SGF-Quantales and their Groupoids

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Quantales. The term quantale was introduced by Mulvey as the 'quantum' counterpart of the term locale. Locales can be thought of as pointfree topological spaces, and hence can be considered in the context of a Gelfand-type duality for commutative C^*-algebras [2]. Quantales were considered in the context of the quest for a Gelfand-type duality for general C^*-algebras.

In the Gelfand duality, the algebra-to-space direction consists in associating any commutative C^*-algebra with its maximal ideal space. This construction was extended to general C^*-algebras by considering the spectrum $MaxA$ of any unital C^*-algebra A, i.e. the unital involutive quantale of closed linear subspaces of A. This gives rise to a functor Max which was extensively studied for more than a decade as it was considered the best candidate for the C^*-algebra-to-quantale direction of a noncommutative Gelfand duality. Remarkably, $MaxA$ is a complete invariant of A, i.e. if A and A' are C^*-algebras such that $MaxA$ and $MaxA'$ are isomorphic, then A and A' are isomorphic. However, there are several problems with Max: 1) it has no adjoint, which is a necessary condition for its providing one direction of a duality; 2) it is not *full* on isomorphisms, i.e. some isomorphisms of spectra of C^*-algebras do not arise from C^*-algebra morphisms [7]; 3) there is no purely algebraic characterization of the class of quantales isomorphic to quantales of type $MaxA$; 4) there is no canonical way of constructing A from $MaxA$. These difficulties motivate the quest for alternative ways of linking C^*-algebras and quantales.

Besides their interest w.r.t. C^*-algebras, quantales have been extensively studied in logic and theoretical computer science: not only do they provide the standard algebraic semantics for various resource-sensitive logics such as linear logic [4,14], they have also been applied to the study of the semantics of concurrent systems and their observable behaviour, described in terms of finite observations [1]. Finite observations are formalized as semidecidable properties, and can therefore be identified with open sets of a topological space [13]; however, this perspective does not account for those (quantum-theoretic) situations where performing finite observations on a system produces changes in the system itself. In those cases, the set of the finite observations that can be performed on a system has a natural noncommutative structure of quantale. The basic view on quantales as generalized topologies can be retrieved also in this context. In [1], this perspective was applied on quantales to provide a uniform algebraic framework for process semantics and develop a systematic study of various notions of observational equivalence between processes.

Merging perspectives: the case study of Penrose tilings. Recently, investigation has focused on ways to integrate the two perspectives on quantales as noncommutative topologies and as algebras of experimental observations on computational (or physical) systems, and use them to investigate the connection between quantales and C^*-algebras. In [8], an important example was studied, which concerns a classification of Penrose tilings using quantales. This classification is alternative to the one previously introduced by Connes [3] (consisting in associating a certain C^*-algebra A_K with the space $(K, \sim)$ of Penrose tilings). The classification in [8] is based on a logic of finite observations performed on Penrose tilings, the Lindenbaum-Tarski algebra of which is a quantale (denoted by **Pen**). This classification arises from a canonical representation of **Pen** as a quantale Pen of relations on $(K, \sim)$. In [8], the exact connection between Pen and A_K was left as an open problem, but since Pen is not isomorphic to $MaxA_K$, the case study of Penrose tilings was considered pivotal in finding the alternative connection between quantales and C^*-algebras in the restricted but geometrically significant setting in which they both arise from *groupoids*.

Étale groupoids and their quantales. In [12], Resende generalized the example of **Pen** to a bijective correspondence between localic étale groupoids and certain unital involutive quantales referred to as *inverse quantal frames* (indeed, their underlying sup-lattice structure is a frame). An important feature of inverse quantal frames $\mathcal{Q}$ is that, denoting the unit of $\mathcal{Q}$ by e, the restriction of the product to the subquantale $\mathcal{Q}_e = e{\downarrow}$ coincides with the lattice meet. The groupoid-to-quantale direction of this correspondence arises from observing that, for every étale localic groupoid $\mathcal{G} = (G_0, G_1)$, the groupoid structure-maps induce a structure of unital involutive quantale on the locale

G_1, which becomes an inverse quantale frame. Conversely, the étale localic groupoid associated with an inverse quantal frame $\mathcal{Q}$ is based on the locales $G_0 := \mathcal{Q}_e$ and $G_1 := \mathcal{Q}$.
Towards a non étale generalization of Resende's correspondence. In [9], a unital involutive quantale is associated with any topological groupoid in a way alternative to Resende's but compatible with it when the topological groupoid is étale. This route makes it possible to account for the connection between the quantale *Pen* and the C^*-algebra A_K, which was left as an open problem in [8]. The quantale associated with a topological groupoid $\mathcal{G}$ is the sub sup-lattice of $\mathcal{P}(G_1)$ generated by the inverse semigroup $\mathcal{S}$ of the images of the local bisections of $\mathcal{G}$.
Spatial SGF-quantales. Building on [9], in [10], a bijective correspondence is established between certain unital involutive quantales referred to as *spatial SGF-quantales* and *topological* groupoids s.t. G_0 is sober. This class of groupoids includes equivalence relations arising from group actions, and significantly extends the class of étale topological groupoids. Dually, inverse quantal frames are exactly those SGF-quantales the underlying sup-lattice of which is a frame. The correspondence defined in [10] extends the theory of [12] to a point-set, non étale setting. Interestingly, this correspondence also forms the basis of a representation theorem for SGF-quantales into unital involutive quantales of relations [11], similar to the one for relation algebras in [6].
Étale vs. non étale. The comparison between the correspondences in [12] and [10] is facilitated by the observation that a topological groupoid is étale iff the images of its local bisections form a base for the topology on G_1. The étale topological setting can be shown to be exactly the one in which the groupoid-to-quantale routes in [12] and in [10] can be identified.
Work in progress. The proposed talk reports on ongoing work [5] aimed at generalizing the correspondence of [10] from a topological to a localic setting. This amounts to defining a bijective correspondence between general SGF-quantales and localic (non étale) groupoids. For any localic groupoid $\mathcal{G} = (G_1, G_0)$ satisfying mild additional hypotheses, one defines its local bisections within the theory of locales and then, observing that the locale $LC(G_1)$ generated by the locally closed sublocales of G_1 has a natural quantale structure, one defines the quantale $\mathcal{Q}(\mathcal{G})$ as the subquantale of $LC(G_1)$ generated by the images of the local bisections of $\mathcal{G}$. As to the quantale-to-groupoid direction, in the point-set case [10], given $\mathcal{Q}$, one defines G_0 as the topological space arising from the spatial frame $\mathcal{Q}_e$, and obtains the points of G_1 as equivalence classes of tuples (p, f) such that p is a point of G_0 and f is a partial unit of $\mathcal{Q}$ (partial units being the quantale counterparts of local bisections), satisfying an algebraic condition which geometrically means that "f is defined on p". Any such equivalence class can be understood a posteriori as the value $f(p)$ of f at p. The pointfree counterpart of the condition "f is defined on p" is one in which points are replaced by locally closed sublocales of the locale G_0 associated with the frame $\mathcal{Q}_e$. The main strategy consists in defining the restrictions of local bisections to locally closed sublocales of G_0 in the theory of SGF-quantales. This makes it possible to extend in a unique way any SGF-quantale $\mathcal{Q}$ to a quantale $\tilde{\mathcal{Q}}$ s.t. $\tilde{\mathcal{Q}}_e$ is the frame associated with the locale $LC(G_0)$, and then derive from the SGF-axiomatization that $\tilde{\mathcal{Q}}$ is an inverse quantal frame, which hence gives rise to an étale localic groupoid $\tilde{\mathcal{G}} = (\tilde{G}_0, \tilde{G}_1)$. Finally, the localic (non-étale) groupoid associated with $\mathcal{Q}$ will be defined as the push-forward of $\tilde{\mathcal{G}}$ induced by the canonical projection $LC(G_0) \to G_0$.

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