Shells and arches

Developing a new method to calculate shells and arches through graphic statics

P4 report

by

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Chapter 1 – Introduction

1.1 Background

Shell structures have been around for ages. The dome of the Pantheon for instance, built around the year 120, is one of the oldest examples of a shell structure. It is only in the recent history however that shell structures would take a more complex form. The shells by Heinz Isler for instance, or the Candela shells are freeform shells from the 20th and 21st century.

Figure 1 left: Interior of the Pantheon, Rome, painted by Panini (1734). right: L'Oceanogràfic designed by Felix Candela, 2003 (Gabaldón, 2010)

Figure 2 Heinz Isler, 1968, Laboratory and research facility for the Gips union (Töffpix, 2011)

Shell structures are usually calculated using the Finite Element Method (FEM). In this method, a structure is split up in small parts and for each part the forces working on it are calculated, including the forces working in between parts. Using this method, all the forces in the structure can be determined.

Using this method has two main disadvantages. The first of them being the efficiency of this method. These calculations can only be done for an almost complete structural design and are time-consuming. If the design turns out not to be strong enough or inefficient in material use, a new design has to be made, which can then be checked, after which the cycle probably needs to be repeated. The feedback on the structural performance is slow resulting in an inefficient design process.

The second, probably even more important disadvantage of FEM for shell structures is that it does not give any insight in the mechanics of shell structures in general. The calculations give numerical results, which can tell us something about a specific shell structure, and whether or not the structure will fail, but does not show the connection between the geometry and the structural performance, making it a kind of black-box method. The lack of knowledge on how these two are connected makes it hard to design a shell structure, it can only be done intuitively and by trial and error.

These problems can be solved if an alternative way of calculating shells is found. Some research is done already on this subject by several graduating students at the TU Delft, focusing on a way to calculate shell structures using graphic statics. The general aim of this research is to (i) find a graphical method to calculate these shells and to (ii) model these methods in a computational program linked to a 3D visualization, which results in a tool that can be used early in the design process and gives direct feedback on the structural performance of a shell. Some results of this research are discussed in chapter 2.

Part of this research focuses on a simple graphical method of calculating arches, with the idea that once a simple and accurate method is found for arches, this can be translated into a method for calculating shells. A method to calculate arches in a graphical way is already found in the minimum complementary energy method (section 2.4). This is an iterative method which uses the thrust line to find the stresses in an arch.

1.2 Problem statement

The current method to calculate a shell does not give any insight in the relation between the geometry and the structural performance. An input can generate an output, but why shells transport forces the way they do isn't explained through this method.

Another problem with the current method of calculating a shell is that it is a time-consuming process because it only can be done on an almost finished structural design.

This problem is already partially solved for arches, using the minimum energy method, which provides a graphical way to calculate arches. This results in a better insight in how they transport forces. The problem with this method is that it is only known how to use it as an iterative method. For a structure, several thrust lines can be generated and checked whether it is the correct one. By changing the thrust line, the correct one can be found. This holds the same problem as the FEM. Even though it provides more insight in the mechanics, it is still a time-consuming method which does not give the opportunity to play in an intuitive way with different structures to see how they transport forces.

1.3 Objective

The main aim of this research is to find a method to calculate shell structures in a quick and graphical way, so that the relation between the geometry and the structural performance is preserved.

To be able to get direct feedback on changes in geometry, a computational algorithm will be designed and connected to a 3D visualization program.

In order to solve this problem for shells, it needs to be solved for arches first. Section 2.5 describes a promising hypothesis on how this works for arches, which still needs to be proven though.

1.4 Research question

The problem statement and objectives stated in previous paragraphs lead to the research question:

How can the structural performance of a shell structure be calculated in such a way that the relation between the geometry and the structural performance is shown?

To answer this questions, the sub questions that need to be answered are:

- *How can the method of equal areas be proven for arches?*
- *How can this method be made applicable to shell structures?*
- *How can this calculation method be translated into a computational algorithm?*

To get started on this subject, a literature study is done, starting with the question:

What methods can be used to calculate shell and arch structures?

1.5 Relevance

Societal relevance

This research aims to provide in a tool for designers which gives them earlier in the design process insight in the structural performance of a shell structure. This will lead to a less time-consuming design process, but also to a more direct feedback on the design changes. It will probably lead to more efficient structural design, in which less material can be used for a similar performance.

Scientific relevance

Currently there is not a lot of insight in the mechanics behind shell structures. This research aims to give more insight in these mechanics.

1.6 Approach and methodology

Literature study

The whole research will be done within the field of structural mechanics. For this reason, the literature to be studied is mainly in the field of structural mechanics. To get the research started, a literature study on several subjects needs to be done. Part of these subjects are studied already. The following subjects will be studied:

- Force density method
- Complementary energy method
- Graphic statics in arches

Method development

From the literature study hypotheses will emerge. From these hypotheses a method to calculate arch and shell structures will be developed.

Design computational algorithm

The found method will be translated into a computational algorithm. For this algorithm, the 3D program Rhino will be used, with the Grasshopper-plugin.

Validate method

The computational algorithm will be compared to FEM calculations for several case studies. Differences in results from these calculations will show whether or not the method is valid.

Chapter 2 - Theory

In this chapter some theory regarding the calculation of beams and shells is summarized. Most of this research is done before starting the method development, and deals with the basics of calculating beams and shells. The first section explains a method developed by Calladine for calculating shells. The second and third section give an overview of methods of designing arches, using a chain line to construct a thrust line. It explains what a thrust line and its force polygon are, and how it relates to an arch and its loads. The fourth section explains a method to determine which thrust line is the correct one and the fifth section summarizes a graduation report by van Dijk (2014) which is partially the starting point for this research. The sixth and seventh paragraph summarize some insights and some research done alongside the development of the calculation method. Lopshits equation is introduced, a way to calculate the area of any polygon. Chapter three will explain how this can be used in relation to arches. The final section of this chapter discusses the topics of research and how they relate to the calculation method to be developed.

2.1 Calculating shells through the split in surfaces

To be able to calculate a shell, C.R. Calladine (1977) describes an element of a shallow shell which is conceptually split into two elements, one element which can only stretch and one of which can only bend. The stretching element is similar to for instance a bar network with hinges only, it can only transport normal forces. For the bending element there is no well-known equivalent, but it can be visualized as the beam shown in figure Figure 3(d).

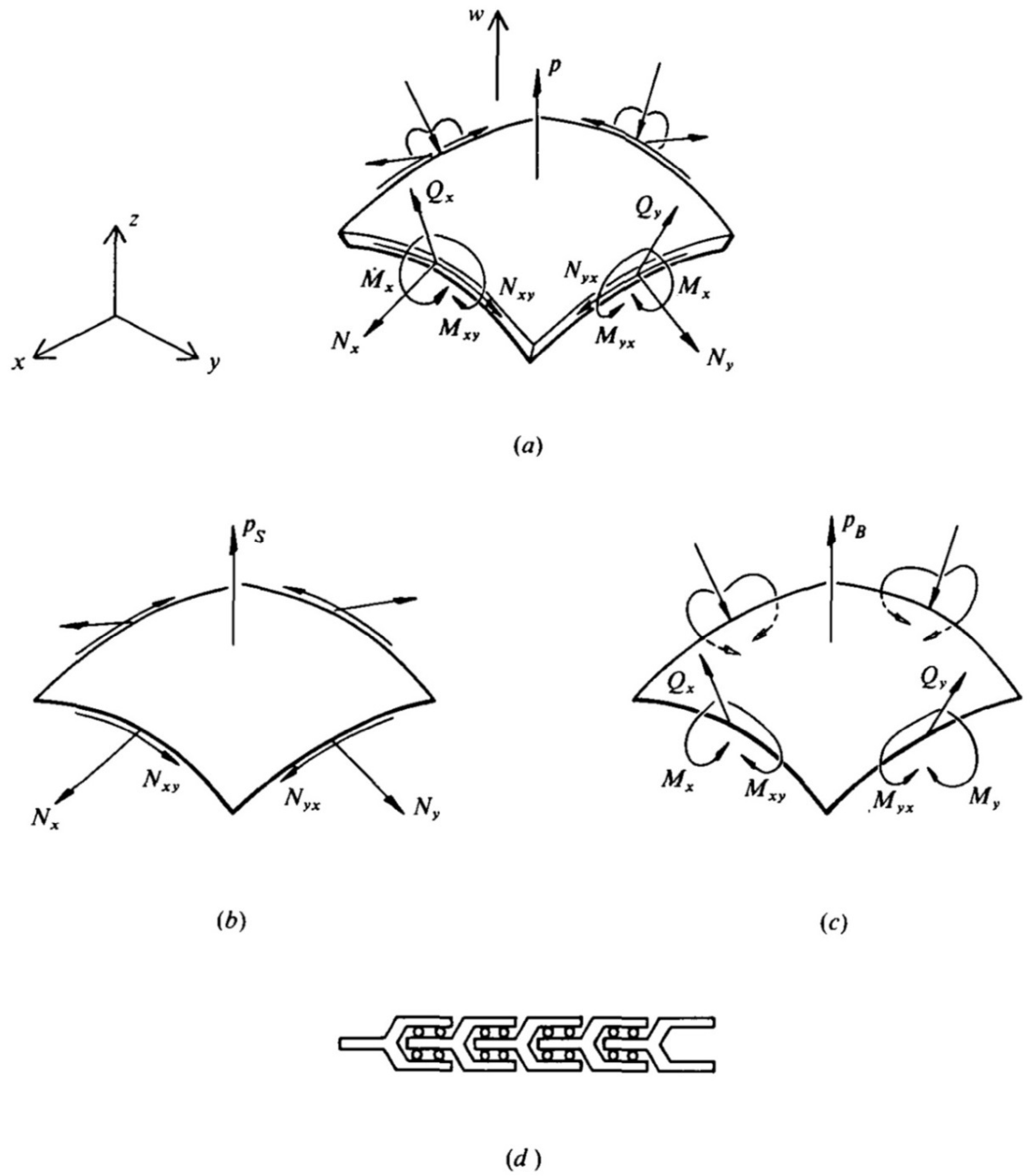


Figure 3 A representation of Calladine's split of a surface (a) into a stretching (b) and bending (c) surface. (d) shows a 1-dimensional equivalent of a bending surface. (Calladine, 1977)

Figure 3.a-c visualize this conceptual split. Surface (a) is the actual shell surface with all stress resultants. Figure 3.a shows the stretching surface (S-surface), with all the stress resultants working in plane. The external load is represented by a pressure p. Figure 3.c shows the bending surface (B-surface) with the bending and twisting stress resultants and the shear stress resultants working out of plane.

These two surfaces are separated so the equilibrium equations can be written out separately. To relate these two surfaces together, it is stated that the sum of the pressure of each surface should equal the pressure in non-split surface. So:

$p = p_S + p_B$

The two surfaces are also related through the equation

$g_S = g_B$

which means that the geometry of the S-surface should always be equal to the geometry of the B-surface, so the deformations of the two surfaces should be the same.

Pavlovic (1984) made a scheme to use this theory to solve an element of a shell (Figure 4). In this scheme, a value for p_S is chosen and with this value, the deformation is calculated. The new geometry g_S is set equal to g_B and from g_B the value for p_B is calculated. If the resulting p_B and the chosen p_S together are equal to p, the solution is reached. If not, the value for p_S should be changed and the cycle repeated until the correct solution is found. So the solution for each element in this case should be found through trial and error.

So the ratio between p_B and p_S is unknown. Once the ratio between these is known for each part of the structure, it is possible to make a relation between the geometry and the structural mechanics of a shell.

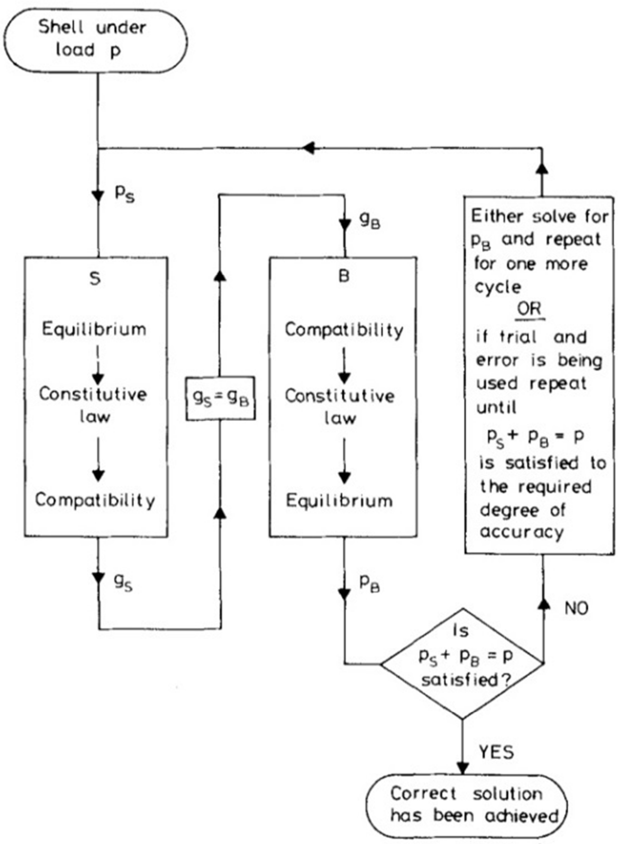


Figure 4 The flowchart by Pavlovic, used to determine the split in bending and normal forces in a shell structure (Pavlovic, 1984)

2.2 Arches and thrust lines

A thrust line is the line in which a 1-dimensional structure (arch or beam) with a given load (own weight for instance) would make equilibrium using compression only. This line can be found for masonry arches using the analogy of the chain models as shown inFigure 5. The loads resulting from the own weight are discretized in a point load and projected on a chain (a). This chain line inverted in (a) shows the thrust line, the line along which all the masonry blocks make equilibrium through compression only.

In (b) all the forces working on a masonry block are drawn. To prove that these make equilibrium, a force polygon can be drawn, using the head-tail method. In this method all the forces working on the block are joined together, head to tail. If the forces make a closed polygon, it means that the sum of the vectors is equal to 0, proving that the block is in equilibrium. Since two of the three forces on block 6 have their reaction forces on other blocks (5 and 7 in this case), all the closed force polygons of the different blocks can be joined together, resulting in the force polygon in (d). This force polygon and the thrust line in (a) are each other's reciprocal figures, which means that if one of them is changed, the other one will change too.

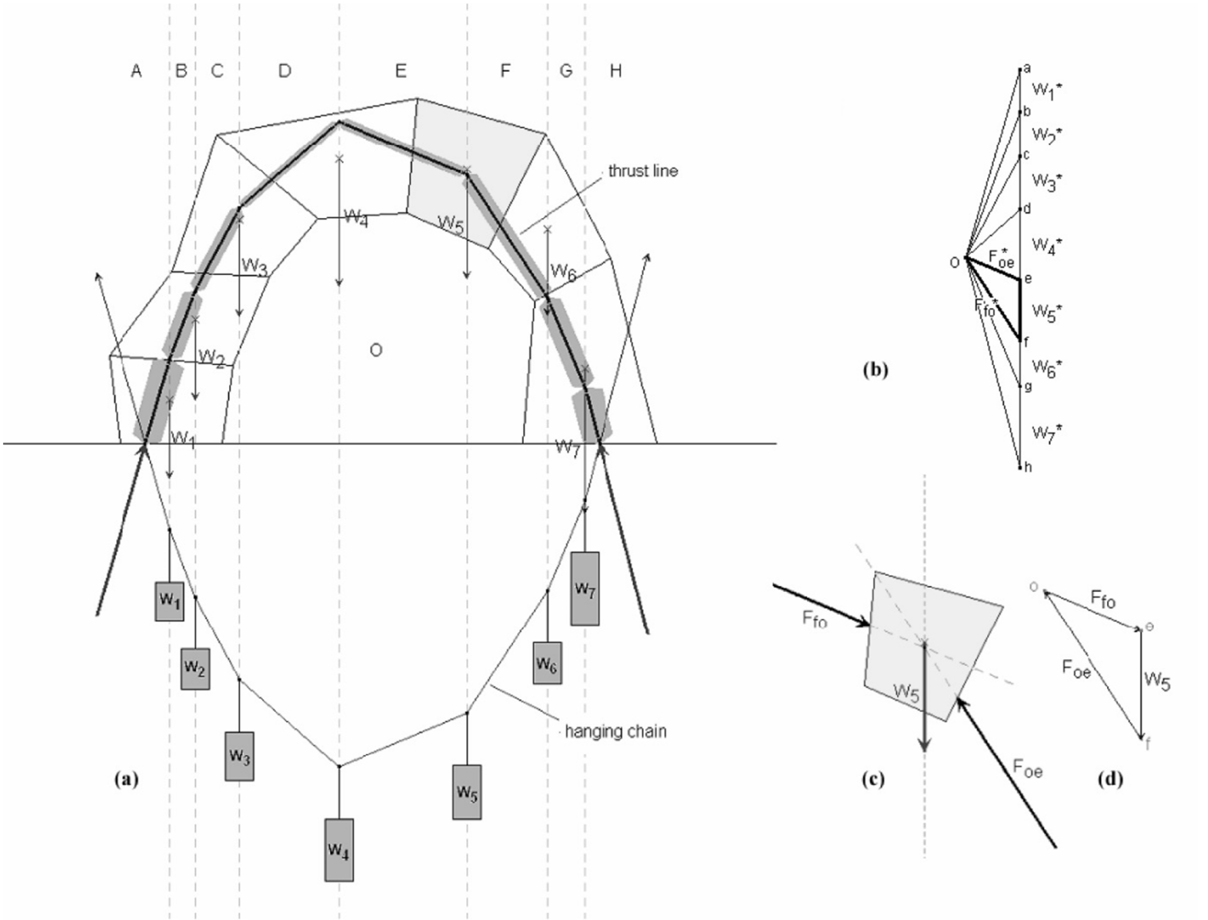


Figure 5 A thrust line in masonry blocks. (Block, 2006)

So other correct thrust lines can be drawn for the same load, simply by moving point O (called the polar coordinate) for instance closer to the own weight lines. If the point moves closer, the thrust line will become steeper (see Figure 6). So for each load an infinite amount of thrust lines can be drawn.

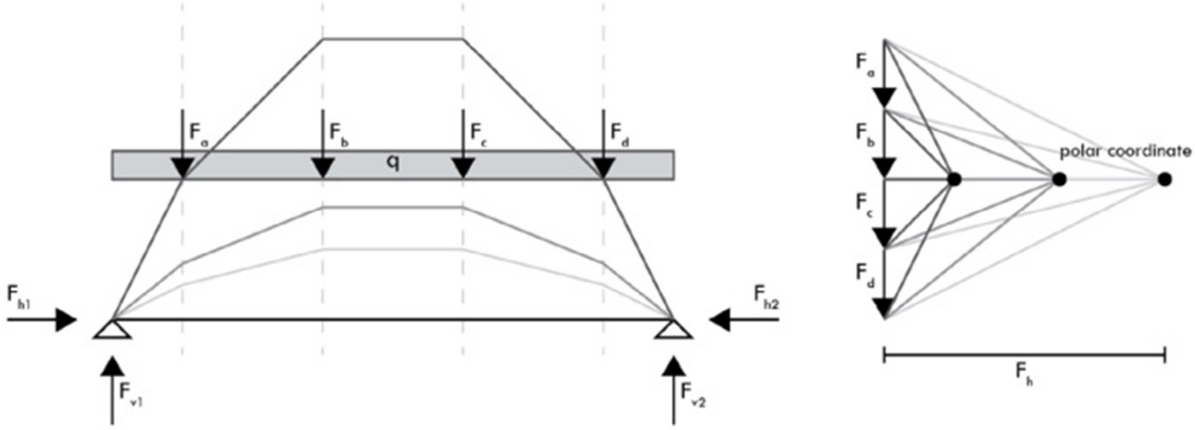


Figure 6 A set of loads can give an infinite number of thrust lines by moving the polar coordinate to the left or right. (van den Dool, 2012)

As long as the line lies within the masonry blocks, the blocks will make equilibrium through compression. Since these blocks can't handle tension or bending moments, the line has to lie within the structure, otherwise the structure will fail. However, when a structure can handle bending moments, reinforced concrete for instance, the line of thrust can lie outside the material without the structure failing. In this case the thrust line does not represent the actual transport of loads through normal forces anymore, because it is a combination of normal forces and bending moments.

Consider the example in Figure 7 to see what a line of thrust outside the material represents. The force in the thrust line in point A can be found in the reciprocal grid as F_A. So the force F_A can be drawn on the thrust line. (a) This force can be translated onto the structure by decomposing F_A in two forces, one shear force and one normal force, and adding a bending moment equal to the eccentricity *e* times F_N. (c) This can be done for the whole structure resulting in a combination of bending moment, shear force and normal force. In this case again, an infinite amount of thrust lines can be drawn. For both the thrust line in- and outside the structure, section 2.4 will explain which one the correct one is.

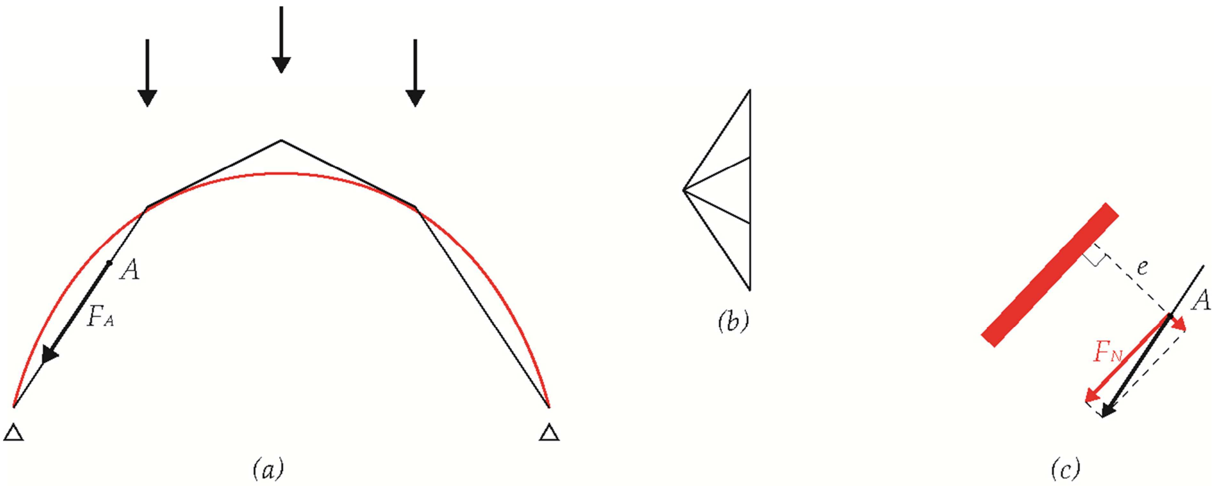


Figure 7 Projecting the force in a thrust line outside the material.

The thrust line has its 2D equivalent for plates and shells in the thrust network or thrust surface (Figure 8). The thrust network also has a reciprocal figure as can be seen in Figure 9. More on this thrust network and how to calculate it can be found in the graduation report of Tiggeler (2009) and the research of Block (2007).

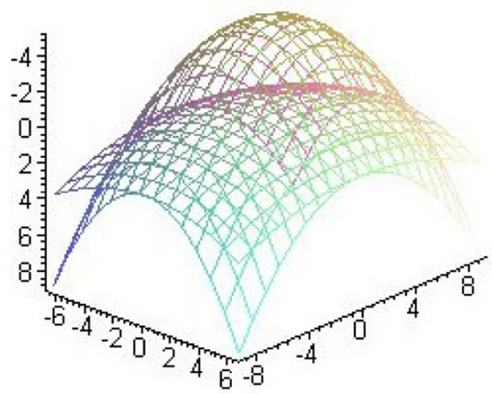


Figure 8 The 2D equivalent of a thrust line: a thrust surface (Tiggeler, 2009)

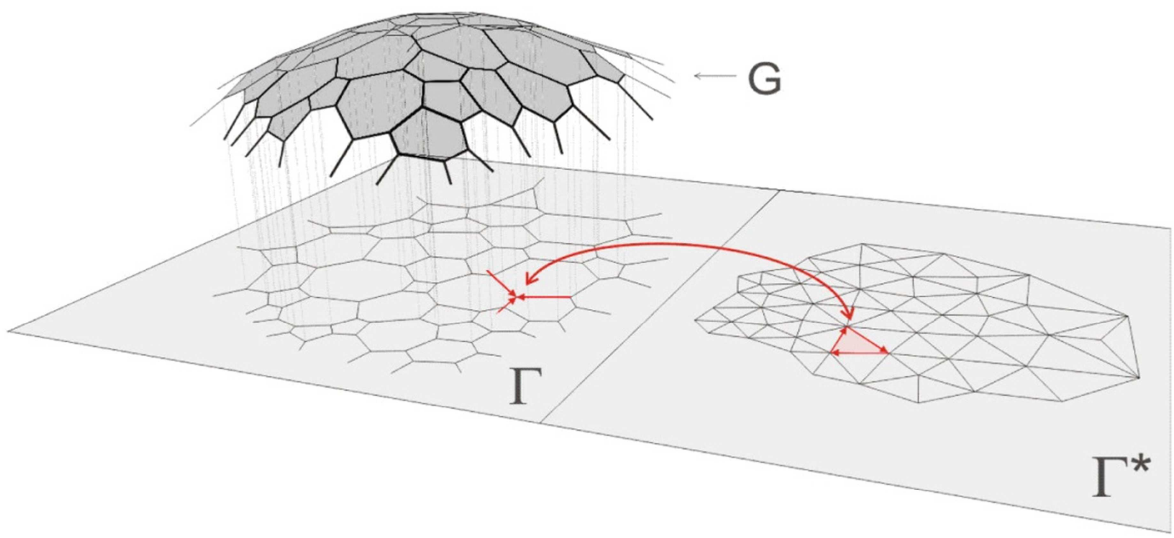


Figure 9 The reciprocal grid of the thrust surface (Block, 2009)

2.3 Uneven supports

In the previous paragraph, only situations with supports in a horizontal line (or loads perpendicular to the line between the two supports) are shown. The reciprocal figure can be used for uneven supports as well. Consider a situation as in Figure 10, four loads and two supports (a) are drawn, one of them being position lower than the other. For the force polygon the loads are drawn and a random position is chosen for the polar coordinate (b). If the corresponding thrust line is drawn, chances are that the thrust line will not fit to the supports. (c) To find the situations for which the thrust line will fit, the following steps need to be taken: draw a line between the first support and the end of the non-fitting thrust line (the blue line in c). Draw a line parallel to this one through the first random chosen polar coordinate and mark the position where this line intersects the loads in the force polygon drawing. Draw a line through the two supports, called the closing line (the red line in c). Finally draw a line parallel to the closing line through the point marked in the force polygon (d).

Any point on this line chosen as polar coordinate will result in a thrust line which begins and ends in the two supports (e/f). The angle of this closing line will be used in this research and will be called θ_c. (Beranek, 1980)

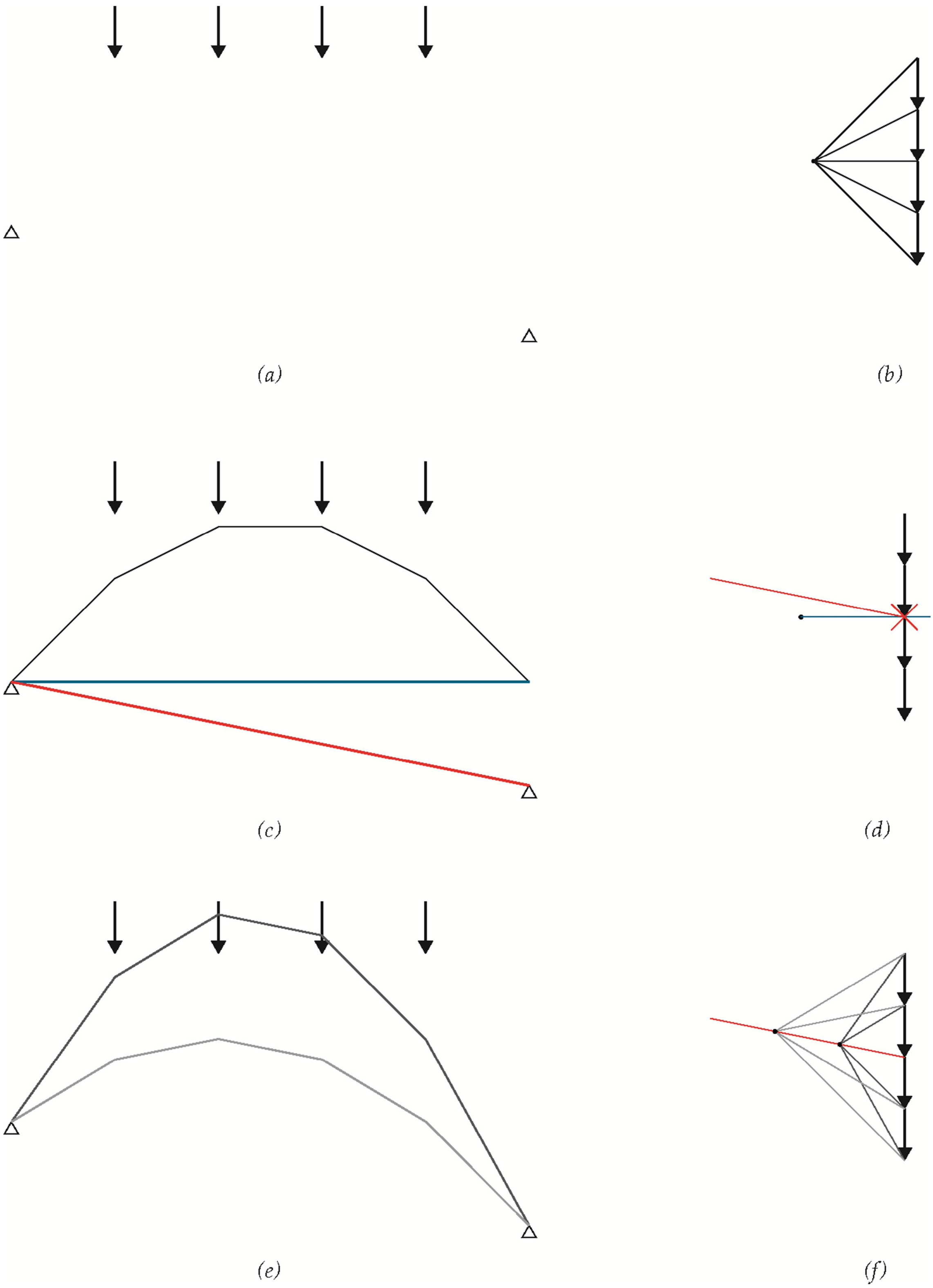


Figure 10 Using the closing line

2.4 Complementary energy method

Each set of loads can generate an infinite number of thrust lines. To determine which the correct one is, the complementary energy method can be used. This section will give a brief explanation of this method. For more examples and a more extensive explanation of the rewriting of these equations, see the paper by Borgart and Liem (2011) and the graduation report by van Dijk (2014).

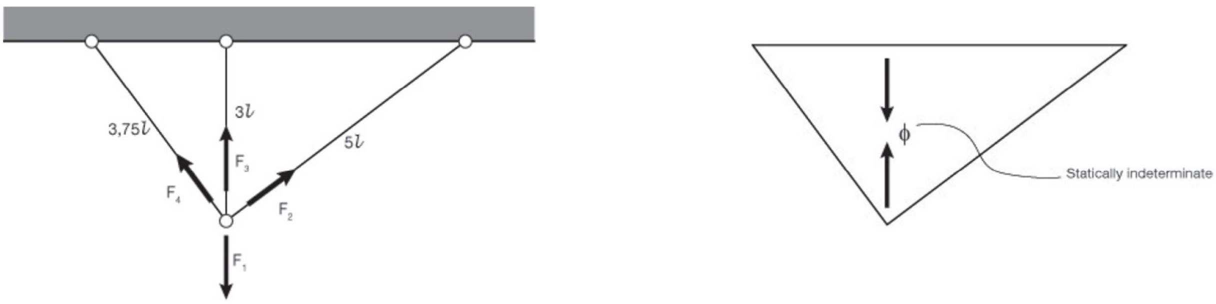
When a normal force works on a single bar the stress in the bar will cause it to strain. The work needed for this deformation is called strain energy. From the strain energy the complementary energy can be calculated by:

$$E_{c;N} = \frac{1}{2}\frac{N^2 l}{EA}$$

The complementary energy due to bending moments can be calculated by:

$$E_{c;M} = \frac{1}{2}\frac{M^2 l}{EI}$$

This complementary energy can be used to determine the flow of forces through a statically indeterminate structure. It is based on the premise that nature will always strive for the situation containing the least energy. A simple example is shown in Figure 11, in which a statically indeterminate structure can be seen. The structure consists of three bars, three supports and one load called F_1 is applied. To calculate the forces in this structure, one force is replaced by a force called ϕ. From this, all the normal forces can be calculated as shown in the table. From the normal force and the length of the bars, E_c can be calculated as well, with ϕ as the only unknown. The three values for E_c added together is the total complementary energy. Taking the derivative of this total and setting it equal to zero will result in the ϕ for which the complementary energy is the lowest, and all normal forces can be calculated.



member	N_i	l_i	E_c
1	$0{,}8(F-\phi)$	3,75l	$1{,}2\frac{l}{EA}(F-\phi)^2$
2	$0{,}6(F-\phi)$	5l	$0{,}9\frac{l}{EA}(F-\phi)^2$
3	ϕ	3l	$1{,}2\frac{l}{EA}\phi^2$

Figure 11 Calculating a statically indeterminate structure by finding the least complementary energy (Liem, 2011)

This principle can be applied to thrust lines as well. For a certain set of loads and a structure, the correct thrust line can be found by looking for the one with the least complementary energy.

Since we are not interested in the actual energy but only in the situation in which the complementary energy is minimal and we consider E to be equal throughout the structure, the equations for the complementary energy can be simplified to:

$$E_{c;N} = N^2 l$$

$$E_{c;M} = \frac{12}{t^2} M^2 l$$

Both these energies together will give the total complementary energy. Finding the thrust line for a structure containing the lowest total energy will result in the correct flow of forces through the structure.

2.5 Graphic statics in arches

In the graduation report by van Dijk (2014), an iterative graphical way to calculate beams and arches is described, and an iterative tool is made in Grasshopper to calculate arches in a quick and easy way.

The calculation method is summarized in Figure 12. The first step is to draw an arch, which can be irregular shaped and with uneven supports. The program discretizes both the projected load and the load due to own weight of the arch. This discretized load can lead to an infinite number of force polygons and their corresponding thrust lines as shown in Figure 6. From a thrust line and the structure together, the complementary energy is calculated as shown in the previous paragraph. With the height of the thrust line as a variable, the solution with the least complementary energy can be found, simply by changing the horizontal support until the lowest value is found.

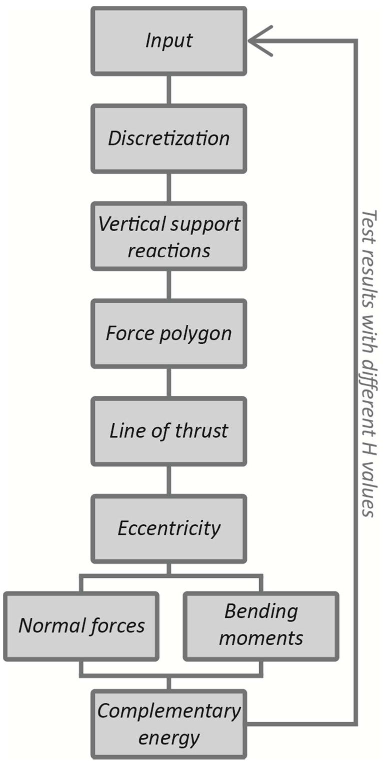


Figure 12 Calculating a thrust line by changing the horizontal support reaction (van Dijk, 2014)

In this report, the discovery is described that the area under the correct line of thrust equals the area under the structure (Figure 13). This leads to a shorter iteration loop (Figure 14) for calculating the correct thrust line. Even though there is not a mathematical proof for these areas being equal yet, this reports shows some arches (Figure 15) for which the forces are calculated using three different methods: the method of equal areas, the Finite Element Method and the method using the lowest complementary energy. This leads to results with a maximum deviation of 4%. These deviations are attributed to different ways of discretizing loads in different calculation methods.

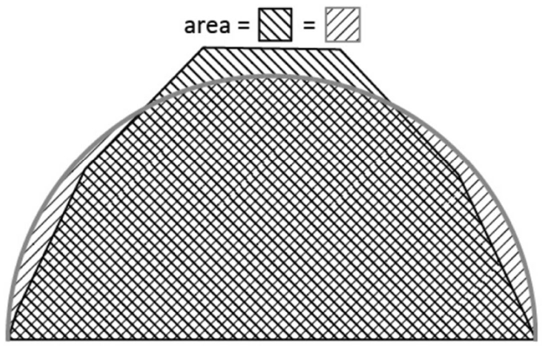


Figure 13 The area under the structure (grey) equals the area under the thrust line (black). (van Dijk, 2014)

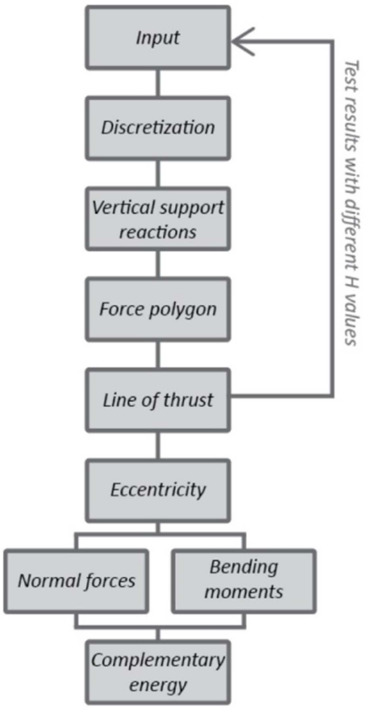


Figure 14 The shorter iteration loop uses the method of equal areas (van Dijk, 2014)

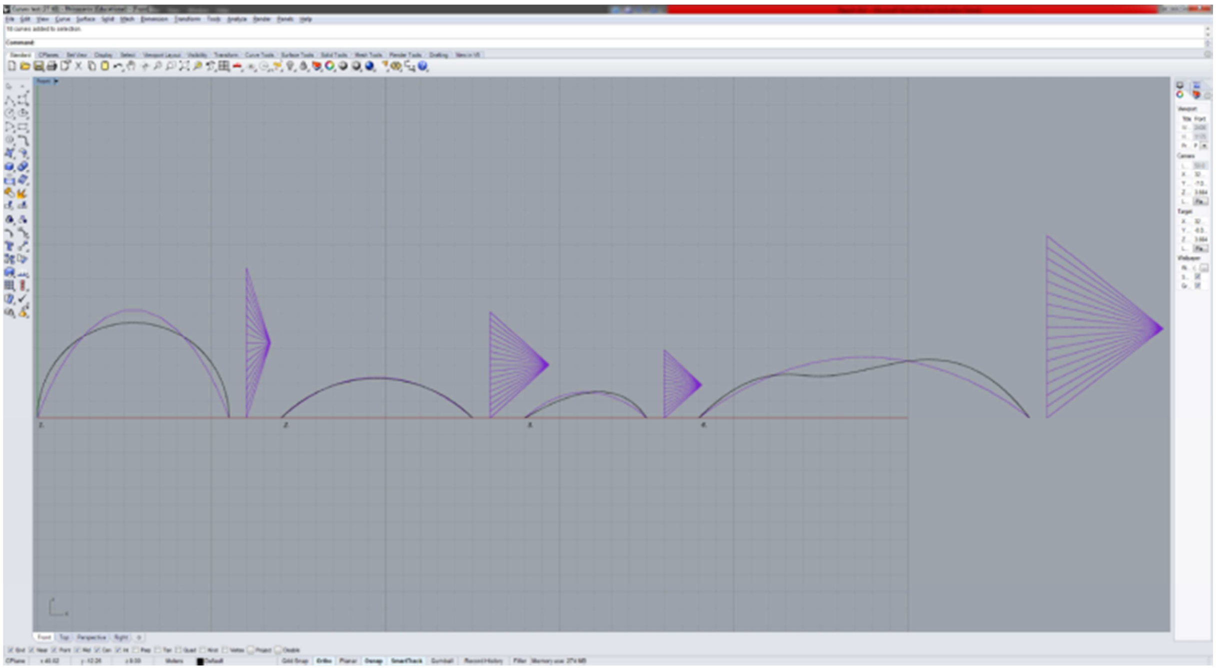


Figure 15 Arches for which the method of equal areas was tested (van Dijk, 2014)

2.6 Force density

Using the thrust line and its reciprocal figure, the force polygon, some observations are made on how these work together. This section summarizes these observation, not to prove a theory, but to make the reader acquainted with how these relate to each other. In this section the force density is introduced as well, a ratio between the force in a bar and the length of that bar.

If we consider a force polygon as shown in Figure 16 corresponding to a structure, the force F and the length l of a bar can be drawn in one figure. If both are projected onto a horizontal line, the ratio between F and l will stay the same, so:

$$\frac{F}{l} = \frac{F'}{l'}$$

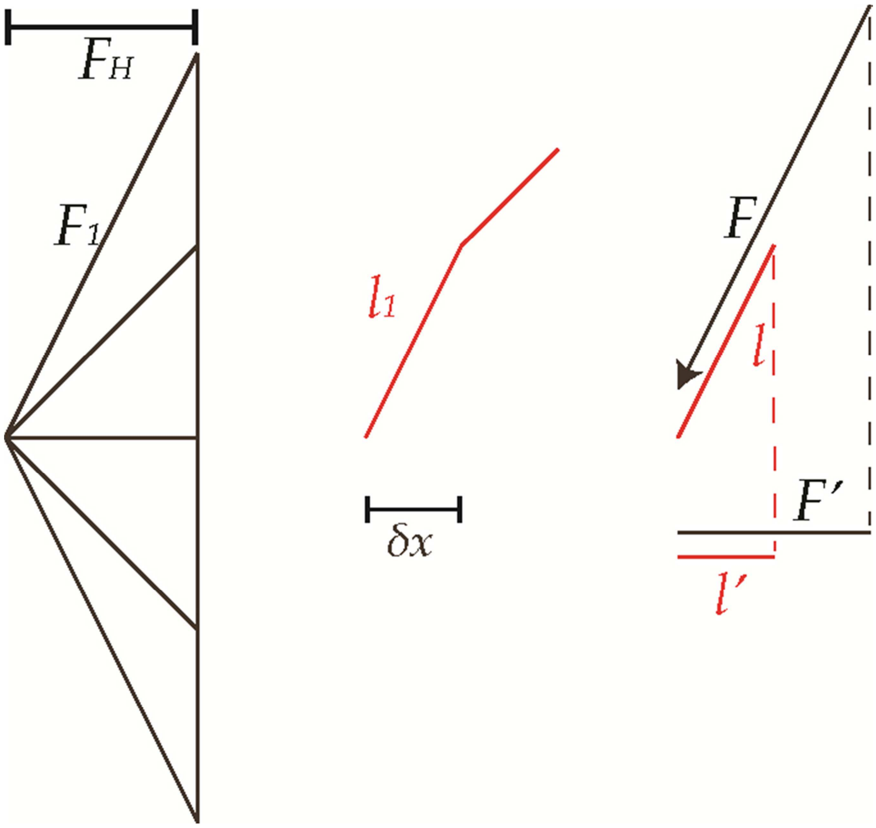


Figure 16 Relation between force polygon and structure

The horizontal projection of each force in the dual grid equals F_H as can be seen in the dual grid figure. The horizontal projection of each bar in the structure equals δx. The force density (*FD*) equals the force divided by the length of a bar, so:

$$FD = \frac{F_n}{l_n} = \frac{F_H}{\delta x}$$

In a similar way can be shown that:

$$FD = \frac{F_V}{\delta h}$$

This explains why if the polar coordinate of a reciprocal grid is moved horizontally, the FD will change in a similar way. If F_H doubles, FD doubles as well. This also explains why the FD in each bar is the same when a constant δx is chosen. The F_H for each bar is equal resulting in an equal FD.

Figure 17 shows a force and a bar length in one figure. For the F, the F_V stays the same, for the l, δx stays the same. F_H is changed by steps of 1. If $\delta x = 2$, this figure shows that the FD will increase by ½.

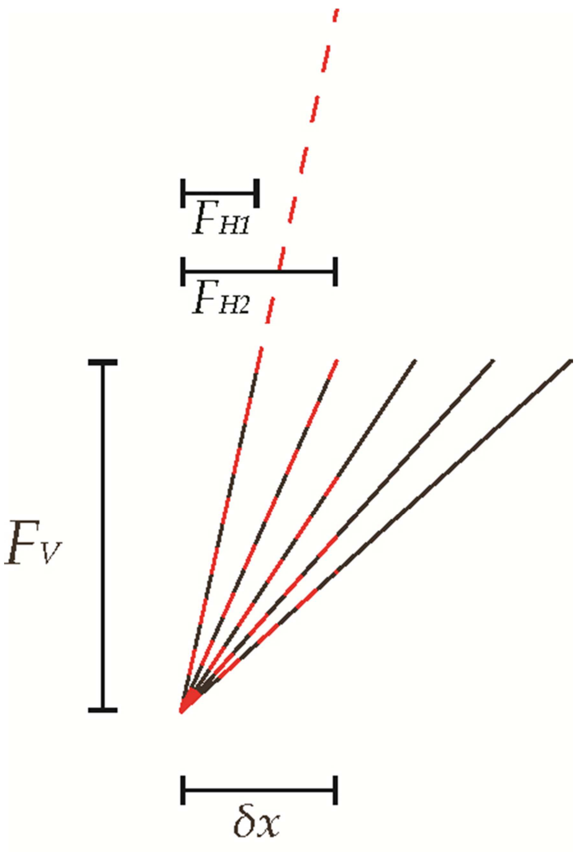


Figure 17 The force in a bar and the length of that bar in one figure

2.7 Calculating the area of a polygon

The main part of this research deals with trying to prove that the area under the structure equals the area under the correct thrust line. In this process Lopshits (1956) way to calculate the area of any force polygon is used. This paragraph will explain how this method can be used.

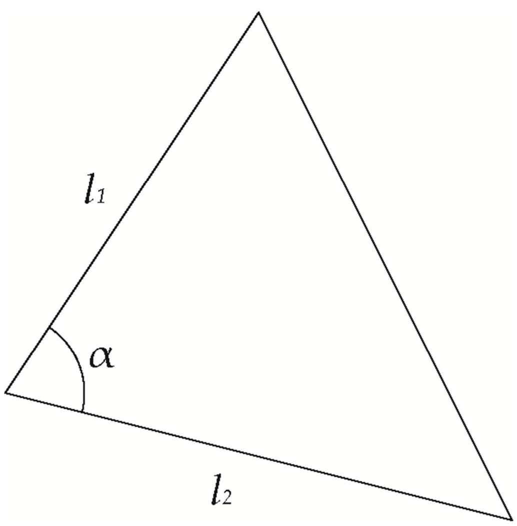


Figure 18 Calculating the area of a triangle

This method calculates the area by using only the length of each segment and the inner angle of each two segments. The area of for example the triangle in Figure 18 can be calculated by:

$$A = \frac{1}{2} l_1 l_2 \sin \alpha$$

To understand the way Lopshits works, the following concept needs to be understood. Consider any parallelogram, the one in Figure 19 (a) for instance. Two points on the same place on two opposing sides are moved (the drawing may seem like a 3D representation, it is a 2D drawing though), creating two new parallelograms (b). The area of the two parallelograms together (2 and 3) equal the area of the first parallelogram (c). This can be seen if we look at the areas marked as A, A' and B and B' in (d). The area removed at the bottom is added at the top of the figure, resulting in the same area. This goes for the triangles marked in (e) as well, since each triangle equals half the parallelogram.

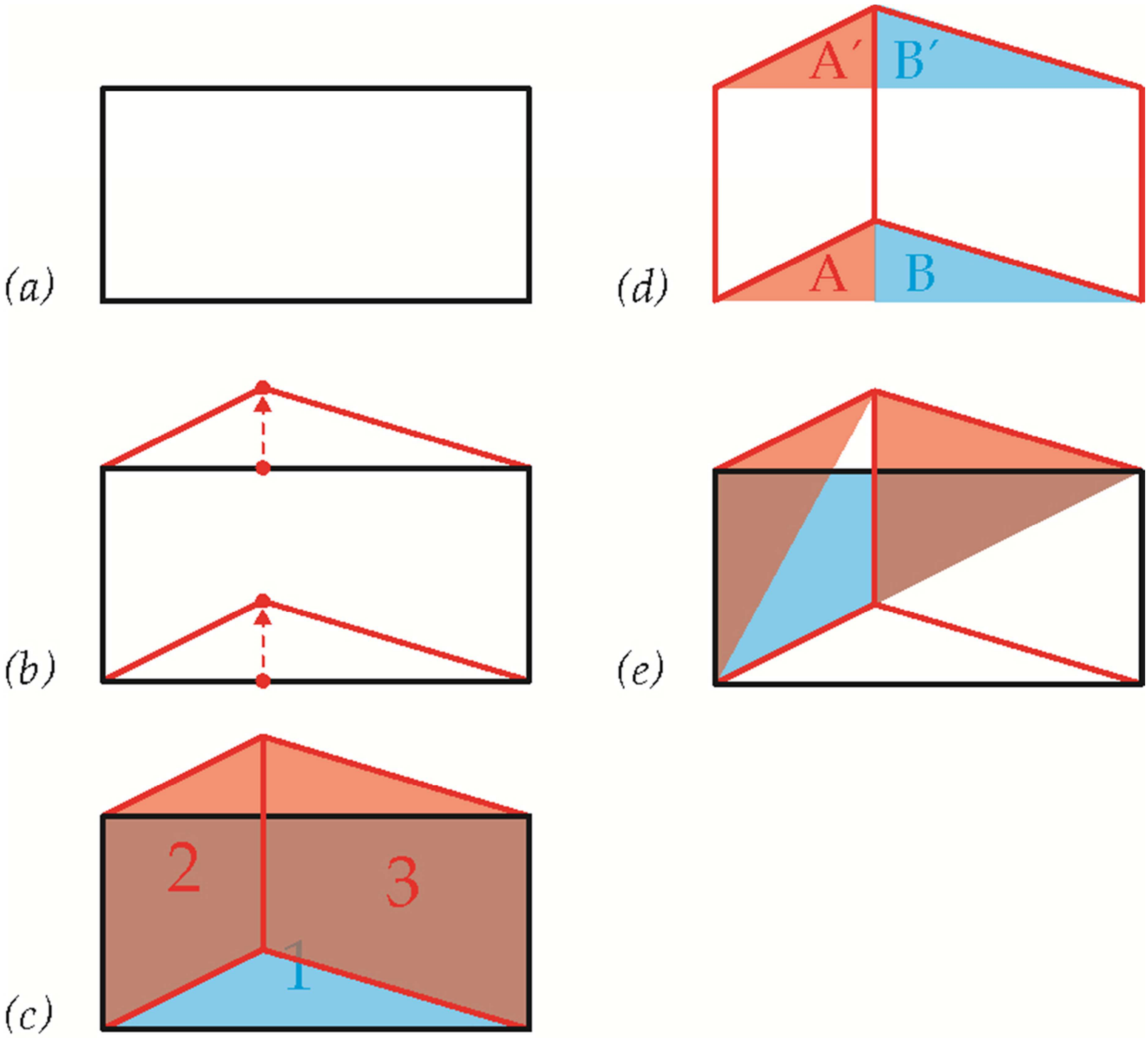


Figure 19 The area of rectangle 1 equals the areas of the parallelograms 2 and 3 together

Consider as an example for Lopshits calculation the polygon drawn in Figure 20, which consists of five points. The coordinates of the points are unknown, only the lengths of l_1 to l_5 are known and the angles α_1 to α_5. To calculate the total, Lopshits adds up the areas Figure 21:

$$A_1A_2A_3A_4A_5 = A_1A_2A_3 + A_1A_3A_4 + A_1A_4A_5$$

The first step is quite simple since all the variables are known:

$$A_1A_2A_3 = \frac{1}{2} l_1 l_2 \sin \alpha_1$$

For the second step, calculating $A_1A_3A_4$, Lopshits translates the figure (or at least the points needed for this calculation) over l_3, creating several parallelograms (Figure 23). Using the theory above, we can see that:

$$A_1A_3A_4 = A_1A_2B_1 + A_2A_3B_2$$

The same is done for the final triangle. Even though the final triangle can be calculated by using l_4, l_5, and α_5, in this example it will be calculated using Lopshits equation for consistency. In this case the figure is translated over l_4. Since this is not the biggest triangle of the ones used (like triangle 1 in Figure 19), not all the triangles need to be added, one needs to be subtracted. An easy way to determine this is to check in which direction the points rotate from low to high, A to B. The areas of the triangles with the same direction as the triangle to be calculated need to be added (Figure 24), the other one subtracted. This results in:

$$A_1A_4A_5 = -A_1A_2C_1 + A_2A_3C_2 + A_3A_4C_3$$

Combining these equation we see that:

$$A_1A_2A_3A_4A_5 = A_1A_2A_3 + A_1A_2B_1 + A_2A_3B_2 + -A_1A_2C_1 + A_2A_3C_2 + A_3A_4C_3$$

To calculate the total area, only the angles shown in Figure 25 needs to be known. These can be calculated using the known angles.

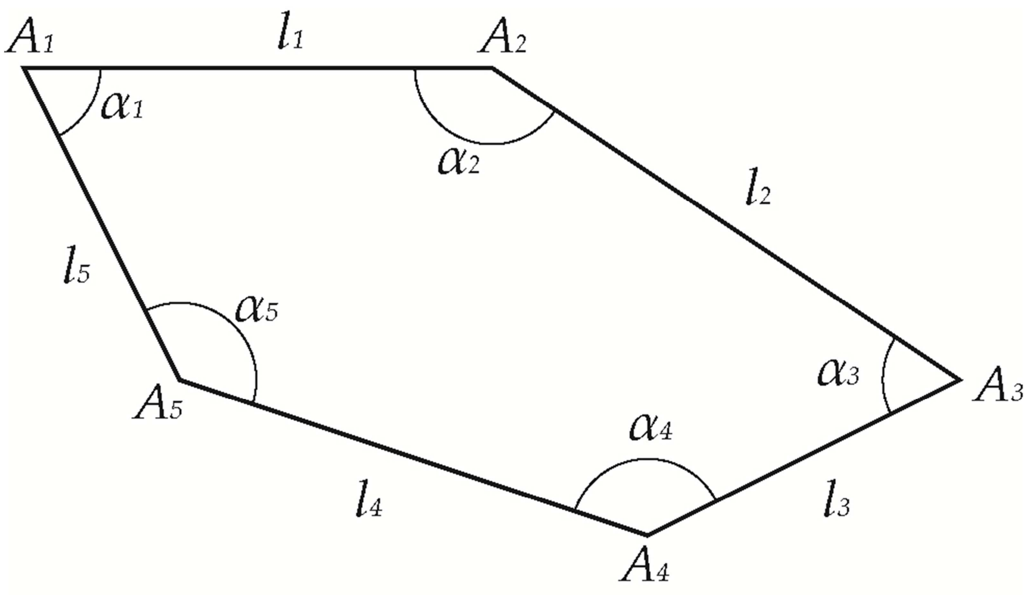


Figure 20 A random chosen polygon with only length and inner angels known

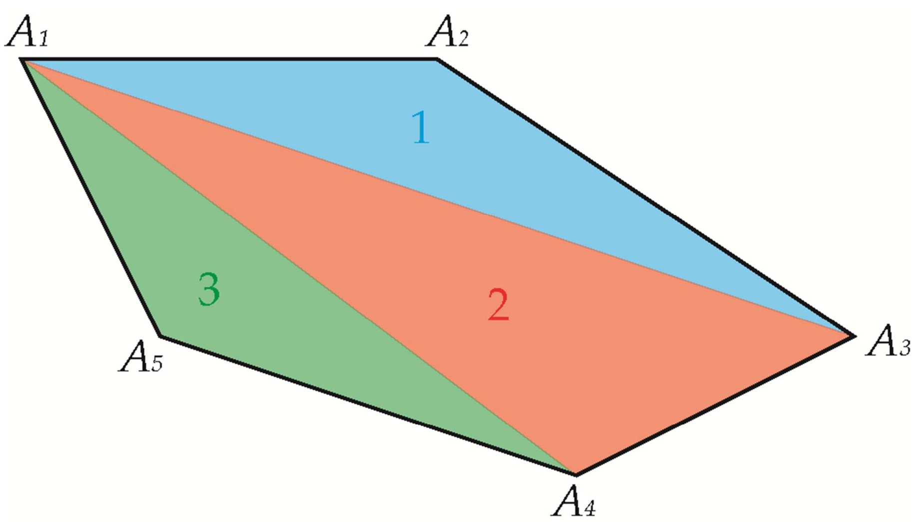


Figure 21 The polygon is split up in triangles

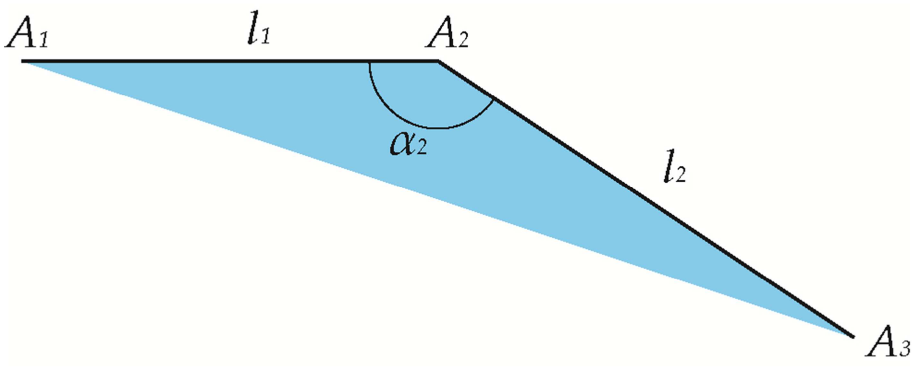


Figure 22 Calculating $A_1A_2A_3$

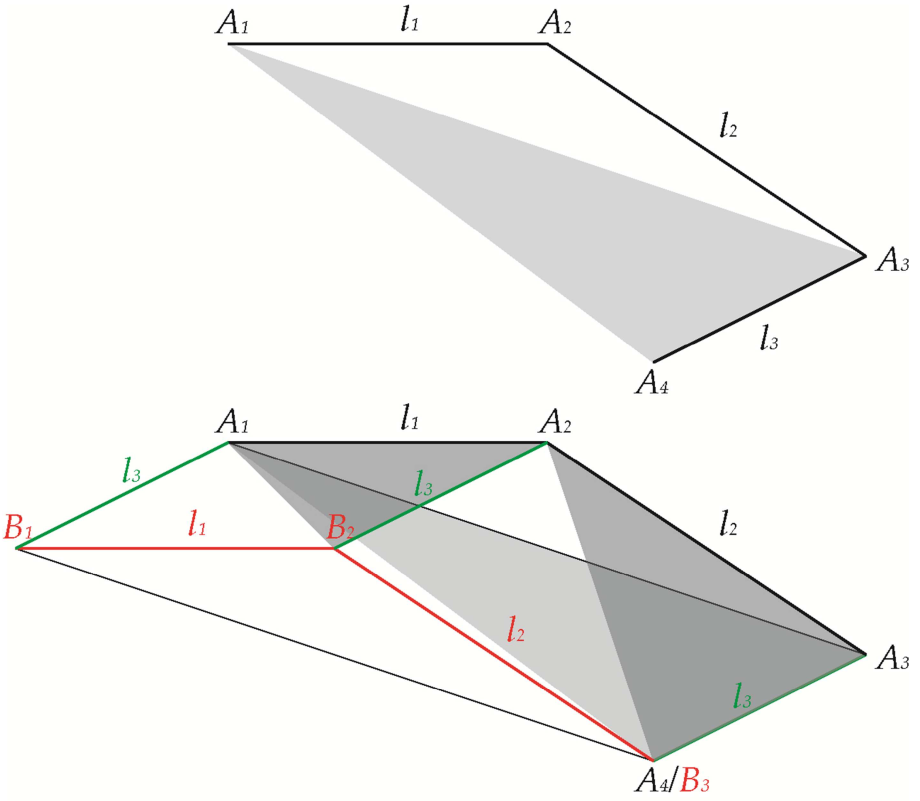


Figure 23 Calculating $A_1A_3A_4$

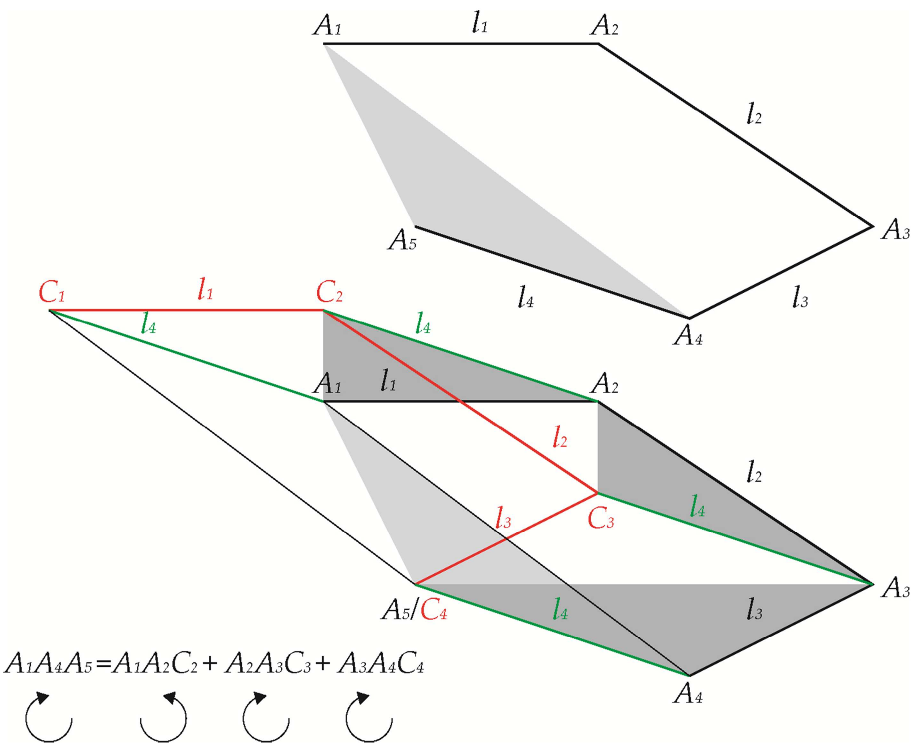


Figure 24 Calculating $A_1A_4A_5$

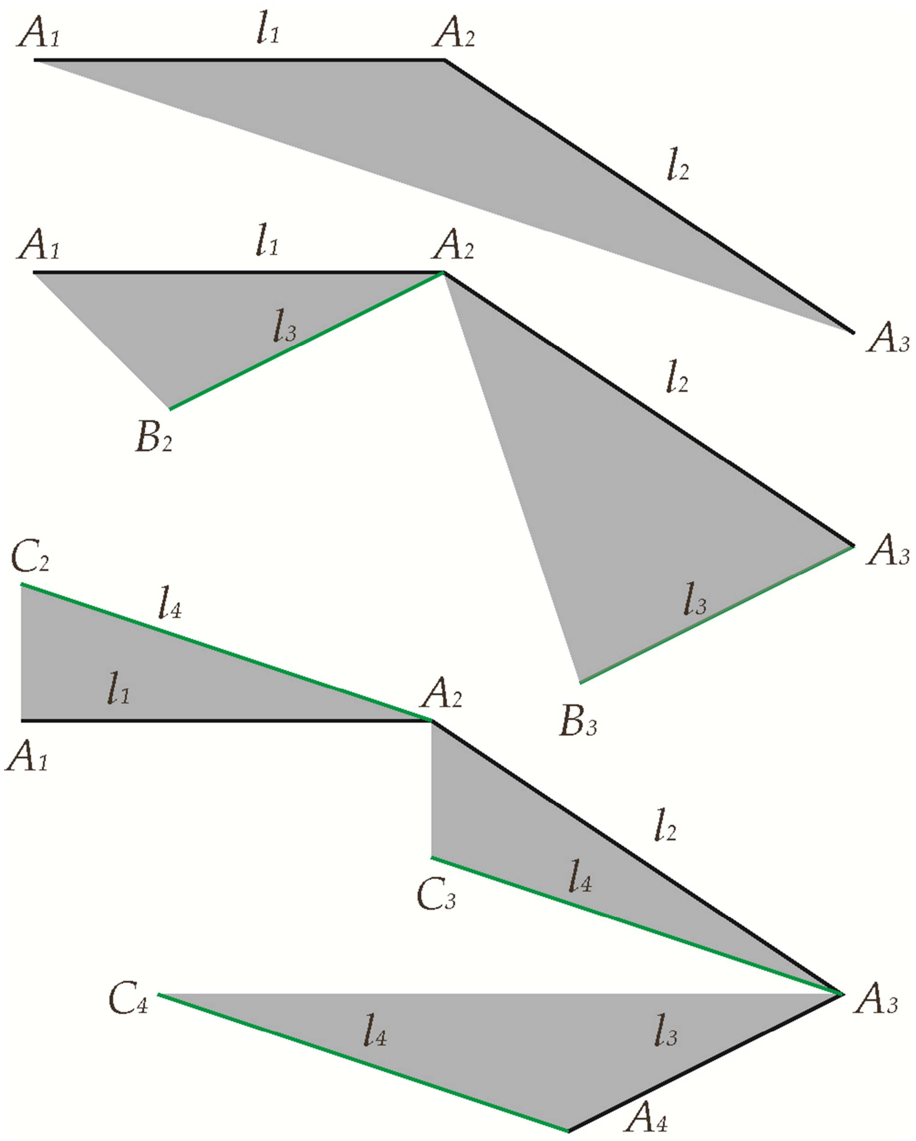


Figure 25 All angles and lengths needed to calculate the total surface of the polygon

2.7.1 Lopshits for thrust lines

In Figure 26 a simple thrust line is drawn and Lopshits is used to calculate the area. In this case not the inner angles are known but only the differences between the slopes is known and used to calculate the area. The first triangle can simply be calculated by:

$$A_1 = \frac{1}{2} l_1 l_2 \sin(180 - \beta_1)$$

For the second triangle the two red triangles in Figure 27 are calculated. The lengths are all known. The figure shows that the first inner angle equals 180 - β_1 - β_2. In a similar way all the areas can be calculated:

$$A_2 = \frac{1}{2} l_1 l_3 \sin(180 - \beta_1 - \beta_2) + \frac{1}{2} l_2 l_3 \sin(180 - \beta_2)$$

$$A_3 = \frac{1}{2} l_1 l_4 \sin(180 - \beta_1 - \beta_2 - \beta_3) + \frac{1}{2} l_2 l_4 \sin(180 - \beta_2 - \beta_3) + \frac{1}{2} l_3 l_4 \sin(180 - \beta_3)$$

$$A_4 = \frac{1}{2} l_1 l_5 \sin(180 - \beta_1 - \beta_2 - \beta_3 - \beta_4) + \frac{1}{2} l_2 l_5 \sin(180 - \beta_2 - \beta_3 - \beta_4) + \frac{1}{2} l_3 l_5 \sin(180 - \beta_3 - \beta_4) + \frac{1}{2} l_4 l_5 \sin(180 - \beta_4)$$

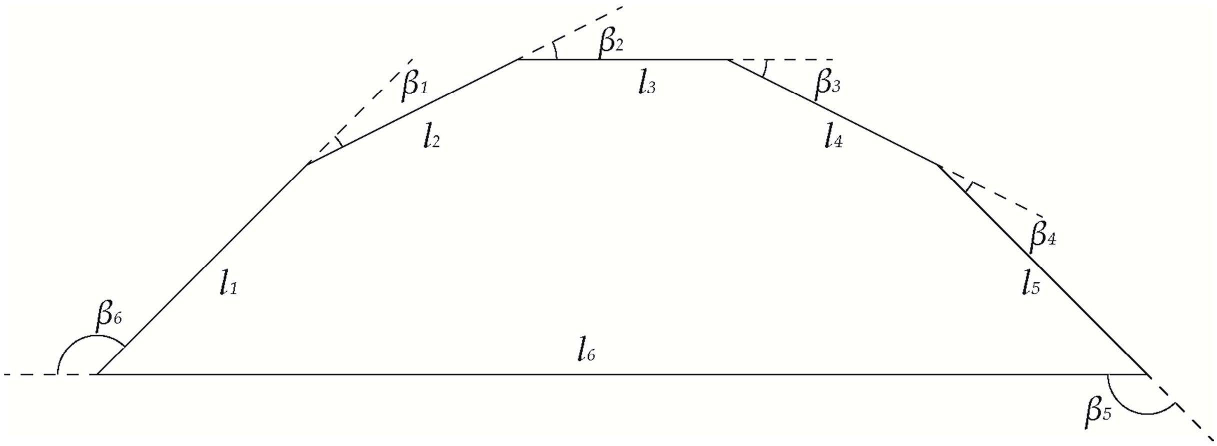


Figure 26 A thrust line with the lengths and change in slope known

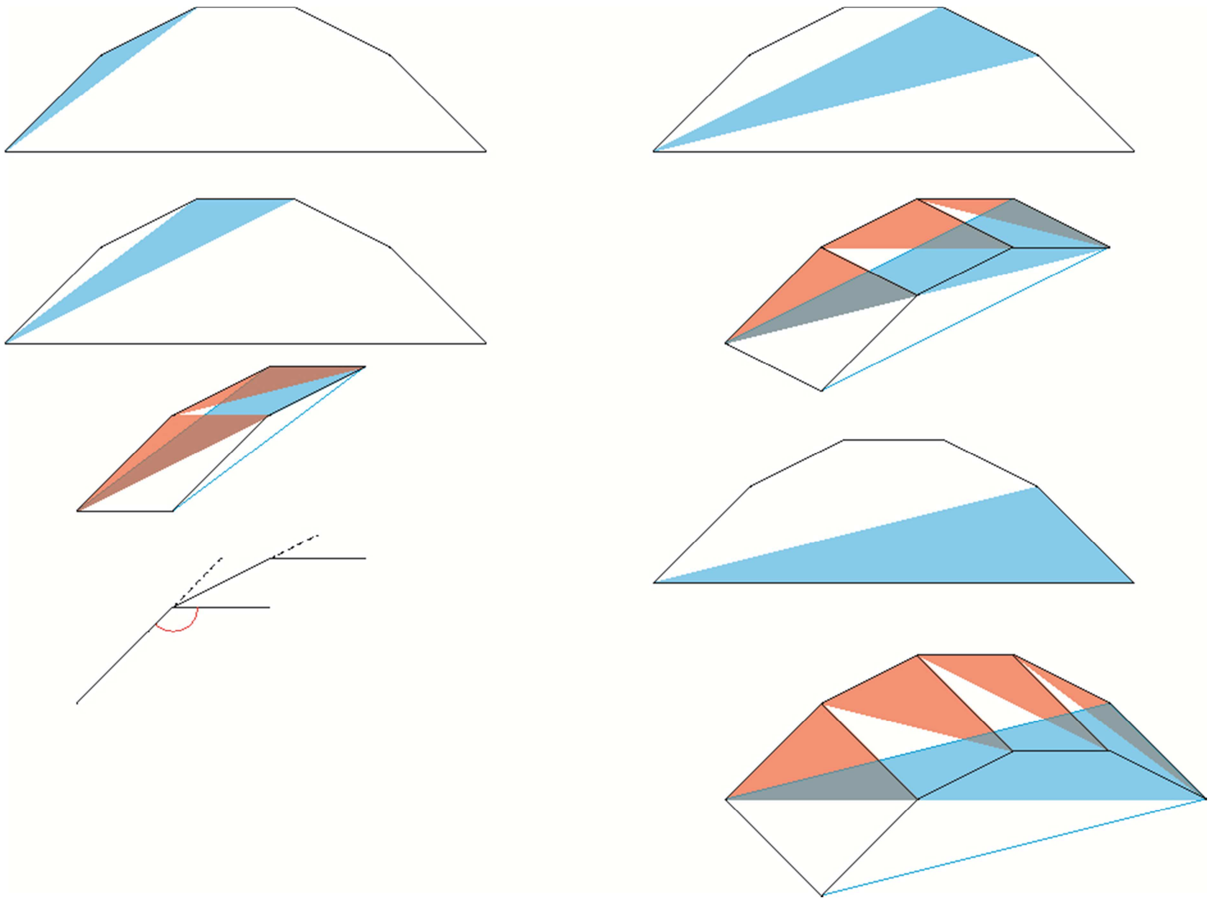


Figure 27 Calculating the area of thrust line with Lopshits equation

2.7.2 Relating the angles of a thrust line

If in a force polygon the polar coordinate moves, the thrust line changes in a uniform way. The angles of all the bars change in a uniform way as well. This shows that they are all related to each other.

Most of the attempts to relate the angles which are described in chapter 3 use a way of relating them through F_H. Another attempt in which this could be used is if we consider the Lopshits theory which uses the outer angle α or the inner angle β. These angles are of course related:

$$\beta_n = 180 - \alpha_n$$

This section shows how the angles between two forces in the force polygon can be seen as the outer angles in the thrust line.

In Figure 28 can be seen how these are related. The β angles in the force polygon can be seen in the thrust line as well. β_1 and β_6 can be found in the force polygon if we extend the horizontal supports (or the closing line in the force polygon). This line is not an actual force in this direction but it shows where this angle can be found.

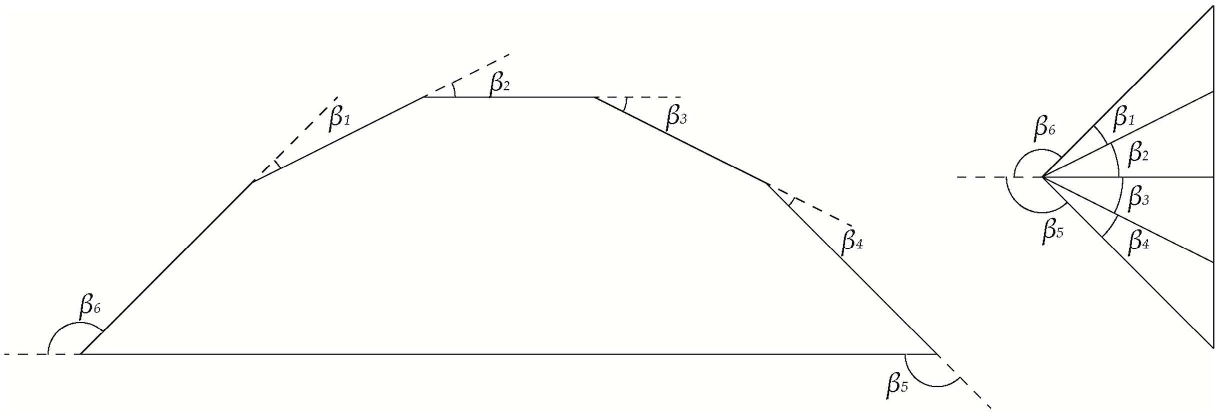


Figure 28 The outer angles of a thrust line equal the angles in the force polygon

2.8 Conclusions

The research in this paragraph gives an overview of the current status. The explanation of arches and thrust lines is well known in structural mechanics and nothing new. The only new theory is the method using equal areas, described in section 2.5. The results in this report look promising, however the theory still needs to be proven.

The next step in this research is to find a way of directly calculating the correct thrust line for arches. This thrust line holds all the information to calculate the forces in the structure. This can be done by minimizing the total energy in the thrust line. The resulting thrust line will be the correct one. Another approach is to prove the equal area method. Once one of these two methods is proven, the step to shell structures can be taken.

Chapter 3 - Calculating arches through a direct method

In section 2.5, a method is described to calculate the correct thrust line for a certain load in an iterative way. The research described in this chapter is aimed to find a direct method of calculating the correct thrust line. Section 3.1 describes an attempt to find the thrust line using the least complementary energy by normal forces only. Since this doesn't lead to a final result, an attempt to prove the theory of equal areas is described in section 3.2. In this section an equation is derived which allows for a direct calculation of the thrust line instead of an iterative one. The results of these calculations however led to some doubts as to whether the theory of equal areas is actually valid. For this reason, section 3.3 describes a comparison of different calculation methods, including a wider variety in topology of structures. From this section, the conclusion is drawn that the theory of equal areas does not always give the correct result. In section 3.4, the expectation is described that minimizing the energy resulting from a bending moment will give the correct thrust line, and is further explored.

3.1 Minimizing complementary energy from normal forces

To start off with a simple situation, the first attempt to find a direct method is done for normal forces only. The starting point is not a structure which needs to be calculated, but just a set of loads and two supports. The first subsection will discuss more on the starting point of this research. The second subsection discusses the aim of this part of research. In the third subsection an equation is derived using θ_n as the only known variable. Since this means that there are still multiple variables, the θ of each bar, the next paragraph expresses all the different angles in one single angle, θ_1. The final section discusses the results.

3.1.1 Starting situation

The aim is to calculate a thrust line for a given load and supports. These supports are not necessarily even. See for instance Figure 29, two supports and four loads and their positions together define the starting situation. For the reciprocal figure, only the loads are known, which make up the line at the most right of the figure. As described in section 2.3, the closing line can be determined as well and drawn in this partial reciprocal figure. The result is a line along which the polar coordinate can move to make a thrust line which touches both the supports. One point on this line represents the solution containing the least complementary energy.

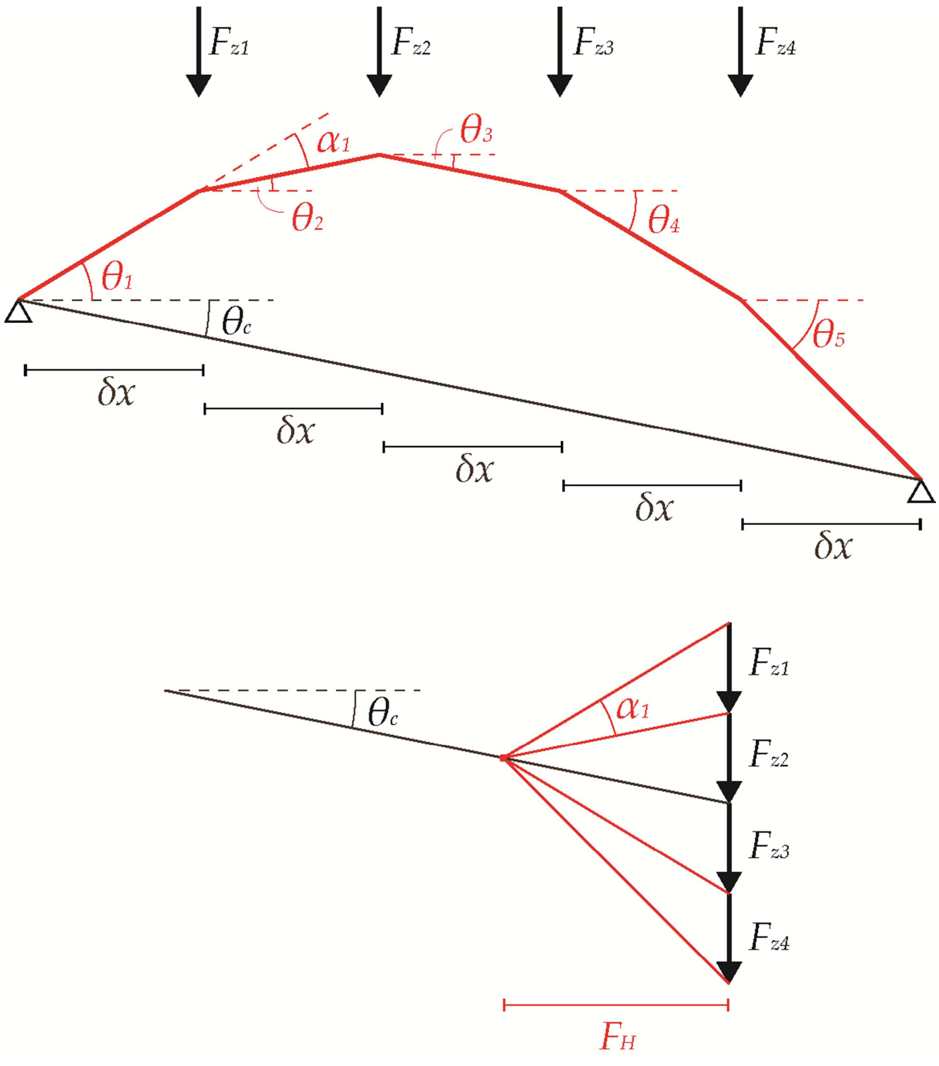


Figure 29 The initial situation in black, with the unknown values in red

In this situation, several variables (which are all dependant on each other) are unknown, others are known.

Unknown variables	*Known variables*
F_H	Fz_1 - Fz_n
$\boldsymbol{\alpha}$	$\delta x_1 - \delta x_n$
$\boldsymbol{\theta_1}$	θ_c
$\boldsymbol{\theta_n}$	

3.1.2 Aim

In this part of the research a structure is not introduced yet. The aim is to find the thrust line which can transport a certain set of load in the most efficient way. This question can be answered by finding the thrust line with the least complementary energy due to normal forces.

This complementary energy for each bar can be calculated by:

$$E_{c;N} = N^2 \cdot l$$

Moving the polar coordinate closer to the loads will result in a small normal force and a big bar length. Moving it further away will do the opposite. So in this case we are looking for the best trade-off between *N* and *l*. Complicating factor is that this trade-off will probably be different for each bar of the structure. But since the shape of all bars are related together through the reciprocal grid, there is probably a way to relate them in an analytical way.

The most logical way for calculating the correct thrust line in a direct way is by setting one variable, which defines where the polar coordinate is placed. With only this one variable the complementary energy must be calculated. Once an equation is found with only one variable, the derivative can be taken from this equation and can be set equal to zero. The solution for this equation will result in the correct thrust line.

3.1.3 Energy dependant on the angles

This section explains how the complementary energy can be calculated with only the slopes of the bars (θ) as variables. Figure 30 is a part of a polar figure in which only F_n, F_{n+1} and Fz_n are shown. In this drawing, the slope of the forces and the difference between the two slopes (α_n) are visible. F_{n+1} is extended and a new line called A is introduced. The upper angle of this triangle is called β_n. Since the two opposite angles are equal and the two right angles are equal, the angle underneath α_n is equal to β_n. From this drawing follows:

$$F_n = \frac{A}{\sin \alpha_n}$$

$$A = \cos \beta_n \cdot F_{zn}$$

$$\beta_n = \theta_n - \alpha_n$$

Since α_n is the difference between two slopes

$$\alpha_n = \theta_n - \theta_{n+1}$$

These equations combined give the equation for the force in a member expressed in the slope of that member, the member next to it:

$$F_n = \frac{\cos\ \theta_{n+1} \cdot F_{zn}}{\sin(\theta_n - \theta_{n+1})}$$

The length of a member can be expressed in the slope and the δx:

$$l_n = \frac{\delta x_n}{\cos \theta_n}$$

Substituting these equations in the simplified equation for the complementary energy as described in section 2.4 gives:

$$E_{c;Nn} = {N_n}^2 \cdot l_n$$

$$E_{c;Nn} = \left(\frac{\cos\ \theta_{n+1} \cdot F_{zn}}{\sin(\theta_n - \theta_{n+1})}\right)^2 \cdot \frac{\delta x_n}{\cos \theta_n}$$

$$= \frac{cos^2\theta_{n+1} \cdot {F_{zn}}^2}{sin^2(\theta_n - \theta_{n+1})} \cdot \frac{\delta x_n}{\cos \theta_n}$$

Which means for the complementary energy in the whole structure:

$$E_{c;N} = \sum \frac{cos^2\theta_{n+1} \cdot {F_{zn}}^2}{sin^2(\theta_n - \theta_{n+1})} \cdot \frac{\delta x_n}{\cos \theta_n}$$

The result is an equation which uses the loads and distance between the loads as input (F_z and δx) and has the θ as a variable.

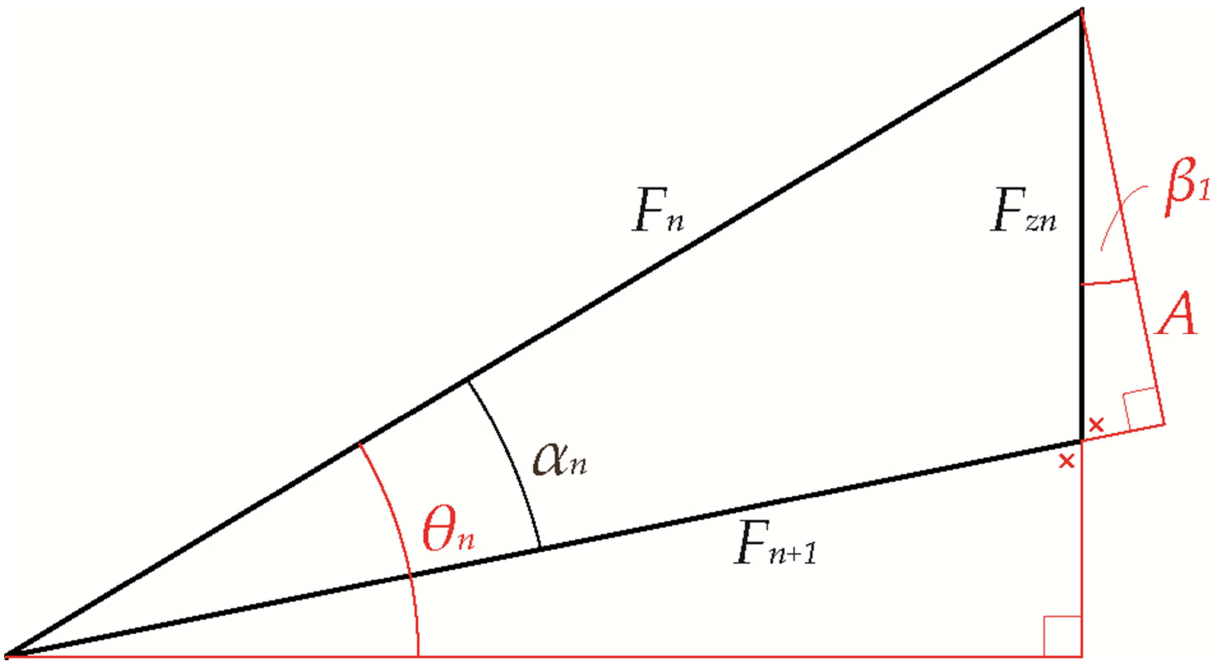


Figure 30 A part of a polar figure

This equation is tested in Excel for a simple situation and a situation with uneven supports and produces the same result as a calculation by hand. (See excel file on CD [E1504])

The next step would be taking the derivative of this function and setting it equal to zero to find the situation with the least complementary energy. There are two problems however which need to be solved in order to be able to take the derivative. The first one being the fact that there are (for each member) two variables, θ_n and θ_{n+1}. The second problem is the fact that for the whole structure, there are as many variables as there are members, from θ_1 up to θ_n. The polar figure shows that changing the polar coordinate affects all the slopes, so all the slopes are related to each other and supposedly can be expressed in one variable. This way the energy can be calculated with only one variable, which should lead to a direct method of calculating the correct thrust line. The next section describes an attempt to change this equation into having only one variable

3.1.4 Relating all angles

The first attempt was to use θ_1 as the only variable. In the initial situation (Figure 29), θ_1 determines the length and position of F_1. From this the position of the polar coordinate follows which results in all the slopes of all the forces. This means that the energy of the whole structure can be expressed in θ_1 (or in any other θ).

In Figure 31 can be seen how the horizontal support reaction F_H can be calculated from the known values. In this figure the values F_{vA} and F_{vB} are introduced, which are only temporarily needed.

$$F_H = \frac{F_{vA}}{\tan\theta_1}$$

$$F_H = \frac{F_{vB}}{\tan\theta_c}$$

And if we define F_{vn} as the vertical distance from the point where the closing line intersects the loads to the point where F_n crosses the loads:

$$F_{v1} = F_{vA} + F_{vB} = (\tan\theta_1 + \tan\theta_c) \cdot F_H$$

$$\tan\theta_n = \frac{F_{Vn}}{F_H} = F_{Vn} \cdot \frac{\tan\theta_1 + \tan\theta_c}{F_{v1}}$$

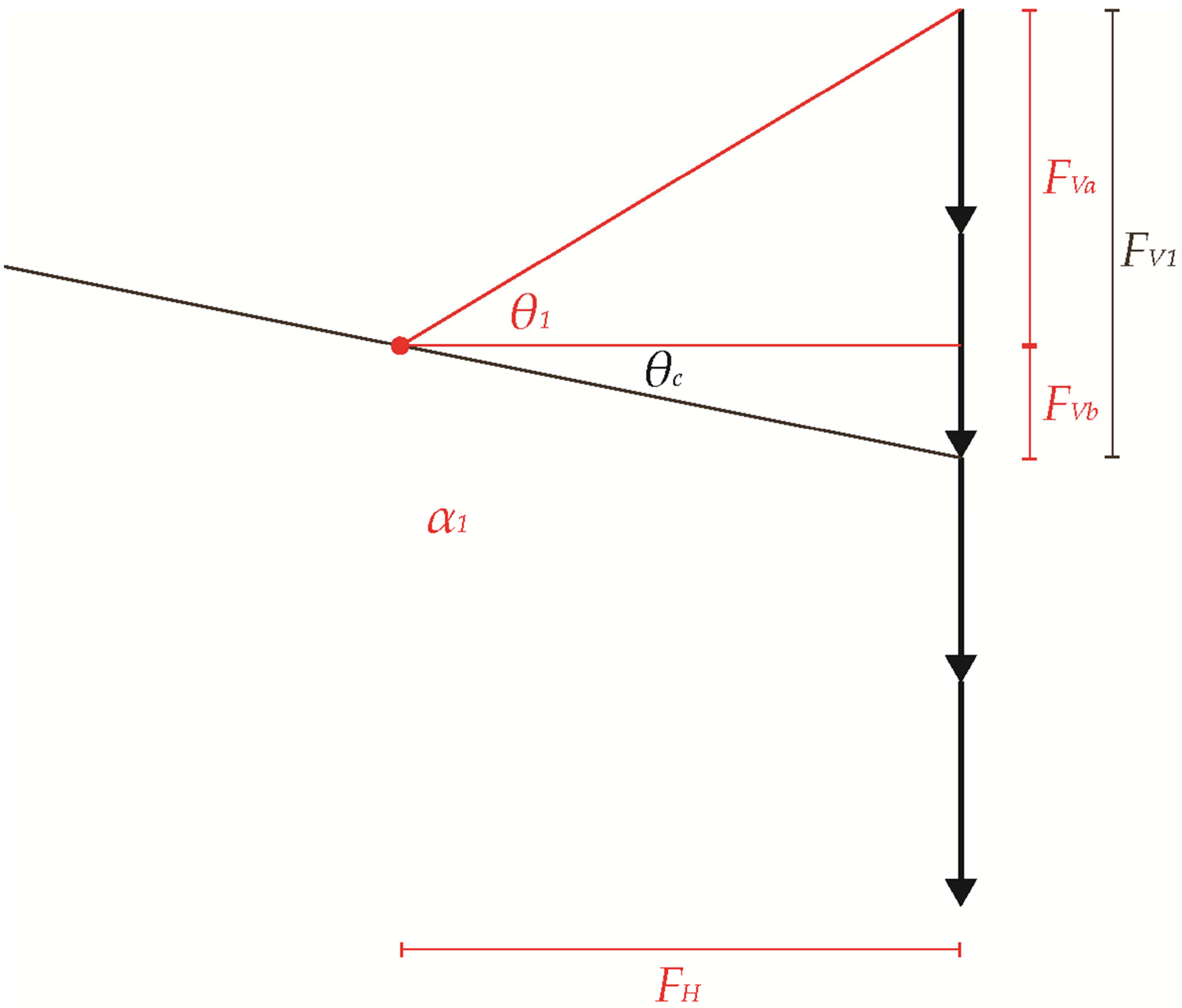


Figure 31 Calculating all angles from θ_1

3.1.5 Conclusions

A method is found to calculate the energy in a certain thrust line, depending on all the angles of the structure bars. It is obvious that these angles are all related, it can be seen from the way the structure changes with a change in the force polygon. If one angle is set, there is only one possibility of what the whole thrust line could look like. This relation is described in the previous section. These two section provide two equations:

$$E_c = \sum \frac{cos^2\theta_{n+1} \cdot F_{zn}{}^2}{sin^2(\theta_n - \theta_{n+1})} \cdot \frac{\delta x_n}{\cos\theta_n}$$

$$\tan\theta_n = \frac{F_{Vn}}{F_H} = F_{Vn} \cdot \frac{\tan\theta_1 + \tan\theta_c}{F_{v1}}$$

So once θ_n is now expressed in known values, the θ-values in the E_c equation can be substituted resulting in a complete equation for the complementary energy of a thrust line. The next step would be taking the derivative of this new equation and setting it equal to zero. This should result in a direct method of calculating the minimum complementary energy. However it turns out that this equation a very extensive one and not very insightful as well. Since the aim of this research is to find a direct and insightful method, this path of research is not continued. For this reason, the extensive equation is not included in this report, and the search continued for a simpler way of calculating the minimum energy.

3.2 Calculating arches using areas of thrust line and structure

Since minimizing the normal energy didn't lead to a method of calculating the thrust line, another approach is applied. This time it is based on the presumption described in chapter 2.5, that the thrust line with an equal area to the area under the structure is the thrust line containing the least complementary energy. The idea is that if both areas are expressed in a mathematical way and related together, the result might be a new starting point from which this theory can be proven.

The first section shows a direct method of calculating the thrust line with an equal area. In the second section is described how this is translated into a Grasshopper algorithm.

3.2.1 Calculating the area under a structure

The area under the structure can be represented by the rectangles shown in Figure 32. All the areas can be calculated by

$$A_{str} =$$

$$\delta x_1 \cdot \frac{1}{2} \cdot \delta y_1$$

$$+ \delta x_2 \left(\delta y_1 + \frac{1}{2} \cdot \delta y_2 \right)$$

$$+ \delta x_3 \left(\delta y_1 + \delta y_2 + \frac{1}{2} \cdot \delta y_3 \right)$$

$$+ \delta x_4 \left(\delta y_1 + \delta y_2 + \delta y_3 + \frac{1}{2} \cdot \delta y_4 \right)$$

$$+ \delta x_5 \left(\delta y_1 + \delta y_2 + \delta y_3 + \delta y_4 + \frac{1}{2} \cdot \delta y_5 \right)$$

or more general:

$$A_{str} =$$

$$\delta x_1 \cdot \frac{1}{2} \cdot \delta y_1$$

$$+ \delta x_2 \left(\delta y_1 + \frac{1}{2} \cdot \delta y_2 \right)$$

$$+ \delta x_3 \left(\delta y_1 + \delta y_2 + \frac{1}{2} \cdot \delta y_3 \right)$$

$$+ \ \ldots$$

$$+\delta x_n\left(\delta y_1+\ldots+\delta y_{n-1}+\frac{1}{2}\cdot\delta y_n\right)$$

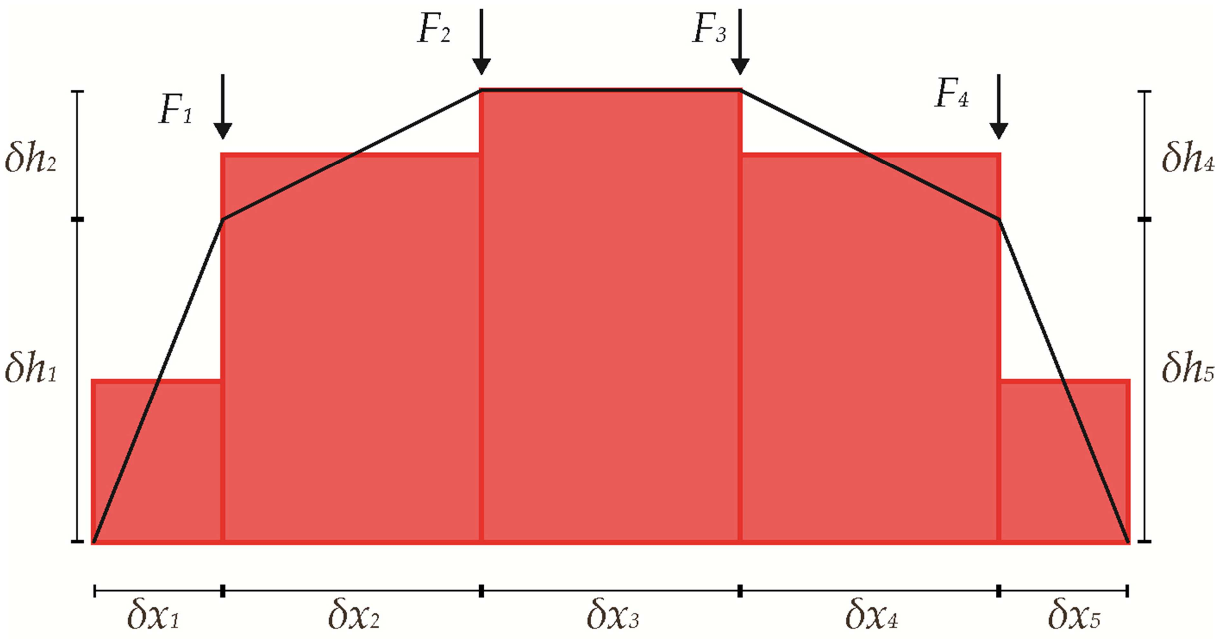


Figure 32 Calculating the area of a structure

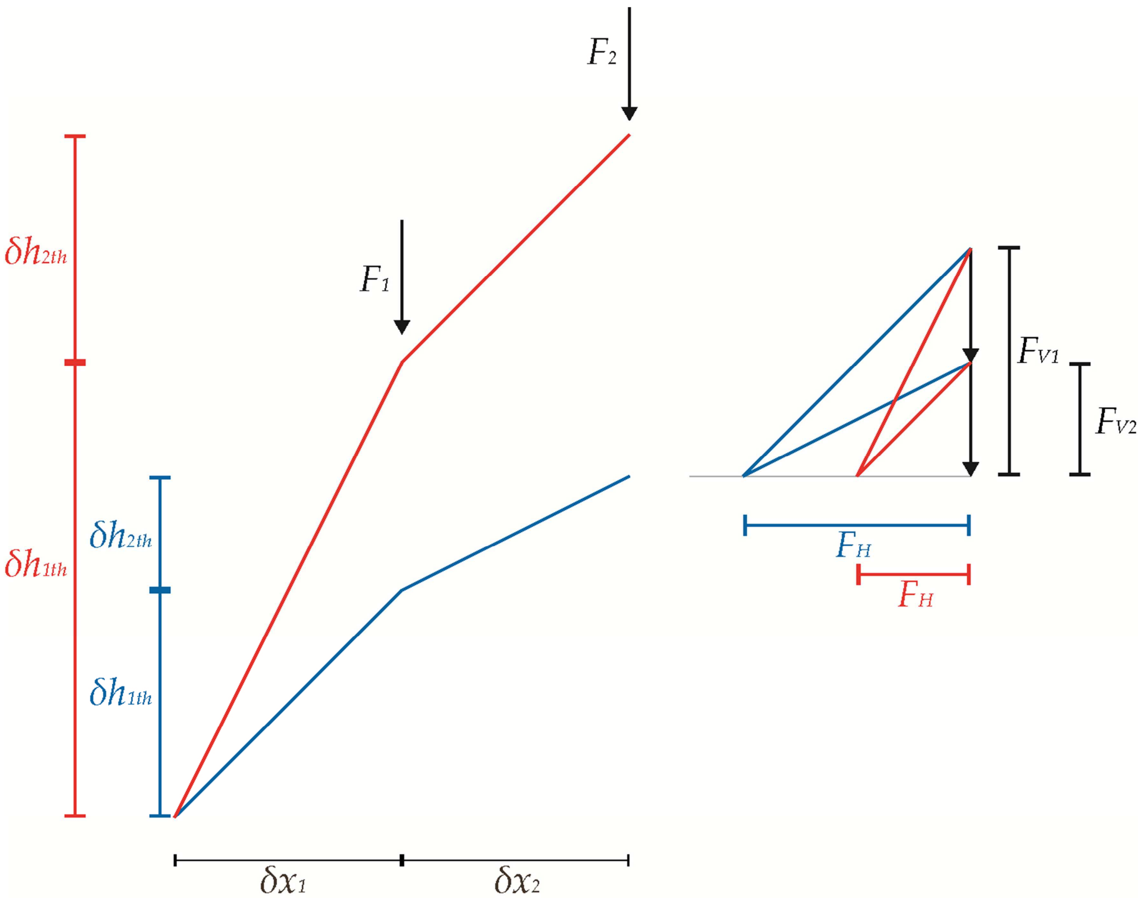


Figure 33 The force polygon related to the thrust line

In Figure 33 is shown that the area under the thrust line depends on the variable F_H. The other variables, such as F_{Vn} are determined by the loads, which are not variable. $\delta y_{th\,n}$ is variable as well but this variable can be expressed in F_H:

$$\frac{F_{Vn}}{F_H} = \frac{\delta y_{th\,n}}{\delta x_n}$$

which means that:

$$\delta y_{th\,n} = \frac{F_{Vn} \cdot \delta x_n}{F_H}$$

The area under the thrust surface can be calculated in a similar way as the area under the structure. δy is substituted using the equation above.

$$A_{th} =$$

$$\delta x_1 \cdot \frac{1}{2} \cdot \frac{F_{V1} \cdot \delta x_1}{F_H}$$

$$+ \delta x_2 \left(\frac{F_{V1} \cdot \delta x_1}{F_H} + \frac{1}{2} \cdot \frac{F_{V2} \cdot \delta x_2}{F_H} \right)$$

$$+ \delta x_3 \left(\frac{F_{V1} \cdot \delta x_1}{F_H} + \frac{F_{V2} \cdot \delta x_2}{F_H} + \frac{1}{2} \cdot \frac{F_{V3} \cdot \delta x_3}{F_H} \right)$$

$$+ \ldots$$

$$+ \delta x_n \left(\frac{F_{V1} \cdot \delta x_1}{F_H} + \ldots + \frac{F_{V(n-1)} \cdot \delta x_{n-1}}{F_H} + \frac{1}{2} \cdot \frac{F_{Vn} \cdot \delta x_n}{F_H} \right)$$

Since every term contains the factor $\frac{1}{H}$ the equation can be rewritten as:

$$F_H = \frac{1}{A_{th}} \cdot \left(\delta x_1 \cdot \frac{1}{2} \cdot F_{V1} \cdot \delta x_1 + \delta x_2 \left(F_{V1} \cdot \delta x_1 + \frac{1}{2} \cdot F_{V2} \cdot \delta x_2 \right) + \cdots + \delta x_n \left(F_{V1} \cdot \delta x_1 + \ldots + F_{V(n-1)} \cdot \delta x_{n-1} + \frac{1}{2} \cdot F_{Vn} \cdot \delta x_n \right) \right)$$

Since we assume $A_{th} = A_{str}$, the equation for the area of the structure can be substituted, resulting in:

$$F_H = \frac{\delta x_1 \cdot \frac{1}{2} \cdot F_{V1} \cdot \delta x_1 + \delta x_2 \left(F_{V1} \cdot \delta x_1 + \frac{1}{2} \cdot F_{V2} \cdot \delta x_2 \right) + \ldots + \delta x_n \left(F_{V1} \cdot \delta x_1 + \ldots + F_{V(n-1)} \cdot \delta x_{n-1} + \frac{1}{2} \cdot F_{Vn} \cdot \delta x_n \right)}{\delta x_1 \cdot \frac{1}{2} \cdot \delta y_1 + \delta x_2 \left(\delta y_1 + \frac{1}{2} \cdot \delta y_2 \right) + \delta x_3 \left(\delta y_1 + \delta y_2 + \frac{1}{2} \cdot \delta y_3 \right) + \ldots + \delta x_n \left(\delta y_1 + \ldots + \delta y_{n-1} + \frac{1}{2} \cdot \delta y_n \right)}$$

With this equation, the correct F_H can be calculated from a given structure resulting in the correct thrust line.

3.2.2 Generating the thrust line

The method above gives a direct way to calculate the F_H of a certain structure. Until now it was only possible to do this in an iterative way as described in section 2.5. This method is written in Grasshopper, a Rhino-plugin. Figure 34 shows the input and the output of the component. The actual algorithm is summarized in Figure 35. As a first step, a curve is drawn and set as input of for the algorithm. The end points are marked as supports. For two situations the load is determined. In one situation a q-load is projected onto the surface. The other situation has its own weight as input. The algorithm allows choosing one of these. In the next step the load is discretized in point loads. This discretization is based on the chosen δx or the segment length, depending on whether it is projected q-load or the own weight of the structure. In the next step the area of the structure is determined, using a Grasshopper function. After that, F_H is calculated by the equation which is shown in the previous section. With a known F_H the force polygon is drawn and the thrust line is generated.

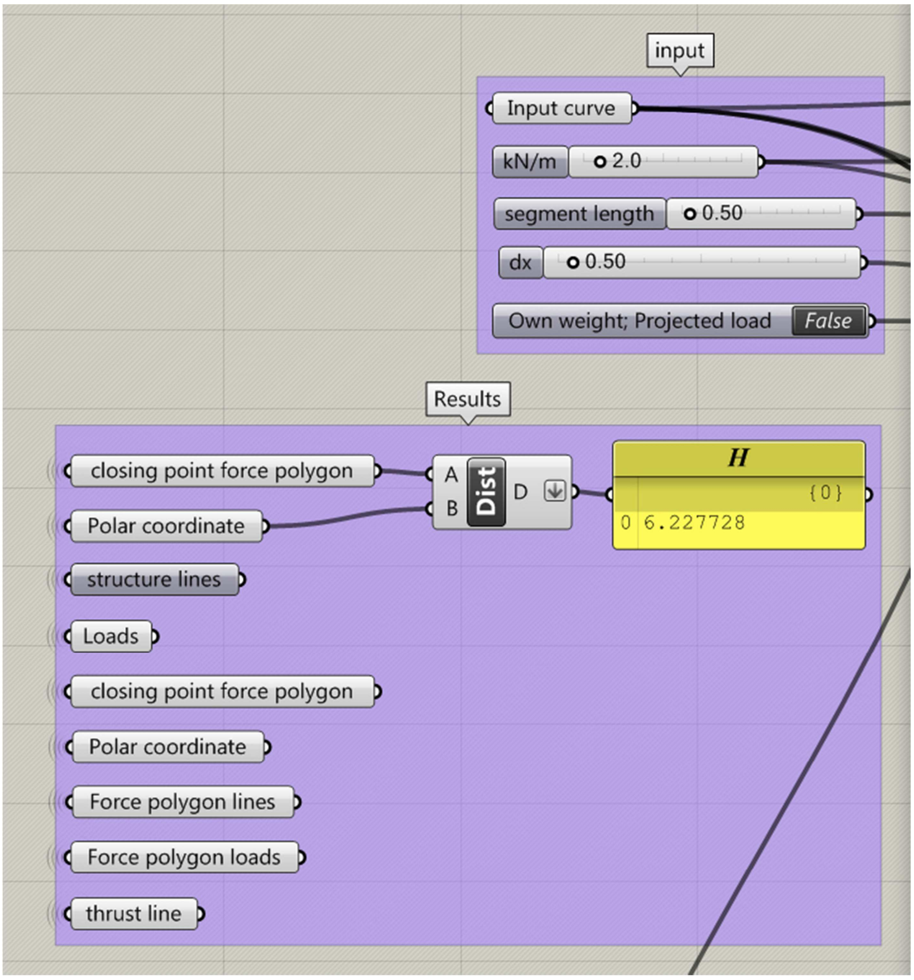


Figure 34 The in- and output of the Grasshopper algorithm

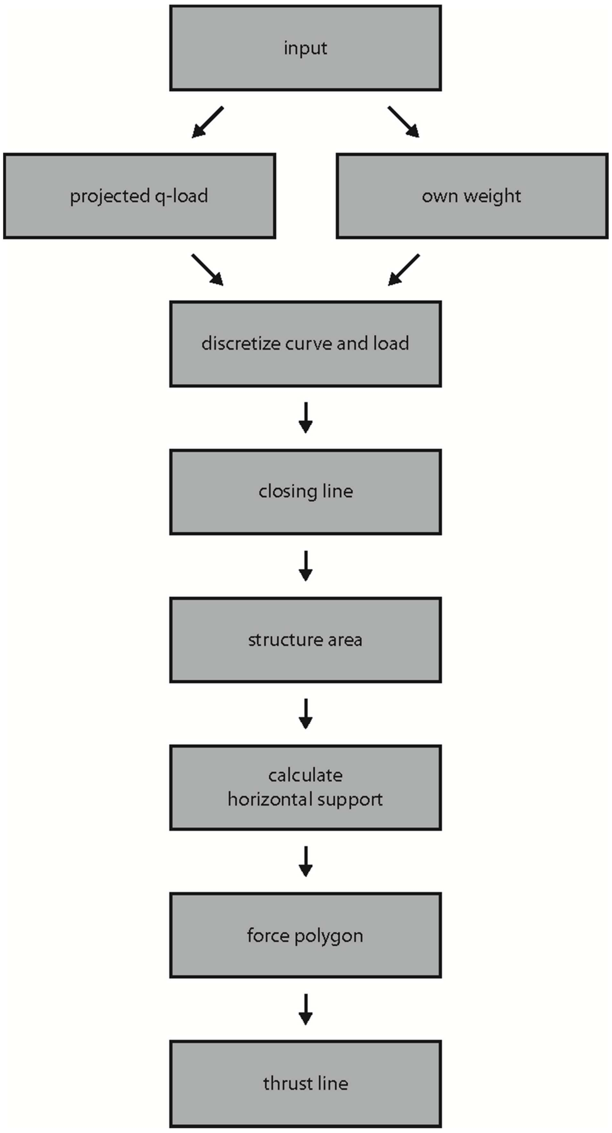


Figure 35 The algorithm summarized

3.2.3 Comparing to FEM

To verify whether or not this equation works, three calculations were done. These calculations only deal with the question whether or not this equation can predict the thrust line containing the lowest complementary energy in a linear way. The control calculations are done in a non-linear way. The energy of a certain structure combined with a thrust line is calculated in excel, based on a randomly chosen F_H. The value for F_H is changed until the lowest complementary energy is found. Subsequently the direct method as described in the previous section is applied to the same structure. Finally the calculations are done using the Finite Element Method as well. If these calculations lead to the same result for F_H, this equation can be considered a valid method to determine the thrust line with the least complementary energy. The calculations in this section can be found in a spreadsheet and GSA files on the CD with the file name [0306].

Calculation 1: a thrust line as structure

For the first calculation, a structure and loading is used from which is known that the thrust line will coincide with the structure (Figure 36). This way, bending moments don't play a role in the calculation, making it a first step in verifying the validity of this method.

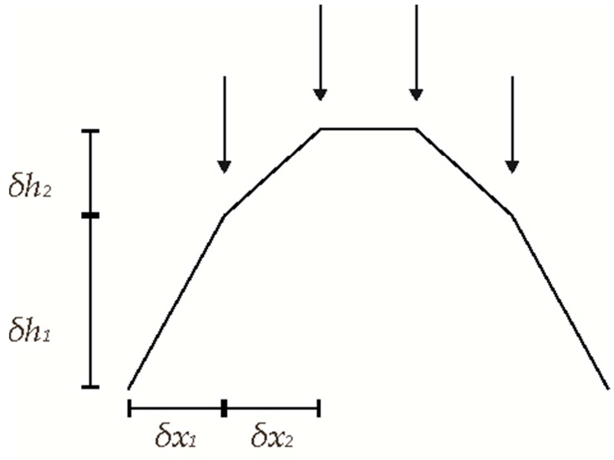


Figure 36 A structure with coincides with a thrust line

First the energy is calculated by using $E_{c;N} = N^2 \cdot l$ to calculate the energy for each bar. Bending moments are not included in this calculations because they don't occur in this structure. The energy of these bars are added together for the total complementary energy of the structure. In the table below, the results of changing F_H can be seen. This shows that, according to this method, the H is 2,22, making the thrust line coincide with the structure.

F_H	$\Sigma E_{c,N}$
2,21	230,4354
2,22	230,4348
2,23	230,4409

The F_H is calculated in excel as well using above method. This results in a F_H of 2,22 N. The horizontal support calculated using FEM also results in 2,22 N.

Calculation 2: a symmetrical structure

For the second verification, a random structure and loading is chosen, expecting the thrust line to be outside the structure (Figure 37).

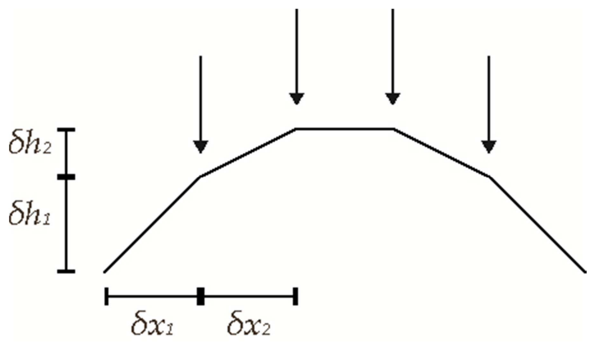


Figure 37 A symmetrical structure with a thrust line outside the material

In section 2.2 is described how, using the thrust line outside the material, the forces can be projected on the structure by adding the eccentricity of the forces to the equation. This is needed for the non-linear calculations, used to check the new method. Since this is not easily done in excel, an approximation is done. The bending moments should be projected on the point perpendicular to the thrust line. This results in a change in bending moment throughout the bar. To keep the calculations simple, for each bar the bending moment is calculated from the thrust line within the same δx. The eccentricity is calculated from the thrust line halfway each bar. This way the results will lose accuracy but it is a simple first calculation to see whether the results are promising. Once the forces and bending moments are determined, a check can be done. Using $E_{c;N,M} = \left(N^2 + \frac{12(N \cdot e)^2}{t^2}\right) l$ the complementary energy due to normal forces and bending moments is calculated. The overall thickness is considered to be 0,1 m. The results in the table below show that the non-linear calculation gives 4,00 N as the F_H with the lowest complementary energy. The direct calculation gives 4,00 as a result as well. The FEM-calculations result in a horizontal support of 3,99 N.

F_H	$\Sigma E_{c,N,M}$
3,99	307,6804
4,00	302,4621
4,01	309,6312

Calculation 3: an irregular shaped arch

The third calculation is conducted in a way similar to the second. The only difference is that this structure is more irregular shaped (Figure 38). The results of the direct calculation method gives a F_H of 1,89 N. The table below shows that the indirect method results in a F_H of 1,70 N, making this result a little bit off. The FEM calculations result in a horizontal support of 1,70 N as well.

F_H	$\Sigma E_{c,N,M}$
1,69	58004,4
1,70	57993,66
1,71	58016,29

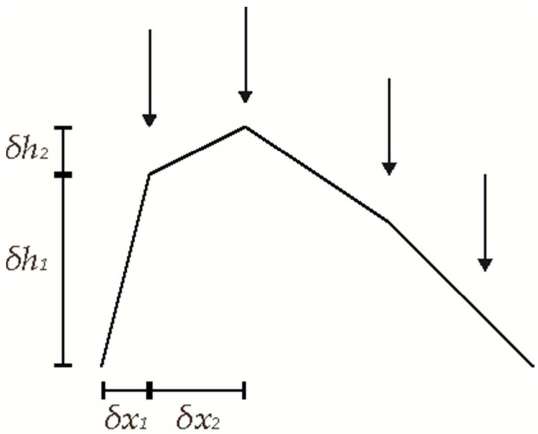


Figure 38 A irregular shaped structure

3.2.3 Conclusions

A new method is described in which the equal area method can be applied in a direct way, instead of an iterative way. Since the resulting equation only demonstrates how the theory of equal areas can be used, but does not prove it, research is done into relating this to the minimum energy using Lopshits equation (section 2.7). Since both equations look a lot alike, there is supposed to be a link.

The equation is translated into a Grasshopper algorithm, resulting in a 2D program in which any 2-dimensional structure can be drawn which will instantly output the thrust line with the same area. This can quite easily be extended with functions like outputting the bending moment diagram for instance. The way this is calculated is described in section 2.2, one can imagine that building such an extension is a simple next step.

The third section is used to check the validity of this method. The first two calculations give a very accurate result, the third is slightly off. There is no simple explanation for this, the minimum energy method does work, since it outputs the same results as the FEM method. An explanation could be that this method is only valid for simple symmetrical situations. These doubts about the validity of this method will be tested in the next section.

Since these doubts arose, the research to connect the equation through Lopshits, and the research to extend the Grasshopper algorithm was not continued.

3.3 Comparing different methods

To see whether the equal area method works, it is compared to other methods in the previous section. During this comparison, doubts arose about the validity of the method using equal areas. For this reason, the comparison was extended to cases with point loads as well. For the comparison, two structures are chosen as can be seen in Figure 39. These structures are chosen with a constant δx and only four segments because they can be used for all the methods. The first shape is chosen as a regular shape which is not a thrust line for an even distribution of loads. The second shape is chosen as a shape with a bigger difference from such a thrust line, to see how good each method is when larger bending moments are needed.

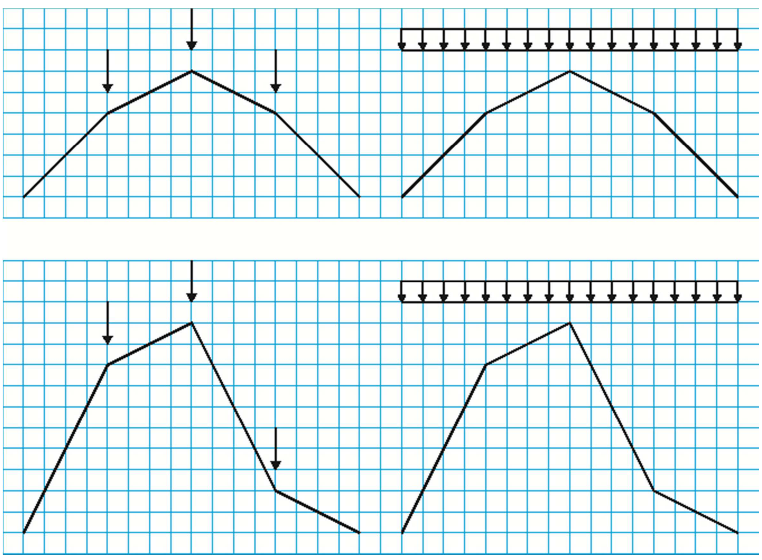


Figure 39 The two structures used to test different calculation methods

3.3.1 Methods used

FEM

The Finite Element Method calculations are done using GSA Suite. This method is a reliable method which is used in current practice. It is used as a method from which we can be sure that it is correct. The results of both other methods are compared with this method to see how well they perform.

Minimum E_c

The method using the minimum E_c is for the point loads calculated in an excel file as described in section 3.2.3. Through trial and error (so in an iterative way) the situation with the lowest complementary energy is found. [excel files on CD 0209(1) en 0209(2)]

Since this excel file doesn't allow for q-loads to be entered, the loadcase with the q-load is calculated using a grasshopper algorithm which is an adaptation of an algorithm created by van Dijk (2014). This is done through an iterative method as well by changing F_H until the lowest complementary energy is found. [GH file on CD 0209 (improved)]

Equal areas

The equal area method is calculated for the q-load using the equation derived in section 3.2.1. This is done in Grasshopper using the file described in section 3.2.2

The case with the point loads is calculated with the same excel files which are used for the minimum E_c method. The situation with equal areas is found in an iterative way.

3.3.2 Results

The results are shown in the tables below. All the minimum E_c calculations and FEM calculations give a very good result. For the first shape, the equal area method gives a slight difference of only 1,1% and 2,3%. The second shape gives bigger differences of 11,7% and 13,1%.

Shape 1 – point loads

method	**H**
equal A	2,86
min. E_c	2,83
FEM	2,83

Shape 1 – projected q-load

method	**H**
equal A	12,19
min. E_c	11,96
FEM	11,96

Shape 2 – point loads

method	**H**
equal A	2,00
min. E_c	1,79

FEM	1,79

Shape 2 – projected q-load

method	**H**
equal A	8,53
min. E_c	7,55
FEM	7,54

3.3.2 Conclusions

The differences, especially for the second shape, are too big to be explained by discretization errors or rounding errors. From these calculations can be concluded that the method of equal areas actually does not work for calculating the correct thrust line. The reason why it seems to work can be explained with the help of Figure 40. The correct thrust line is the one with the lowest complementary energy. This energy is dependent on the bending moment and the normal force.

$$E_c = N^2 l + \frac{12}{t^2} M^2 l$$

Since the thickness t of the structure will often be quite small in comparison to the width of the total structure and thus the $\sum l$ as well, the bending moment will have a much bigger influence on the total energy then the normal force. The bending moment can be calculated by:

$$M = e_V \cdot H$$

The figure shows that for symmetrical arches with not too irregular geometry, the thrust line calculated by equal areas (shown in red) lies very close to the thrust line calculated through minimum energy (shown in green), because the eccentricity e is very small for equal areas. So as long as the arches to be calculated are chosen quite conventional, the equal area method will result in small errors. The structure to the left shows that for more irregular shapes the lines are further apart.

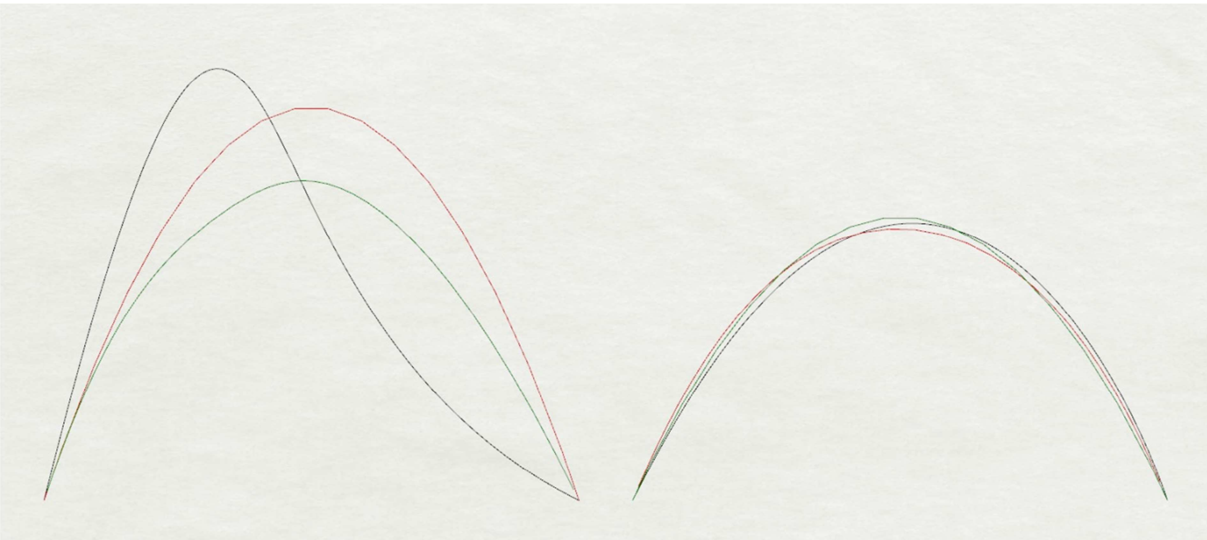


Figure 40 Structures (in black) with both the thrust line calculated using equal areas (red) and minimum energy (green)

3.4 Minimizing complementary energy from bending moment

Since the method of equal areas does not give the correct thrust line, a new method needs to be found. This section focuses on the idea that the thrust line with a minimum total energy is approximately the same as the thrust line with a minimum bending energy. The reason for this is the equation for the total energy as introduced in chapter 2.4:

$$E_c = \left(N^2 + \frac{12}{t^2}M\right)l$$

Since the thickness *t* of the arch or shell will be quite small in relation to the length *l*, and it is squared as well, the bending moment will often be of a much bigger influence on the total than the normal force will be.

The first subsection explains in more detail than the equation above how the bending energy can be calculated. The second subsection is a description of a proof written to show that for a certain situation, the thrust line with a minimum bending energy is the same thrust line as the one having an equal area to the structure. This proof was written when trying to prove the method of equal areas. It is however included in this chapter because it also illustrates a way of finding the situation with the minimum bending energy. The third section shows how this proof can be translated into a method of minimizing bending energy for a more complex situation, giving a more general method. The fourth subsection compares the results of a thrust line with a minimum bending energy to thrust line with a minimum total energy, to see how accurate this method is.

3.4.1 Calculating the bending energy

The energy due to bending moments can be calculated, as mentioned before, by $E_{c;M} = \frac{12}{t^2}M^2 l$. There are more steps involved to get from a drawing with a force polygon, a thrust line and a structure to the energy from a bending moment in a bar. This is somewhat more complex than the energy due to the normal forces, especially because the bending moment changes throughout a bar and the normal forces doesn't. Consider for instance the situation in Figure 41. A part of a structure is shown together with a part of the corresponding thrust line. The force drawn in this thrust line can usually be obtained by the corresponding force polygon which is not shown in this case. There are several different ways of determining the bending moment in a certain point of the structure, from which three will be discussed here.

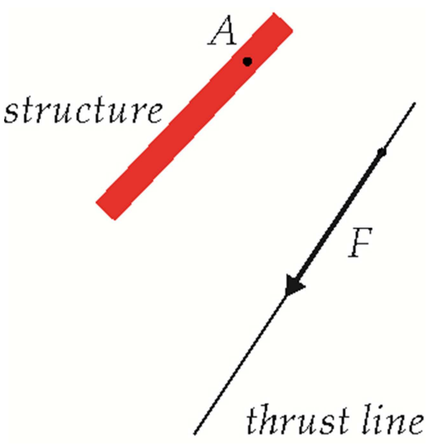


Figure 41 Thrust line outside the material

The first method (Figure 42.a) is to draw a line perpendicular to the force in the thrust line through point A. The bending moment equals the length of the eccentricity e times the force in the thrust line. The advantage of this method is that the force can be directly obtained from the force polygon.

$$M = e_{\perp F} \cdot F$$

The second method (Figure 42.b) shows a line perpendicular to the structure from point A to the thrust line. The force needs to be decomposed into one perpendicular to the eccentricity and one parallel to the eccentricity, resulting in:

$$M = e_{\perp str} \cdot F_N$$

In the third method (Figure 42.c) the eccentricity is chosen vertically. The corresponding component of the force is the horizontal component, which can be obtained from the force polygon as well.

$$M = e_V \cdot F_H$$

The advantage of this method is that for a vertical load, the force needed to calculate this bending moment will be equal throughout the whole structure.

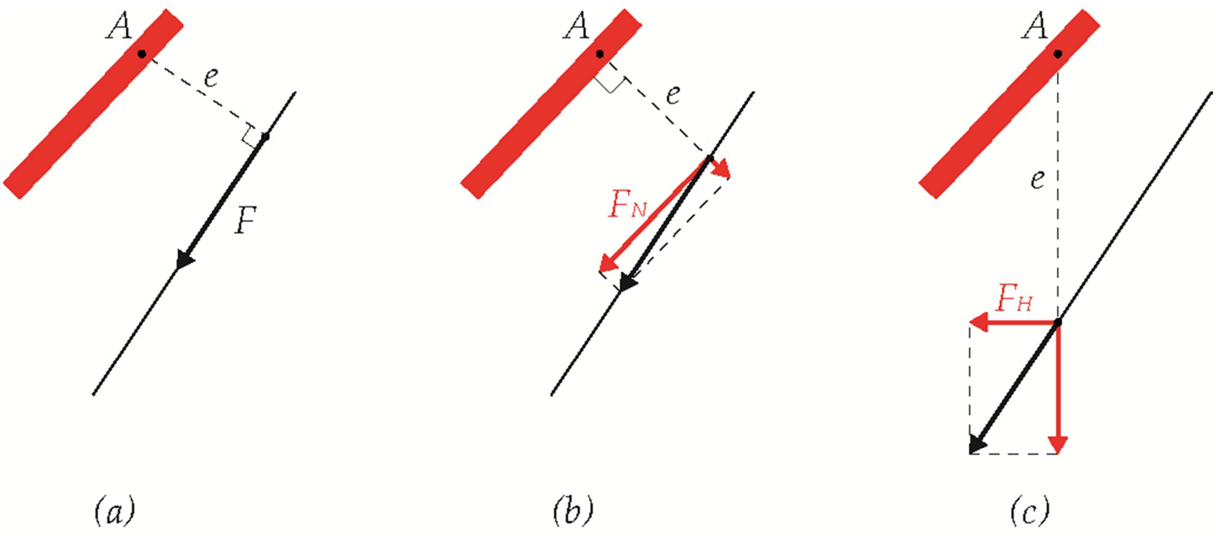


Figure 42 Three different ways to calculate the bending moment from the thrust line

To calculate the energy for a structure, the example of Figure 43 is used. The structure, force polygon and thrust line are shown. The first step is calculating the e_V, the vertical difference between the thrust line and the structure. This difference is plotted in Figure 43.b. Figure 43.c shows the bending moment throughout the structure which equals the first plot times F_H. Since these graphs show the bending moment changing over δx, and the change over the length is needed to calculate the energy, the x-axis is changed to the length of each bar, resulting in Figure 43.d. To get a better understanding of what this step does, one can also consider this as being the bending moment diagram, which is usually drawn directly on the structure, in which the structure is taken as the x-axis (Figure 44). In the next step, the graph with the bending moments is squared, since the energy consists of the bending moment squared times the length. This result in the Figure 43.e in which the area between the x-axis and the graph equals the total $E_{c;M}$.

This method will be used in the following sections to find an analytical way for calculating the minimum complementary bending energy.

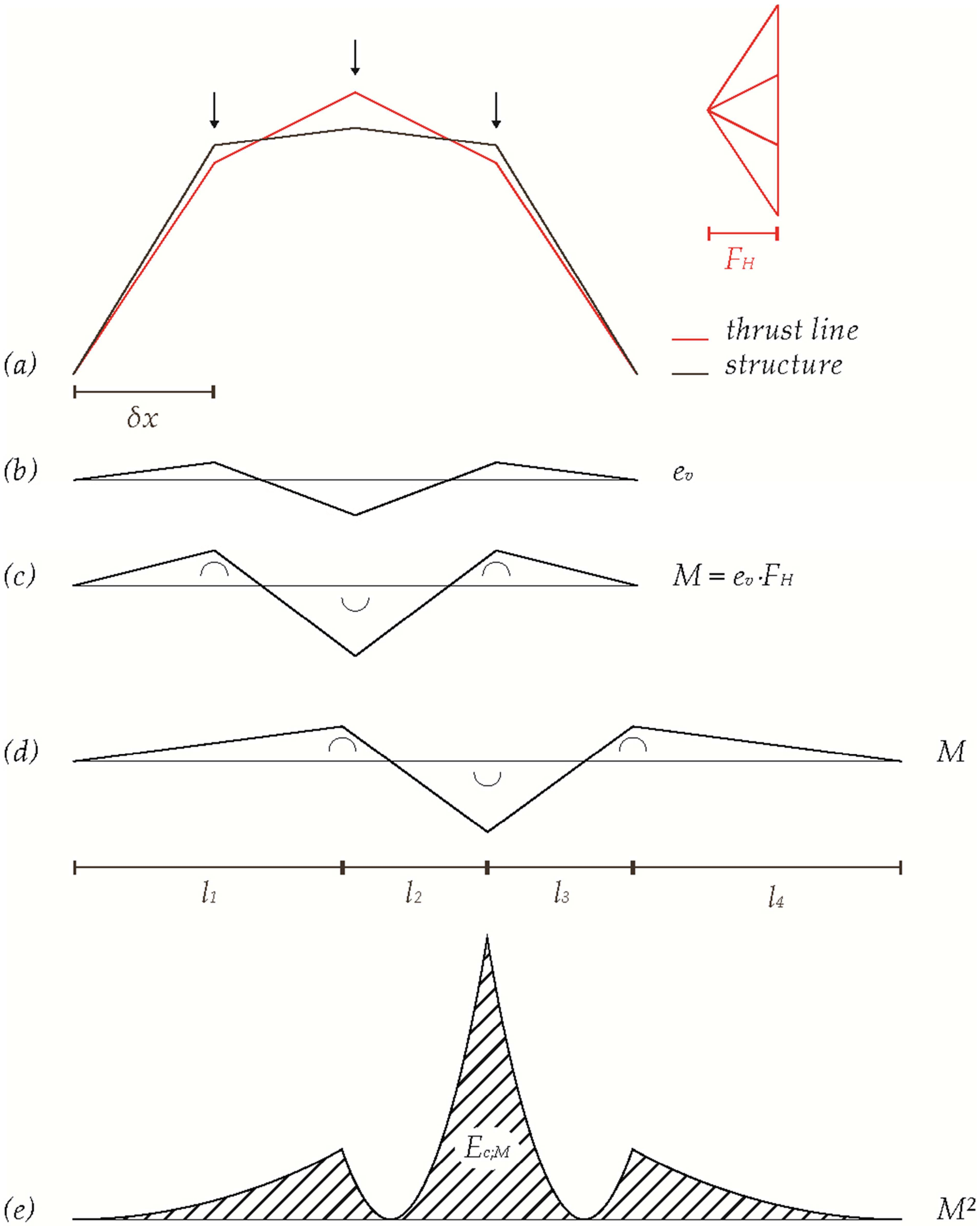


Figure 43 Calculating the bending energy of a structure

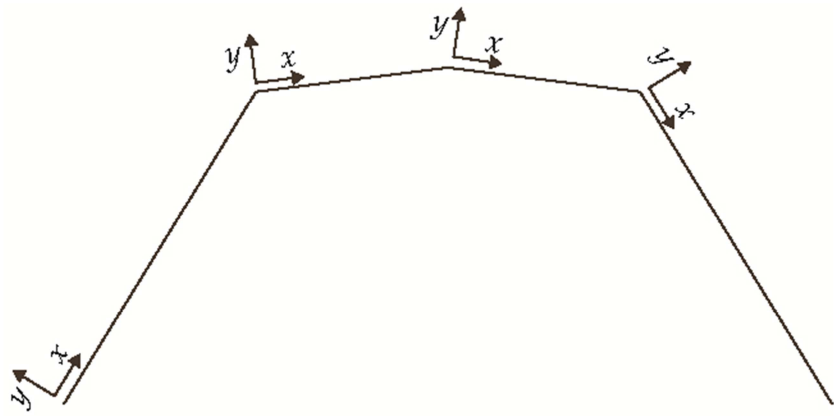


Figure 44 The structure is used as the x-axis

3.4.2 Proof of equal areas

In an attempt to prove in an analytical way that a thrust line with an area equal to the structure will be the line with the lowest $E_{c;M}$, the proof is written for a very simple situation (Figure 45), one load, two supports, two bars. In red, the force polygon and the thrust line are drawn. It is obvious that the lowest energy in this case will be the case where the areas are equal, since the thrust line will, for a correct F_H be equal to the structure, resulting in the bending moment being zero. So the result of this proof is not very surprising, however it is a basis on which a more extended proof is written later.

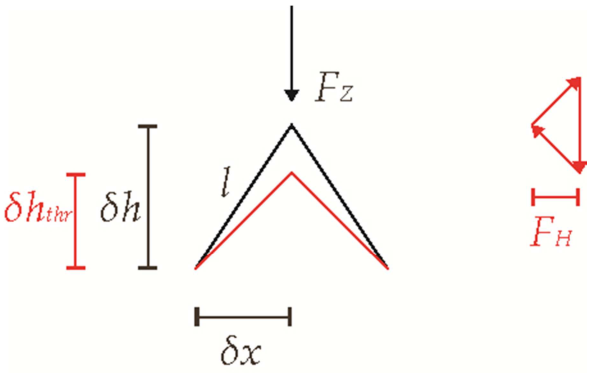


Figure 45 A structure consisting of two bars and one point load

The area of this structure can be calculated by $\delta h \cdot \delta x$. The area of the thrust line by $\delta h_{thr} \cdot \delta x$. So to prove that both areas are equal the following needs to be proven:

$$\delta h \cdot \delta x = \delta h_{thr} \cdot \delta x$$

The height of the thrust line is dependent on the F_H and can be described as:

$$\delta h_{thr} = \frac{1}{2}\frac{F_z}{F_H}\delta x$$

Since $E_{c;N}$ is not included in this calculation, $\frac{12}{t^2}$ does not influence the position of the lowest point of the complementary energy, it can be omitted from the calculations.

$$E_{c;M} = M^2 l$$

The vertical eccentricity at any point in the structure can be calculated by:

$$e_V = h - h_{thr}$$

The left half of the structure can be described as

$$h = \frac{\delta h}{\delta x} x \text{ for domain } [0, \delta x]$$

Since this can be applied to the left half of the thrust line as well, the vertical eccentricity can be calculated

$$e_V = \frac{\delta h - \delta h_{thr}}{\delta x} x \text{ for domain } [0, \delta x]$$

If the length of the bar is taken as x-axis instead of δx, the following equation applies:

$$e_V = \frac{\delta h - \delta h_{thr}}{l} x \text{ for domain } [0, l]$$

Substituting the equation for δh_{thr} and rewriting the equation results in

$$e_V = \left(\delta h - \frac{1}{2}\frac{F_z}{F_H}\delta x\right)\frac{x}{l}$$

Multiplying this with F_H gives the equation for the bending moment:

$$M = \left(F_H \delta h - \frac{1}{2}F_z \delta x\right)\frac{x}{l}$$

$$M^2 = \left(F_H \delta h - \frac{1}{2}F_z \delta x\right)^2 \frac{x^2}{l^2} \text{ for domain } [0, l]$$

The *M*^{2} is the equation which represents a graph similar to Figure 43.e for the left bar of the structure. Integrating this function over *x* for domain $[0, l]$ will give the complementary energy for this part.

$$E_{c;M} = \int_0^l \left(F_H\, \delta h - \frac{1}{2}F_z\, \delta x\right)^2 \frac{1}{l^2} x^2\ \delta l$$

$$E_{c;M} = \frac{1}{3}\left(F_H\, \delta h - \frac{1}{2}F_z\, \delta x\right)^2 \frac{1}{l^2} l^3$$

$$= \frac{1}{3}\left(F_H\, \delta h - \frac{1}{2}F_z\, \delta x\right)\left(F_H\, \delta h - \frac{1}{2}F_z\, \delta x\right) l$$

$$= \frac{1}{3}\left({F_H}^2 \delta h^2 - F_z\, F_H\, \delta x\, \delta h + \frac{1}{4}{F_z}^2 \delta x^2\right) l$$

$$= \frac{1}{3}\delta h^2 l\, {F_H}^2 - \frac{1}{3}F_z\, \delta x\, \delta h\, l\, F_H + \frac{1}{12}{F_z}^2 \delta x^2 l$$

Taking the derivative of this equation with F_H as the only variable, and setting that equal to zero should result in the minimum $E_{c;M}$.

$$E'_{c;M} = \frac{2}{3}\delta h^2 l\, F_H - \frac{1}{3} F_z\, \delta x\, \delta h\, l$$

$$\frac{2}{3}\delta h^2 l\, F_H - \frac{1}{3} F_z\, \delta x\, \delta h\, l = 0$$

$$\frac{2}{3}\delta h^2 l\, F_H = \frac{1}{3} F_z\, \delta x\, \delta h\, l$$

$$2\, \delta h\, F_H = F_z\, \delta x$$

$$\delta h = \frac{1}{2}\frac{F_z}{F_H}\, \delta x$$

Substituting the equation for δh_{thr} shows:

$$\delta h = \delta h_{thr}$$

This shows that through calculating the bending moment and setting the derivative of that bending moment equal to zero, it can be proven that these two areas are equal when the bending moment is minimal.

During the making of this proof, the first doubts arose about the equal area method. For this reason no attempt was made to prove this for a more complex structure. However this method is expanded in the next paragraph, not to prove the areas to be equal, but to find another method through which the complementary energy can be calculated in a direct way.

3.4.3 Minimizing the bending energy for a three-bar structure

The approach described in the previous section will be applied to a more advanced and slightly more general structure in this section. This time however the aim is not to prove the areas being equal, the method is only used to provide a direct method of calculating the structure with the minimum bending energy. The structure calculated is shown in Figure 46. This time the structure can be non-symmetrical (if δh_2 does not equal zero) so it can have bending moments.

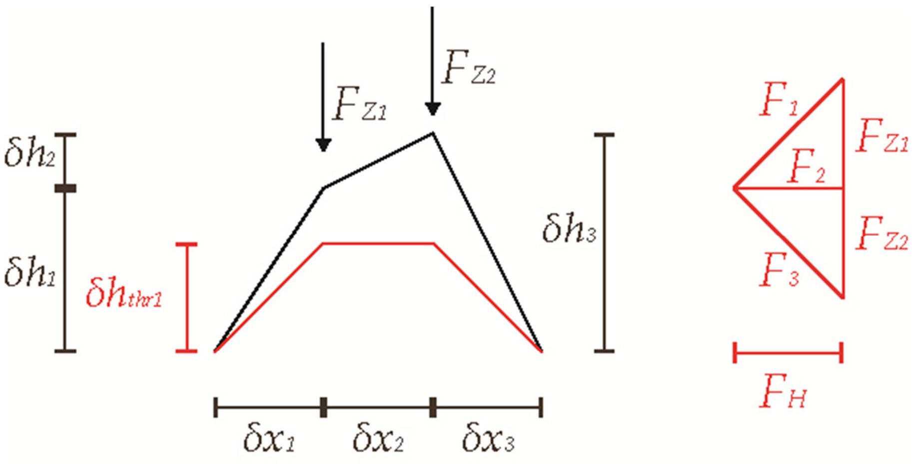


Figure 46 A structure consisting of three bars (in black) with the force polygon an thrust line (red)

To make the calculation simpler, some values are set equal to each other:

$$\delta x_1 = \delta x_2 = \delta x_3 = \delta x$$

$$\delta x_{thr1} = \delta x_{thr2} = \delta x_{thr3} = \delta x$$

$$F_{Z1} = F_{Z2} = F_Z$$

$$F_1 = F_3$$

$$\delta h_{thr1} = -\delta h_{thr3}$$

$$\delta x_{thr2} = 0$$

The height of δh_{thr1} can be calculated by

$$\delta h_{thr1} = \frac{F_z}{F_H}\delta x$$

Just like in the previous section, the bending energy can be calculated by

$$E_{c;M} = M^2 l$$

Again, the e_V is calculated for each bar.

$$e_{V1} = \frac{\delta h_1 - \delta h_{thr1}}{l_1}x \text{ for domain } [0, l_1]$$

$$e_{V2} = \frac{\delta h_2}{l_2}x + \delta h_1 - \delta h_{thr1} \text{ for domain } [0, l_2]$$

$$e_{V3} = \frac{\delta h_3 + \delta h_{thr1}}{l_3}x + \delta h_1 + \delta h_2 - \delta h_{thr1} \text{ for domain } [0, l_3]$$

This can be written as

$$e_{V1} = \frac{\delta h_1}{l_1}x - \frac{F_z}{F_H}\frac{\delta x}{l_1}x \text{ for domain } [0, l_1]$$

$$e_{V2} = \frac{\delta h_2}{l_2}x + \delta h_1 - \frac{F_z}{F_H}\delta x \text{ for domain } [0, l_2]$$

$$e_{V3} = \frac{\delta h_3}{l_3}x + \frac{F_z}{F_H}\frac{\delta x}{l_3}x + \delta h_1 + \delta h_2 - \frac{F_z}{F_H}\delta x \text{ for domain } [0, l_3]$$

These eccentricities are multiplied by F_H and then squared to get the M^2. Rewriting this equation in the form of $ax^2 + bx + c$ results in

$${M_1}^2 = {F_H}^2\left(\frac{\delta h_1}{l_1} - \frac{F_z}{F_H}\frac{\delta x}{l_1}\right)^2 x^2 \text{ for domain } [0, l_1]$$

$${M_2}^2 = {F_H}^2\frac{{\delta h_2}^2}{{l_2}^2}x^2 + 2\frac{\delta h_2}{l_2}F_H(F_H\,\delta h_1 - F_z\,\delta x)x + {\delta h_1}^2{F_H}^2 + {F_Z}^2\delta x^2 - 2\,\delta h_1\,F_Z\,\delta x\,F_H \text{ for domain } [0, l_2]$$

$$M_3{}^2 = F_H{}^2\left(\frac{\delta h_3}{l_3} + \frac{F_z}{F_H}\frac{\delta x}{l_3}\right)^2 x^2 + 2\,F_H{}^2\left(\frac{\delta h_3}{l_3} + \frac{F_z}{F_H}\frac{\delta x}{l_3}\right)\left(\delta h_1 + \delta h_2 - \frac{F_z}{F_H}\delta x\right)x + \delta h_1{}^2 F_H{}^2 + 2\,\delta h_1\,\delta h_2\,F_H{}^2 - 2\,F_H\,F_z\,\delta x\,\delta h_1 + \delta h_2{}^2 F_H{}^2 - 2\,F_H\,F_z\,\delta x\,\delta h_2 + \delta x^2 F_Z{}^2 \text{ for domain } [0, l_3]$$

Calculating these areas, as shown in Figure 43 can be done by integrating these equations. This results in the bending energy.

$$E_{c;M1} = \frac{1}{3}F_H{}^2\left(\frac{\delta h_1}{l_1} - \frac{F_z}{F_H}\frac{\delta x}{l_1}\right)^2 l_1{}^3$$

$$E_{c;M2} = \frac{1}{3}F_H{}^2\delta h_2{}^2\,l_2 + \delta h_2\,F_H(F_H\,\delta h_1 - F_z\,\delta x)l_2 + \delta h_1{}^2F_H{}^2 l_2 + F_Z{}^2\delta x^2 l_2 - 2\,\delta h_1\,F_Z\,\delta x\,F_H\,l_2$$

$$E_{c;M3} = \frac{1}{3}F_H{}^2\left(\frac{\delta h_3}{l_3} + \frac{F_z}{F_H}\frac{\delta x}{l_3}\right)^2 l_3{}^3 + F_H{}^2\left(\frac{\delta h_3}{l_3} + \frac{F_z}{F_H}\frac{\delta x}{l_3}\right)\left(\delta h_1 + \delta h_2 - \frac{F_z}{F_H}\delta x\right)l_3{}^2 + \delta h_1{}^2F_H{}^2 l_3 + 2\,\delta h_1\,\delta h_2\,F_H{}^2 l_3 - 2\,F_H\,F_z\,\delta x\,\delta h_1\,l_3 + \delta h_2{}^2F_H{}^2 l_3 - 2\,F_H\,F_z\,\delta x\,\delta h_2\,l_3 + \delta x^2 F_Z{}^2 l_3$$

These equations are rewritten in the form of $aF_H{}^2 + bF_H + c$ from which c is omitted since these terms do not influence the position of the minimum bending energy

$$E_{c;M1} = \frac{1}{3}\delta h_1{}^2 l_1 F_H{}^2 - \frac{2}{3}\delta h_1\,F_z\,\delta x\,l_1 F_H + c$$

$$E_{c;M2} = \left(\frac{1}{3}\delta h_2{}^2 + \delta h_1\delta h_2 + \delta h_1{}^2\right) l_2\,F_H{}^2 - (2\,\delta h_1 + \delta h_2)\,F_z\,\delta x\,l_2 F_H + c$$

$$E_{c;M3} = \left(\frac{1}{3}\delta h_3{}^2 + 2\,\delta h_1\delta h_2 + \delta h_1{}^2 + \delta h_2{}^2 + \delta h_1 + \delta h_2\right) l_3\,F_H{}^2 - \left(\delta h_1 + \delta h_2 + \frac{1}{3}\delta h_3\right) F_z\,\delta x\,l_3 F_H + c$$

Taking the derivative of the sum of these equations and setting that equal to zero and solving it will result in the point for the least complementary energy.

$$E'_{c;M} = \frac{2}{3}\delta h_1{}^2 l_1 F_H - \frac{2}{3}\delta h_1\,F_z\,\delta x\,l_1 + \left(\frac{2}{3}\delta h_2{}^2 + 2\,\delta h_1\delta h_2 + 2\,\delta h_1{}^2\right) l_2\,F_H - (2\,\delta h_1 + \delta h_2)\,F_z\,\delta x\,l_2 + 2\left(\frac{1}{3}\delta h_3{}^2 + 2\,\delta h_1\delta h_2 + \delta h_1{}^2 + \delta h_2{}^2 + \delta h_1 + \delta h_2\right) l_3\,F_H - \left(\delta h_1 + \delta h_2 + \frac{1}{3}\delta h_3\right) F_z\,\delta x\,l_3 = 0$$

Rewriting this gives

$$F_H\left(\frac{2}{3}\delta h_1{}^2 l_1 + \frac{2}{3}\delta h_2{}^2 l_2 + 2\,\delta h_1\delta h_2 l_2 + 2\,\delta h_1{}^2 l_2 + \frac{2}{3}\delta h_3{}^2 l_3 + 4\,\delta h_1\delta h_2 l_3 + 2\,\delta h_1{}^2 l_3 + 2\,\delta h_2{}^2 l_3 + 2\,\delta h_1 l_3 + 2\,\delta h_2 l_3\right) = F_z\,\delta x\,\left(\frac{2}{3}\delta h_1 l_1 + 2\,\delta h_1 l_2 + \delta h_2 l_2 + \delta h_1 l_3 + \delta h_2 l_3 + \frac{1}{3}\delta h_3 l_3\right)$$

$$F_H = \frac{F_z\,\delta x\left(\frac{2}{3}\delta h_1 l_1 + 2\,\delta h_1 l_2 + \delta h_2 l_2 + \delta h_1 l_3 + \delta h_2 l_3 + \frac{1}{3}\delta h_3 l_3\right)}{\frac{2}{3}\delta {h_1}^2 l_1 + \frac{2}{3}\delta {h_2}^2 l_2 + 2\,\delta h_1 \delta h_2 l_2 + 2\,\delta {h_1}^2 l_2 + \frac{2}{3}\delta {h_3}^2 l_3 + 4\,\delta h_1 \delta h_2 l_3 + 2\,\delta {h_1}^2 l_3 + 2\,\delta {h_2}^2 l_3 + 2\,\delta h_1 l_3 + 2\,\delta h_2 l_3}$$

In this way, the situation with the minimum bending energy can be calculated directly.

3.4.4 Comparing minimum bending energy method to FEM calculations

A few first calculations are done to check to what extent the result of the minimum bending energy equals the result of FEM calculations. The first results look promising. However, for the final report at the P5 presentation, more extensive calculations will be done and be included in this report.

Chapter 4 – Conclusions

4.1 Results

Some of the first results of this research are the steps taken to minimize the energy due to normal forces. In this case, the question was how to minimize the bending energy due to normal forces in a structure which equals a thrust line. This research had two equations as a result, which however are to extensive to be used easily. They give however some insight in how the normal energy can be calculated and how the angles of the force polygon an thrust line are related.

The method of equal areas is also researched more in-depth. The first result of that is the equation which allows for a direct calculation of the thrust line according to this method. This is also translated in an algorithm which, if a structure is drawn, instantly displays the thrust line with an area equal to that structure. This gives the possibility to change a design and see directly what the influence of this change is.

However, the research after that showed that the equal area method is not a valid method. It gets more inaccurate when the arch is more irregularly shaped. This makes the equation and algorithm designed using this method not suitable for practical application.

The equal area method is proven to be the same as minimizing bending energy for a very simple case, consisting of two bars and one load. It may be obvious that both of these methods can be applied to this particular case, since the most efficient way of transferring loads would be through mainly normal forces, which in this case excludes any bending moments and which also happens to be the case in which the thrust line equals the structure, thus resulting in equal areas. However this is a proof upon which can be built further and it also leads to the idea that minimizing bending energy might work for more complex situations as well.

This method is extended to situations where bending moments have to occur, since no thrust line will coincide with the structure. For this situation, a three-bar structure, the minimum bending energy is described in an analytical way. The resulting equation still needs to be compared to other calculations to see whether it actually works.

[add conclusions 3.4.4]

4.2 Process

The process started off with a lot of background research. But only when the first calculations are done, one can really understand what the equations mean.

A lot of time went into trying to prove the equal area method. The process could have been more efficient if a more extensive calculation was done at the beginning of the research. This might have resulted in the conclusion that the equal area method is not valid for all arches. This way less time would have been put into trying to prove this theory

Most of the time, the research was kept on track, focusing on solving the way to find the correct thrust line. However, from every part of research, new ideas and hypotheses emerge, making it sometimes hard to keep focus and not directly test every new idea on whether it works or not. Sometimes this led to shifting off track.

During the process, the focus of research was once adjusted. The research started off with the aim to prove the equal area method directly at the beginning and then extend this method to shells. When this method turned out not to be correct, the focus had to be adjusted to finding a new method of finding the correct thrust lines. This is inherent to doing this kind of research. One can never know what results will be found and when they will be found. This requires a flexibility to change focus if needed.

4.3 Recommendations

There is still a lot which needs to be researched in this particular area of mechanics. Following are a few recommendations on whether and how this research can be continued:

- Research needs to be done into the method of minimizing bending moments. A study could be done to how accurate this method is. What does the accuracy exactly depend on? It is suspected that it will be applicable to most of the situations in today's building practice, since the thickness is always quite small in relation to the span, but this needs to be calculated first.
- The method of minimizing bending energy as described in section 3.4.3 with a three-bar structure can be translated in a spreadsheet or Grasshopper algorithm. This way it can instantly calculate the thrust lines from a structure and load. This would also allow for easy testing of this method, which also needs to be done.
- Since the equal area method does not hold for situations with irregular shapes, it is recommended not to continue this research. However, to be completely sure, a research could be done to determine for which this method does give an accurate result and for which it does not.
- The research from section 3.1 could be continued, however, if the hypothesis of minimizing bending energy turns out to be true, it is a less interesting path to follow, since it will be of less use.
- Once a method is found through which the thrust line can be calculated, this method should be translated to shell structures. An interesting question is whether the minimum bending energy method would work for shell structures as well.

List of symbols

δh or δx	vertical component of the length of a part of a structure or thrust line
δx	horizontal component of the length of a part of a structure or thrust line
θ_c	angle between the closing line of a structure and the horizontal axis
θ_n	angle between the *n*th bar of a structure and the horizontal axis
A	area
A_{str}	the area between a structure and its closing line
A_{th}	the area between a thrust line and its closing line
E	Young's modulus
e_V	the vertical eccentricity of a force in a thrust line
FD	force density
F_H	horizontal component of a force
F_n	force in the *n*th bar of a structure
F_V	vertical component of a force
F_z	gravitational force
I	area moment of inertia
l	length
M	bending moment
t	thickness

List of figures

Note: all images in this report are created by the author, unless stated otherwise

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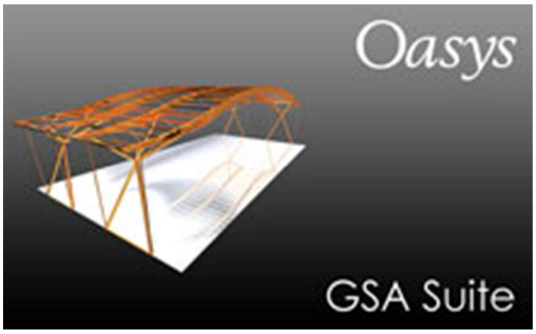
Software used

Grasshopper, extension for Rhinoceros 3D, Robbert McNeel & Assoc., Seattle, WA, USA

Rhinoceros 3D, NURBS-based 3D-modeling software, Robert McNeel & Assoc., Seattle, WA, USA



GSA Suite, structural design and analysis software, Oasys software, Newcastle-Upon-Tyne, UK

WolframAlpha

Wolfram|Alpha, Wolfram Alpha LLC, Champaign, IL, USA

Appendix I Excel and Grasshopper calculations

The final report will have a CD with all the spreadsheets and grasshopper algorithms included, as well as some prints of spread sheet results in the appendix. For now these are available from http://goo.gl/VozHN6

Appendix II Reflection

Research proposal

In the research proposal the methods to be used are described. These methods will be described here as well.

Approach and methodology

Literature study

The whole research will be done within the field of structural mechanics. For this reason, the literature to be studied is mainly in the field of structural mechanics. To get the research started, a literature study on several subjects needs to be done. Part of these subjects are studied already. The following subjects will be studied:

- Complementary energy method
- Arches and thrust lines
- Graphic statics in arches
- Hoop forces
- Split in surfaces
- Curvature

Method development

From the literature study hypotheses will emerge. From these hypotheses a method to calculate shell structures will be developed.

Design computational algorithm

The found method will be translated into a computational algorithm. For this algorithm, the 3D program Rhino will be used, with the Grasshopper-plugin.

Validate method

The computational algorithm will be compared to FEM calculations for several case studies. Differences in results from these calculations will show whether or not the method is valid.

Relevance

Societal relevance

This research aims to provide in a tool for designers which gives them earlier in the design process insight in the structural performance of a shell structure. This will lead to a less time-consuming design process, but also to a more direct feedback on the design changes. It will probably lead to more efficient structural design, in which less material can be used for a similar performance.

Scientific relevance

Currently it is still unknown what the mechanics are behind shell structures. This research aims to give more insight in these mechanics.

Methods during the research

During the research, some of the research was conducted as planned but not all of it. This section will reflect on how these methods worked out.

Literature study

All the subjects in the literature study were researched extensively. However, alongside this study, the first hypotheses emerged. Some of the topics studied turned out to be less relevant to the hypotheses which were to be tested.

Method development

The method development turned out to be a bit less structured then imagined in the first instance. Even though from the literature study some ideas emerged, only when the first calculations are done, you really understand how the theories work. This results in constantly changing of ideas of what might work and how to test it. Some hypotheses could be tested quite quickly (in a day or two) and if they didn't seem to work out, they were not included in any report.

Design computational algorithm

The design of the algorithm was done before the theory was proven, making it more of a research tool than a final product.

Validate method

The theories that seemed to be promising were always compared to a FEM calculation to see whether it was accurate or not.

Results

These methods resulted in several products:

- A summary of some of the methods to calculate the bending energy in arches (chapter 2 and section 3.4.1)
- An equation for calculating the energy due to normal forces
- An equation for directly calculating the thrust line using the equal area method
- An algorithm using the equal area equation
- FEM calculations compared to the algorithm, which prove that the equal area method is not valid for a lot of situations
- A proof of the equal area method for one situation in which it is valid, the situation in which the thrust line coincides with the structure
- A hypothesis on how to calculate the correct thrust line, by only minimizing the bending energy
- An equation using the minimizing of bending energy to find the correct thrust line for three-bar structures
- FEM calculations to show whether or not this method works [to be added in P5 report]

Conclusions

As can be seen from the results, the methods were adequate for this type of research. The aim of the research however is not fully achieved, finding a method to calculate shells. It turned out that for arches, there was still so much to be discovered that the step to shells could not be made in this time frame. This results for instance in some of the subjects (*curvature* and *hoop forces*) which are included in the literature study but not in the final report, since they mainly deal with shell structures. The process might have been more efficiently if the literature study was more fragmented, by studying arch related literature first, conducting that part of the research after that and wait with the second part of the literature study until the problem is solved for arches.

Since the equal area method is proven to be invalid, the second and third product in the list are less relevant than they would have been if it turned out to be valid. This inefficiency could have been prevented if some more extensive calculations were done on this subject. This way the theory would probably have been proven wrong earlier in the process, making sure that less time was spent on trying to prove this subject.

Apart from these two inefficiencies, the methods turned out to fit the problem quite well. Even though the scope of the research was during the process limited to arches, the methods could be applied to this part of the subject as well.