodynamics*. That is to say, if Hatsopoulos and Keenan introduce a thermodynamic topic or definition important to establishing the axiomatic approach to classical thermodynamics, I also develop a corresponding economic analog or address it in an appropriate manner.

To create this analogy, first Section 3-2 establishes economic analogs to basic concepts in thermodynamics. Then in Section 3-3 the economic analogs to work and to the first law are derived. Equilibrium is analyzed in Section 3-4 and the economic analog to stable equilibrium is derived. The economic analogs to the second law and state principle are derived in Section 3-5. Section 3-6 develops the economic analogs of heat and temperature. Section 3-7 develops the economic analog to heat engines and perpetual motion machines. Section 3-8 derives the economic analogs to reversibility and irreversibility. Section 3-9 develops the economic analog to how temperature scales are determined in thermodynamics. Then Section 3-10 derives the economic analogs to entropy. Section 3-11 develops the economic analog to three different material properties. The chapter is concluded in Section 3-12.

3-2 Fundamental Concepts

Before economic analogs to work or thermodynamic laws can be derived, first fundamental analogs between thermodynamics and economics must be established.

Bodies and Primitive Properties - Economic Bodies and Conditions

Hatsopoulos and Keenan (HK) begin by introducing the general notion of something thermodynamic, a body. The term body refers to something that is "enclosed by a well-defined surface which may be either material or imaginary and through which particles of matter do not pass" [11, pp. 7]. The notion of "particles of matter" is similar to the assets that constitute a society—consisting of goods, inputs, and services. Economists consider endogenous variables to be part of an economic model, whereas exogenous variables come from outside a model [2]. However, these terms—model and variable—are too specific for the notion of an economic body.

I will use this economic terminology to conceive a "well-defined surface" for an economic body as the "well-defined" distinction between what is endogenous to the economic body and what is exogenous to the economic body. This description of an economic body is similar to how economists describe an economic system, defined by economists as "a system of production, resource allocation and distribution of goods and services within a society or a given geographic area" [17]. The term economic system in economics is used to refer to the type of economy in question—capitalist, socialist, left-wing, etc. Economists use the word system to refer to what I describe as a body. The economic definition of a system captures the idea of the assets that constitute a society while only defining a surface as something geographical, which is limiting.

Next, HK introduce the conditions bodies can assume. These conditions, called primitive properties, are properties that can be observed of a body by performing a suitable test without knowing any history of said system [11, pp. 7]. In thermodynamics examples of such properties are volume, weight, and color. A parallel concept in economics are economic conditions, defined in economics as "the state of macroeconomic variables and trends in a country at a point in time" [18]. An example of these economic conditions are measurements published by groups such as the government, usually weekly, monthly, or quarterly. In economics examples of these conditions are an unemployment rate, prevailing interest rates, or productivity.

State

Economists use terms like conditions or variables often, but the notion of a state is vague in economics. HK define a state in order to identify the condition of a body, so now I define a state in economics so I can describe the condition of an economic body.

Thermodynamics	**Economics**
State	Economic State
"the condition of a *body* which is identified by all its *primitive properties*" [11, pp. 8]	The condition of an *economic body* which is identified by all its *economic conditions.*

Analog 3.1: Economic State

I establish the notion of an economic state in Analog 3.1 in the way an engineer or thermodynamicist would use the term. Two economic states will be the same if they have the same economic conditions.

HK state a corollary of the definitions of primitive property and state, noting that "it is possible to determine the value of a primitive property without changing the state of the body" [11, pp. 8]. It is not immediately clear if this is possible in economics. As an example, if a government hired nine people to make measurements of economic conditions, but the entirely economy only consisted of ten people, then the act of measuring economic conditions would affect the economic body. I will assume that for large enough economic systems that this corollary applies such that I too can determine economic conditions of economic bodies without altering their state.

Systems

HK state what must be done to fully describe a body: enumerating all the "inherent limitations in the variations of its state" [11, pp. 8]. For an economic body, this list would be all the limitations of how a current economic state could vary. This list could be very long and impossible to analyze given the potential size of economic bodies.

HK now introduce the term system, a more specific notion of a body, from which I define an economic system in Analog 3.2.

Thermodynamics	**Economics**
System	Economic System
"an idealized *body* which is further restricted in that it *can be isolated from everything else.*" [11, pp. 8]	An idealized *economic body* which is further restricted in that *all endogenous economic conditions can be discriminated.*

Analog 3.2: Economic System

By determining which economic conditions are endogenous, an economic body can be isolated and idealized as an economic system. The definition of an economic system as defined by economists is defined similarly to an economic body and not restrictive enough to capture the notion of a system as used in thermodynamics. My definition of an economic system—and economic state–is precise enough for me to proceed with my thermodynamic analysis of economic systems, consistent with how an engineer would use these terms. This economic system will be a simpler, idealized version of an economic body, but is still more general than how economists use models and variables.

Some examples of economic systems are economies—such as national and local economies, businesses, and individual economic agents. In these examples, endogenous conditions can be specified as to isolate these systems from the changes that happen in other economic systems.

Environment of a System

HK consider the environment of a system as a "collection of bodies which includes everything external to system A which may at times have a detectable influence on system A, but which can at will be isolated from system A." [11, pp. 8]. Therefore, the environment of economic system A is the collection of economic bodies—that may or may not be economic systems themselves—that are not economic system A and can have an influence on economic system A or be isolated from it. The environment is responsible for all economic conditions exogenous to the economic system. As an example, consider an isolated economy that views all neighboring countries as its environment but does not consider a country across the world as its environment.

Allowed States and Laws of Matter - Economic Allowed States and Conservation of Assets

Like a physical system, an economic system is not able to take on every state. HK define allowed states as all the possible states a system can "assume consistent with the laws of matter, passive resistances, or constraints" [11, pp. 9]. It stands to reason that an allowed economic state will follow the economic analogs to the laws of matter, passive resistances, and constraints. In this section I derive the economic analog to relevant laws of matter. In the next two sections I derive the economic analogs to passive resistances and constraints.

HK present three laws of matter they deem relevant to thermodynamics. These are the three laws of matter given by HK: the law of continuity, the law of conservation of electrical charge, and the law of conversation of chemical species. The economic analogs I derive are a law of conservation of ownership, a law of conservation of intangible assets, and a law of conservation of resources.

A. The law of continuity states that "matter cannot change from one position to another without passing through the intervening space". This law of continuity is similar to the notion of the continuity of ownership of tangible assets in economics. One way to define assets in finance is as "a resource controlled by the entity as a result of past events and from which future economic benefits are expected to flow to the entity" [19] , and tangible simply requires that the asset can be touched such as inventory, machinery, or laborers. It is a requirement in economics that ownership cannot disappear. A tangible asset will not change ownership unless an interaction occurs. For example, apples will not appear in my cupboard unless I buy them from the store, and the ownership of my car will not change hands unless I sell it. Labor can be a prepaid expense, which is a type of tangible asset.

B. The law of conservation of electric charges states that electrical charge is conserved in all processes. In economics this notion is parallel to the conservation of intangible assets. Mass is a tangible thing like a tangible asset, but an intangible asset is more like a charge, something that can be observed but is not tangible. In economics, intangible assets usually include things like financial capital, services, software, and intellectual property [20]. These intangible assets are conserved in the sense that use of them can be provided or received. As an example, consider services received or rendered. A tally or tab would indicate whether these services are received or owned—a positive charge—or are required or owed—a negative charge. Another example is financial capital where a negative charge is analogous to a debt and a positive charge is analogous to a credit.

C. The law of conservation of chemical species states that chemical species are conserved in chemical reactions. This law in economics is analogous to a law of the conservation of resources and inputs. For example, the creation of consumer goods require the input of raw resources and the use of labor and capital. These inputs are all conserved in some way, including the use of labor and capital.

The economic analogs to the three given conservation laws given above imply that matter in economics itself is all the assets of an economic system.

Passive Resistances - Laws

HK characterize passive resistances by "their ability to prevent certain modes of behavior of the system regardless of what state the system may assume or to what influences it may be subjected" [11, pp. 9]. These passive resistances are internal to the system and are "go or no go" types of things—meaning they do not slow down changes, they prevent them. Similarly in economic systems, laws—requirements put in place by legal systems regarding economic behavior—are a barrier internal to economic systems analogous to passive resistances. These laws do not slow down types of behavior; they are either followed or broken. The laws in place in an economy can change what allowed economic states are possible and are one way of changing an economic system.

Constraints - Scarcity of Resources

HK state that constraints are external to a system and defined as "restrictions imposed by the environment on the states a system can assume" [11, pp. 11]. I found no clear definition of constraints in economics as would be used by an engineer or thermodynamicist. However, the "basic economic problem", the scarcity of resources, describes this situation where limitless wants are balanced—constrained—by limited resources [21]. Since the scarcity of resources is exogenously imposed on an economic system, I define a constraint in economics as a restriction exogenously imposed on the states of an economic system due to the scarcity of resources.

I analyze three relevant constraints listed by HK in this section.

1.) No Constraint. For no constraints, "Allowed states are limited only by the laws of matter and passive resistances." An economic system with no constraints would need unlimited resources but would still not violate any conservation law.

2.) Upper Limit to Volume. A "rigid, impermeable wall" is an example of a limit to the volume of a system. For an economic system, this limits the amount of resources it can obtain. For example, an economic system has an upper limit of available labor—their population—the same way a thermodynamic system can have an upper limit in volume.

3.) An Electrically Insulating Wall. If a wall prevents electrical charges from moving, then this insulating wall in economics would prevent the movement of intangible assets.

HK state that "constraints limit the states a system can assume." [11, pp. 9] Then similarly in economics, constraints—constraints involving the scarcity of resources—will determine what economic states are possible. One hard constraint on resources, inputs, and goods is that they can never be negative. Constraints do not limit interactions the environment; rather they prohibit certain states from existing. So, as an example for an economic system, a limited labor pool would not imply that an economy would or would not interact with other economic systems.

Isolated Systems and their Allowed States

HK introduce a way to determine if a system is isolated. Their definition of isolation in turn allows me to determine if and when an economic system is isolated.

Thermodynamics	**Economics**
Isolated System "For any period of time during which all changes in the *system* are independent of all changes that can occur in the environment." [11, pp. 11]	Isolated Economic System For any period of time during which all changes in the *economic system* are independent of all changes that can occur in the environment.

Analog 3.3: Isolated Economic Systems

The definition of isolation from HK needs no modification to be used by me. For an isolated economic system, exogenous conditions are either fixed or exogenous conditions that change have no effect on the economic system. A country can make its economy isolated by placing trade barriers between it and other countries. Exogenous conditions, such as the labor and output of other countries has no influence on the isolated country.

HK note that while isolation can prevent allowed states from being reached, it does not determine which states are allowed states. To the same extent, isolated economic systems may be limited from reaching certain economic states due to isolation. For example, an isolated economy may experience a shortage of certain resources, unable to obtain said resources due to its isolation.

The way I have defined isolation and exogenous conditions allows me to isolate certain elements within an economy as economic systems that economists would consider tangled with other inputs of production. Consider a group of manual laborers as an isolated economic system. They interact with the economy (their environment) only by providing manual labor, they own no capital, and they never (or very slowly) increase their skill set. In thermodynamics, an analogous example is a completely isolated piston with a barrier held in place by walls and a spring as pictured in Fig. 3-1 [11, pp. 12].

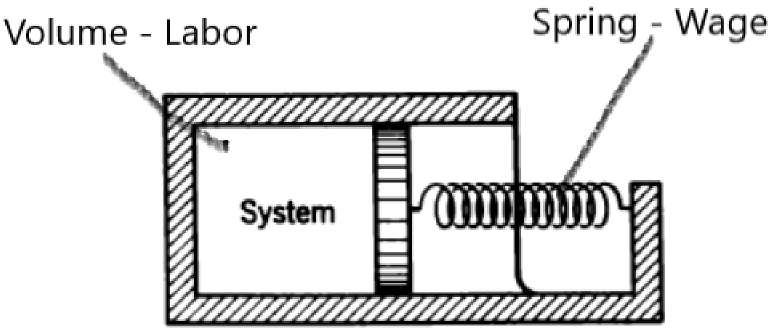

Figure 3-1: Group of blue collar workers modeled as a piston.

The only exogenous condition that could change is the spring in Fig. 3-1, otherwise the system of laborers is unaffected by all other exogenous conditions.

Change of State

HK define a change of state as the change of least one primitive property of a system [11, pp. 12], so for economic systems a change of economic state is when at least one of its economic conditions change. My definition of an economic state allows the growth or decline in all economic conditions to be analyzed during a change of economic state instead of looking at the change of a single variable in a model.

Path - Steps of Interactions

HK define a path as the "complete series of states a system assumes during a change of state" [11, pp. 12]. For an economic system, each change of economic state can be called an interaction, and the steps of interactions is analogous to a path.

Process

Having defined economic states, change of states, and interactions, I can define the economic analog to a thermodynamic process. HK define a process as the involvement of the changes of state, a path, the end state of that path, and the phenomena that occur at system boundaries [11, pp. 13]. In economics, economic activity is defined by economists as " activity of making, providing, purchasing, or selling goods or services" [22]. Economic activity involves the changes of economic states, different steps representing interactions, a final step, and a description of what happened. Economic activity is in contrast to non-economic activity, such as going to the park for a leisurely stroll. Non-economic activity involves nothing like a process. Thus only economic activity is of interest to my analysis since it will describe the economic analog to a process.

Cycle

HK state that if that a process whose end states are identical it is a cycle [11, pp. 13]. A typical definition of a cycle in economics a cycle is "fluctuation of the economy between periods of expansion (growth) and contraction (recession)" [23]. The economic definition of a cycle is restrictive compared to the concept of a cycle in thermodynamics, so I use the term cycle more generally than in economics to refer economic activity where the economic end state is the same as beginning state without any notion of growth or decline.

Interaction Between Systems - Transactions

HK indicate that the interaction between systems can be described by what happens at the boundary between the systems and the end state of either system [11, pp. 13]. For economic systems, this description means interactions between economic systems can be described by what happens at the boundary between the systems and the end economic state of either system. If what happens inside the boundary is the interaction, then the transaction is what occurs across the boundary in the other economic system.

The economic concept of a transaction allows me to differentiate between types of systems and interactions that may occur between different economic systems, such as between customers and vendors, businesses of one sector with businesses of another sector, or imports and exports between economies. Within an economic system, such as a business or household, there may be communication between partners, associates, or family members, but this is not a transaction since whatever occurs is within the system boundaries.

Derived Properties

Not all properties of a system are primitive properties. HK state that a process is required to define the change in a derived property, defined as a state function which is not a primitive property [11, pp. 13]. Based on HK's definition, in economics a derived economic condition must be determined during economic activity, and suitable economic activity can be devised to measure the change in a derived economic condition. On the other hand, economists make no distinction between derived economic conditions and economic conditions. For example, measuring the GDP requires keeping track over a period of time of all final sales and services, making it analogous to a derived property.

HK require that "the states of a system must be identifiable in terms of primitive properties before derived properties can be measured" [11, pp. 13]. So if economists can measure derived conditions such as GDP, then they do so by first measuring economic conditions that are observable and not derived; yet there exists no distinction between these conditions in economics. HK say that these derived properties are important, and list energy and entropy as two important derived properties in thermodynamics [11, pp. 14]. The economic analogs to these derived properties will be important for developing my application of thermodynamics to economics, as seen in Section 3-3-6 for energy and Section 3-10 for entropy.

Allowed States in Terms of Primitive Properties

HK make the point that the term "the values of primitive properties" could be substituted for the word "states" in the definition of allowed state because they are only concerned in defining the state in terms of primitive properties and not derived properties [11, pp. 14]. As a consequence, I will only be concerned in defining an economic state in terms of economic conditions, from which derived economic conditions can be obtained. The types of economic conditions I need for the analysis of economic systems will be extensive and intensive properties defined later on in this section.

General Definition of a Property

For completeness I transliterate HK's general definition of a property into economics. HK define a property as any quantity whose value depends only on the state of the system [11, pp. 14]. The corollaries are: A change of the value of a property is determined by the end state of a system and independent of the process, the sum of changes of a property during steps of the process is equal to the change of a property for the overall process, a change of a property during a cycle is zero, and any quantity whose change is zero during a cycle is a property.

I use this definition of a property and its corollaries from HK to establish a definition of an economic property. An economic property is any quantity whose value depends on the state of the economic system, is fixed by the end states of the system and independent of the process, has the change of quantity of each step sum to the total change, and change is zero during a cycle. As a consequence of this definition of an economic property, an economic property is more specific than an economic condition, and only economic conditions that are economic properties will be analyzed.

By establishing the notion of an economic property, I can create a list of standard economic properties used to systematically analyze economic systems the same way that the analysis of most thermodynamic systems would start by looking at the properties of volume, pressure, and number of particles. These properties will be extensive and intensive properties, defined later in this section.

Independent Properties

The properties that are required to define a state are independent properties. According to HK properties are independent if "at least one variation of state can be found such that the one property varies by a finite amount while the other remains fixed and vice versa" [11, pp. 15]. An example of dependent properties is the position of a spring and the force exerted by the spring. If the position is varied by a finite amount, the force of the spring changes. On the other hand, there exists constant pressure processes in thermodynamics where the volume can change, making volume and pressure in that process independent properties.

In economic systems, I assume that it will be possible to vary properties independently—like capital and labor—such that an economic system can be analyzed by varying properties independently. It follows from HK's definition of an independent property that an independent property in economics is any economic property that can be varied by a finite amount while keeping other properties fixed.

HK state the importance of independent properties is that a complete set is required to define the state of a system [11, pp. 16]. I then need a set of independent economic properties to define the state of an economic system. It will not be necessary to use every economic condition measured by economists if I instead stipulate that what is only required are independent economic properties.

Extensive and Intensive Properties - Stocks and Rates

An engineer or thermodynamicist uses extensive and intensive properties to analyze systems, so I too will use these extensive and intensive properties to analyze an economic system. HK define an extensive property of a system as being the sum of the value of that property for each subdivided part of the system [11, pp. 19]. HK define an intensive variable as a property "which approaches a finite limit as the size of a system which includes a selected point approaches zero" [11, pp. 13]. Analog 3.4 compares extensive and intensive variables in economics to thermodynamic counterparts.

Extensive Property		Intensive Property	
Thermodynamic	Economic	Thermodynamic	Economic
Volume (V)	Labor (L)	Pressure (p)	Wage (w)
Angle (θ)	Financial Share (F)	Torque (τ)	Return (κ)
# of Particles (N)	Capital (K)	Potential (μ)	Rent (r)
Angular Momentum (L)	Financial Capital (M) [$]	Angular Velocity (ω)	Interest Rate (ρ)
Momentum (p)	Price Index (p)	Velocity (v)	Quantity Demanded ($\dot{q}$)
Position (q)	Basket of Goods (q)	Force (F)	Inventory Cost (c)

Analog 3.4: Extensive and Intensive Properties in Economics and Thermodynamics

Examples of extensive properties in thermodynamics are volume and mass, and examples of intensive properties are pressure and density. Dividing one extensive property by another typically results in an intensive property, such as deriving density by dividing mass by volume.

In economic engineering, we consider extensive properties analogous to stocks [1]. These stock properties are measured in [#] or [θ] if it shares (% of ownership) of financial capital [1]. These stock variables depend on the size of an economic system. If an economy was cut in half, each half would only have a fraction of the labor and capital. For an economic good, if it was cut in half, hypothetically each half would carry half the price. These extensive stock variables are used in my analysis of economic systems in the same way that one might start the analysis of a thermodynamic system by first considering the volume and number of particles.

In economics, a concept related to extensiveness is that of extensive growth. It is defined as the growth of use of inputs to production [24]. The inputs related to production and extensive growth—such as capital, labor, and raw inputs—are what economic engineers consider as extensive variables.

We consider intensive variables analogous to rates in economic engineering [1], carrying the units $\frac{\$}{\text{time}\,\#}$ if it is a force or $\frac{1}{\text{time}}$ if it is a rate. In the Newtonian approach to economics, these forces are considered analogous to spring forces, such that by Hooke's law the wage (or rent) associated with labor (or capital) is related by $w = kL$ for some spring constant k. However, for my thermodynamic approach, I generally will not be able to use such a relationship, and these intensive variables will form another set of properties used to analyze economic systems as one might use pressure and chemical potential to analyze a thermodynamic system.

Conservative Properties

Conservative properties are defined by HK for properties such that "the sum of the values of that property for the system and its environment is fixed for all possible states of the system and its environment" [11, pp. 20]. Conserved property is a term commonly used in physics but not used in economics. However, the concept of a conserved property exists in economics. In economics, an analogous example to the conserved property of mass is the conserved property of resources, inputs, and goods. Goods are conserved in the sense that they cannot be produced without using appropriate amounts of raw or intermediary resources and inputs. The goods are eventually consumed and returned to the environment, and resources are limited. The sun is included to be part of the environment as otherwise the mass of an economic system would appear to be increasing.

Laws of Thermodynamics - Economic Laws

HK state that thermodynamics is based on premises called laws. Economics bases its theory on theorems, which are sometimes referred to as laws in economics. The law of supply and demand are considered analogous to Hooke's law and Newton's second law in economic engineering [1], but it remains to be seen what other laws in economics would be considered analogous to physical laws.

The method this chapter takes to developing the first law, second law, and state principle is not considered by Hatsopoulos and Keenan to be the "tidiest" approach. There is another approach by HK which generalizes thermodynamics to two independent laws [11, pp. 20]. Nonetheless, it would interesting to see if all economic theorems could be generalized to a few essential laws—such as those developed in this chapter.

3-3 The First Law of Thermodynamics and its Formulation in Economics

3-3-1 Weight in a Gravity Field - The Labor Theory of Value

To define the first law of thermodynamics, Hatsopoulos and Keenan first define an interaction between two systems called work. This definition requires an external measure of work such as a weight in a gravity field [11, pp. 22].

Historically in thermodynamics, Joule and Carnot expressed work as equivalent to the change of height of a weight in a gravity field. I searched historically in economics to find an economic concept or theory that would allow me to define the economic analog to a weight in a gravity field. I determined that Adam Smith’s labor theory of value best described an external measure of economic value. The labor theory of value states that the value of a good is determined by the value of work needed to produce that good [25]. *The Oxford Handbook of Adam Smith* states that Adam Smith reasoned

> "A good measurement unit must be invariable , and gold and silver ’like every other commodity, vary in their value, are sometimes cheaper and sometimes dearer, sometimes of easier and sometimes of more difficult purchase’ (WN I.v.7: 49-51). On the contrary, such an invariability Smith believes may be recognised in labour which: never varying in its own value, is alone the ultimate and real standard by which the value of all commodities can at all times and places be estimated and compared. It is their real price; money is their nominal price only." (WN I.v.7: 51) [26, pp. 296]

Smith argues that labor is the "ultimate and real standard" by which any other commodity can be judged by exogenously. Instead of gold or silver, he believes a "standard" amount of labor working at a "standard" wage can be used as the external measure of value. This standard wage could be a minimum or median wage and standard labor a set number of men with a standard skill set, but the point here is that instead of a gold standard an economy could have a labor standard whereby money (the basic unit of account) is based on.

One might believe that this measurement of value is old-fashioned and not applicable to modern times, but I argue that given modern economies no longer use the gold standard and base their currency on their economic performance, money is in part based on the faith in the value of standard labor in a modern economy. Likewise in modern physics, mass and work are not defined by a weight moving in a gravity field, but are instead defined by the electron-volt. This point leads me to define standard labor and a standard wage as analogous to height and weight in a gravity field in Analog 3.5.

Thermodynamics	**Economics**
Weight in Gravity Field	Standard Wage and Standard Labor
A change of height of a weight in a gravity field.	*A standard amount of labor working at a standard wage.*
$\int mg\,dH$	$\int w^*\,dL^*$

Analog 3.5: Standard Labor as Exogenous Measure of Economic Value

Here w^* is a standard wage and L^* is some amount of standard labor.

3-3-2 Work Interactions - Expenditures

At this point I can develop the economic analog to a work interaction since I have defined my exogenous measure of economic value. I refer to this phenomenon in economics as an expenditure in Analog 3.6.

Thermodynamics	**Economics**
Work Interaction	Expenditure
"An interaction between two *systems* such that what happens in each system at the interaction boundary could be repeated while the sole effect external to each system was *the change in level of a weight*" [11, pp. 22]	An interaction between two *economic systems* such that what happens in each system at the interaction boundary could be repeated while the sole effect external to each system was *that of the provision of a standard amount of labor at a standard wage.*

Analog 3.6: Work Interaction and Expenditure

The provision of a factor means that some factor—defined as a resource or input such as labor, land, capital, or financial capital—has been provided for use. If two economic systems interact such that only factors are provided, then the economic activity in each system can be measured exogenously by the provision of standard labor working at a standard wage.

I borrow the term expenditure from accounting to refer to the payments that are made to the factors of production for their use or procurement. In accounting, the payment or cost involved for the use or procurement of an asset is typically called an expenditure [27]. Business assets such as fixed assets, inventory goods, and prepaid expenses are factors of production. An expenditure is required to acquire factors needed for the operation of the business. While from the perspective of the factor, such as a laborer, this expenditure constitutes income, the business sees this payment as a cost. I use the term factor income to refer to the income associated with the use of any factor. I also use the term factor payment to refer to an expenditure. The use of the term expenditure will be synonymous with the use of the term factor income or factor payment in this thesis, depending on whether a cost is incurred or income is received.

Many interactions in thermodynamics fit this description of a work interaction, such as interactions like mechanical work, electrical work, or magnetic work. In the same manner, many types of factors—also called factors of production—can be provided in economics such as labor, machinery, land, or financial capital. The implication for both definitions is that there will be interactions that will not be work interactions in thermodynamics and will not be expenditures in economics. This other type of interaction is introduced in Section 3-6.

Analog 3.7 lists some types of work in thermodynamics and some factor uses in economics.

Thermodynamics	**Economics**
Interactions Equivalent to Work	Types of Factor Income
$\text{Pressure-Volume Work} = \int -p\,dV$	$\text{Wage Income} = \int w\,dL$
$\text{Electrical Work} = \int V\,dQ$	$\text{Rental Income} = \int r\,dK$
$\text{Angular Work} = \int \tau\,d\theta$	$\text{Dividend Income} = \int \rho\,dF$

Analog 3.7: Factor Uses Equivalent to Provision of Standard Labor

w is any kind of wage, L is any kind of labor, r is a rent, K is capital, ρ is a return, and F is financial ownership. In Analog 3.7 the provision of a factor results in an income associated with the use of that factor. Thus all types of income associated with the use of a factor is equivalent to the provision of some standard labor at a standard wage.

3-3-3 Adiabatic Processes and Adiabatic Surfaces - Standard Wage Labor Activity and Contracts

HK define an adiabatic process and use it later to define the change in a system property called energy, so in Analog 3.8 I derive the economic analog to adiabatic processes, the use of standard wage labor.

Thermodynamics	**Economics**
Adiabatic Process	Standard Wage Labor Activity
"Any *process involving only work interactions*" [11, pp. 22]	Any *economic activity involving only expenditures.*

Analog 3.8: Adiabatic Process and Standard Wage Labor Activity

Wage labor is the income a laborer receives working at an hourly or daily rate [28]. Then standard wage labor is the income a standard labor receives working at a standard hourly or daily wage. Since the income associated with any factor use is equivalent to the provision of standard labor at a standard wage, it follows any factor income is equivalent to the use of standard wage labor.

An example in economics a boundary where only factor income may pass is a contract. Similarly in thermodynamics a system boundary where only work interactions can occur is called an adiabatic surface. The contract has the effect of specifying the transaction between laborer and employer or customer and vendor. A contract specifies exactly how much work can be provided and can stipulate that there are no benefits other than the income associated with the labor. On the other hand, if a job were to have benefits besides the wage labor income, then the contract is not necessarily adiabatic.

3-3-4 The First Law of Thermodynamics - The First No-Free-Lunch Law

At this point, HK are in place define the first law of thermodynamics. Since the first law is indeed a law of nature, I develop a law fundamental to economics in Analog 3.9.

Thermodynamics	**Economics**
The First Law	First No-Free-Lunch Law
"For any *process* involving no effects *external* to the system except *the change in position of a number of standard weights between specified levels*, this *number of standard weights* is determined by the end states of the *system* and is independent of the details of the *process*" [11, pp. 22]	For any *economic activity* involving no effects *exogenous* to the system except *that of the provision of a specified amount of standard labor at a multiple of the standard wage*, this *multiple of the standard wage* is determined by the end states of the *economic system* and independent of the details of the *economic activity*.

Analog 3.9: The First Laws

To define this economic law, I juxtapose a multiple of a standard wage exogenous to economic activity to a number of standard weights moving in a gravity field. Given a certain amount of standard labor, the value of economic activity involving standard wage labor can be judged exogenously by a "multiple of the standard wage". This "multiple of the standard wage" gives the relative value for the use of a specified amount of standard labor and is independent of the economic activity.

If this first no-free-lunch law did not exist, then economics as we know would not exist in its current form. Imagine that lunches—or anything worth obtaining—could be obtained without the use of any kind of factors or by repeating a cycle. It is reasonable to ask why even study economics at that point? If the first law of thermodynamics did not exist, then engineers would not exist as there would be no need to design anything besides a singular perpetual motion machine. This law might as well be built into economics since we know nothing of value can be obtained for free and that resources are scarce.

The idea behind this first law—that nothing of value is free in economics—ties directly into the economic concept of opportunity cost. The opportunity cost is the benefit missed out on by choosing one option out of all possible alternatives [29]. The economist Milton Friedman once summarized all of economics by "There ain't no such thing as a free lunch" [30]. The first law is the "free" part: nothing can be obtained in economics without first giving something else up—the opportunity cost. For example, a laborer has to 'give up' his labor in order to receive an income. The economic activity needed to obtain anything of value is why nothing is free because exogenous to the economic activity required to obtain something of value is the use of some standard labor working at a multiple of a standard wage.

Measure of Work - Measure of an Expenditure

HK state that as a result of the first law, a multiple of a standard weight can be assigned to a work interaction as its magnitude [11, pp. 22]. For economics, this result implies that a multiple of a standard wage—such as a multiple of a median wage or median income—can be assigned to any expenditure. In everyday use people discus wages and incomes in relation to median wages. Someone might say "I make three times the median wage".

In economics, this multiple of the standard wage is a way to express labor effectiveness from the Solow growth model (see Section 2-3 for background information on Solow model). Since HK use a multiple of a standard weight for a given change of height, I can use a multiple of a standard wage for the use of a single standard laborer. In Equation (3-1) I analyze the income of a single standard laborer working at a multiple of the standard wage to demonstrate its relation to labor effectiveness from the Solow model. I denote the multiple of the standard wage N. The standard wage is w^*, and an ordinary wage is w. Standard labor is L^*, the factor of labor effectiveness is A, and effective labor is $AL^* = L_{\text{eff}}$. I propose there exists a constant α such that $\alpha A = N$.

$$\int Nw^* \, dL^* = Nw^*L^* = (\alpha w^*)(AL^*) = wL_{\text{eff}} \tag{3-1}$$

This equation shows how the use of a multiple of a standard wage N can be related to effective labor and a nonstandard wage. Given that all factor usage is equivalent to the use of standard labor at a multiple of the standard wage, this multiple of the standard wage tells how effective or valuable the factor in question is compared to standard labor. The Solow model chooses to use a multiple of standard labor instead of a multiple of the standard wage, but it is not specified if the Solow model uses a standard wage or an effective wage. If the Solow model uses a standard or median wage, then this issue is resolved by letting $\alpha = 1$. The multiple of a standard wage is a way to express labor effectiveness from the Solow model.

3-3-5 Concerning Work - Concerning Expenditures

There are interactions in thermodynamics that are equivalent to work. Similarly in economics, there are interactions that are equivalent to expenditures. HK introduce this work-energy equivalence by demonstrating how the effects of different forms of work e.g. mechanical or electrical work can be equated with each other or to the change of extensive flow variables like momentum or angular momentum. HK do not use the term work-energy equivalence since they have yet to define energy. However, this section by HK is fundamentally about this work-energy equivalence. I use examples of these equivalences given by HK in order to derive these equivalences for economic systems.

The first example is taken from HK and can be seen in Fig. 3-2 [11, pp. 23]. A piston is connected to a flywheel and allowed to accelerate the flywheel once the restraint is removed.

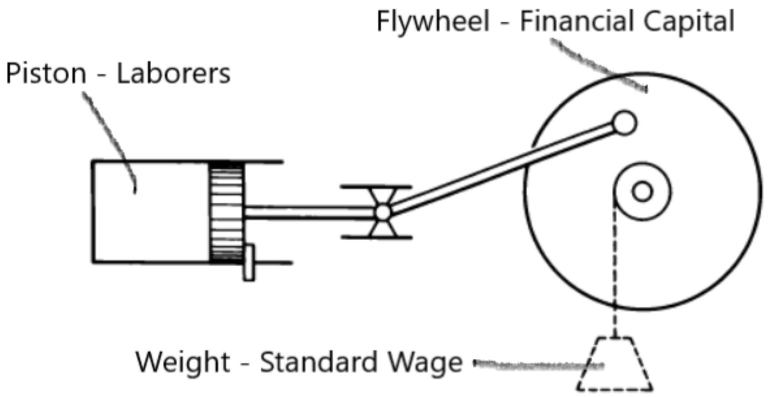

Figure 3-2: Financial capital's equivalence to labor analogous to the acceleration of a flywheel due to work of piston.

Imagine that this piston is a group of laborers (not necessarily standard labor) and that the flywheel is representative of some amount of financial capital owned by a rentier—a person who lives off interest. The rentier's received interest is equivalent to the use of labor, which is suggestive why the phrase "make your money work for you" exists in everyday language. Of course, external to the flywheel accelerating or the entire system can be some amount of standard labor working at a standard wage indicated by the dashed outline of a weight attached to the flywheel.

Another example given by HK is the movement of a disk in a viscous substance with a weight externally attached in Fig. 3-3 [11, pp. 23]. Now imagine this example as a group of consumers.

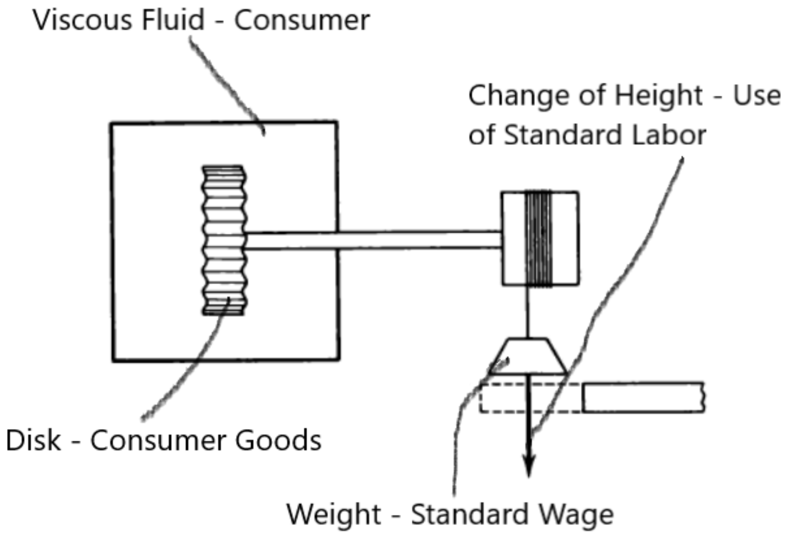

Figure 3-3: Consumption's equivalence to a factor income, analogous to disk rotating in viscous liquid.

The disk inside will eventually come to a stop because the viscous fluid takes away momentum from the disk, and external to this process is the change of height of a weight. Similarly, a group of consumers will consume some goods until the goods no longer have a price (a good loses price as it is consumed and has zero price when it is fully consumed [1]). This consumption is equivalent to the provision of standard labor in Fig. 3-3 due to the movement of the weight. This equivalence results in the the knowledge that consumption on a macroeconomic level is analogous to an expenditure for consumer goods.

Knowing that consumption is analogous to a factor payment on consuming, so too can profits be analogous to a factor payment on entrepreneurship. Profits (P) is defined as a change of price at a given quantity demanded in Equation 3-2.

$$P = \int \dot{q}\, dp \tag{3-2}$$

Here $\dot{q}$ is the quantity demanded of the good or service q provided by the economic agent. A positive change in price results in profits, and a negative change results in losses—this change is also negative in the case of consumption. The factor income for entrepreneurship is a measure of the price-raising capabilities of a company. The equivalence of Equation (3-2) to work done is known as the work-energy equivalence in mechanics, and in economics this relationship gives the equivalence between extensive stock variables and extensive flow variables.

3-3-6 Energy - Earnings

Until this point, energy has been absent from HK's treatment of thermodynamics, making no mention of energy during the definition of the first law. At this point, HK state that as a consequence of the first law, a change in a property called energy can be defined by the work involved in any adiabatic process [11, pp. 27]. In Analog 3.10 I define a change of a property I call the earnings of an economic system by the expenditure involved in economic activity.

Thermodynamics	**Economics**
Change of Energy $E_2 - E_1 = -W_{12}$	Change of Earnings $E_2 - E_1 = -W_{12}$

Analog 3.10: Change of Earnings and Change of Energy

W_{12} represents the expenditure involved in the change of earnings from state one to two the same way it represents the work in an interaction. According to economic engineering, energy and work both have the units of a cash flow ($\frac{\$}{yr}$) in economics [1].

The way I use the term "earnings" differs from its use in accounting or business. A typical definition of earnings as used in business or accounting is "the amount of money a business expects to earn assuming there are no changes in its ability or capacity to produce its products or services" [31]. I can extend this definition of earnings for more general use by replacing "business" with "economic system". However, this definition still implies that an economic system with no income has no earnings. According to the first law, notice that an economic system still has earnings even when there is no income. An economic system receives income but has earnings.

The concept of a economic system still having something without an income exists in economics. This concept is called opportunity cost and results as a consequence of the first law. The idea behind the free lunch is that by obtaining anything of economic value one forgoes another opportunity or something else of value. For example, if I choose not to work by not providing labor I can stay home to fix my garage, forgoing an income. However, the earnings that I have do not change. If instead I choose to work, I can increase my earnings by providing my labor for an income, and then I can hire someone else to fix my garage. The opportunity cost—implicit or explicit—is always captured by the earnings of an economic system, whether or not income is present. The same way a system has energy, earnings is a cash flow that a system owns independent of income.

The way economists define economics by its relation to income is analogous to the way thermodynamics approaches analyzing systems by changes of energy. HK state that in thermodynamics "energy can be identified only for a set of states which can be connected by adiabatic processes" but that "no such state has yet been found" [11, pp. 28]. HK is only concerned with states that have measurable energy the same way that economics is only concerned with economic states that have measurable earnings. Economists analyze economics by the way people earn an income—changes of earnings. The economist Alfred Marshal describes economics as "A study of man in the ordinary business of life. It enquires how he gets his income and how he uses it" [32]. He also stated that "man earns money to get material welfare" [32]. Marshal directly states that economics is concerned with income and changes of earnings. Adam Smith describes economics as "a branch of the science of a statesman or legislator [with the twofold objectives of providing] a plentiful revenue or subsistence for the people" [33]. Here "a plentiful revenue or subsistence" refers to an income. Both thermodynamics and economics approach their respective sciences by looking at the analogous changes of energy and earnings.

3-4 Thermodynamic Equilibrium - Economic Equilibrium

HK introduce the concept of equilibrium states—as well as define types of equilibrium states such as stable equilibrium—for the purpose of establishing the second law of thermodynamics. For this same purpose, I derive the economic analog to equilibrium states in this section, and of particular importance to my axiomatic approach I develop the economic analog to stable equilibrium states.

Equilibrium State - Economic Equilibrium State

The way economists define equilibrium is parallel to how it is defined in thermodynamics. In Analog 3.11, I use the same definition for both thermodynamic and economic equilibrium states.

Thermodynamics	**Economics**
Equilibrium State	Economic Equilibrium State
"A *system* is in a state of equilibrium if a change of state cannot occur while the system is not subject to interactions. " [11, pp. 30]	An *economic system* is in a state of equilibrium if a change of state cannot occur while the system is not subject to interactions.

Analog 3.11: Equilibrium States in Economics and Thermodynamics

No modifications are needed to define the economic analog to an equilibrium state. In economics, a typical definition of equilibrium as used by economists is "a situation in which economic forces such as supply and demand are balanced and in the absence of external influences the values of economic variables will not change" [34]. This definition is a bit vague since it uses the term economic forces, which is poorly defined in economics. A "situation" can be considered as a "state" without changing the meaning. The second change I make to the definition is the removal of "economic forces" to focus on the idea of external influences. The "external influences" are then analogous to "interactions".

Stable Thermodynamic Equilibrium - Pareto-Efficient Equilibrium

Establishing the economic analog to a stable state is needed to derive the economic analog to the second law of thermodynamics. HK explicitly state that stable equilibrium is "by far the most important kind" [11, pp. 30]. In Analog 3.12 I introduce the economic analog to stable equilibrium states as Pareto-efficient equilibrium states.

Thermodynamics	**Economics**
Stable Equilibrium State	Pareto-Efficient Equilibrium State
"A *system* is in a state of *stable equilibrium* if a finite change of state of the *system* cannot occur without leaving a corresponding finite alteration in the state of the environment. " [11, pp. 30]	An *economic system* is in a state of *Pareto-efficient equilibrium* if a finite change of state of the *economic system* cannot occur without leaving a corresponding finite alteration in the state of the environment.

Analog 3.12: Pareto-Efficient Equilibrium States and Stable Equilibrium States

The terminology used to define the economic concept of Pareto optimality is consistent with the terminology I have developed for the definition of Pareto-efficient equilibrium states. Pareto optimality in economics is defined as

> "a state of allocation of resources from which it is impossible to reallocate so as to make any one individual or preference criterion better off without making at least one individual or preference criterion worse off." [35, pp. 15]

I substitute an "allocation of resources" with "economic state" in my use of Pareto optimality. I could use the term "allocation" in my definition of Pareto-efficient equilibrium states, but I chose to stay consistent with HK. The way economists use allocation suffices as another way to refer to an economic state. It follows that economists use the term "reallocation" in the way I would refer to a change of state. Then the operational implication of "making people better off" is that people only accept reallocations that are beneficial in some way. The definition of Pareto optimality in economics does not mention an environment, but if an economic system "reallocates", then based on my definition of an environment a "finite reallocation" would be found in the environment. Taking everything into account, the way Pareto-efficient equilibrium states is defined in Analog 3.12 is consistent with the terminology economists use to describe Pareto optimality.

To argue why a stable equilibrium state and a Pareto-efficient equilibrium state are analogs, consider a situation where someone is external to an economic system assumed to be at a Pareto-efficient equilibrium state. First I imagine myself exogenous to an economic system that is at a Pareto-efficient equilibrium state. I find a person that I can work for in this system, which would result in both me and him be better off. Now I interact with the system, resulting in a reallocation, making them and me better off. Then the system could have not been at a Pareto-efficient equilibrium state since the assumption that the system was at Pareto-efficient equilibrium state was violated. The person inside the system benefited from a reallocation, and the benefit I received constitutes a finite reallocation in the environment. There can be no exogenous effects produced by a reallocation of this economic system by the definition of a Pareto-efficient equilibrium state. It follows then that these finite reallocations in the system and environment occur because the economic system is not at a Pareto-efficient equilibrium state.

Neutral, Metastable, and Mutual Equilibrium

For completeness, I analyze the other types of equilibrium HK introduce. The definition of a neutral equilibrium is required to define the economic analogs to neutral and thermodynamics property, which is done in the next subsection. A neutral equilibrium is defined as an equilibrium state where one change of state can be made by a temporary change to the environment [11, pp. 31].

Metastable equilibrium is similar to a type of Pareto optimality. Metastable equilibrium are defined as equilibrium points which could be made into stable equilibrium if passive resistances are introduced to a system [11, pp. 30]. This concept is similar to constrained Pareto optimality whereby a planner such as the government acts on a market [36]. The resulting state is a Pareto-efficient equilibrium state after the planner takes action or places law, incentives, etc. The equilibrium state is either constrained from reaching a more efficient state, making the current state equilibrium Pareto-efficient, or the equilibrium state is pushed or forced—such as by government action—towards a new Pareto-efficient equilibrium state.

Mutual equilibrium is defined by bringing two systems in equilibrium together into communication and seeing what happens. If no changes occur to both systems, then the systems are in mutual equilibrium [11, pp. 31]. Similarly in economics, two economic systems, originally in economic equilibrium and both separate, can suddenly be made into contact. If no changes occur in these economic systems or their environment then economic systems are in mutual equilibrium.

An example of economies not being in mutual equilibrium is the opening of Japan to trade in the 19^{th} century. Japan, originally isolated and presumably in some type of equilibrium, was made into contact with the United States. This interaction did not result in mutual equilibrium since any one around the globe could now make a profit off of interacting with Japan directly or through the United States.

Thermodynamic Property and State

HK define the properties of a system that can be changed by temporary changes in the environment as neutral properties. They define all properties that are not neutral properties as thermodynamic properties [11, pp. 31]. The thermodynamic properties are those that are of interest for analyzing thermodynamic systems. For an economic system, properties such as labor and capital are thermodynamic, and a property such as the stock or type of goods consumed might be a neutral property depending on the situation. If inventory costs are important, then inventory quantities of goods are clearly not neutral properties.

Stable State - Pareto State

As HK hereafter refer to stable thermodynamic equilibrium states as stable states, I too will refer to Pareto-efficient equilibrium states as Pareto states, Pareto optimal states, or Pareto-efficient states.

Work and Allowed States

A point HK make about equilibrium is that in thermodynamics work interactions can only occur if at least one state is not in a stable state. Similarly, based on my definition of a Pareto state, I infer that no expenditure will occur in an interaction unless one of the two economic systems involved is not at a Pareto-efficient state.

3-5 The Second Law of Thermodynamics and its Formulation in Economics

The conditions that would guarantee the existence of stable equilibrium states cannot be proven within a finite amount of time. HK state that the existence of stable equilibrium must be a fundamental law of nature, leading HK to introduce the second law of thermodynamics. This section likewise introduces a second fundamental law to economics concerning the existence of Pareto-efficient states.

3-5-1 The Second Law of Thermodynamics - The Second No-Free-Lunch Law

Here I introduce this second fundamental law of economics in addition to the first: the second no-free-lunch law.

Thermodynamics	**Economics**
The Second Law	Second No-Free-Lunch Law
"A *system* having specified allowed *states* and an upper bound in *volume* can reach from any given state a *stable state* and leave no net effect on the environment." [11, pp. 34]	An *economic system* having specified allowed *economic states* and upper bound in *factor inputs* can reach from any given state a *Pareto-efficient state* and leave no net effect on the environment.

Analog 3.13: The Second Laws

The meaning of "having specified allowed economic states" and an "upper bound in factor inputs" is that an economic system has a finite amount of resources and factors and that it follows the laws of economic matter—conservation of ownership, conservation of intangible assets, and conservation of resources and inputs. Any economic system that satisfies these conditions—in our world this is all of them—can reach Pareto-efficient states without leaving a net effect on their environment according to the second law. The importance of this second law is that Pareto states exist for economic systems and can be reached.

This law must also be fundamental to economics because, like the case of the first law, a world where this second law did not apply would result in a field of economics completely different than our own. There are two parts to the second law that both must be fundamental to economics. The first is that Pareto states exist, and the second is that an economic system can reach these Pareto states without leaving a net effect on the environment.

First, imagine a world where Pareto-efficient states could never be reached. This case would require a completely different field of economics compared to ours since the concept of equilibrium is central to economics as we know it. The conditions that hold for equilibrium states would also not exist, implying that situations such as supply equaling demand could never exist.

Second, imagine a world where these Pareto-efficient states can be achieved by economic systems at Pareto states continuously leaving net changes in the environment. Then a system in a Pareto state could earn a free lunch by carefully devising economic activity such that it could continuously reclaim this free lunch back from the environment, bringing the economic system back to a Pareto state. This formulation is known as a perpetual motion machine in thermodynamics, and in economics it would result in a world where waste products had the same value as the goods they came from.

Economics is about how constraints force economic systems to make choices on how to use scare resources. One definition of economics is "Economics is the study of how societies use scarce resources" [37]. This terse definition of economics is equivalent to the no-free-lunch laws. Both imply that there is always opportunity cost required to obtain something of value and the benefit of the use of it results in something of less value.

Corollary to the Second Law - Corollary to the Second No-Free-Lunch Law

There exists economic theorems related to the existence of equilibrium and conditions that govern equilibrium. These theorems then must be derived from the second law such that they are applicable to economic systems only through the existence of a second law. HK introduce a particular corollary to the second law, which I develop an economic analog to in Analog 3.14 before discussing the existence of other economic corollaries to the second law.

Thermodynamics	**Economics**
Corollary to the Second Law	Corollary to the Second No-Free-Lunch Law
"If a system is in a *stable state*, no change to an allowed state can have as sole effect *external to the system the rise of a weight.*" [11, pp. 35]	If an economic system is in a *Pareto-efficient state*, no change to an allowed state can have as sole effect *exogenous to the economic system that of the provision of standard labor.*

Analog 3.14: Corollary to The Second laws

This particular corollary in Analog 3.14 implies in economics that economic activity involving the provision of factors indicates that at least one of the systems involved is not at a Pareto-efficient state. This corollary is applicable to economic systems because of the existence of the second law and the definition of a Pareto state. If a system provides a factor, it is attempting to obtain something of value—such as providing labor for an income. Then in a Pareto-efficient state an economic system no longer tries to obtain something of value. Seeing when an economic system provides factors can determine when it is not in a Pareto state, granting the ability to recognize which economic states are Pareto states through a process of elimination.

The conditions that dictate when an economic system is an Pareto state must follow from the second law. That is to say, the second law states that these Pareto states exist but does not explicitly derive what conditions are present during equilibrium. Conditions such as boundary or internal forces balancing, chemical or electrical potential being equal, or stresses balancing in statics are conditions that result from the second law in physical systems. In the same manner for economic systems, conditions such as diminishing marginal utility leading to equilibrium, demand equaling supply, or marginal cost equaling marginal benefit must be consequences of the second law.

In economics there exists theorems on equilibrium that must follow from the existence of a second law. These theorems demonstrate the attainability of equilibrium and conditions that govern equilibrium for particular situations. One example are the two fundamental theorems of welfare economics that guarantee the existence and reachability of Pareto-Efficient states under certain conditions for markets [35]. These fundamental welfare theorems are indeed theorems, not laws, and they must result from the application of the second law to welfare economics and markets. Another example are Varian's theorems [38], which again are another set of theorems of welfare economics and must result from a particular application of the second law. There are many theorems and conditions in economics that result from the existence of equilibrium, but it is beyond the scope of this thesis to explicitly show how they follow from the second law.

3-5-2 The State Principle - Economic Welfare

HK introduce the state principle to connect energy and independent properties to stable states. I provide a statement of this principle in economics in Analog 3.15 and connect it to economic welfare.

Thermodynamics	**Economics**
State Principle	Economic Welfare Principle
"The *stable state* of a *system* bounded by a fixed surface is fully determined by the set of allowed states and the *energy*." [11, pp. 35]	The *Pareto-efficient state* of an *economic system* bounded by a fixed surface is fully determined by the set of allowed states and the *earnings*.

Analog 3.15: State Principle in Thermodynamics and Economics

This "fixed surface" serves as a boundary in economics where nothing is exchanged across. The "surface" in economics is the distinction between what is endogenous to the economic system and what is exogenous, and it being "fixed" means that it does not change or let anything pass. For example, if an economic system, namely a country, was to be isolated, its borders could serve as a fixed surface where it would not trade goods, capital, or labor across this border.

The concept of economic welfare demonstrates how this state principle allows for the ranking of the desirability of economic states and showing which states will be Pareto states. Economic welfare is the concept that people undertake economic activity in order to improve their material well-being—their economic welfare [32]. If earnings is the sole independent property of this economic system with a fixed surface, then the system would naturally reallocate to a state with the highest possible earnings. The highest obtainable state of earnings—or set of states with the highest earnings—would be the most desirable. So if an economic system reallocated to a state with the highest earnings, there would be no other state worth reallocating to. Thus, this state would be a Pareto state since the economic system would not reallocate to a state with lower earnings. The earnings serves as a money metric utility[1], ranking the desirability of the outcome of any reallocation according to earnings.

The first law, second law, and state principle make the basis for the classical "exposition" of thermodynamic theory according to HK [11, pp. 35]. So, of consequence for economic engineering theory, the economic analogs I have developed thus far—the no-free-lunch laws and economic welfare principle—form the axiomatic basis to the thermodynamics of economic engineering.

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[1] Utility measured in terms of money—in the case with earnings, a cash flow—instead of unitless utils.

3-6 Heat, Temperature, and their Formulation in Economics

The formulation of heat and temperature in economics in this section will lead to the development of the economic analog to entropy in Section 3-10.

3-6-1 Heat Interactions - Expenses

HK describe a pure heat interaction as the flow of energy that may occur between systems at stable states without work being done by the environment. Similarly in economics, the economic analog to a heat interaction will result in variations in earnings between systems without any expenditure. In Analog 3.16 I derive the economic analog to a heat interaction, which I refer to as an expense.

Thermodynamics	**Economics**
Heat Interaction "An interaction which may occur between two *systems*, originally isolated and in *stable states*, when they are brought into communication without altering their allowed states and without *work being done by the environment*." [11, pp. 37]	Expense An interaction which may occur between two *economic systems*, originally isolated and in *Pareto-efficient states*, when they are brought into communication without altering their allowed states and without *the use of factors by the environment.*

Analog 3.16: Heat Interaction and Expense

While the terms expense and expenditure may be used interchangeably in everyday language, an expense is defined in contrast to an expenditure in accounting [39]. An expense refers to the consumption or use of item or asset that reduces its value as it is used to generate revenue [40]. For example, a salary for the services of a skilled worker is considered an expense payment incurred immediately when the services are provided [40].

The term "communication" is an appropriate term for two economic systems brought together, referring to the exchange of information and earnings between two economic systems. If communication is cut between economic system while a service is being provided, the expense ceases and both system will still be at Pareto states. So if the earnings vary when two economic systems at Pareto states are in communication, then the only interaction that can occur is an expense since they will not provide the use of factors to each other.

An economic system may incur petty expenses for items such as office supplies like pencils or paper. As soon as these items are bought they are considered an expense not an expenditure. These items are required for a business to run but are not assets. An example of another expense for a business is when it pays another economic system for services in order to increase its productivity, such as expenses for professional services, advice, or consultation.

While the use of labor is an expenditure, the use of services is an expense. The distinction between labor and services can be understood by analyzing how Adam Smith develops the use of his terms "productive" and "unproductive" labor. Adam Smith defines the two terms as

> "There is one sort of labour which adds to the value of the subject upon which it is bestowed; there is another which has no such effect. The former, as it produces a value, may be called productive; the latter, unproductive labour. Thus the labour of a manufacturer adds, generally, to the value of the materials which he works upon, that of his own maintenance, and of his master's profit. The labour of a menial servant, on the contrary, adds to the value of nothing." [33, Book 2 Chapter 3]

Adam Smith defines the "productive" labor as the factors of production that produce tangible goods or assets. The result of productive labor for him results in physical commodities. So if a laborer builds a chair, the laborer transforms nails, wood, and his labor into a commodity. Smith sees the "unproductive" labor as services that have no productive capability. Smith states the results of unproductive labor "perish in the very instant of their performance" [26, pp. 275]. Smith is not saying there are not skilled services; however, he considers the effect of skilled services to also be capital producing as if it were labor. The value added by skilled services to the earnings of an economic system is large enough in comparison to a "menial servant" to be considered "productive" by Smith. However, my use of an expense implies that no factor is provided with the services of either the skilled worker or menial servant. I argue from Smith's use of these terms that both the "menial servant" and skilled services provide unproductive labor that incurs an expense. To quote Maxwell, "All heat is of the same kind" [41], which can be interpreted in economics that all services are of the same kind and distinct from the goods and labor.

What Smith did not analyze about these expenses—that neoclassical economics emphasizes—is that they can increase the productivity of the factors of production or contribute to output. Neoclassical economics recognizes the existence of productivity and effectiveness of the factors of production. The contribution of Solow himself to economics was to recognize that capital accumulation and labor growth alone did not explain economic growth, whereas before Solow economists thought labor and capital primarily explained economic growth [2, 42]. He introduced a residual term that accounted for half of economic growth, which he attributed to increases in technological innovation [2]. Neoclassical economics does not make a hard delineation between "productive" and "unproductive" labor that Smith makes, but instead considers how expensing services—even if the service of a menial servant—contributes to the growth of earnings of an economy.

Services make up a large portion of a modern economy. For instance, the service sector now makes up over 67% of the United States GDP [43]. Throughout history this service portion of any nation's economy has grown, and most modern economies are service-based economies. These services constitute the heat while the labor and goods constitute work. Consider how many thermodynamic systems start off with availability to do work but eventually heat up and reach stable equilibrium, unable to do work. Economies act the same way, starting from a large group of laborers and raw resources and eventually reaching a point where services are needed to better facilitate the use of the factors of production.

Heat Interactions between Systems not in Stable Equilibrium

HK state that if systems that are in equilibrium but not necessarily stable equilibrium interact, then the resulting interaction is a heat interaction if variation in states results in allowed states that are not allowed states that can be eliminated to make nonstable states stable [11, pp. 38] For economic systems, this statement means that expenses can be exchanged between systems that are at economic equilibrium, and particular allowed states can eliminated to make these economic equilibrium states into Pareto-efficient states.

Quantitative Definition of Heat - Quantitative Definition of Expense

HK now quantify the magnitude in change of energy during a heat interaction, which can be seen in Equation (3-3). E_A is the earnings of system A, and $(Q_A)_B$ is positive for a flow of heat from B to A.

$$E_{A_2} - E_{A_1} \coloneqq (Q_A)_B \tag{3-3}$$

Similarly in economics, an expense will result in a change of earnings. As a consequence of the economic welfare principle, the changes of economic state that occurs during expenses involves a change of earnings. The economic welfare principle states that earnings is the only independent property of these interacting economic systems both at Pareto states. Thus expenses results in the movement of welfare between the two economic systems.

The cash flow associated with the expense is $(Q_A)_B$. This cash flow serves as a measure of value of the services provided in an expense. This cash flow is equal in magnitude to the change of earnings of each system. The system expensing the services pays receives services and the system providing a service receives a payment.

These expense payments can be service charges, fees, commissions, penalties, premiums, tips, or salaries. A fee is defined as "a charge or payment for professional services " and "a sum paid or charged for a privilege " [44]. For example, I might have to pay a late fee for returning a book to a library. The fee is the service charge that represents the magnitude of earnings lost by the library for not having the book in stock. A salary is defined in contrast to wage labor; a salary is paid periodically and not based on hours worked or tasks completed. A salary inherently cannot be exogenously measured by the use of standard labor at standard wage because a salary is paid regardless of work done. This reasoning applies to all types of service charges, where the expense payment is for consultation, service, or a right that cannot be expressed as the use of standard labor.

3-6-2 Zeroth Law of Thermodynamics - Law of One Price

The zeroth law that HK introduce is a law of mutual equilibrium between multiple systems. Similarly in economics, a law for mutual equilibrium between multiple economic systems exists and is called the law of one price. Consider three stable (economic) systems A, B, and C:

Thermodynamics	**Economics**
Zeroth Law of Thermodynamics	Law of One Price
"if A and C are each in mutual equilibrium with B when communication with B is established so as to allow variations in their *energy*, they are in mutual equilibrium with each other when brought into communication in the same manner." [11, pp. 37]	If *economic systems* A and C are each in mutual equilibrium with *economic system* B when communication with B is established so as to allow variations in their *earnings*, they are are in mutual equilibrium with each other when brought into communication the same manner.

Analog 3.17: The Zeroth Laws

The law of one price arises from the assumption that arbitrage is eventually eliminated between economic systems [45]. One interpretation of the law of one price is that a good must have the same price in different markets or otherwise arbitrage can occur [45]. Arbitrage is commonly taken to be taking advantage of a price difference between two market for the same asset [46]. If two systems are brought together and transactions between the systems eventually die down, the intuition is that arbitrage has been eliminated an can no longer occur in this mutual equilibrium.

The law of one price must be a fundamental law of economics the same way the zeroth law is a fundamental law of thermodynamics. If the law of one price as stated in Analog 3.17 did not exist, then it would be possible for arbitrage to occur for situations in economics where prices were the same or systems were in mutual equilibrium. For three economic systems—A, B, and C—take A to be in mutual equilibrium with B, and A to also be in mutual equilibrium with C. If B is brought into communication with C and an exchange of earnings occurs, then arbitrage occurred. Thus this law of one price must guarantee the existence of a no-arbitrage condition between economic systems such that the result of B being brought into communication with C results in no exchange of earnings.

3-6-3 Temperature - Price Level

HK use the zeroth law to define a property called temperature, which leads me to develop a property called price level as the economic analog to temperature in Analog 3.18.

Thermodynamics	**Economics**
Temperature	Price Level
"When two finite *systems can be made to undergo a heat interaction* at a finite rate, they are said to be unequal in temperature. When eventually the systems reach equilibrium and *the interaction ceases*, the systems are said to be equal in *temperature.* " [11, pp. 40]	When two finite *economic systems undergo an expense transaction* at a finite rate, they are said to be unequal in *price level.* When eventually the systems reach equilibrium and the *expenses cease*, the systems are said to be equal in *price level.*

Analog 3.18: Temperature and Price Level

In the same way that the zeroth law allows HK to define this property called temperature, the law of one price allows me to define a property called a price level. A corollary to the law of one price is then if economic system A is equal in price level to systems B and C, then if B is brought into communication with C no interaction will occur because they are too equal in price level. It follows then that price level is analogous to temperature because the law of one price is analogous to the zeroth law and the existence of both properties are consequences of their respective law.

The reason I chose the term price level is that it represents an average price across an entire economic system. A price level can be described as "the average of current prices across the entire spectrum of goods and services produced in the economy" [47]. Price level will determine whether earnings will vary between economic systems if they are brought into communication. If two markets are unequal in price levels, then arbitrage of goods and services is possible. The use of services can be described as a type of arbitrage between economic systems, where one economic system takes advantage of a system at a higher price level to increase the welfare of their own economic system through incurring an expense.

The units of a price level can be determined by considering how intensive properties are 'prices' of extensive properties. Temperature is an intensive property, which means in economic engineering a price level is analogous to a rate [1]. The units of a rate are generally units of \$ over other units. The exception is for money which is already measured in \$ so an interest rate is in units of one over time. Consider that the 'price' of labor is its wage, the 'price' of capital is its rent, and the 'price' of money is its interest rate. The 'price' of services in an expense is then its price level, which can be measured in $\frac{\$}{hr}$, $\frac{\$}{mo}$, or $\frac{\$}{yr}$.

Economists use price indices to approximate price levels. These price indices serve as 'pricemometers', which can be considered analogous to thermometers, but are not perfect and suffer from biases [48]. There are many types of price indices, such as price indices for stock markets, a GDP price index that measures for all domestic goods and services, or a PCE index that measures the inflation rate. It follows that if inflation is defined as a change in the price level [49], then inflation is analogous to a change of temperature. The construction of a more sophisticated 'pricemometer' would lead to better measures of inflation by measuring a true price level instead of an approximation by a price index.

3-7 Heat Engines and their Formulation in Economics

This section formulates how idealized businesses are analogous to engines. The perpetual motion machine of type 1 is analyzed in this section for completeness since a perpetual motion machine of type 2 is defined in this section in terms of a heat engine.

The concepts of this section and Section 3-8 are used by HK in their derivation of entropy, which I derive the economic analog to in Section 3-10.

3-7-1 Perpetual Motion Machine of Type 1 - Perpetual Activity Businesses of Type 1

HK state that perpetual motion machines (PMM) are devices that could theoretically create infinite energy or do infinite work [11, pp. 46]. An economic perpetual activity business (PAB) of type 1 is characterized by its ability to produce free profits by going through a cycle.

Thermodynamics	**Economics**
PMM1	PAB1: Free Profits
"any *system* which undergoes a cycle and has no external effect except *the rise of a weight.*" [11, pp. 46]	Any *economic system* which undergoes a cycle and has no external effect except *that of the provision of standard labor at a standard wage.*

Analog 3.19: PMM1 and PAB1

The PAB1 is a business that can generate profits without any revenue or costs and by operating in a cycle. One example is a trader who gains profit in a cycle by trading away his inventory and trading back for the same inventory for the same stock and price.

This PAB eliminates the scarcity of resources, which is not possible by the first no-free-lunch law. Since this PAB can supply free standard labor, it can supply an infinite amount of anything equivalent to standard labor. The machine goes perpetually through economic activity, such as through a cycle, and continuously delivers standard labor at some standard wage. Anything of value can be obtained without putting any work in, meaning that resources become limitless.

3-7-2 Heat Engines and PMM2 - Businesses and PAB2

Heat Engines - Businesses

HK introduce heat engines, a device that can produce work by using two sources of heat. I introduce a business as the analog to a heat engine in Analog 3.20.

Thermodynamics	**Economics**
Heat Engines	Businesses
"*a system* which operates in a cycle while only *heat and work* cross its boundaries" [11, pp. 121]	*An economic system* that operates in a cycle while only *expenses and expenditures* cross its boundaries.

Analog 3.20: Heat Engines and Businesses

Businesses bring together different economic systems in order to receive profits. They have revenues, expenses, and profits that move between them and other systems. Revenues are opposite in sign to an expense since a revenue flows into a business. Sources of revenue include customers, lender, or savers, and sources of expenses includes investors, borrowers, or vendors. Other economic systems such as business sectors or entire countries can also act like businesses.

Profits can only be achieved by bringing together two different systems. As a consequence of the law of one price, someone cannot continuously achieve profits by bringing together similar economic systems. For example, if someone tries to do arbitrage between two different vendors with similar goods, the vendors' price levels will eventually equalize. In order to achieve long-term profits, a business must bring together different types of systems such as bringing customers together with vendors.

I use a business as the analogy to a heat engine in a general sense. Some business are open systems, meaning that goods can enter and leave the system. In this section I will analyze businesses that are closed such that they are analogous to heat engines. In general, businesses are analogous to open engines like an internal combustion engine. For example, a store can have inventory assets that move through it, which constitutes an open system. A business can also grow by acquiring more assets. The goal of this section is to analyze an idealized business and start by analyzing a business as a heat engine.

The operation of a business can be understood by analyzing its components. Fig. 3-4 is an example of a heat engine given by HK [11, pp. 121].

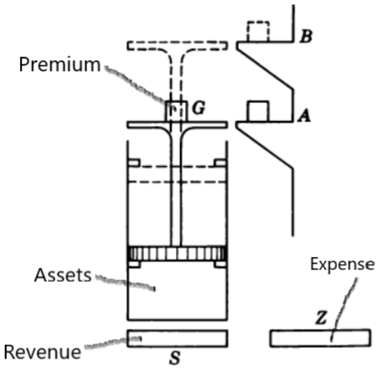

Figure 3-4: A business has certain assets. While receiving revenue, the business uses its assets at a premium. Once expenses must be paid, the assets are compensated at the lower rate and the business pays other expenses such as interest on a loan.

The piston in Fig. 3-4 represents the assets the business owns. For example, consider a bike rental company that rents out bikes only for a day at a time. For this example I assume this business took out a loan to purchase this capital.

Say that the business keep this initial amount of bikes fixed. When in contact with the revenue source—customers—the bikes are rented out for the day but do not leave the balance sheet of the company. The bikes are still owned by the business and are rented out at a higher rent compared to their initial rent. This difference in rents is referred to as a spread [50]. Since the spread is positive, it can be considered a premium [51]. If the spread were negative, it would be a discount [52]. The bikes are rented out to customers at a premium for the convenience of using the bike without the liability of owning it.

Then the bikes are returned at the end of the day and the business pays certain expenses. The source of expenses can be interest on the loan used to purchase the bikes or required maintenance and insurance on the bikes that the business needs to pay.

This process can be visualized with a phase diagram [11, pp. 122]. An example of a business cycle is shown in Fig. 3-5.

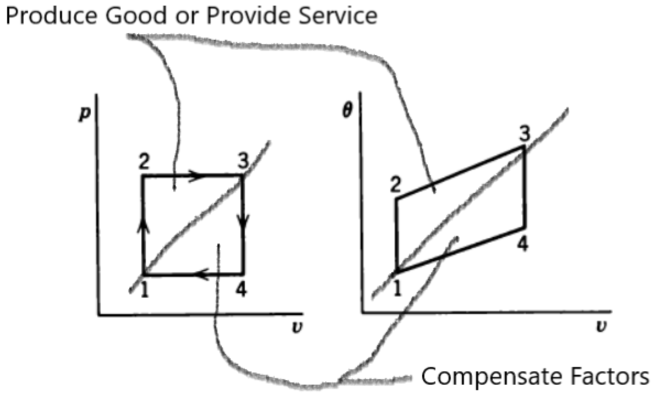

Figure 3-5: Business cycles represented by a phase diagram.

Consider a group of laborers that go to work everyday. The cycle starts at point 1 where the labors arrive at work. Now at work management steps in and implements their business plan that creates a wage spread, making the laborers more productive. From steps 2 to 3 the business sells a product or provides a service as these laborers work. Revenue Q_1 flows into the business between these points.

Now from points 3 to 1 the business must compensate its laborers. The work place is swapped for the home environment. The laborers go home and are no longer as productive. In this example the business compensates its laborers at a lower rate than the wage at which the laborers provided their labor. The premium on their labor dissipates, and they are compensated at their normal wage. The expense the business pays is Q_2.

The difference between the revenues Q_1 of a business and its expenses Q_2 is the net profit received by a business. This net profit is the factor income for entrepreneurship for taking the risk to bring together two different economic systems. Equation (3-4) shows the relationship between profits W, revenue, and expense. Q_2 is negative since it leaves the business.

$$W = Q_1 + Q_2 \tag{3-4}$$

The profits are a factor income and are not the same as earnings. These profits may or may not be retained as earnings by the company.

The net profit as defined in Equation (3-4) is an accounting profit. Another type of profit is economic profit, which is commonly taken to be the accounting profit minus opportunity costs [53]. This net profit is the opportunity cost of entrepreneurship.

Efficiency of a Heat Engine - Net Profit Margin

HK define the efficiency of a heat engine, and I derive the analog to the the efficiency of a heat engine in Analog 3.21.

Thermodynamics	**Economics**
Efficiency of a Heat Engine $\eta := \frac{W}{Q_1} = \frac{Q_1 + Q_2}{Q_1}$	Net Profit Margin of a Business $\eta := \frac{W}{Q_1} = \frac{Q_1 + Q_2}{Q_1}$

Analog 3.21: Efficiency and Net Profit Margin

Here the revenue that flows into the company is measured by Q_1 and sometimes called the top line in accounting [54]. W is called the bottom line [54]. The top line is the revenue of the business that it wants to maximize, and the business attempts to maximize net accounting profit by also reducing expenses Q_2.

$$\begin{array}{r l} q_1 & \text{— Top Line} \\ -q_2 & \text{— Expenses} \\ \hline W & \text{— Bottom Line} \end{array}$$

Here q_1 and q_2 are the unsigned magnitudes of Q_1 and Q_2. The net profit margin is defined as net profit divided by total revenue, which is parallel to how the efficiency of a heat engine is defined.

Perpetual Motion Machine Type 2 (PMM2) - Perpetual Activity Business Type 2 (PAB2)

HK introduce a PMM2 as a theoretical heat engine that could produce work by extracting heat from a single system. Such a device violates the second law. It follows such as business should also not exist in economics.

Thermodynamics	**Economics**
PMM2	PAB2: Ponzi Scheme
"*A perpetual motion* machine of the second kind is *any heat engine* which exchanges *heat* with a single *thermodynamic system in a stable state* and delivers *net work.*" [11, pp. 124]	*An economic perpetual activity* machine of the second kind is any *business* which exchanges *expenses* with a single *economic system in a Pareto state* and delivers *net profits.*

Analog 3.22: PMM2 and PAB2

A Ponzi scheme relies on receiving payments from new customers to pay back other customers. The business has expenses that it pays to its customers, but is then able to recover those expenses back by interacting with the customers again. Such a business cannot produce net profits and is not sustainable.

The economic PAM2 is any business that can create net profits without any source of revenue. The business would only have a bottom line and expenses, but it would be able to make net profits. The business somehow recovers these expenses by interacting with their source of expenses. With the use of a PAM2, anything of value could be enjoyed in economics but then recovered to be enjoyed again. It is easy to incur costs in economics, but it takes effort to generate a revenue.

It follows from the second law violation by a PAB2 that the impossibility of a PAB2 is an equivalent statement to the second law. Thus it is impossible in economics to receive profits without both a revenue and expense. Any business that can be formulated as a Ponzi scheme is a violation of the second law and cannot be sustained.

3-8 Reversibility and Irreversibility

The concepts of reversibility and irreversibility are needed for a proper description of entropy.

Reversible Process - Reversible Activity

HK define the reversibility of an isolated system as "A process of an isolated system undergoing a finite change of state is reversible if the system can be restored to its initial state, except for changes (DP) of an order smaller than the maximum changes (ΔP) that occur during the process in question" [11, pp. 124]. In Analog 3.23 I compare HK's extended definition of reversibility to reversible activity in economics.

Thermodynamics	**Economics**
Reversible Process	Reversible Activity
"*A process of any system is reversible if the process* could be performed in at least one way such that the *system* and all elements of its environment can be restored to their respective initial states, except for differences of smaller order than the maximum changes that occur during the *process* in question. " [11, pp. 124]	*Economic activity of an economic system is reversible* if it could be performed in at least one way such that the *economic system* and all elements of its environment can be restored to their respective initial states, except for differences of smaller order than the maximum changes that occur during the *activity* in question.

Analog 3.23: Reversibility

In thermodynamics, movement along a curve that represents an adiabatic process is reversible as long as the process does not deviate from the curve. An example of such a curve from HK [11, pp. 124] can be seen in Fig. 3-6.

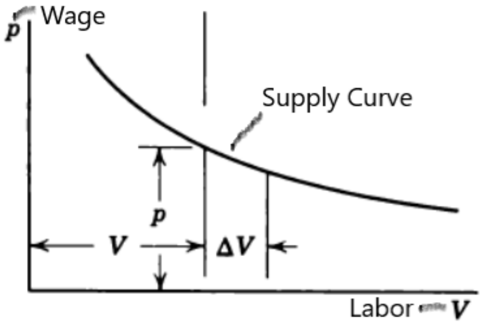

Figure 3-6: Movement along a curve representing an adiabatic process is an example of reversibility in thermodynamics. Reinterpreting this graph in economics, I consider the pressure analogous to a wage and the volume analogous to labor. Then movement along this curve is analogous to moving along a supply curve. It follows that movement along this supply curve is reversible. If this supply curve shifts or the economic activity in question moves between different supply curves then the activity is not reversible.

It follows in economics that movements along a demand or supply curve is one example of a reversible process as long as activity moves along the curve. The activity that causes movement along a supply or a demand curve can be repeated in reverse to bring the states of the environment and system back to its original states. For example, if a vendor changes the price of its good and quantity demanded for this good moves along a demand curve, the vendor could change the price back to its old price and the quantity demanded would shift back to its old value.

However, it is possible to move away from a supply or demand curve. Supply and demand curves can shift in economics for numerous reasons, making it difficult to determine by which activity the curve shifted. Even if the activity required to shift back to the old supply and demand curve can be determined, it is possible this activity results in changes of state in the environment or system large in magnitude compared to the magnitude of the activity. There can be loses present such as taxes, depreciation, or consumption that make it impossible to shift back to the old demand and supply curve without incurring changes of a large order.

Another example of reversibility in economics is a clawback. A clawback is defined as money or benefits that have been given out, but are required to be returned—clawed back—due to special circumstances or events or where there is a clawback provision in a contract [55]. A clawback in economics results in the return of paid cash but often with a fee. Small changes to the original state much smaller than the magnitude of the activity could be expected with a clawback. If this fee satisfies the condition of being small in magnitude compared to the initial transaction then the original transaction can be considered clawback-able.

HK give an example of a slowly expanding piston as a reversible process [11, pp. 124]. However, authors Gyftopoulus and Baretta in *Thermodynamics Foundations and Application* state that adiabatic expansion must happen extremely quickly to be a reversible process [56]. I infer from both statements that for an adiabatic process to be reversible time cannot be a factor. A process happening infinitely quickly or infinitely fast both imply that time was not a factor in the process. I conclude that if time is not a factor in economic activity equivalent to the use of standard wage labor then the activity is reversible.

Heat Reservoir - Competitive Markets

HK introduce heat reservoirs as a type of system with a temperature that remains constant in interactions. These heat reservoirs are usually large in comparison to the systems they interact with. In Analog 3.24 I derive a competitive market as the analog to heat reservoirs.

Thermodynamics	**Economics**
Heat Reservoir	Competitive Market
"*a system* in a *stable equilibrium state* such that, when subjected to finite *heat interactions*, its *temperature* remains constant." [11, pp. 130]	"*an economic system* in a *Pareto state* such that, when subjected to finite *expenses*, its *price level* remains constant. "

Analog 3.24: Heat Reservoir and Competitive Market

A competitive market is commonly taken to be a market that satisfies the conditions that it is in equilibrium, there are a large amount of buyers and sellers, and an individual or business cannot affect prices through either supply or demand [57]. A business that accepts the price level of a competitive market is considered a price taker. A price setter would also be analogous to a heat reservoir if such an economic system could always set its price level.

An example of a competitive market is a consumer market or producer market for commodities. These markets are large and price levels remain relatively constant. A business operating between these markets would have to accept the price levels of each market. Service markets also can be competitive markets. For example, consider a large group of skilled workers such as plumbers that have unionized. Any economic system hiring the services of a plumber would have to accept the same price level for the services of any plumber.

Carnot Engine - An Efficient Business

The Carnot engine is a particular example of a reversible heat engine given by HK. The phase diagram of the reversible processes involved in a Carnot cycle from HK [11, pp. 130] can be seen in Fig. 3-7.

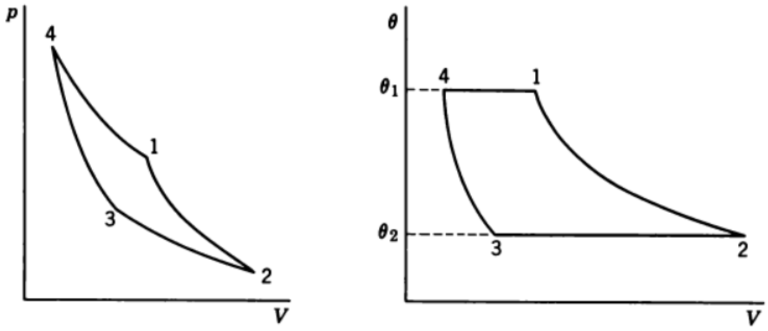

Figure 3-7: The Carnot cycle for a Carnot engine. Two adiabatic processes are connected by two isothermal processes. For this efficient business, the business operates between two fixed supply and demand curves. These two supply and demand curves detail how their assets are used during each cycle. These two supply and demand curves are connected by iso-price level activity. The business is a price taker, accepting the price levels of each market.

I will refer this idealized business that is analogous to reversible heat engines an efficient business. As the Carnot engine is one type of reversible heat engine, the efficient business pictured in Fig. 3-7 is one type of efficient business. This efficient business can repeat its activity in reverse by supplying a factor income analogous to how a Carnot refrigerator can be supplied work to move heat from a cold to hot reservoir.

The efficient business operates between economic systems that are competitive markets such as a consumer base and producer base. This efficient business is always a price taker. It can have assets such as laborers or capital that are provided in such a manner to always move along supply and demand curves. If these assets move away from their original supply and demand curves while being used then the business ceases to be a efficient business.

The efficient business also does not pick up miscellaneous expenses while undergoing economic activity. If certain expenses were required during an expenditure, then the business is no longer efficient. For example, if some asset is consumed or depreciates in some way while it is being used than a business is no longer operating as efficiently as it could be operating. Any form of irreversibility during the operations of an efficient business will lead to it to no longer being efficient. If these miscellaneous expenses are small in magnitude compared to the magnitude of activity the business engages in, then a business can approximately be an efficient business by the definition of reversibility in Analog 3.23.

Irreversibility

A process that is not reversible is irreversible. HK list criteria that can be met to eliminate the possibility that a process is reversible.

Thermodynamics	**Economics**
Irreversible Process Criteria	Irreversible Activity Criteria
"*A process* is irreversible if as a result of its being reversible, either (1) a system in a stable state could be made to change to another allowed state with *the sole external effect being the rise of a weight*, or (2) a *PMM2* could be devised. " [11, pp. 132]	*Economic activity* is irreversible if as a result of its being reversible, either (1) *an economic system in a Pareto-Efficient state* could be made to change to another allowed state with *the sole exogenous effect being the provision of standard labor*, or (2) an *economic PAM2* could be devised.

Analog 3.25: Criteria of Irreversibility

The following are a few examples of irreversible economic processes. If supply and demand curves shift, then the economic activity will be irreversible since deviation from these curves cannot solely be the exogenous provision of standard labor. If price levels of a competitive market of price setters were to fluctuate during a cycle, then a business is not an efficient business. If a business was to be formulated to be an efficient business but found to be acting as a Ponzi scheme, then it actually is not an efficient business. If large expenses are occurred during the use of a factor then the activity is not reversible.

Internal and External Irreversibility

HK introduce two notions of reversibility. The first is internal reversibility that HK define as "a process in a particular system is internally reversible if it can be performed in at least one way with an arbitrarily selected environment such that the system and all elements of its environment can be restored to their respective initial states" [11, pp. 136]. Thus, in economics, if an arbitrary environment can be selected that will result in reversible activity being possible, then that economic system is internally reversible. For example, a company could have an internal clawback to take back salary from a group of employees that acted in bad faith.

The second notion of reversibility is external reversibility, which HK state is "merely the application of the definition of internal reversibility to the environment as a system" [11, pp. 137]. In an economic system it means that the environment could be treated as a system and reversible activity could be devised to restore it to its original state. As an example, if some customers want a refund on a service, the business can provide this refund such that the customer and business are reset to their original states.

HK make the point that "if a process involves internal reversibility alone or external reversibility alone, it will be an irreversible process as regards an isolated system" [11, pp. 137]. For an economic system to be able to reverse any economic activity, it must then be able to do so internally and externally. In the example of the refund to the customer, the vendor sitting at the other side of the business cannot reclaim this service from the environment, meaning that the refund for the service is irreversible.

Efficiencies of Reversible and Irreversible Heat Engines - Net Profit Margins of Reversible and Irreversible Businesses

HK state that a reversible process is a limiting process. What they mean is that most processes are irreversible and only can be idealized as reversible processes such as by reducing friction [11, pp. 138]. Some examples of analogous idealizations in economics are having no transaction costs, perfect spread of information, or a market being perfectly competitive.

Since a reversible process is a limiting process, it follows that a reversible heat engine is a limiting heat engine. HK prove a theorem on the efficiencies of heat engines: "The efficiency of an irreversible heat engine operating between a given pair of reservoirs at stable states is less than that of any reversible engine operating between the same reservoirs " [11, pp. 139]. Then reversible heat engines can produce the largest amount of work given two reservoirs. I call businesses analogous to reversible heat engines efficient businesses for this reason. Efficient businesses produce the largest net profit possible between two markets. An irreversible heat engine is analogous to an inefficient business since the inefficient business is unable to make this highest net profit because they engage in economic activity that is irreversible.

Economic profit will no longer be zero for businesses that undergo irreversible activity. The economic profit of an efficient business will be zero since its accounting profit will be equal to the opportunity cost of entrepreneurship of any other type of efficient business operating between the same markets. Then for a business with irreversible activity, it is possible for them to have a nonzero economic cost because they will not make this maximum net profit. These inefficient businesses could have theoretically created an efficient business. The inefficient businesses then incurs a negative economic profit.

Second Law in Terms of Reversible Cycles

HK state that the concept of reversibility allows them to restate the second law in terms of reversible cycles. This statement will be needed to prove the existence of a property called entropy. The second law can be restated as "For any reversible heat engine which may exchange heat with a single system in a stable state, the net work and net heat in a cycle are zero" [11, pp. 140]. It is stated mathematically in Equation (3-5).

$$\oint_{\text{rev}} \delta W = \oint_{\text{rev}} \delta Q = 0 \tag{3-5}$$

Here δW and δQ are inexact differentials indicating they are path dependent. If Equation (3-5) did not hold for efficient businesses without a source of revenue, then they could incur expenses over a cycle while making a net profit. Since these efficient businesses without a revenue cannot expense costs and make a profit without being a Ponzi scheme, it follows that the net profit and net expense over a cycle has to be zero.

3-9 Thermodynamic Temperature Scales - Economic Price Level Scales

HK present a way to define a temperature scale independent of the characteristics of materials present by using a reversible heat engine. In this section I derive how an efficient business can determine a price level scale.

Heats and Temperatures of a Reversible Cycle - Expenses and Price Levels of a Reversible Cycle

HK prove a theorem relating the ratio of heat supplied to the temperature of two reservoirs. These quantities of heat can be seen in Fig. 3-8 as their unsigned magnitudes q_{AX} and q_{BX} [11, pp. 145].

Thermodynamics	**Economics**
Efficiency Ratio	Ratio of Expenses
"The ratio of the *heat quantities* Q_{AX} *and* Q_{BX} which any *reversible heat engine* X exchanges in a cycle when operating between any two *heat reservoirs* A and B is fixed by the *temperatures of the reservoirs*." [11, pp. 145]	"The ratio of the *revenue* Q_{AX} *and expense* Q_{BX} which any *efficient business* X exchanges in a cycle when operating between any two *competitive markets* A and B is fixed by the *price level of the competitive markets*."

Analog 3.26: Ratio of Heats and Efficiency Ratio

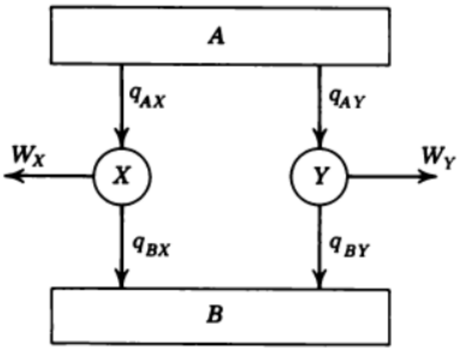

Figure 3-8: Two efficient business X and Y operating between two competitive markets A and B. Net profits W_X and W_Y are the same.

The proof of this theorem in economics lies in the impossibility of a second efficient business Y making a greater net profit than the original efficient business X. If this situation could occur, then a Ponzi scheme could be developed where the businesses could exchange payments with each other through competitive market B.

The theorem of Analog 3.26 is expressed in terms of symbols in Equation (3-6). The ratio of heats is equal to an arbitrary function f of their temperatures.

$$\frac{Q_{BX}}{Q_{AX}} = f(\theta_B, \theta_A) \tag{3-6}$$

In economics, Q_{AX} represents the revenue of an efficient business, and Q_{BX} represents the expenses. The ratio of expenses to revenue in Equation (3-6) is an example of an efficiency ratio in business economics [58]. The lower the efficiency ratio the better the business is operating.

Analog 3.26 implies that efficiency ratios for efficient business are solely determined by the markets they operate between. This efficiency ratio exists independent of the existence of any business operating between these markets. The price levels are the only properties that determine what efficiency ratio is obtainable for any efficient business that operates between these price levels. If a business cannot obtain this efficiency ratio, then it is an inefficient business engaging in some type of irreversible activity.

Thermodynamic Temperature Scales - Price Level Scales

An efficient business will be used in this subsection to define temperature scales by examining its efficiency ratio.

HK introduce a way to define a temperature scale independent of the substance of the system due to the second law as seen in Equation (3-7). θ is the temperature of system A, R is a heat reservoir, and F_R is a function of the ratio of heats exchanged.

$$F_R\left(\frac{Q_{AX}}{Q_{RX}}\right) = \theta \tag{3-7}$$

Following Section 3-8, I interpret heat reservoir R as a group of vendors that act as price setters. Then take economic system A as a group of consumers with an unknown price level. Q_{AX} is the revenue of business X, and Q_{BX} is its expenses. Then the function F_R of the efficiency ratio is a price level scale that determines the price level of the consumers A. If this scale is known, then this business X can be used to determine the price levels of many different groups of consumers on this price level scale. The efficiency ratio is determined independent of the business only by the price level of A since R is a price setter.

Kelvin Temperature Scale - Absolute Price Level Scale

HK state that an absolute scale can be defined using a linear function, shown below in Equation (3-8).

$$T = -T_R\left(\frac{Q_{AX}}{Q_{RX}}\right) \tag{3-8}$$

Here T represents the absolute temperature of a system, and T_R equals 273.16, the arbitrarily assigned temperature of a mixture of water in solid, liquid, and gaseous forms.

Based on Equation (3-8), it follows an absolute price level scale exists in economics. An absolute price scale can be defined up to a constant given the efficiency ratio of an efficient business operating between a competitive market R and arbitrarily chosen competitive market A.

As an example, take a particular choice of an economic system such as the United States service market as a point of reference. Then assign a particular value of its absolute price level, T_R. Then take any efficient business and have it operate between the United States services market and another country's service market. The absolute price level of any country's service market can be determined in this manner assuming the price level of the United States' remains relatively constant.

An economic system could have a zero price level, meaning that it could provide no useful services—though it could still receive services. HK state that zero Kelvin cannot be maintained [11, pp. 151], so it is reasonable that an absolute zero price level cannot be maintained in economics.

Ratios of Temperature and Heat for a Reversible Engine - Ratios of Price Level and Expenses for Efficient Businesses

Fig. 3-9 will serve as a reference for this section [11, pp. 149]. The ratio between price levels of A and B have a relationship to the efficiency ratio Q_B/Q_A.

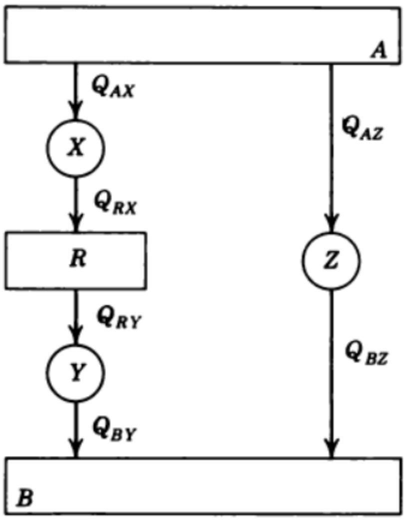

Figure 3-9: The revenues and expenses that move through the system are indicated by arrows. The heat engines in this figure can be imagined as efficient businesses. Economic system A is a competitive market and is a customer base. Economic system B is also a competitive market and is a vendor base. Price setter R is an intermediary system that acts as both a vendor and consumer. Business Z does wholesale between the customers A and vendor B, skipping the need for an intermediary R. The most profit business Z can make must be equal to the sum of profits of businesses X and Y in the absence of transaction costs.

The systems X, R, and Y together act as a single efficient business between A and B like efficient business Z. It follows then that the efficiency ratios must be the same

$$\frac{Q_{AX}}{Q_{BY}} = \frac{Q_{AZ}}{Q_{BZ}}.$$

Since system R is a price setter, it must be that $Q_{RX} = -Q_{RY}$, and from the existence of an absolute price scale

$$\frac{T_A}{T_B} = -\frac{Q_{AX}}{Q_{BY}}.$$

Then it must be that the ratios of the price levels related

$$\frac{T_A}{T_B} = -\frac{Q_A}{Q_B}. \tag{3-9}$$

This result means that the efficiency ratio of any efficient business is determined by the ratio of absolute price levels of the markets they operate between.

Efficiency of a Reversible Heat Engine - Net Profit Margin of an Efficient Business

From the results of the previous subsection, the net profit margin of an efficient business from Section 3-7-2 can be restated in terms of price levels. It follows that

$$\eta = \frac{Q_{AX} + Q_{BX}}{Q_{AX}} = 1 + \frac{Q_{BX}}{Q_{AX}} = 1 - \frac{T_B}{T_A} \tag{3-10}$$

which means that the highest obtainable net profit margin for any business is determined by the absolute price levels of the markets it operates between, T_A and T_B.

3-10 Entropy and its Formulation in Economics

Existence of Entropy - Existence of Human Capital

HK introduce the existence of a property involved in the exchange of heat and call it entropy. I establish an analogous property for economic systems in Analog 3.27 that I call human capital. Here δQ represents an expense at a price level T.

Thermodynamics	**Economics**
Entropy	Human Capital
"The quantity δQ/T in any *reversible process of a system* represents the change in the value of a property of the *system.*" [11, pp. 153]	The quantity δQ/T in any *reversible economic activity* represents the change in the value of a property of the *economic system.*

Analog 3.27: Entropy and Human Capital

I now follow HK's proof that entropy is a property of a system. By following their proof analogously in economics, I show that there must exist a property of an economic system that I call human capital. I use Fig. 3-10 as reference.

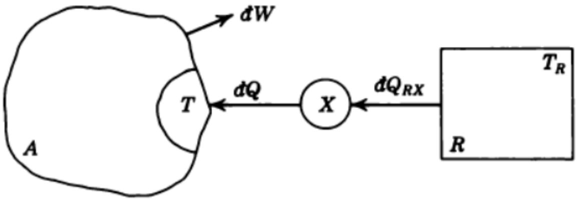

Figure 3-10: Two economic systems A and R are brought together by an efficient business X. R is a reference competitive market. A is in a Pareto state.

A is an economic system executing a reversible cycle while receiving expense δQ at price level T. δQ is supplied to A by an efficient business X. Efficient business X receives revenue δQ_{RX} during each cycle from competitive market R at price level T_R. Now consider the combined economic system of X and A. According to Equation (3-5), the net profit must be zero since system AX is exchanging an expense with a single reservoir and that

$$\oint_{\text{rev}} \delta Q_{RX} = 0.$$

It is also true that from the definition of an absolute price scale that

$$\frac{\delta Q_{AX}}{T} = -\frac{\delta Q_{RX}}{T_R}.$$

Knowing that $\delta Q = -\delta Q_{AX}$, it results that

$$T_R \oint_{\text{rev}} \frac{\delta Q}{T} = 0$$

which equates to

$$\oint_{\text{rev}} \frac{\delta Q}{T} = 0.$$

Since the change in this quantity of the ratio of expenses to price level $\delta Q/T$ is always zero over this cycle, there must exist an economic property S of the economic system such that

$$dS = (\frac{\delta Q}{T})_{\text{rev}}$$

that I call human capital. I have shown that this property of human capital must exist in economic systems and captures the effects unmeasured by the factors of production. The exchange of human capital alongside an expense measures the quantity of services exchanged dS.

I use the term human capital as a collective term to refer to the effects of services unmeasured by other factors. Analogously in thermodynamics, entropy is interpreted as the "one aggregate variable for the unexamined degrees of freedom" where "energy shared among the unmeasured degrees of freedom of these subparts is called heat" [59]. If in a specific instance the effects of an expense could be explained and considered an expenditure, the human capital involved in the expense would actually be a factor or asset by implication.

Business economics, macroeconomics, and classical economics all have different ways to refer to the effects unaccounted for by the factors of production. In Table 3-1, I list what each respective field (Adam Smith represents classical economics) calls what I have decided to call human capital for all economic systems.

Field of Economics	Name for Human Capital
Business Economics	Goodwill
Macroeconomics	Human Capital
Adam Smith	"acquired and useful abilities"
Solow Growth	Technology Level/Labor Effectiveness

Table 3-1: The right column lists the term the left column would use to refer to the effects unmeasured by factors of production or assets.

In business and accounting economics, goodwill is a type of human capital relevant for businesses. Goodwill is considered to be an intangible asset associated with the purchase of a business [60]. When a business purchases another business for a value higher than the fair value, goodwill is recorded. This goodwill can represent a company's brand name, good relations with customers and employees, and any patents or proprietary technology owned by the company. Thus goodwill serves a singular measure for businesses to account for all the unmeasured effects left over and not found in its assets and liabilities.

Human capital, though used in broad terms in economics, serves a singular measure in Analog 3.27 for the quantity of services provided with an expense. Human capital is considered in modern times to be "the stock of habits, knowledge, social and personality attributes (including creativity) embodied in the ability to perform labour so as to produce economic value" [61]. This statement about human capital can be interpreted as human capital is a stock property that measures the effect of many unexamined degrees of freedom that can add to the earnings of an economic system. There are many types of intangible capital that are all considered to be types of human capital. Some examples are intellectual capital, social capital, or cultural capital [61]. Sometimes human capital is split into general and specific human capital [61]. The economic term human capital serves as a catchall economic property in this thesis to account for what is not measured by other independent properties in an economic system.

Adam Smith saw an increase in productivity as stemming from the acquisition and creation of "acquired and useful abilities", which he defined as

> "Fourthly, of the acquired and useful abilities of all the inhabitants or members of the society. The acquisition of such talents, by the maintenance of the acquirer during his education, study, or apprenticeship, always costs a real expense, which is a capital fixed and realized, as it were, in his person. Those talents, as they make a part of his fortune, so do they likewise that of the society to which he belongs. The improved dexterity of a workman may be considered in the same light as a machine or instrument of trade which facilitates and abridges labor, and which, though it costs a certain expense, repays that expense with a profit." [33]

The human capital is all "the acquired and useful abilities of the inhabitants or members of society" that Smith considers the "fourth" type of capital. The service provided that exchanges human capital is this "real expense", and the resulting increase of human capital that leads to higher earnings "repays that expense with a profit". Smith sees the greatest increase in earnings from the division of labor, stating that "the greatest improvement in the productive powers of labour, and the greater part of the skill, dexterity, and judgement with which it is any where directed, or applied, seem to have been the effects of the division of labour." [33] Smith is analogously stating that the largest increase in entropy—human capital—comes from the division of labor. He saw how the creation of more human capital through specialization led to a large increase in the earnings of economic systems not explained by the labor itself.

Modern growth theories focus on a level of technology in economic growth. A technology level, or technological innovation, is another form of human capital that accounts for effects not explained by capital accumulation or labor growth. Capital can be made more efficient through an increase in the level of technology, which Solow called technological innovation. For example, a computer today is more efficient today than a similar computer a few years ago. Software upgrades can increase the efficiency of existing computers. Nowadays, personal data has become a new type of human capital that is being used to create economic value [62].

Inequality of Clausius - Inefficiency Inequality

HK discuss an increase in entropy over each cycle of an irreversible process. This result is called the inequality of Clausius, which I extend to economic systems in Analog 3.28.

Thermodynamics	**Economics**
Clausius Inequality	Inefficiency Inequality
"For an internally irreversible cycle the integral, $\oint \delta \mathrm{Q}/\mathrm{T}$, of the ratio of the heat $\delta \mathrm{Q}$ received by a system to the temperature T at which the heat is received is always less than zero. " [11, pp. 154]	For an internally irreversible cycle the integral, $\oint \delta \mathrm{Q}/\mathrm{T}$, of the ratio of the *revenue* $\delta \mathrm{Q}$ received by a *business* to the *price level* T at which the *revenue* is received is always less than zero.

Analog 3.28: Clausius Inequality and Creation of Human Capital

Here I discuss an inefficient business that receives some revenue δQ at price level T. This business goes through irreversible activity, such as having demand curves shift during economic activity, having price levels of markets fluctuate, or having unplanned expenses such as transaction costs. The result is a permanent increase each cycle of human capital dS. This permanent increase of human capital makes it possible to earn a non-zero economic profit since some human capital is created each cycle the business goes through.

Given that real thermodynamic heat engines are never truly reversible, it is reasonable to assume that not all business are efficient business. A business is guaranteed to incur additional expenses over what is required such as through transaction costs and taxes that prevent it from being an efficient business. Any irreversible economic activity will prevent the maximum net profit from being reached. Then every cycle that a business goes through results in a permanent increase in human capital. For a business, this can specifically be the creation of goodwill.

For example, this permanent increase in human capital can occur because of research and development. A company can invest in research and development that leads to the creation of new patents and proprietary technology. At the business level, this leads to an increase of earnings of the company. At a societal level, the increase of human capital generated by many businesses can lead to an increase of a technology level.

The factors of the business gain permanent increases in productivity too. Many employees gain knowledge while working at their job, either task specific or general, which makes them more knowledgeable and skilled workers in the future. If the employees are selected from the same market, then the business can be thought of as slowly heating up a finite skilled workers market by supplying human capital to employees.

Entropy and Heat in Irreversible Processes - Human Capital and Expenses in Irreversible Activity

For irreversible processes that do not involve cycles, there still is a greater increase in entropy than the quantity dQ/T.

Thermodynamics	**Economics**
Increase in Entropy	Increase in Human Capital
$\int_{\text{irr}} dS > \int_{\text{irr}} \frac{\delta Q}{T}$	$\int_{\text{irr}} dS > \int_{\text{irr}} \frac{\delta Q}{T}$

Analog 3.29: Permanent Changes in Human Capital and Entropy

For economic activity not involving cycles, it is still the case that permanent increases in human capital can occur for irreversible economic activity. Human capital is not a conserved quantity the same way that entropy is not a conserved quantity.

One form of this irreversibility is innovation. Innovation results in the creation of new ideas that once imagined, realized, or implemented are not easily taken back. For example, innovation leads to the generation of new human capital or technology through research and development [2].

If interactions in thermodynamics can lead to the creation of entropy, then it follows that in economics transactions between two systems can result in net increases of human capital. Consider that during interactions between two economic systems, knowledge or ideas may be passed between them—such as person to person, and this interaction increases the human capital of one system while not necessarily affecting the other system. It would be strange if the act of educating reduced the human capital level of an instructor.

Arbitrariness of Zero Level of Entropy - Arbitrariness of Zero Level of Human Capital

HK mention that entropy is defined by its change between states, meaning that a zero level of entropy is arbitrary [11, pp. 155]. So, in economics a ground level of human capital is arbitrary and can be decided on in the analysis of an economic system. For example, if the earnings of an economic system is determined and not fully explained by the factors of production, a ground level of human capital can be set to explain the level of earnings.

Entropy as a Coordinate - Human Capital as a Coordinate

Entropy can be used on phase diagrams as a coordinate just like any other property. Here I include a T-S diagram from HK in Fig. 3-11.

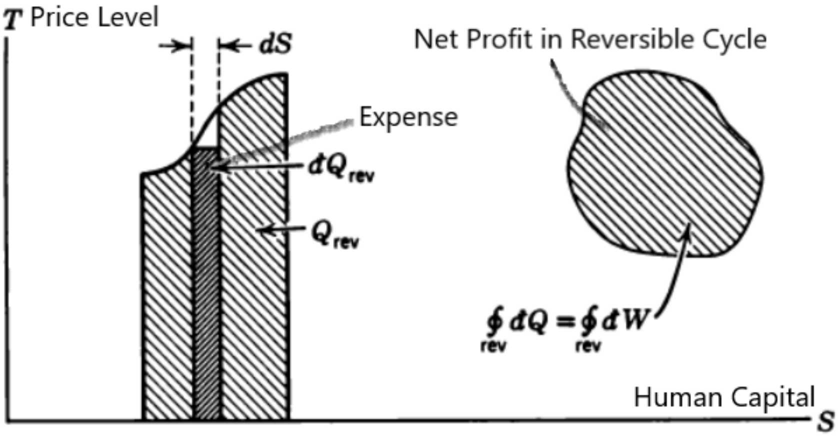

Figure 3-11: Entropy and temperature can be graphed together. Similarly in economics, a human capital supply graph can be created to show the change of human capital for activity that may vary in its price levels. The area under the activity (the T-S curve) represents a service charge, and the area enclosed by a path integral represents the net profit in reversible activity.

I interpret from Fig. 3-11 that T-S diagrams analogously form human capital supply schedules the same way that labor and wage form supply graphs for economic activity. The area under a path that represents economic activity can be integrated to determine the expense involved in the activity. The area enclosed by a path on these graphs represents the net profit associated with an efficient business.

I also include a T-S diagram of a Carnot cycle in Fig. 3-12.

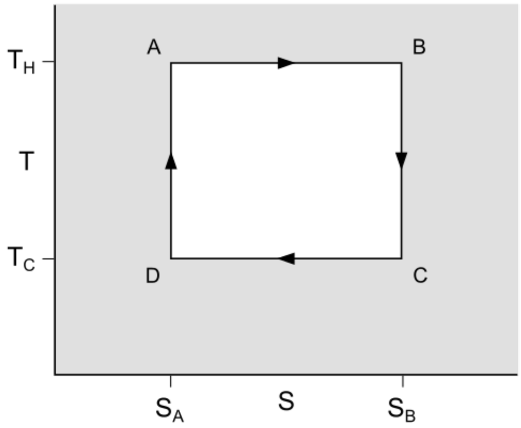

Figure 3-12: The T-S diagram of a Carnot cycle. This graph can be interpreted as the price level human capital graph of an efficient business. Notice there is no net increase in human capital over a cycle. The area enclosed by this cycle is the net profit.

I interpret Fig. 3-12 as the human capital supply schedule of an efficient business. Iso-price level activity can be seen from points A to B and from C to D. Then the demand for human capital is perfectly elastic when its price level is fixed. For activity involving only standard wage labor, the demand for human capital is perfectly inelastic since no human capital is supplied while a change in price levels occurs. This standard wage labor activity can be seen between points D and A and points B and C. No expenses will be incurred between these points. This activity must be inelastic since if standard wage labor supplies any type of human capital or expenses are incurred the activity is irreversible.

The supply curve for human capital such as between points A and B does not have to be a horizontal line. For example, in a reversible Rankine cycle when the water is in its steam phase its temperature increases along with its entropy. When the water is in its liquid form and boiling, the temperature remains constant as entropy increases or decreases. Then it follows in economics it is possible for an efficient business to have a human capital supply graph with a nonzero slope.

Human capital demand and supply graphs exist in economics but seem to be focused specifically on skilled labor [63, 64]. From my use of the term human capital it follows that human supply graphs can be used to study any economic system where there is a demand for human capital. Since the way I use human capital refers to more than just skilled services, human capital supply graphs can be created for any economic system such as for business a demand for goodwill or for an economy a demand for a level of technology.

Principle of the Increase of Entropy - Principle of the Increase of Human Capital

HK introduce the principle of the increase of entropy for isolated systems, which reveals the inevitability of the increase of human capital for an isolated economic system in Analog 3.30.

Thermodynamics	**Economics**
Increase of Entropy for Isolated System	Increase of Human Capital for Isolated System
$dS_{\text{isol}} \geq 0$	$dS_{\text{isol}} \geq 0$

Analog 3.30: Principle of Increase of Entropy and The Increase of Human Capital

It follows in economics that the increase of human capital for an isolated economic system results in an isolated system eventually achieving a maximum amount of human capital. At equilibrium, an isolated economic system will have reached a maximum amount of human capital. In thermodynamics this principle is analogously known as the principle of maximum entropy [56].

HK note that this principle can be applied generally because any process can be included within an isolated system. If an economic system is isolated its earnings cannot increase or decrease, which means that if it undergoes irreversible activity the result must be the increase of human capital. If we consider the entire world one isolated economic system, it can be inferred that human capital is increasing overall.

Entropy Changes for a Simple System - Human Capital Changes for a Simple Economic System

HK use the first law to write the changes in internal energy for a simple system that moves between equilibrium states. This well-known equation can be written as

$$dU = TdS - pdV \tag{3-11}$$

which holds for any process connecting equilibrium states. Only in a reversible process does TdS represent δQ_{rev} and pdV represent δW_{rev}. U is the notation used for internal energy but E could have been used if preferred.

This equation can be written analogously in economics. In Equation (3-12) I consider the change of human capital dS for a change of earnings dE and use of labor dL. The fixed wage is represented by w and T is a fixed price level. The result is

$$dE = TdS - pdL \quad . \tag{3-12}$$

Equation (3-12) can be used to analyze changes between economic equilibrium states of simple economic systems. Simple economic system can be considered systems that only make use of labor and human capital. As Equation (3-11) is referred to as the fundamental thermodynamic relationship [11, pp. 159], Equation (3-12) can be referred to as the fundamental economic relationship.

Energy, Entropy, and Temperature Visualized - Earnings, Human Capital, and Price Level Visualized

I use the text of Gyftopoulus and Baretta (GB) [56] in this subsection so I can visualize the relationship between earnings and human capital. There is insight that is gained by analyzing the energy-entropy graphs GB introduce.

GB introduce the fundamental thermodynamic relationship, which is the relationship between entropy, energy, and the other extensive properties of the system e.g. volume as seen in Analog 3.31.

Thermodynamics	**Economics**
Fundamental Thermodynamic Relationship	Fundamental Economic Relationship
$E = E(S, V, N, ...)$ [56]	$E = E(S, L, K, F, ...)$

Analog 3.31: Fundamental Relationship

The way GB introduce the fundamental thermodynamic relationship is reminiscent of how the production function is written in Solow growth. Consider a production function that relates output to its inputs

$$Y = Y(A, L, K) \quad .$$

Here L is labor, K is capital, and A is some type of factor effectiveness. In Solow growth models if human capital is used as an input—typically denoted as H—to the production function it is treated like a factor such as labor [2].

If Y is a type of earnings, then I propose that human capital S as developed in this section serves the role of total factor productivity A[2]. Total factor productivity is a measure of output not explained by labor or capital inputs [65]. In order for human capital to serve as factor productivity in a production function, it cannot have diminishing returns normally assumed for labor and capital [2]. Then this lack of diminishing returns should analogously be seen in thermodynamics between the relationship of energy and entropy.

The relationship between entropy, energy, and temperature can be observed in Fig. 3-13. This graph is not a phase diagram but is instead a projection onto a two dimensional space from a hypersurface that intersects a multidimensional space [56, pp. 186].

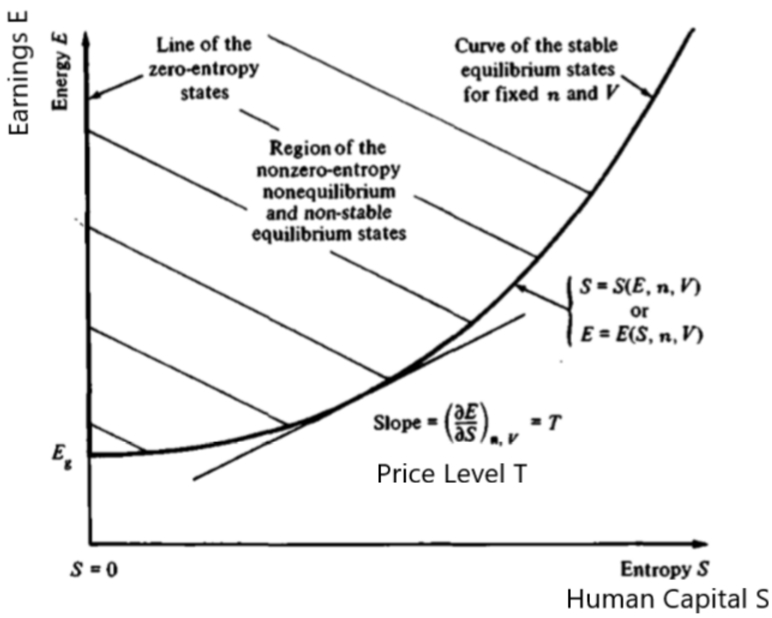

Figure 3-13: Energy-Entropy graph where earnings is energy and human capital is entropy. The price level is temperature, the slope tangent to an Earnings-Human Capital curve. Note that this relationship is convex. [56, pp. 187]

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[2] When the technology levels enters the production function as a measure of total factor productivity and not a measure of labor or capital effectiveness it is called Hick-neutral [2, pp. 10]

The temperature of a system can be seen in Fig. 3-13 as the slope of the line tangent to a stable equilibrium state curve. This curve represents fixed values of independent properties such as volume and number of particles. As a slope, temperature is defined as $\partial E/\partial S$. Note the relationship between entropy and energy is convex.

Fig. 3-13 can be interpreted analogously in economics that as human capital S grows, so too do the earnings E along a curve representing a fixed allocation of factor inputs—same amount of labor, capital, or financial capital input. For a fixed allocation, if the earnings go up then the economic system must have gained human capital. It follows the relationship between earnings and human capital is convex. The condition for a production function that the input A does not have diminishing returns is satisfied if human capital is used as this input A.

Fig. 3-13 can also be interpreted as showing the earnings that can achieved by an investment into human capital for any economic system. An expression in common use regarding human capital is "invest in yourself", which can be understood that as an individual you can increase your earnings by increasing your human capital through acquisition of education, skills, or talents. Just as an economic system can invest in labor or capital, it can also invest in human capital.

The allocation curve itself in Fig. 3-13 can also shift to a new allocation, which may or may not increase human capital. Thus, there are two ways an economic system may become more productive, either by increasing human capital for a given allocation, or by reallocating appropriately—the reallocation that results in an state that makes "everyone better off".

For economic systems such as economies, the increase in human capital measures the intensive growth rate of the economy. Intensive growth measures the capability of an economic systems to translate increases in productivity into increases in output [24]. Intensive growth specifically measures growth not accounted for by extensive growth of labor and capital. Output, such as GDP, is the earnings for an economy, and the intensive growth rate is how fast productivity is changing. When analyzing economic systems, the intensive growth of the system can be determined by its rate of human capital accumulation $\dot{S}$ the same way extensive growth can be measured by labor and capital accumulation.

Being able to create earnings-human capital graphs could serve as a way to analyze and manage economic systems from the perspective of a planner like the government or management for a business. The energy-entropy graph is itself a projection of the Maxwell surface, meaning that analysis of the thermodynamic system can be done in higher dimensions. It is likely Maxwell surfaces can be constructed for simple economic systems to aid in the analysis of the system.

3-11 Material Properties and Economic Asset Properties

I derive economic analogs to various material properties in thermodynamics in this section. By developing economic analogs to material, I develop a way a way to classify economic systems like a thermodynamic system could be classified by the properties of the materials it is composed of.

3-11-1 Compressibility - Factor Elasticity

Compressibility in thermodynamics is expressed qualitatively as the change of volume expected for a given change of pressure at a constant temperature [66]. An increase of pressure means that volume will decrease.

Thermodynamics	**Economics**
Compressibility	Wage Elasticity
$\beta = -\frac{1}{V}\frac{\partial V}{\partial p}$	$\beta = \frac{1}{L}\frac{\partial L}{\partial w}$

Analog 3.32: Compressibility and Wage Elasticity

Substituting a wage for pressure and labor for volume, compressibility is seen as analogous to the (labor specific) wage elasticity of labor in Analog (3.32) [67]. Labor specific means at a given supply of labor (the division by L). The wage elasticity as defined in Analog (3.32) is technically a linear elasticity, meaning that in order to obtain an elasticity [%] some change in wage needs to be known. Multiplying a change of wage Δw by the wage elasticity of supply results in an expected change of labor (in %).

Capital and rent could have been used instead of labor and wage, and the result would have been the rent elasticity of capital. Any extensive-intensive property pair of factors can be used to generate the corresponding economic elasticity. Thus elasticities in economics are analogous to different types of compressibility.

3-11-2 Heat Capacity - Inflation Capacity

Heat capacity gives the relationship between a transfer of heat to an object and increase of temperature at a given volume and mass [56, pp. 254] .

Thermodynamics	**Economics**
Heat Capacity	Inflation Capacity
$C_V, \; C_p = T\frac{\partial S}{\partial T}$	$C_L, \; C_w = T\frac{\partial S}{\partial T}$

Analog 3.33: Heat Capacity and Inflation Capacity

Price level is analogous with temperature and human capital is analogous with entropy in Analog 3.33. The result is a measure of the change of a price level of an economic system in response to some expense.

This concept of receptiveness to inflation is not explicit defined in economics, so I will refer to it as inflation capacity. This inflation capacity of an economic system tells how receptive it is to changes in price levels for a given expense. Inflation itself is defined as a change of price levels [49]. The equation relating an expense δQ to a change of price level ΔT is

$$\delta Q = C\,\Delta T \quad . \tag{3-13}$$

C is the inflation capacity for a constant amount of labor or a fixed wage. Analyzing Equation (3-13), it can be observed that the smaller the inflation capacity of an economy, the greater the inflation an economy experiences for some expense. The larger the inflation capacity of an economy, the less the inflation it experiences for the same expense.

Inflation can be accounted for in economic growth. For example, box office sales are often readjusted by inflation since ticket prices have steadily rose over time. The growth of an economic system can be evaluated both by increases in earnings and changes in price levels as in Equation (3-14).

$$\frac{GDP_{t_2}}{P_2} - \frac{GDP_{t_1}}{P_1} \Longleftrightarrow \frac{E_{t_2}}{T_2} - \frac{E_{t_1}}{T_1} \quad . \tag{3-14}$$

Here the GDP is analogous to earnings, and the price level P is analogous to temperature T. t_2 is a time later than t_1. The change in earnings is normalized by the price level measured at two different times to evaluate earnings growth in the context of inflation.

3-11-3 Coefficient of Thermal Expansion - Effectiveness

The coefficient of thermal expansion relates the expansion of a material due to an increase of temperature at a constant pressure [68].

Thermodynamics	**Economics**
Coefficient of Thermal Expansion	Coefficient of Labor Effectiveness
$\alpha = \frac{1}{V}\frac{\partial V}{\partial T}$	$\alpha_L = \frac{1}{L}\frac{\partial L}{\partial T}$

Analog 3.34: Coefficient of Thermal Expansion and Labor Effectiveness

Volume is analogous with labor, and a price level is analogous with temperature in Analog 3.34. An increase of a price level results in an increase of labor at a specific level of labor.

The coefficient of labor effectiveness in Analog 3.34 gives the relative increase of effective labor given a change in price level. Capital could have also been used in Analog 3.34, which would give the relative increase in effective capital due to an increase in a price level. Thus, the relative increase in a factor of production as if it grew extensively is the effectiveness of that factor multiplied by the increase (or decrease) of price level. For example, if a laborer receives some task-specific training, they might be considered to be as effective as two standard laborers.

This method of finding labor (or any factor) effectiveness is in contrast to how effective labor was defined in Section 3-3-4. The method in Section 3-3-4 is more so extensive effectiveness—how valuable some labor is relative to standard labor—than how effectiveness is defined intensively here—related to changes of productivity and price level of laborers.

3-12 Conclusions

This chapter establishes the basis for the thermodynamics of economic engineering by identifying key analogs between economics and thermodynamics. The overall analogy that this chapter develops fits within the economic engineering framework. Concepts from accounting economics, business economic, macroeconomic, classical economics, and growth economics are used in conjunction to form a framework for the thermodynamic analysis of economic systems.

An important result in this chapter is the clear definition of economic analogs to energy, work, and heat that is generally applicable anywhere in economics where there are cash flows. A consistent use of accounting terminology is useful for describing the expenditure and expenses that serve as the analogy to work and heat respectively. Earnings is established as a property of system independent of an income that it may receive.

The economic analog to entropy is not specifically consumption or utility. The interpretation of entropy developed in this chapter as human capital presents an appropriate economic realization of the concept of entropy only by working through an axiomatic approach to the thermodynamic-economic analogy.

All of classical thermodynamics is available as a tool for those whom wish to do future research in developing thermodynamic economic growth theories. The applicability of this theory extends to all types of economic systems. Changes of earnings along with changes of price levels can be used in evaluating economic growth of many types of economic systems.

The axiomatic approach developed in this chapter is only the start to the thermodynamics of economic engineering. I did not analyze chapters in HK such as bulk flow, boundary forces, enthalpy, or open thermodynamic systems. Businesses can be open economic systems, meaning they buy goods and sell products or they grow in size by acquiring more factors of production. There are also chapters in HK after the chapter on the existence of entropy that can lead to the development of more economics theory. This chapter succeeded in laying the foundation for the thermodynamics of economic engineering such that these topics can be analyzed in the future.

Future research should be done in this field to develop an analogy between material sciences and economics. Different phases of matter have different properties. It would be interesting if certain types of economies (Malthusian, Developing, Modern) displayed similar magnitudes of these material properties. A theory of phase transitions could then explain why certain economies meet the criteria to experience growth miracles or what criteria needs to be met to be considered a modern slow-growing economy.

A relationship between consumption and expenses has not been developed yet. Future research should be done to see how consumption can lead to an expense. The relationship between consumption and expenses can be determined by understanding how the entropy generation of a dynamic mechanical system leads to heat accumulation. Thus the economic interpretation of the relationship between consumption and expense will follow from the development of thermodynamic theory in controls. Future research should be done to determine the relationship between human capital and microeconomic utility. Future research should also be done to see how thermodynamics can be included in control analysis. Chapter 6 proposes a way to include thermodynamic energy in an energy-based approach to controls.

Contributions

1. A thermodynamic textbook was transliterated into economics, establishing the thermodynamics of economic engineering
2. Economic analogs to work and heat are derived
3. Analogs to the first, second, and zeroth laws of thermodynamics established for economics
4. Economic analog to entropy defined as human capital
5. Idealized businesses developed as economic analog to heat engines
6. Temperature derived as analogous to price level, changes in temperature shown to be analogous to inflation
7. Material properties of economic systems derived

Chapter 4

Applications to Economic Growth

4-1 Introduction

The thermodynamic-economic theory developed in this thesis has applications to existing problems in macroeconomics and business economics. In this chapter I develop applications to the theory I have developed in Chapter 3. Section 4-2 applies the theory to the empirical growth measurement of the growth of economies. Section 4-3 develops a method to measure total potential earnings of an economy. Then the section analyzes how productivity is related to the allocation of factors over a homogeneous economy. Section 4-4 applies the theory of Chapter 3 to the valuation of a business for sale. Section 4-5 concludes the chapter.

4-2 Growth Accounting

In this section I show how the fundamental thermodynamic relationship can be used to empirically measure the growth of an economy.

Empirically measuring the growth of an economy is called growth accounting. Solow's contribution to economics was to account for growth not accounted by labor and capital accumulation [2, 42]. He considered a production function Y that was a function of labor L, capital K, and factor productivity A. Then he considers the continuous time derivative of output due to each input to the production function for the empirical calculation

$$\dot{Y} = \frac{\partial Y}{\partial K}\dot{K} + \frac{\partial Y}{\partial L}\dot{L} + \frac{\partial Y}{\partial A}\dot{A} \quad . \tag{4-1}$$

The growth rates of Y, K, and L are "straightforward" to measure [2, pp. 30].

Solow denoted the residual R as the part of the growth not explain by capital accumulation or labor growth as

$$R = \frac{A}{Y}\frac{\partial Y}{\partial A}\frac{\dot{A}}{A} \quad . \tag{4-2}$$

The output Y is commonly measured by using the gross domestic product (GDP) as the measure of an economy's output [69]. Other measures of output such as gross nation product or gross output still measure a quantity in the same units of GDP in \$/yr but may differ in magnitude. Since economists measure an output by a cash flow, I will use earnings to measure an output. Using the theory developed in Chapter 3, I express growth accounting with a change of earnings

$$dE = \frac{\partial E}{\partial K}dK + \frac{\partial E}{\partial L}dL + \frac{\partial E}{\partial S}dS \quad . \tag{4-3}$$

In Equation (4-3) the economy's earnings E is used to measure output Y. The rent is $\frac{\partial E}{\partial K}$, and the wage is $\frac{\partial E}{\partial L}$. The price level is $\frac{\partial E}{\partial S}$. Human capital S is used in place of factor productivity A.

Then I consider the rate of change of the extensive properties

$$\dot{E} = \frac{\partial E}{\partial K}\dot{K} + \frac{\partial E}{\partial L}\dot{L} + \frac{\partial E}{\partial S}\dot{S} \tag{4-4}$$

where again $\dot{E}$, $\dot{L}$, and $\dot{K}$ are presumably measurable.

The residual term can be expressed in terms of symbols used through Chapter 3 as

$$R = \frac{S}{E}\frac{\partial E}{\partial S}\frac{\dot{S}}{S} \quad . \tag{4-5}$$

Notice that the application of earnings to growth accounting shows how the Solow residual is a measure of the accumulation of human capital and the price level. The Solow residual is sometimes interpreted as the measure of contribution of technological progress [2]; however, I interpret it as a measure of how the accumulation of all human capital contributes to the growth of earnings of an economy. Technological progress is one way that human capital can accumulate. Then human capital as developed in Chapter 3 measures all the effects not accounted for by labor or capital in the growth accounting of an economy.

The Solow residual combines the effects of human capital accumulation and price level, obscuring the economic interpretation of $\frac{\partial Y}{\partial A}$. Following the theory developed in Section 3-6, the term $\frac{\partial Y}{\partial A}$ is the price level of an economy, but in the context of growth accounting it would be considered the marginal product of human capital. To the best of my knowledge, this term $\frac{\partial Y}{\partial A}$ has not been defined or interpreted as a marginal product in economic literature. However, from my analysis it follows that this marginal product of human capital is its price level in the same manner that a wage is the marginal product of labor [2].

Growth accounting measures the contribution of the accumulation of labor, capital, and human capital to growth of an economic system. The residual term is not an unaccounted measure of growth but rather the measure of how human capital accumulates at a given price level. Earnings growth accounting can be used to measure the growth rate of human capital in developing countries to determine when a country might be in the position for a growth miracle or growth disaster. The style of growth accounting in this section could also be applied towards a business looking to expand its operations.

4-3 Macroeconomic Growth and Production Functions

In this section I apply the theory of Chapter 3 to macroeconomic growth by analyzing production functions. The total potential earnings of an economy is determined in Section 4-3-1. This total potential earnings is shown to be a linear production function. Then the productivity of factors is derived in Section 4-3-2 by examining the work needed to identify opportunities in an economy. The relationship between a Cobb-Douglas and linear production function is derived in Section 4-3-3. The Cobb-Douglas production function is shown to take different inputs and measure a different output compared to a linear production function.

4-3-1 Total Potential Earnings and Linear Production Functions

For the growth accounting in Section 4-2, only changes in output were considered to empirically measure economic growth. However, in economics a production function commonly expresses a total amount of output from inputs. A linear production is of the form

$$Y = Y(X_1, X_2, \dots, X_n) = a_0 + a_1 X_1 + a_2 X_2 + \dots + a_n X_n \tag{4-6}$$

where a_i is a coefficient relating each input X_i to the total output Y [70]. The linear production function can be expressed in terms of labor and capital

$$Y = Y(L, K) = aL + bK \tag{4-7}$$

where a and b are coefficients to be determined.

I now summarize HK's determination of total potential energy analogously in economics to determine total potential earnings of an economic system. A detailed explanation of how total potential energy is determined in thermodynamics can be followed in HK [11, pp. 474-477]. In order to measure a total amount of earnings of an economy, some assumptions about the economy are needed. Appropriate activity will also be needed to determine the total earnings.

One assumption needed about the economy is that it is completely homogeneous. The economy must be homogeneous such that intensive properties are identical at any point of the economy. That is to say, if the wage and rent are measured at one place in the economy, it should be the same as the rent and wage measured at another part of the economy.

If the economy is homogeneous, then certain activity can be devised to determine the total amount of earnings. An assumption needed about this activity is that it is reversible. Consider an infinitesimal amount of some arbitrary economy dn that has no earnings. Then reversibly introduce dn into a finite part of the homogeneous economy such that is exposed to the economy's wage, rent, and price level fields. An expenditure and expense will occur such that the material dn now has infinitesimal earnings dE. Now repeat this activity in different parts of the economy and integrate through to determine the total potential earnings E.

The total earnings E is then expressed as

$$E = E(L, K, S) = wL + rK + TS \quad . \tag{4-8}$$

Equation (4-8) is a linear production function that relates inputs labor L, capital K, and human capital S to the output measured by E. The coefficients that relate the inputs to the output are their respective intensive properties wage w, rent r, and price level T. Equation (4-8) is also identical to a national accounting identity commonly used in economics such as used by the economist Ramsey [71]. The only difference between Equation (4-8) and Ramsey's accounting identity (and accounting identities in general) is that human capital is included in Equation (4-8).

I conclude from this application that a linear production function expresses total potential earnings for an economy—assuming the economy is homogeneous enough for a total potential earnings to be determined. The total potential earnings is a measure of output in terms of a cash flow. Instead of seeing a cash flow like GDP as a measure of output, the output itself is the measured cash flow.

4-3-2 Expenditure of Identification and Factor Productivity

Now I consider the expenditure needed to find an opportunity in an economy. In thermodynamics, the work needed to identify a state is called the work of identification. For a detailed explanation of the work of identification see HK [11, pp. 600-604]. I will summarize HK's explanation of the work of identification analogously in economics to show how opportunities can be located in an economy. These opportunities at the microeconomic scale will be found by taking a macroeconomic system and subdividing it down into smaller prats.

To identify where opportunities G_i exist in an economy, some expenditure must be taken to find where these opportunities lie. Consider an outsider at price level T looking for opportunities in an economy not necessary at a Pareto state. First assume that there is a minimum expenditure $\mathbf{w}$ needed to find an opportunity G_i. The response $\mathbf{w}_A$ indicates G_i is found, and the response $\mathbf{w}_B$ indicates G_i is not found.

To find the opportunities, some procedure must be taken. The procedure I consider is subdividing the economy into pieces and checking to see whether opportunity G_i lies in either piece. When the piece of the economy that has G_i is identified, that piece is again subdivided into further pieces until G_i is found. If the expenditure for both responses A and B is the same, then the expenditure of identification is on average

$$f_i = \mathbf{w} \log_2 \left(\frac{1}{x_i} \right) \tag{4-9}$$

for an economy that is continuously divided into two pieces each step of the procedure. $\frac{1}{x_i}$ is the number of members of this economy that are divided in two in each step.

Equation (4-10) can be written as

$$f_i = -\frac{1}{2\ln(2)}\mathbf{w}\ln(x_i) \quad . \tag{4-10}$$

Here the subdivision at each step results in a piece being cut into e number of pieces.

Now I consider a homogeneous economy at a Pareto state to determine how factors of production are allocated in this economy.

Assume that as an outsider to this homogeneous economy I could interact with the economy to make a profit. Then I could follow the procedure of splitting my search of the economy and searching each part to find an opportunity. At at each step there is an expenditure of identification required from me to identify if there is or is not an opportunity. However, since this economy is homogeneous, if I found an opportunity to make a profit I would have violated the assumption that the economy was homogeneous. Since the economy is homogeneous, I would know if there is or is not an opportunity just by interacting with the entire economy. If I could then interact with the economy to make a profit, I violate the second assumption that the economy was at a Pareto state. It follows that it is not possible for an outsider to interact with a homogeneous economy at a Pareto state and make a profit.

In order for opportunities to be evenly distributed at the microeconomic level, the factors of production and human capital must be evenly allocated at the microeconomic level. If the factors of production and human capital were not evenly distributed, it follows someone could make a profit by using an expenditure of identification to find discrepancies in allocations. Under one more assumption that this homogeneous economy at a Pareto state is isolated, the distribution of human capital can be expressed as

$$S = \ln(\Omega) \quad . \tag{4-11}$$

In statistical mechanics, Equation (4-11) is the definition of entropy for an isolated homogeneous system at stable equilibrium [11, pp. 609]. It relates the number of possible configurations of microstates Ω to the entropy of a system. In economics, it follows from Equation (4-11) that if human capital is larger then Ω must be larger too. Then it reasonably follows that this quantity Ω measures the productivity of human capital. In economics, total factor productivity is considered the part of growth that is not explained by the growth of labor or capital [65]. Since human capital measures effects not captured by labor and capital, Ω serves the role of total factor productivity in economics. Ω measures the productivity of everything not captured by the productivity of other factors of production. The greater the productivity of human capital at a microeconomic level, the greater the total amount of human capital is measurable at the macroeconomic level.

In thermodynamics, entropy is evenly spread over the volume of a system that is homogeneous, isolated, and at equilibrium. Then in economics, human capital must be evenly allocated over something in the economy. This something is standard labor. I argue that non-standard labor and capital are also allocated over standard labor in an economy. I propose that labor and capital can be defined like Equation (4-11) for an isolated homogeneous economy at a Pareto state where

$$K = c_K \ln(\mathcal{K}) \qquad\qquad L = c_L \ln(\mathcal{L}) \quad . \tag{4-12}$$

c_K is a constant in units of capital, and c_L is a constant in units of labor. $\mathcal{K}$ is the productivity of capital, and $\mathcal{L}$ is the productivity of non-standard labor. Non-standard labor and capital are allocated over standard labor such that the further non-standard labor and capital can be spread over an economy the larger the productivities $\mathcal{K}$ and $\mathcal{L}$ must be.

While Equation (4-12) is not something that is defined analogously in thermodynamics, it is conceivable for macroeconomics states there exists many possible microeconomic allocations that result in the same amount of capital or labor. Then in an isolated homogeneous economy at a Pareto state, labor and capital are related to their productivities through the relation in Equation (4-12).

4-3-3 Relationship between Total Earnings and Cobb-Douglas Utility

For an isolated homogeneous economy at a Pareto state, a relationship between a linear production function and Cobb-Douglas production exists.

The Cobb-Douglas function in microeconomics measures utility in the form of

$$u(x) = \prod_{i}^{N} x_i^{\lambda_i} \tag{4-13}$$

where λ_i is the preference given to each input x_i, and the utility $u(x)$ is maximized [72]. In macroeconomics, a Cobb-Douglas production function gives an output

$$Y = Y(A, K, L) = AK^{\alpha}L^{\beta} \quad . \tag{4-14}$$

First consider an output measured by a cash flow. Below I restate Equation (4-8) that measures the total potential output in terms of earnings of an economy

$$E = wL + rK + TS \quad .$$

Here L and K are labor and capital. w is a wage, r is a rent, T is a price level, and S is a level of human capital.

Given Equations (4-11) and (4-12), Equation (4-8) can be written as

$$E = w\, c_L \ln(\mathcal{L}) + r\, c_K \ln(\mathcal{K}) + T \ln(\Omega)\,. \tag{4-15}$$

Both sides are exponentiated after dividing through by price level T

$$\exp\left(\frac{E}{T}\right) = \mathcal{L}^{\frac{w\, c_l}{T}} \mathcal{K}^{\frac{r\, c_k}{T}} \Omega. \tag{4-16}$$

Then the expression is restated such that

$$G = \exp\left(\frac{E}{T}\right) = \Omega \mathcal{K}^{\alpha} \mathcal{L}^{\beta}. \tag{4-17}$$

Here $\alpha = \frac{r\, c_K}{T}$ and $\beta = \frac{w\, c_L}{T}$. G is the resulting measure of growth in Equation (4-17). The output elasticity of labor is β, and the output elasticity of capital is α [2].

Now Equation (4-17) is of the form of a Cobb-Douglas production function (Equation (4-14)) but instead measures a unitless quantity that is not the same output measured by the linear production function. This unitless quantity more so measures utility for macroeconomic systems by a Cobb-Douglas utility function than an output measured by a cash flow. Since growth is considered important for macroeconomic systems, the unitless quantity G from Equation (4-17) serves as a measure of growth and only serves as a proxy measure of the output if given a certain price level T.

If the output E and price level T grow at the same rate, then G is a constant. This situation is related to the concept of balanced growth in the Solow growth model, where the economy grows and allocates to the point where economic growth follows first-order exponential growth [2]. Thus G is a constant for economies on a balanced growth path.

The quantity $\exp\left(\frac{E}{T}\right)$ in Equation (4-17) is the inverse of the Boltzmann factor. The exponentiated quantity E/T is known as an earnings-price ratio in finance, and it measures expected but not realized growth [73]. The Boltzmann factor is proportional to a probability [11] where as the quantity G in Equation (4-17) expresses growth of an economy. It is the case that

$$\exp\left(-\frac{E}{T}\right) \exp\left(\frac{E}{T}\right) = 1 \quad . \tag{4-18}$$

I interpret that since G measures growth, its inverse measures shrinkage. Since the inverse of G is the Boltzmann factor, it follows that if G is large than the relative probability of finding an economy with the corresponding values of E and T are lower.

The output elasticises α and β are defined by a ratio of the respective rent or wage to the price level of the economy in Equation (4-17). If the output elasticities of labor and capital allocations are less than 1 in Equation (4-17), then labor and capital have diminishing marginal returns [2]. When the wage or rent is equal to a price level, then labor or capital has constant returns to scale. Then if the wage or rent is greater than the price level, there are increasing returns to scale for labor and capital. Considering both α and β, if the sum of the elasticities $(\alpha + \beta)$ is equal to one then the economy is said to be competitive, and if greater than 1 returns are increasing to scale and less than 1 returns are decreasing to scale [2]. If these coefficients α and β can be determined for an economic system by measuring wage, rent and price level, then the competitiveness of that system and the type of return to scale can be determined.

Observe from the definition of the exponents α and β that increasing price level T leads to the values of these exponents decreasing. An increase of price level—inflation—leads to a decrease of G if all else stays constant. However, if say Ω also increased during this inflation, then the net result can still be an increase of G.

Human capital can be explicitly measured if G, L, and K can be measured. Solving for human capital in Equation (4-17) results in

$$\ln(\Omega) = \ln(G) - \alpha\ln(\mathcal{K}) - \beta\ln(\mathcal{L}) \quad . \tag{4-19}$$

Comparing Equation (4-4) to Equation (4-19), both equations present a way to measure human capital in an economy depending on whether total amounts of labor and capital are used or their rates of growth are used.

Equation (4-17) and the assumptions needed to derive it address criticisms of Cobb-Douglas production functions. One criticism is that the Cobb-Douglas production function only captures an underlying accounting identity [74]. This criticism is indeed true, but there are assumptions in Section 4-8 and 4-3-2 needed to derive Equation (4-17) from Equation (4-8). Another criticism in economics of the Cobb-Douglas production function is that that while Cobb-Douglas production functions may hold at the microeconomic level, the conditions that specify when it holds at the macroeconomic level are still not specified [75]. The assumptions I made are that the economy is homogeneous, in a Pareto state, and isolated. These assumptions form the conditions needed for a Cobb-Douglas production function to hold for macroeconomic conditions. Another criticism of the Cobb-Douglas function is that the units do not make sense and that capital cannot be measured—the so called "capital controversy" [76]. The Cobb-Douglas production function derived in Equation (4-17) is consistently unitless, and there is one measurement of capital in Equation (4-12).

A link between the total potential earnings—a linear production function—and Cobb-Douglas production functions has been shown in this section. Though Cobb-Douglass production functions and linear production are considered together as types of production functions, the results of this section suggest the fundamental thermodynamic relationship only corresponds to linear production functions and accounting identities where output can be measured by earnings while the Cobb-Douglass function corresponds to a unitless measure of growth with productivities as inputs.

4-4 Business Valuation

The thermodynamic theory developed in Chapter 3 can be applied to the valuation of a business. A business is valuated to determine its total worth during the purchase of the business. When a business is purchased, it is crucial the business is correctly valued so the buyer does not overpay or underpay for the business. There is much at stake when valuating the worth of a business as a faulty valuation can lead to the loss of many millions of dollars.

There currently are many different methods to valuate the worth of a business. The three main approaches are an asset, income, or market based approach [77]. By analyzing the total earnings of a business, an earnings based approach to valuate the worth of a business during its sale can be created with the theory from Chapter 3.

A hypothetical earnings based approached to valuation would start by determining the total earnings of a company. The earnings is a property of the business, a cash flow the business owns distinct from income streams it receives. Then the net present value of those earnings can then be used to determine a buying price for the company at the moment of the sale.

Determining Earnings

An earnings based approach to business valuations can be developed from the theory of Chapter 3 and application of Section 4-3-1 as follows. First consider business A that is being purchased by business B. Business B wants to purchase A at a fair market value. The earnings of business A are then determined by a third party.

A ground state of earnings E_0 of business A can be agreed upon by all parties. The ground level of earnings would be the earnings of the company if it had zero assets. The sale of businesses with little to no assets occurs often, and they are valued well above zero [78]. One way to determine this ground state of earnings would be to take the minimum earnings necessary for a hypothetical business similar to business A to survive and not go bankrupt.

A ground level of human capital[1] S_0 can also be determined during this step. In thermodynamics, a system can have an arbitrary ground level of entropy S_0 [56]. Then the type of business for sale may have an associated nominal amount of human capital. The value of S_0 of business A can also be considered the human capital business B would purchase in the acquisition of business A excluding the human capital assets business A has itself accumulated. If this purchase is irreversible activity, then the human capital generated in this purchase is an asset gained by business B. Then business B would have to pay a fair value for this human capital generated by the purchase of business A.

The next step would be to determine the earnings associated with all the factor assets and liabilities business A owns. If a prevailing market wage or rent is known for the assets and liabilities then the earnings associated with these factors are

$$E_a = r_1 a_1 + r_2 a_2 + ... + r_n a_n$$

where r_i is the rate, wage, or rent of factor a_i. These factor assets are useful factors like any financial capital, buildings, or inventory assets. Liabilities are included too and have a negative rent. Labor is generally seen as an expense, but it is sometimes listed an asset [79]. The value of existing employees may be seen as an asset to Business B, so it should be included in the valuation of the earnings.

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[1] I use the term human capital here, but the term goodwill can be used interchangeably in this context

The next step is to determine the value added by any human capital Business A may own. Business A may own intellectual property such as patents and proprietary technology. The skilled employees of the business may also contribute to the level of human capital owned by business A.

$$E_s = T_A(dS_{acc} + S_0 + dS_{irr})$$

Here T_A is a price level of the business A, and dS_{acc} captures the total human capital accumulated by business A. dS_{irr} represents the creation of human capital that may occur when business B purchases business A. The human capital comes from intellectual property, value added by skilled employees, brand recognition, and any other form of human capital. One method to determine the level of human capital is to determine what increases in income for business A have been associated with the purchase, development, or creation of human capital over the lifespan of the business. Income and accounting statements from the beginning of operations to the current time could be analyzed to determine the present level of human capital accumulated by business A. A fair price level of business A can be determined by analyzing the price level of other similar business to business A that operate between the same markets with similar types of human capital. Ideally business A already catalogues its acquisition of human capital in its assets, which would make the valuation at this step easier.

The final valuation of business A's earnings results in

$$E_{val} = E_a + E_s + E_0.$$

The method developed for determining earnings has similarities to an asset and market based approach to the valuation of a business [77]. It takes into account tangible and intangible assets while also consider the market value of the business.

These earnings are not the value or purchase price of the business. If the business was purchased at this price, the new owner would run the business for a year to make his money back then sell it again. In order to determine the value of the company, some assumptions are needed about the lifetime of the company. The next step to this earnings method is to determine the value of these earnings.

Laplace Transformation and Net Present Value

I will use the net present value (NPV) of the earnings of a business to determine the value of the business and its price of sale. The net present value of a cash flow is the present value of a cash flow in the future [80]. The idea behind net present value is that a dollar in the future has less value than a dollar today due to factors like inflation. It has already been noted in finance that the net present value of a cash flow can be determined with the use of a Laplace transform [81]. I will use the earnings of business as this cash flow that will be discounted over a period of time to determine the net present value of a company.

Going back to the example, the earnings determined in the evaluation of business A will have a certain value to business B at the present. This value includes the discounted value of future earnings. The net present value of business A's earnings can be written as

$$NPV = \int_0^{\tau} e^{-rt} E_{val}(t)\, dt.$$

The earnings $E_{val}(t)$ are a function of time and need not be a constant year to year. For instance, there can be projected growth for the earnings. The discount rate is r. This discount rate is usually taken to be a rate of a safer opportunity such as government bonds [77]. τ is the lifetime of the company. The lifetime can be assumed to be infinity, making the NPV equivalent to the Laplace transform, or it can be assumed to be some period of time like 10 years. The NPV of business A is the value of business A that it should be purchased at given the assumption of its lifetime and discount rate.

This step of the earnings approach makes use of a discounted cash flow, similar to the income method of business valuation [77]. The method of determining earnings in the previous subsection has similarities to the assets and market approaches to business valuation. This earnings approach to business valuation captures the effects of three different approaches to valuating business. It is possible this earnings approach could be used in conjunction with an asset, income, and market approach for a holistic business valuation approach.

Example of a Business Valuation

I now provide a sample valuation for a fictional business. Take a business with a yearly prepaid cost of 5 laborers that all work at the same wage of 10,000 \$/yr-man. There is a single manager that is salaried, and there is no intellectual property. The business owns a single piece of capital with rent 20,000\$/yr-machine. Now a second business wants to buy this business.

First they decide on a ground level of earnings. They agree that this business has a minimum earnings of 30,000 \$/yr based on market research into similar businesses. They then decide on a base level of human capital associated with this business, say $S_0 = 0$. The manager's contribution to the business' human capital is decided as 0.4. The human capital gained by the purchaser is 0.1. There is no intellectual property or other forms of human capital. No growth is projected. The price level for such a business is determined to be 20,000 \$/yr.

Then this business can be valuated as

$$E_{val} = 10,000 \times 5 + 20,000 \times 1 + 20,000 \times (0 + 0.4 + 0.1) + 30,000$$

that results in $E_{val} = 110,000\$/yr$.

The two parties agree the lifetime of the business is 5 years and that there is some risk involved in this business. The earnings of the business is then discounted at a given rate r = 15% over a time interval of five years

$$NPV = \int_0^{5} e^{-0.15t}\, 110,000\, dt$$

that gives the net present value of the business. The purchaser agrees to buy the business at this dollar amount, which comes out to be approximately 387 thousand dollars.

4-5 Conclusions

The fundamental thermodynamic relationship can be used to empirically measure the growth of real world economies given the theory of Chapter 3. A linear production function is derived from the creation of a total potential of an economy. While the linear production function still follows from the fundamental thermodynamic relationship, a production function of a Cobb-Douglas type does not take the same inputs as the linear form. Instead, it is a function of the productivities of the factors of production and human capital given that opportunities are evenly spread out through an economy. Human capital can be measured from overall economic growth and the growth of the factors of production.

Statistical mechanics was briefly used in this section. Future research should be done to see how statistical mechanics can explain how microeconomic behavior contributes to macroeconomic properties and behavior. Only homogeneous economic systems were considered in this section. If an economy is not homogeneous, then capital, labor, and human capital are not evenly spread through an economy. The relationship between a linear production function and Cobb-Douglas production function will not follow as it does in Section 4-3-3. Statistical mechanics can be applied to determine how uneven allocations at the microeconomic level lead to macroeconomic proprieties. Statistical mechanics also introduces probability through the Boltzmann factor, so applications of statistical mechanics to economics may bring concepts of risk and uncertainty into the economic engineering framework.

There are promising applications in real world economic scenarios for the thermodynamic-economic theory developed in this thesis. Businesses would be interested in this theory for its ability to determine operating efficiency, determine the value of a business, and accurately account the growth of a business. Economies can account for the factors that can influence economic growth and better determine the conditions that are necessary for economies to undergo rapid growth or sudden decline. The concept of human capital will be important in macroeconomics to explain productivity and a level of technology not measured by the factors of production. The applications to growth in this chapter demonstrate that it is possible to measure and analyze economic growth with the theory of Chapter 3.

The applications developed in this chapter are not inclusive. Other students can take the initiative to figure how the theory I have developed in Chapter 3 can be further applied in economic engineering. The applications developed in this chapter can be developed further and matured in future applications. As other masters students in the economic engineering group start their master's thesis, they hopefully will find the foundational theory of Chapter 3 as one tool in their solution to a problem in economics.

Contributions

1. The fundamental thermodynamic equation can be used to empirically measure economic growth

2. Linear production functions shown to be measure of total potential earnings

3. Cobb-Douglas production function takes productivities as inputs and measures growth

4. Relationship between linear production functions and Cobb-Douglas clarified; the two functions measure two different quantities

5. Method created to measure total potential earnings of a business to value it for sale

Chapter 5

Conclusions

The goal of this thesis was to include thermodynamic modeling of economic systems in the field of economic engineering. To that effect, this thesis has one main contribution.

The main contribution of the thesis is the development of the axiomatic theory needed to model economic systems as thermodynamic systems. The foundational principles of the thermodynamics of economics fit within the economic engineering framework. The thermodynamics of economic engineering developed in this thesis can be used in conjunction with a Newtonian method to increase the scope and insight of analysis of economic systems.

Key analogs were established in this thesis between economics and thermodynamics. The use of standard labor as an exogenous measure of value leads to the development of an expenditure of a factor being analogous to work. The first no-free-lunch law forms a fundamental law in economics where nothing of economic value is obtained for free. There exists earnings of an economic system analogous to energy independent of any income the system may receive. Pareto states are established as the economic analog to stable states. The derived second no-free-lunch law is another fundamental law in economics that establishes the attainability and existence of Pareto states for economic systems. Expenses are analogous to heat and are defined in contrast to expenditures. By determining the economic analog to the zeroth law is the law of one price, temperature is analogous a price level. Business can be analyzed as engines that produce an income by bringing together different economic systems at different price levels. Reversibility and irreversibility are concepts in thermodynamics that can be directly applied to transactions in economics. Establishing human capital as the economic analog to entropy serves as way to capture the magnitude of services unmeasured by the factors of production. The economic analogs to three material properties—compressibility, heat capacity, and coefficient of thermal expansion—describe an economy by its factor elasticity, inflation capacity, and factor effectiveness, respectively. These key analogs between economics and thermodynamics form the foundational principles of the thermodynamics of economic engineering.

Additionally, applications developed in the thesis demonstrate the applicability of the theory of Chapter 3 to economic growth. Chapter 4 mainly demonstrates the theory's applicability to macroeconomic growth but also applies the thermodynamics of economic engineering to the valuation of a business for sale. Empirical growth accounting for economies is measured by the fundamental thermodynamic relationship. The total potential earnings of an economy serves as a linear production function for an economy that is homogeneous. For an isolated homogeneous economy at equilibrium, the Cobb-Douglas production function is derived from a linear production function. The Cobb-Douglas function is a function of the productivities of the factors of production and human capital at a microeconomic level, while the linear production function relates total potential earnings to the factors of production and human capital. The method for determining total potential earnings is also used to determine the earnings of a business for sale. The total potential earnings of the business are discounted to determine the value of the business for its assumed lifetime. These applications are not exhaustive, and further applications of the theory will be developed within the economic engineering framework.

Chapter 6

Recommendations

6-1 General Recommendations

Three economic applications of the thermodynamic theory developed in this thesis were presented in Chapter 4. However, there are many potential applications of this theory in economics that can be used together with a Newtonian approach. The thermodynamics of economic engineering will benefit from further developments and refinements to the theory. Particular applications—such as from other masters students—will likely have the effect of contributing to this theory.

Although this thesis develops the thermodynamics of economic engineering, it does not show how the thermodynamics of economic engineering is related to the Newtonian or analytical mechanic approach of economic engineering. The task of showing how they are all related is tantamount to asking to show how dynamics and thermodynamics can be understood in the same framework. Thus a way forward for further developing economic engineering theory would follow the development of thermodynamic modeling in control formalisms or vice versa.

Some statistical mechanics were used in Chapter 4, but its use not as developed compared to the thermodynamics of economic engineering. Statistical mechanics could be used to model the emergent behavior of the interactions between many individual Newtonian economic agents. Future research should be done to develop statistical mechanic theory for economic engineering for its potential to model risk and uncertainty.

6-2 Putting the Thermodynamics Back into Controls

Thermodynamic effects are not often considered in controls. Control formalisms such as PID, LQR, or MPC control do not build in the thermodynamic effects that may result from the design, implementation, and operation of a controller. An implemented controller may result in the build up of heat, resulting in the need for ad hoc cooling solutions or a different controller design.

In order to account for thermodynamic effects, I argue that an energy based approach to controls is needed. Thermodynamics is inherently about the exchange of energy between systems. In order to account for energy like heat, the thermodynamic energy of a dynamical control system needs to be considered.

The effort to bring energy back into controls has been a topic of research within the economic engineering group. With the work of Coenraad Hutters, damping was included within a Hamiltonian formalism in order to model economic surplus and consumption, see [14]. A complex Hamiltonian was developed that modeled the equations of motion of a complex state. Control was introduced by analyzing port Hamiltonian theory. However, this complex Hamiltonian models the change of mechanical energy and does not model thermodynamic energy.

To move forward and put the thermodynamics back into controls, I argue that higher dimensional analysis needs to be done to model the thermodynamic effects of damping in a second-order system. Developments have been made in the economic engineering group for the extension of energy-based control to four dimensions. I will briefly discuss preliminary results, the direction the research of this topic is heading, and how I would apply a developed formalism of this topic to controls.

The Second-Order System as a Hyperbolic Top

A breakthrough made in the economic engineering group is to see a second-order spring-mass-damper system as a hyperbolic top. While a regular top has different types of angular movement, the damping of a second-order system results in a mixture of angular and hyperbolic movements. Fig. 6-1 shows the trajectory in phase space of an underdamped system and the vector field at selected points associated with the geometry of a higher dimensional system.

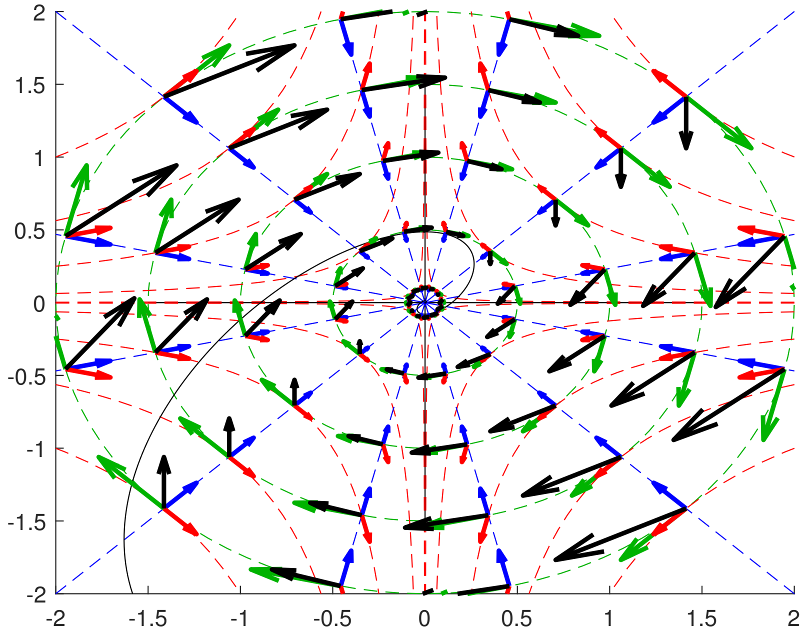

Figure 6-1: x axis is the position, y axis is the momentum. The green arrows are the spin at a point associated with the green dashed line forming circles. The blue arrows are the precession at a point associated with the dashed blue expansion lines. The red arrow is the nutation at a point associated with the red dashed lines forming hyperbolas. The resulting vector at a point is the black arrow that indicates the resulting motion at that point. The natural frequency is 0.25 rad/s and the damping is placed on the spring (damping ratio of 0.5). Credit to Coenraad Hutters for generating the graph.

A top has a spin, precession, and nutation. The spin is analogous to the natural frequency of a second-order system and is an angular velocity. The geometry associated with this movement is a circle, pictured in green in the figure.

The precession is the change of spin. For a second-order system, this precession is hyperbolic due to the damping and results in the decrease of mechanical energy in the system. The geometry associated with the precession is a line, pictured in blue in the figure.

The nutation is the change of precession. For second-order systems, the nutation gives the causality of damping. The damping can be placed only the mass such as in the A-matrix formalism, the damping can be placed on the spring, or a mixture of damping can be placed with some damping on the spring and some on the mass. The geometry associated with the nutation is a hyperbola, pictured in red in the figure.

Notice in Fig. 6-1 that I placed the damping on the spring. No damping occurs when the position is zero and there is some momentum. The total damping of the system is equal to the sum of the damping on the spring and damping on the mass. The nutation of the system is the difference of the damping on the mass and damping on the spring.

There is a fourth piece of geometry associated with the forced solution of a system. There is another hyperbola not pictured in Fig. 6-1 associated with the interaction of say a controller that can supply a force. Research needs to be done to see how this degree of freedom can be used to do control.

The Thermodynamic Part

The precession of the system can be thought of as the thermodynamic part of the system. Its geometry is not of a circle or hyperbola; the precession is a line. The circles and hyperbolas will indicate how mechanical energy will change but not how it will grow or diminish.

For isolated systems with positive damping, the precession indicates the contraction of mechanical energy available to the system at any point. These expansion lines indicate how the system will lose mechanical energy. However, this mechanical energy is still conserved as thermodynamic energy and must make up the energy that can be dissipated as heat.

Research is being done into using the Euler-Lagrange equations for rigid body rotations to model the mechanical energy of a second-order system. The precession—thermodynamic part—would represent the real part in this analysis while the mechanical part would be the vector part of a higher-dimensional number.

Projective Geometry and Visualization

The trajectory and geometry can be visualized with projective geometry. Fig. 6-2 shows the trajectory from Fig. 6-1 projected on other surfaces.

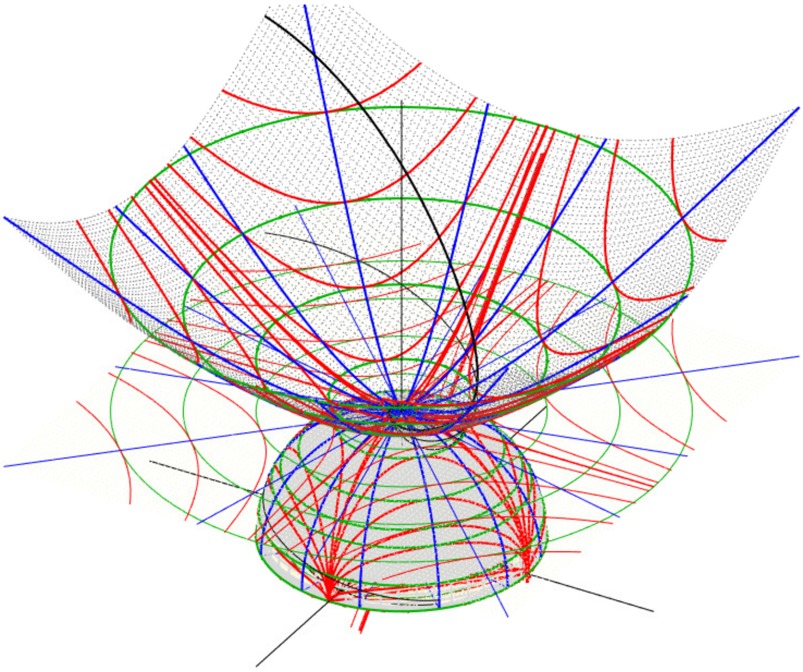

Figure 6-2: The geometry from Fig. 6-1 projected onto different surfaces. The top surface is a pseudosphere, the bottom surface is the hemisphere model, the phase plane lies between the two surfaces, and the Poincare disc lies below the hemisphere (not visible in this picture). The trajectory in black is most visible on the pseudosphere. Credit to Coenraad Hutters for the image.

Fig. 6-2 shows the trajectory of a second-order order system in higher dimensions spaces. Projective geometry will be needed for analyzing second-order systems in higher dimensions. The inclusion of energy in the analysis of a second-order system results in the need of at least one more dimension to model the wiggle in energy due to the causality of damping.

Application

If the theory is developed enough, it can be applied to account for heat accumulation in controlled dynamical systems. The goal of the theory is being able to account the for the build up of heat in the design process of a controller. A controller can be evaluated not only on conventional control design parameters—like overshoot, settling time, etc.—but also on its rate of heat generation or energy use. Optimal controllers can be developed where heat is accounted for. This application is for any situation in controls where heat can be a concern.

Accounting for the generation of heat implies that the generation of entropy can also be accounted for in second-order systems. The heat dissipated by a dynamical system is representative of entropy generated from friction. If this energy based approached is fully developed, it not only puts thermodynamic modeling into control but also puts entropy into control as a measure of the mechanical dissipation.

Given that a damper is a thermodynamic system, statistical mechanics can also be applied to analyze the behavior of the damper as the source of entropy generation. The particles of the damper can be thought as a distribution of particles in a higher-dimensional system. A Liouville theorem can be developed for dissapative systems such that a distribution of damper particles is conserved in a higher dimensional phase space. The entropy generation from the damper can be analyzed to determine the heat that it generates.

The development of the theory needed to develop thermodynamics analysis into control formalisms was beyond this thesis, but some promising results from research being done within the economic engineering group was presented to show there is a path forward to modeling thermodynamic energy within mechanical or electrical systems. Future research needs to be done on how the effects of entropy generation can be brought into control formalisms. There should also be research done on how to apply statistical mechanics to ensembles of particles that are modeled by higher-dimensional mechanics in order to measure properties such as entropy generation.

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