Report LR-451

On the conceptual design of subsonic transport aircraft for cruising flight optimized for different merit functions

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E. Torenbeek

TH Delft

Department of Aerospace Engineering

Delft University of Technology

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SUMMARY

The present report describes an analytical approach to aircraft design optimization, which can provide a useful alternative and/or supplement to complex computer-aided and automated optimization programs. The emphasis is on conceptual design optimization of medium- and long-range turbine-engine powered transport aircraft. Consequently, the derived equations apply basically to the case where engines are sized to balance the cruise drag at a specified (constant) rating.

In all cases considered the design payload and the fuselage shape are considered to be specified and remain constant. The design variables are mainly associated with wing design and powerplant sizing. Several figures of merit are studied as a function of the design variables and analytical expressions are derived for partial as well as combined ("absolute") optima.
Merit functions considered are L/D-ratio, specific range (V/F), fuel plus powerplant weight as a fraction of the take-off weight and of the design payload, fuel plus powerplant plus wing weight fraction, payload weight fraction, fuel burned per seat-km, and Direct Operating Costs. The design variables studied are: engine size, (mean) cruise wing loading (W/S), pressure altitude, lift coefficient, Mach number and wing aspect ratio. In general the attention is concentrated on unconstrained optima for conventional aircraft configurations, i.e. aircraft with non-integrated combinations of fuselage (with specified volume) and wings. The wing taper ratio, sweepback angle and spanwise section shape variation are held constant.

CONTENTS

NOTATIONS AND ABBREVIATIONS

A	wing aspect ratio ($A = b^2/S$)
b	wing span
C	(specific) cost factor
C_D	drag coefficient ($C_D = D/(q/S)$)
C_{D_o}	zero-lift drag coefficient
C_{D_i}	lift-induced drag
C_{D_p}	profile drag coefficient
C_L	lift coefficient ($C_L = L/(qS)$)
C_T	specific fuel consumption ($C_T = F/T$)
C_η	coefficient in expression for powerplant efficiency
D	drag (no index: total cruise drag)
DOC	Direct Operating Costs
F	fuel mass flow/unit time
F_C, F_P, F_Q	powerplant merit functions
g	acceleration due to gravity
H	specific calorific value of fuel
L	lift
M	(cruise) Mach number
M_{crit}	drag critical Mach number
MTOW	Maximum Take-Off Weight
N_{fl}, N_S	number of flights, seats
m	mass (component) (no index: all-up mass)
p	atmospheric pressure
p_o	p at sea level ISA
q	dynamic pressure ($q = \frac{1}{2}\gamma p M^2$)
q_o	q at sea level ($q_o = \frac{1}{2}\gamma p_o M^2$)
R, R_{eq}	range , equivalent range
$\bar{R}_b$	average sector distance
R_H	range-equivalence of H ($R_H = H/g$)
S	(wing) area
T	thrust (no index: cruise thrust)
T_{to}	take-off thrust, static conditions, SL/ISA
U	total aircraft utilization (hrs)
V	True Air Speed
V_b	block speed
W	weight (component)(no index: All-Up Weight)
$\bar{W}$	mean weight during cruising flight
x_A, x_S	exponents of A and S in wing weight equation

β	dC_{D_i}/dC_L^2
γ	ratio of specific heats of atmospheric air
Δ	parasite drag area: $\Delta = \Sigma(C_D S)_{C_L=S=o}$
δ	relative ambient pressure ($\delta = p/p_o$)
η	overall powerplant efficiency
η_M	$d \log \eta / d \log M$
μ	mass or weight fraction
μ_T	power plant specific weight (W_{en}/T_{to})
τ	engine thrust lapse factor ($\tau = T/\delta T_{to}$)
ω	non-dimensional wing weight sensitivity

Indices

a	(fixed) airframe
en	engine(s) plus nacelle(s)
f	fuel
fix	fixed aircraft components
fl	flight
h	hourly
pℓ	payload
MD	minimum drag condition for given aircraft
net	relating to net thrust definition
s	Seat(s)
to	take-off
tot	total
var	variable weight components
w	wing

symbols with ^ refer to a reference (baseline) aircraft design

1. INTRODUCTION

Historically the conceptual design of aircraft relied on mission and economical analysis of a limited range of aircraft geometries in combination with selected engine types. The advent of computer-based design systems has made it possible to consider a much greater range of design alternatives for both the airframe and the powerplant cycle. Computer Aided Engineering (CAE) systems are now in use in design offices and laboratories which combine the capabilities of sophisticated computer graphics facilities with automated or semi-automated multivariable aircraft sizing and optimization programs. Certain issues in such a design environment, however, require continuous attention from the developers and users of CAE systems to the following aspects:

a) The selection of suitable criteria to assess the quality of a design, the so-called Merit Functions, and the art of balancing the aircraft so that it is near-optimum with respect to all important Merit Functions.
b) The proper representation of sensitivities to design variations, in particular the interaction between the numerical routines and the geometrical representation of the design.
c) Certain numerical problems of convergence in the case of weak or multiple optima.
d) The interpretation of results from a 'black-box' like approach such as Multi-Variate Optimization (MVO), cf. for example Ref. 10.

The present report forms an alternative and systematic approach to conceptual design optimization, which can provide useful testcases for comprehensive CAE programs and help the designers to understand and verify their results. The analytical results presented are also useful to start an optimization process with a first-order approximation of an unconstrained optimum design, resulting in a reduced number of iterations.

Attention is paid mainly to the sizing of aircraft designs for optimum cruising. The emphasis of the present method is therefore on transport aircraft designed for medium and long ranges. However, performance constraints such as take-off and landing distances, buffet limits, fuel tank volume constraints, etc., can be imposed afterwards, cf. for example Ref. 1. The derivations apply to turbojet and turbofan aircraft, but can be modified readily for turboprop or propfan-powered aircraft. In that case particular attention must be paid to derivations which involve the variation of engine thrust and efficiency with speed.

In all cases considered the design payload and the fuselage shape are considered to be specified and they remain constant. The design variables are mainly associated with wing design and powerplant sizing. Several figures of merit are studied as a function of the design variables, and analytical expressions are derived for partial as well as combined ("absolute") optima. Merit functions ("Criteria") considered are lift-to-drag (L/D) ratio, specific range (V/F), fuel plus powerplant installation weight as a fraction of both the take-off and the payload weight, fuel plus powerplant plus wing weight fraction, payload weight fraction, fuel burned per seat-km, and Direct Operating Costs. The design variables studied are: engine size, (mean) cruise wing loading (W/S), cruise (pressure) altitude, lift coefficient (C_L), Mach number (M) and wing aspect ratio (A). The wing taper ratio, sweepback angle and spanwise section shape variation are held constant.

2. GENERAL ASPECTS OF THE OPTIMIZATION

The optimization of a conceptual design involves basically the sizing of the aircraft for an assigned design-mission (payload/range) so that it achieves the best yield with respect to a predefined objective (merit) function. The following categories of properties will be distinguished:

a) Assigned characteristics: in the present case the combination of a design-payload and a specified design-range. In addition to this, certain characteristics will not be varied in this study, such as wing sweep angle, thickness/chord ratio and taper ratio.

b) Independent variables: the wing area (or wing loading), aspect ratio, design-cruise altitude, and Mach number.

c) Dependent variables: lift and drag coefficients, the mass distributions, the Maximum Take-Off Weight (MTOW), engine take-off thrust (T_{to}), etc. Several of these dependent variables will be used as merit functions; in those cases they will be referred to as CRITERIA, according to which the independent variables are optimized, so that these criteria achieve an extremal value.

d) Constraints: since this article deals basically with unconstrained optimization, the introduction of constraints is avoided, with the following exceptions:

   1. There is a constraint on the fuselage size in the sense that when the wing size is varied the fuselage geometry remains constant. The result of this constraint is that only conventional wing/fuselage combinations are considered. Configurations where a constraint is imposed on the total wing plus fuselage volume, for example, result in generally lower optimum wing loadings, hence larger wings or even all-wing configurations.

   2. An essential constraint is the engine rating during cruise. Since it will be assumed that engines are sized to balance the cruise drag, the constraint on rating results in an engine rubberizing proces. An exception is the case where the specific range of a given aircraft is maximized for constant altitude or Mach number.

   3. A special case of aspect ratio optimization will be treated where combinations of wing area and aspect ratio are considered resulting in constant wing weight fraction.

The general approach to the design optimization is illustrated in Fig. 1 and can be explained as follows.
In the present example wing loading and cruise altitude are independent variables. The Merit Function, which can be L/D, V/F or payload fraction, etc., is expressed in terms of these variables, bearing in mind that the condition of vertical equilibrium in cruising flight is satisfied:

$$\frac{W}{S} = \frac{1}{2}\gamma p_o \delta M^2 C_L \tag{1}$$

This equation interrelates the four independent variables (δ, C_L, M and W/S) so that only three are left. In Fig. 1 the Mach number is held constant, but it is truly an independent variable and will be treated as such in several cases. Lines of constant C_L are also indicated.
Contours of constant Merit Function are also indicated in Fig. 1. The partial optimum of this "criterion" with respect to wing loading is the interconnecting curve of the horizontal tangents to these "isomerit contours", curve I. Curve II defines the partial optimum for the cruise altitude. Curves I and II intersect in point A, referred to as an "absolute" or unconstrained optimum, but this refers strictly to the case of M = constant only. Clearly, when curves I and II are combined with a partial optimum of the Mach number, a more general "absolute" optimum is found, defining the maximum (or minimum) of the Merit Function with respect to all three variables. Again, this case may be further generalized when other variables, such as the wing aspect ratio, are introduced.

Instead of the wing loading, the cruise-C_L may be considered as an independent variable. In Fig. 1 lines of constant C_L are drawn, satisfying eq. (1). Line III indicates the points of tangency to the isomerit contours and defines therefore the partial optimum of C_L. Obviously, curve III intersects curves I and II in point A. The equations for curves I, II and IIIwill be derived separately in the following chapters, since they may be useful in cases where constraints are imposed on any of these variables.

It should be mentioned that an absolute optimum cannot always be found. In cases where the partial optima are incompatible with each other, they do not intersect. In that case only a constrained optimum yields a useful condition. In Fig. 1 an example of a constrained optimum is point B, defining an optimum combination of wing loading, altitude and C_L for a given engine at specified cruise rating.
The relevance of partial, constrained or "absolute" optima depends entirely on the design case considered. Therefore a unique answer to the general optimization problem of aircraft can never be found. In the present report the attention will be focussed to partial and absolute optima, since they result in ultimate goals for the design. Constrained optima can be very useful and practical, but are more specific in character. They apply in particular to short-range aircraft, where limitations on low-speed performances dominate the specification.

With respect to the definition of thrust and drag the convention will be adopted that the installed propulsive thrust is the engine's standard net thrust reduced by the drag of the engine plus nacelle (en) installation:

$$T = T_{net} - D_{en} \qquad (2)$$

Accordingly, the airframe drag (D) is the total aircraft drag minus the engine plus nacelle installation drag:

$$D = D_{tot} - D_{en} \qquad (3)$$

This subdivision has been adopted in order to take account of the influence of the airflow variation associated with the rubberizing process on the size of the engine, and hence on nacelle size, drag and weight. All values of engine efficiency, specific weight, etc. are therefore referred to the podded propulsion system installation.

In the following sections several different design criteria (merit functions) will be considered in order to investigate the main differences between aircraft designed for different merit functions. Such generalized results can help the designer to interprete the optimization process.
The results will be illustrated for a long-haul passenger aircraft, with design data summarized in Table 1.

3. CRITERION 1: CRUISE LIFT/DRAG RATIO

3.1. Drag polar representation for subcritical speeds

For a given low-speed Mach number the drag area of the airplane is represented, within a range of practical cruise lift coefficients, as a linear function of the wing area and the cruise lift coefficient squared:

$$C_D S = \Sigma(C_D S)_{C_L = S = o} + \frac{d(C_{D_p} S)}{dS} S + \frac{d\, C_{D_i}}{d(C_L^2)} C_L^2 S \tag{4}$$

Introducing the notations:

$$\left.\begin{aligned} \Delta &\triangleq \Sigma(C_D S)_{C_L = S = o} && \text{(parasite drag area)} \\ C_{D_p} &\triangleq \frac{d(C_{D_p} S)}{dS} && \text{(profile drag coefficient)} \\ \beta &\triangleq \frac{dC_{D_i}}{dC_L^{\,2}} && \text{(induced drag factor)} \end{aligned}\right\} \tag{5}$$

the airframe drag coefficient is, referred to the wing area,

$$C_D = C_{D_o} + \beta C_L^{\,2} = C_{D_p} + \frac{\Delta}{S} + \beta\, C_L^{\,2} \tag{6}$$

The profile drag coefficient C_{D_p} represents mainly the wing contribution; a contribution of the horizontal tail can be included to allow for the proportional variation of its area with wing area variation. The parasite drag area Δ represents mainly the fuselage plus vertical tail contributions. The induced drag factor β is equal to $(\pi A)^{-1}$, multiplied by a suitable factor to account for a non-elliptic lift distribution, wing/fuselage interference, trim drag, etc. It is assumed that variations in design C_L are matched by wing shape (camber) and wing/fuselage angle variation, so that for each C_L the fuselage has approximately its minimum drag. All drag contributions include roughness effects.

3.2. Lift/drag ratio for given Mach number

The airframe L/D ratio according to (6) can be rewritten as:

$$L/D = (C_{D_o}/C_L + \beta\, C_L)^{-1} = (C_{D_p}/C_L + \Delta/C_L S + \beta C_L)^{-1} \tag{7}$$

Assuming steady horizontal cruising flight (eq. 1) the wing loading W/S and pressure altitude δ are introduced explicitely into (7), as follows:

$$L/D = \left\{\delta\left(C_{D_p} \frac{q_o}{W/S} + q_o \frac{\Delta}{W}\right) + \frac{\beta}{\delta} \frac{W/S}{q_o}\right\}^{-1} \tag{8}$$

Setting the derivatives with respect to the wing loading and δ, representing the cruise altitude, equal to zero, yields two partial optima for given Mach number, in terms of optimum lift coefficients:

$$\text{opt. } W/S \rightarrow C_L = \sqrt{C_{D_p}/\beta} \quad \text{(I)} \quad \text{(9a)}$$
$$\text{opt. altitude} \rightarrow C_L = \sqrt{C_{D_o}/\beta} \quad \text{(II)} \quad \Big\} \; M < M_{crit} \quad \text{(9b)}$$

The first condition (I) defines the maximum L/D for the wing (plus hor. tail-plane) alone, since for constant altitude and Mach number $q\Delta/W$ is constant in eq. (8). The second condition (II) defines the altitude for max. L/D of the whole aircraft. Clearly, these two partial optima are incompatible for non-zero values of Δ (the all-wing configuration) or the wing loading (not practical). The example on Fig. 2 shows indeed that an unconstrained maximum of L/D w.r.t. altitude and wing loading is not obtained, provided Reynolds number effects on C_{D_p} due to altitude and wing chord variation are ignored.

A constrained optimum is obtained when the condition of horizontal equilibrium is introduced:

$$\frac{T}{\delta} = \frac{D}{\delta} = \frac{1}{2}\gamma p_o M^2 C_D S = C_{D_p} q_o S + q_o \Delta + \frac{\beta}{\delta^2}\frac{W^2}{q_o S} \qquad (10)$$

For constant engine cruise rating T/δ can be taken as an approximately constant value. For flight in the standard stratosphere, which is isothermal, T/δ is theoretically constant, provided Reynolds number effects are ignored. For given T/δ the maximum of L/D is obtained for

$$C_{D_p} S = \frac{1}{2}\left(\frac{T/\delta}{q_o} - \Delta\right) \quad \text{or:} \quad \frac{W/S}{C_{D_p} q_o} = \frac{2W}{T/\delta - q_o\Delta} \qquad (11a)$$

$$\frac{\delta}{\sqrt{\beta C_{D_p}}} = \frac{2W}{T/\delta - q_o\Delta} \qquad (11b)$$

while C_L is given by eq. (9a). Correspondingly, the maximum L/D is:

$$(L/D)_{max} = \frac{1}{2}\left(1 - \Delta\frac{q_o}{T/\delta}\right) / \sqrt{\beta C_{D_p}} \qquad (12)$$

In Fig. 2 this condition is denoted as point A, for $T/\delta = 0.30\ W$.

3.3. Effects of Mach number variation

Equation (12) shows that increasing the Mach number (hence $q_o \triangleq \frac{1}{2}\gamma p_o M^2$) results in a decreasing $(L/D)_{max}$, provided the wing area is reduced according to (11a). High values of $(L/D)_{max}$ can be obtained for low Mach numbers and low wing loadings, but for turbofan aircraft this is not in the interest of good engine efficiency, low structural weight and high productivity. Increasing the cruising speed to high-subsonic Mach numbers results in compressibility effects on the aerodynamic characteristics as illustrated in Fig. 3, which shows drag polars for subcritical speeds ($M < M_{crit}$) and high-subsonic Mach numbers. An alternative representation (Fig. 4) shows C_L/C_D (= L/D) in the C_L - M plane, for a given aircraft (constant S). Compressibility effects (drag rise) are apparent for $M > M_{crit}$, which is a function of C_L. Point C is the cricital Mach number for $C_L = C_{L_{MD}}$; the corresponding altitude is defined by $(W/\delta)_C$. Curves of constant W/δ are characterized by $C_L M^2$ = constant, according to eq. (1). For $W/\delta < (W/\delta)_C$ variation of altitude and Mach number independently results in the same value ($C_{L_{MD}}$) for max L/D.

For $W/\delta > (W/\delta)_C$ optimization (w.r.t. L/D) for given Mach number or given W/δ results in different C_L-values, as indicated by the two diverging curves for $C_{L_{MD}}$ and the curve for constant W/δ. A constraint on engine rating, characterized simply by T/δ = constant (hence $C_D M^2$ = constant), results in a constrained maximum L/D in point B. It is therefore that, as opposed to flying at subcritical Mach numbers, the maximum L/d-condition at high speeds must be defined for specified M, altitude (W/δ) or engine rating (T/δ).

3.4. Conclusions

a) Maximum values of L/D at given subcritical Mach numbers result in a useful optimum condition for wing loading only in the case when constraints are imposed on the engine thrust or the altitude. Both cases indicate an optimum C_L equal to the C_L for maximum wing-alone L/D.
b) For given aircraft design (S constant) maximum values of L/D for subcritical Mach numbers are identical for Mach number and altitude variation. For $M > M_{crit}$ different contrained maxima are obtained, which are all below the subcritical value.
c) For given operating conditions (altitude and speed) the optimum wing loading is defined by the wing-alone optimum C_L, according to:

$$C_L = \sqrt{C_{D_p}/\beta} \quad , \text{ or } \quad \frac{W/S}{C_{D_p} q_o} = \frac{\delta}{\sqrt{\beta C_{D_p}}} \tag{13}$$

Hence, the optimum C_L is independent of the operating conditions.
d) Since high L/D-values are obtained for low Mach numbers and high altitudes, it is obvious that powerplant characteristics (efficiency and weight) will have to be included in all optimizations where the operating conditions are not specified.

4. CRITERION 2: SPECIFIC RANGE

4.1. Engine characteristics

In all considerations where operational conditions (altitude and speed or Mach number) are varied, attention should be paid to the variation of engine characteristics. Specifically the overall powerplant efficiency is an important variable. It is defined as follows:

$$\eta = \frac{\text{propulsive thrust power}}{\text{fuel heat input}} = \frac{TV}{FH} \tag{14}$$

In the case of turbojet or turbofan powered aircraft the engine fuel consumption is usually expressed in terms of Thrust Specific Fuel Consumption (TSFC):

$$C_T = \frac{F}{T} = \frac{V}{\eta H} \tag{15}$$

In the case of propeller-powered aircraft the overall efficiency is the product of the engine efficiency and the propeller efficiency:

$$\eta = \eta_{prop} \cdot \frac{P_{br}}{FH} = \frac{TV}{P_{br}} \frac{P_{br}}{FH} \tag{16}$$

In many classical theories the aircraft flight performance is optimized either for constant η (propeller aircraft) or for $\eta \doteqdot M$ (pure jet aircraft). A more general approach in Ref. 12 will be summarized here, since it is valid for both propeller- and jet-powered aircraft, and takes into account compressibility effects as well.

The distance travelled per unit fuel mass consumed is denoted "specific range" and is given by (Ref. 12):

$$\frac{V}{F} = \eta \frac{L}{D} \frac{H}{W} \tag{17}$$

In this expression the non-dimensional parameter η L/D is a useful non-dimensional quantity, defining the specific range for a given All-Up Weight. The overall efficiency is affected by engine rating (r.p.m.), altitude and speed (or Mach number). In the case of turbojet and turbofan aircraft the variation of η with Mach number is most significant, although it is also affected by engine rating and altitude. The analysis in Ref. 12 involves the logarithmic derivative of η with respect to M,

$$\eta_M \mathrel{\hat{=}} \left.\frac{d \log \eta}{d \log M}\right|_{T/\delta} = \left.\frac{M}{\eta} \frac{d\eta}{dM}\right|_{T/\delta} \tag{18}$$

The numerical value of η_M for subsonic speeds is normally between 0 and 1. It depends to a high degree on the specific engine thrust, and is generally of the order of 0.8 fur pure jets, 0.5 to 0.6 for high bypass engines and close to zero for turboprops.

4.2. Maximum specific range for $M < M_{crit}$

Partial optima derived in Ref. 12, which can be simplified for the case of a parabolic drag polar and when η is a function of the Mach number only. In this case constrained optima are obtained for specified altitude or engine rating.

a) Given altitude (W/δ): The speed for max η L/D is obtained from Ref. 12:

$$C_L = C_{L_{MD}} \left(\frac{2 - \eta_M}{2 + \eta_M}\right)^{\frac{1}{2}} \tag{19}$$

For the classical case of pure jet aircraft, when C_T is assumed constant (i.e. independent of M), it is found that $\eta \doteq M$, hence $\eta_M = 1$. This results in:

$$C_L = C_{L_{MD}}/\sqrt{3} \qquad (C_T = \text{constant}) \tag{19a}$$

For the classical case of propeller aircraft, we get $C_L = C_{L_{MD}}$, since η is assumed constant.
Eq. (19) may be combined with the optimum C_L defining the wing loading for max. L/D. This makes sense, because the engine characteristics are not directly related to the aircraft L/D. From eqs. (13) and (19) we obtain an optimum wing area:

$$C_{D_p} S = \frac{2 - \eta_M}{2\eta_M} \Delta \tag{20a}$$

and an optimum Mach number:

$$M = 2 \left(\frac{1}{\gamma} \frac{W/\delta}{p_o \Delta} \frac{\eta_M}{2 - \eta_M} \sqrt{\beta C_{D_p}}\right)^{\frac{1}{2}} < M_{crit} \tag{20b}$$

Although η_M is a function of M, it does usually not vary strongly in the range of Mach numbers of interest, and eq. (20) is therefore an explicit solution of remarkable simplicity. It says that for $\eta_M < 1$ (C_T = constant) and $\eta_M > 0$ (propeller a/c) the optimum wing profile drag area should be between 50% and 100% of the parasite drag area, resp., dependent on η_M (type of engine). However, this case has limited validity, since the engine rating is also variable and a check should be made to varify that M according to (20b) is not greater than the max. Mach number in horizontal flight.

b) Given engine rating (T/δ): The optimum altitude and speed are obtained from Ref. 12 in terms of an optimum C_L:

$$C_L = C_{L_{MD}} (1 + \eta_M)^{-\frac{1}{2}} \tag{21}$$

Again this can be combined with eq. (13) to find an optimum wing area:

$$C_{D_p} S = \Delta/\eta_M \tag{22a}$$

The corresponding optimum wing loading, altitude, Mach number and L/D ratio are:

$$\frac{W/S}{C_{D_p} q_o} = 2(2 + \eta_M) \frac{W}{T/\delta} \tag{22b}$$

$$\frac{\delta}{\sqrt{\beta\, C_{D_p}}} = (2 + \eta_M)\,\frac{W}{T/\delta} \tag{22c}$$

$$M = \left(\frac{1}{\gamma}\,\frac{T/\delta}{p_o\Delta}\,\frac{2\eta_M}{2 + \eta_M}\right)^{\frac{1}{2}} < M_{crit} \tag{22d}$$

$$L/D = \left\{(2 + \eta_M)\,\sqrt{\beta\, C_{D_p}}\right\}^{-1} \tag{22e}$$

and the range parameter is obtained from the overall efficiency which corresponds to eq. (22d).

4.3. Maximum Specific Range for $M \geq M_{crit}$

The general case of aircraft performance at Mach numbers above the critical value is significant for high-speed turbojet and turbofan aircraft because their optimum cruise conditions are found in the drag rise (Ref. 12). This is illustrated in Fig. 5, which shows curves of constant range parameters in a C_L vs. M plane. The various partial optima (curves I through IV) as derived in Ref. 12 are defined as follows:

I : optimum C_L, given M
II : optimum M, given C_L
III : optimum C_L and M, constant W/δ (hence $C_L M^2$)
IV : optimum C_L and M, constant T/δ (hence $C_D M^2$)

Point A denotes the unconstrained optimum flight condition, while points B and C refer to optima with constraints on the altitude (W/δ) and engine rating (T/δ). Approximate expressions for curves I and II are:

$$\text{I} : \left.\frac{\partial C_D}{\partial C_L}\right|_M = \frac{C_D}{C_L} \quad \text{or} \quad C_L = C_{L_{MD}} = \sqrt{C_{D_o}(M)/\beta(M)} \tag{23a}$$

$$\text{II} : \left.\frac{\partial C_D}{\partial M}\right|_{C_L} = \eta_M\,\frac{C_D}{M} \tag{23b}$$

where it is to be noted that $C_{L_{MD}}$ is a function of M: it decreases significantly in the drag rise.

In addition to the case of constant wing area, a condition can be derived for the optimum wing loading. Similar to eq. (13) we get for each Mach number and altitude:

$$C_L = \sqrt{C_{D_p}(M)/\beta(M)} \tag{24}$$

This result is again incompatible with eq. (23a) and therefore we have to consider either a constrained optimum condition, or the effect of altitude on the engine size required. The latter case will be treated in the following chapter.

4.4. Conclusions

a) Conditions for maximum specific range result in high Mach numbers for jet aircraft due to the favourable effect of Mach number increase on the overall efficiency. Unconstrained optimum flight conditions are found in the drag rise.

b) For subcritical speeds a combination of optimum wing loading and optimum flight conditions can only be obtained by constrained optimization. The optimum wing profile drag area appears to be proportional to the parasite drag area. The factor of proportionality, which depends mainly on the term η_M, varies widely for the different cases of W/δ or T/δ constant. Only the constraint on T/δ ensures flight within available thrust limitations.

c) The optimum Mach number and altitude increase with growing engine size (T/δ), resulting in a steadily increasing V/F. Compressibility effects will ultimately reverse this trend.

5. CRITERION 3: FUEL PLUS POWERPLANT WEIGHT FRACTION

5.1. Derivation of the powerplant function F_P

In the previous paragraph it was concluded that for subcritical speeds the specific range increases with engine size, provided wing loading, altitude and Mach number are optimized. However, engine weight, nacelle size and drag also increase with increasing engine size and it becomes desirable to account properly for the engine(s) plus nacelle(s) installations penalties. In this paragraph the simple case will be considered of the combined fuel plus powerplant installation weight (including nacelle structure) as a fraction of the mean All-Up Weight ($\bar{W}$) during cruising flight.

Matching the powerplant to the airframe drag is effected by 'rubberizing' the engines, assuming a constant thermodynamic cycle (rating) and variable air mass flow. The variation of the engine and nacelle size are taken into account by subdividing the aircraft drag into airframe drag and installed engine nacelle drag as discussed in chapter 2.
A non-dimensional thrust lapse parameter τ is introduced to allow for the rubberizing process:

$$\tau = \frac{T/\delta}{T_{to}} \qquad (25)$$

In accordance with eq. (2) this parameter can be calculated from the net engine thrust and the installation drag, as follows:

$$\tau = \frac{T_{net}/\delta}{T_{to}} - q_o \frac{(C_D S)_{en}}{T_{to}} \qquad (26)$$

For an isothermal atmosphere (e.g. the standard stratosphere) T_{net}/δ, and hence τ, are independent of the pressure altitude. Therefore, when the pressure altitude is varied at constant Mach number $d\tau/d\delta = 0$ for the rubberized engines. The weight of fuel and engine/nacelle installation are as follows:

a) Fuel weight fraction: a good approximation is found by taking the specific range according to eq. (17) for the mean aircraft weight. The fuel weight fraction is thus approximately:

$$\frac{W_f}{\bar{W}} = \frac{R/R_H}{\eta} \frac{C_D}{C_L} \qquad (R_H \mathrel{\hat{=}} \frac{H}{g} \approx 4300 \text{ km for jet engine fuel}) \qquad (27)$$

b) Powerplant/nacelle weight fraction: this is determined by the specific weight μ_T:

$$\mu_T \mathrel{\hat{=}} \frac{\text{engine plus nacelle weight}}{\text{take-off thrust (ISA, S.L.)}} = \frac{W_{en}}{T_{to}} \qquad (28)$$

Hence, for engines sized for the steady cruising condition:

$$\frac{W_{en}}{\bar{W}} = \mu_T \frac{T_{to}}{T} \frac{T}{\bar{W}} = \frac{\mu_T}{\tau\delta} \frac{C_D}{C_L} \qquad (29)$$

The factor μ_T is assumed independent of the engine size, and eq.(29) indicates clearly the increase of the powerplant weight factor with increasing altitude (decreasing δ) associated with the thrust lapse of the turbine engine.

Summation of eqs. (27) and (29) yields:

$$\bar{\mu}_{f,en} \mathrel{\hat{=}} \frac{W_f + W_{en}}{\bar{W}} = F_P \frac{C_D}{C_L} = F_P \left(\frac{C_{D_w}}{C_L} + \frac{q\Delta}{\bar{W}}\right) \tag{30}$$

where the powerplant function F_P is defined as follows:

$$F_P \mathrel{\hat{=}} \frac{R/R_H}{\eta} + \frac{\mu_T}{\tau\delta} \tag{31}$$

and the wing drag-to-lift ratio is:

$$C_{D_w}/C_L = C_{D_p}(M)/C_L + \beta(M)\, C_L \tag{32}$$

For a specified flight condition the function F_P can be used as a figure of merit for weighing the relative importance of the overall efficiency η (or C_T) specific weight (μ_T), thrust lapse with altitude and speed, and engine installation drag. If, for example, a study is made of bypass ratio variation effects on aircraft propulsion, the minimum value of F_P can be useful as a first indication of an 'optimum' bypass ratio, independent of the application of the engines in a particular aircraft design. This criterion does apply only when the engines are sized for cruising flight.

5.2. Optimum conditions for given Mach number

Fig. 6 gives an example of 'isomerit' contours for $\mu_{f,en}$ versus reduced wing loading and pressure altitude, calculated according to eqs. (30) through (32). Optimum values of wing loading, pressure altitude and Mach number, resulting in minimum $\mu_{f,en}$, can be found by setting the respective partial derivatives equal to zero. The results are as follows:

a) Optimum wing loading. Since for a given flight condition F_P is fixed, this partial optimum is identical to eq. (9a):

$$\left.\frac{\partial\bar{\mu}_{f,en}}{\partial(W/S)}\right|_{\delta,M} = 0 \rightarrow C_L = \sqrt{C_{D_p}/\beta} \qquad \text{(Curve I)} \tag{33a}$$

The wing loading is obtained from multiplication by the dynamic pressure q.

b) Optimum altitude. Differentiation of eq. (30) w.r.t. δ, after substitution of $C_L = \bar{W}/qS$ yields:

$$\left.\frac{\partial\bar{\mu}_{f,en}}{\partial\delta}\right|_{M,\bar{W}/S} = 0 \rightarrow C_L = C_{L_{MD}} \left(1 + 2\, W_{en}/W_f\right)^{-\frac{1}{2}} \tag{33b}$$

The pressure altitude can be obtained from this equation:

$$\delta = \delta_{MD} \left(1 + \frac{2}{\delta}\frac{\mu_T}{\tau}\frac{\eta}{R/R_H}\right)^{\frac{1}{2}} \qquad \text{(Curve II)} \tag{33c}$$

with $\delta_{MD} \mathrel{\hat{=}} \dfrac{\bar{W}/S}{q_o C_{L_{MD}}} = \dfrac{\bar{W}/S}{q_o \sqrt{C_{D_o}/\beta}}$

The solution of this equation can be speeded up by means of a first-order approximation:

$$\delta \approx \delta_{MD} + 0.7 \frac{\mu_T}{\tau} \frac{\eta}{R/R_H} \tag{33d}$$

The result of the present partial optimization (for given Mach number and wing loading) is essentially similar to the criterion of Küchemann and Weber derived in Ref. 8. It applies to subcritical speeds as well as in the drag rise, provided C_{D_o} and β are adapted to each Mach number.

c) Optimum altitude and wing loading for given C_L. Differentiation of eq. (30) w.r.t. δ, for given C_L yields:

$$\left.\frac{\partial \bar{\mu}_{f,en}}{\partial \delta}\right|_{C_L,M} = 0 \quad \rightarrow \quad \delta = \left\{\frac{\bar{W}}{q_o \Delta} \frac{\mu_T}{\tau} \frac{\eta}{R/R_H} (C_{D_p} + \beta C_L)\right\}^{\frac{1}{2}} \tag{33e}$$

(Curve III)

and the corresponding wing loading is obtained from eq. (1).

d) Unconstrained optimum wing loading and altitude. The partial optima derived above can be combined in several ways. A useful result is obtained when eqs. (33a) and (33c) or (33e) are combined. The resulting optimum wing loading and altitude for minimum $\bar{\mu}_{f,en}$ are:

$$\frac{\bar{W}/S}{C_{D_p} q_o} = \frac{\delta}{\sqrt{\beta C_{D_p}}} = \left(2 \frac{\bar{W}}{q_o \Delta} \frac{\mu_T}{\tau} \frac{\eta}{R/R_H} / \sqrt{\beta C_{D_p}}\right)^{\frac{1}{2}} \tag{34a}$$

From combination of these equations an optimum wing area is obtained:

$$C_{D_p} S = \left(2\Delta \frac{\bar{W}}{q_o} \frac{\tau}{\mu_T} \frac{R/R_H}{\eta} \sqrt{\beta C_{D_p}}\right)^{\frac{1}{2}} \tag{34b}$$

and the engine sized to this condition has the following thrust:

$$\tau \frac{T_{to}}{\bar{W}} = \frac{T/\delta}{\bar{W}} = \frac{q_o \Delta}{\bar{W}} + \left(2 \frac{q_o \Delta}{\bar{W}} \frac{\tau}{\mu_T} \frac{R/R_H}{\eta} \sqrt{\beta C_{D_p}}\right)^{\frac{1}{2}} \tag{34c}$$

Finally, the corresponding minimum $\mu_{f,en}$ is:

$$(\bar{\mu}_{f,en})_{min} = \frac{W_f + W_{en}}{\bar{W}} = \left\{\left(2 \frac{R/R_H}{\eta} \sqrt{\beta C_{D_p}}\right)^{\frac{1}{2}} + \left(\frac{\mu_T}{\tau} \frac{q_o \Delta}{\bar{W}}\right)^{\frac{1}{2}}\right\}^2 \tag{34d}$$

In Fig. 6 this optimum is denoted point A.

This elegant closed-form analytical solution is characterized by a pronounced sensitivity of the design characteristics to the design range. For long range aircraft the result is a high cruise altitude, low wing loading and large engines. For example, pont A in Fig. 6 is characterized by a mean altitude of 14.200 m, ($\bar{\delta}$ = 0.14) a mean wing loading of 3067 N/m^2 and a thrust-to-weight ratio $T_{to}/\bar{W}$ = 0.425, or T_{to}/W_{to} = 0.36. It will be clear that more realistic results are obtained when wing structural weight variation is also taken into consideration. From Fig. 6 it is also clear, however, that eq. (34d) gives a rather good approximation of actual values for $\mu_{f,en}$ within a fairly wide range of wing loadings and altitudes away from the optimum values. This result is therefore useful for a first approximation of $\mu_{f,en}$, even if ultimately the design parameters will be different from eq. (34).

It is worth noting that in all the equations defining the optimum conditions, except (34d), the combination η/τ occurs. Since both the installed overall efficiency and the cruise thrust are reduced by the same percentage installation (drag) losses, these losses cancel in η/τ and therefore in most formulae η and the thrust may be obtained directly from the uninstalled engine performances. Obviously, losses have to be accounted for in the final result for $\bar{\mu}_{f,en}$, eq. (34d).

5.3. Effects of Mach number variation

Mach number variation has the following major effects:

a) The aerodynamic characteristics of the airframe are modified above M_{crit}, see section 3.3.
b) The engine efficiency η varies with Mach number. For jet-propelled a/c it increases steadily at subsonic speeds. As a consequence, the fuel burned generally reaches a minimum near the drag-critical Mach number.
c) The engine size required attains a minimum value for L/D-max, provided thrust does not vary with speed. For most turbofan engines thrust does not vary much at high subsonic speeds, and we will therefore assume $d\tau/dM = 0$. A more general solution can be found in Ref. 1.

a) Constant wing loading and C_L, or constant q. In this case variation in Mach number is accompanied by altitude variation, so that δM^2 = constant. The airframe L/D is constant and $\bar{\mu}_{f,en}$ has a minimum value for F_P minimal:

$$\left.\frac{\partial\bar{\mu}_{f,en}}{\partial M}\right|_{W/S,C_L} = 0 \rightarrow \left.\frac{\partial F_P}{\partial M}\right|_q = 0 \rightarrow \frac{W_{en}}{W_f} = \frac{1}{2}\eta_M \qquad (35a)$$

This remarkably simple result is in accordance with Pearson's criterion (Ref. 4), stating that for pure jets (i.e. $\eta_M = 1$) the optimum engine weight equals half the optimum fuel weight. The optimum cruise altitude is obtained from substitution of (35a) into (31):

$$\delta = \frac{2}{\eta_M}\frac{\mu_T}{\tau}\frac{\eta}{R/R_H} \qquad (35b)$$

and for given C_L the optimum Mach number is:

$$M = \left(\frac{\bar{W}/S}{\gamma p_o C_L}\,\frac{\eta_M R/R_H}{\eta}\,\frac{\tau}{\mu_T}\right)^{\frac{1}{2}} < M_{crit} \qquad (35c)$$

Obviously the last equation must be solved iteratively, since $\eta = f(M)$.

b) Constant altitude and wing loading. For parabolic drag polars and subcritical speeds the following result is found (Ref. 1):

$$\left.\frac{\partial\bar{\mu}_{f,en}}{\partial M}\right|_{\delta,W/S} = 0 \rightarrow C_L = C_{L_{MD}}\left(\frac{1-\frac{1}{2}\eta_M + W_{en}/W_f}{1+\frac{1}{2}\eta_M + W_{en}/W_f}\right)^{\frac{1}{2}} \qquad (35d)$$

where

$$C_{L_{MD}} = \sqrt{C_{D_o}/\beta} \quad \text{and} \quad W_{en}/W_f = \frac{\mu_T}{\tau\delta}\,\frac{\eta}{R/R_H}$$

Comparing this result with eq. (19) reveals that the optimum flight condition is now closer to the minimum drag condition.

c) Optimum Mach number, wing area and altitude. Combination of eqs. (34) and (35) results in an unconstrained optimum for $M < M_{crit}$. However, an explicit solution for M is only possible when η is explicitly expressed as f(M). Assuming, for example, an exponential relationship:

$$\eta = C_\eta M^{\eta_M} \tag{36}$$

the following unconstrained optimum is found:

$$C_{D_p} S = \Delta/\eta_M \tag{37a}$$

$$M = \left(\frac{1}{\gamma}\,\frac{\bar{W}}{p_o\Delta}\,\frac{\tau}{\mu_T}\,\frac{R/R_H}{C_\eta}\,\eta_M{}^2\,\sqrt{\beta C_{D_p}}\right)^{1/(2+\eta_M)} < M_{crit} \tag{37b}$$

Correspondingly the optimum L/D ratio and mass fraction are:

$$L/D = \left\{(2+\eta_M)\,\sqrt{\beta\, C_{D_p}}\right\}^{-1} \tag{37c}$$

$$\bar{\mu}_{f,en} = \frac{W_f + W_{en}}{\bar{W}} = 2\,\frac{R/R_H}{\eta}\,\left(1 + \frac{1}{2}\,\eta_M\right)^2 \sqrt{\beta\, C_{D_p}} \tag{37d}$$

where η is obtained from eqs. (36) and (37b). The optimum cruise altitude is given by eq. (35b) and finally, the optimum engine thrust is:

$$\frac{T_{to}}{\bar{W}} = \frac{T/\delta}{\tau\bar{W}} = \eta_M\left(1 + \frac{1}{2}\,\eta_M\right)\,\frac{R/R_H}{\mu_T\eta}\,\sqrt{\beta C_{D_p}} \tag{37e}$$

5.4. Conclusions

a) All optimum values are sensitive to the design range. For long ranges the optimum altitude and Mach number are high, the wing loading is low, and big engines are required.

b) As opposed to the case of max. V/F an unconstrained optimum Mach number is found also if the results of section 5.3 are strictly valid for subcritical Mach numbers and a check should always be made whether eq. (37b) results in a subcritical Mach number. On the other hand, numerical examples have shown that the optimization of altitude and wing loading for given Mach number (Section 5.2) remains essentially valid in the drag rise, on the provision that the drag polar is also varied with Mach number.

c) The example on Fig. 6 shows that low wing loadings are found in the absolute optimum. It becomes therefore logical to take into account the weight penalty of a large wing structure; this will be done in the next chapter.

d) There is obviously an important effect of Mach number variation on the aircraft's productivity and therefore on its economic performance. In addition the aerodynamic design of the wing is very sensitive to the cruise Mach number and parameters such as wing section relative thickness and sweep angle will be affected. In practical design the selection of a cruise Mach number is therefore based on considerations which have not been taken into account in this chapter.

e) The sensitivity for deviations from the optimum given by eqs. (34) is not very large. In particular eq. (34d) appears to be a useful first approximation for the fuel and engine mass fraction, for given M, if no decision has been taken as regards wing loading and altitude.

6. CRITERION 4: WING, FUEL PLUS POWERPLANT WEIGHT FRACTION

In view of the effect of wing area variation on wing structural weight, it appears desirable to consider a case where the wing weight fraction has been added to those of the fuel and powerplant installation:

$$\bar{\mu}_{w,f,en} = \frac{W_w + W_f + W_{en}}{\bar{W}} = W_w/\bar{W} + F_P\, C_D/C_L \tag{38}$$

where $\bar{\mu}_{f,en} = F_P\, C_D/C_L$ is defined by eqs. (30) through (32).

6.1. The wing weight fraction

The wing weight fraction is a complex function of many variables - mainly wing-related - and cannot be calculated accurately in the conceptual design stage, since many details of the structural design have to be settled downstream in the design process. In the present chapter only variations of the wing area are considered and it will be assumed that within a reasonable range of wing areas the wing structural weight can be linearized around a baseline design, denoted by an "^":

$$W_w = \hat{W}_w + \frac{dW_w}{dS}(S - \hat{S}) \tag{39a}$$

and hence:

$$\bar{\mu}_w \triangleq \frac{W_w}{\bar{W}} = \hat{\mu}_{w_{ref}} + \frac{dW_w/dS}{\bar{W}/S} \tag{39b}$$

where $$\hat{\mu}_{w_{ref}} = \frac{\hat{W}_w}{\bar{W}} - \frac{dW_w}{dS}/(\bar{W}/\hat{S}) \tag{39c}$$

The order of magnitude of dW_w/dS can be obtained by assuming an exponential relationship:

$$W_w = \text{constant} \times S^{x_S} \tag{40}$$

Differentiation yields:

$$\frac{dW_w}{dS} = x_S \frac{W_w}{\bar{W}} \frac{\bar{W}}{S} \approx x_S \frac{\hat{W}_w}{\bar{W}} \cdot \frac{\bar{W}}{\hat{S}} \tag{41a}$$

hence: $$\bar{\mu}_{w_{ref}} \approx \frac{\hat{W}_w}{\bar{W}}(1 - x_S) \tag{41b}$$

Theoretically the exponent of S for the primary structure is ½, for constant wing shape and stress levels. In practice it is somewhat higher when secondary weight terms are included. The wing weight for a modern airliner is typically 10 % of the MTOW. If we assume: $\hat{W}_w = 0.12\ \bar{W}$, and $x_S = 0.6$, then we obtain: $dW_w/dS \approx 0.072\ \bar{W}/\hat{S}$, and $\bar{\mu}_{w_{ref}} \approx 0.048$ as typical numbers.

6.2. Optimum wing loading and altitude for given M

The weight fraction under consideration becomes, after subsitution of eq. (39):

$$\bar{\mu}_{w,f,en} = \bar{\mu}_{w_{ref}} + \frac{dW_w/dS}{\bar{W}/S} + F_P\, C_D/C_L \tag{42}$$

a) Optimum wing loading for constant altitude.
The optimum wing loading is expressed in terms of an optimum lift coefficient as follows:

$$\left.\frac{\partial\bar{\mu}_{w,f,en}}{\partial(W/S)}\right|_{\delta} = 0 \rightarrow C_L = \left\{\frac{1}{\beta}\left(C_{D_P} + \frac{dW_w/dS}{\delta q_o F_P}\right)\right\}^{\frac{1}{2}} \quad \text{(Curve I)} \tag{43}$$

The wing loading is obtained by multiplying C_L with δq_o.

b) Optimum altitude for constant wing loading.
Since for given wing loading the wing weight fraction is constant, the same optimum altitude is obtained as for criterion 3: eq. (33b), curve II.

c) Optimum altitude and wing loading for given C_L
Eliminating the wing loading in eq. (42) with eq. (1) and differentiation results in the optimum altitude:

$$\left.\frac{\partial\bar{\mu}_{w,f,en}}{\partial\delta}\right|_{C_L} = 0 \rightarrow \delta = \left[\frac{\bar{W}}{q_o\Delta}\,\frac{\eta}{R/R_H}\left\{\frac{\mu_T}{\tau}\left(\frac{C_{D_P}}{C_L} + \beta C_L\right) + \frac{dW_w/dS}{q_o C_L}\right\}\right]^{\frac{1}{2}} \tag{44}$$

(Curve III)

d) Unconstrained ("absolute") optimum wing loading and altitude.
Combination of the partial optima derived above into an absolute (unconstrained) optimum results in unwieldly expressions. However, a good approximation is found by substituting $C_L = \sqrt{C_{D_P}/\beta}$ into eq. (44), with the following result:

$$\frac{\delta}{\sqrt{\beta C_{D_P}}} = \left\{\frac{\bar{W}}{q_o\Delta}\,\frac{\eta}{R/R_H}\left(2\,\frac{\mu_T}{\tau} + 0.8\,\frac{dW_w/dS}{q_o C_{D_P}}\right)/\sqrt{\beta C_{D_P}}\right\}^{\frac{1}{2}} \tag{45}$$

(point A)

If necessary a second approximation can be obtained by substituting this relative ambient pressure into eqs. (33) and (43), and the resulting C_L is substituted into eq. (44). The optimum wing loading is given by:

$$\frac{\bar{W}}{S} = q_o\,\delta\,C_L \tag{46}$$

Substitution of these optima for $\bar{W}/S$, C_L and δ into eq. (42) yields the minimum value of $\bar{\mu}_{w,f,en}$, which is approximately:

$$(\bar{\mu}_{w,f,en})_{min} = \bar{\mu}_{w_{ref}} + (\bar{\mu}_{f,en})_{min} +$$

$$+ 0.5\,\frac{dW_w/dS}{C_{D_P} q_o}\left(\frac{q_o\Delta}{\bar{W}}\,\frac{R/R_H}{\eta}\,\frac{\tau}{\mu_T}\sqrt{\beta C_{D_P}}\right)^{\frac{1}{2}} \tag{47}$$

where $(\bar{\mu}_{f,en})_{min}$ is the minimum fuel plus engine weight fraction, defined by eq. (34d).

6.3. Conclusions

a) For the case under consideration the optimum C_L and δ are both noticeably higher than for the case of minimum fuel plus engine weight. The difference between the two cases varies between 20% and 50%, typically, dependent on the range. Since for short-range aircraft the wing weight dominates, the unconstrained optimum wing loading is higher than for long-range aircraft. Relative to the case of minimum fuel plus engine installation weight the optimum wing loading increases by 80 to 120 %, typically.

b) For large aircraft dW_w/dS is considerably larger than for small aircraft. As a result of this there is a size effect, resulting in higher optimum wing loading for large aircraft, as compared to small aircraft cruising at the same Mach number. This effect is counteracted by the Reynolds number effect on C_{D_p}, which reduces the optimum C_L for large aircraft somewhat.

7. CRITERION 5: PAYLOAD WEIGHT FRACTION

The ratio of payload weight to take-off weight is used frequently to optimize aircraft designs. Apart from the weight contributions considered in the previous criteria, which are directly affected by changes in the design variables, certain other components must also be considered which vary indirectly as a consequence of take-off weight variation. Typical structural items with weights dependent on the All-Up Weight are, for example, the undercarriage, the wing box, and wing/fuselage connections. The All-Up Weight at take-off (MTOW) is therefore written as:

$$W_{to} = W_{fix} + W_{var} + W_{en} + W_f + W_{p\ell} \tag{48}$$

W_{fix} denotes the summation of weight components, which are considered independent of the variables considered and of the MTOW. For specific payload this could be the major part of the fuselage structure, systems, equipment, services and operational items.
W_{var} denotes the summation of all weight components that are considered proportional to the MTOW, and dependent on wing size and shape.
For cruise-sized engines the fuel plus powerplant weight is written analogously to eq. (30) as follows:

$$W_f + W_{en} = F_P \left(\frac{C_{D_w}}{C_L} W_{to} + q\Delta\right) \tag{49}$$

In this case the cruise altitude, wing loading and Mach number are to be defined at the initiation of the cruising flight, when AUW ≈ MTOW. The function F_P is consequently defined as:

$$F_P = \frac{R_{eq}/R_H}{\eta} + \frac{\mu_T}{\tau\delta} \tag{50}$$

Since the fuel and engine installation weight are now expressed as fractions of the MTOW, instead of the mean cruising weight ($\bar{W}$), an equivalent range is now used instead of the actual range. Its value can be calculated with Appendix A of Ref. 1.

Combination of (48) and (49) yields the payload fraction:

$$\frac{W_{p\ell}}{W_{to}} = \frac{1 - \left\{W_{var}/W_{to} + F_P(C_D/C_L)_w\right\}}{1 + W_{fix}/W_{p\ell} + F_P\, q\, \Delta/W_{p\ell}} \tag{51}$$

On the basis of eqs. (50) and (51) the effects of varying wing size and shape, initial cruise altitude and Mach number can be studied analytically or numerically, provided suitable data for W_{var}/W_{to} and $W_{fix}/W_{p\ell}$ are available. Fig. 7 shows an example of a typical result, in the form of a diagram with contours of constant payload fraction. Such a diagram can be enriched by plotting practical design constraints, such as buffet limits or wing tank volume constraints. Moreover, it clearly illustrates the sensitivity of a design w.r.t. the design variables.

7.1. Optimum wing loading and altitude for given M

Analytically derived optima for δ, C_L and W_{to}/S, resulting in minimum MTOW, can be found in Refs. 1 and 2. They appear to be almost identical to the expressions derived in Chapter 6, with the following exceptions:

a) Instead of the wing weight sensitivity (dW_w/dS) the variable weight sensitivity must now be used: dW_{var}/dS. This term also includes tailplane weight variation. If this is limited to the horizontal tailplane only, we have:

$$dW_{var}/dS = dW_w/dS + S_h/S \; dW_h/dS_h \tag{52}$$

b) In all expressions derived in Chapter 6 $\bar{W}$ must be replaced by W_{to} and δ refers to the initial instead of the mean cruise altitude.

7.2. Range for a given payload and MTOW

A closed form expression for the range obtainable with a given payload and MTOW (hence: payload fraction) is obtainable from eq. (51), using the optimum wing loading and cruise altitude. The result is as follows:

$$R_{eq} = \frac{\eta R_H}{2\sqrt{\beta C_{D_p}}} \left[\left\{ 1 - \mu_{var} - \mu_{p\ell}\left(1 + \frac{W_{fix}}{W_{p\ell}}\right) \right\}^{\frac{1}{2}} + M \left\{ \frac{\gamma}{2} \frac{\mu_T}{\tau} \frac{\mu_{p\ell}\Delta}{W_{p\ell}/p_o} \right\}^{\frac{1}{2}} \right]^2$$

$$\text{with } \mu_{var} \mathrel{\hat{=}} \frac{W_{var}}{W_{to}} \text{ and } \mu_{p\ell} \mathrel{\hat{=}} \frac{W_{p\ell}}{W_{to}} \tag{53}$$

The variable weight fraction μ_{var} consists of contributions of items not directly related to the wing weight such as the landing gear weight fraction, and an optimized wing weight term. The latter is obtained in a similar fashion as discussed in par. 6.2, replacing dW_w/dS, $\bar{W}$ and R by dW_{var}/dS, W_{to} and R_{eq}, resp.

7.3. Conclusions

a) All optimum conditions for maximum payload fraction are sensitive to the design range. The results are very similar to the case of minimum $\mu_{w,f,en}$ and knowledge of $W_{fix}/W_{p\ell}$ and W_{var}/W_{to} are not required, except the weight sensitivity dW_{var}/dS.

b) The simple closed-form expression derived for the payload fraction is very useful for a first estimate of the MTOW for given payload. This first estimate is in fact required for application of all design criteria optimization used heretofore.

8. CRITERION 6: MEAN FUEL BURNOFF PER SEAT-KM

8.1. Basic equations

In the previous chapters a number of weight fractions has been used to compare different optima. Although these fractions give useful first-order results, they have the disadvantage that the mean All-Up Weight ($\bar{W}$) or the MTOW (W_{to}) are both not fixed for given payload. In the present and following chapters we will therefore relate certain cost-related items directly to the design-payload, since this is normally a specified quantity.

One of the quality indices used frequently for comparing the efficiency of transport aircraft is the fuel weight consumed per unit distance flown by the aircraft. According to eq. (17) this is, for the mean All-Up Weight:

$$\frac{\overline{Fg}}{V} = \frac{\bar{W}}{\eta\ L/D\ R_H} \qquad (R_H = H/g \approx 4300 \text{ km for jet fuel}) \tag{54}$$

Alternatively, in terms of the mean fuel burnoff per seat km:

$$\frac{\overline{Fg}}{VN_s} = \frac{\bar{W}}{W_{p\ell}} \frac{1}{\eta\ L/D} \frac{W_{p\ell}}{N_s\ R_H} \tag{55}$$

Hence, for given design payload and number of seats, the minimum fuel per seat-km produced corresponds to the maximum value of the parameter $\eta\ L/D * W_{p\ell}/\bar{W}$. On the basis of eq. (51) the payload fraction related to the mean AUW can be written alternatively,

$$\frac{W_{p\ell}}{\bar{W}} = \frac{1 - (W_{var}/\bar{W} + F_P\ C_D/C_L)}{1 + W_{fix}/W_{p\ell}} \tag{56}$$

where the C_D/C_L ratio refers to the whole airframe.

We are therefore interested in the maximum value of:

$$\eta \frac{C_L}{C_D}\left(1 - \frac{W_{var}}{\bar{W}}\right) - \eta\ F_P = \eta \frac{C_L}{C_D}\left(1 - \frac{W_{var}}{\bar{W}}\right) - \left(\frac{R}{R_H} + \eta \frac{\tau\ \mu_T}{\mu\delta}\right) \tag{57}$$

The optima of this term will be derived for constant Mach number, and since η is assumed independent of the altitude (for an isothermal atmosphere), the maximum is to be found of the following term:

$$\frac{C_L}{C_D}\left(1 - \frac{W_{var}}{\bar{W}}\right) - \frac{\mu_T}{\tau\delta} \tag{58}$$

8.2. Optimim wing loading and altitude

The derivation of optimum conditions appears to be simplified by introducing a non-dimensional term for the wing weight sensitivity to variations in the wing area:

$$\omega \mathrel{\hat{=}} \frac{dW_{var}/dS}{q_o C_{D_p}(1 - W_{var}/\bar{W})} \tag{59}$$

Although $W_{var}/\bar{W}$ is not precisely constant due to variation in the wing weight, this is not objectionable, since ω can be found by means of iteration once a value for $\bar{W}/S$ has been found.

a) Optimum wing loading for constant altitude.
In this case only the first term of eq. (58) is varying. The following result is found:

$$\left.\frac{\partial(Fg/VN_s)}{\partial(W/S)}\right|_{\delta} = 0 \rightarrow C_L = \left\{\frac{C_{D_p}}{\beta}\left(1 + \frac{C_D}{\delta C_L}\,\omega\right)\right\}^{\frac{1}{2}} \quad \text{(Curve I)} \tag{60}$$

This is not a solution in closed form, but a generally satisfactory first approximation is found by assuming that C_D/C_L is equal to the value for $\beta C_L{}^2/C_{D_p} = 1$. Series expansion results in

$$\frac{\bar{W}/S}{C_{D_p} q_o} \approx \frac{\delta}{\sqrt{\beta C_{D_p}}}\left(1 + \frac{q_o\Delta}{\bar{W}}\,\omega\right)^{\frac{1}{2}} + \omega \tag{60a}$$

If necessary this wing loading can be used to find a new value for C_D and a second approximation is than obtained with eq. (60).

b) Optimum altitude for constant wing loading
In this case the term $W_{var}/\bar{W}$ is essentially constant and maximization of expression (58) results in:

$$\left.\frac{\partial(Fg/VN_s)}{\partial\delta}\right|_{\bar{W}/S} = 0 \rightarrow C_L = C_{L_{MD}}\left\{1 - \frac{\mu_T}{\tau}\,\frac{C_D{}^2}{C_{D_o}}\,\frac{q_o}{\bar{W}/S\,(1 - W_{var}/\bar{W})}\right\}^{\frac{1}{2}} \tag{61}$$

(Curve II)

If required a first-order solution of this implicit equation can be used by assuming $C_D \approx 1.8\, C_{D_o}$, with the following result:

$$\delta \approx \delta_{MD} + 1.7\,\frac{\mu_T\sqrt{\beta C_{D_o}}}{\tau(1 - W_{var}/\bar{W})} \tag{61a}$$

where δ_{MD} is defined by eq. (33d). Comparing this result with eq. (33d) it is found that the design range is not involved in this optimum condition.

c) Optimum altitude and wing loading for given C_L
For given C_L the wing loading and altitude are directly coupled by means of eq. (1). The maximum of expression (58) occurs when:

$$\left.\frac{\partial(Fg/VN_s)}{\partial\delta}\right|_{C_L} = 0 \rightarrow \frac{\delta}{\sqrt{\beta C_{D_p}}} = \left[\frac{\bar{W}}{q_o\Delta}\,\frac{C_D}{\beta C_L{}^2}\left\{\omega + \frac{C_D}{C_{D_p}}\,\frac{\mu_T}{\tau(1 - W_{var}/\bar{W})}\right\}\right]^{\frac{1}{2}} \tag{62}$$

(Curve III)

d) Unconstrained optimum altitude and wing loading.
Intersection of eq. (60) with (61) or (62) defines the unconstrained optimum. The solution is not readily obtained in a closed-form analytical ex-

pression. Moreover, it is generally found that a very high "optimum" cruise altitude is obtained, similar to the case of minimum $\bar{\mu}_{f,en}$.

e) Optimum altitude and wing loading with constraint on thrust.
Since the present criterion does not consider the consequences of flying at very high altitudes for the installed engine size, it is logical to derive a constrained optimum for the case of a given powerplant installation (hence thrust). For this case the altitude to be obtained is approximately equal to the case considered in par. 3.2.:

$$\frac{\delta}{\sqrt{\beta C_{D_p}}} = \frac{2\bar{W}}{T/\delta - q_o\Delta} \tag{63}$$

Here it has been assumed that the optimum L/D-ratio, corresponding to eq. (59), deviates only slightly from the maximum value, given by eq. (12). Substitution of eq. (63) into the optimum wing loading (eq. 60) results in:

$$C_L \sqrt{\beta/C_{D_p}} = \left(1 + \omega \frac{T/\delta}{\bar{W}}\right)^{\frac{1}{2}} \tag{64}$$

and the wing loading is obtained by multiplication of (63) with q_o and (64) This result can be compared with eq. (11a), which shows that the optimum wing loading has increased relative to the value for L/D-max by a value of about 18%, typically (for $\omega \approx 1.3$ and $T/\delta\bar{W} = 0.3$).

8.3. Conclusions

a) The optimum wing loading for given altitude is, according to eq. (59), always between the values for maximum L/D (eq. 9b) and for maximum payload fraction (eq. 43).

b) As opposed to most previous cases the various optima for minimum fuel burnoff per seat-km are not affected by the design range.

c) The most significant optimum condition is one with a constraint on the thrust, i.e. minimum fuel burnoff per seat-km for given engine.

d) It can be shown that, for given Mach number, the unconstrained optimum is identical to the case of minimum drag and thrust per unit payload.

9. CRITERION 7: FUEL PLUS ENGINE WEIGHT IN RELATION TO PAYLOAD

In the previous chapter it was concluded that the fuel burnoff in relation to payload should be optimized with a constraint on the engine thrust. It is therefore a logical next step to consider the fuel plus engine weight for given payload, when the engines are "rubberized", so that for each drag level the thrust balances the drag, at a constant engine rating. This case is analoguous to criterion 3 (Chapter 5), but here we relate the fuel plus engine installation weight to the payload, which is a fixed and specified quantity, as opposed to the All-Up Weight.

For a specified design payload the fuel plus engine installation weight is obtained from:

$$\frac{W_f + W_{en}}{W_{p\ell}} = \mu_{f,en} \frac{\bar{W}}{W_{p\ell}} = \frac{1 + W_{fix}/W_{p\ell}}{\left(1 - \frac{W_{var}}{\bar{W}}\right) \frac{C_L/C_D}{F_P} - 1} \tag{65}$$

The minimum value of this term can be obtained by maximizing the expression:

$$\frac{C_L}{C_D}\,(1 - W_{var}/\bar{W})/F_P \tag{66}$$

For specified (constant) Mach number the results are as follows:

a) Optimum wing loading for given altitude.
For this case the value of F_P is constant and eq. (60) applies to this case as well (Curve I).

b) Optimum altitude for given wing loading.
In this case $W_{var}/\bar{W}$ is constant and the same expression is found as for criterion 3: eqs. (33b) and (33c), (Curve II).

c) Optimum altitude and wing loading for given C_L
Minimization of expression (65) gives:

$$\partial(W_f + W_{en})/W_{p\ell}\Big|_{C_L} = 0 \rightarrow \delta = \left\{\frac{W}{q_o}\,\frac{C_D}{C_L}\left(\frac{\mu_T/\tau}{F_P} + \omega\,\frac{C_{D_p}}{C_L}\right)\right\}^{\frac{1}{2}} \tag{67}$$

(Curve III)

d) Unconstrained optimum altitude and wing loading (point A).
The combination of cases a), b) and c) above can be approximated by introducing a new term f(R):

$$f(R) \cong \left\{1 + \left(\frac{q_o\Delta}{2\bar{W}}\,\frac{\mu_T}{\tau}\,\frac{\eta}{R/R_H}\,/\,\sqrt{\beta C_{D_p}}\right)^{\frac{1}{2}}\right\}^2 \tag{68a}$$

The resulting approximations are:

$$\frac{\delta}{\sqrt{\beta C_{D_p}}} = \left\{\frac{2\bar{W}}{q_o\Delta}\left(\frac{\mu_T}{\tau}\,\frac{\eta}{R/R_H}\,/\,\sqrt{\beta C_{D_p}} + f(R)\omega\right)\right\}^{\frac{1}{2}} \tag{68b}$$

where ω is defined according eq. (59). The optimum wing loading is obtained by substitution of (68b) into eq. (60):

$$\frac{\bar{W}/S}{C_{D_p} q_o} = \left\{2\left(\frac{\bar{W}}{q_o\Delta} + \omega\right)\left(\frac{\mu_T}{\tau}\frac{\eta}{R/R_H}/\sqrt{\beta C_{D_p}} + f(R)\right)\right\}^{\frac{1}{2}} + \omega \qquad (68c)$$

From this result it appears that due to addition of the terms, which are proportional to ω, the optimum relative pressure and wing loading are considerably higher than for the case of minimum fuel plus engine weight fraction, but lower than for the case of maximum payload fraction (minimum W_{to}).

Conclusions.

a) The optimum conditions are much more realistic as compared to the case of minimum $\bar{\mu}_{f,en}$, and probably also more significant than for maximum payload fraction.

b) The presence of the ω-related term makes the optimum altitude and wing loading less sensitive to the design range as compared to $\mu_{f,en}$. Practical application to long-range aircraft learns that eqs. (68) result in quite realistic values for optimum δ and $\bar{W}/S$.

An objection against the present criterion is that fuel and engine installation weight are simply added, whereas in practical design optimization the significance of fuel consumption and engine size variations should be expressed in terms of an economical criterion. This will be worked out in the next chapter.

10. CRITERION 8: DIRECT OPERATING COSTS (DOC)

10.1. Derivation of the basic expression to be minimized

The DOC are very often used as a means to compare the qualities of different designs, designed for the same mission. As such, they are a useful tool to weigh the relative differences in fuel consumption, empty weight and cost of production, engine costs, etc.
There is no generally accepted method for defining DOC, nor for its computation. The relationship used is therefore a very general and basic expression, featuring a number of statistical factors which are assumed to be known, at least approximately.

The Direct Operating Costs (DOC) during the useful service life (U hours) of a transport aircraft can be expressed as follows:

$$DOC = C_a N_s + C_f g \frac{\Sigma R}{V/F} + C_{en} T_{to} + C_h U + C_{fl} N_{fl} \tag{69}$$

where it has been assumed that the mean value of V/F is a fixed proportion of the cruise value at the cruise design point for which the engines are sized. The following symbols have been used:

- C_a : airframe cost (\$/seat), including spare parts, insurance and interest.
- C_f : average specific fuel price (\$/unit weight) divided by the reduction in mean V/F relative to the design value
- C_{en} : specific engine installation costs (\$/unit thrust), including spares, insurance and interest
- N_s : number of seats
- ΣR : total distance travelled ($\Sigma R = U \bar{V}_b$)
- $\bar{V}_b$: mean block speed
- C_h : maintenance, flight crew and other hourly costs (\$/hour)
- C_{fl} : trip-related costs for landing, maintenance, etc. (\$/flight)
- N_{fl} : total number of flights ($N_{fl} = \Sigma R/\bar{R}_b$)
- $\bar{R}_b$: average sector distance

It is to be noted that eq. (69) considers the aircraft cost as a fixed cost, proportional to the number of seats installed, plus a variable cost which is porportional to the installed thrust. In a more accurate analysis one might also take into account variations of the aircraft cost due to variations in the airframe geometry, noticeably variations in wing design.

Per seat-km the DOC according to (69) can be rewritten as,

$$\frac{DOC}{N_s \; \Sigma R} = \frac{1}{U \bar{V}_b} \left\{ C_a + \frac{W_{p\ell}}{N_s} F_Q \frac{C_D}{C_L} \frac{W_{p\ell}}{W_{to}} \right\} + \frac{C_h}{\bar{V}_b N_s} + \frac{C_{fl}}{\bar{R}_b N_s} \tag{70}$$

where the powerplant cost function F_Q is defined as:

$$F_Q = \frac{U \bar{V}_b C_f}{\eta R_H} + C_{en} \frac{T_{to}}{T} \tag{71}$$

Substitution of the payload fraction according to eq. (56) into eq. (70) yields:

$$\frac{DOC}{N_s\ \Sigma R} = \frac{1}{U\ \bar{V}_b}\left\{C_a + \frac{W_{p\ell} + W_{fix}}{N_s}\ \frac{F_Q}{(1-\mu_{var})\frac{L}{D} - F_P}\right\} + \frac{C_h}{\bar{V}_b\ N_s} + \frac{C_{fl}}{\bar{R}_b\ N_s} \tag{72}$$

with $\mu_{var} = \frac{W_{var}}{W_{to}}$.

Equation (72) indicates that with respect to the DOC associated with fuel and engine installation the following penalty function appears to be a useful basis for assessing different engine types:

$$F_C \mathrel{\hat{=}} \frac{F_Q}{(1-\mu_{var})\ L/D - F_P} \tag{73}$$

For a given class of (long-range) aircraft, the airframe lift/drag ratio and the variable weight fraction are approximately known and F_C can be used to compare different powerplant options with regard to the cost and weight factors involved in F_P (eq. 31) and F_Q (eq. 71), provided their maintenance cost properties are equal.

Using the methods derived in this report the derivation of conditions for minimun DOC is straightforward, provided suitable cost factors are available. If certain approximations are accepted, analytical results can be obtained as demonstrated hereafter. The optimization of eq. (69) is equivalent to minimization of (73), provided the Mach number is fixed.

10.2. Optimum altitude and wing loading for constant M

Fig. 8 gives an example of an unconstrained optimization w.r.t. DOC, resulting in an unconstrained optimum (point A) between those for criteria 3 and 5.

a) Optimum wing loading (constant altitude).
Although the airframe acquisition cost per seat C_a will be to a certain extent a function of the airframe weight, this is a minor effect and probably insignificant in preliminary design optimization. Therefore the maximum value of $(1 - \mu_{var})$ L/D minimizes the DOC with respect to wing area variation. This partial optimum is identical to the case of minimum fuel used per seat-km and minimum engine plus fuel weight per unit payload, eq. (60) (Curve I).

b) Optimum altitude (constant wing loading).
For given wing loading and M the wing weight fraction is approximately constant. The altitude for minimum cost will therefore be located between the two altitudes where $F_P\ C_D/C_L$ and F_Q/C_L have their minimum values.
Analoguous to eq. (33b) one would expect the solution to be approximately:

$$C_L = C_{L_{MD}}\left\{1 + \frac{W_{en}}{W_f} + \frac{\text{engine cost}}{\text{fuel cost}}\right\}^{-\frac{1}{2}} \tag{74}$$

The altitude is then to be solved from:

$$\delta = \delta_{MD}\left\{1 + \frac{\eta}{\tau\delta\ R_{eq}/R_H}\left(\mu_T + \frac{C_{en}}{C_f}\ \frac{R_{eq}}{U\ \bar{V}_b}\right)\right\}^{\frac{1}{2}} \qquad \text{(Curve II)} \tag{74b}$$

Practical application of this result shows that the magnitude of engine cost/fuel cost is similar to W_{en}/W_f.

c) Optimum altitude and wing loading combined (point A).
Exact solution of the combined optimum for δ and W/S is not feasible. However, a reasonable accurate approximation is, analoguous to eq. (68):

$$\frac{\delta}{\sqrt{\beta\, C_{D_p}}} = \left[2\,\frac{W_{to}}{q_o\Delta}\left\{\frac{\eta\, R_H}{2\tau}\left(\frac{\mu_T}{R_{eq}} + \frac{C_{en}}{C_f\, U\, \bar{V}_b}\right) / \sqrt{\beta C_{D_p}} + f(R)\omega\right\}\right]^{\frac{1}{2}} \tag{75a}$$

$$\frac{W_{to}/S}{q_o C_{D_p}} = \omega + \frac{\delta}{\sqrt{\beta\, C_{D_p}}}\left(1 + \omega\,\frac{q_o\Delta}{W_{to}}\right)^{\frac{1}{2}} \tag{75b}$$

$$\text{with } \omega = \frac{dW_{var}/dS}{q_o\bar{C}_{D_p}(\,1 - W_{var}/W_{to})} \tag{75c}$$

where δ according to (75a) is to be substituted into eq. (75b) to find an explicit solution.

10.3. Optimum Mach number

The occurrence of the block speed in the denominator of all terms of eq. (72), except the fuel cost contribution and the last term, results in a much higher cruising speed as compared to all previously considered criteria. For a given design the following optimum block speed for minimum DOC is derived in Ref. 1.

$$\frac{d\,\log W_f}{d\,\log V_b} = \frac{V_b}{W_f}\frac{dW_f}{dV_b} = \frac{\text{DOC/trip}}{\text{fuel cost/trip}} - 1 \tag{76}$$

This can be converted into an optimum value of the drag rise:

$$\left.\frac{\partial\,\log C_D}{\partial\,\log M}\right|_{C_L} = \eta_M + \frac{\text{DOC/trip}}{\text{fuel cost/trip}} - 1 \tag{77}$$

For a given aircraft the optimum cruise Mach number is therefore in the drag rise, provided the engine thrust available allows this to be achieved. For example, if the fuel costs are 40% of the DOC, and $\eta_M = 0.6$ (turbofans), the optimum cruise M is the value for which $dC_D/dM \approx 2.1\,C_D/M$.
The shape of the drag rise is determined primarily by the detailed wing design, such as the thickness/chord ratio progression, sweepback angle and applied supercritical wing technology. Variations in the design-Mach number should be accompanied by variations in these parameters and will therefore result in variations in wing weight. The selection of an optimum cruise Mach number is therefore considerably more complicated than simply finding a minimum of one of the criteria considered in this report.

10.4. Conclusions

a) Since we have assumed the aircraft price to be insensitive to variations in the wing size, the optimum lift coefficient for minimum DOC is insensitive to cost terms (C_a, C_f and C_{en}). It is therefore possible to optimize this important coefficient on a basis of purely technical considerations. The wing loading and altitude, however, are sensitive to variations in C_f and C_{en}.

b) The optimum altitude according to both equations (74) and (75) is much less sensitive to the design range compared to the optimum values based on weight fractions. Therefore, as regards DOC, long-range aircraft with different design ranges, but similar speeds, have more similar optimum conditions as compared to the previous criteria.

c) Optimum conditions for altitude and wing loading, resulting in minimum DOC, are between the optima for minimum $\mu_{f,en}$ and those for maximum payload fraction, criteria 3 and 5, resp.

11. OPTIMUM ASPECT RATIO

11.1. Basic equations

Inspection of the expressions found for the maximum L/D-ratio, the specific range and the fuel plus engine installation weight ratio (criteria 1 through 3) indicates that in all cases reduction of $\beta \triangleq dC_{D_i}/d\,C_L^2$ continues to improve these figures. Since for a near-elliptic lift distribution we have β = factor/ πA, it is clear that on the basis of these criteria there is no optimum aspect ratio. Inclusion of the effects of wing weight variation due to variation of A is therefore essential.
For given wing tapes, section shape, sweep angle and load factor, the wing weight fraction may be written generally as follows:

$$\frac{W_w}{\bar{W}} = \hat{\mu}_w \left(\frac{A}{\hat{A}}\right)^{x_A} \left(\frac{S}{\hat{S}}\right)^{x_S} \tag{78}$$

Actually, there may be other wing weight terms, which are insensitive to A and S, which do not effect the optimization and are not included in eq. (78). The term $\hat{\mu}_w$ denotes the wing weight fraction for a baseline design with aspect ratio $\hat{A}$ and wing area $\hat{S}$. The exponents x_A and x_S can be theoretically or empirically derived quantities. For example, for bending material it can be shown that x_A = 3/2 and x_S = 1/2, provided stress levels remain constant. For other structural members, such as shear webs, ribs, leading edges, flaps and controls, other exponents are found. For the total wing weight it is generally found that x_A is close to 1 and x_S is close to 1/2.

The induced drag factor variation with aspect ratio is written as follows:

$$\beta = \hat{\beta}\,\frac{\hat{A}}{A} \tag{79}$$

where $\hat{\beta}$ and $\hat{A}$ refer to a baseline design.

11.2. Minimum wing plus fuel plus engine weight fraction

Substitution of eqs. (78) and (79) in eq. (38) and differentiation w.r.t. A, for constant wing area, results in:

$$\left.\frac{\partial\,\bar{\mu}_{w,f,en}}{\partial A}\right|_S = 0 \rightarrow \frac{A}{\hat{A}} = \left[\frac{F_P}{x_A\,\hat{\mu}_w}\,\beta C_L \left(\frac{C_L}{\hat{C}_L}\right)^{x_B}\right]^{\frac{1}{x_A+1}} \tag{80}$$

If it is also assumed that the wing area is optimized, so that $C_L/\hat{C}_L = \sqrt{\hat{\beta}/\beta} = \sqrt{A/\hat{A}}$, while the optimum C_L is approximately 120% of the value for L/D-max, the following result is found for the unconstrained optimum aspect ratio:

$$A = \hat{A}\left(1.2\,\frac{F_P}{x_A\,\hat{\mu}_w}\,\sqrt{\hat{\beta} C_{D_p}}\right) \exp\left(\frac{2}{2x_A\,x_B + 1}\right) \tag{81}$$

Realistic values for $\hat{\beta}$, $\hat{\mu}_{var}$, x_A and x_S can be obtained from a weight distribution for an existing aircraft or project, with specified aspect ratio $\hat{A}$. If, for example, the following typical values are used:

$\hat{A} = 8$; $\hat{\beta} = 0.045$; $\hat{\mu}_w = 0.10$; $x_A = 1.2$ and $x_B = 0.6$, a first approximation is found:

$$A_{opt} \approx 13.7 \left\{\left(\frac{R_{eq}/R_H}{\eta} + \frac{\mu_T}{\tau\delta}\right) \sqrt{C_{D_p}}\right\}^{0.714} \qquad (81a)$$

It is obvious that this result is sensitive to the design range and, to a lesser extent, the cruise altitude.

11.3. Effect of a constraint on the wing weight fraction

It can be argued that a simple addition of wing weight and fuel plus engine weight does not result in a realistic criterion, since the effects of structural weight variation are quite different from fuel weight and engine thrust variation. It then makes sense to minimize the payload fraction with a constraint on the variable wing weight fraction. This implies that

$$A^{x_A} S^{x_S} = \text{constant, or } C_L/(A^{x_A/x_S}) = \text{constant} \qquad (82)$$

From eq. (38) it appears that simply the ratio C_D/C_L must be minimized, simultaneously satisfying eq. (82). The result is

$$\left.\frac{\partial \bar{\mu}_{w,f,en}}{\partial A}\right|_{W_w/W} = 0 \rightarrow \frac{A}{\hat{A}} = \left(1 - \frac{x_S}{x_A}\right) \frac{\hat{\beta} C_L{}^2}{C_{D_p}} \qquad (83)$$

and correspondingly the optimum aspect ratio can be obtained from eq. (79). Eq. (83) can be written alternatively as follows:

$$C_L = \left\{\frac{C_{D_p}}{\beta} / (1 - x_s/x_A)\right\}^{\frac{1}{2}} \qquad (83a)$$

This remarkably simple result is to be compared with eq. (43), for example. Although the two results are comparable in magnitude, equation (83) does not exhibit any dependence from the design range and altitude.
Since eq. (83) contains C_L and A, it must be combined with eq. (78) to solve the individual optima of C_L and with the constraint on W_w/W.

11.4. Aspect Ratio for minimum DOC

A first approximation for the optimum aspect ratio can be obtained by assuming that for given wing area the sensitivity of μ_{var} to aspect ratio variations can be written similar to eq. (78) as follows:

$$\mu_{var} = \hat{\mu}_w \left(\frac{A}{\hat{A}}\right)^{x_A} + \mu_{ref} \qquad (84)$$

For given cruise altitude the DOC are minimum if, according to eq. (73) the term $(1 - \mu_{var})L/D$ reaches a maximum. As a result it is found that

$$\frac{A}{\hat{A}} = \left[\frac{1 - \mu_{ref}}{\hat{\mu}_w \left\{1 + x_A \left(1 + \frac{C_{D_0}}{\beta C_L{}^2}\right)\right\}}\right]^{1/x_A} \qquad (85)$$

This is an implicit equation, since β is inversely proportional to A. As a first-order approximation we may take $C_{D_o}/\beta C_L{}^2 \approx 2C_{D_p}/\beta C_L{}^2 \approx 1.5$. Hence,

$$\frac{A}{\hat{A}} = \left\{\frac{1 - \mu_{ref}}{\hat{\mu}_w\ (1 + 2.5\ x_A)}\right\}^{1/x_A} \qquad (85a)$$

If, for example, $\hat{\mu}_w = \mu_{ref} = 0.1$ for $\hat{A} = 8$ and $x_A = 1.2$, we find an optimum aspect ratio of 15.7.
It should be noted that, although $C_{D_o}/(\beta C_L{}^2)$ is strictly speaking a function of the range, the sensitivity of A_{opt} to the design range is very weak. On the other hand, the sensitivity to x_A, and therefore to the method of calculating the wing weight, is very strong. Practical constraints will therefore have to be imposed on A in order to avoid aspect ratios that result in too flexible, flutter-prone wings. In particular a constraint on the aspect ratio may be imposed by the fuel volume available in a wing.

11.5. Conclusions

a) All cases considered, except the case of minimum DOC, indicate that high values of cruise-C_L require high aspect ratios. Incidentally, the reverse has been found to be also true.

b) Substantial differences between optimum aspect ratios are found when different design criteria are used. In the cases of minimum $\mu_{w,f,en}$ and maximum payload fraction as design criteria, the optimum aspect ratio is sensitive to the range. In the case of a constraint on the wing weight fraction and for minimum DOC very simple results are found, which are not sensitive to the design range.

c) In all cases the results are very sensitive to the expressions used for the wing weight fraction, in particular the exponent of A.

d) The optimization of aspect ratio with a constraint on wing weight is considered as a useful procedure, since
   - for constant wing weight the production cost of the wing can be considered as constant,
   - the optimization is limited to maximization of L/D, for specified flying conditions (altitude, Mach number). This is a well-defined problem.

e) Since the selection of the aspect ratio has many consequences outside the field of weight considerations, it is recommendable to bias the results of any optimization by practical constraints.

f) It is likely that in many cases low-speed performance requirements, in particular climb performances after engine failure, will have a profound influence on the selection of A. This subject is treated in Ref. 1.

12. SUMMARY OF RESULTS

In the previous chapters eight different figures of merit ("criteria) have been evaluated with regard to their effect on optimum wing loading, cruise altitude, lift coefficient, Mach number and engine thrust required. In most cases analytical closed-form expressions have been derived for partial optima as well as for unconstrained ("absolute") optima. From these results some significant conclusions can be drawn.

12.1. Optimum wing loading and altitude

Figures 2 through 7 show examples of iso-meritcontours for various criteria, as well as partial optima, defined as follows:
Curve I : optimum wing loading, constant δ and M
Curve II : optimum altitude, constant W/S and M
Curve III: optimum wing loading and altitude, constant C_L and M.
Effects of Mach number and aspect ratio variations have been considered separately. Combined optima are indicated as "point A". Figure 9 summarizes the various partial and unconstrained optima for a given Mach number and aspect ratio. For example, curve II-3 refers to the optimum altitude for given W/S, according to criterion 3 (minimum fuel plus engine installation weight fraction, $\mu_{f,en}$). From inspection of the equations it was concluded that the independent variables involved are preferably generalized as follows:

wing loading : $\dfrac{W/S}{q_o\, C_{D_p}}$

(pressure) altitude : $\delta/\sqrt{\beta C_{D_p}}$

lift coefficient : $C_L\,\sqrt{\beta/C_{D_p}}$

Since the DOC-criterion (no. 8) is rather sensitive to specific cost data, such as fuel price and specific engine cost, these optima have been deleted in Fig. 9. However, for the special case of:

$$\frac{C_{en}}{C_f}\,\frac{R_{eq}}{U\,\bar{V}_b} = \mu_T$$

they are identical to criterion 7: cf. eqs. (74) and (75), compared with eqs. (33a) and (68).

a) Partial optimum wing loading (Curve I)
This optimum is consistently characterized by an expression for the lift coefficient of the following appearance:

$$C_L\,\sqrt{\beta/C_{D_p}} = \sqrt{1 + \text{factor}}$$

For max. L/D, V/F and $\mu_{f,en}$ the factor is zero. The factor is generally highest when the payload fraction and $\mu_{w,f,en}$ are maximized, because then the effect of wing weight is large, in particular for short ranges. Intermediate values for C_L are found for minimum fuel per seat-km, fuel plus engine weight as a fraction of the payload, and DOC. Fig. 9 shows that points A for criteria 4 through 7 have quite similar C_L-values, and it is felt that the most useful result appears to be eq. (60) for minimum engine plus fuel weight, rewritten as follows:

$$C_L\,\sqrt{\beta/C_{D_p}} \approx \left\{1 + \left(\frac{2\sqrt{\beta C_{D_p}}}{\delta} + \frac{q_o\Delta}{\bar{W}}\right)\cdot\omega\right\}^{\frac{1}{2}}$$

For the present example this becomes approximately 1.175. Assuming $\beta = 1.1/\pi A$ and $C_{D_p} = 0.009$, we find $C_{L_{opt}} \approx 0.19\sqrt{A}$; hence a high-aspect-ratio wing requires a high cruise lift coefficient. However, constraints on C_L must be observed.

b) Partial optimum altitude (Curve II)
Application of criterion 3 results in a useful expression for the optimum altitude (eq. 23), which applies to both low and high Mach numbers, provided the drag polar for each Mach number can be approximated by a parabola. It is even more general if $C_{L_{MD}}$ is derived numerically or graphically from a general drag polar representation. Equation (33) appears to be valid for criteria 3, 4, 5 and 7, while criterion 6 results in a considerably higher altitude. However, it has been argued that the fuel used per seat-km ignores a thrust limitation, which will generally result in a constrained optimum. It is thus concluded that the following equation can be used to define the optimum altitude:

$$\delta = \delta_{MD}\left(1 + \frac{2}{\delta}\frac{\mu_T}{\tau}\frac{\eta}{R/R_H}\right)^{\frac{1}{2}} \; ; \; \delta_{MD} = \frac{\bar{W}/S}{q_o\sqrt{C_{D_o}/\beta}}$$

This result is relatively sensitive to the design range.

c) Optimum altitude and wing loading for given C_L (Curve III)
Fig. 8 shows that the various criteria result in different equations. For $C_L\sqrt{\beta/C_{D_p}} > 1$ these curves generally define an almost invariable altitude level for each criterion. Since criterion 7 results in an itermediate altitude, which takes the design not too far from other optima, we obtain from eq. (67) for $C_L\sqrt{\beta/C_{D_p}} = 1$:

$$\delta = \left[\left(\delta + 2\sqrt{\beta C_{D_p}}\frac{\bar{W}}{q_o\Delta}\right)\left\{\left(\frac{\tau}{\mu_T}\frac{R/R_H}{\eta} + \frac{1}{\delta}\right)^{-1} + \omega\sqrt{\beta C_{D_p}}\right\}\right]^{\frac{1}{2}}$$

From this equation δ can be solved by iteration.

d) Unconstrained optimum altitude and wing loading (point A)
From the previous considerations it can be concluded that this point should be defined by criterion 7 or, if data are available, by criterion 8 (DOC). For the case of minimum fuel plus engine weight the solution can be found either by intersection of the equations found for cases a), b) and c) above, or by eqs. (68), which give a direct, closed form solution. A simplified result is found by assuming $f(R) \approx 2$:

$$\frac{\delta}{\sqrt{\beta C_{D_p}}} = \left\{\frac{2\bar{W}}{q_o\Delta}\left(\frac{\mu_T}{\tau}\frac{\eta}{R/R_H} \Big/ \sqrt{\beta C_{D_p}} + 2\omega\right)\right\}^{\frac{1}{2}}$$

$$\frac{\bar{W}/S}{C_{D_p}\,q_o} = \left\{2\left(\frac{\bar{W}}{q_o\Delta} + \omega\right)\left(\frac{\mu_T}{\tau}\frac{\eta}{R/R_H} \Big/ \sqrt{\beta C_{D_p}} + 2\omega\right)\right\}^{\frac{1}{2}} + \omega$$

e) Optimum wing loading with a constraint on the thrust.
For the case of given thrust level (T/δ = constant and specified) the altitude is generally constrained according to eq. (63):

$$\frac{\delta}{\sqrt{\beta C_{D_p}}} = \frac{2\bar{W}}{T/\delta - q_o\Delta}$$

In this case an optimum wing loading is defined by:

$$C_L \sqrt{\beta/C_{D_p}} = \left(1 + \omega \frac{T/\delta}{\bar{W}}\right)^{\frac{1}{2}}$$

Again this result complies with the observations made in par. a) of this section.

12.2. Optimum Mach number

This appears to be the most fundamental parameter in all cases considerd. The effect of Mach number is evident in all optima considered in the previous sections, either explicitly in the term $q_o = \frac{1}{2}\gamma\, p_o\, M^2$, or implicitly in the dependence of η upon Mach number.
For a jet aircraft of given design the optimum cruise Mach number can be analyzed consistently, either numerically or analytically (Ref. 12). For maximum specific range (V/F) an unconstrained optimum is found in the drag rise, characterized by:

$$\left.\frac{\partial C_D}{\partial M}\right|_{C_L} = \eta_M \frac{C_D}{M}$$

For minimum DOC a somewhat higher Mach number in the drag rise is found (eq. 77).
The maximum L/D-ratio as a criterion for aircraft design optimization contradicts the above observations, and points in the direction of low speeds. The obvious reason is that low Mach numbers are not in favour of a good overall engine efficiency. The isolated factor L/D is therefore not useful to optimize M.
Simple criteria for the combination of
- optimum Mach number for given W/S and altitude
- optimum W/S and altitude for given M

can be obtained using V/F and $\mu_{f\ en}$ as criteria, and ignoring compressibility effects. These equations are derived in section 4.2 and 5.3, resp. In both cases the optimum wing area is simply:

$$C_{D_p} S = \frac{\Delta}{\eta_M} \quad \text{with} \quad \eta_M \triangleq \left.\frac{d \log \eta}{d \log M}\right|_{T/\delta}$$

and the optimum engine size and altitude are:

$$\frac{W_{en}}{W_f} = \frac{1}{2}\eta_M \quad ; \quad \delta = \frac{2}{\eta_M}\frac{\mu_T}{\tau}\frac{\eta}{R/R_H}$$

This extremely simple result is not very realistic since it results in very low wing loadings and high thrust levels. In Fig. 9 it represents point A-3, although at another Mach number. As a result, the "optimum" Mach number is found in the medium to high subsonic speed bracket, dependent on the cruise range. Generally speaking this is not in the interest of low DOC, which is improved by increasing the productivity of the aircraft through high speeds. More interesting is therefore the simultaneous optimization of cruise Mach number, wing loading and altitude for the criterion fuel used per seat-km with a constraint on T/δ (given engine). This results in eq. (22) for a given wing loading and eq. (69) for the optimum wing loading. Combination of these two equations yields:

$$C_{D_p}S = \frac{\Delta}{\eta_M + \omega \frac{T/\delta}{\bar{W}} (1 + \eta_M)} \quad ; \quad \frac{\delta}{\sqrt{\beta C_{D_p}}} = \frac{2(2 + \eta_M)}{\frac{T/\delta}{\bar{W}}} \frac{1 + \omega \frac{T/\delta}{\bar{W}}}{2 + \omega \frac{T/\delta}{\bar{W}}}$$

for an optimum Mach number:

$$M = \left[\frac{2}{\gamma} \frac{T/\delta}{p_o \Delta} \left\{1 + \frac{2 + \omega \frac{T/\delta}{\bar{W}}}{\eta_M + \omega \frac{T/}{\bar{W}} (1 + \eta_M)}\right\}^{-1}\right]^{\frac{1}{2}} < M_{crit}$$

These equations have to be solved by iteration, since $\omega = f(M)$

In view of the effects of Mach number on the wing aerodynamic design, which are outside the scope of the present report, all conclusions drawn in this section should be considered to be of limited value only.

12.3. Powerplant selection

a) Selection of an engine cycle.
Engine cycle studies have not been carried out in this report. However, several criteria have been derived which can be useful for selecting an engine cycle: The powerplant function F_P:

$$F_P = \frac{R/R_H}{\eta} + \mu_T \frac{T_{to}}{T}$$

is, for given altitude and speed, a figure of merit for the aircraft AUW. A minimum value of F_P results in a maximum payload fraction, or in minimum AUW for given payload. As such it provides a method of weighing the relative importance of overall powerplant efficiency, thrust lapse, specific engine weight and installation drag.
The powerplant cost function F_Q provides a different method of weighing these quantities:

$$F_Q = \frac{U \bar{V}_b C_f}{\eta R_H} + C_{en} \frac{T_{to}}{T}$$

The following combined function is a useful basis for assessing powerplant cycles on a basis of DOC, for equal maintenance cost properties:

$$F_C \triangleq \frac{F_Q}{(1 - W_{var}/\bar{W})L/D - F_P}$$

For a given class of aircraft typical value of $W_{var}/\bar{W}$, U, $\bar{V}_b$, and L/D can be used to provide a useful criterion to assist in the powerplant cycle selection.

12.4. Thrust level required and MTOW

Since for most cases considered it was found that the optimum wing L/D is nearly equal to the maximum L/D, we can write:

$$\tau \frac{T_{to}}{W_{to}} = \frac{T/\delta}{W_{to}} = \frac{2\sqrt{\beta C_{D_p}}}{\delta} + \frac{\Delta q_o}{W_{to}}$$

The MTOW can be obtained from the payload fraction:

$$\frac{W_{p\ell}}{W_{to}} = \frac{1 - (\mu_{var} + \mu_{f,en})}{1 + W_{fix}/W_{p\ell}}$$

A first approximation for $\mu_{f,en}$ can be obtained by using the minimum value of this function:

$$(\mu_{f,en})_{min} = \left\{\left(2\,\frac{R/R_H}{\eta}\,\sqrt{\beta C_{D_p}}\right)^{\frac{1}{2}} + \left(\frac{\mu_T}{\tau}\,\frac{q_o\Delta}{\bar{W}}\right)^{\frac{1}{2}}\right\}^2$$

An increment of about 4 percent will generally be adequate to find the actual weight fraction. For given payload the MTOW is than obtained and the corrected cruise thrust level is found. For a known value of τ this results in a take-off thrust level required. Once the engine has been selected, optimization of the altitude and wing loading may be carried out according to Section 12.1, par. e).

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Table 1: Design data of a long-haul passenger aircraft	
Number of seats	N_s = 180
Equivalent design range	R_{eq} = 8600 km
Cruise Mach number	M = 0.80
Profile drag coefficient (incl. hor. tail)	C_{D_p} = .009
Aspect ratio (baseline)	$\hat{A}$ = 8.50
Induced drag factor (baseline)	$\hat{\beta}$ = 0.045
Parasite drag area - excluding nacelles	Δ = 2.0 m^2
- including nacelles	Δ = 2.20 m^2
Powerplant specific weight	μ_T = 0.25
Thrust lapse factor	τ = 0.85
Installed overall powerplant efficiency	η = 0.32
Design payload	$W_{p\ell}$ = 170 kN
Fixed empty weight	W_{fix} = 210 kN
Variable empty weight fraction	W_{var}/W_{to} = 0.20
Wing weight sensitivity	dW_w/dS = 350 N/m^2
Generalized weight sensitivity factor	ω = 1.30

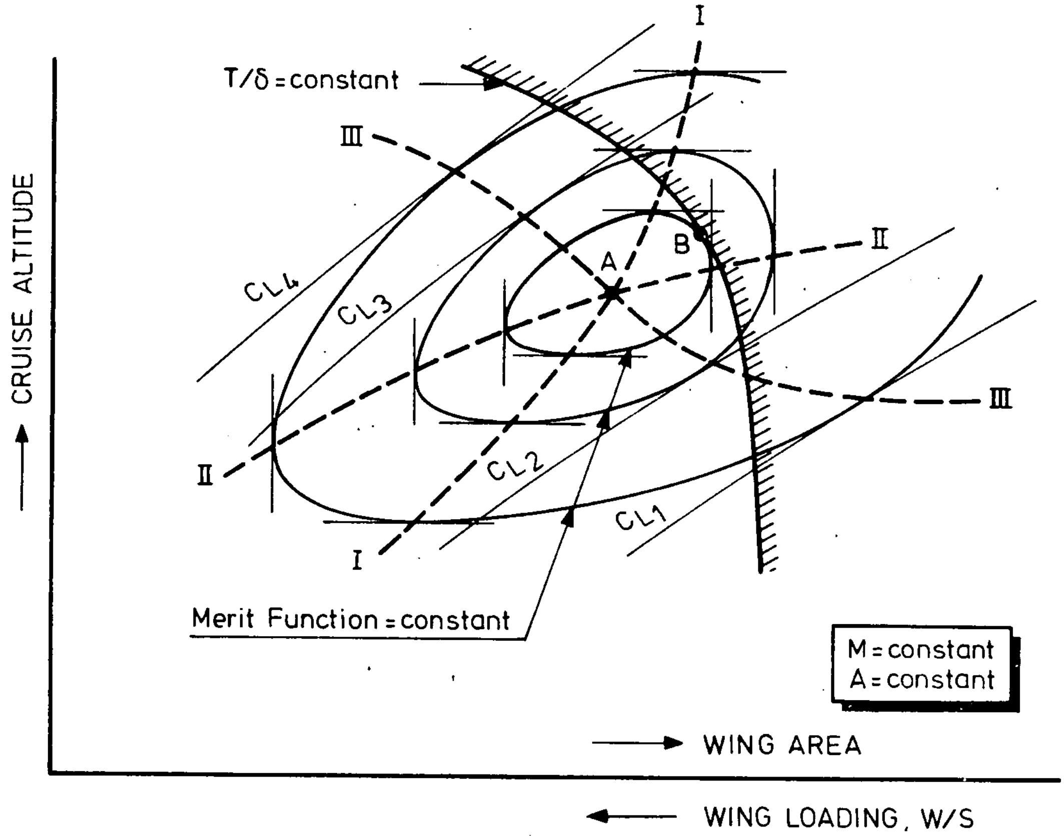

Fig. 1: Partial, absolute and constrained optima.

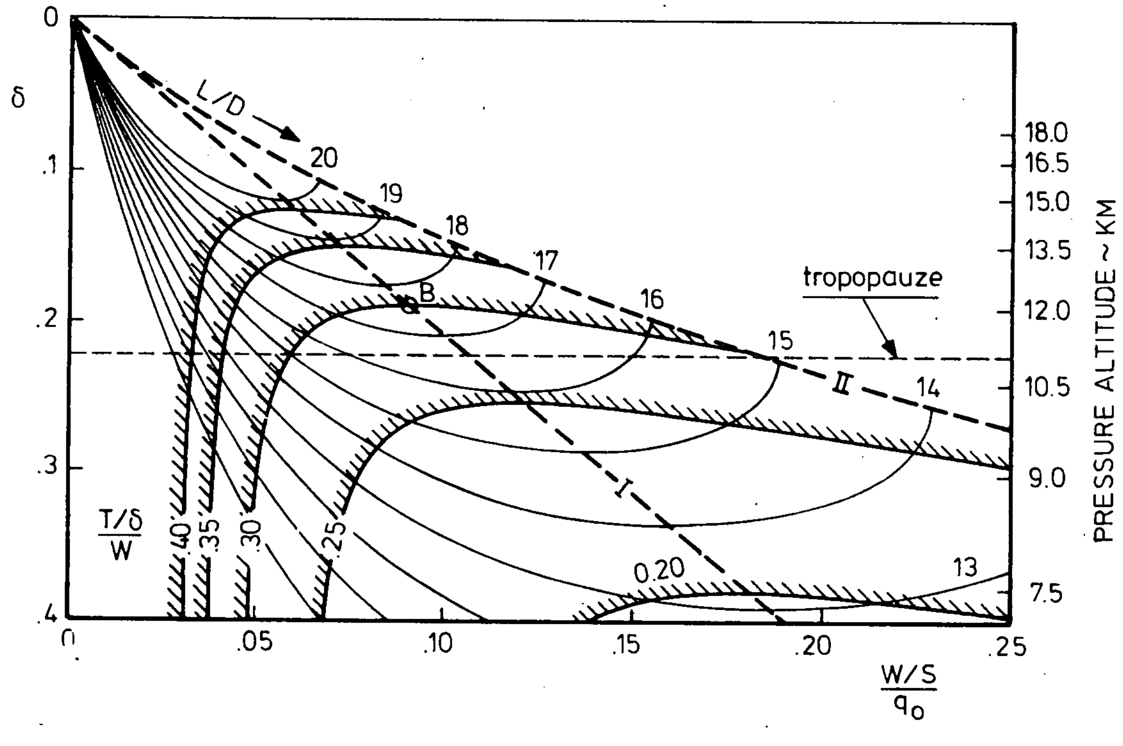

Fig. 2: Lift/Drag ratio vs. wing loading and pressure altitude.

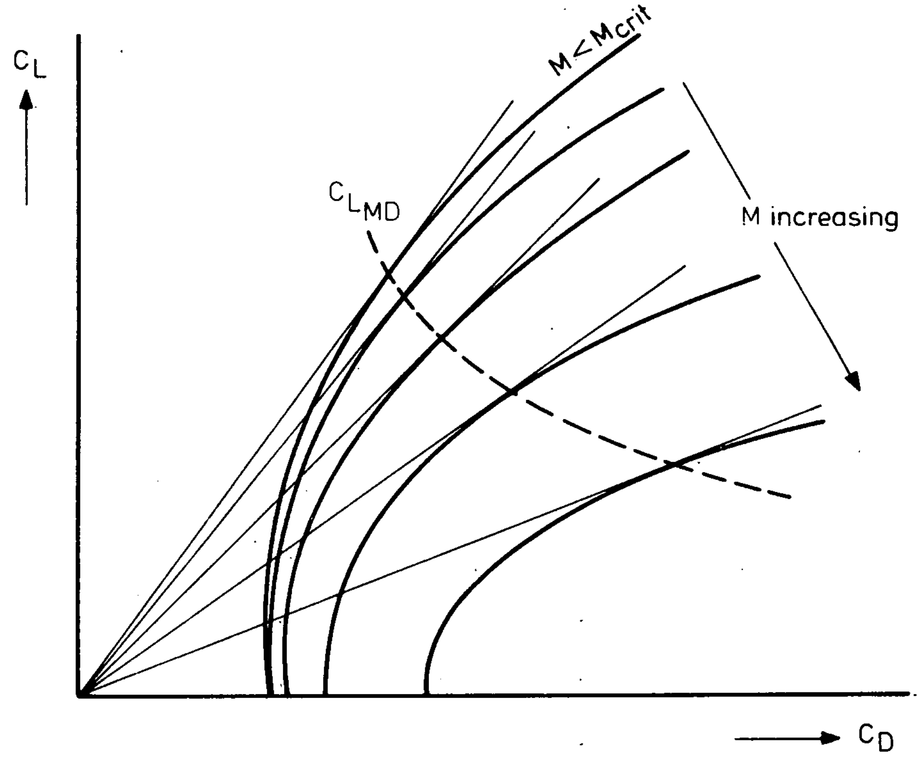

Fig. 3: Drag polars at low and high subsonic Mach numbers.

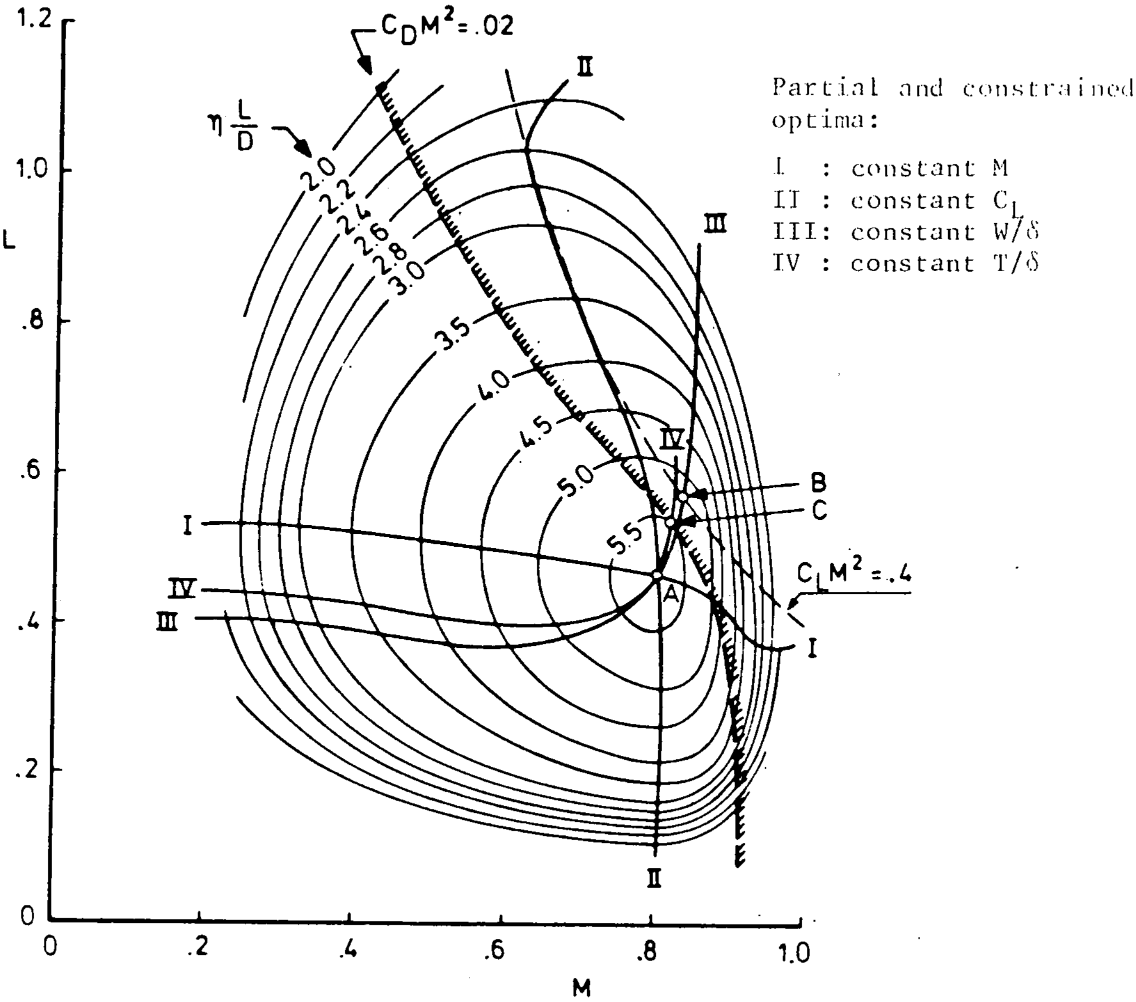

Fig. 5: Contours of constant range parameter.

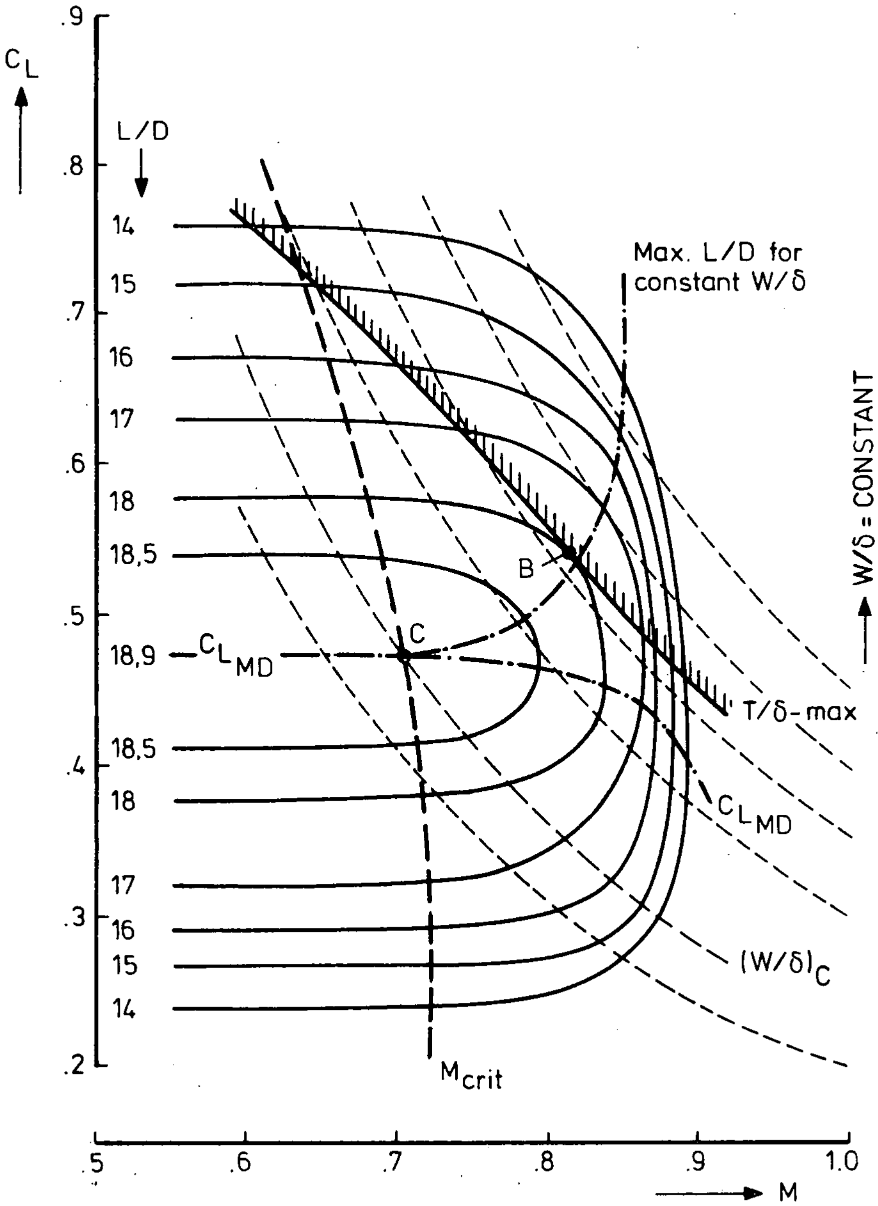

Fig. 4: Compressibility effects on L/D and conditions for $(L/D)_{max}$ for a given aircraft.

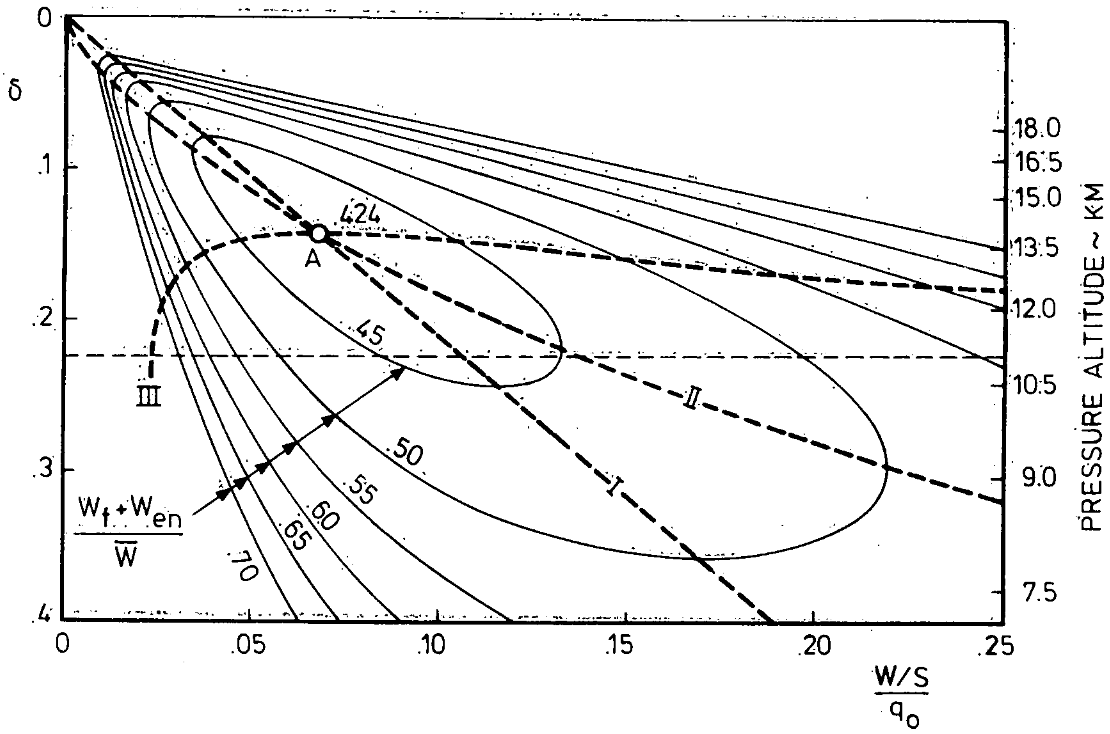

Fig. 6: Fuel plus powerplant fractions vs. wing loading and pressure altitude.

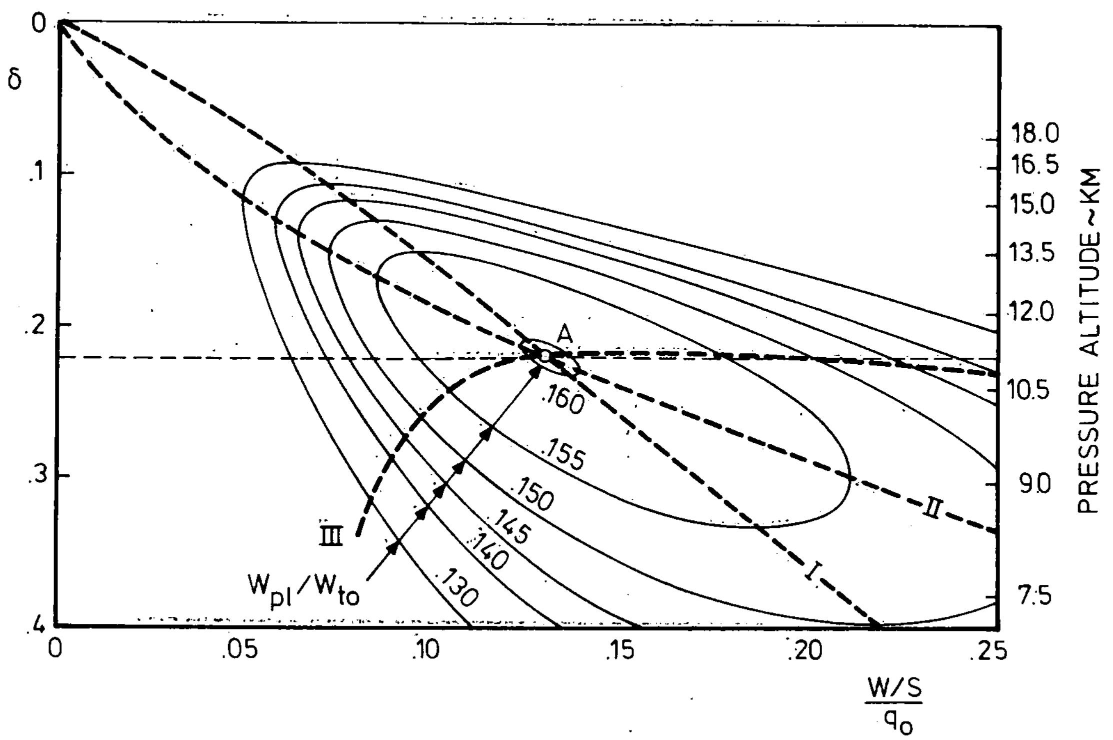

Fig. 7: Payload fraction vs. wing loading and pressure altitude.

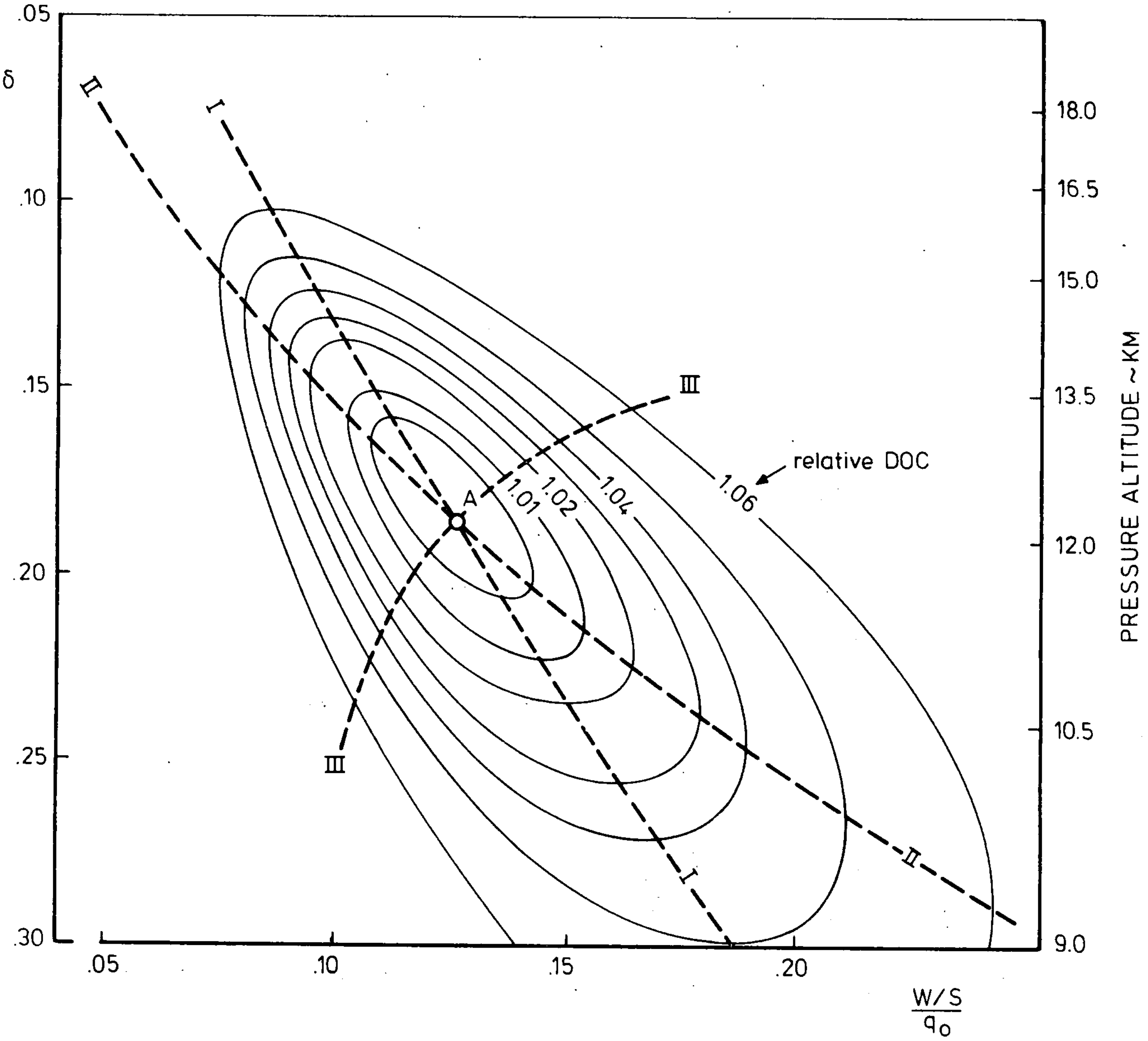

Fig. 8: Relative DOC vs. wing loading and pressure altitude.

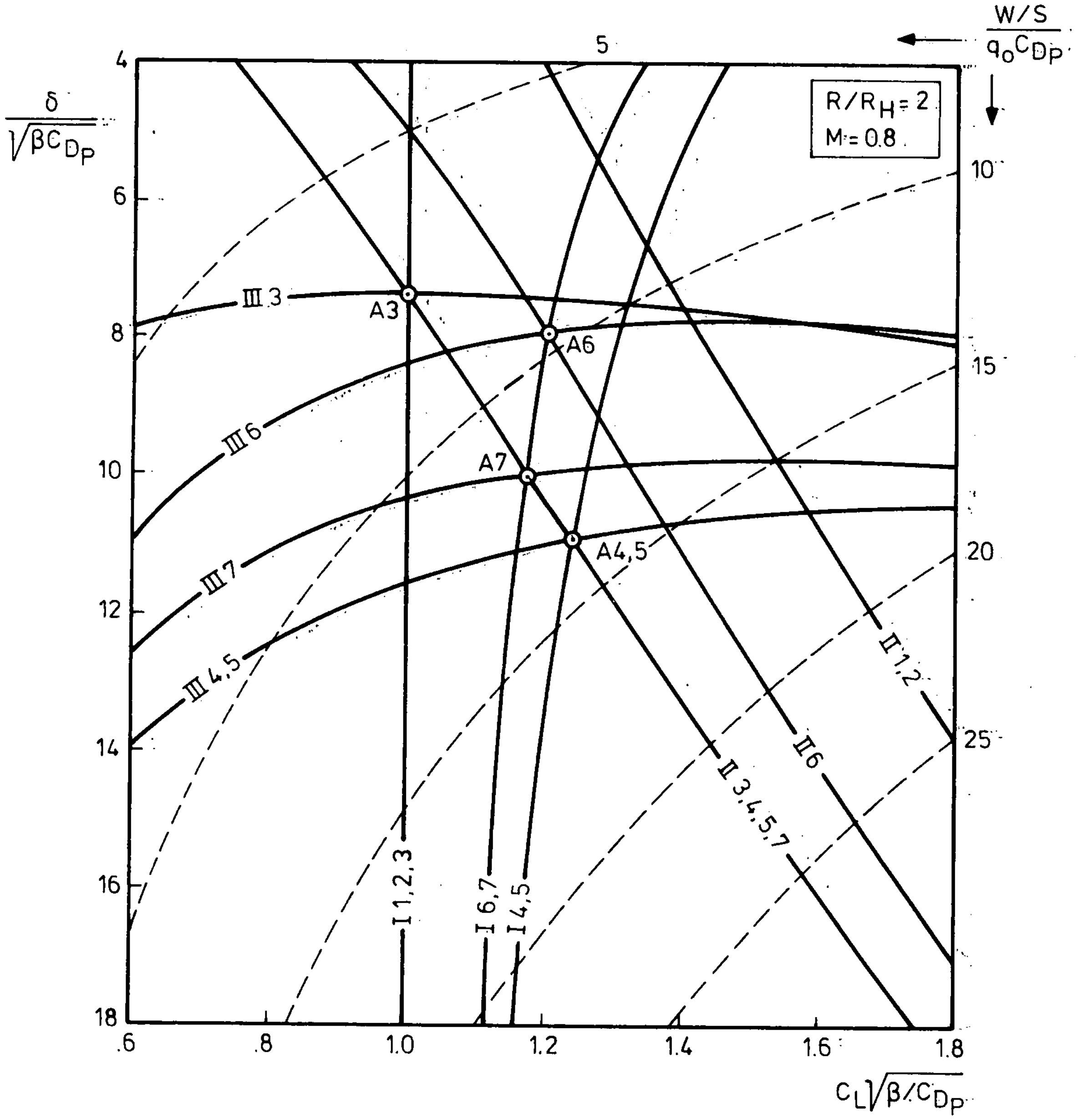

Fig. 9: Partial and unconstrained optima for various Merit Functions.

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