On the basic representation of the double affine Hecke algebra at critical level

Van Diejen, J. F.; Emsiz, E.; Zurrián, I. N.

DOI
10.1142/S0219498824500610

Publication date
2022

Document Version
Accepted author manuscript

Published in
Journal of Algebra and its Applications

Citation (APA)
Van Diejen, J. F., Emsiz, E., & Zurrián, I. N. (2022). On the basic representation of the double affine Hecke algebra at critical level. *Journal of Algebra and its Applications*, *23 (2024)*(3), Article 2450061. https://doi.org/10.1142/S0219498824500610

ON THE BASIC REPRESENTATION OF THE DOUBLE AFFINE HECKE ALGEBRA AT CRITICAL LEVEL

J.F. VAN DIEJEN, E. EMSIZ, AND I.N. ZURRIÁN

Abstract. We construct the basic representation of the double affine Hecke algebra at critical level $q = 1$ associated to an irreducible reduced affine root system R with a reduced gradient root system. For R of untwisted type such a representation was studied by Oblomkov [O04] and further detailed by Gehles [G06] in the presence of minuscule weights.

1. Introduction

Our aim is to construct a monomorphism of the double affine Hecke algebra at critical level $\mathcal{H}$ associated to any irreducible reduced affine root system R with a reduced gradient root system. To this end we adapt corresponding results of Cherednik [C05] and Macdonald [M03] (but see also [H06]) for general Macdonald parameter values q (not equal to a root of unity) to the critical level $q = 1$, with the aid of techniques from [O04, Sec. 3], [G06, Ch. 2.1] and [M03, Ch. 4.3].

For R of untwisted type the basic representation at critical level was considered by Oblomkov [O04, Secs. 3 and 5] in his study of the center and the ring-theoretical properties of $\mathcal{H}$; details of this construction were worked out in [G06, Sec. 2.1] for the case of a nontrivial Ω (where Ω denotes the subgroup of the extended affine Weyl group consisting of elements of length zero).

The underlying idea in the later case is that the generator of $\mathcal{H}$ associated to the affine simple reflection can be suppressed in the defining relations because it can be expressed as a conjugation of a generator associated to a finite simple reflections by an element in Ω. In this note we use a different technique, which is based on the reduction to the subalgebras associated with the parabolic subgroups of the affine Weyl group that fix a vertex of the Weyl alcove (see Remark 3.7 for further details).

One reason that double affine Hecke algebras at critical level associated to affine root systems and their representations are useful is that they can be used to compute affine Pieri rules for Hall-Littlewood polynomials and also govern the underlying symmetry structures of certain discretizations of quantum integrable systems with delta-potentials on the root hyperplanes of an affine root system [DE13, DEZ18] (see Remark 3.4 for more details).

The material is organized as follows. After recalling preliminaries concerning the definition of the double affine Hecke algebra at critical level in Section 2, the basic representation is constructed and its proof is provided in Section 3.

Date: September, 2022.

2010 *Mathematics Subject Classification.* Primary: 20C08; Secondary: 17B22, 17B67, 33D80.

Key words and phrases. affine Hecke algebras, affine Lie algebras, root systems, representation theory.

This work was supported in part by the *Fondo Nacional de Desarrollo Científico y Tecnológico (FONDECYT)* Grant # 1210015.

2. The double affine finite Hecke algebra at critical level

2.1. The affine root system. Let V be a real finite-dimensional Euclidean vector space with inner product $\langle \cdot, \cdot \rangle$. We identify the set $\hat{V}$ of affine-linear functions on V with $V + \mathbb{R}$ via $(v + r)(x) := \langle v, x \rangle + r$. We also need the gradient map from $\hat{V}$ to V, defined by $(v + r)' := v$. We fix an irreducible reduced affine root system $R \subset \hat{V}$ with a reduced gradient root system $R_0 := R' = \{a' \mid a \in R\} \subset V$ of full rank in V. We write Q, P, and W_0, for the root lattice, the weight lattice, and the Weyl group associated with R_0. We also fix a choice of positive roots R_0^+. The semigroup of the root lattice generated by R_0^+ is denoted by Q^+ whereas P^+ stands for the corresponding cone of dominant weights.

We fix a simple basis $\alpha_1, \ldots, \alpha_n$ for R_0^+ and extend it to a basis $a_0, \ldots, a_n$ of R, with $a_0 := \alpha_0 + c$, $a_j := \alpha_j$ $(j = 1, \ldots, n)$ and $c > 0$. Here $\alpha_0 = -\vartheta$ if R is *twisted* and $\alpha_0 = -\varphi$ if R is *untwisted*, where φ and ϑ denote the highest root and the highest short root of R_0^+, respectively. If R_0 is simply-laced, then we will call all roots long and short. (So if R_0 is simply-laced, then R is twisted and untwisted). The affine root system R consists of the set of elements $\alpha + m_\alpha rc$ $(\alpha \in R_0, r \in \mathbb{Z})$, where $m_\alpha = 1$ if α is short root and $m_\alpha = \frac{\langle \varphi, \varphi \rangle}{\langle \alpha_0, \alpha_0 \rangle}$ if α is long.

2.2. The affine Weyl group. An affine root $a = \alpha + m_\alpha rc \in R$ gives rise to an affine reflection $s_a : V \to V$ across the hyperplane $V_a := \{x \in V \mid a(x) = 0\}$ given by $s_a(x) := x - a(x)\alpha^\vee$. The *affine Weyl group* W^a is defined as the group generated by the affine reflections s_a, $a \in R$ and contains the finite Weyl group W_0 as the subgroup fixing the origin. For any $y \in V$ we denote by $t_y : V \to V$ the translation determined by the action $t_y(x) := x + y$. Let $\hat{R}_0 := \{\hat{\alpha} \mid \alpha \in R_0\}$ with $\hat{\alpha} := m_\alpha \alpha^\vee$ (so $\hat{R}_0 = \frac{2}{\langle \vartheta, \vartheta \rangle} R_0$ in the twisted case and $\hat{R}_0 = R_0^\vee$ in the untwisted case, i.e. $\hat{R}_0 = R_0^\vee$ for simply-laced R_0). Then the elements of W^a can be written as $vt_{c\lambda}$ with $v \in W_0$ and λ in $\hat{Q}$, i.e. $W = W_0 \ltimes t(c\hat{Q})$, where $\hat{Q}$ denotes the root lattice of $\hat{R}_0$. By extending the lattice from $\hat{Q}$ to the weight lattice $\hat{P}$ of $\hat{R}_0$ one arrives at the *extended affine Weyl group* $W = W_0 \ltimes t(c\hat{P})$ consisting of group elements of the form $vt_{c\lambda}$ with $v \in W_0$ and $\lambda \in \hat{P}$. The action of W on V induces a dual action on the space $\mathcal{C}(V)$ of functions $f : V \to \mathbb{C}$ given by $(wf)(x) := f(w^{-1}x)$ $(w \in W, x \in V)$. The affine root system $R \subset \mathcal{C}(V)$ is stable with respect to this dual action.

We denote by Ω the subgroup of elements of W that permutes the simple affine roots. So in particular $ua_j = a_{u_j}$ $(u \in \Omega)$, and thus $us_ju^{-1} = s_{u_j}$, where $j \to u_j$ encodes the corresponding permutation of the indices $j = 0, \ldots, n$. The upshot is that $W = \Omega \ltimes W^a$, with W being normal in W, whence Ω is a finite abelian subgroup: $\Omega \cong W/W^a \cong \hat{P}/\hat{Q}$. The extended affine Weyl group W can now be presented as the group generated by the commuting elements from Ω and the simple reflections $s_0, \ldots s_n$, subject to the relations

$$us_ju^{-1} = s_{u_j} \quad u \in \Omega,\ j = 0, \ldots, n, \tag{2.1a}$$

and

$$(s_js_k)^{m_{jk}} = 1 \quad j, k = 0, \ldots, n. \tag{2.1b}$$

Here $m_{jk} = 1$ if $j = k$ and $m_{jk} \in \{2, 3, 4, 6\}$ if $j \neq k$ (with the provision that for $n = 1$ the order $m_{10} = m_{01} = \infty$). It follows that any $w \in W$ can be decomposed (nonuniquely) as

$$w = us_{j_1}\cdots s_{j_\ell}, \tag{2.2}$$

with $j_1,\ldots,j_\ell \in \{0,\ldots,n\}$, and $u\in\Omega$. The length $\ell(w)$ is defined as the minimum number of reflections s_j ($j=0,1,\ldots,n$) involved in any decomposition (2.2) of w. Any decomposition (2.2) with $\ell=\ell(w)$ is called a reduced decomposition of w. The group Ω can thus be also characterized as the group of all elements W that have length zero.

2.3. **The double affine Hecke algebra at critical level.** Let τ_α ($\alpha\in R_0$) be a family of indeterminates such that $\tau_{w\alpha}=\tau_\alpha$ for all $\alpha\in R_0$. We denote by $\mathbb{L}$ the complex algebra of Laurent polynomials in the indeterminates τ_α ($\alpha\in R_0$). It will be also useful to extend the definition of indeterminates by setting $\tau_a := \tau_{a'}$ for $a\in R$ and $\tau_j := \tau_{\alpha_j}$ for $j=0,1,\ldots,n$.

The *double affine Hecke algebra at critical level* $\mathcal{H}$ associated to the affine root system R is the unital associative $\mathbb{L}$-algebra with invertible generators T_u ($u\in\Omega$), T_j ($j=0,\ldots,n$) and the commuting elements X^λ ($\lambda\in P$), such that the following relations are satisfied

$$T_uT_{\tilde u} = T_{u\tilde u} \quad\text{and}\quad T_uT_j = T_{u_j}T_u \qquad (u,\tilde u\in\Omega,\ 0\le j\le n), \tag{2.3a}$$

$$(T_j-\tau_j)(T_j+\tau_j^{-1}) = 0 \qquad (0\le j\le n), \tag{2.3b}$$

$$\underbrace{T_jT_kT_j\cdots}_{m_{jk}\ \text{factors}} = \underbrace{T_kT_jT_k\cdots}_{m_{jk}\ \text{factors}} \qquad (0\le j\ne k\le n), \tag{2.3c}$$

$$T_uX^\lambda = X^{u'\lambda}T_u \qquad (u\in\Omega,\ \lambda\in P), \tag{2.3d}$$

$$T_jX^\lambda = X^{s_j'\lambda}T_j + (\tau_j-\tau_j^{-1})\frac{X^\lambda - X^{s_j'\lambda}}{1-X^{-\alpha_j}} \qquad (0\le j\le n,\ \lambda\in P), \tag{2.3e}$$

$$X^\lambda X^{\tilde\lambda} = X^{\lambda+\tilde\lambda} \qquad (\lambda,\tilde\lambda\in P), \tag{2.3f}$$

where the number of factors m_{jk} on both sides of the braid relation (2.3c) is the same as the order of the corresponding braid relation (2.1b) for W. Here, for any element in W of the form $vt_{c\lambda}$ ($v\in W_0$, $\lambda\in\hat P$) we denote $(vt_{c\lambda})' := v$. Hence $s_j' = s_j$ for $j=1,\ldots,n$ and s_0' is the orthogonal reflection associated with α_0.

Remark 2.1. Although the results in the note are formulated in terms of the algebra $\mathbb{L}$, all statements (and proofs) are also valid when the indeterminants τ_α are specialized to non-zero complex values.

We now turn to the construction of the basic representation.

3. The basic representation

We denote the group algebra of P by $\mathbb{L}[P]$ and the corresponding natural basis elements (also) by X^λ, $\lambda\in P$. The finite Weyl group W_0 acts on $\mathbb{L}[P]$ via $wX^\lambda = X^{w\lambda}$. We will also need the localization $\mathcal{A}$ of $\mathbb{L}[P]$ by the Weyl denominator $\delta := \delta(R_0) = \prod_{\alpha\in R_0}(1-X^\alpha)$. Then the action of W_0 on $\mathbb{L}[P]$ extends to an action on $\mathcal{A}$ via $\mathbb{L}$-isomorphisms. We write f^w for the result of the action of $w\in W_0$ on $f\in\mathcal{A}$.

Definition 3.1. *The* smash product $\mathcal{A}\#W$ *is the associative unital* $\mathbb{L}$*-algebra characterized by the following properties:*

i) $\mathcal{A}\#W$ *contains* $\mathcal{A}$ *and the group algebra* $\mathbb{L}[W]$ *as subalgebras,*

ii) *the multiplication map defines an isomorphism* $\mathcal{A} \otimes_{\mathbb{L}} \mathbb{L}[W] \to \mathcal{A} \# W$ *of* $\mathbb{L}$*-modules, and*

iii) *we have the cross relations* $(fv)(gw) = fg^{v'}vw$ *for all* $f, g \in \mathcal{A}$ *and for all* $v, w \in W$.

We will also need the following Demazure-Lusztig-type elements in $\mathcal{A} \# W$:

$$\check{T}_j := \tau_j s_j + \frac{\tau_j - \tau_j^{-1}}{1 - X^{-\alpha_j}}(1 - s_j) \qquad (j = 0, \ldots, n). \tag{3.1}$$

Theorem 3.2 (The Basic Representation of $\mathcal{H}$)**.** *The assignments* $T_j \mapsto \check{T}_j$ $(j \in \{0, \ldots, n\})$, $T_u \mapsto u$ $(u \in \Omega)$ *and* $X^\lambda \mapsto X^\lambda$ $(\lambda \in P)$ *uniquely extend to an injective homomorphism* $\jmath : \mathcal{H} \to \mathcal{A} \# W$ *of* $\mathbb{L}$*-algebras.*

Remark 3.3. The above monomorphism can be lifted to an isomorphism after a suitable localization. For this consider the extension $\mathcal{H}_{\delta_\tau}$ of $\mathcal{H}$ obtained by adjoining the inverses of $1 - X^\alpha$ and $\tau_\alpha - \tau_\alpha^{-1} X^\alpha$ $(\alpha \in R_0)$. In other words, $\mathcal{H}_{\delta_\tau}$ is the Ore extension obtained by localizing $\mathcal{H}$ at the τ-deformed Weyl denominator $\delta_\tau = \prod_{\alpha \in R_0}(1 - X^\alpha)(\tau_\alpha^{-1} - \tau_\alpha X^\alpha)$. Then the assignments in Theorem 3.2 uniquely extend to an $\mathbb{L}$-isomorphism $\jmath : \mathcal{H}_{\delta_\tau} \xrightarrow{\sim} \mathbb{L}[P]_{\delta_\tau} \# W$.

Remark 3.4 (The Polynonomial Representation of $\mathcal{H}$). For $c \in \frac{\langle \alpha_0, \alpha_0 \rangle}{2}\mathbb{N}$ the affine Weyl group W acts on $\mathbb{L}[P]$ via $wX^\lambda = X^{w(\lambda)}$; this gives rise to a corresponding action of the Demazure-Lusztig operators (3.1) on $\mathbb{L}[P]$. By letting the monomials $X^\lambda \in \mathcal{H}$ $(\lambda \in P)$ act as multiplication operators the basic representation then reduces at such values for c to a (unfaithful) representation of $\mathcal{H}$ on $\mathbb{L}[P]$. In [DE13, Sect. 4] this polynomial representation was, for R of twisted type and the indeterminants τ_α specialized to non-zero complex values, dualized so as to arrive at a representation of $\mathcal{H}$ given by translation operators and integral-reflection operators used to build quantum integrable systems in a Hilbert space of complex functions on the weight lattice P (cf. also [DEZ18, Sect. 4 and 5] for a corresponding construction associated with the double affine Hecke algebra of type $C_n C_n^\vee$ at critical level). These quantum integrable systems can be interpreted as lattice regularizations of quantum models with delta potentials related to the Cherednik algebra at critical level [EOS09].

The rest of this section is devoted to the proof of Theorem 3.2. We start with the construction of the homomorphism. For this we have to show that the relations (2.3) holds when T_j is replaced by $\check{T}_j$ $(j = 0, \ldots, n)$, T_u by u $(u \in \Omega)$ and $X^\lambda \in \mathcal{H}$ by $X^\lambda \in \mathbb{L}[P]$ $(\lambda \in P)$. For any affine root a with gradient α let $\check{T}(a) = \tau_a s_a + \frac{\tau_a - \tau_a^{-1}}{1 - X^{-\alpha}}(1 - s_a)$, so in particular $\check{T}_j = \check{T}(a_j)$. Relations (2.3d) and (2.3f) are trivial while relation (2.3a) is a consequence of $w\check{T}(a)w^{-1} = \check{T}(wa)$ $(w \in W)$ and $ua_j = a_{u_j}$ $(j = 0, \ldots, n)$. Relation (2.3e) follows from a direct calculation.

We shall now concentrate on the quadratic relations (2.3b) and the braid relations (2.3c). For this we consider the affine Hecke algebra H associated to R, i.e. the unital associative $\mathbb{L}$-algebra with invertible generators $T_0, T_1, \ldots, T_n$ subject to the relations Eqs. (2.3b) and (2.3c). The relations (2.3b) and (2.3c) are an immediate consequence of the following lemma.

Lemma 3.5. *The assignments* $T_j \mapsto \check{T}_j$ $(j \in \{0, \ldots, n\})$ *uniquely extend to a homomorphism* $\jmath : H \to \mathcal{A} \# W$ *of* $\mathbb{L}$*-algebras.*

The idea of the proof of the above lemma is to reduce it to the subalgebras associated with the parabolic subgroups of the affine Weyl group that fix a vertex of the Weyl alcove. These subalgebras are of finite type and this leads us to recall the definition of the finite Hecke algebra $H(R_0)$ associated to R_0 (in the rest of this paragraph we will also allow R_0 not to be irreducible): it is the unital associative $\mathbb{L}$-algebra with invertible generators $T_1, T_2, \ldots, T_n$ subject to the relations Eqs. (2.3b) and (2.3c) that do not involve T_0. We will also need the smash product $\mathcal{A}_Q \# W_0$, where $\mathcal{A}_Q = \mathbb{L}[Q]_{\delta(R_0)}$, and which is defined analogously as $\mathcal{A} \# W$ but with the cross relations iii) in Definition 3.1 replaced by $(fv)(gw) = fg^v vw$ for $f, g \in \mathcal{A}_Q$ and $v, w \in W_0$. The elements $\check{T}_j$ $(j \in \{1, \ldots, n\})$ can be naturally interpreted as elements of $\mathcal{A}_Q \# W_0$. The proof of Lemma 3.5 makes use of the following lemma:

Lemma 3.6. *The assignments $T_j \mapsto \check{T}_j$ $(j = 1, \ldots, n)$ uniquely extend to a homomorphism $H(R_0) \to \mathcal{A}_Q \# W_0$ of $\mathbb{L}$-algebras. Moreover, this holds for any (not necessarily irreducible) reduced crystallographic root system R_0 of full rank in V.*

This lemma is an immediate consequence of the Bernstein-Lusztig relations [L89] in the affine Hecke algebra, see e.g. [M03, (4.3.3)]. Although in loc. cit. it was assumed that R_0 was irreducible, it actually holds for all finite crystallographic root systems R_0 of full rank: for this observe that if R_0 is decomposed into disjoint, irreducible and orthogonal subsystems $\Sigma_1 \cup \cdots \cup \Sigma_\ell$, then $H(R_0)$ decomposes as the tensor product of the $H(\Sigma_k)$, $k = 1, \ldots, \ell$ and similarly for $\mathcal{A}_Q \# W_0$.

Let us now return to the proof of Lemma 3.5. For any $k \in \{0, 1, \ldots, n\}$ consider the parabolic $\mathbb{L}_k$-subalgebra H_k of H generated by T_j $(j \neq k)$, where $\mathbb{L}_k$ denotes the complex subalgebra of $\mathbb{L}$ generated by $\tau_j^{\pm 1}$ $(j \neq k)$. It suffices to show that, for any (fixed) k, the assignment $T_j \mapsto \check{T}_j$, $j \neq k$ extends to an homomorphism $H_k \to \mathcal{A} \# W$ of $\mathbb{L}_k$-algebras. For this we introduce the parabolic subgroup W_k of W generated by s_j $(j \neq k)$ and set $R_k = \{w' \alpha_j \mid w \in W_k, j \neq k\}$. Then $R_k \subset R_0$ is a finite root system of rank n in V (although in general not irreducible) with a basis of simple roots given by α_j, $j \neq k$. The map $s_j \mapsto s_j'$, $j \neq k$ defines a Weyl group isomorphism $W_k \to W_0(R_k)$ and this isomorphism induces a natural homomorphism $\imath_k : H_k \to H(R_k)$ of $\mathbb{L}_k$-algebras defined by $T_j \mapsto T_j'$ where T_j', $j \neq k$ denote the natural generators of $H(R_k)$. Let $Q_k \subset Q$ be the root lattice of R_k in V. If we denote the $\mathbb{L}_k$-subalgebra of $\mathcal{A} \# W$ generated by $\mathcal{A}_{Q_k} := \mathbb{L}_k[Q_k]_{\delta(R_k)}$ and W_k by $(\mathcal{A} \# W)_k$, then $fw \mapsto fw'$ $(f \in A_{Q_k}, w \in W_k)$ extends to an $\mathbb{L}_k$-isomorphism $\ell_k : (\mathcal{A} \# W)_k \to A_{Q_k} \# W_0(R_k)$. By applying Lemma 3.6 (with R_0 replaced by R_k) we obtain an algebra isomorphism $\jmath_k : H(R_k) \to \mathcal{A}_{Q_k} \# W_0(R_k)$. The proof of Lemma 3.5 now follows by observing that the homomorphism $\ell_k^{-1} \circ \jmath_k \circ \imath_k$ of $\mathbb{L}_k$-algebras maps T_j to the Demazure-Lusztig-type elements $\check{T}_j$ (3.1) for $j \neq k$. So in particular $\jmath_k \circ \imath_k = \ell_k \circ \jmath_{|H_k}$. The situation can be visualized in the following commutative diagram:

$$\begin{array}{ccc} H_k & \overset{\jmath_{|H_k}}{\dashrightarrow} & (\mathcal{A} \# W)_k \\ \big\downarrow{\scriptstyle \imath_k} & & \big\downarrow{\scriptstyle \ell_k} \\ H(R_k) & \xrightarrow{\jmath_k} & \mathcal{A}_{Q_k} \# W_0(R_k) \end{array} \tag{3.2}$$

This proves Lemma 3.5 and therefore also the existence of the homomorphism $\jmath$.

We now turn to the injectivity of $\jmath$. For a $w \in W$ with a reduced expression $w = u\, s_{j_1} \cdots s_{j_\ell}$, let $T_w := T_u\, T_{j_1} \ldots T_{j_\ell}$. From the defining relations of $\mathcal{H}$ it follows that the elements $X^\mu T_w$ (or alternatively $T_w X^\mu$) span $\mathcal{H}$ over $\mathbb{L}$. To show injectivey it therefore suffices to show that the elements $X^\mu \check{T}_w$ (or alternatively $\check{T}_w X^\mu$), where $\check{T}_w := \jmath(T_w)$ with $\mu \in P$ and $w \in W$, are linearly independent in $\mathcal{A} \# W$ over $\mathbb{L}$. This can be proven by mimicking Macdonald's proof in [M03, (4.3.11)] (but see also [DEZ18, Prop. A.4]) of a corresponding statement for $q \neq 1$. It is essentially based on the fact that for any $w \in W$ one has $\check{T}_w = \sum_{v \leq w} f_{vw}(X) v$, where the coefficients $f_{vw}(X)$ belong to $\mathcal{A}$ and that the leading coefficient $f_{ww}(X)$ is non-zero (and where $\leq$ refers to the Bruhat partial order on W, i.e. $v \leq w$ iff a reduced expression for v can be obtained from a reduced expression for w by deleting simple reflections [B68, H90]). Furthermore, this triangularity property can be inverted to yield the decomposition $w = \sum_{v \leq w} \tilde{f}_{vw}(X) \check{T}_v \in \mathbb{L}[P]_{\delta_\tau} \# W$, where the coefficients $\tilde{f}_{vw}(X) \in \mathbb{L}[P]_{\delta_\tau}$ and with non-zero leading coefficient $\tilde{f}_{ww}(X) = f_{ww}(X)^{-1}$. This implies that the homomorphism from Remark 3.3 is an isomorphism.

Remark 3.7. If the finite group Ω is non-trivial, then there is an alternative proof of the existence of the homomormphism $\jmath$. It is based on the fact that the generator T_0 and the relations (2.3a)-(2.3e) involving T_0 are *redundant* in this case because there exist an $u \in \Omega$ and $j \in \{1, 2, \ldots, n\}$ such that $s_0 = u s_j u^{-1}$, and therefore also $T_0 = T_u T_j T_u^{-1}$ and $\check{T}_0 = u \check{T}_j u^{-1}$. Since the quadratic relations (2.3b) and braid relations (2.3c) not involving T_0 follow from Lemma 3.6, the existence of the homomorphism $\jmath$ follows in a straightforward way. This argument was employed by Oblomkov [O04] and Gehles [G06] to arrive at Theorem 3.2 in the case that R is of untwisted type. In [DEZ18] it was shown for the five-parameter double affine Hecke algebra at critical level of type $C_n C_n^\vee$ that the proof in question can be adapted to the case of a trivial Ω via a conjugation/translation trick borrowed from [EOS09] (where it was used to construct integral-reflection representations of the trigonometric Cherednik algebra at critical level). The idea of the present proof should be viewed as a uniform technique that works for all affine root systems R, irrespective of whether Ω is trivial or not.

As a consequence of (the proof of) Theorem 3.2 we obtain as an immediate corollary the following PBW property of Cherednik for $\mathcal{H}$:

Corollary 3.8. *The elements $X^\lambda T_w$ (or alternatively $T_w X^\lambda$), with $\lambda \in P$ and $w \in W$, form an $\mathbb{L}$-basis for $\mathcal{H}$.*

Together with the Bernstein-Zelevinsky relations in the affine Hecke algebra [L89, Prop. 3.7] this corollary also yields the PBW property $\mathcal{H} \simeq \mathbb{L}[\hat{P}] \otimes_{\mathbb{L}} H(R_0) \otimes_{\mathbb{L}} \mathbb{L}[P]$ (isomorphism as $\mathbb{L}$-modules), cf. [C05, Thm. 3.2.1 (p. 310)].

Acknowledgments. This work was supported in part by the *Fondo Nacional de Desarrollo Científico y Tecnológico* (FONDECYT) Grant # 1210015 and VI PPIT-US. We thank the referee for the helpful constructive remarks and suggesting some improvements concerning the presentation.

References

B68. N. Bourbaki, *Groupes et Algèbres de Lie, Chapitres 4–6*, Hermann, Paris, 1968.

C05. I. Cherednik, *Double Affine Hecke Algebras*, London Mathematical Society Lecture Note Series **319**, Cambridge University Press, Cambridge, 2005.

DE13. J.F. van Diejen and E. Emsiz, Discrete harmonic analysis on a Weyl alcove, J. Funct. Anal. **265** (2013), 1981–2038.

DEZ18. J.F. van Diejen, E. Emsiz and I.N. Zurrián, Completeness of the Bethe Ansatz for an open q-boson system with integrable boundary interactions, Ann. Henri Poincaré **19**, 1349-1384 (2018)

EOS09. E. Emsiz, E.M. Opdam, and J.V. Stokman, Trigonometric Cherednik algebra at critical level and quantum many-body problems, Selecta Math. (N.S.) **14** (2009), 571–605.

G06. K.E. Gehles, *Properties of Cherednik Algebras and Graded Hecke Algebras*, PhD Thesis, University of Glasgow, 2006.

H06. M. Haiman, Cherednik algebras, Macdonald polynomials and combinatorics. *International Congress of Mathematicians. Vol. III*, 843–872, Eur. Math. Soc., Zürich, 2006

H90. J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, Cambridge, 1990.

L89. G. Lusztig, Affine Hecke algebras and their graded version. J. Amer. Math. Soc. **2** (1989), 599–635.

M03. I.G. Macdonald, *Affine Hecke Algebras and Orthogonal Polynomials*, Cambridge University Press, Cambridge, 2003.

O04. A. Oblomkov, Double affine Hecke algebras and Calogero-Moser spaces, Represent. Theory **8** (2004), 243–266.

Instituto de Matemáticas, Universidad de Talca, Casilla 747, Talca, Chile
Email address: diejen@inst-mat.utalca.cl

Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands
Email address: e.emsiz@tudelft.nl

Departamento de Matemática Aplicada II, Universidad de Sevilla. EPS c/ Virgen de Africa 7, 41011 Sevilla, Spain
Email address: ignacio.zurrian@fulbrightmail.org