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Een optimale investeringsstrategie aan de hand van twee types onzekerheid

(Engelse titel: An optimal investment strategy tailored to two types of uncertainties

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" Een optimale investeringsstrategie aan de hand van twee types onzekerheid"

(Engelse titel: "An optimal investment strategy tailored to two types of uncertainties")

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1 Introduction

1.1 The uncertainty in investments

People invest in projects to make a profit. If the cost of investment is larger than the revenue, a rational person would not invest. Though these two statements may seem trivial, the reality is that it is often difficult to know what the expected costs and payoffs are. Like with any prediction in the future, there is always a level of uncertainty. The amount of available literature which qualitively deals with risk, greatly outnumbers the available literature on quantitative risk. Through mathematic modelling, we can attempt to quantify this uncertainty and bring it to a minimum. In addition, this allows us to find an optimal investment strategy (Should we invest now? And if so, how much? Or should we invest later or drop the project altogether?). The relevance and importance of such a strategy is obvious to anyone considering investing into a project whether it is in researching a new medicine, planning construction for a new library or producing a new electronics device. Though each mentioned example deals with different types of project-specic problems, it will become apparent that the model we find has a wide range of application.

This paper is divided into three parts. First there is a finance part which gives some background information and applies stochastic calculus to derive the main equation for investment opportunities. In the second part, the equation is solved numerically, and the role of the various parameters are discussed. This is then applied to a fictional company considering a large investment project.

For now we will consider the payoff, or profit, as a function max $[0, V - K]$ where V is the *given* value of the completed project and K its expected cost. Pindyck considers investing into a project as a put option in [1]. You are paying an undetermined amount of money (K) to own (or sell) an asset of value V. Similarly, the cost of the put option K is what determines whether you will partake in the opportunity. The motivation behind this is that much of the theory in call and put options can be applied to this situation.

1.2 Diversification

Before explaining the characteristics of the uncertainties, it is important to understand the concept of diversification. In general, diversification is a strategy for your portfolio to reduce risk as much as possible. This can be achieved by having a wide range of (financial) products, possibly correlated to one another. For instance, having stocks in two leading companies in the car industry would potentially reduce risk because if one company is not performing so well, it is likely the other is doing better and so your portfolio evens out. On the other hand, one could argue that if there is a recession, both car firms will be performing poorly, leading to a far greater loss. This shows that the right choice of products is essential to create a risk free portfolio.

1.3 Two types of uncertainty

In the first step to quantifying uncertainty, two types will be distinguished: technical uncertainty and input cost uncertainty. The first is related to the physical difficulty of completing a project. This is the 'internal' uncertainty of a project. There is always risk in price changes due to the specific nature of the project. For example, when building a house one might have to deal with a volatile cement market. However, this is risk can be minimized through diversification. To truly eliminate the uncertainty would be to simply undertake or invest in the project. Contrary to technical uncertainty, the input cost uncertainty is the uncertainty due to possible changes in the environment of the project or overall market. It is the 'external' uncertainty. Using the house example again, if there is likely to be a recession soon, this would have an effect on the purchase of all the required matierals. Hence, this risk is undiversifiable.

To further characterise the difference between the two uncertainties, we can look at how they affect an investment strategy. Consider a project with technical uncertainty. Investing in the project will reveal more information about cost while not investing has 0 value because you are no closer to completion and no follow-up information is obtained. Hence technical uncertainty makes investing attractive in the sense that, next to being one step closer to completion, you gain (the value of) the information on further costs. Now consider a project with input cost uncertainty. Because costs could always change, it is more economic to wait for new information than immediately spending money. So if a project has a large input cost uncertainty it is less attractive to invest.

2 The basic model

2.1 No uncertainty

The remaining cost of completion is defined as a random variable $\tilde{K}$ with $E[\tilde{K}] = K$. Only when the project is completed, does the investor gain an asset of known value V . If the project is dropped all costs are sunk, which means there is no way to get back the invested money. The maximum rate of investment is constant k. As a preliminary step we first consider a model that has no uncertainty over the total cost. In that case the time to complete is given by $T = \frac{K}{k}$. The payoff, $F(K)$ is given by

$$F(K) = \max \left[Ve^{-r\frac{K}{k}} - \int_0^K ke^{-rt}dt \ , \ 0 \right]$$

This can be roughly translated to being equal to the difference between V after being adjusted due to interest rate r (assumed constant) and the total invesetment made with rate k and time T (hence the integral sign). Clearly, one should invest as long as $F(K) > 0$. Elementary operations lead to the following demand for K in order for the payoff to be positive:

$$K < \frac{k}{r} \log(1 + \frac{rV}{k})$$

2.2 Introducing uncertainty

Now uncertainty is introduced by letting the change in K follow a controlled diffusion process. Consider that the value of $K(t)$, is given by:

$$dK = -Idt + g(I, K)dz$$

where $I(t)$ is the rate of investment, dz is the increment of a Wiener process (possibly correlated to the economy) for which $z \sim N(0, \sigma^2)$. The equation states that although the expected remaining cost to completion declines with investment, it also fluctuates stochastically. Analogous to the no uncertainty model, the value of the investment opportunity becomes:

$$F(K) = \max E \left[Ve^{-\mu\tilde{T}} - \int_0^{\tilde{T}} I(t)e^{-\mu t}dt \ , \ 0 \right]$$

where $I(t)$ is the rate which maximizes F (which will be discussed later) and $\tilde{T}$ is the stochastic time of completion. $E[..]$ is the expectation operator and μ is a risk adjusted discount rate. To specify the function $g(I, K)$ attention must be paid to the following properties:

1. For the partial derivatives $g_I \geq 0$ and $g_{II} \leq 0$. To explain this, consider the effect of investing in a project with uncertainty in cost. By investing, we will now know better what the (new) expected cost of completion is. Hence investing has an effect on the stochastic movement of K, next to the decrease in cost represented by $-Idt$. Further, $g_K \geq 0$. This is the observation that an increase in the remaining cost also increases the magnitude of Brownian motion.

2. As far as technical uncertainty is concerned, when there is no investment, there is no change in the remaining cost.

3. However, for input cost uncertainty there will be a stochastic change, regardless of investment. Hence $g(0, K)$ is a function of only K.

4. $F_K < 0$: An increase in the expected cost will reduce the payoff.

5. dK is bounded for all finite K and approaches 0 as $K \to 0$. Once a project is completed, the remaining cost stays at 0.

All these conditions can be satisfied if we let $g(I, K) = cK\frac{I}{K}\alpha$ with $0 \leq \alpha \leq 0.5$ and $c < 0$. This is suggested by Pindyck in [1] and limiting to $\alpha = 0$ and $\frac{1}{2}$ is mainly a choice of practicality. Having already described the nature of $g(I, K)$, it makes sense that $\alpha = \frac{1}{2}$ responds to technical uncertainty and $\alpha = 0$ to input cost uncertainty. Altogether we have:

$$dK = -Idt + \beta(IK)^{\frac{1}{2}}dz + \gamma K dw$$

where β and γ are the 'measure' of their respective uncertainty. The change in expected remaining cost is equal to a stochastic change due to technical uncertainty and input cost uncertainty minus the investment rate. An important point for the following subsection is that dz and dw are uncorrelated Wiener processes and that the risk of dz is diversifiable (uncorrelated to the economy to the economy and stock market). In contrast, dw is related to the economy (input cost) and therefore not diversifiable.

2.3 Finding a differential equation

The problem with $F(K)$ is that the risk adjusted discount rate μ cannot be the risk free rate of interest. This is because dw is correlated to the market (non diversifiable). It would therefore be wise to eliminate dw from the equation. First, we allow x to be the price of an asset or portfolio perfectly correlated to w such that:

$$dx = \alpha_x x dt + \sigma_x x dw \tag{1}$$

According to the capital asset pricing model (CAPM), explained very and concisely in [4] the risk adjusted expected return on x is

$$r_x = r + \theta \rho_{xm} \sigma_x$$

where θ is the overall market price of risk (assumed to be around 0.4) and ρ_{xm} is the instantaneous correlation of x with the market portfolio. Just to clarify, the subscript of

x is merely notational and not indicating a partial derivative. Now consider we have a portfolio with the investment opportunity (worth $F(K)$). We then short sell n units of the asset for price x. The portfolio is then worth $\Phi = F(K) - nx$. We can also say that a marginal change in the portfolio value is equal to the marginal change in F and x:

$$d\Phi = dF - ndx$$

However, holding this portfolio obliges us to invest at rate $I(t)$. Also, because we are selling short, the portfolio brings an extra cost of $n(r_x - \alpha_x)x$. Letting $r_x - \alpha_x = \delta$, We can formulate the return of the portfolio over an interval dt as:

$$d\Phi = dF - ndx - n\delta x dt - I(t)dt \tag{2}$$

Applying Ito's Lemma, explained in [2], we are able to rewrite dF as:

$$dF = -IF_k dt + \beta(IK)^{\frac{1}{2}} F_K dz + \gamma K F_K dw + \frac{1}{2}\beta^2 IKF_{KK}dt + \frac{1}{2}\gamma^2 K^2 F_{KK} dt \tag{3}$$

Substituting (1) and (3) into (2) and setting $n = \frac{\gamma K F_K}{\sigma_x x}$ we then obtain:

$$\begin{aligned} d\Phi &= -IF_k dt + \beta(IK)^{\frac{1}{2}} F_K dz + \gamma K F_K dw + \frac{1}{2}\beta^2 IKF_{KK}dt + \frac{1}{2}\gamma^2 K^2 F_{KK} dt \\ &\quad -\gamma K F_K \frac{\alpha_x}{\sigma_x} dt - \gamma K F_K dw - \gamma K F_K \frac{\delta}{\sigma_x} - Idt \end{aligned}$$

Notice now that dw is removed from the equation. Now there is only diversifiable risk and that means the expected return on the portfolio will be the risk free rate r. Rewriting $d\Phi = r(F - nx)dt$ we now have:

$$\begin{aligned} r(F - nx)dt &= -IF_K dt + \beta(IK)^{\frac{1}{2}} F_K dz + \frac{1}{2}\beta^2 IKF_{KK}dt + \frac{1}{2}\gamma^2 K^2 F_{KK} dt \\ &\quad -\gamma K F_K (\frac{\alpha_x + \delta}{\sigma_x})dt - Idt \\ r(F - \frac{\gamma K F_K}{\sigma_x}) &= -IF_K + \beta(IK)^{\frac{1}{2}} F_K \frac{dz}{dt} + \frac{1}{2}\beta^2 IKF_{KK} + \frac{1}{2}\gamma^2 K^2 F_{KK} \\ &\quad -\gamma K F_K (\frac{r_x}{\sigma_x}) - I \\ rF &= -IF_K + \beta(IK)^{\frac{1}{2}} F_K \frac{dz}{dt} + \frac{1}{2}\beta^2 IKF_{KK} + \frac{1}{2}\gamma^2 K^2 F_{KK} \\ &\quad -\gamma K F_K (\frac{r_x - r}{\sigma_x}) - I \\ rF &= -IF_K + \frac{1}{2}\beta^2 IKF_{KK} + \frac{1}{2}\gamma^2 K^2 F_{KK} - \gamma K F_K (\frac{r_x - r}{\sigma_x}) - I \\ rF &= -IF_K + \frac{1}{2}\beta^2 IKF_{KK} + \frac{1}{2}\gamma^2 K^2 F_{KK} - \gamma K F_K \theta \rho_{xm} - I \qquad (4) \end{aligned}$$

Equation (4) will be examined in the folllowing section. Given that θ is an economic constant, that means only ρ_{xm}, the correlation between fluctuations in cost of the project and the stock market, needs to be predetermined. For now we will let $\lambda = \theta\rho_{xm}$ Having done so, there is the freedom to experiment with the values of γ and β.

2.4 An optimal investment stategy

Given eqation (4), note that the order of I is 1 (i.e. linear). Rewriting the equation as:

$$F = I \underbrace{\left(-F_K + \frac{1}{2}\beta^2 K F_{KK} - 1\right)}_{(*)} \frac{1}{r} + \frac{1}{2r}\gamma^2 K^2 F_{KK} - \frac{\gamma\lambda}{r} K F_K$$

it becomes clear that in order to maximize F, then I should be as large as possible (which is k) only if $(*)$ is positive. If $(*)$ is nonpositive the investment should be 0. In other words, the rate of investment should always be the maximum possible; investing at half the maximum rate would not lead to a maximized F.

3 Model Characteristics

In order to clearly demonstrate the differences in technical and input cost uncertainty, we will first consider them separately and afterwards combine them. First we define K^* as the critical value. Given the optimal investment rule, it is the value for which:

$$-F_K(K^*) + \frac{1}{2}\beta^2 K^* F_{KK}(K^*) - 1 = 0 \tag{5}$$

If K exceeds this, then the payoff will be 0. This also means that the investment rate I is simply k as long as $K < K^*$. If this is not the case, the investment rate is 0. The main matter is now to find such a K^*. There are two more boundary conditions:

$$F(0) = V \tag{6}$$

$$\lim_{K\to\infty} F(K) = 0 \tag{7}$$

Condition (6) is from the definition that a completed project is worth V. Condition (7) means that when the expected cost of completion becomes extremely large, it becomes unlikely that the payoff would ever become positive.

Each subsection consists of a numerical solution and how it was achieved. In addition, some conclusions are drawn relating to the behaviour of the parameters.

3.1 Technical uncertainty

If we only consider technical uncertainty, we let $\gamma = 0$ in equation (4) leaving us with:

$$rF = \frac{1}{2}\beta^2 I K F_{KK} - I F_K - I$$

subject to the previously stated conditions (5), (6) and (7). Remembering that $I = 0$ for $K > K^*$, this makes it impossible to determine an analytical solution.

3.1.1 Numerical solution approach

In order to solve this numerically we will treat the problem as an linear complementarity problem, explained in [3]. The motivation for this is that we do not yet know the value of K^* but are still required to apply its boundary condition. Once formulated as a non-linear system, it is possible to apply the projected Gauss-Seidel method. There are two situations:

$$\begin{aligned}&\text{If } K < K^*\text{: } \frac{1}{2}\beta^2 kKF_{KK} - kF_K - k - rF = 0\\&\text{If } K \geq K^*\text{: } F = 0\end{aligned}$$

In this case, the equation $F = 0$ act as the 'obstacle function'. A formulation that combines the two situations is to find a function $F(K)$ such that:

$$\begin{aligned}\left(\frac{1}{2}\beta^2 kKF_{KK} - kF_K - k - rF\right)(F) &= 0\\ -\left(\frac{1}{2}\beta^2 kKF_{KK} - kF_K - k - rF\right) &\geq 0\\ F \geq 0\ ,\ F(0) = V\ ,\ F(S) &= 0\end{aligned}$$

where S is suffciently far away from K^*. The inequality is based on the fact that if k was 0, then $F \geq 0$ would still need to hold. It is more an educated guess than a sound proof. Notice now that K^* is not explicitly mentioned. To discretize the problem, the following notation will be used:

$$\begin{aligned}x_i &= ih\ ,\ i = 0..N\\ w_i &= F(x_i)\end{aligned}$$

This leads to the following discretization for the equality $i = 1 \ldots N-1$:

$$\begin{aligned}\left[\frac{1}{2}\beta^2 kx_i \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} - k\frac{w_i - w_{i-1}}{h} - k - rw_i\right][w_i] &= 0\\ \left[\frac{\beta^2 ki}{2h}(w_{i-1} - 2w_i + w_{i+1}) - \frac{k}{h}(w_i - w_{i-1}) - k - rw_i\right][w_i] &= 0\\ \left[w_{i-1}\left(\frac{\beta^2 ki}{2h} + \frac{k}{h}\right) + w_i\left(-\frac{\beta^2 ki}{h} - \frac{k}{h} - r\right) + w_{i+1}\left(\frac{\beta^2 ki}{2h}\right) - k\right][w_i] &= 0\\ \left[w_{i-1}\left(-\frac{\beta^2 ki}{2h} - \frac{k}{h}\right) + w_i\left(\frac{\beta^2 ki}{h} + \frac{k}{h} + r\right) + w_{i+1}\left(-\frac{\beta^2 ki}{2h}\right) + k\right][w_i] &= 0\end{aligned}$$

The discretization for the inequality is:

$$\begin{aligned}-\left[\frac{1}{2}\beta^2 kx_i \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} - k\frac{w_i - w_{i-1}}{h} - k - rw_i\right] &\geq 0\\ \left[w_{i-1}\left(-\frac{\beta^2 ki}{2h} - \frac{k}{h}\right) + w_i\left(\frac{\beta^2 ki}{h} + \frac{k}{h} + r\right) + w_{i+1}\left(-\frac{\beta^2 ki}{2h}\right) + k\right] &\geq 0\end{aligned}$$

The reason for this choice of discretization is that backward differentiation will be stable. Also, notice that A is now strictly positive in the diagonal, and negative in the adjacent diagonals. Combined with the boundary values, the discretization leads to finding vector $w \geq 0$ such that:

$$\begin{aligned}w^T(Aw - b) &= 0\\ Aw - b &\geq 0\end{aligned}$$

where:

$$A_{ij} = \begin{cases} -\frac{\beta^2 ki}{2h} - \frac{k}{h} & \text{if } i-1=j \\ \frac{\beta^2 ki}{h} + \frac{k}{h} + r & \text{if } i=j \\ -\frac{\beta^2 ki}{2h} & \text{if } i+1=j \end{cases}$$

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \end{bmatrix}$$

$$b = \begin{bmatrix} -k + V(\frac{\beta^2 k+2k}{2h}) \\ -k \\ \vdots \\ -k \end{bmatrix}$$

Applying the transformation $x = w$ and $y = Aw - b$ the problem can be formulated as finding x, y such that:

$$\begin{aligned} x^T y &= 0 \\ Ax - y &= \hat{b} = b \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

This is identical to the Cryer problem from [3] and can be solved using a 'pointwise' iteration process. The idea is to take an initial guess $x^{(0)}$ and pointwise calculate a correction vector $x^{(t)} - x^{(t-1)}$. The algorithm looks as follows:

Outer Loop $t = 1, 2..$

Inner Loop $i = 1, 2..N-1$

$$r_i^{(t)} = -\hat{b}_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(t)} - a_{ii} x_i^{(t-1)} - \sum_{j=i+1}^{N-1} x_j^{(t-1)}$$

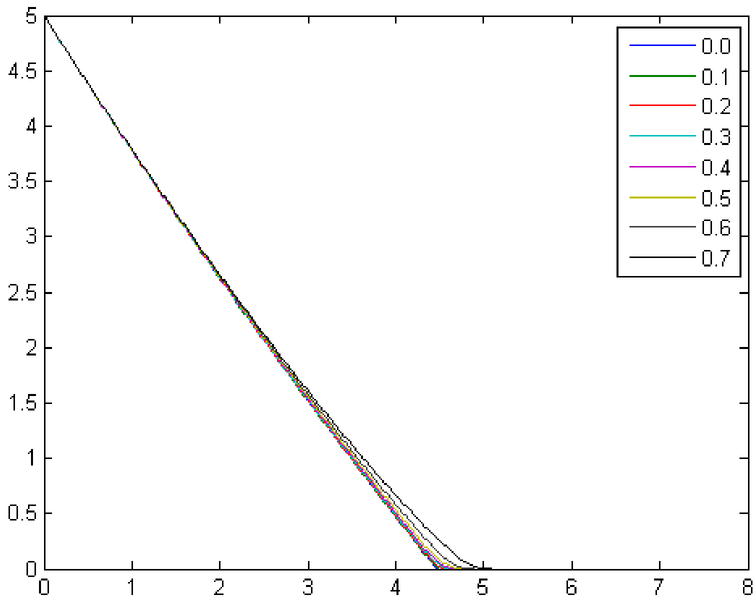
$$x_i^{(t)} = \max \left\{ x_i^{(t-1)} + \frac{r_i^{(k)}}{a_{ii}}, 0 \right\}$$

The advantage to this approach is that the amount of iterations can be chosen, and gives much 'smoother' functions, as opposed to calculating $Aw - b = 0$ explicitly. Choosing $w = A^{-1}b$ as an initial guess may seem odd because calculating inverses is always a heavy job for MATLAB. However, in practice it meant that less iterations are needed to be done (so t stays relatively low) because the initial guess was already quite close to the final solution. This saved time in the long run.

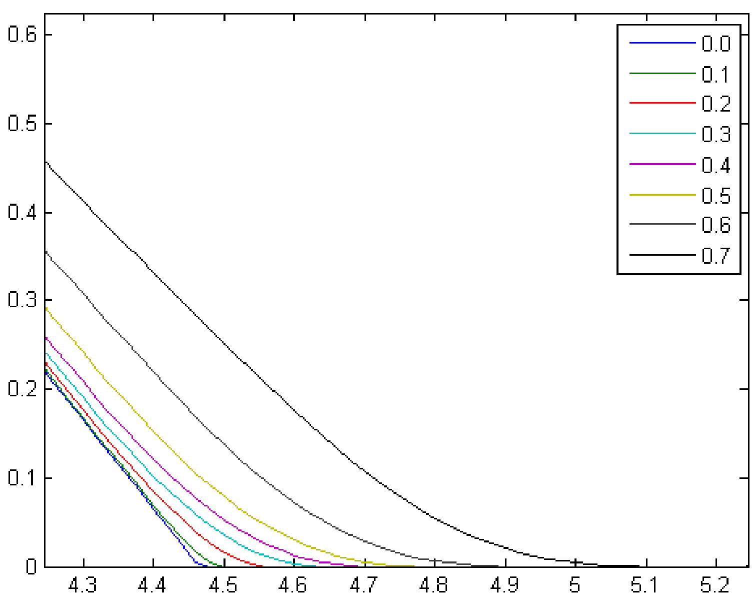
3.1.2 Variation in parameters: β, V, k and r

Having determined an algorithm, this allows us to explore the effects of varying the parameters: β, V, k and r. The most relevant one is β; what is the effect of a large degree of technical uncertainty on the payoff of a project?

As seen in Figure 1, a larger degree of technical uncertainty *raises* the payoff at any given K and also increases K^*. Note that K^* is the K intercept of each line. The idea that 'more uncertainty' will actually increase your profit may seem counterintuitive, but this can be explained by the fact that this equation is based on a risk free portfolio. While investing, there is the possibility that K may drop (because K acts stochastically) and thus may be more profitable than an opportunity with no uncertainty, where there is no chance that K may drop. Note that the effects of β only become apparant once F approaches the K-axis. When K is small, the differences in β are negligible.



(a) Value of $F(K)$ as a function of K



(b) Value of $F(K)$ as a function of K, zoomed in

Figure 1: Various values of β. Each value of β is given by a different colour in the legend. Other parameters are $V = 5, k = 1, r = 0.05$. Note that K^* is given by the K-intercept

Variation in the second parameter V, results in the line shifting horizontally, as seen in Figure 2. These shifts are roughly equidistant for every increment in V. Increasing V results in a translation to the right. This can be tied to the natural observation that given a larger value of a completed project, you would be willing to take on larger costs (so K^* increases).

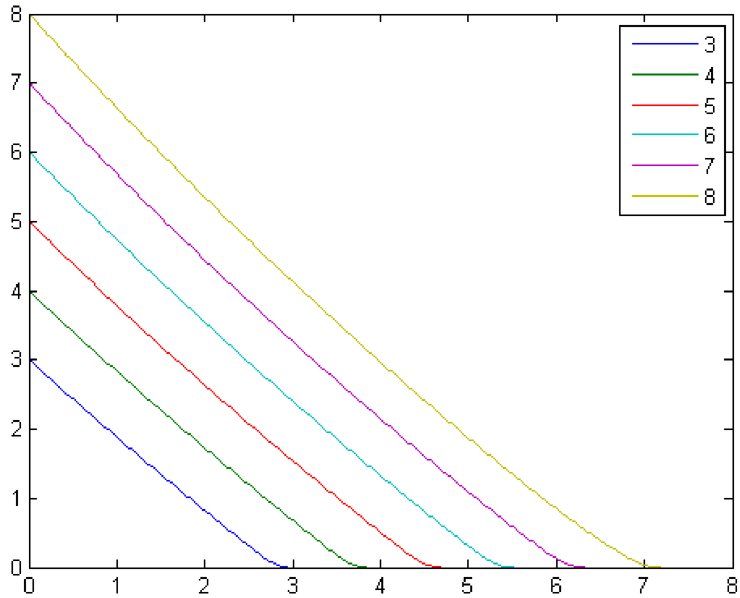
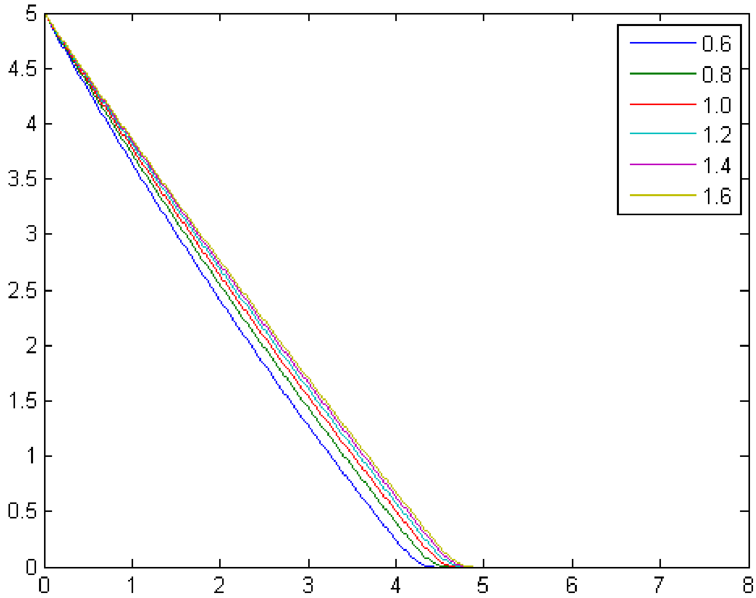
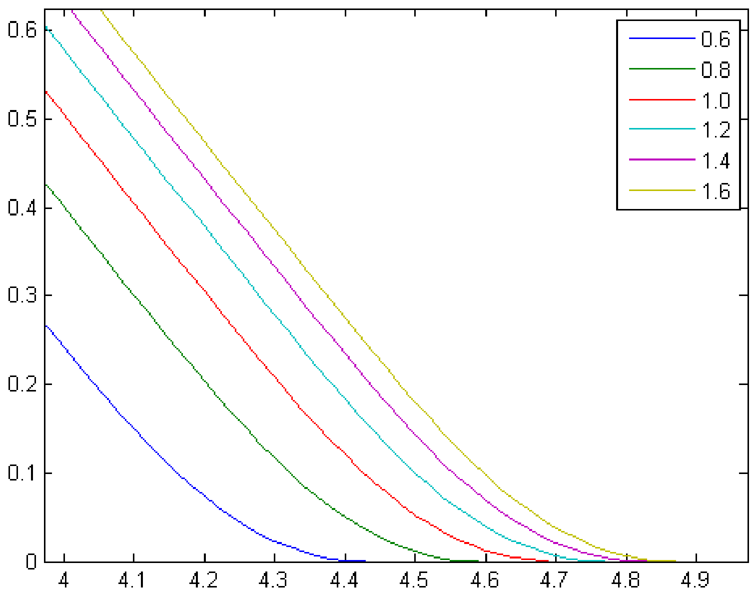


Figure 2: Value of $F(K)$ as a function of K for various values of V. Each separate V is given by a different colour in the legend. Other parameters are $\beta = 0.4, k = 1, r = 0.05$. Note that K^* is given by the K-intercept

The third parameter k shows slightly different behaviour than V, as seen in Figure 3. The reason for this is that V acts as a boundary condition whereas k is a part of the equation that needs to be solved. Choosing several values for k we see that the payoff and K^* will be larger, as k increases. However, notice that an increment for larger k's has a smaller effect compared to an increment for a small k. I.e. there is a larger spacing between $k = 0.6, 0.8$ than $k = 1.6, 1.4$



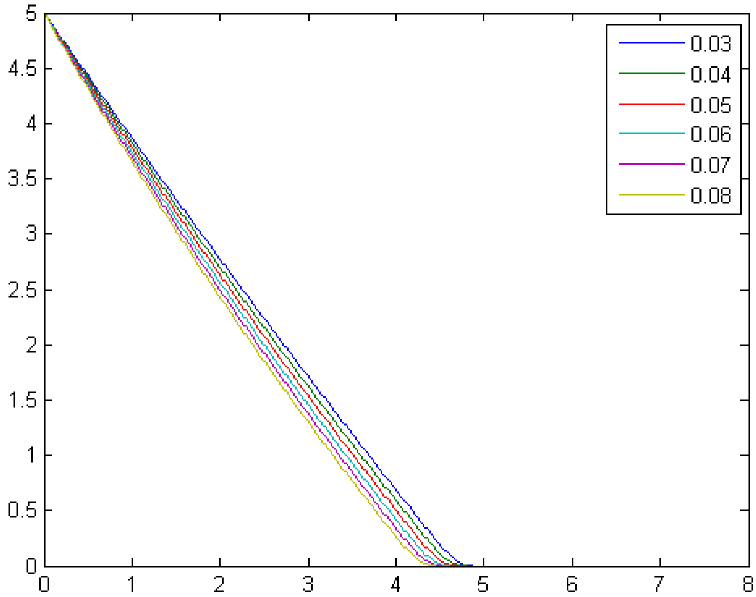
(a) Value of $F(K)$ as a function of K



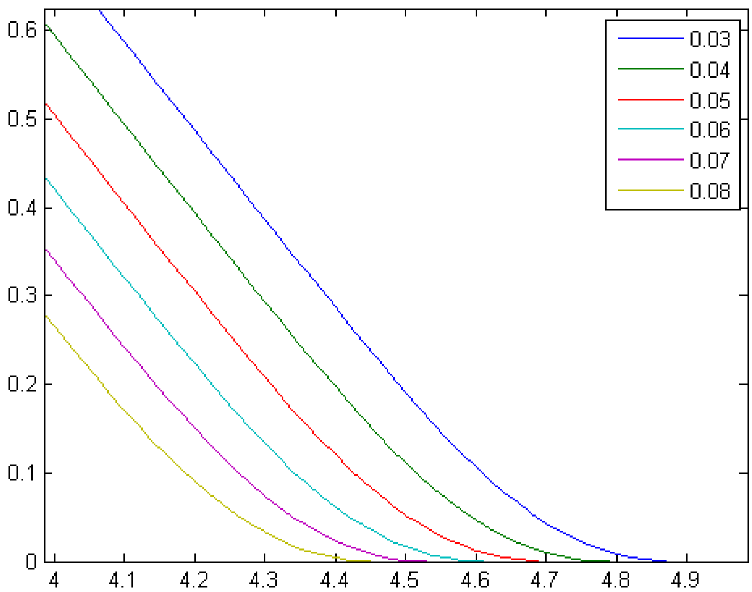
(b) Value of $F(K)$ as a function of K, zoomed in

Figure 3: Various values of k. Each value of k is given by a different colour in the legend. Other parameters are $\beta = 0.4, V = 5, r = 0.05$. Note that K^* is given by the K-intercept

The final parameter, r shows that a *lower* rate of interest increases the payoff and K^* (Figure 4). A higher rate of interest means that profits are worth less because it has to compete with getting a higher revenue from saving the money. It also confirms the classis economic concept that a lower interest rate will encourage investment (here confirmed by the payoff and K^*).



(a) Value of $F(K)$ as a function of K



(b) Value of $F(K)$ as a function of K, zoomed in

Figure 4: Various values of r. Each value of r is given by a different colour in the legend. Other parameters are $\beta = 0.4, V = 5, k = 1$. Note that K^* is given by the K-intercept

3.2 Input cost uncertainty

For input cost uncertainty we now consider $\beta = 0$ and are interested in the following:

$$rF = -IF_K + \frac{1}{2}\gamma^2 K^2 F_{KK} - \gamma\lambda K F_K - I$$

3.2.1 Numerical solution approach

There are two situations concerning K^*:

$$\text{If } K < K^*\text{:} \quad -rF - kF_K + \frac{1}{2}\gamma^2 K^2 F_{KK} - \gamma\lambda K F_K - k = 0$$
$$\text{If } K \geq K^*\text{:} \quad -rF + \frac{1}{2}\gamma^2 K^2 F_{KK} - \gamma\lambda K F_K = 0$$

For technical uncertainty we knew the lower bound function. The problem is now that the lower bound function is unknown. All we have is a differential equation. Trying to solve the equation analytically leads to the following. This function $L(K)$ which act as the lower bound of $F(K)$, must satisfy the second situation, which was:

$$-rL - \gamma\lambda K L_K + \frac{1}{2}\gamma^2 K^2 L_{KK} = 0$$

Letting $L(K) = pK^q$ where q is the solution of $-r - \gamma\lambda d + \frac{\gamma^2}{2} d(d-1)$, therefore:

$$\begin{aligned}
q &= \frac{\gamma\lambda + \frac{\gamma^2}{2} \pm \sqrt{\left(\gamma\lambda + \frac{\gamma^2}{2}\right)^2 + 4r\frac{\gamma^2}{2}}}{\gamma^2} \\
&= \frac{\lambda}{\gamma} + \frac{1}{2} \pm \frac{1}{\gamma}\sqrt{\frac{\left(\gamma\lambda + \frac{\gamma^2}{2}\right)^2 + 4r\frac{\gamma^2}{2}}{\gamma^2}} \\
&= \frac{\lambda}{\gamma} + \frac{1}{2} \pm \frac{1}{\gamma}\sqrt{\frac{\gamma^2\left(\lambda + \frac{\gamma}{2}\right)^2}{\gamma^2} + 2r} \\
&= \frac{\lambda}{\gamma} + \frac{1}{2} \pm \frac{1}{\gamma}\sqrt{\left(\lambda + \frac{\gamma}{2}\right)^2 + 2r}
\end{aligned}$$

If there is a positive solution for d, it can be ignored. This is because if that *were* the lower bound, then $F(K)$ would increase as K does, and condition (7) would be violated. The problem we now face is that c_1 cannot be solved. Boundary condition (6) does not apply (for surely $K^* > 0$) and (7) does not provide any extra information. Boundary condition (5) is of no use; finding K^* using our equation was the goal in the first place, not using K^* to solve the equation. Despite not explicitly knowing the obstacle function we can still continue in a similar fashion, although it does get more complicated. We are looking for a function $F(K) \geq 0$ such that:

$$\begin{aligned}
(-rF - kF_K + \frac{1}{2}\gamma^2 K^2 F_{KK} - \gamma\lambda K F_K - k)\cdot & \\
(-rF + \frac{1}{2}\gamma^2 K^2 F_{KK} - \gamma\lambda K F_K) &= 0 \\
-(-rF - kF_K + \frac{1}{2}\gamma^2 K^2 F_{KK} - \gamma\lambda K F_K - k) &\geq 0 \\
-(-rF + \frac{1}{2}\gamma^2 K^2 F_{KK} - \gamma\lambda K F_K) &\geq 0 \\
F(0) = V\ ,\ F(S) = 0 &
\end{aligned}$$

Again, the inequalities are mainly making an educated guess. Discretization of the equality, for $i = 1 \dots N$ looks as follows:

$$\begin{aligned}
(-rw_i - k\frac{w_i - w_{i-1}}{h} + \left(\frac{\gamma^2 (x_i)^2}{2}\right) \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} - \gamma\lambda x_i \frac{w_i - w_{i-1}}{h} - k) \cdot & \\
(-rw_i + \frac{\gamma^2 (x_i)^2}{2} \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} - \gamma\lambda x_i \frac{w_i - w_{i-1}}{h}) &= 0 \\
\left[w_{i-1}\left(\frac{k}{h} + \frac{\gamma^2 i^2}{2} + \gamma\lambda i\right) + w_i \left(-r - \frac{k}{h} - \gamma^2 i^2 - \gamma\lambda i\right) + w_{i+1}\left(\frac{\gamma^2 i^2}{2}\right) - k\right] \cdot & \\
\left[w_{i-1}\left(\frac{\gamma^2 i^2}{2} + \gamma\lambda i\right) + w_i \left(-r - \gamma^2 i^2 - \gamma\lambda i\right) + w_{i+1}\left(\frac{\gamma^2 i^2}{2}\right)\right] &= 0 \\
\left[w_{i-1}\left(-\frac{k}{h} - \frac{\gamma^2 i^2}{2} - \gamma\lambda i\right) + w_i \left(r + \frac{k}{h} + \gamma^2 i^2 + \gamma\lambda i\right) + w_{i+1}\left(-\frac{\gamma^2 i^2}{2}\right) + k\right] \cdot & \\
\left[w_{i-1}\left(-\frac{\gamma^2 i^2}{2} - \gamma\lambda i\right) + w_i \left(r + \gamma^2 i^2 + \gamma\lambda i\right) + w_{i+1}\left(-\frac{\gamma^2 i^2}{2}\right)\right] &= 0
\end{aligned}$$

The first inequality is discretized as follows:

$$\begin{aligned}
-(-rw_i - k\frac{w_i - w_{i-1}}{h} + \left(\frac{\gamma^2 (x_i)^2}{2}\right) \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} - \gamma\lambda x_i \frac{w_i - w_{i-1}}{h} - k) &\geq 0 \\
\left[w_{i-1}\left(-\frac{k}{h} - \frac{\gamma^2 i^2}{2} - \gamma\lambda i\right) + w_i \left(r + \frac{k}{h} + \gamma^2 i^2 + \gamma\lambda i\right) + w_{i+1}\left(-\frac{\gamma^2 i^2}{2}\right) + k\right] &\geq 0
\end{aligned}$$

The second inequality is discretized as follows:

$$\begin{aligned}
(-rw_i + \frac{\gamma^2 (x_i)^2}{2} \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} - \gamma\lambda x_i \frac{w_i - w_{i-1}}{h}) &\geq 0 \\
\left[w_{i-1}\left(-\frac{\gamma^2 i^2}{2} - \gamma\lambda i\right) + w_i \left(r + \gamma^2 i^2 + \gamma\lambda i\right) + w_{i+1}\left(-\frac{\gamma^2 i^2}{2}\right)\right] &\geq 0
\end{aligned}$$

Using the boundary conditions, this can all be summarized as follows:

$$\begin{aligned}
(Aw - b)^T (Cw - d) &= 0 \\
Aw - b &\geq 0 \\
Cw - d &\geq 0
\end{aligned}$$

where:

$$
\begin{aligned}
A_{ij} &= \begin{cases} -\frac{k}{h} - \frac{\gamma^2 i^2}{2} - \gamma\lambda i & \text{if } i-1=j \\ r + \frac{k}{h} + \gamma^2 i^2 + \gamma\lambda i & \text{if } i=j \\ -\frac{\gamma^2 i^2}{2} & \text{if } i+1=j \end{cases} \\
b &= \begin{bmatrix} -k - V(-\frac{k}{h} - \frac{\gamma^2}{2} - \gamma\lambda) \\ -k \\ \vdots \\ -k \end{bmatrix} \\
C_{ij} &= \begin{cases} -\frac{\gamma^2 i^2}{2} - \gamma\lambda i & \text{if } i-1=j \\ r + \gamma^2 i^2 + \gamma\lambda i & \text{if } i=j \\ -\frac{\gamma^2 i^2}{2} & \text{if } i+1=j \end{cases} \\
d &= \begin{bmatrix} -V(-\frac{\gamma^2}{2} - \gamma\lambda) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
w &= \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \end{bmatrix}
\end{aligned}
$$

If we apply the transformation:

$$
\begin{aligned}
x &= Cw - d \\
y &= Aw - b \\
G &= AC^{-1}
\end{aligned}
$$

we can now use the previous algorithm to solve:

$$
\begin{aligned}
x^T y &= 0 \\
Gx - y &= \hat{b} = -Gd + b \\
x &\geq 0 \\
y &\geq 0
\end{aligned}
$$

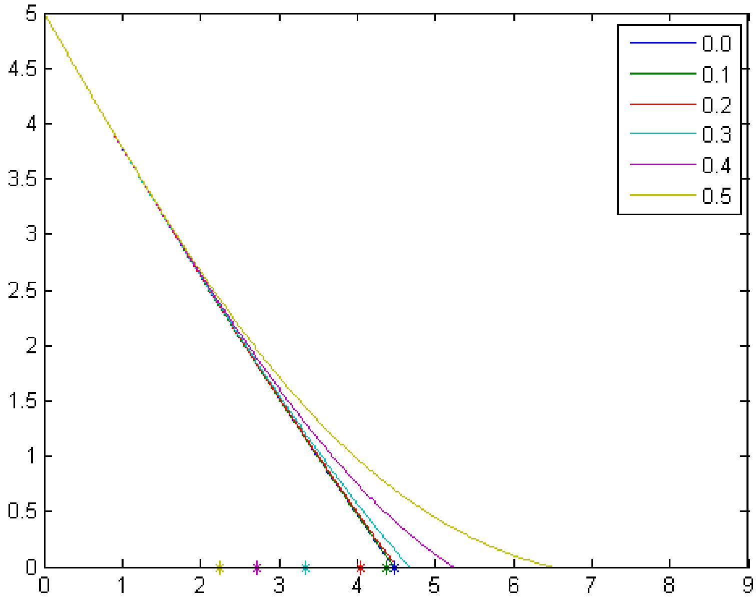
Once x is calculated, we convert it back to w with the reverse transformation (so C^{-1} will have to be computed). To find K^* an iteration along F can be taken to find the first point for which:

$$
\begin{aligned}
-F_k - 1 &= 0 \\
-\frac{w_i - w_{i-1}}{h} &= 1
\end{aligned}
$$

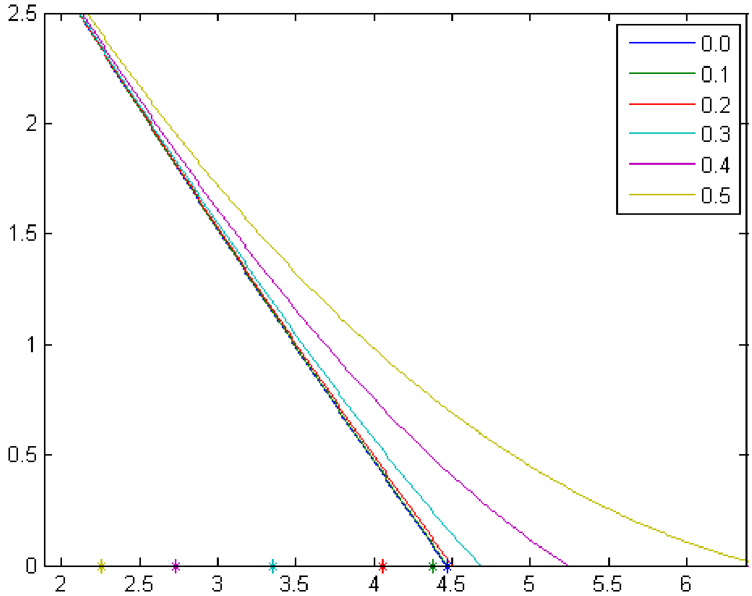
The critical costs for the respective F is marked with a * on the K-axis.

3.2.2 Variation in parameters: γ and λ

First the influence of γ on the shape of F will be demonstrated. As opposed to β, it is clear that a larger γ reduces the critical cost K^* in Figure 5. Unlike the nature of β, if the remaining costs are larger than K^*, that does not necessarily mean the payoff will be 0. The interval from K^* to the K for which $F(K)$ becomes 0 is where investment should be stopped, but the project not aborted. Qualitavely speaking, there is the possibility that the remaining cost will decrease by itself and so the project should not be too hastily aborted.



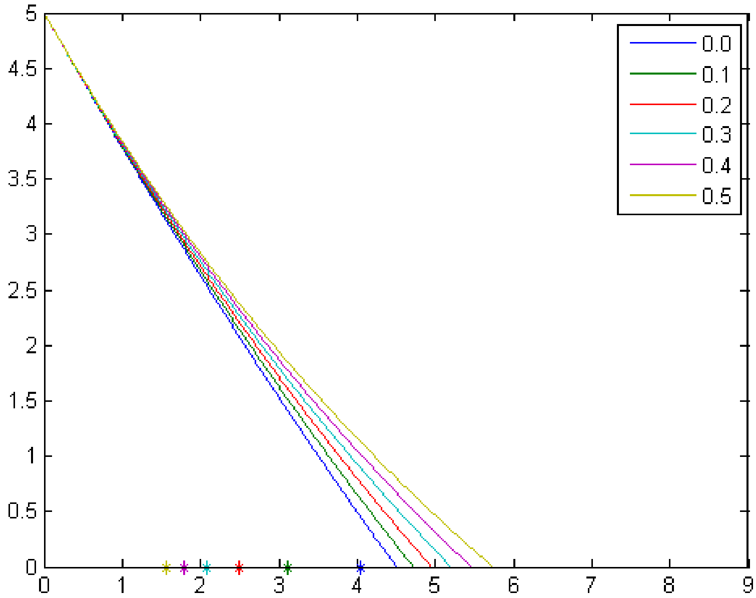
(a) Value of $F(K)$ as a function of K



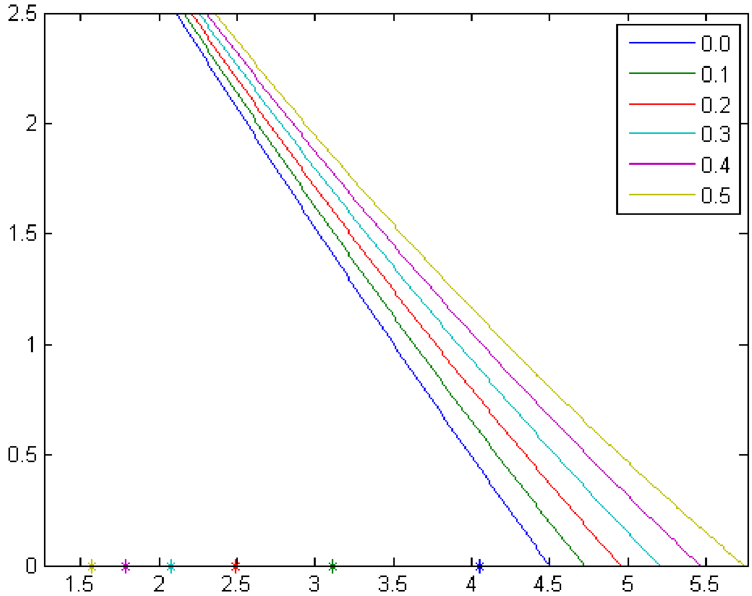
(b) Value of $F(K)$ as a function of K, zoomed in

Figure 5: Various values of γ. Each value of γ is given by a different colour in the legend. Other parameters are $\lambda = 0, V = 5, k = 1, r = 0.05$. The critical cost K^* is marked on the K-axis

Notice that increasing λ has a similar effect as γ in Figure 6. However, the spread of the F's is now much more even.



(a) Value of $F(K)$ as a function of K



(b) Value of $F(K)$ as a function of K, zoomed in

Figure 6: Various values of λ. Each value of λ is given by a different colour in the legend. Other parameters are $\gamma = 0.2, V = 5, k = 1, r = 0.05$. The critical cost K^* is marked on the K-axis

3.3 Combining the two uncertainties

Having discussed the two uncertainties separately, we now return to equation (4), which combines both types of uncertainty:

$$-rF - IF_K + \frac{1}{2}\beta^2 IKF_{KK} + \frac{1}{2}\gamma^2 K^2 F_{KK} - \gamma\lambda K F_K - I = 0$$

There are two situations concerning K^*:

$$\text{If } K < K^*: \ -rF - kF_K + \frac{1}{2}\beta^2 kKF_{KK} + \frac{1}{2}\gamma^2 K^2 F_{KK} - \gamma\lambda KF_K - k = 0$$
$$\text{If } K \geq K^*: \ -rF + \frac{1}{2}\gamma^2 K^2 F_{KK} - \gamma\lambda KF_K = 0$$

Again, we would like an explicit obstacle function. However, this is impossible to find, so we will use the second equation as the obstacle function. Hence we are looking for a $F \geq 0$ such that

$$\begin{aligned}
(-rF - kF_K + \frac{1}{2}\beta^2 kKF_{KK} + \frac{1}{2}\gamma^2 K^2 F_{KK} - \gamma\lambda KF_K - k)\cdot & & \\
(-rF + \frac{1}{2}\gamma^2 K^2 F_{KK} - \gamma\lambda KF_K) &=& 0 \\
-(-rF - kF_K + \frac{1}{2}\beta^2 kKF_{KK} + \frac{1}{2}\gamma^2 K^2 F_{KK} - \gamma\lambda KF_K - k) &\geq& 0 \\
-(-rF + \frac{1}{2}\gamma^2 K^2 F_{KK} - \gamma\lambda KF_K) &\geq& 0 \\
F(0) = V\ ,\ F(S) = 0 & &
\end{aligned}$$

Discretization of the equality, for $i = 1 \dots N$ looks as follows:

$$\begin{aligned}
&(-rw_i - k\frac{w_i - w_{i-1}}{h} + \left(\frac{\beta^2 kx_i + \gamma^2 (x_i)^2}{2}\right)\frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} - \gamma\lambda x_i \frac{w_i - w_{i-1}}{h} - k) \\
&\times(-rw_i + \frac{\gamma^2 (x_i)^2}{2}\frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} - \gamma\lambda x_i \frac{w_i - w_{i-1}}{h}) = 0 \\
&\Big[w_{i-1}\left(\frac{k}{h} + \frac{\beta^2 ki}{2h} + \frac{\gamma^2 i^2}{2} + \gamma\lambda i\right) \\
&+w_i\left(-r - \frac{k}{h} - \frac{\beta^2 ki}{h} - \gamma^2 i^2 - \gamma\lambda i\right) + w_{i+1}\left(\frac{\beta^2 ki}{2h} + \frac{\gamma^2 i^2}{2}\right) - k\Big] \\
&\cdot\left[w_{i-1}\left(\frac{\gamma^2 i^2}{2} + \gamma\lambda i\right) + w_i\left(-r - \gamma^2 i^2 - \gamma\lambda i\right) + w_{i+1}\left(\frac{\gamma^2 i^2}{2}\right)\right] = 0 \\
&\Big[w_{i-1}\left(-\frac{k}{h} - \frac{\beta^2 ki}{2h} - \frac{\gamma^2 i^2}{2} - \gamma\lambda i\right) + w_i\left(r + \frac{k}{h} + \frac{\beta^2 ki}{h} + \gamma^2 i^2 + \gamma\lambda i\right) \\
&+w_{i+1}\left(-\frac{\beta^2 ki}{2h} - \frac{\gamma^2 i^2}{2}\right) + k\Big] \\
&\times\left[w_{i-1}\left(-\frac{\gamma^2 i^2}{2} - \gamma\lambda i\right) + w_i\left(r + \gamma^2 i^2 + \gamma\lambda i\right) + w_{i+1}\left(-\frac{\gamma^2 i^2}{2}\right)\right] = 0
\end{aligned}$$

The first inequality is discretized as follows:

$$\begin{aligned}
-(-rw_i - k\frac{w_i - w_{i-1}}{h} + \left(\frac{\beta^2 kx_i + \gamma^2 (x_i)^2}{2}\right)\frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} & & \\
-\gamma\lambda x_i \frac{w_i - w_{i-1}}{h} - k) &\geq& 0 \\
\Big[w_{i-1}\left(-\frac{k}{h} - \frac{\beta^2 ki}{2h} - \frac{\gamma^2 i^2}{2} - \gamma\lambda i\right) + w_i\left(r + \frac{k}{h} + \frac{\beta^2 ki}{h} + \gamma^2 i^2 + \gamma\lambda i\right) & & \\
+w_{i+1}\left(-\frac{\beta^2 ki}{2h} - \frac{\gamma^2 i^2}{2}\right) + k\Big] &\geq& 0
\end{aligned}$$

The second inequality is discretized as follows:

$$\begin{aligned}
(-rw_i + \frac{\gamma^2 (x_i)^2}{2}\frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} - \gamma\lambda x_i \frac{w_i - w_{i-1}}{h}) &\geq 0 \\
\left[w_{i-1}\left(-\frac{\gamma^2 i^2}{2} - \gamma\lambda i\right) + w_i\left(r + \gamma^2 i^2 + \gamma\lambda i\right) + w_{i+1}\left(-\frac{\gamma^2 i^2}{2}\right)\right] &\geq 0
\end{aligned}$$

Using the boundary conditions, this can all be summarized as follows:

$$\begin{aligned}
(Aw - b)^T (Cw - d) = 0 \\
Aw - b \geq 0 \\
Cw - d \geq 0
\end{aligned}$$

where:

$$\begin{aligned}
A_{ij} &= \begin{cases} -\frac{k}{h} - \frac{\beta^2 ki}{2h} - \frac{\gamma^2 i^2}{2} - \gamma\lambda i & \text{if } i-1=j \\ r + \frac{k}{h} + \frac{\beta^2 ki}{h} + \gamma^2 i^2 + \gamma\lambda i & \text{if } i=j \\ -\frac{\beta^2 ki}{2h} - \frac{\gamma^2 i^2}{2} & \text{if } i+1=j \end{cases} \\
b &= \begin{bmatrix} -k - V(-\frac{k}{h} - \frac{\beta^2 k}{2h} - \frac{\gamma^2}{2} - \gamma\lambda) \\ -k \\ \vdots \\ -k \end{bmatrix} \\
C_{ij} &= \begin{cases} -\frac{\gamma^2 i^2}{2} - \gamma\lambda i & \text{if } i-1=j \\ r + \gamma^2 i^2 + \gamma\lambda i & \text{if } i=j \\ -\frac{\gamma^2 i^2}{2} & \text{if } i+1=j \end{cases} \\
d &= \begin{bmatrix} -V(-\frac{\gamma^2}{2} - \gamma\lambda) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
w &= \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \end{bmatrix}
\end{aligned}$$

Applying the transformation:

$$\begin{aligned}
x &= Cw - d \\
y &= Aw - b \\
G &= AC^{-1}
\end{aligned}$$

we can use the previous algorithm to solve:

$$\begin{aligned}
x^T y &= 0 \\
Gx - y &= \hat{b} = -Gd + b \\
x &\geq 0 \\
y &\geq 0
\end{aligned}$$

To find K^* we iterate along w and finding the first point for which:

$$\begin{aligned} -F_k + \frac{\beta^2}{2} K F_{KK} - 1 &= 0 \\ -\frac{w_i - w_{i-1}}{h} + \frac{\beta^2}{2} x_i \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} &= 1 \\ -\frac{w_i - w_{i-1}}{h} + \frac{\beta^2 i}{2} \frac{w_{i-1} - 2w_i + w_{i+1}}{h} &= 1 \end{aligned}$$

To demonstrate the interaction of β and γ a table is created showing the value of K^* for various β and γ: The table confirms that as γ rises, the critical cost K^* falls significantly.

		γ				
		0	0.1	0.2	0.3	0.4
	0	4.47	4.38	4.05	3.35	2.73
	0.1	4.47	4.38	4.04	3.34	2.73
β	0.2	4.48	4.39	4.01	3.32	2.71
	0.3	4.49	4.39	3.96	3.28	2.70
	0.4	4.51	4.39	3.89	3.24	2.68

Table 1: Value of K^* for various β and γ. Other parameters are $V = 5, k = 1, r = 0.05, \lambda = 0$

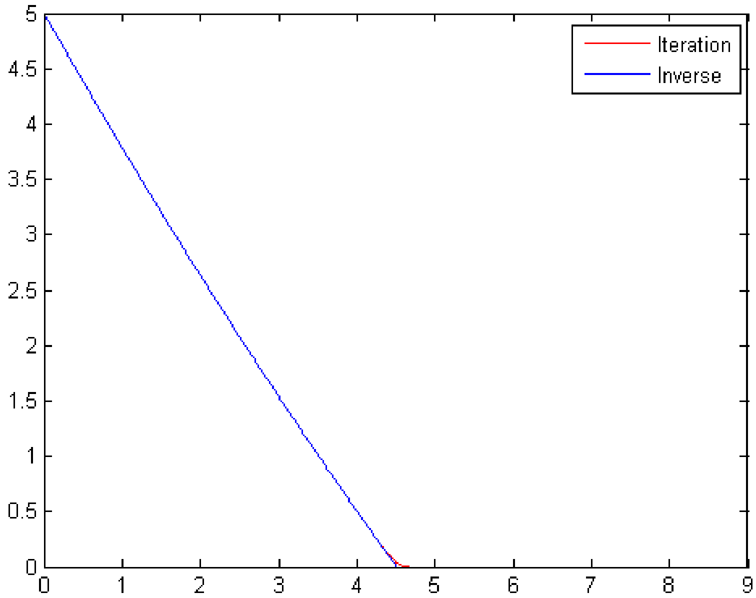
These changes are much smaller in comparison to β. In this particular case, an increase in β results in larger K^* (which was the case in section 3.1.2) in the first two columns. However, for $\gamma = 0.2, 0.3, 0.4$ it is the opposite - an increase in β reduces K^*. This unexpected behaviour is discussed in the next section.

3.4 Notes and Recommendations

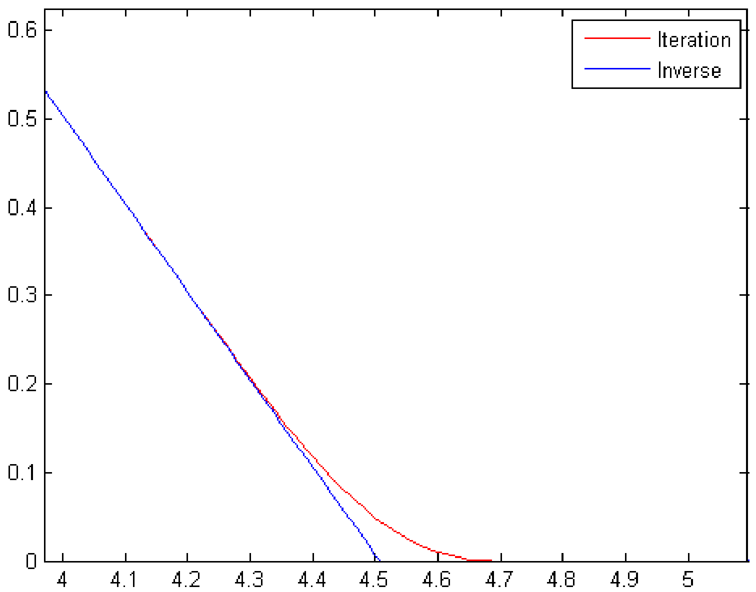
Seydel proves that the method used for technical uncertainty is correct and it is interesting to see how much of an improvement the algorithm ('Iteration' in Figure 7) is compared to the less elegant solution of solving $Aw = b$ ('Inverse' in Figure 7) and then taking 0 as the minimum value of each point of w. For the most part the values are the same for both methods. However, once F reaches the K axis (or more specifically, when K approaches K^*) the iteration method is much smoother and shows $F(K)$ is larger than the 'Inverse' method. This shows that while the 'Inverse' method is a reasonable approximation of F, it does not offer a truly correct solution.

For the methods involving input cost uncertainty, it is not proven that the method will work. It is an extension of the method described by Seydel, though not mentioned by him. It is expected that F would not go below 0. After all, we previously found a function pK^q with $q < 0$ that was the lower bound of F. However, F did go below 0 often when working with larger γ. As no better option has presented itself, we take the solution and apply $\max[w, 0]$.

In terms of computation, the algorithm was quite heavy. Using a vector w with more than 800 points started to slow MATLAB down, especially when also needing to do more than 200 iterations. Also, calculating the inverse of matrices A and C took more time. This also posed a challenge on boundary condition (7). If the furthest K was not significantly far away from K^* the line of F could become distorted. There is no conclusive



(a) Function $F(K)$ solved using the iteration method and inverse method



(b) Value of $F(K)$ as function a function of K, zoomed in

Figure 7: Solution of $F(K)$ using the iteration and inverse method. Other parameters are $\beta = 0.4, V = 5, k = 1, r = 0.05$

way to test if the range of K is far away enough, so it is a method of trial and error. The choice for h did not seem to cause the system to be unstable (the usual choice was $h = 0.01$). The above table does not fully agree with the findings. When β increases, then so does K^*. This is not the case for the third, fourth and fifth column. The nature of this numerical error is hard to tell. This could possibly remedied with a smaller h. Another option is to run the iterations longer. Both these solutions will require more computing power. Also, because calculating the inverse of matrices was inevitable I also used that in the initial guesses, which did cut the amount of iterations needed to be done. After around 100 iterations, the vector seemed to converge to its final solution. Seydel does explain a method to use y as a means of seeing if the vector has converged but this would also be at the cost of more computer processing power.

4 Application of the model

So far this model has been derived from an fairly academic point of view. The question is how applicable this model is to real life cases. In order to demonstrate this, we first look at the process of how cost estimates are made. The next subsection describes how to link β and γ to physical data, and concludes with an example of the model on terminals.

4.1 Cost estimates, uncertainty and risk

In the light of this paper, we will say uncertainty is the inability to precisely state the outcome of an event. Large uncertainty means the scope of possibilities that are not excluded is large. The risk of an outcome X is the expected gain or loss as a consequence of X, so:

$$\text{Risk} = P(X) \cdot \text{Consequence}(X)$$

where the consequence is an assigned value to the outcome X. This is one way risk can be quantified. Risk involves uncertainty but uncertainty does not necessarily involve risk. One example that clarifies this is the following.
Suppose I say that Andy Schleck will win the Tour de France in 2012. There is no way to be sure what I say is true. In this case, there is uncertainty, but no risk because whether the prediction turns out to be true or not, the outcome does not matter to me. Mathematically, all the possible outcomes are of consequence 0. However, suppose I decide to place a bet on my prediction. Now there would be risk, because there is a negative outcome (my prediction was wrong and I lose my bet) and a positive outcome (I win the bet).

Considering the above explanation of uncertainty and risk one can reach a cost estimate as follows. First one must identify all the elements required to complete the project. The next step is to make an estimation of the elements' price. After that, one can start considering all the possible events, and their respective probability and consequence. How one finds a cost estimate is not the focus of this study but it would be careless to not be aware of the possibilities. For the model we are dealing with, the cost estimate is most useful if it is given in a (normally) distributed probability function.

4.2 Determining β and γ

So far β and γ have been arbitrary values, without any physical meaning to them. Pindyck shows that relationship between the variance of $\tilde{K}$ and β is given by:

$$\text{Var}(\tilde{K}) = \left(\frac{\beta^2}{2-\beta^2}\right)$$

The explanation behind this is yet beyond my knowledge level. Further, γ can be interpreted as the standard deviation of percentage changes per period (for example, a year or month) in K. For this, an educated guess need to be made on what sort of interval is reasonable.

4.3 Container terminals - a worked example

Let us consider a fictional container terminal operating firm deciding on the following investment opportunity. Company CFC currently has several container terminals in Europe and North America. Realising the enormous potential of the growing economies in Asia, CFC would like to construct a terminal in Shanghai. A combination of engineers, business developers and managers give the following data:

- The completed terminal will have a value of €340 million
- Initially, the expected cost of completion will be €270 million. The construction period is expected to be 3 years so the maximum level of investment is set at €90 million per year.
- At any given point, engineers can give a probability density function of the remaining costs. The mean of the remaining cost will have a standard deviation of around 15% to 25%.
- Market researchers in Shangai believe that the project is of political and strategic importance to the city, so that the project is unlikely to be hindered by government regulation. Also, there is no evidence that suggests price fluctuations in cost are correlated to the stock market.
- Over the last few years where similar projects have taken place, standard deviations of percentage changes per year in K have been between 0.05 and 0.20
- The risk free rate of interest is 5%

CFC would like to know what their investment strategy should be, and if given the opportunity to modify an aspect of the project, which should it be? Normalising to €100 million, we then have the following parameters:

- $V = 3.4$
- $k = 0.9$
- β will be between 0.209 and 0.343
- ρ_{xm} is 0, hence $\lambda = \theta\rho_{xm}$ will be 0.
- γ will be between 0.05 and 0.20.
- $r = 0.05$

Solving $F(K)$ for these parameters gives the following values for K^*:

		γ					
		0.05	0.07	0.09	0.11	0.13	0.15
	0.20	3.11	3.10	3.08	3.06	3.04	3.01
	0.23	3.11	3.10	3.09	3.06	3.04	3.01
	0.26	3.12	3.10	3.09	3.07	3.04	3.01
β	0.29	3.12	3.11	3.09	3.07	3.04	3.01
	0.32	3.12	3.11	3.09	3.07	3.04	3.01
	0.35	3.12	3.11	3.09	3.07	3.04	3.01

Table 2: Values of K^* in €100 million for various β and γ. Other parameters are $V = 3.4, k = 0.9, \lambda = 0, r = 0.05$

Although it may seem a risky investment due to the cost being so close to the value of the completed project, CFC should start investing under all the ranges of β and γ because €270 million is less than all the K^* seen in Table 2. The investment strategy is to invest at the maximum rate unless the expected cost is greater than the given critical cost.

One of the engineer suggests that, given this information on K^*, the company should only use $K^* = 3.01$. If costs ever go above this, the project will be aborted. This strategy, the engineer argues, would be much simpler for CFC to use and assumes the 'worst case,' in which there is maximum uncertainty. In reality, this is a poor strategy. If there is actually less input cost uncertainty (I.E. $\gamma = 0.07$), the project would be falsely aborted if the remaining costs were €307 million. Also, one should also ask how confident the interval of β and γ are. Could it be that there is actually a higher level of uncertainty? A similar argument holds if someone suggests chosing a high as possible K^*.

Notice that the range of β does not have much effect on the critical cost. However, changes in γ influence the critical cost more. Varying with the parameters lead to some interesting observations:

- Increasing or decreasing the value V within 10% has a roughly equivalent effect on the critical cost. In practical terms, it is unlikely that V would change throughout the course of the project (and it is also an assumption of the model). However, Pindyck does offer an extension to the model where the completion cost is also subject to uncertainty.
- Only drastic increases in k have a noticable effect on the critical cost. Doubling k usually leads to K^* increasing about €10 to €20 million. This shows that a larger k is always better, even though the added benefit is marginal.
- The change of the risk free rate of interest is not in the hands of CFC, but it does slightly affect the critical cost (around €4 million decrease per increment of 0.01 in r.

4.4 Summary

Translated to business terms, CFC could receive the following recommendations:

1. Decreasing γ should be made a priority, as this will surely increase K^* significantly. The critical cost is dangerously close to the initial cost for $\gamma = 0.15$

2. If possible, raising the value of the completed project would increase the critical cost reasonably.

3. While increasing β is an option, the benefit is relatively small.

This section dealt with a fictional example of an investment opportunity (albeit simplified). In this particular case, the company should start investing. In practice, reaching the stage where there are accurate estimates on the distribution of K is a long step, and trying to modify β and γ requires great expertise to properly address. Still, dealing with both uncertainties separately has proven to offer valuable insight - it shows that uncertainty is not necessarily a bad thing, but does need to be analyzed correctly.

5 Conclusion

In this study we have considered an investment opportunity, whose nature was mainly specified by the uncertainty in its remaining cost of completion. An optimal investment strategy has been derived and consequently used to numerically calculate the final payoff function of the investment opportunity. Using this model, several factors of the opportunity have been explored, and what their effect is on the payoff and investment strategy. Further, the model has been demonstrated on a fictional business opportunity. One advantage is that the model can offer clear and concrete advice on the investment strategy for an investor. Although it is not the case in this paper's example, the model could also demonstrate why aborting a project is sometimes better than completing it.

In this paper, the numerical solutions for input cost uncertainty may not be flawless, but the obstacle problem methodology has shown to be a shining example of clever numerical methods. Investing more time in the numerical aspect would definitely be interesting from an academic perspective. However, it is my belief that the model becomes more valuable once certain assumption could be let go, such as letting V be uncertain (see [1]), or incorporate the effect of shocks in the market (see [4])or that there *is* a price that must be paid when not investing or quitting the project. A final extension could be to let β and γ be functions of K - that the uncertainty changes during the course of project.

References

[1] R. S. Pindyck *Investments of uncertain cost.* 1993, Journal of Financial Economics 34, pg 53-76,

[2] P. A. Forsyth *An Introduction to Computational Finance without Agonizing Pain.* 2011, University of Waterloo, Waterloo, Ontario, Canada

[3] R. U. Seydel *Tools for Computational Finance.* 2008, Springer

[4] C. van Buren *Analysis of costs in new terminal investments.* 2011, Delft University Press

[5] Roberts, Kevin and M. L. Weitzman *Funding criteria for research, development, and exploration projects.* 1981, Econometrica 49, pg 1261-1288

A MATLAB code - Technical uncertainty

This function returns a vector with F and K, taking N steps of size h. The amount of iterations is *itsteps*.

```
function y=f(B,V,k,r,h,N,itsteps)
clc
%Construct A
left=zeros(1,N-2);
right=zeros(1,N-2);
centre=zeros(1,N-1);

for i=1:N-2
    left(i)=-(B^2*k*(i+1))/(2*h)-k/h;
    right(i)=-(B^2*k*(i))/(2*h);
    centre(i)=(B^2*k*(i))/(h)+ k/h + r;
end
centre(N-1)=(B^2*k*(N-1))/(h)+ k/h + r;

A=zeros(N-1,N-1);
A=diag(left,-1)  + diag(right,+1) + diag(centre,0);

%Construct b
b=zeros(N-1,1)-k;
b(1)=-k+ V*((B^2*k)/(2*h)+k/h);

%Initial guess x

x=(A^(-1))*b;
x=max(0,x);

% Set R
R=zeros(N-1,2);

for t=2:itsteps
    for i=1:N-1
        sum1=0;
        sum2=0;
        for j=1:i-1
            sum1=sum1+ A(i,j)*x(j,t);
        end
        for s=i+1:N-1
            sum2=sum2+ A(i,s)*x(s,t-1);
        end

        R(i,t)=b(i)-sum1 -A(i,i)*x(i,t-1) -sum2;
        x(i,t)=max(0,x(i,t-1)+ 1* R(i,t)/A(i,i));
    end
end
```

```
this=x(:,itsteps);
F=[V;this];
K=(0:h:(N-1)*h)';

y=[K,F];
```

B MATLAB code - Input cost uncertainty

This function returns a vector with F and K, taking N steps of size h. The amount of iterations is *itsteps*.

```
function e=funct(y,lambda,V,k,r,h,N,itsteps)
clc
%Construct A
left=zeros(N-2,1);
right=zeros(N-2,1);
centre=zeros(N-1,1);

for i=1:N-2
    left(i)=-k/h-(y^2*(i+1)^2)/2-y*lambda*(i+1);
    right(i)=-(y^2*i^2)/2;
    centre(i)=r+k/h +(y^2*i^2)+y*lambda*i;
end
centre(N-1)=r+k/h +(y^2*(N-1)^2)+y*lambda*(N-1);

A=zeros(N-1,N-1);
A=diag(left,-1)  + diag(right,+1) + diag(centre,0);

%Construct b
b=zeros(N-1,1)-k;
b(1)=-k+V*(k/h+(y^2)/2 + y*lambda);

%Construct C
left=zeros(N-2,1)+k/h;
centre=zeros(N-1,1)-k/h;

C=zeros(N-1,N-1);
C=diag(left,-1) + diag(centre,0);
Cinv=C^(-1);

%Construct d
d=zeros(N-1,1);
d(1)=-V*k/h;

%Construct G
G=A*C^(-1);

%Construct bhat
bhat=-G*d+b;
```

```
%Initial guess x
x=C*(A^(-1)*b)-d;
x=max(0,x);

% Set R
R=zeros(N-1,2);

for t=2:itsteps
    for i=1:N-1
        sum1=0;
        sum2=0;
        for j=1:i-1
            sum1=sum1+ G(i,j)*x(j,t);
        end
        for s=i+1:N-1
            sum2=sum2+ G(i,s)*x(s,t-1);
        end

        R(i,t)=bhat(i)-sum1 -G(i,i)*x(i,t-1) -sum2;
        x(i,t)=max(0,x(i,t-1)+ 1* R(i,t)/G(i,i));
    end
end

w=Cinv*(x(:,itsteps)+d);

wafg(1,1)=w(1)-V;
for i=2:N-1
    wafg(i,1)=w(i)-w(i-1);
end
wafg=wafg/h;
position=find(wafg>=-1,1);
Kstar=h*position

F=[V;w];

F=max(0,F);

K=0:h:(N-1)*h;

e=[v1,this];
```

C MATLAB code - Both uncertainties

This function returns a vector with F and K and also K^* along with $F(K^*)$, taking N steps of size h. The amount of iterations is *itsteps*.

```
function e=functi(B,y,lambda,V,k,r,h,N,itsteps)
clc
```

```
%Construct A
left=zeros(N-2,1);
right=zeros(N-2,1);
centre=zeros(N-1,1);

for i=1:N-2
    left(i)=-k/h-(B^2*k*(i+1))/(2*h)-(y^2*(i+1)^2)/2-y*lambda*(i+1);
    right(i)=-(B^2*k*(i))/(2*h)-(y^2*i^2)/2;
    centre(i)=r+k/h + (B^2*k*(i))/(h)+(y^2*i^2)+y*lambda*i;
end
centre(N-1)=r+k/h + (B^2*k*(N-1))/(h)+(y^2*i^2)+y*lambda*(N-1);

A=zeros(N-1,N-1);
A=diag(left,-1)  + diag(right,+1) + diag(centre,0);

%Construct b
b=zeros(N-1,1)-k;
b(1)=-k+V*(k/h+(k*B^2)/(2*h)+(y^2)/2 + y*lambda);

%Construct C
left=zeros(N-2,1);
right=zeros(N-2,1);
centre=zeros(N-1,1);

for i=1:N-2
    left(i)=-(y^2*(i+1)^2)/2-y*lambda*(i+1);
    right(i)=-(y^2*i^2)/2;
    centre(i)=r+(y^2*i^2)+y*lambda*i;
end
centre(N-1)=r+(y^2*(N-1)^2)+y*lambda*(N-1);

C=zeros(N-1,N-1);
C=diag(left,-1)  + diag(right,+1) + diag(centre,0);

%Construct d
d=zeros(N-1,1);
d(1)=V*((y^2)/2+y*lambda);

%Construct G
G=A*(C^(-1));

%Construct bhat
bhat=-G*d+b;

%Initial guess x

x=(C*(A^(-1))*b)-d;
x=max(0,x);
```

```
% Set R
R=zeros(N-1,2);

for t=2:itsteps
    for i=1:N-1
        sum1=0;
        sum2=0;
        for j=1:i-1
            sum1=sum1+ G(i,j)*x(j,t);
        end
        for s=i+1:N-1
            sum2=sum2+ G(i,s)*x(s,t-1);
        end

        R(i,t)=bhat(i)-sum1 -G(i,i)*x(i,t-1) -sum2;
        x(i,t)=max(0,x(i,t-1)+ 1* R(i,t)/G(i,i));
    end
end

w=A^(-1)*b;

%Find Kstar
wafg=zeros(1,N-1);
wafg(1)=-(w(1)-V)/h + (B^2)/2*(V-2*w(1)+w(2))/(h)-1;
for i=2:N-2
   wafg(i)=-(w(i)-w(i-1))/h + (B^2)/2*i*(w(i-1)-2*w(i)+w(i+1))/(h)-1;
end
wafg(N-1)=-(w(N-1)-w(N-2))/h + (B^2)/2*(N-1)*(w(N-2)-2*w(N-1))/(h)-1;
Kpos=find(wafg<=0,1);
Kstar=Kpos*h;
KValue=max(0,w(Kpos));


F=[V;w];
F=max(0,F);
K=(0:h:(N-1)*h)';


e=[K,F];
e(1,3)=Kstar;
e(2,3)=KValue;
```