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Adaptive Prescribed Performance Asymptotic Tracking for High-Order Odd-Rational-Power Nonlinear Systems

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***Abstract*— Practical tracking results have been reported in the literature for high-order odd-rational-power nonlinear dynamics (a chain of integrators whose power is the ratio of odd integers). Asymptotic tracking remains an open problem for such dynamics. This note gives a positive answer to this problem in the framework of prescribed performance control (PPC), without approximation structures (neural networks, fuzzy logic, etc.) being involved in the control design. The unknown system uncertainties are first transformed to unknown but bounded terms using barrier Lyapunov functions, and then these terms are compensated by appropriate adaptation laws. A method is also proposed to extract the control terms in a linear-like fashion during the control design which overcomes the difficulty that virtual or actual control signals appear in a non-affine manner. A practical poppet valve system is used to validate the effectiveness of the theoretical findings.**

***Index Terms*— High-order odd-rational-power nonlinear systems, Asymptotic tracking, Prescribed performance.**

I. Introduction

Over the last decade, high-order nonlinear dynamics have been attracting great attention. The reason is twofold: first, high-order nonlinear dynamics generalize strict-feedback and pure-feedback dynamics by including more general integrators (with odd integer powers [1]-[3] or ratios of odd integer powers [4]-[11]) in the dynamics; second, high-order nonlinear dynamics appear in some practical systems such as in dynamical boiler-turbine units [12], in classes of hydraulic dynamics [13], or in classes of under-actuated, weakly coupled mechanical systems [1]-[2]. It is well documented in the literature that high-order nonlinear systems are intrinsically more challenging than strict-feedback and pure-feedback systems, as feedback linearization and backstepping methods fail to work [1]-[2]. A parametric nonlinear adaptive control methodology called adding-one-power-integrator technique, originally proposed in [2], has been successfully applied in stabilizing high-order nonlinear systems [3]-[11]. In the following, let us distinguish and refer to such high-order nonlinear dynamics as high-order odd-integer-power and high-order odd-rational-power nonlinear systems (with high-order odd-integer power being a special case of high-order odd-rational-power).

For high-order odd-rational-power nonlinear systems, both stabilization to zero [5]-[12] and output tracking [3]-[4] have been studied. It is worth remarking that, while stabilization (regulation to zero) can be obtained at the price of imposing growth conditions on the system nonlinearities [5]-[12], no asymptotic tracking results have been reported for these dynamics. All reported results achieve practical tracking in a residual set, either by imposing the aforementioned growth conditions [3]-[4] (see also recent works considering rational or irrationals powers [14]-[15]), or by removing growth conditions via the use of universal approximators (e.g. neural networks) [16]. Therefore, two open problems appear for high-order odd-rational-power nonlinear systems: asymptotic tracking is the first one, and avoiding the use of universal approximators is the second one.

The main contribution of this note is to give positive answers to these problems. To this purpose, the unknown system uncertainties are first transformed to some unknown but bounded terms via barrier Lyapunov functions and then these terms are compensated by designing appropriate adaptation laws. To overcome the difficulty that virtual and actual control signals of odd-rational-power dynamics appear in a non-affine manner and cannot be designed directly, the proposed design is achieved in combination with a newly proposed lemma that allows to deal with the control terms in a "linear-like" fashion. Because the proposed solution is given in the prescribed performance control (PPC) framework, as a further evidence of effectiveness, we show that the proposed result is in line with the-state-of-the-art on PPC, since it can also handle the recently studied problem of input quantization [17].

This paper is organized as follows: the problem formulation and some useful lemmas are given in Section 2. Sections 3 and 4 present the proposed prescribed performance quantized control scheme and asymptotic tracking analysis, respectively. Simulation results are provided in Section 5 and Section 6 draws the conclusions.

Notations: Notations adopted in the paper are: $\mathbf{R}_{\geq 0}$ denotes the set of non-negative real numbers, $\mathbf{R}^i$ represents the Euclidean space with dimension i, and $\mathbf{R}_{\text{odd}} \triangleq \left\{\frac{p}{q}\middle|p \text{ and } q \text{ are positive odd integers}\right\}$. The symbol " $\triangleq$ " means "equal by definition". Similarly to [10], we define the notation $\lceil\sigma\rfloor^\tau \triangleq |\sigma|^\tau \text{sign}(\sigma)$, $\forall\sigma \in \mathbf{R}$. For compactness and whenever unambiguous, some variable dependencies might be dropped, e.g. ε, μ_i, and ϑ_i can be used to denote $\varepsilon(x_1, x_2)$, $\vartheta_i(x_1, x_2)$, and $\mu_i(x_1, x_2)$, respectively.

II. Preliminaries

Let us consider the following uncertain odd-rational-power nonlinear system with input quantization:

$$\begin{cases} \dot{x}_i = \phi_i(\bar{\boldsymbol{x}}_i, t) + \psi_i(\bar{\boldsymbol{x}}_i, t)x_{i+1}^{\frac{p_i}{q_i}}, \ i = 1, \cdots, n-1, \\ \dot{x}_n = \phi_n(\bar{\boldsymbol{x}}_n, t) + \psi_n(\bar{\boldsymbol{x}}_n, t)\big(Q(u)\big)^{\frac{p_n}{q_n}}, \\ y = x_1, \end{cases} \tag{1}$$

where $y \in \mathbf{R}$ is the system output; $u \in \mathbf{R}$ and $Q(u) \in \mathbf{R}$ are the control input (to be designed) and the quantized control input; $\bar{\boldsymbol{x}}_i = [x_1, \ldots, x_i]^T \in \mathbf{R}^i$ is an intermediate state, with the full state being $\bar{\boldsymbol{x}}_n$. We assume that $\frac{p_i}{q_i} \in \mathbf{R}_{\text{odd}}$, $i = 1, \ldots, n$, are known odd-rational-powers. The system nonlinearities $\phi_i(\cdot, \cdot) : \mathbf{R}^i \times \mathbf{R}_{\geq 0} \to \mathbf{R}$ are locally Lipschitz in $\bar{\boldsymbol{x}}_i$. The control-gain functions $\psi_i(\cdot, \cdot) : \mathbf{R}^i \times \mathbf{R}_{\geq 0} \to \mathbf{R}$ are locally Lipschitz in $\bar{\boldsymbol{x}}_i$ and are either strictly positive or strictly negative, and their signs are assumed to be known. Without loss of generality, in the following we assume $\text{sign}(\psi_i) = 1$, $i =$

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$1,\ldots,n$. In line with [18]-[20], we assume that there exist continuous and non-negative functions $\bar{\phi}_i(\cdot):\mathbf{R}^i\to\mathbf{R}_{\geq 0}$, $i=1,\ldots,n$, such that $|\phi(\bar{\boldsymbol{x}}_i,t)|\leq\bar{\phi}_i(\bar{\boldsymbol{x}}_i)$, $\forall\ (\bar{\boldsymbol{x}}_i,t)\in\mathbf{R}^i\times\mathbf{R}_{\geq 0}$.

Assumption 1 [19]: The desired trajectory $y_{\mathrm{r}}(\cdot)$ is known and bounded, and $\dot{y}_r(\cdot)$ is bounded but its bound is not necessarily known.

Remark 1: Assumption 1 implies that only the desired trajectory (none of its derivatives) can be used for control design.

Remark 2: System (1) generalizes the classes of systems considered in literature for PPC: more specifically, (1) reduces to the strict-feedback classes of [17]-[20] when $r_i=1$, $i=1,\ldots,n$, while it reduces to the high-order integer-power classes of [1]-[3] when $q_i=1$ and $p_i\neq 1$, $i=1,\ldots,n$.

Let us consider the asymmetric hysteresis quantizer (2) originally proposed in [21] (see Remark 3 for details of this choice). As typical in literature (cf. [22]), we denote such quantizer simply as $Q(u)$, even though the quantizer formally depends on both u and its derivative. In (2), $\nu_+^k=\hbar_+^{1-k}\nu_{\min}^+$ and $\nu_-^k=\hbar_-^{1-k}\nu_{\min}^-$, $k=1,2,\ldots$, with $\hbar_+=\frac{1-\varrho_+}{1+\varrho_+}$ and $\hbar_-=\frac{1-\varrho_-}{1+\varrho_-}$; $Q(u(t^-))$ is the latest status prior to $Q(u(t))$, and $\nu_{\min}^+$ and $\nu_{\min}^-$ denote the size of the dead-zone for $Q(u)$. The constants ϱ_+, $\varrho_-\in(0,1)$ determine the quantization density, i.e., the larger ϱ_+ and ϱ_-, the coarser the quantizer.

Remark 3: The interest in considering an asymmetric hysteresis quantizer is that it generalizes the uniform quantizer [21], logarithmic quantizer [21], and symmetric hysteresis quantizer [22], while its hysteresis property is of paramount importance in guaranteeing the absence of chattering and Zeno behavior. These issues have been thoroughly discussed in [17, Remark 8 and Lemma A.1] and are not further discussed here due to space limitations.

In line with [21], let us decompose (2) as

$$Q(u)=\varsigma(u)u+d(u), \tag{3}$$

where $\varsigma(u)=\frac{Q(u)}{u}$ and $d(u)=0$ when $Q(u)\neq 0$, and $\varsigma(u)=1$ and $d(u)=-u$ when $Q(u)=0$.

Before presenting the proposed prescribed performance quantized control, the following lemmas are useful to derive the main results.

Lemma 1 [21]: The control coefficient $\varsigma(u)$ and input quantization error $d(u)$ in (3) are such that

$$\underline{\varsigma}_{\min}\leq\varsigma(u)\leq\bar{\varsigma}_{\max},\quad\text{and}\quad|d(u)|\leq\bar{d}, \tag{4}$$

where $\underline{\varsigma}_{\min}=1-\max\{\varrho_+,\varrho_-\}$, $\bar{\varsigma}_{\max}=1+\max\{\varrho_+,\varrho_-\}$ and $\bar{d}=\max\{\nu_{\min}^+,|\nu_{\min}^-|\}$.

Lemma 2 [10]: Suppose $\frac{p}{q}\in\mathbf{R}_{\mathrm{odd}}$, then for any $x_1\in\mathbf{R}$ and $x_2\in\mathbf{R}$, it holds that

$$\left|x_1^{\frac{p}{q}}-x_2^{\frac{p}{q}}\right|\leq 2^{1-\frac{1}{q}}\left|\lceil x_1\rceil^p-\lceil x_2\rceil^p\right|^{\frac{1}{q}}. \tag{5}$$

Lemma 3 [23]-[27]: The following inequality holds for any $\eta>0$ and for any $\hbar\in\mathbf{R}$:

$$0\leq|\hbar|-\frac{\hbar^2}{\sqrt{\hbar^2+\eta^2}}<\eta. \tag{6}$$

Lemma 4 [1]: For any x_1, $x_2\in\mathbf{R}$, any positive integers b_1, b_2 and any real-valued function $\delta(\cdot,\cdot)$ with $\varepsilon(x_1,x_2)>0$, it holds that

$$|x_1|^{b_1}|x_2|^{b_2}\leq\frac{b_1\varepsilon|x_1|^{b_1+b_2}}{b_1+b_2}+\frac{b_2\varepsilon^{-\frac{b_1}{b_2}}|x_2|^{b_1+b_2}}{b_1+b_2}. \tag{7}$$

Lemma 5: For any x_1, $x_2\in\mathbf{R}$ and positive odd integers p and q, there exist real-valued functions $\mu(\cdot,\cdot)$ and $\vartheta(\cdot,\cdot)$, such that

$$(x_1+x_2)^{\frac{p}{q}}=\left(\vartheta(x_1,x_2)x_1^p+\mu(x_1,x_2)x_2^p\right)^{\frac{1}{q}}, \tag{8}$$

where $\vartheta(x_1,x_2)\subseteq\left[1-\bar{\epsilon},\max\{1+\bar{\epsilon},2^{p-1}\}\right]$ with $\bar{\epsilon}=\sum_{k=1}^{p-1}\frac{k}{p}\varepsilon^{\frac{p}{k}}$ a constant that can be made to take value in $(0,1)$ by selecting some appropriately small positive constant ε, and where $\mu(x_1,x_2)$ satisfies $|\mu(x_1,x_2)|\leq\bar{\upsilon}$ with $\bar{\upsilon}=\max\{1+\omega,2^{p-1}\}$ and $\omega=\sum_{k=1}^{p-1}\frac{p-k}{p}\begin{pmatrix}p\\k\end{pmatrix}\varepsilon^{\frac{-p}{p-k}}$ positive constants that are independent of x_1 and x_2.

Proof. See appendix. ■

III. Adaptive Prescribed Performance Control Design

Let us begin the control design by defining the state errors [17]:

$$e_1(t)=x_1(t)-y_r(t), \tag{9}$$

$$e_i(t)=x_i(t)-\alpha_{i-1}(t),\quad i=2,\ldots,n, \tag{10}$$

where α_{i-1} denotes a virtual control law whose design will be explained later. Define the normalized error variables

$$\zeta_i(t)=\frac{e_i(t)}{\kappa_i(t)}, \tag{11}$$

where $\kappa_i(t)=(\kappa_{i,0}-\kappa_{i,\infty})\exp(-\iota_i t)+\kappa_{i,\infty}$, $i=1,\ldots,n$, is the so-called *prescribed performance function* [28], where $\kappa_{i,0}>0$, $\kappa_{i,\infty}>0$, and $\iota_i>0$ are design constants, and $|e_i(0)|<\kappa_{i,0}$.

The goal is to design a control u for (1) such that the system output y asymptotically tracks the reference signal y_r, while having e_i satisfying the prescribed performance. Since existing literature [21] and [23] has shown that asymptotic tracking can be realized for some classes of dynamics in the presence of input quantization, we set an asymptotic tracking goal for dynamics (1) in our paper.

Hereafter is the proposed design for the virtual control laws and for the actual control law. The motivation behind this design is explained

$$Q(u)=\begin{cases}\nu_+^k, & \text{if }\begin{cases}\frac{\nu_+^k}{1+\varrho_+}<u<\nu_+^k,\dot{u}<0,\text{or},\\ \nu_+^k<u<\frac{\nu_+^k}{1-\varrho_+},\dot{u}>0,\end{cases}\\ \nu_+^k(1+\varrho_+), & \text{if }\begin{cases}\nu_+^k<u\leq\frac{\nu_+^k}{1-\varrho_+},\dot{u}<0,\text{or},\\ \frac{\nu_+^k}{1-\varrho_+}<u\leq\frac{(1+\varrho_+)\nu_+^k}{1-\varrho_+},\dot{u}>0,\end{cases}\\ 0, & \text{if }\begin{cases}0\leq u<\frac{\nu_{\min}^+}{1+\nu_+^k},\dot{u}>0,\text{or},\\ \frac{\nu_{\min}^+}{1+\varrho_+}\leq u\leq\nu_{\min}^+,\dot{u}>0,\end{cases}\end{cases}\quad\text{and}\quad\begin{cases}\nu_-^k, & \text{if }\begin{cases}\nu_-^k\leq u<\frac{\nu_-^k}{1+\varrho_-},\dot{u}>0,\text{or},\\ \frac{\nu_-^k}{1-\varrho_-}\leq u<\nu_-^k,\dot{u}<0,\end{cases}\\ \nu_-^k(1+\varrho_-), & \text{if }\begin{cases}\frac{\nu_-^k}{1-\varrho_-}\leq u<\nu_-^k,\dot{u}>0,\text{or},\\ \frac{(1+\varrho_-)\nu_-^k}{1-\varrho_-}\leq u<\frac{\nu_-^k}{1-\varrho_-},\dot{u}<0,\end{cases}\\ 0, & \text{if }\begin{cases}\frac{\nu_{\min}^-}{1+\varrho_-}<u\leq 0,\text{or},\\ \nu_{\min}^-\leq u\leq\frac{\nu_{\min}^-}{1+\varrho_-},\dot{u}<0,\end{cases}\\ Q(u(t^-)),\dot{u}=0\end{cases} \tag{2}$$

via the stability analysis in Sect. IV. Specifically, we devise the virtual and actual control laws as follows:

$$\alpha_i = -\bar{\vartheta}_i^{-\frac{1}{p_i}}\left(k_i\varpi_i + \frac{c_i\varpi_i\widehat{\Xi}_i}{\sqrt{\varpi_i^2+\sigma^2(t)}}\right)^{\frac{q_i}{p_i}},\ i=1,\ldots,n-1, \tag{12}$$

$$\triangleq \alpha_i^\star(\zeta_i,\widehat{\Xi}_i,t), \tag{13}$$

$$u = -\varsigma_{\max}^{-1}\bar{\vartheta}_n^{-\frac{1}{p_n}}\left(k_n\varpi_n + \frac{c_n\varpi_n\widehat{\Xi}_n}{\sqrt{\varpi_n^2+\sigma^2(t)}}\right)^{\frac{q_n}{p_n}} \tag{14}$$

$$\triangleq \alpha_n^\star(\zeta_n,\widehat{\Xi}_n,t), \tag{15}$$

where $\varpi_i = \frac{\zeta_i+\zeta_i^3}{(1-\zeta_i^2)^3}$, $\bar{\vartheta}_i = \max\{1+\bar{\epsilon}_i, 2^{p_i-1}\}$ with $\bar{\epsilon}_i$ being an arbitrary constant in $(0,1)$, $k_i>0$, and $c_i>0$ are design constants. The terms $\widehat{\Xi}_i$ in (12) and (14) are updated by the adaptation laws

$$\dot{\widehat{\Xi}}_i = \frac{\gamma_i\varpi_i^2}{\sqrt{\varpi_i^2+\sigma^2(t)}} \triangleq \beta_{n+i}(\zeta_i,t) \geq 0,\ i=1,\ldots,n. \tag{16}$$

with initial conditions $\widehat{\Xi}_i^0 = \widehat{\Xi}_i(0) \geq 0$, where $\gamma_i>0$ is a design constant, and $\sigma(\cdot)$ is a positive integrable function satisfying $\int_0^t \sigma(\tau)\mathrm{d}\tau \leq \bar{\sigma} < \infty$ and $|\dot{\sigma}(t)| \leq \sigma^*$ for $\forall t\geq 0$ with constants $\bar{\sigma}>0$ and $\sigma^*>0$.

Remark 4: Common forms adopted in the literature for the positive integrable function $\sigma(\cdot)$ include $\varpi\exp(-\lambda t)$ as in [23]-[26], and $\frac{1}{\varpi+t^{2\iota t}}$ as in [20], [27], with design constants $\varpi>0$, $\lambda>0$, and $\iota>0$. The numerical simulations in these works typically select small values for λ and ι, yielding a slow decay rate of $\sigma(\cdot)$. This helps avoiding numerical integration problems that might arise when $\sigma(\cdot)$ becomes smaller and smaller.

IV. Asymptotic tracking analysis

The following theorem summarizes the main results of the paper.

Theorem 1: Let Assumption 1 hold. Consider the closed-loop odd-rational-power nonlinear system (1) with hysteresis quantizer (2), control laws (12)-(15), and adaptation law (16). Then, it holds that:

- The state errors $e_i(t)$, $i=1,\ldots,n$, are such that $|e_i(t)|<\kappa_i(t)$ for all $t\geq 0$;
- The output tracking error $e_1(t)=y(t)-y_r(t)$ satisfies $e_1(t)\to 0$ as $t\to+\infty$;
- All closed-loop signals remain bounded.

PROOF: (Time dependence will be kept only for the functions κ_i and y_r, and will be otherwise omitted whenever unambiguous). It follows from (9)-(11) that the states $x_1,\ldots,x_n$ can be rewritten as

$$x_1 = \zeta_1\kappa_1(t)+y_r(t) \triangleq \chi_1(\zeta_1,t), \tag{17}$$

$$x_i = \zeta_i\kappa_i(t)+\alpha_{i-1} \triangleq \chi_i(\zeta_{i-1},\zeta_i,t),\ i=2,\ldots,n. \tag{18}$$

Differentiating the normalized errors ζ_i in (11) with respect to time and using (12)-(16) and the dynamics in (1) gives

$$\dot{\zeta}_1 = \frac{1}{\kappa_1(t)}\left[\phi_1(\chi_1,t)+\psi_1(\chi_1,t)\chi_2^{\frac{p_1}{q_1}} - \dot{y}_r(t) - \dot{\kappa}_1(t)\zeta_1\right] \triangleq \beta_1(\zeta_1,\zeta_2,\widehat{\Xi}_1,t), \tag{19}$$

$$\begin{aligned}\dot{\zeta}_i =& \frac{1}{\kappa_i(t)}\Big[\phi_i(\bar{\chi}_i,t)+\psi_i(\bar{\chi}_i,t)\chi_{i+1}^{\frac{p_i}{q_i}} - \frac{\partial\alpha_{i-1}^\star}{\partial t} - \frac{\partial\alpha_{i-1}^\star}{\partial\zeta_{i-1}}\beta_{i-1} \\ &- \frac{\partial\alpha_{i-1}^\star}{\partial\widehat{\Xi}_i}\dot{\widehat{\Xi}}_i - \dot{\kappa}_i(t)\zeta_i\Big] \\ \triangleq& \beta_i(\zeta_1,\ldots,\zeta_{i+1},\widehat{\Xi}_1,\ldots,\widehat{\Xi}_i,t),\ i=2,\ldots,n-1,\end{aligned} \tag{20}$$

$$\begin{aligned}\dot{\zeta}_n =& \frac{1}{\kappa_n(t)}\Big[\phi_n(\bar{\chi}_n,t)+\psi_n(\bar{\chi}_n,t)(Q(u))^{\frac{p_n}{q_n}} - \frac{\partial\alpha_{n-1}^\star}{\partial t} \\ &- \frac{\partial\alpha_{n-1}^\star}{\partial\zeta_{n-1}}\beta_{n-1} - \frac{\partial\alpha_{n-1}^\star}{\partial\widehat{\Xi}_n}\dot{\widehat{\Xi}}_n - \dot{\kappa}_n(t)\zeta_n\Big] \\ \triangleq& \beta_n(\zeta_1,\ldots,\zeta_n,\widehat{\Xi}_1,\ldots,\widehat{\Xi}_n,t),\end{aligned} \tag{21}$$

where $\bar{\chi}_i \triangleq [\chi_1,\ldots,\chi_i]^T$, $i=1,\ldots,n$. For compactness, let us define $\boldsymbol{\xi} = [\zeta_1,\ldots,\zeta_n,\widehat{\Xi}_1,\ldots,\widehat{\Xi}_n]^T$ and let us rewrite (16) and (19)-(21) in the form of

$$\begin{aligned}\dot{\boldsymbol{\xi}} =& \boldsymbol{\beta}(\boldsymbol{\xi},t) = \Big[\beta_1(\bar{\boldsymbol{\zeta}}_2,\widehat{\Xi}_1,t),\ldots,\beta_i(\bar{\boldsymbol{\zeta}}_i,\ldots,\bar{\widehat{\boldsymbol{\Xi}}}_i),\ldots, \\ &\beta_n(\bar{\boldsymbol{\zeta}}_n,\ldots,\bar{\widehat{\boldsymbol{\Xi}}}_n),\beta_{n+1}(\zeta_1,t),\ldots,\beta_{2n}(\zeta_n,t)\Big]^T,\end{aligned} \tag{22}$$

where $\bar{\boldsymbol{\zeta}}_i = [\zeta_1,\ldots,\zeta_i]^T$, $\bar{\widehat{\boldsymbol{\Xi}}}_i = [\widehat{\Xi}_1,\ldots,\widehat{\Xi}_i]^T$, $i=2,\ldots,n$. Define the open set $\Theta_{\boldsymbol{\xi}} = \Theta_{\boldsymbol{\xi},1}\times\cdots\times\Theta_{\boldsymbol{\xi},i}\times\cdots\times\Theta_{\boldsymbol{\xi},n}\times\mathbf{R}^{n\times 1}$ with $\Theta_{\boldsymbol{\xi},i}=(-1,1)$, $i=1,\ldots,n$. It is straightforward to verify that $\boldsymbol{\xi}(0) = [\zeta_1(0),\ldots,\zeta_n(0),\widehat{\Xi}_1^0,\ldots,\widehat{\Xi}_n^0]^T \subseteq \Theta_{\boldsymbol{\xi}}$ due to $|e_i(0)|<\kappa_{i,0}$. Note that $\boldsymbol{\beta}(\cdot,\cdot):\Theta_{\boldsymbol{\xi}}\times\mathbf{R}_+\to\mathbf{R}^{2n\times 1}$ is piecewise continuous in t and locally Lipschitz in $\Theta_{\boldsymbol{\xi}}$; ϕ_i and ψ_i are piecewise continuous in t and locally Lipschitz in $\bar{\boldsymbol{x}}_i$; $y_r(\cdot)$ and $\kappa_i(\cdot)$ are bounded and differentiable. Then, it follows from [29, Thm. 54] that there exists a unique maximal solution $\boldsymbol{\xi}(\cdot)$ of (22) on the time interval $[0,\tau_{\max})$, where $\tau_{\max}<+\infty$ is chosen such that $\boldsymbol{\xi}(t)\in\Theta_{\boldsymbol{\xi}}$ for all $t\in[0,\tau_{\max})$. In what follows, we first suppose $\tau_{\max}<+\infty$, and eventually we prove by a contradiction that $\tau_{\max}$ must be extended to $+\infty$.

Let us consider the barrier Lyapunov function candidates

$$\mathcal{L}_i = \frac{\zeta_i^2}{2(1-\zeta_i^2)^2} + \frac{1}{2\gamma_i}c_i\varrho_i\widetilde{\Xi}_i^2,\ i=1,\ldots,n \tag{23}$$

which are positive definite and continuously differentiable over $\Theta_{\boldsymbol{\xi}}$, where $\widetilde{\Xi}_i = \Xi_i - \widehat{\Xi}_i$, $\varrho_i>0$, Ξ_i are unknown constants whose specific expressions are given after (30), and $\widehat{\Xi}_i$ is the estimate of Ξ_i. Consider the following induction steps on the time interval $[0,\tau_{\max})$.

Step 0**:** Note from (17) that $\alpha_0 \triangleq y_r(t)$, $\dot{\alpha}_0$, and x_1 are bounded on $[0,\tau_{\max})$ as a result of ζ_1, $\kappa_1(t)$, $y_r(t)$, and $\dot{y}_r(t)$ being bounded on $[0,\tau_{\max})$.

Step i ($i\in\{1,\ldots,n-1\}$)**:** Consider that at step $i-1$ we have shown $x_1(\cdot),\ldots,x_{i-1}$, α_{i-1}, and $\dot{\alpha}_{i-1}(\cdot)$ to be bounded on $[0,\tau_{\max})$. From (18) we further have that $x_i(\cdot)$ is bounded on $[0,\tau_{\max})$. Then, it follows from (1), (10), (18), and (20) that the time derivative of $\mathcal{L}_i$ is

$$\begin{aligned}\dot{\mathcal{L}}_i =& \frac{\varpi_i}{\kappa_i(t)}\Big[\phi_i(\bar{\boldsymbol{x}}_i,t)+\psi_i(\bar{\boldsymbol{x}}_i,t)(e_{i+1}+\alpha_i)^{\frac{p_i}{q_i}} - \dot{\kappa}_i(t)\zeta_i \\ &- \dot{\alpha}_{i-1}\Big] - \frac{1}{\gamma_i}c_i\varrho_i\widetilde{\Xi}_i\dot{\widehat{\Xi}}_i,\quad t\in[0,\tau_{\max}),\end{aligned} \tag{24}$$

Applying Lemma 5 to $x_{i+1}^{\frac{p_i}{q_i}}$ gives

$$x_{i+1}^{\frac{p_i}{q_i}} = \left[\vartheta_i(e_{i+1},\alpha_i)\alpha_i^{p_i} + \mu_i(e_{i+1},\alpha_i)e_{i+1}^{p_i}\right]^{\frac{1}{q_i}}, \tag{25}$$

where $\vartheta_i(\cdot,\cdot)$ and $\mu_i(\cdot,\cdot)$ are some real-valued functions satisfying $1-\bar{\epsilon}_i \leq \vartheta_i(\cdot,\cdot) \leq \max\{1+\bar{\epsilon}_i, 2^{p_i-1}\}$ with $\bar{\epsilon}_i$ being an arbitrary constant taking value in $(0,1)$, and $|\mu_i(\cdot,\cdot)|\leq\bar{v}_i$, with $\bar{v}_i>0$ being a constant which is independent of e_{i+1} and α_i.

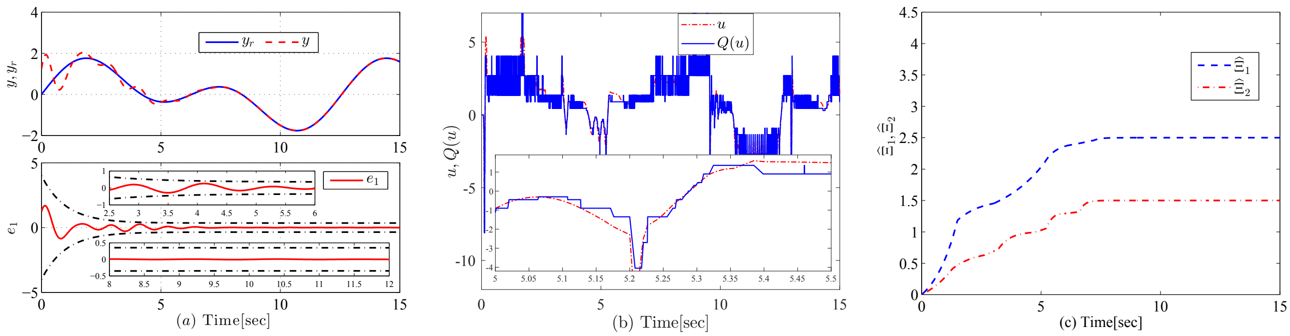

Fig. 1: Numerical Example: (a) Evolution of y, y_r, and e_1; (b) Evolution of the actual control signal u and the quantized control signal $Q(u)$; (c) Evolution of $\widehat{\Xi}_1$ and $\widehat{\Xi}_2$.

The following inequality results from applying Lemma 2 to (25):

$$\left|x_{i+1}^{\frac{p_i}{q_i}} - \left(\vartheta_i(e_{i+1},\alpha_i)\alpha_i^{p_i}\right)^{\frac{1}{q_i}}\right| \le 2^{1-\frac{1}{q_i}}\left|\mu_i(e_{i+1},\alpha_i)e_{i+1}^{p_i}\right|^{\frac{1}{q_i}} \le \bar{E}_i,\quad t\in[0,\tau_{\max}) \qquad (26)$$

where $\bar{E}_i > 0$ is an upper bound of $E_i(e_{i+1},\alpha_i) \triangleq 2^{1-\frac{1}{q_i}} \cdot \left|\mu_i(e_{i+1},\alpha_i)e_{i+1}^{p_i}\right|^{\frac{1}{q_i}}$ which is bounded due to the boundedness of $\mu_i(e_{i+1},\alpha_i)$, and $e_{i+1}(t)$ on $[0,\tau_{\max})$. Hence, it follows that

$$x_{i+1}^{\frac{p_i}{q_i}} = \left(\vartheta_i(e_{i+1},\alpha_i)\alpha_i^{p_i}\right)^{\frac{1}{q_i}} + \ell_i\bar{E}_i,\quad t\in[0,\tau_{\max}) \qquad (27)$$

for some function $\ell_i \subseteq (-1,1)$. Substituting the virtual control law α_i (12) and (27) into (24) yields

$$\dot{\mathcal{L}}_i \le |F_i(t)|\,|\varpi_i| - k_iH_i(t)\varpi_i^2 - \frac{c_i\psi_i(\bar{\boldsymbol{x}}_i,t)\varpi_i^2\widehat{\Xi}_i}{\kappa_i(t)\sqrt{\varpi_i^2+\sigma^2(t)}} - \frac{1}{\gamma_i}c_i\varrho_i\widetilde{\Xi}_i\dot{\widehat{\Xi}}_i,\quad t\in[0,\tau_{\max}), \qquad (28)$$

where $|F_i(t)| \triangleq \frac{1}{\kappa_i(t)}\Big[\left|\phi_i(\bar{\boldsymbol{x}}_i,t)\right| + \left|\dot{\alpha}_{i-1}\right| + \left|\ell_i\psi_i(\bar{\boldsymbol{x}}_i,t)\right|\bar{E}_i + \left|\dot{\kappa}_i(t)\zeta_i\right|\Big]$ and $H_i(t) = \frac{\psi_i(\bar{\boldsymbol{x}}_i,t)}{\kappa_i(t)}$, $i=1,\ldots,n-1$. Using the fact that $|\zeta_i(t)| < 1$ for all $t\in[0,\tau_{\max})$, $i=1,\ldots,n$, and that $y_r(\cdot)$, $\kappa_i(\cdot)$, $\dot{\kappa}_i(\cdot)$, $x_1(\cdot),\ldots,x_i(\cdot)$, $\dot{\alpha}_{i-1}(\cdot)$ are bounded on $[0,\tau_{\max})$, we get from the Extreme Value Theorem that there exist unknown constants $\bar{\psi}_i > 0$, $\underline{\psi}_i > 0$, $\underline{F}_i > 0$, $\bar{F}_i > 0$, $\underline{H}_i > 0$, and $\bar{H}_i > 0$ such that

$$\underline{\psi}_i \le |\psi_i(\cdot,\cdot)| \le \bar{\psi}_i,\ \ \underline{F}_i \le F_i(\cdot) \le \bar{F}_i,\ \ \underline{H}_i \le H_i(\cdot) \le \bar{H}_i \qquad (29)$$

on $[0,\tau_{\max})$. Substituting (29) and adaptation law (16) into (28) gives

$$\dot{\mathcal{L}}_i \le -k_i\underline{H}_i\varpi_i^2 + \bar{F}_i\,|\varpi_i| - \frac{c_i\psi_i(\bar{\boldsymbol{x}}_i,t)\varpi_i^2\widehat{\Xi}_i}{\kappa_i(t)\sqrt{\varpi_i^2+\sigma^2(t)}} + \frac{c_i\varrho_i\varpi_i^2\widehat{\Xi}_i}{\sqrt{\varpi_i^2+\sigma^2(t)}} - \frac{c_i\varrho_i\varpi_i^2\Xi_i}{\sqrt{\varpi_i^2+\sigma^2(t)}},\quad t\in[0,\tau_{\max}) \qquad (30)$$

Note from (16) that $\widehat{\Xi}_i(t) \ge 0$, $\forall t \ge 0$. After defining $\varrho_i = \frac{\underline{\psi}_i}{\kappa_{i,0}}$ and $\Xi_i = \frac{\bar{F}_i}{c_i\varrho_i}$, and applying Lemma 3 to (30) results in

$$\dot{\mathcal{L}}_i \le -k_i\underline{H}_i\varpi_i^2 + \bar{F}_i\sigma(t),\quad t\in[0,\tau_{\max}). \qquad (31)$$

Integrating (31) over $[0,t)$ leads to

$$\mathcal{L}_i(t) + \int_0^t k_i\underline{H}_i\varpi_i^2(s)\mathrm{d}s \le \mathcal{L}_i(0) + \bar{F}_i\bar{\sigma} \triangleq \bar{\delta}_i,\quad t\in[0,\tau_{\max}), \qquad (32)$$

which, combined with (23), implies that

$$\frac{\zeta_i^2}{2\left(1-\zeta_i^2\right)^2} \le \mathcal{L}_i(t) \le \bar{\delta}_i,\ \text{ and }\ \frac{c_i\varrho_i\widetilde{\Xi}_i^2}{2\gamma_i} \le \mathcal{L}_i(t) \le \bar{\delta}_i \qquad (33)$$

$\forall t\in[0,\tau_{\max})$. Solving (33) results in

$$|\zeta_i| \le \bar{\zeta}_i \triangleq \sqrt{1-\frac{\sqrt{8\bar{\delta}_i+1}-1}{4\bar{\delta}_i}} < 1,\quad t\in[0,\tau_{\max}) \qquad (34)$$

$$\left|\widehat{\Xi}_i\right| \le \widehat{\Xi}_i^* \triangleq \Xi_i + \sqrt{\frac{2\gamma_i\bar{\delta}_i}{c_i\varrho_i}},\qquad t\in[0,\tau_{\max}). \qquad (35)$$

Note that (34) implies the boundedness of ϖ_i, which together with (35) ensures the boundedness of α_i and x_{i+1} on $[0,\tau_{\max})$ according to (12) and (10), respectively. Then, it can be derived that the time derivative of ϖ_i can be bounded by

$$|\dot{\varpi}_i| \le \frac{4\zeta_i^2+1}{k_i(t)\left[1-\zeta_i^2\right]^2}\Big[|\phi_i(\bar{\boldsymbol{x}}_i,t)| + \bar{\psi}_i\vartheta_i^{\frac{1}{q_i}}\left|\alpha_i^{\frac{p_i}{q_i}}\right| + \bar{\psi}_i|\ell_i|\bar{E}_i + |\dot{\alpha}_{i-1}| + |\dot{\kappa}_i(t)\zeta_i|\Big],\quad t\in[0,\tau_{\max}). \qquad (36)$$

Invoking (12), (36), and the boundedness of ϖ_i, the time derivative of virtual control law α_i can be bounded by

$$|\dot{\alpha}_i| \le \frac{q_i}{p_i}\vartheta_i^{\frac{-1}{p_i}}\left(k_i|\varpi_i| + \frac{c_i\widehat{\Xi}_i|\varpi_i|}{\sqrt{\varpi_i^2+\sigma^2(t)}}\right)^{r_i}\Bigg[\frac{c_i\widehat{\Xi}_i|\dot{\varpi}_i|}{\left(\varpi_i^2+\sigma^2\right)^{\frac{1}{2}}} + k_i|\dot{\varpi}_i| + \frac{c_i\gamma_i|\varpi_i^3|}{\varpi_i^2+\sigma^2} + \frac{c_i\widehat{\Xi}_i\varpi_i^2|\dot{\varpi}_i|}{\left(\varpi_i^2+\sigma^2\right)^{\frac{3}{2}}} + \frac{c_i\widehat{\Xi}_i\varpi_i\sigma|\dot{\sigma}|}{\left(\varpi_i^2+\sigma^2\right)^{\frac{3}{2}}}\Bigg]$$

for $t\in[0,\tau_{\max})$, (37)

where $r_i = \frac{q_i-p_i}{p_i}$.

Step n: Similarly to Step i, we obtain the derivative of $\mathcal{L}_n$ as

$$\dot{\mathcal{L}}_n = \frac{\varpi_n}{\kappa_n(t)}\Big[\phi_n(\bar{\boldsymbol{x}}_n,t) + \psi_n(\bar{\boldsymbol{x}}_n,t)\Big(\vartheta_n^{\frac{1}{q_n}}\varsigma^{\frac{p_n}{q_n}}(u)u^{\frac{p_n}{q_n}} + \ell_n\bar{E}_n\Big) - \dot{\kappa}_n(t)\zeta_n - \dot{\alpha}_{n-1}\Big] - \frac{c_n\varrho_n\widetilde{\Xi}_n\dot{\widehat{\Xi}}_n}{\gamma_n},\quad t\in[0,\tau_{\max}), \qquad (38)$$

where the function $\ell_n \subseteq (-1,1)$, and $\bar{E}_n = 2^{1-\frac{1}{q_n}}\left|\mu_n\bar{d}^{p_n}\right|^{\frac{1}{q_n}}$. Substituting actual control u as in (14) into (38) yields

$$\dot{\mathcal{L}}_n \le |F_n(t)|\,|\varpi_n| - k_nH_n(t)\varpi_n^2 - \frac{c_n\psi_n(\bar{\boldsymbol{x}}_n,t)\varpi_n^2\widehat{\Xi}_n}{\kappa_n(t)\sqrt{\varpi_n^2+\sigma^2(t)}} - \frac{1}{\gamma_n}c_n\varrho_n\widetilde{\Xi}_n\dot{\widehat{\Xi}}_n,\quad t\in[0,\tau_{\max}), \qquad (39)$$

where $|F_n(t)| \triangleq \frac{1}{\kappa_n(t)}\left[\left|\phi_n(\bar{\boldsymbol{x}}_n,t)\right|+\left|\dot{\alpha}_{n-1}\right|+\left|\ell_n\psi_n(\bar{\boldsymbol{x}}_n,t)\right|\bar{E}_n+\left|\dot{\kappa}_n(t)\zeta_n\right|\right]$ and $H_n(t)=\frac{\psi_n(\bar{\boldsymbol{x}}_n,t)}{\kappa_n(t)}$. Similarly to the analysis after (28), there exist unknown constants $\bar{\psi}_n>0$, $\underline{\psi}_n>0$, $\underline{F}_n>0$, $\bar{F}_n>0$, $\underline{H}_n>0$, and $\bar{H}_n>0$ such that

$$\underline{\psi}_n\leq|\psi_n(\cdot,\cdot)|\leq\bar{\psi}_n,\ \underline{F}_n\leq F_n(\cdot)\leq\bar{F}_n,\ \underline{H}_n\leq H_n(\cdot)\leq\bar{H}_n \tag{40}$$

on $[0,\tau_{\max})$. Substituting (40) and adaptation law (16) into (39) and conducting the same steps as (31)-(35), it is possible to obtain:

$$|\zeta_n|\leq\bar{\zeta}_n\triangleq\sqrt{1-\frac{\sqrt{8\bar{\delta}_n+1}-1}{4\bar{\delta}_n}}<1,\quad t\in[0,\tau_{\max}) \tag{41}$$

$$\left|\widehat{\Xi}_n\right|\leq\widehat{\Xi}_n^*\triangleq\Xi_n+\sqrt{\frac{2\gamma_n\bar{\delta}_n}{c_n\varrho_n}},\qquad t\in[0,\tau_{\max}) \tag{42}$$

where $\bar{\delta}_n=\mathcal{L}_n(0)+\bar{F}_n\bar{\sigma}$ and $\Xi_n=\frac{\bar{F}_n}{c_n\varrho_n}$ with $\varrho_n=\frac{\underline{\psi}_n}{\kappa_{n,0}}$.

Consequently, one can obtain that $\zeta_n\in\left[-\bar{\zeta}_n,\bar{\zeta}_n\right]\subseteq(-1,1)$. Following reasonings similar to (36)-(37), the boundedness of u, and $\dot{u}$ can be achieved on the time interval $[0,\tau_{\max})$. Therefore, all closed-loop signals, including states x_i in (18), $i=1,\ldots,n$, intermediate control laws α_i and their derivatives $\dot{\alpha}_i$, $i=1,\ldots,n-1$, and actual control u are bounded for all $t\in[0,\tau_{\max})$. Moreover, from the above analysis, one can conclude that there exists a compact set $\Theta_{\boldsymbol{\xi}}^+=\left[-\bar{\zeta}_1,\bar{\zeta}_1\right]\times\cdots\times\left[-\bar{\zeta}_n,\bar{\zeta}_n\right]\times\mathbf{R}^{n\times1}\subset\Theta_{\boldsymbol{\xi}}$ such that the maximal solution of (22) satisfies $\boldsymbol{\xi}(t)\in\Theta_\zeta^+$ for all $t\in[0,\tau_{\max})$. This contradicts the argument of [29, pp. 481 Proposition C. 3.6] (i.e. there exists a time instant $t^\star\in[0,\tau_{\max})$ such that $\boldsymbol{\xi}(t^\star)\notin\Theta_{\boldsymbol{\xi}}^+$), which implies that $\tau_{\max}=+\infty$. Therefore, all closed-loop signals are bounded and $\boldsymbol{\xi}(t)\in\Theta_{\boldsymbol{\xi}}^+\cup\Theta_{\boldsymbol{\xi}}$ for all $t\geq0$, and $|e_i(t)|<\kappa_i(t)$, $i=1,\ldots,n$, holds for all $t\geq0$. In addition, it can be concluded from (32) and (37) that $\int_0^t k_1\underline{H}_1\varpi_1^2(s)\mathrm{d}s\leq\bar{\delta}_1$ holds and $|\dot{\varpi}_1|$ is bounded, respectively. This implies that $\lim_{t\to+\infty}\varpi_1(t)=0$ according to Barbalat lemma [24], which eventually implies $\lim_{t\to+\infty}e_1(t)=0$. This completes the proof. ■

Remark 5: Barrier Lyapunov functions have been used in the literature [30]-[34] for constraints satisfaction, whereas the barrier Lyapunov function (23) in our design serves to transform the unknown system nonlinearities in (1) to some unknown but bounded terms (cf. (29) and (40)). Then, these terms are compensated by adaptation laws (cf. (30)-(31)) without imposing growth conditions on system nonlinearities (such as [4, Assumption 2], [5, Assumption 2], [6, Assumption 2], [7, Assumption 2], [8, Assumption 1], [9, Assumption 1], [10, Assumption 1], [11, Assumption 1], [14, Assumption 3], and [15, Assumptions 1 and 3]) and without universal approximators.

Remark 6: The main innovation of Lemma 5 is to allow handling the control terms in a linear-like manner (cf. (27) and (38)). With this tool, Theorem 1 shows that prescribed performance asymptotic tracking can be achieved for the challenging class of dynamics (1).

V. SIMULATIONS

A. Numerical Example: To illustrate the validity of the proposed control method, consider the following dynamics:

$$\begin{cases}\dot{x}_1=2.5x_1^2\cos(x_1)+(1.5+\sin(x_1))x_2^{\frac{5}{3}},\\ \dot{x}_2=1.25\sin(x_1x_2)+(2.5-\cos(x_1x_2))Q(u),\\ y=x_1,\end{cases} \tag{43}$$

with desired trajectory $y_r(t)=\sin(t)+\sin(0.5t)$ and initial conditions $[x_1(0),x_2(0)]^T=[1.25,0.25]^T$. We select the prescribed performance functions $\kappa_i(t)=(4-0.35)\exp(-t)+0.35$, $i=1,2$, the quantizer parameters $v_{\min}^+=0.025$, $v_{\min}^-=-0.035$, $\varrho_+=0.2$ and $\varrho_-=0.25$, and the design parameters $k_1=1.5$, $k_2=2.5$, $c_1=3$, $c_2=3.5$, $\gamma_1=1.75$, $\gamma_2=1.5$, $\bar{\epsilon}_1=0.275$, and $\bar{\epsilon}_2=0.75$. The initial conditions of adaptive parameters are set as $\widehat{\Xi}_1(0)=\widehat{\Xi}_2(0)=0$. The positive function $\sigma(\cdot)$ is chosen as $\sigma(t)=\frac{1}{0.15+2t^4}$. The simulation results are shown in Fig. 1. Fig. 1-(a) reveals that the system output y tracks the desired trajectory y_r asymptotically, while ensuring that output tracking error e_1 evolves within the prescribed performance interval $(-\kappa_1(t),\kappa_1(t))$ all the time. Fig. 1-(b) depicts the evolution of the actual control signal u and of the quantized control $Q(u)$. Notably, asymptotic tracking is achieved in spite of quantized information. Compared with bounded tracking for similar dynamics, e.g. [35], the output of the quantizer of Fig. 1-(b) seems to require higher bandwidth. This is expected since asymptotic tracking results for other input-quantized dynamics, e.g. strict-feedback dynamics [21] and [23] have shown that asymptotic tracking may require faster inputs. Fig. 1-(c) shows the evolution of adaptation parameters $\widehat{\Xi}_1$ and $\widehat{\Xi}_2$. To further investigate the effect of the quantizer, the mean absolute control actions (MACA) $\frac{1}{T}\int_0^T|u|$ and $\frac{1}{T}\int_0^T|Q(u)|$ for three different sets of initial conditions are given in Table I where u_{M} and $Q(u)_{\mathrm{M}}$ respectively represent the MACA of u and $Q(u)$ (the latter resulting slightly smaller than the former).

TABLE I: MACA for three different sets of initial conditions

Signal / $\boldsymbol{x}(0)$	u_{M}	$Q(u)_{\mathrm{M}}$	$u_{\mathrm{M}}-Q(u)_{\mathrm{M}}$
$[1.25, 0.25]$	2.8631	2.4767	$\downarrow$ 0.3864
$[2.75, -1.75]$	2.9743	2.5132	$\downarrow$ 0.4611
$[-2.25, 3.25]$	3.0124	2.6659	$\downarrow$ 0.3465

B. Practical Example: A poppet valve is one of the most common components in hydraulic systems [13]. It is typically used to control the timing and quantity of gas or vapor flow into an engine, and its behavior can be modeled by the annular leakage equation. According to [13, pp. 54], the input force F drives the poppet to move for regulating the volumetric flow rate $Q_{\mathrm{vol}}=\lambda c^3$ of oil from the high-pressure to the low-pressure chamber, where $\lambda=\frac{\pi r}{6\mu L}\Delta P$ is a lumped coefficient, $c=\alpha y$ is the effective clearance of the annular passage with α a constant and y the displacement of poppet, and where r, μ, and L are constants independent of the axial motion of poppet, and ΔP is the pressure drop between two chambers. The dynamics of oil volume V in upper chamber is given by

$$\dot{V}(t)=Q_{\mathrm{vol}}-R(t), \tag{44}$$

where R is the lumped reduction rate of oil attributed to consumption and other leakages. The equation of motion of the poppet is

$$m\ddot{y}(t)=-k\dot{y}(t)+T(t)+F(t), \tag{45}$$

where m is the mass of the poppet, k is the viscous friction coefficient, T is the lumped elastic force, and F is the input force. At this point, let us introduce the following notation substitutions:

$$x_1=V,\quad x_2=y,\quad x_3=\dot{y},\quad u=F. \tag{46}$$

Then, the dynamics of systems (46) becomes

$$\dot{x}_1=\phi_1+\psi_1x_2^3,\quad \dot{x}_2=x_3,\quad \dot{x}_3=\phi_3+\psi_3Q(u), \tag{47}$$

where $\psi_1=\lambda\alpha^3$, $\phi_1=-R$, $\psi_3=\frac{1}{m}$, and $\phi_3=\frac{1}{m}(T-kx_3)$. We take the desired trajectory $y_r(t)=\sin(t)+\sin(0.5t)$ and initial conditions $[x_1(0),x_2(0),x_3(0)]^T=[2.5,1.5,-0.75]^T$. We take $m=7.5\mathrm{kg}$, $k=2.5\mathrm{N/m}$, $R=5\mathrm{L/min}$, $\Delta P=10\mathrm{N/m^2}$, $T=5\mathrm{N}$, $\mu=2.5$, $L=5$, $r=1.25$, $\alpha=4.5$, and the prescribed performance function defined by $\kappa_i(t)=(6-0.25)\exp(-t)+0.25$, $i=1,2,3$, the quantizer parameters $v_{\min}^+=0.25$, $v_{\min}^-=-0.05$, $\varrho_+=0.2$ and

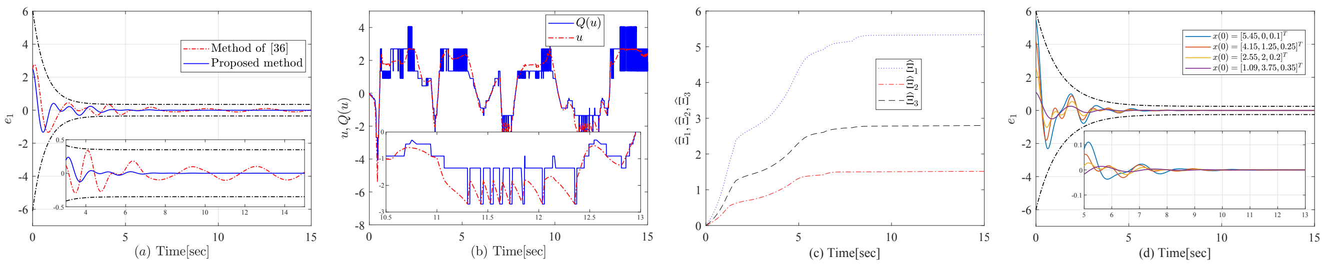

Fig. 2: Practical Example: (a) Evolution of tracking error e_1 under two schemes; (b) Evolution of the actual control signal u and the quantized control signal $Q(u)$; (c) Evolution of adaptation parameters $\widehat{\Xi}_1$, $\widehat{\Xi}_2$, and $\widehat{\Xi}_3$; (d) Evolution of tracking error e_1 under four different sets of initial conditions.

TABLE II: MACA for four different sets of initial conditions

$\boldsymbol{x}(0)$ \ Signal	$u_{\rm M}$	$Q(u)_{\rm M}$	$u_{\rm M} - Q(u)_{\rm M}$
$[5.45, 0, 0.1]$	2.3653	2.0125	$\downarrow$ 0.3528
$[4.15, 1.25, 0.25]$	2.1134	1.9269	$\downarrow$ 0.1867
$[2.55, 2, 0.2]$	2.0863	1.7267	$\downarrow$ 0.3596
$[1.09, 3.75, 0.35]$	1.9867	1.6235	$\downarrow$ 0.3632

$\varrho_- = 0.25$, and the design parameters $k_1 = 5$, $k_2 = 3.5$, $k_3 = 15$, $c_1 = 2.5$, $c_2 = 5$, $c_3 = 10$, $\gamma_1 = 1.25$, $\gamma_2 = 0.75$, $\gamma_3 = 1.5$, $\bar\epsilon_1 = 0.5$, $\bar\epsilon_2 = 0.25$, and $\bar\epsilon_3 = 0.75$. The initial conditions of adaptive parameters are set as $\widehat{\Xi}_1(0) = \widehat{\Xi}_2(0) = \widehat{\Xi}_3(0) = 0$. The integral function $\sigma(\cdot)$ is chosen as $\sigma(t) = \frac{1}{0.25+t^4}$. The simulation results are shown in Fig. 3, where the standard PPC approach as in [36] is taken as a means of comparison. Fig. 3-(a) shows that the proposed approach exhibits asymptotic tracking differently from the standard PPC approach. Fig. 3-(b) and (c) show the profiles of the control signal u and the quantized control signal $Q(u)$, and adaptation parameters $\widehat{\Xi}_1$, $\widehat{\Xi}_2$, and $\widehat{\Xi}_3$, respectively. Fig. 3-(d) shows that the proposed method achieves asymptotic tracking for different initial conditions, despite quantized information and even with reduced control effort in terms of MACA (cf. Table II).

VI. Conclusions

This paper has addressed asymptotic tracking for uncertain high-order odd-rational-power nonlinear systems without imposing growth restrictions on the nonlinearities. The proposed result extends the class of dynamics for which asymptotic tracking is possible with minimum knowledge of the system dynamics. In line with [37], an interesting open problem deserving future investigation is to further reduce the knowledge of the system dynamics by considering completely unknown control directions.

Appendix

Proof of Lemma 5. The aim is to first find an upper and lower bound in the form

$$\left(\underline{\vartheta}(x_1,x_2)x_1^p + \underline{\mu}(x_1,x_2)x_2^p\right)^{\frac{1}{q}} \le (x_1+x_2)^{\frac{p}{q}} \le \left(\bar\vartheta(x_1,x_2)x_1^p + \bar\mu(x_1,x_2)x_2^p\right)^{\frac{1}{q}}, \tag{48}$$

for some appropriately bounded functions $\underline{\mu}(\cdot,\cdot)$, $\underline{\vartheta}(\cdot,\cdot)$, $\bar\mu(\cdot,\cdot)$, and $\bar\vartheta(\cdot,\cdot)$. Using the binomial theorem [38, Sect. 3.1, page 10], the following inequalities can be derived for $\forall x_1$, $x_2 \in \mathbf{R}$:

$$\begin{aligned}(x_1+x_2)^{\frac{p}{q}} &\le \left(x_1^p + x_2^p + \sum_{k=1}^{p-1}\binom{p}{k}|x_1|^k|x_2^{p-k}|\right)^{\frac{1}{q}}\\
&\le \left(x_1^p + x_2^p + \sum_{k=1}^{p-1}\left(\frac{k}{p}\varepsilon^{\frac{p}{k}}|x_1|^p + \frac{p-k}{p}\binom{p}{k}\varepsilon^{\frac{-p}{p-k}}|x_2|^p\right)\right)^{\frac{1}{q}}\\
&\le \left(x_1^p + x_2^p + \sum_{k=1}^{p-1}\varepsilon_k|x_1|^p + \sum_{k=1}^{p-1}\omega_k|x_2|^p\right)^{\frac{1}{q}}\\
&\le \left(\big[1+\bar\epsilon\cdot\text{sign}(x_1)\big]x_1^p + \big[1+\omega\cdot\text{sign}(x_2)\big]x_2^p\right)^{\frac{1}{q}},\end{aligned} \tag{49}$$

where the second inequality relies on Lemma 4, and where $\varepsilon_k = \frac{k}{p}\varepsilon^{\frac{p}{k}}$, $\omega_k = \frac{p-k}{p}\binom{p}{k}\varepsilon^{\frac{-p}{p-k}}$, $\omega = \sum_{k=1}^{p-1}\omega_k$, and $\bar\epsilon = \sum_{k=1}^{p-1}\varepsilon_k$ can be made to satisfy $0 < \bar\epsilon < 1$ by appropriately selecting the small positive constant ε.

A lower bound will be sought along the following three situations.

Situation 1: When $x_1 < 0$ and $x_1 + x_2 \ge 0$, we immediately have $(x_1+x_2)^{\frac{p}{q}} \ge 0 \ge x_1^{\frac{p}{q}}$ as p is a positive odd integer.

Situation 2: When $x_1 < 0$ and $x_1 + x_2 < 0$, it follows that

$$\begin{aligned}&(x_1^p + x_2^p)^{\frac{1}{q}} =\\
&2^{\frac{1}{q}}\left[\sum_{m=1}^{\frac{p-1}{2}}\binom{p}{2m-1}\underbrace{\overbrace{\left(\frac{x_1+x_2}{2}\right)^{2m-1}}^{<0}\overbrace{\left(\frac{x_1-x_2}{2}\right)^{p-2m+1}}^{>0}}_{<0}\right.\\
&\left.+\left(\frac{x_1+x_2}{2}\right)^p\right]^{\frac{1}{q}} \le 2^{\frac{1-p}{q}}(x_1+x_2)^{\frac{p}{q}},\end{aligned} \tag{50}$$

which indicates that $(x_1+x_2)^{\frac{p}{q}} \ge \left(2^{p-1}x_1^p + 2^{p-1}x_2^p\right)^{\frac{1}{q}}$.

Situation 3: When $x_1 \ge 0$ and $x_2 \in \mathbf{R}$, then following similar derivations to (48), it holds that $(x_1 - x_2)^{\frac{p}{q}} = \big[x_1 + (-x_2)\big]^{\frac{p}{q}} \le \left(\big[1+\bar\epsilon\cdot\text{sign}(x_1)\big]x_1^p - \big[1-\omega\cdot\text{sign}(x_2)\big]x_2^p\right)^{\frac{1}{q}}$. Besides, note that $(x_1+x_2)^p + (x_1-x_2)^p = 2\Big[x_1^p + \sum_{k=1}^{\frac{p-1}{2}}\binom{p}{2k-1}\overbrace{x_1^{2k-1}}^{\ge 0}\overbrace{x_2^{p-2k+1}}^{\ge 0}\Big] \ge 2x_1^2$. Thus, we have $(x_1+x_2)^{\frac{p}{q}} \ge \left([1-\bar\epsilon\cdot\text{sign}(x_1)]x_1^p + [1+\omega\cdot\text{sign}(x_2)]x_2^p\right)^{\frac{1}{q}}$.

Having derived all the upper and lower bounds in the form of (48), we conclude that the equality $\big[x_1^p\vartheta(x_1,x_2) + x_2^p\mu(x_1,x_2)\big]^{\frac{1}{q}} = (x_1+x_2)^{\frac{p}{q}}$ holds for any x_1, x_2, for some function $\vartheta(\cdot,\cdot) \subseteq \big[1 -$

$\bar{\epsilon}, \max\{1+\bar{\epsilon}, 2^{p-1}\}]$ with $\bar{\epsilon}$ being a constant that can be made to take value in $(0,1)$ by selecting an appropriately small constant ε, and $|\mu(\cdot,\cdot)| \leq \bar{\upsilon}$ with $\bar{\upsilon} = \max\{1+\omega, 2^{p-1}\}$ a positive constant that is independent of x_1 and x_2. This completes the proof. ■

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