Stochastic Evolution Equations with Adapted Drift

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1 Introduction

This thesis treats some aspects of the theory of non-autonomous stochastic evolution equations with a random drift

$$\begin{cases} du(t) = (A(t)u(t) + F(t,u(t)))\,dt + B(t,u(t))\,dW(t), & t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{1.0.1}$$

A large class of stochastic partial differential equations, which are models of problems in for instance mathematical physics, biology or finance, can be formulated as a stochastic evolution equation (1.0.1).

The main goal of this thesis is to extend the theory of stochastic evolution equations to the setting where the generator is time-dependent *and random*, i.e., $A = A(t,\omega)$, and adapted to a filtration $(\mathscr{F}_t)_{t\in[0,T]}$. Throughout the thesis we will assume that each operator

$$A(t,\omega) : E \supset D(A(t,\omega)) \to E$$

is a closed (sometimes unbounded) densely defined linear operator on a Banach space E.

A formula that is commonly used in the theory of stochastic evolution equations is the variation of constants formula

$$u(t) = S(t,0)u_0 + \int_0^t S(t,s)F(s,u(s))\,ds + \int_0^t S(t,s)B(s,u(s))\,dW(s). \tag{1.0.2}$$

Here, $(S(t,s))_{0\le s\le t\le T}$ is the random evolution system that is uniquely determined by the requirements $\frac{d}{dt}S(t,s) = A(t)S(t,s)$ and $S(t,t) = I$, where I is the identity operator.

An extension to the setting where the drift A may be random is not a trivial one. Indeed, already in the case $E = \mathbb{R}$ difficulties arise. Consider for instance the situation where $A = a$ with $a : [0,T]\times\Omega \to \mathbb{R}$ progressively measurable and $F = 0$, $B = 1$. The random evolution system is then given by

$$S(t,s) = \exp\Big(\int_s^t a(\sigma)\,d\sigma\Big).$$

From this formula it is clear that $S(t,s)$ is $\mathscr{F}_t$-measurable, but certainly not $\mathscr{F}_s$-measurable. Therefore, the stochastic integral appearing in the variation of constants formula

$$u(t) = S(t,0)u(0) + \int_0^t S(t,s)\, dW(s),$$

does not exist as an Itô integral.

Observe that in this situation, and more generally in the situation where $A(t,\omega)$ is a bounded linear operator, a solution can be found directly by integrating equation (1.0.1) and using Banach's fixed point theorem. For a proof in the deterministic case, see [104, Theorem 5.1].

The remainder of the introduction is organized as follows: first, we start with an example of a stochastic partial differential equation from filtering theory, and we show how to formulate it as a stochastic evolution equation with random drift $A(t,\omega)$; second, we discuss general theory of stochastic evolution equations with a random drift.

1.1 An example from filtering theory

The example in this section, taken from filtering theory, serves to show how dependence on the probability space in the drift occurs naturally in applications.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathscr{F}_t)_{t\in[0,T]}$. Suppose there is given a real-valued signal process $X = X(t,\omega)$, $(t,\omega) \in [0,T]\times\Omega$. The signal process is assumed to be unobservable, and instead what is observed is a function $h(X(t,\omega))$ perturbed by an observational error $n(t,\omega)$ called the noise. As common in the literature, we suppress the letter ω in the notation. The observation $y = y(t,\omega)$ can be described by

$$y(t) = h(X(t)) + n(t). \tag{1.1.1}$$

Heuristically, the noise $n(t)$ is thought of as the time derivative of a Brownian motion $W(t)$, although in fact $W(t)$ is nowhere differentiable a.s. To overcome this problem, one can interpret equation (1.1.1) in an integrated sense, and consider the accumulated observation process $Y(t) = \int_0^t y(s)\, ds$. This is described by

$$Y(t) = \int_0^t h(X(s))\, ds + W(t). \tag{1.1.2}$$

Usually, one writes equation (1.1.2) as

$$dY(t) = h(X(t))\, dt + dW(t).$$

We will make the following assumptions on the signal process X. Let us assume that it is an $\mathbb{R}$-valued process satisfying the stochastic differential equation

$$dX(t) = b(t, X(t))\, dt + \sigma(t, X(t))\, dB(t).$$

Here, B is a Brownian motion independent of W, and we assume $b : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ are both measurable, adapted and Lipschitz continuous functions. Finally, we assume $h : \mathbb{R} \to \mathbb{R}$ is bounded and Lipschitz continuous.

The goal is to estimate the signal $X(t)$ based on the available information up to time t, i.e., based on the σ-algebra $\mathscr{G}_t := \sigma(Y(s) : \ s \in [0,t])$. This means that we need to 'filter out' the noise, and therefore this example is commonly referred to as the filtering equation or the filtering model.

Recall that for a sub-σ-algebra $\mathscr{G} \subset \mathscr{F}$, the conditional expectation $\mathbb{E}(\cdot|\mathscr{G})$ is the orthogonal projection of $L^2(\Omega; \mathscr{F})$ onto $L^2(\Omega; \mathscr{G})$. It follows that for a random variable $\xi \in L^2(\Omega; \mathscr{F})$, the conditional expectation $\mathbb{E}(\xi|\mathscr{G})$ is the minimum of the mean square error among all $\mathscr{G}$-measurable square-integrable random variables. That is,

$$\mathbb{E}(\xi - \mathbb{E}(\xi|\mathscr{G})^2) = \min\{\mathbb{E}(\xi - \eta)^2 : \ \eta \in L^2(\Omega; \mathscr{G})\}.$$

This means that if we want to find the best estimate for $X(t)$, in the sense of the mean square error, we need to find $\mathbb{E}(X(t)|\mathscr{G}_t)$. However, in most cases we would like to estimate $f(X(t))$, where f is in a suitable class of test functions, and therefore one would like to calculate $\mathbb{E}(f(X(t))|\mathscr{G}_t)$. It turns out that

$$\mathbb{E}(f(X(t))|\mathscr{G}_t) = \int_{\mathbb{R}} f(x)\, d\pi_t(x),$$

where π_t is the conditional probability distribution of X_t given $\mathscr{G}_t$. This conditional probability distribution π_t is called the optimal filter. Suppose that the filtration is generated by the Brownian motions B and W, i.e., $\mathscr{F}_t := \sigma(B_s, W_s, \ s \in [0,t])$. Suppose that $\mathscr{F} = \mathscr{F}_T$. Set

$$M_t := \exp\Big(- \int_0^t h(X(s))\, dW(s) - \frac{1}{2} \int_0^t |h(X(s))|^2\, ds \Big).$$

Note that M_t is the stochastic exponential of $\xi(t) := -\int_0^t h(X(s))\, dW(s)$. Consider the measure $\hat{\mathbb{P}}_t := M_t\ d\mathbb{P}$, and set $\hat{\mathbb{P}} = \hat{\mathbb{P}}_T$. Then, as an application of Girsanov's theorem, Y is a Brownian motion with respect to the probability measure $\hat{\mathbb{P}}$. If we write $\hat{\mathbb{E}}$ for the expectation with respect to $\hat{\mathbb{P}}$, and $\int_{\mathbb{R}} f(x)\ d\pi_t(x) = \langle \pi_t, f\rangle$, then we obtain the following theorem [140, Theorem 5.3], which is Bayes' formula in filtering theory, and is called the Kallianpur-Striebel formula.

Theorem 1.1. *For all $f \in C_b(\mathbb{R})$, the optimal filter π_t satisfies the identity*

$$\langle \pi_t, f\rangle = \frac{\hat{\mathbb{E}}(M_t f(X_t)|\mathscr{G}_t)}{\hat{\mathbb{E}}(M_t|\mathscr{G}_t)} = \frac{\langle V_t, f\rangle}{\langle V_t, 1\rangle},$$

where $\langle V_t, f\rangle := \hat{\mathbb{E}}(M_t f(X_t)|\mathscr{G}_t)$.

This means that, to compute the optimal filter, it suffices to compute the right hand side $\frac{\langle V_t, f\rangle}{\langle V_t, 1\rangle}$. This is helpful, since it is possible to derive a stochastic evolution equation for $\langle V_t, f\rangle$, where $f \in C_b^2(\mathbb{R})$, using Itô's formula.

Theorem 1.2. *Let $f \in C_b^2(\mathbb{R})$. The filter V_t satisfies the stochastic differential equation*

$$d\langle V_t, f\rangle = \langle V_t, Lf\rangle \, dt + \langle V_t, fh\rangle dY(t), \tag{1.1.3}$$

where

$$Lf = \frac{1}{2}c^2 f'' + bf'.$$

If we consider V as a process taking values in the space of finite measures, then we can make the assumption that V_t has a density $\tilde{V}_t$ that is twice continuously differentiable. Under this assumption, we write

$$\langle V_t, f\rangle = \int_{\mathbb{R}} f(x)\tilde{V}_t(x)\, dx.$$

In this case, one can formally rewrite (1.1.3) as a stochastic evolution equation for $\tilde{V}_t$, given by

$$d\tilde{V}_t = L^*\tilde{V}_t \; dt + h\tilde{V}_t \; dY(t).$$

Here, L^* is formally the adjoint of L and is given by

$$(L^*\varphi) = \frac{d^2}{dx^2}(c^2\varphi) - \frac{d}{dx}(b\varphi)$$

As the functions c and b are assumed to be random and time-dependent, one notices that the above evolution equation is a special case of (1.0.1).

1.2 Stochastic integration in Banach spaces: the Itô integral

In [52], Itô extended Wiener's theory of stochastic integration [139] in such a way that random processes $\phi : \Omega \times [0, T] \to \mathbb{R}$ that are adapted to a filtration $(\mathscr{F}_t)_{t\in[0,T]}$ can be integrated with respect to a Brownian motion $(W(t))_{t\in[0,T]}$ adapted to the same filtration. The latter means that not only $W(t)$ is $\mathscr{F}_t$-measurable, but also that $W(t) - W(s)$ is independent of $\mathscr{F}_s$ if $s \le t$. In the same paper, he proved that for all $t \in [0, T]$,

$$\mathbb{E}\Big|\int_0^t \phi(s)\; dW(s)\Big|^2 = \mathbb{E}\int_0^t |\phi(s)|^2 \; ds.$$

which by now is generally referred to as the Itô isometry.

The Itô integration theory extends to the Hilbert case setting in the following way. The Brownian motion W may be replaced by an H-cylindrical Brownian motion W_H, where H is a Hilbert space, and the adapted process may be $\mathscr{L}_2(H,K)$-valued, where K is another Hilbert space. The isometry should then be regarded in a Hilbert-Schmidt setting, i.e.,

$$\mathbb{E}\Big\| \int_0^T \phi(s)\, dW_H(s)\Big\|_K^2 = \mathbb{E}\|R_\phi\|^2_{\mathscr{L}_2(L^2(0,T;H),K)}, \tag{1.2.1}$$

where $R_\phi : L^2(0,T;H) \to K$ is given by

$$R_\phi(f) = \int_0^T \phi(t)f(t)\, dt. \tag{1.2.2}$$

To extend this theory to the Banach space setting, new tools are needed. Firstly, let $\mathscr{H}$ be a separable Hilbert space and E be a Banach space. For $h \in \mathscr{H}$ and $x \in E$, we denote by $h \otimes x$ the *rank one operator* in $\mathscr{L}(\mathscr{H}, E)$ given by $(h\otimes x)(h') = [h,h]_{\mathscr{H}} x$, $h' \in \mathscr{H}$. We will call an operator $R \in \mathscr{L}(\mathscr{H}, E)$ a *finite rank operator* if it is a linear combination of rank one operators. For any finite rank operator, we define the norm

$$\Big\|\sum_{n=1}^N h_n \otimes x_n\Big\|_{\gamma(\mathscr{H},E)} := \Big(\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n x_n\Big\|^2\Big)^{1/2}.$$

Here, the sequence $(\gamma_n)_{n=1}^N$ is a sequence of independent standard Gaussian random variables. The Banach space $\gamma(\mathscr{H}, E)$ is defined as the completion of all finite rank operators with respect to the norm $\|\cdot\|_{\gamma(\mathscr{H},E)}$, and is called the space of γ-radonifying operators. More on γ-radonifying operators can be found in [85] and the references therein.

Secondly, we call a Banach space E a UMD$_p$ Banach space, $p \in (1,\infty)$, whenever the following geometric property holds: there exists a constant C such that for all martingale difference sequences $(d_n)_{n=1}^N$ in E and all $\varepsilon_n = \pm 1$,

$$\mathbb{E}\Big\|\sum_{n=1}^N \varepsilon_n d_n\Big\|^p \le C^p \Big\|\sum_{n=1}^N d_n\Big\|^p.$$

The term UMD stands for "unconditionality of martingale differences". It turns out that the property is independent of $p \in (1,\infty)$, and is therefore referred to as the UMD property. More on UMD spaces can be found in the papers by Burkholder [23] and by Rubio de Francia [114]. The following Banach spaces are UMD spaces:

- every Hilbert space,
- every $L^p(S,\mu)$-space for $p \in (1,\infty)$ and (S,μ) a σ-finite measure space,
- Reflexive Sobolev spaces, Besov spaces and Hardy spaces,
- Reflexive Orlicz space

Also:

- Every closed subspace of a UMD space is again a UMD space,
- Every quotient space of a UMD space is again a UMD space,
- The dual of a UMD space is again a UMD space,
- For an interpolation couple (E_0, E_1) of UMD spaces, the real and complex interpolation spaces $(E_0, E_1)_{\theta,p}$ and $[E_0, E_1]_\theta$, for $1 < \theta < \infty$, $1 < p < \infty$, are again UMD spaces,
- The space $L^p(E)$, $1 < p < \infty$ is a UMD space whenever E is a UMD space.

In [91], van Neerven and Weis extended the stochastic integral to $\mathscr{L}(H,E)$-valued functions $\phi : [0,T] \to \mathscr{L}(H,E)$ for which the operator $R_\phi : L^2(0,T;H) \to E$ given by (1.2.2) belongs to $\gamma(L^2(0,T;H),E)$. An isometry similar to (1.2.1) holds:

$$\mathbb{E}\Big\| \int_0^T \phi(s)\, dW_H(s)\Big\|_E^2 = \mathbb{E}\|R_\phi\|^2_{\gamma(L^2(0,T;H),E)}$$

Later, van Neerven, Veraar and Weis [86] extended this to adapted processes $\phi : \Omega \times [0,T] \to \mathscr{L}(H,E)$, with the assumption that E is a UMD Banach space. Here, an isometry fails, but an Itô isomorphism still holds, i.e.,

$$\mathbb{E}\Big\| \int_0^T \phi(s)\, dW_H(s)\Big\|_E^p \eqsim \mathbb{E}\|R_\phi\|^p_{\gamma(L^2(0,T;H),E)}, \qquad 1 < p < \infty.$$

1.3 The Skorohod integral and Malliavin calculus

The Skorohod integral is one of several possible extensions of the Itô integral and it allows us to integrate non-adapted processes. These are interesting as we consider equations where the stochastic integral is not well-defined as an Itô integral due to the non-adaptedness of the integrand.

The Skorohod integral was first introduced by Skorohod [129] and is well connected to Malliavin calculus. The Malliavin calculus is a mathematical theory introduced by Malliavin [76], originally called “the stochastic calculus of variations” and designed to give an independent proof of a theorem by Hörmander [49]. In [128], Shigekawa reformulates the theory of Malliavin calculus and introduces the notion of a derivative D on a Wiener space. The adjoint of the derivative, denoted by δ, can be identified with the Skorohod integral, as proved in [46].

Since the paper [76], Malliavin calculus has played an influential role in probability theory (see the monographs [12,15–18,31,53,77,95,98,137] and references therein). In particular, it has played an important role in the study of stochastic (partial) differential equations (S(P)DE) and mathematical finance (see the monographs [24,38,43,51,79,123] and references therein).

As one considers the Skorohod integral operator δ as an adjoint operator, it is not clear whether for a Skorohod integrable process f the truncated process

$\mathbf{1}_{[0,t]}f$ is again in the domain of δ. Such a property is evidently true for the Itô integral and for the Lebesgue integral, but it turns out to be false in the case of the Skorohod integral. This is already stated as an exercise in [95, Exercise 3.2.1]. In Chapter 3, which is based on [109], we give an alternative proof. This proof is based on the construction of a counterexample, which is the main result of [109].

Theorem 1.3. *There exists a process $u \in Dom(\delta)$ such that $\mathbf{1}_{[0,\frac{1}{2})}u \notin Dom(\delta)$.*

From Meyer's inequalities, it follows that the Sobolev space of Malliavin differentiable $L^2(0,1)$-valued random variables, denoted by $\mathbb{D}^{1,2}(L^2(0,1))$, is included in the domain of δ. Consequently, considering the fractional Sobolev spaces introduced by Watanabe [138], one readily obtains that $\mathbb{D}^{s,2}(L^2(0,1)) \subset \mathrm{Dom}(\delta)$ for all $s \geq 1$. As a corollary of the construction made in Chapter 3, we find that the above result is sharp in the sense that $\mathbb{D}^{s,2}(L^2(0,1)) \subset \mathrm{Dom}(\delta)$ if and only if $s \geq 1$.

Malliavin calculus can be generalized to the setting of Hilbert space valued stochastic processes, see [24,43,47,68] and references therein. As an application of the theory of γ-radonifying operators, Maas [74] and Maas and van Neerven [75] extended Malliavin calculus and Skorohod integration to the UMD Banach space setting and proved that the UMD-valued Skorohod integral is again an extension of the Itô stochastic integral. In Chapter 2 we present further extensions of this theory. In particular we obtain a non-adapted version of the chain rule for Lipschitz functions and an Itô formula.

An Itô formula for a Hilbert spaces valued Itô process can be found in [32]. A version of Itô's formula for adapted processes taking values in a 2-uniformly smooth Banach spaces was proved in [92], and where the processes take values in a UMD Banach space in [21]. In the anticipating case, a finite-dimensional Itô's formula for the Skorohod integral and Stratonovich integral is stated and proved in [95, 96]. A generalization of the Itô formula for the Skorohod integral to Hilbert space valued processes can be found in [47]. The Itô formula is one of the main results from Chapter 2, which is based on [112], and we will present its formulation here.

Let E be a UMD space with type 2, U be a separable Hilbert space and let W_U be a U-cylindrical Brownian motion. Consider the following assumptions:

$$\begin{aligned} &\zeta_0 \in \mathbb{D}^{1,2}(E), && D\zeta_0 \in L^2(\Omega; L^2(0,T;\gamma(U,E))) \\ &u \in \mathbb{D}^{2,2}(L^2(0,T;\gamma(U,E))), && Du \in L^2(0,T;\mathbb{D}^{1,2}(\gamma(U,\gamma(H,E)))), \\ &v \in \mathbb{D}^{1,2}(L^2(0,T;E)), && Dv \in L^1(0,T;L^2((0,T)\times\Omega;\gamma(U,E))). \end{aligned} \tag{1.3.1}$$

Consider the process $\zeta : \Omega \times [0,T] \to E$ given by

$$\zeta_t = \zeta_0 + \int_0^t v(r)dr + \int_0^t u(r)\, dW_U(r). \tag{1.3.2}$$

Then we have the following theorem:

Theorem 1.4 (Itô's formula). *Let E be a* UMD *Banach space with type* 2*. Suppose that the conditions* (1.3.1) *hold and let $\zeta : [0,T] \times \Omega \to E$ be as in* (1.3.2)*. Assume ζ has continuous paths. Let $F : E \to \mathbb{R}$ be a twice continuously Fréchet differentiable function. Suppose that F' and F'' are bounded. Then*

$$F(\zeta_t) = F(\zeta_0) + \int_0^t F'(\zeta_s)(v(s))\,ds + \delta(\big\langle F'(\zeta), \mathbf{1}_{[0,t]}u\big\rangle) + \frac{1}{2}\int_0^t \langle u(s), F''(\zeta_s)(u(s))\rangle_{\mathrm{Tr}}\,ds + \int_0^t \big\langle u(s), F''(\zeta_s)((D^-\zeta)(s))\big\rangle_{\mathrm{Tr}}\,ds. \tag{1.3.3}$$

Here, the pair $\langle\cdot,\cdot,\rangle_{\mathrm{Tr}}$ is the trace duality pairing defined in equation (2.2.1). Compared to the Itô formula in the adapted setting, the extra term

$$\int_0^t \big\langle u(s), F''(\zeta_s)((D^-\zeta)(s))\big\rangle_{\mathrm{Tr}}\,ds$$

appears. The operator D^- is defined by

$$(D^-\zeta)(s) = (D\zeta_0)(s) + \int_0^s (Dv(r))(s)dr + \delta(\mathbf{1}_{[0,s]}I_{U,H}((Du)(s))),$$

where $I_{U,H}$ is the isomorphism

$$I_{U,H} : \gamma(U,\gamma(H,E)) \to \gamma(H,\gamma(U,E)).$$

In the case that ζ is adapted, $D^-\zeta = 0$ and (1.3.3) becomes the Itô formula for adapted processes.

1.4 The forward integral

In [117, 118], Russo and Vallois initiated a theory of stochastic integration via regularization procedures. This theory has been further developed by the same authors (see [119–122]) and by other authors (see [25, 42, 54, 58, 93, 116, 133] and the references in [122]). Their theory allows integration with respect to integrators more general than semimartingales, and it also allows integration of non-adapted processes. Applications arise for instance in the theory of fractional Brownian motion.

One of the stochastic integrals defined in [118] is the forward integral. In the case that the integrator is a Brownian motion, the definition is as follows. For a measurable process $G : [0,T] \times \Omega \to \mathbb{R}$ that belongs to $L^2(0,T)$ a.s., we define the sequence $(I^-(G,n))_{n=1}^\infty$ given by

$$I^-(G,n) = n\int_0^T G(s)(W(s+1/n) - W(s))\,ds.$$

If the sequence $(I^-(G,n))$ converges in probability, then G is called forward integrable. Its limit is called the forward integral and is denoted by

$$\delta^-(G) := \int_0^T G\,dW^- = \int_0^T G(s)\,dW^-(s).$$

Applications of the forward integral can be found in mathematical finance, see Chapter 8 of [38] and its references. In [36,37,99] the authors consider the forward integral in the setting of Lévy processes.

Although in the case of the Skorohod integral many authors have considered the infinite dimensional setting, in the case of the forward integral only few results are available. In [34], the integration via regularization has been generalized to separable Banach spaces. In [68], León and Nualart have introduced the forward integral in the operator valued setting.

In chapter 4, which is based on [110], we examine several properties of the forward integral in the operator valued setting, where the integrator is a cylindrical Brownian motion. We will prove that in this setting, as in the real valued case, the forward integral is an extension of the Itô integral. In particular, for adapted processes G, the forward integral of $\mathbf{1}_{[0,t]}G$ exists for all $t \in [0,T]$. For arbitrary G, let us write $J^-(G,n)(t)$ for $I^-(G\mathbf{1}_{[0,t]}, n)$. For adapted G, we write $I(G)(t) = \int_0^t G\,dW^-$. One of the main results of Chapter 4 is the following:

Theorem 1.5. *Let E be a* UMD *Banach space with type 2, and let $p \in [2,\infty)$.*

(1) *If $G \in L^p(\Omega; L^p(0,T;\gamma(H,E)))$ is adapted, then the sequence of processes $(J^-(G,n))_{n=1}^\infty$ converges to $I(G)$ in $L^p(\Omega; W^{\alpha,p}(0,T;E))$ for all $\alpha \in (0,\frac{1}{2})$, as $n \to \infty$.*
(2) *If $G \in L^0(\Omega; L^p(0,T;\gamma(H,E)))$ is adapted, then the sequence of processes $(J^-(G,n))_{n=1}^\infty$ converges to $I(G)$ in $L^0(\Omega; W^{\alpha,p}(0,T;E))$ for all $\alpha \in (0,\frac{1}{2})$, as $n \to \infty$.*

1.5 Methods for solving stochastic evolution equation

There are many different methods for solving equations of the form (1.0.1). We would like to point out some of them. First, there is the theory using random fields due to Walsh [136]. Second, the theory of monotone operators, which is studied in the case $B = 0$ by Lions [72] and in the case $B = I$ by Bensoussan [13]. It was further developed and generalized to the stochastic case by Pardoux [101], Rozovskii [113], Krylov and Rozovskii [63] (see also [108]). Third, there is an L^p-theory due to Krylov [60], in which the drift is a second order operator on $\mathbb{R}^d$ which is allowed to be dependent on time and on the probability space. Krylov shows existence and uniqueness in the space $W^{s,p}(\mathbb{R}^d)$ for $p \geq 2$.

The method we are most interested in, is called the semigroup approach. This has been studied by Curtain and Falb [30], Dawson [33] and then by Da Prato an Zabczyk [32] and their collaborators. Here, it is assumed that the drift $A(t)$

is the generator of an evolution system $S(t,s)_{0\leq s\leq t\leq T}$. In the autonomous case $A(t) = A$ this translates into A being the generator of a semigroup, and hence the name 'semigroup approach'.

Crucial in the semigroup approach is the definition of a mild solution. A mild solution is an adapted and strongly measurable process $u : [0,T] \times \Omega \to E$ such that for all $t \in [0,T]$, almost surely,

$$u(t) = S(t,0)u_0 + \int_0^t S(t,s)F(s,u(s))\,ds + \int_0^t S(t,s)B(s,u(s))\,dW(s).$$

The above cited works are all in the Hilbert space setting, since the stochastic integral is, in these works, defined only on Hilbert spaces. Generalizations of (1.0.1) to Banach spaces started with Neidhardt (to 2-smooth Banach spaces) [92], Brzeźniak (to martingale type 2 Banach spaces) [19], and more recently van Neerven, Veraar and Weis (to UMD Banach spaces) [86], [87], and Veraar [135]. In [86] a theory of stochastic integration in UMD Banach spaces is set up, and in [87] stochastic evolution equations in UMD Banach spaces are considered, with autonomous drift $A(t) = A$. In [135], the non-autonomous case is covered, where beside UMD it is also assumed that the space has type 2.

As discussed before, the semigroup approach does not work when the stochastic integral is an Itô integral and when A is random. A new approach to solve problem (1.0.1) is treated in Chapter 5 and is explained in the next section. Another approach, due to León and Nualart [68], which is based on Malliavin calculus and forward integration, is improved in Chapter 6. This is explained briefly in Section 1.7. Let us also mention that in [88] a maximal regularity approach to (6.7.1) with random A has been developed.

1.6 A new solution concept for stochastic evolution equations

The method described here is treated in Chapter 5 and is based on the paper [111]. As in [135], we consider UMD spaces with type 2. For fixed $\omega \in \Omega$, we assume the Acquistapace-Terreni conditions on the drift $A = A(t,\omega)$, see (AT1), (AT2) below in Subsection 5.2.1. Evolution equations with these assumptions have been studied, among others, by Acquistapace and Terreni [1–4] and Schnaubelt [125].

We introduce a new solution concept, and compare this to other solution concepts. An adapted process $u \in L^0(\Omega; L^p(0,T;E))$ is called a pathwise mild solution to problem (1.0.1) if for all $t \in [0,T]$, a.s.,

$$\begin{aligned}u(t) = S(t,0)u_0 &+ \int_0^t S(t,s)F(s,u(s))\,ds \\ &- \int_0^t S(t,s)A(s)(I(\mathbf{1}_{(s,t)}B(\cdot,u(\cdot)))\,ds + S(t,0)I(\mathbf{1}_{(0,t)}B(\cdot,u(\cdot)))\end{aligned}$$

where

$$I(G) = \int_0^T G(s)\, dW(s).$$

Note that this solution does not involve any anticipating stochastic integral, as opposed to the variation of constants formula for the mild solution.

Let us state the first main theorem, Theorem 5.28. The hypotheses can be found in Chapter 5.

Theorem 1.6. *Assume (H1)–(H5), (HF) and (HB). Let* $\delta, \lambda > 0$ *be such that* $a + \delta + \lambda < \min\{\frac{1}{2} - \theta_B, 1 - \theta_F, \eta_+\}$. *Assume that* $u_0 : \Omega \to E$ *is* $\mathscr{F}_0$*-measurable and* $u_0 \in E^0_{a,1}$ *a.s. Then the following holds:*

1. *There exists a unique adapted pathwise mild solution* $u \in L^0(\Omega; C([0,T]; \tilde{E}_a))$ *to* (1.0.1). *Moreover,* $u - S(t,0)u_0 \in L^0(\Omega; C^\lambda(0,T; \tilde{E}_{a+\delta}))$.
2. *If additionally,* $u_0 \in E^0_{a+\beta,1}$ *a.s. with* $\beta > 0$ *and* $\lambda + \delta < \beta$, *then* $u \in L^0(\Omega; C^\lambda(0,T; \tilde{E}_{a+\delta}))$.

Assumption (H5) is of special importance here. This assumption implies for instance that the random evolution system $S(t,s)$ is bounded in the $\mathscr{L}(E)$-norm uniformly in ω, i.e., there exists a $C > 0$ such that for all $\omega \in \Omega$ one has $\|S(t,s)\|_{\mathscr{L}(E)} \leq C$. Without condition (H5) the constant would depend on ω. Many other estimates are uniformly in ω as a consequence of (H5).

In Subsection 5.5.3 we prove the following existence and uniqueness result without the uniformity condition (H5), based on a localization argument. Due to technical reasons we assume (H5)$'$, a slightly more restrictive condition than (AT2).

Theorem 1.7. *Assume (H1)–(H4), (H5)$'$, (HF) and (HB). Let* $\delta, \lambda > 0$ *be such that* $a + \delta + \lambda < \min\{\frac{1}{2} - \theta_B, 1 - \theta_F, \eta_+\}$. *Assume that* $u_0 : \Omega \to E$ *is* $\mathscr{F}_0$*-measurable and* $u_0 \in E^0_a$ *a.s. Then the assertions (1) and (2) of Theorem 1.6 hold.*

1.7 Forward mild solutions to stochastic evolution equations

When one considers the notion of a mild solution to problem (6.1.1), it seems that the forward integral and not the Skorohod integral is the right choice for the extension of the Itô integral. This is mainly because if one interprets the stochastic integral appearing in the variation of constants formula (1.0.2) as a forward integral, then a mild solution is always a weak solution (see [68, Proposition 5.3]). In fact, if one interprets the stochastic integral as a Skorohod integral, a complementary term appears (see [67]). A mild solution where the stochastic integral is a forward integral will be called a forward mild solution.

The proof of the fact that a forward mild solution is a weak solution relies on a maximal inequality for forward integration. The authors of [68] prove the latter by first proving a similar result for the Skorohod integral, and then by comparing the two stochastic integrals. The proof of the maximal inequality for the Skorohod integral relies on functional analytic techniques and the Itô formula for the Skorohod integral.

In Chapter 6, we present a new proof of the maximal inequality mentioned below, in UMD Banach spaces with type 2 and some additional assumptions. We prove the inequality without using the Itô formula for the Skorohod integral, but instead we use an Itô formula for the forward integral. Apart from the fact that the proof is more direct, the same result is obtained with fewer assumptions on the evolution system: the evolution system is only assumed to be in $\mathbb{D}^{1,p}(\mathscr{L}(E))$ instead of $\mathbb{D}^{2,p}(\mathscr{L}(E))$. This is summarized in the next theorem. Let us denote by $\mathscr{S}_a$ the set of all $\gamma(H,E)$-valued processes of the form

$$G = \sum_{k=1}^{N}\sum_{n=1}^{N} f_{kn}(W(\varphi_1^n),\dots,W(\varphi_N^n)) \otimes \mathbf{1}_{(t_n,t_{n+1}]} \otimes R_k,$$

where $f_{kn} \in C_b^\infty(\mathbb{R}^N)$, $0 \le t_1 < t_2 < \ldots < t_{n+1} \le T$, $\varphi_j^n \in L^2(0,T)\otimes H$ with $\operatorname{supp}(h_j^n) \subset [0,t_n]$, and $R_k = \sum_{i=1}^{N_k} h_i \otimes x_i$, $h_i \in H$, $x_i \in E$.

Theorem 1.8. *Let E be a* UMD *Banach space that has type 2 and satisfies property (D) from section 6.5. Let $S(t,s)$ be a random evolution system satisfying (H) from section 6.5, and let $G \in \mathscr{S}_a$. For all $p \in (2,\infty)$, we have the estimate*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\Big\|\int_0^t S(t,s)G(s)\,dW^-(s)\Big\|_E^p\Big) \le C\mathbb{E}\int_0^T \|G(s)\|_{\gamma(H,E)}^p\,ds. \tag{1.7.1}$$

Moreover, the operator $S^\diamond : \mathscr{S}_a \to L^p(\Omega;C([0,T];E))$ defined by

$$S^\diamond(G)(t) = \int_0^t S(t,s)G(s)\,dW^-(s)$$

extends uniquely to a linear bounded operator $S^\diamond : L_a^p(\Omega\times[0,T];\gamma(H,E)) \to L^p(\Omega;C([0,T];E))$ for which (1.7.1) holds.

Next, we will formulate the concept of a weak solution and a forward mild solution. Under suitable assumptions on A, F and B, we prove that any mild solution is a weak solution, and that problem (6.1.1) has a unique mild solution. These two results use the above theorem, and are the main results of Chapter 6:

Proposition 1.9. *Assume hypotheses (A.1) – (A.3). If u is a mild solution, then it is a weak solution.*

Theorem 1.10. *Assume hypotheses (A.1) – (A.3). Then problem (6.7.1) has a unique forward mild solution.*

1.8 Application

Consider the stochastic partial differential equation

$$\begin{aligned} du(t,s) &= \big(A(t,s,\omega,D)u(t,s) + f(t,s,u(t,s))\big)\,dt \\ &\qquad + g(t,s,u(t,s))\,dW(t,s),\ t\in(0,T],\ s\in S, \\ C(t,s,\omega,D)u(t,s) &= 0,\ t\in(0,T],\ s\in\partial S, \\ u(0,s) &= u_0(s),\ s\in S. \end{aligned} \tag{1.8.1}$$

Here, S is a bounded domain in $\mathbb{R}^n$ with C^2-boundary and outer normal vector $n(s)$. The drift operator A is of the form

$$\begin{aligned} A(t,s,\omega,D) &= \sum_{i,j=1}^{n} D_i(a_{ij}(t,s,\omega)D_j) + a_0(t,s,\omega), \\ C(t,s,\omega,D) &= \sum_{i,j=1}^{n} a_{ij}(t,s,\omega)n_i(s)D_j, \end{aligned}$$

where D_i stands for the derivative in the i-th coordinate. Under suitable assumptions on a_{ij}, a_0, f and g (see Section 5.6 for precise formulation), one can rewrite equation (1.8.1) into a stochastic evolution equation of the form (1.0.1), where each $A(t,\omega)$ is a closed linear operator on $L^p(S)$ for $p\geq 2$. We obtain the following existence and uniqueness result, which is Theorem 5.33.

Theorem 1.11. *Let $p\in[2,\infty)$ and suppose $u_0:\Omega\to L^p(S)$ is $\mathscr{F}_0$-measurable. The following holds under the assumptions just mentioned.*

1. *There exists a unique adapted pathwise mild solution u that belongs to the space $L^0(\Omega;C([0,T];L^p(S)))$.*
2. *If $u_0\in W^{1,p}(S)$ a.s., and $\delta,\lambda>0$ such that $\delta+\lambda<\frac{1}{2}$, then u belongs to $L^0(\Omega;C^\lambda(0,T;B^{2\delta}_{p,p}(S)))$.*

1.9 Outline of the thesis

This thesis consists of two parts. The first part consists of Chapter 2-4, and is dedicated to the theory of stochastic integration of non-adapted processes. In particular, in Chapter 2 we discuss the theory of Malliavin calculus in UMD Banach spaces, and Itô's formula for the Skorohod integral. Chapter 3 contains a construction of a process u that is Skorohod integrable on an interval $[0,1]$ but which is not Skorohod integrable on $[0,1/2]$. In Chapter 4 we obtain convergence results of the approximating sequence of processes converging to the forward integral process in the vector valued setting.

The second part consists of Chapter 5 and Chapter 6 and contains the theory of stochastic evolution equations and applications to stochastic partial differential equations. Chapter 6 is essentially an extension to the paper by León and

Nualart [68]. Originally the starting point for this thesis was to extend [68] from the Hilbert space setting to the UMD Banach space setting. However, along the way we have found two improvements. First, we found that part of the theory in [68] can be improved in such a way that less Malliavin differentiability needs to be assumed. For this, we prove Itô's formula for the forward integral and a maximal inequality for the forward integral. This is covered in Chapter 6. Second, we established a different theory for solving stochastic evolution equations with adapted drift which is based on a new representation formula for the mild solution. For that we assume the conditions by Acquistapace and Terreni, and use the convergence results from Chapter 4. The assumptions are milder than those in Chapter 6, and also give better regularity results. This is covered in Chapter 5.

Part I

Stochastic integration of non-adapted processes

2

Tools for Malliavin calculus in UMD Banach spaces

2.1 Introduction

Since the seminal paper [76], Malliavin calculus has played an influential role in probability theory (see the monographs [12, 15–18, 31, 53, 77, 95, 98, 137] and references therein). In particular, it has played an important part in the study of stochastic (partial) differential equations (S(P)DE) and mathematical finance (see the monographs [24, 38, 43, 51, 79, 123] and references therein). For certain models in finance and SPDEs, Malliavin calculus and Skorohod integration can be applied in an infinite dimensional framework. In the setting of Hilbert space-valued stochastic processes details on this matter can be found in [24, 43, 47, 68] and references therein. For Banach space-valued stochastic processes, there are geometric obstacles which have to be overcome in order to extend stochastic calculus to this setting.

In [86] a new Itô type integration theory for processes with values in a Banach space E has been developed using earlier ideas from [45, 82]. The theory uses a geometric assumption on E, called the UMD-property, and it allows two-sided estimates for L^p-moments for stochastic integrals (see Theorem 2.31 below). A deep result in the theory of UMD spaces is that a Banach space E has the UMD-property if and only if the Hilbert transform is bounded on $L^p(\mathbb{R}; E)$ (see [23] and references therein). The class of UMD spaces include all Hilbert spaces, L^q-spaces with $q \in (1, \infty)$ and the reflexive Sobolev spaces, Besov spaces and Orlicz spaces. Among these spaces the L^q-spaces with $q \in (1, \infty)$ are the most important ones for applications to SPDEs. Recently, the full strength of the stochastic integration theory from [86] has made it possible to obtain optimal space-time-regularity results for a large class of SPDEs (see [88, 89]).

In [75], Malliavin calculus and Skorohod integration have been studied in the Banach space-valued setting. In particular, the authors have shown that if the space E has the UMD-property, the Skorohod integral is an extension of the Itô integral for processes with values in E (see Theorem 2.26 below). The main result of [75] is a Clark-Ocone representation formula for E-valued random variables, where again E is a UMD space. A previous attempt to obtain this representation formula was been given in [80], but the proof contains a gap (see [81]). Further developments on Malliavin calculus have been made in [74] and in particular, the connection with Meyer's inequalities has been investigated. Here the UMD-property is needed again in order to obtain the vector-valued analogue of the Meyer's inequalities (see [74] and [106]). In particular, in [74] a vector-valued version of Meyer's inequalities for higher order derivatives has been proved. Also here there has been a previous attempt to show Meyer's inequalities for higher order derivatives in UMD spaces (see [78]), but unfortunately the proof contains a gap since an integral in [78, Theorem 1.17] is not convergent.

In this chapter we proceed with the development of Malliavin calculus in the UMD-valued setting. After recalling some prerequisites, in Section 2.3.2 we will prove a weak characterization of the Malliavin derivative and extend the Meyers-Serrin result for Sobolev spaces to the setting of Gaussian Sobolev spaces. In Section 2.3.3 several calculus facts such as the product and chain rule will be obtained. Under additional geometric conditions, we further extend the chain rule to the case of Lipschitz function in Section 2.3.4. Some new results for the Skorohod integral are derived in Section 2.4. In particular, pathwise properties of the Skorohod integral are studied in Section 2.4.4. In the final Section 2.5 we prove a version of Itô's formula in the non-adapted setting.

This chapter is based on the paper [112].

2.2 Preliminaries

Below all vector spaces are assumed to be real. With minor modifications, most results can be extended to complex spaces as well. For a parameter t, and real numbers A and B, we write $A \lesssim_t B$ to indicate that there is a constant c only depending on t such that $A \leq cB$. Moreover, we write $A \eqsim_t B$ if both $A \lesssim_t B$ and $B \lesssim_t A$ hold.

2.2.1 γ-Radonifying operators

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let H be a separable Hilbert space and let E be a Banach space. In this section we will review some results about γ-radonifying operators. For a detailed overview we refer to [39, 55, 85]. For $h \in H, x \in E$, we denote by $h \otimes x$ the operator in $\mathscr{L}(H, E)$ defined by

$$(h \otimes x)h' := \langle h, h' \rangle \, x, \qquad h' \in H.$$

For finite rank operators $\sum_{j=1}^n h_j \otimes x_j$, where the vectors $h_1, \dots, h_n \in H$ are orthonormal and $x_1, \dots, x_n \in E$, we define

$$\Big\| \sum_{j=1}^n h_j \otimes x_j \Big\|_{\gamma(H,E)} := \Big(\mathbb{E} \Big\| \sum_{n=1}^N \gamma_n x_n \Big\|_E^2 \Big)^{1/2}.$$

Here $(\gamma_n)_{n\geq 1}$ is a sequence of independent standard real-valued Gaussian random variables. The space $\gamma(H,E)$ of *γ-radonifying operators* is defined as the closure of all finite rank operators in the norm $\|\cdot\|_{\gamma(H,E)}$. One can show that a bounded operator $R : H \to E$ belongs to $\gamma(H,E)$ if and only if the sum $\sum_{n\geq 1}\gamma_n R h_n$ converges in $L^2(\Omega;E)$.

By the Kahane–Khintchine inequalities (see [66, Corollary 3.2]) one has

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n x_n\Big\|^p\Big)^{1/p} \eqsim_p \Big(\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n x_n\Big\|^2\Big)^{1/2},$$

and this extends to infinite sums as well, whenever the sum is convergent.

The γ-radonifying operators also satisfy the following ideal property (see [11, 71, 85]).

Proposition 2.1 (Ideal property). *Suppose that H_0 and H_1 are Hilbert spaces and E_0 and E_1 are Banach spaces. Let $R \in \gamma(H_0,E_0)$, $T \in \mathscr{L}(H_1,H_0)$ and $U \in \mathscr{L}(E_0,E_1)$, then $URT \in \gamma(H_1,E_1)$ and*

$$\|URT\|_{\gamma(H_1,E_1)} \leq \|U\|\|R\|_{\gamma(H_0,E_0)}\|T\|.$$

The following lemma is called the γ-Fubini Lemma, and is taken from [86].

Lemma 2.2. *Let (S,Σ,μ) be a σ-finite measure space and let $1 \leq p < \infty$. The mapping $U : L^p(S;\gamma(H,E)) \to \mathscr{L}(H, L^p(S;E))$, given by $((UF)h)s = F(s)h$ for $s \in S$ and $h \in H$, defines an isomorphism $L^p(S;\gamma(H,E)) \eqsim \gamma(H, L^p(S;E))$.*

The following lemma is taken from [55] and for convenience we include a proof.

Lemma 2.3. *Suppose that H_0 and H_1 are Hilbert spaces. Let $R \in \gamma(H_0,E)$ and $S \in \gamma(H_1,E^*)$. Define the operator $\langle R,S\rangle \in \mathscr{L}(H_1,H_0)$ by*

$$\langle R,S\rangle\, h := R^*(Sh),\ \ h \in H_1.$$

Then $\langle R,S\rangle$ is a trace class operator.

Proof. Let $(h_n)_{n\geq 1}$ and $(k_n)_{n\geq 1}$ be orthonormal bases for H_0 and H_1, respectively. Let $(\varepsilon_n)_{n\geq 1}$ be such that $|\varepsilon_n| = 1$ and $\langle Rh_n, Su_n\rangle\varepsilon_n = |\langle Rh_n, Su_n\rangle|$ for every $n \geq 1$. Then by Hölder's inequality

$$\begin{aligned}
\sum_{n\geq 1}\Big|\Big\langle \langle R,S\rangle\, u_n, h_n\Big\rangle\Big| &= \sum_{n\geq 1}\langle \varepsilon_n R h_n, S u_n\rangle \\
&= \mathbb{E}\Big\langle \sum_{n\geq 1}\varepsilon_n\gamma_n R h_n, \sum_{k\geq 1}\gamma_k S u_k\Big\rangle \\
&\leq \Big\|\sum_{n\geq 1}\varepsilon_n\gamma_n R h_n\Big\|_{L^2(\Omega;E)}\Big\|\sum_{k\geq 1}\gamma_k S u_k\Big\|_{L^2(\Omega;E^*)} \\
&= \|R\|_{\gamma(H_0,E)}\|S\|_{\gamma(H_1,E^*)}.
\end{aligned}$$

Now the result follows from [39, Theorem 4.6]. ☐

We define the *trace duality pairing* as

$$\langle R, S\rangle_{\mathrm{Tr}} = \mathrm{Tr}(R^* S) = \sum_{n\geq 1} \langle Rh_n, Su_n\rangle . \tag{2.2.1}$$

From the calculation in the above proof we see that

$$|\langle R, S\rangle_{\mathrm{Tr}}| \leq \|R\|_{\gamma(H_0,E)} \|S\|_{\gamma(H_1,E^*)}.$$

Recall the following facts: (details and references on UMD and type can be found in Subsections 2.2.3 and 2.4.4, respectively).

Facts 2.4.

- If E is a Hilbert space, then $\gamma(H, E)$ coincides with the Hilbert-Schmidt operators $\mathscr{L}_2(H, E)$
- For all $p \in [1, \infty)$, $\gamma(H, E)$ is isomorphic to a closed subspace of $L^p(\Omega; E)$.

Moreover, the following properties of E are inherited by $\gamma(H, E)$: reflexivity, type $p \in [1, 2]$, cotype $q \in [2, \infty]$, UMD, separability.

2.2.2 The Malliavin derivative operator

In this section we recall some of the basic elements of Malliavin calculus. We refer to [95] for details in the scalar situation.

Let $\{W(h), h \in H\}$ be an *isonormal Gaussian process* associated with H, that is $\{Wh : h \in H\}$ is a centered family of Gaussian random variables and

$$\mathbb{E}(Wh_1 Wh_2) = \langle h_1, h_2\rangle, \qquad h_1, h_2 \in H.$$

Under these conditions, $W : H \to L^2(\Omega)$ is a bounded linear operator. We will assume $\mathscr{F}$ is generated by W.

Let $1 \leq p < \infty$, and let E be a Banach space. Let us define the *Gaussian Sobolev space* $\mathbb{D}^{1,p}(E)$ of E-valued random variables in the following way. Consider the class $\mathscr{S} \otimes E$ of *smooth E-valued random variables* $F : \Omega \to E$ of the form

$$F = f(W(h_1), \ldots, W(h_n)) \otimes x,$$

where $f \in C_b^\infty(\mathbb{R}^n)$, $n \geq 1$ $h_1, \ldots, h_n \in H$, $x \in E$, and linear combinations thereof. Since $\mathscr{S}$ is dense in $L^p(\Omega)$ and $L^p(\Omega) \otimes E$ is dense in $L^p(\Omega; E)$, it follows that $\mathscr{S} \otimes E$ is dense in $L^p(\Omega; E)$. For $F \in \mathscr{S} \otimes E$, define the *Malliavin derivative* DF as the random variable $DF : \Omega \to \gamma(H, E)$ given by

$$DF = \sum_{i=1}^{n} \partial_i f(W(h_1), \ldots, W(h_n)) \otimes (h_i \otimes x).$$

If $E = \mathbb{R}$, we can identify $\gamma(H, \mathbb{R})$ with H and in that case for all $F \in \mathscr{S}$, $DF \in L^p(\Omega; H)$ coincides with the Malliavin derivative in [95]. Recall from [95, Proposition 1.2.1] that D is closable as an operator from $L^p(\Omega)$ into $L^p(\Omega; H)$, and this easily extends to the vector-valued setting (see [75, Proposition 3.3]). For convenience we provide a short proof.

Proposition 2.5 (Closability). *For all $1 \leq p < \infty$, the Malliavin derivative D is closable as an operator from $L^p(\Omega; E)$ into $L^p(\Omega; \gamma(H, E))$.*

Proof. Let $(F_n)_{n\geq 1}$ in $\mathscr{S} \otimes E$ and $G \in L^p(\Omega; \gamma(H, E))$ be such that $\lim_{n\to\infty} F_n = 0$ in $L^p(\Omega; X)$ and $\lim_{n\to\infty} DF_n = G$ in $L^p(\Omega; \gamma(H, E))$. We need to show that $G = 0$. Since G is strongly measurable, it suffices to check that for any $h \in H$ and $x^* \in E^*$ one has $\langle Gh, x^*\rangle = 0$. By the closability of D in the scalar case one obtains

$$\langle Gh, x^*\rangle = \lim_{n\to\infty} \langle DF_n h, x^*\rangle = \lim_{n\to\infty} D(\langle F_n, x^*\rangle)(h) = 0.$$

□

The closure of the operator D is denoted by D again. The domain of the closure is denoted by $\mathbb{D}^{1,p}(E)$ and endowed with the norm

$$\|F\|_{\mathbb{D}^{1,p}(E)} := (\|F\|^p_{L^p(\Omega;E)} + \|DF\|^p_{L^p(\Omega;\gamma(H,E))})^{1/p}$$

it becomes a Banach space. Similarly, for $k \geq 2$ and $p \geq 1$ we let $\mathbb{D}^{k,p}(E)$ be the closure of $\mathscr{S} \otimes E$ with respect to the norm

$$\|F\|_{\mathbb{D}^{k,p}(E)} := (\|F\|^p_{L^p(\Omega;E)} + \sum_{i=1}^{k} \|D^i F\|^p_{L^p(\Omega;\gamma^i(H,E))})^{1/p}.$$

Here $\gamma^1(H, E) = \gamma(H, E)$ and recursively, $\gamma^n(H, E) = \gamma(H, \gamma^{n-1}(H, E))$ for $n \geq 2$. Finally let $\mathbb{D}^{\infty,p}(E) = \bigcap_{k\geq 1} \mathbb{D}^{k,p}(E)$.

2.2.3 Ornstein-Uhlenbeck operators and Meyer's inequalities

In this subsection we recall several results from [74] and [95].

Recall the definition of the *n-th Wiener chaos*

$$\mathcal{H}_n := \overline{\text{lin}}\{H_n(W(h)) : \ \|h\| = 1\}.$$

Here, H_n is the n-th Hermite polynomial. Also let $\mathscr{P}$ be the set of random variable of the form $p(W(h_1), \dots W(h_n))$ where p is a polynomial and $h_1, \dots, h_n \in H$. This set is dense in $L^p(\Omega)$ for all $p \in [1, \infty)$ (see [95, Exercise 1.1.7]). A classical result is the following orthogonal decomposition $L^2(\Omega) = \bigoplus_{n\geq 0} \mathcal{H}_n$ (see [95, Theorem 1.1.1]). For each $n \geq 1$, let $J_n \in \mathscr{L}(L^2(\Omega))$ be the orthogonal projection onto $\mathcal{H}_n$. The *Ornstein-Uhlenbeck semigroup* $(P(t))_{t\geq 0}$ on $L^2(\Omega)$ is defined by

$$P(t) := \sum_{n=0}^{\infty} e^{-nt} J_n.$$

Clearly, $P(t)^* = P(t)$. Moreover, $(P(t))_{t\geq 0}$ extends to a C_0-semigroup of positive contractions on $L^p(\Omega)$ for all $1 \leq p < \infty$ (see [95, Section 1.4]).

Let E be a Banach space. By positivity, for any $t \geq 0$ the mapping $P(t) \otimes I_E$ extends to a contraction $P_E(t) \in \mathscr{L}(L^p(\Omega; E))$. Moreover, $(P_E(t))_{t\geq 0}$ is a C_0-semigroup on $L^p(\Omega; E)$. Also note that $\mathscr{P} \otimes E$ is dense in $L^p(\Omega; E)$. Denote the generator of $(P_E(t))_{t\geq 0}$ by L_E, and set $C_E := -\sqrt{-L_E}$. Observe that for all $F \in \mathscr{P} \otimes E$ one has $C_E F = \sum_{n\geq 0} \sqrt{n} J_n F$. Whenever, there is no danger of confusion we will leave out the subscript E from all these expressions. If E has type $p' > 1$, then $P_E(t)$ is an analytic semigroup on $L^p(\Omega; E)$ for all $p \in (1, \infty)$. In this case each J_n is a bounded operator on $L^p(\Omega; E)$ for all $p \in (1, \infty)$. We refer to [105, Theorem 5.5, Remark 5.9(ii), Identity (5.15)] for details.

Next we recall the vector-valued anlagen of Meyer's inequalities from [74, Theorem 6.8]. To do so we need the following Banach space property. A Banach space E is said to have UMD if for some $p \in (1, \infty)$, there is a constant $\beta_{p,E}$ such that for every $N \geq 1$, every martingale difference sequence $(d_n)_{n=1}^N$ in $L^p(\Omega, E)$ and every $\{-1, 1\}$-valued sequence $(\varepsilon_n)_{n=1}^N$, we have

$$\Big(\mathbb{E}\Big\| \sum_{n=1}^N \varepsilon_n d_n \Big\|^p\Big)^{\frac{1}{p}} \leq \beta_{p,X} \Big(\mathbb{E}\Big\| \sum_{n=1}^N d_n \Big\|^p\Big)^{\frac{1}{p}}. \tag{2.2.2}$$

UMD stands for unconditional martingale differences. It can be shown that if (2.2.2) holds for some $p \in (1, \infty)$, then one can show that it holds for all $p \in (1, \infty)$. We refer to [23] for details. The UMD property plays an important role in vector-valued harmonic analysis, due to the fact that the Hilbert transform is bounded if and only if E is a UMD space. We will also use that the UMD property implies several other useful Banach space properties. If X is a UMD space, then it is reflexive, and hence spaces such as $C(K)$, L^1, L^∞ do not have UMD. In the reflexive range many of the classical spaces (Lebesgue space, Sobolev spaces, Besov spaces, Orlicz spaces, Schatten class, etc) are known to be UMD. In applications to SPDEs the most important example is L^q with $q \in (1, \infty)$.

The case $n = 1$ of the following result was proved in [106] and used in [74, Theorem 6.8] to derive the case $n \geq 2$ by induction.

Theorem 2.6 (Meyer's inequalities). *Let E be a UMD Banach space, let $1 < p < \infty$ and $n \geq 1$. Then the domain of the operator C^n on $L^p(\Omega; E)$ equals $\mathbb{D}^{n,p}(E)$. Moreover, for all $F \in \mathbb{D}^{n,p}(E)$ we have*

$$\begin{aligned}\|D^n F\|_{L^p(\Omega;\gamma^n(H,E))} &\lesssim_{p,E,n} \|C^n F\|_{L^p(\Omega;E)} \\ &\lesssim_{p,E,n} \|F\|_{L^p(\Omega;E)} + \|D^n F\|_{L^p(\Omega;\gamma^n(H,E))}.\end{aligned}$$

Also recall the following vector-valued version of Meyer's Multiplier Theorem (see [74, Theorem 6.5], where even an operator-valued version has been obtained).

Theorem 2.7 (Meyer's Multiplier Theorem). *Let $1 < p < \infty$. Let E be a UMD Banach space, and let $(a_k)_{k=0}^\infty$ be a sequence of real numbers such that $\sum_{k=0}^\infty |a_k| N^{-k} < \infty$ for some $N \geq 1$. If a sequence of scalars $(\phi(n))_{n\geq 1}$ satisfies $\phi(n) := \sum_{k=0}^\infty a_k n^{-k}$ for $n \geq N$, then the operator T_ϕ defined by*

$$T_\phi F := \sum_{n=0}^{\infty} \phi(n) J_n F, \qquad F \in \mathscr{P} \otimes E,$$

extends to a bounded operator on $L^p(\Omega; E)$.

Due to the above results, many of the results in the scalar setting can be extended to the UMD-valued setting. Some results have already been derived in [74], and we will obtain several other results which will be needed to present some of the tools in Malliavin calculus in the UMD setting.

The following density result is a consequence of the corresponding result in the scalar case. It will play a minor role in the sequel.

Proposition 2.8. *Let E be a Banach space, $p \in [1, \infty)$ and $k \geq 1$. Then $\mathscr{P} \otimes E$ is dense in $\mathbb{D}^{k,p}(E)$.*

Proof. It follows from [53, Theorem 15.108] and [95, Corollary 1.5.1] that $\mathscr{P}$ is dense in $\mathbb{D}^{k,p}(\mathbb{R})$. Since $\mathbb{D}^{k,p}(\mathbb{R}) \otimes E$ contains the dense subspace $\mathscr{S} \otimes E$, the result follows from the definition of $\mathbb{D}^{k,p}(E)$. □

Below, the space $\mathbb{D}^{1,p}(\gamma(K, E))$, where K is an arbitrary Hilbert space, will play an important role for the divergence operator δ. The following result is a direct consequence of the γ-Fubini lemma 2.2.

Proposition 2.9. *Let K be a separable Hilbert space, E be a Banach space and let $1 \leq p < \infty$. Then the map $\mathrm{Fub} : \mathbb{D}^{1,p}(\gamma(K, E)) \to \gamma(K, \mathbb{D}^{1,p}(E))$ defined by*

$$((\mathrm{Fub}F)k)(\omega) := F(\omega)k, \qquad \omega \in \Omega, k \in H,$$

is an isomorphism $\mathbb{D}^{1,p}(\gamma(H, E)) \eqsim \gamma(H, \mathbb{D}^{1,p}(E))$. Here

$$(D[(\mathrm{Fub}F)k])h = ((DF)h)k, \qquad h \in H, \ k \in K.$$

In particular, this result holds for the case $K = H$.

2.3 Results on Malliavin derivatives

2.3.1 Poincaré inequality and its consequences

The following Poincaré inequality will be useful to us. A similar result for UMD spaces was obtained [84, Theorem 1] in the discrete setting using entirely different methods. We extend the scalar-valued proof from [95, Proposition 1.5.8].

Proposition 2.10 (Poincaré inequality). *Let E be a UMD space and let $p \in (1, \infty)$. For all $u \in \mathbb{D}^{1,p}(\Omega; E)$ one has*

$$\|u - \mathbb{E}(u)\|_{L^p(\Omega;E)} \lesssim_{p,E} \|Du\|_{L^p(\Omega;\gamma(H,E))},$$

Proof. Let R be the operator on $L^p(\Omega;E)$ defined by $R = \sum_{n=1}^{\infty} \sqrt{1+1/n}J_n$. We will prove that R is bounded, using Theorem 2.7. With $\phi(n) := \sqrt{1+1/n}$, observe that

$$\phi(n) = \sum_{k=0}^{\infty} \binom{1/2}{k} n^{-k} = \sum_{k=0}^{\infty} \frac{(1/2)(1/2-1)\dots(1/2-k+1)}{k!} n^{-k}.$$

With Stirling's formula we obtain

$$\sum_{k=1}^{\infty} \Big| \binom{1/2}{k} \Big| = \sum_{k=1}^{\infty} \frac{(2k)!}{(2k-1)(k!)^2 4^k} \sim \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}.$$

It follows that the above series converges, hence R is bounded.

For any smooth $u \in \mathscr{S} \otimes E$ we have

$$\|(I-L)^{-\frac{1}{2}} RCu\|_{L^p(\Omega;E)} = \Big\| \sum_{j=1}^{\infty} J_n u \Big\|_{L^p(\Omega;E)} = \|u - \mathbb{E}(u)\|_{L^p(\Omega;E)}.$$

With approximation, it follows that the equality holds for all $u \in \mathbb{D}^{1,p}(E)$. Using the boundedness of R, $(I-L)^{-1/2}$ and Meyer's inequalities, we obtain

$$\begin{aligned} \|u - \mathbb{E}(u)\|_{L^p(\Omega;E)} &= \|(I-L)^{-1/2} RCu\|_{L^p(\Omega;E)} \le c\|Cu\|_{L^p(\Omega;E)} \\ &\le c'\|Du\|_{L^p(\Omega;\gamma(H,E))}, \end{aligned}$$

□

As a consequence of the Poincaré inequality one has the following:

Corollary 2.11. *Let E be a* UMD *space, $p \in (1,\infty)$ and let $k \ge 1$. For all $u \in \mathbb{D}^{k,p}(\Omega;E)$ one has*

$$\|u\|_{\mathbb{D}^{k,p}(E)} \eqsim_{p,E,k} \|u\|_{L^p(\Omega;E)} + \|D^k u\|_{L^p(\Omega;\gamma^k(H,E))}.$$

Proof. By density it suffices to prove the norm equivalence for all $u \in \mathscr{S} \otimes E$. The part $\gtrsim_{p,E,k}$ is trivial. For the estimate $\lesssim_{p,E}$, by an iteration argument it suffices to show that for all $i \ge 1$,

$$\|D^i u\|_{L^p(\Omega;\gamma^i(H,E))} \lesssim_{p,E,i} \|u\|_{L^p(\Omega;E)} + \|D^{i+1} u\|_{L^p(\Omega;\gamma^{i+1}(H,E))}. \tag{2.3.1}$$

Observe that $\mathbb{E}D^i u = J_0 D^i u = D^i J_i u$ (see [95, Proposition 1.2.2]). Hence by [74, Theorem 5.3] (applied $i-1$ times), one has

$$\|\mathbb{E}D^i u\|_{\gamma^i(H,E)} = \|D^i J_i u\|_{L^p(\Omega;\gamma^i(H,E))} \eqsim_{p,i} \|J_i u\|_{L^p(\Omega;E)} \lesssim_{p,E,i} \|u\|_{L^p(\Omega;E)}.$$

Now Proposition 2.10 and the latter estimate imply that

$$\begin{aligned} \|D^i u\|_{L^p(\Omega;\gamma^i(H,E))} &\le \|\mathbb{E}D^i u\| + \|D^i u - \mathbb{E}D^i u\|_{L^p(\Omega;\gamma^i(H,E))} \\ &\lesssim_{p,E,i} \|u\|_{L^p(\Omega;E)} + \|D^{i+1} u\|_{L^p(\Omega;\gamma^{i+1}(H,E))}. \end{aligned}$$

and hence (2.3.1) follows. □

2.3.2 Independence of p and a weak characterization

The following theorem suggests that for F to be in $\mathbb{D}^{k,p}(E)$, it suffices to check that F is differentiable in a very weak sense. The result is known in the case $E = \mathbb{R}$ (see [53, Theorem 15.64]). However, in this situation the proof below is new as well.

Theorem 2.12. *Let E be a* UMD *Banach space, let $p \in (1,\infty)$ and $k \geq 1$. Let $F \in L^p(\Omega; E)$ be such that for all $x^* \in E^*$ one has $\langle F, x^* \rangle \in \mathbb{D}^{k,1}(\mathbb{R})$. If there exists an $\xi \in L^p(\Omega; \gamma^k(H,E))$ such that for all $x^* \in E^*$*

$$D^k \langle F, x^* \rangle = \langle \xi, x^* \rangle .$$

Then $F \in \mathbb{D}^{k,p}(E)$ and $D^k F = \xi$.

Proof. Since P_E is an analytic semigroup, $P_E(t)F \in \bigcap_{j\geq 1} \mathrm{Dom}(L_E^j) \subset \mathbb{D}^{\infty,p}(E)$ for all $t > 0$. The inclusion follows from Meyer's inequalities (see Theorem 2.6). From [74, Lemma 6.2] we get for all $x^* \in E^*$,

$$\begin{aligned}\left\langle D^k P_E(t)F, x^* \right\rangle &= D^k \langle P_E(t)F, x^* \rangle = D^k P_{\mathbb{R}}(t) \langle F, x^* \rangle = e^{-kt} P_{\gamma^k(H,\mathbb{R})} D^k \langle F, x^* \rangle \\ &= e^{-kt} P_{\gamma^k(H,\mathbb{R})} \langle \xi, x^* \rangle = \left\langle e^{-kt} P_{\gamma^k(H,E)} \xi, x^* \right\rangle .\end{aligned}$$

Hence $D^k P_E(t)F = e^{-kt} P_{\gamma^k(H,E)}(t)\xi$.

Now, let $t_n \downarrow 0$ as $n \to \infty$, and set $F_n = P_E(t_n)F$. Then, by the strong continuity of $(P(t))_{t\geq 0}$, we get $F_n \to F$ in $L^p(\Omega; E)$ and $D^k F_n \to \xi$ in $L^p(\Omega; \gamma^k(H,E))$. From Corollary 2.11 it follows that $(F_n)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{D}^{k,p}(E)$. Hence by the closedness of D, we get $F \in \mathbb{D}^{k,p}(E)$ and $D^k F = \xi$. □

Remark 2.13. A careful check of the above proof of Theorem 2.12 shows that we can replace the assumption $\langle F, x^* \rangle \in \mathbb{D}^{k,1}(\mathbb{R})$ by $\langle F, x^* \rangle \in D^{1,1}(\mathbb{R})$ and iteratively, for all $1 \leq j \leq k-1$, $D^j \langle F, x^* \rangle \in D^{1,1}(\gamma^{j-1}(H,E))$. As a consequence, for UMD spaces E, one obtains that our definition of $\mathbb{D}^{k,p}(E)$ coincides with the definition of [74].

Next we will give another definition of a Gaussian Sobolev space. For $p \in [1,\infty)$ and $k \in \mathbb{N}$ let

$$\mathbb{D}_*^{k,p}(E) := \{F \in \mathbb{D}^{k,1}(E) : F \in L^p(\Omega; E),\ \forall\, 1 \leq j \leq k\ \ D^j F \in L^p(\Omega; \gamma^j(H,E))\}.$$

The next result can be viewed as a Gaussian version of the Meyers-Serrin theorem for Sobolev spaces. For the scalar setting, a proof is provided in [53, Theorem 15.64]. There, as in [18], a different definition of $\mathbb{D}^{k,p}(\mathbb{R})$ is given in terms of differentiability properties. Their definition coincides with our definition, since $\mathscr{P}$ is dense in both spaces (see Proposition 2.8, and [53, Theorem 15.108]).

Corollary 2.14. *Let E be a* UMD *Banach space, let $p \in [1,\infty)$ and $k \in \mathbb{N}$. Then $\mathbb{D}_*^{k,p}(E) = \mathbb{D}^{k,p}(E)$.*

Proof. First note that the case $p = 1$ is trivial. Let $p > 1$. Obviously, one has $\mathbb{D}^{k,p}(E) \subseteq \mathbb{D}_*^{k,p}(E)$. The converse result follows from Theorem 2.12. □

2.3.3 Calculus results

Next, we will prove several calculus results in the vector-valued setting. The following product rule will be useful later on.

Lemma 2.15 (Product rule). *Let E_0, E_1 and E_2 be Banach spaces. Let $b : E_0 \times E_1 \to E_2$ be a bilinear operator with the property that there is a constant C such that for all $x \in E_0$ and $y \in E_1$ one has $\|b(x,y)\| \le C\|x\|\,\|y\|$. Let $1 \le p,q,r < \infty$ be such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. If $F \in \mathbb{D}^{1,p}(E_0)$ and $G \in \mathbb{D}^{1,q}(E_1)$, then $b(F,G) \in \mathbb{D}^{1,r}(E)$ and*

$$D[b(F,G)] = b(DF,G) + b(F,DG). \tag{2.3.2}$$

Here $b(DF,G)h = b((DF)h,G)$ and $b(F,DG)h = b(F,(DG)h)$ for $h \in H$.

Proof. Using Hölder's inequality, one sees that for all $F \in L^p(\Omega;E_0)$ and $G \in L^q(\Omega;E_1)$, $b(F,G) \in L^r(\Omega;E_2)$ and

$$\|b(F,G)\|_{L^r(\Omega;E_2)} \le C\|F\|_{L^p(\Omega;E_0)}\|G\|_{L^q(\Omega;E_1)}. \tag{2.3.3}$$

If $F \in \mathscr{S}\otimes E_0$ and $G \in \mathscr{S}\otimes E_1$, (2.3.2) follows from a straightforward calculation and the product rule for ordinary derivatives. Moreover, observe that

$$\|D[b(F,G)]\|_{L^r(\Omega;\gamma(H,E_2))} \le \|b(DF,G)\|_{L^r(\Omega;\gamma(H,E_2))} + \|b(F,DG)\|_{L^r(\Omega;\gamma(H,E_2))}$$

Now by linearity it follows that pointwise in Ω, we have

$$\begin{aligned}\|b(DF,G)\|_{\gamma(H,E_2)} &= \Big\|b\Big(\sum_{n\ge1}\tilde\gamma_n DFh_n, G\Big)\Big\|_{L^2(\tilde\Omega;E_2)}\\ &\le C\Big\|\sum_{n\ge1}\tilde\gamma_n DFh_n\Big\|_{L^2(\tilde\Omega;E_0)}\|G\|_{E_1} = C\|DF\|_{\gamma(H,E_0)}\|G\|_{E_1}.\end{aligned}$$

Here $(\tilde\gamma_n)_{n\ge1}$ is a sequence of standard Gaussian random variables on a probability space $(\tilde\Omega,\tilde{\mathscr{F}},\tilde{\mathbb{P}})$.

Similarly, one sees that $\|b(F,DG)\|_{\gamma(H,E_2)} \le C\|F\|_{\gamma(H,E_0)}\|DG\|_{\gamma(H,E_1)}$ pointwise in Ω. From Hölder's inequality we obtain

$$\begin{aligned}&\|D[b(F,G)]\|_{L^r(\Omega;\gamma(H,E_2))}\\ &\quad\le C\|DF\|_{L^p(\Omega;\gamma(H,E_0))}\|G\|_{L^q(\Omega;E_1)} + C\|F\|_{L^p(\Omega;E_0)}\|DG\|_{L^q(\Omega;\gamma(H,E_1))}\\ &\quad\le 2C\|F\|_{\mathbb{D}^{1,p}(\Omega;E_0)}\|G\|_{\mathbb{D}^{1,q}(\Omega;E_1)}.\end{aligned} \tag{2.3.4}$$

Now let $F \in \mathbb{D}^{1,p}(E_0)$ and $G \in \mathbb{D}^{1,q}(E_1)$. Choose sequences $(F_n)_{n\ge1}$ and $(G_n)_{n\ge1}$ of smooth random variables such that $\lim_{n\to\infty} F_n = F$ in $\mathbb{D}^{1,p}(E_0)$ and $\lim_{n\to\infty} G_n = G$ in $\mathbb{D}^{1,p}(E_1)$. Then by (2.3.3), $\lim_{n\to\infty} b(F_n,G_n) = b(F,G)$ in $L^r(\Omega;E_2)$. Moreover, by (2.3.4) $(Db(F_n,G_n))_{n\ge1}$ is a Cauchy sequence and hence convergent in $L^r(\Omega;\gamma(H,E_2))$. Since D is closed, we obtain $b(F,G) \in \mathbb{D}^{1,r}(E_2)$. Furthermore, (2.3.2) follows from an approximation argument. ∎

Let E be a Banach space. For a sequence $(x_n)_{n\geq 1}$ in E and $x \in E$ we say that $\lim_{n\to\infty} x_n = x$ weakly if for all $x^* \in E^*$ one has $\lim_{n\to\infty}\langle x_n, x^*\rangle = \langle x, x^*\rangle$. Notation: $x_n \rightharpoonup x$. Recall that if E is reflexive, then for every bounded sequence $(x_n)_{n\geq 1}$ in E there is a subsequence $(n_k)_{k\geq 1}$ and an element $x \in E$ such that $x_{n_k} \rightharpoonup x$ as k tends to infinity. Moreover, in this case $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

Lemma 2.16 (Compactness). *Let E be a reflexive Banach space and let $p \in (1,\infty)$. Let $(F_n)_{n\geq 1}$ be a sequence in $\mathbb{D}^{1,p}(E)$ and $F \in L^p(\Omega; E)$. Assume $F_n \rightharpoonup F$ in $L^p(\Omega; E)$ and that there is a constant C such that for all $n \geq 1$, $\|DF_n\|_{L^p(\Omega;\gamma(H,E))} \leq C$. Then $F \in \mathbb{D}^{1,p}(E)$ and $\|DF\|_{L^p(\Omega;\gamma(H,E))} \leq C$. Moreover, there exists a subsequence $(n_k)_{k\geq 1}$ such that $DF_{n_k} \rightharpoonup DF$.*

Proof. Let $G = \{(\xi, D\xi) : \xi \in \mathbb{D}^{1,p}(E)\} \subseteq L^p(\Omega; E) \times L^p(\Omega; \gamma(H, E))$. Since D is a closed linear operator, G is a closed linear subspace of $L^p(\Omega; E) \times L^p(\Omega; \gamma(H, E))$. As E and $\gamma(H, E)$ are reflexive, the latter space is reflexive, and hence G is reflexive as well.

As $F_n \rightharpoonup F$ in $L^p(\Omega; E)$, the uniform boundedness principle implies that $(F_n)_{n\geq 1}$ is bounded in $L^p(\Omega; E)$. Now this together with the assumptions yields that $(F_n, DF_n)_{n\geq 1}$ is a bounded sequence in G. Since G is reflexive, it follows that there is a $(\zeta, D\zeta) \in G$ and a subsequence $(n_k)_{k\geq 1}$ such that $(F_{n_k}, DF_{n_k}) \rightharpoonup (\zeta, D\zeta)$. Since $F_n \rightharpoonup F$, one has that $\zeta = F$, and hence $F \in \mathbb{D}^{1,p}(E)$ with $DF = D\zeta$. It follows that $DF_{n_k} \rightharpoonup DF$, and in particular, $\|DF\|_{L^p(\Omega;\gamma(H,E))} \leq C$. □

Next we extend the chain rule for the Malliavin derivative to the vector-valued setting.

Proposition 2.17. *Let E_0 be a Banach space, let E_1 be a* UMD *Banach space and let $p \in (1,\infty)$. Suppose $\varphi : E_0 \to E_1$ is Fréchet differentiable and has a continuous and bounded derivative. If $F \in \mathbb{D}^{1,p}(E_0)$, then $\varphi(F) \in \mathbb{D}^{1,p}(E_1)$ with*

$$D(\varphi(F)) = \varphi'(F)DF.$$

Proof. Observe that for all $x, y \in E_0$, $\|\varphi(x) - \varphi(y)\| \leq C\|x - y\|$, where $C = \sup_{y\in E_0} \|\varphi'(y)\|_{\mathscr{L}(E_0,E_1)}$. In particular, for all $x \in E_0$, $\|\varphi(x)\| \leq C(1 + \|\varphi(0)\|)$.

Step 1*:* First assume $E_1 = \mathbb{R}$.
Suppose that F is a smooth random variable of the form

$$F = \sum_{m=1}^{M} f_m(W(h_1), \ldots, W(h_n)) \otimes x_m.$$

Now consider the isomorphism $b : \mathrm{sp}\{x_1, \ldots, x_M\} \to \mathbb{R}^{M'}$, $M' \leq M$, which sends x_i to e_i. Then obviously $\psi : \mathbb{R}^{M'} \to \mathrm{sp}\{x_1, \ldots, x_M\}$ given by $\psi := \varphi \circ b^{-1}$ is Fréchet differentiable and has a continuous and bounded derivative given by $\psi'(x)(y) = \varphi'(b^{-1}(x))b^{-1}(y)$. Moreover, from the finite dimensional chain rule (see [95, Proposition 1.2.3]) we get $\varphi(F) = \psi(b(F))$ is in $\mathbb{D}^{1,p}(\mathbb{R})$ and $D(\varphi(F)) = \varphi'(F)DF$.

Now let $F \in \mathbb{D}^{1,p}(E_0)$. Choose sequence of smooth random variables F_k converging to F in $\mathbb{D}^{1,p}(E_0)$. By going to a subsequence if necessary, we can assume that $F_k \to F$ in E_0 and $DF_k \to DF$ in $\gamma(H, E_0)$ almost surely as $k \to \infty$. Clearly, one has

$$\lim_{k\to\infty} \|\varphi(F_k) - \varphi(F)\|_{L^p(\Omega)} \leq C \lim_{k\to\infty} \|F_k - F\|_{L^p(\Omega)} = 0.$$

Moreover, by the above, $\varphi(F_k) \in \mathbb{D}^{1,p}(\Omega)$ for each $k \geq 1$ and

$$\begin{aligned}
&\|D(\varphi(F_k)) - \varphi'(F)DF\|_{L^p(\Omega;H)} = \|\varphi'(F_k)DF_k - \varphi'(F)DF\|_{L^p(\Omega;H)} \\
&\quad\leq \|\varphi'(F_k)DF_k - \varphi'(F_k)DF\|_{L^p(\Omega;H)} + \|\varphi'(F_k)DF - \varphi'(F)DF\|_{L^p(\Omega;H)} \\
&\quad\leq C\|DF_k - DF\|_{L^p(\Omega;H)} + \|\varphi'(F_k)DF - \varphi'(F)DF\|_{L^p(\Omega;H)}
\end{aligned}$$

The first term clearly converges to zero. The second term converges to zero by the continuity and boundedness of φ' and the Dominated Convergence Theorem. By the closedness of D it follows that $\varphi(F) \in \mathbb{D}^{1,p}(\mathbb{R})$ and $D(\varphi(F)) = \varphi'(F)DF$.

Step 2: Let E_1 be an arbitrary UMD space. Let $F \in \mathbb{D}^{1,p}(E_0)$. Fix an $y^* \in E_1^*$. Consider the function $\Phi_{y^*} : E_0 \to \mathbb{R}$ defined by

$$\Phi_{y^*}(x) = \langle \varphi(x), y^* \rangle, \quad x \in E_0.$$

Applying step 1 to the function Φ_{y^*} we obtain that $\Phi_{y^*}(F) \in \mathbb{D}^{1,p}(\mathbb{R})$ and

$$D\langle \varphi(F), y^* \rangle = D(\Phi_{y^*}(F)) = \Phi'_{y^*}(F)DF = \langle \varphi'(F)DF, y^* \rangle.$$

Since $y^* \in E_1^*$ was arbitrary, and $\varphi'(F)DF \in L^p(\Omega; \gamma(H, E_1))$, we can use Theorem 2.12 to obtain that $\varphi(F) \in \mathbb{D}^{1,p}(E_1)$ and $D(\varphi(F)) = \varphi'(F)DF$. □

Remark 2.18. It is clear from the proof of Proposition 2.17 that it remains true if $E_1 = \mathbb{R}$ and $p = 1$.

2.3.4 A chain rule for Lipschitz functions

In this section we will study the chain rule for the Malliavin derivative for Lipschitz functions $\phi : E_0 \to E_1$, where E_0 and E_1 are Banach spaces with some additional geometric structure.

Proposition 2.19. *Let E_0 be a Banach space that has a Schauder basis, let E_1 be a* UMD *Banach space and let $p \in (1, \infty)$. Let $\phi : E_0 \to E_1$ be a Lipschitz function with*

$$\|\phi(x_1) - \phi(x_2)\| \leq L\|x_1 - x_2\|, \quad x_1, x_2 \in E_0.$$

If $F \in \mathbb{D}^{1,p}(E_0)$, then $\phi(F) \in \mathbb{D}^{1,p}(E_1)$, and furthermore there exists a bounded linear operator $T_F : \gamma(H, E_0) \to L^\infty(\Omega; \gamma(H, E_1))$ with $\|T_F\| \leq L$ and $D(\phi(F)) = T_F(DF)$.

Observe that for $\xi \in L^p(\Omega;\gamma(H,E_0))$, $T_F(\xi) \in L^p(\Omega;\gamma(H,E_1))$ is well-defined as the composition of the mappings $\omega \mapsto (\omega,\xi(\omega)) \in \Omega\times\gamma(H,E_0)$ and $(\omega,R) \mapsto (T_F R)(\omega) \in \gamma(H,E_1)$.

Proof. Let $(x_n)_{n\geq 1}$ be a Schauder basis for E_0 and let $(x_n^*)_{n\geq 1}$ be its associated biorthogonal functionals. We can assume that $\|x_n\| = 1$ for all $n \geq 1$. For each $n \geq 1$ consider the projection $S_n : E_0 \to E_0$ onto the first n basis coordinates. It is well-known that there is a constant C such that for all $n \geq 1$, $\|S_n\| \leq C$. Letting, $S_0 = 0$, we see that $P_n := S_n - S_{n-1}$ satisfies $\|P_n\| \leq 2C$ for all $n \geq 1$.

For $n \geq 1$ fixed, consider the map $l_n : \mathbb{R}^n \to \mathrm{sp}\{x_1,\ldots,x_n\}$ that sends the basis coordinate $e_i \in \mathbb{R}^n$ to x_i. We claim that $\|l_n\| \leq \sqrt{n}$ and $\|l_n^{-1}\| \leq 2C\sqrt{n}$. Indeed, for $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{R}^n$ one has

$$\|l_n\alpha\| = \Big\|\sum_{i=1}^n \alpha_i x_i\Big\| \leq \sum_{i=1}^n |\alpha_i|\|x_i\| \leq \sqrt{n}\|\alpha\|,$$

and the first part of the claim follows. For $x = \sum_{i=1}^n \alpha_i x_i \in \mathrm{sp}\{x_1,\ldots,x_n\}$ one has

$$2C\Big\|\sum_{i=1}^n \alpha_i x_i\Big\| \geq \Big\|P_j\sum_{i=1}^n \alpha_i x_i\Big\| = \|\alpha_j x_j\| = |\alpha_j|, \quad j \in \{1,2,\ldots,n\}.$$

It follows that

$$2\sqrt{n}C\Big\|\sum_{i=1}^n \alpha_i x_i\Big\| \geq \Big(\sum_{j=1}^n |\alpha_j|^2\Big)^{1/2} = \|l_n^{-1}x\|.$$

Hence the second part of the claim follows.

Next, for every $n \geq 1$, let $\zeta_n : \mathbb{R}^n \to \mathbb{R}$ be a $C^\infty(\mathbb{R}^n)$-function such that

$$\zeta_n \geq 0, \quad \mathrm{supp}(\zeta_n) \subset B(0,1), \quad \text{and} \quad \int_{\mathbb{R}^n} \zeta_n(x)\,dx = 1.$$

Fix $n \geq 1$ and fix $\varepsilon > 0$. Let $\zeta_n^\varepsilon : \mathbb{R}^n \to \mathbb{R}$ be given by $\zeta_n^\varepsilon(x) := \varepsilon^{-n}\zeta_n(x/\varepsilon)$. Define $\phi_n : E_0 \to E_1$ by

$$\phi_n(x) := \int_{\mathbb{R}^n} \zeta_n^\varepsilon\big(y - l_n^{-1}(S_n x)\big)\phi(l_n y)\,dy = \int_{\mathbb{R}^n} \zeta_n^\varepsilon(y)\phi\big(S_n(x) + l_n(y)\big)\,dy.$$

It follows that

$$\begin{aligned}
(\mathbb{E}\|\phi_n(F)-\phi(F)\|_{E_1}^p)^{1/p} &= \Big(\mathbb{E}\Big\|\int_{\mathbb{R}^n}\zeta_n^\varepsilon(y)\big[\phi\big(S_n(F)+l_n(y)\big)-\phi(F)\big]\,dy\Big\|_{E_1}^p\Big)^{1/p}\\
&\le \Big(\mathbb{E}\Big(\int_{\mathbb{R}^n}\zeta_n^\varepsilon(y)\|\phi\big(S_n(F)+l_n(y)\big)-\phi(F)\|_{E_1}\,dy\Big)^p\Big)^{1/p}\\
&\le L\Big(\mathbb{E}\Big(\int_{\mathbb{R}^n}\zeta_n^\varepsilon(y)\|S_n(F)+l_n(y)-F\|_{E_0}\,dy\Big)^p\Big)^{1/p}\\
&\le L\Big(\mathbb{E}\Big(\int_{\mathbb{R}^n}\zeta_n^\varepsilon(y)\|S_n(F)-F\|\,dy\Big)^p\Big)^{1/p}+L\int_{\mathbb{R}^n}\zeta_n^\varepsilon(y)\|l_ny\|_{E_0}\,dy\\
&\le L(\mathbb{E}\|S_n(F)-F\|^p)^{1/p}+L\sqrt{n}\int_{B(0,\varepsilon)}\varepsilon^{-n}\zeta_n(y/\varepsilon)\|y\|_{\mathbb{R}^n}\,dy\\
&\le L(\mathbb{E}\|S_n(F)-F\|^p)^{1/p}+L\varepsilon\sqrt{n}.
\end{aligned}$$

By the dominated convergence theorem one has $(\mathbb{E}\|S_n(F)-(F)\|^p)^{1/p}\to 0$ as $n\to\infty$. Therefore, letting $\varepsilon=\frac{1}{n}$, it follows that $\lim_{n\to\infty}\phi_n(F)=\phi(F)$ in $L^p(\Omega;E_1)$.

Clearly, $x\mapsto\zeta_n^\varepsilon(y-l_n^{-1}(S_nx))$ is Fréchet differentiable, and hence ϕ_n is differentiable. We claim that for all $x\in E_0$, $\|\phi_n'(x)\|\le CL$. Indeed, fix $x,h\in E_0$ and note that

$$\phi_n'(x)h=\lim_{t\to 0}\frac{\phi_n(x+th)-\phi_n(x)}{t}.$$

Now for $t\neq 0$ one has

$$\begin{aligned}
&\Big\|\frac{\phi_n(x+th)-\phi_n(x)}{t}\Big\|_{E_1}\\
&=\frac{1}{|t|}\Big\|\int_{\mathbb{R}^n}\zeta_n^\varepsilon(y)[\phi(S_n(x)+S_n(th)+l_n(y))-\phi(S_n(x)+l_n(y))]\,dy\Big\|_{E_1}\\
&\le\frac{L}{|t|}\int_{\mathbb{R}^n}\zeta_n^\varepsilon(y)\|tS_n(h)\|_{E_0}\,dy\le L\|S_nh\|_{E_0}\le LC\|h\|_{E_0}.
\end{aligned}$$

Therefore, $\|\phi_n'(x)\|\le LC$ and the claim follows. By Proposition 2.17, we see that $\phi_n(F)\in\mathbb{D}^{1,p}(E_1)$, with

$$D\phi_n(F)=\phi_n'(F)DF.$$

Moreover, by the above claim one obtains that

$$\|\phi_n'(F)DF\|_{L^p(\Omega;\gamma(H,E_1)}\le LC\|DF\|_{L^p(\Omega;\gamma(H,E_0))}.$$

Since the latter is independent of n, we can use Lemma 2.16 to conclude that $\phi(F)\in\mathbb{D}^{1,p}(E_1)$. Moreover, taking an appropriate subsequence we can assume that

$$\lim_{n\to\infty}\phi_n'(F)DF=D\phi(F)\quad\text{ in the weak topology of } L^p(\Omega;\gamma(H,E_1)).\tag{2.3.5}$$

Next, we will show that there exists an operator

$$T \in \mathscr{L}(\gamma(H, E_0), L^\infty(\Omega; \gamma(H, E_1)))$$

such that $D(\phi(F)) = TDF$. Since E_0 has a basis, there exists a basis $(R_n)_{n\geq 1}$ for $\gamma(H, E_0)$. Set $T_n := \phi_n'(F)$. Replacing $(\Omega, \mathscr{F}, \mathbb{P})$ by the space generated by F and DF, we can assume Ω is countably generated. Moreover, since for each $n, j \geq 1$, $\phi_n'(F)R_j$ is strongly measurable, we can assume E_1 is separable and hence that $\gamma(H, E_1)$ is separable. Since E_1 is reflexive, it follows that $\gamma(H, E_1)$ is reflexive and hence also $\gamma(H, E_1)^*$ is separable. We can conclude that $L^1(\Omega; \gamma(H, E_1)^*)$ is separable. Moreover, once again by the reflexivity of $\gamma(H, E_1)$, one has $L^1(\Omega; \gamma(H, E_1)^*)^* = L^\infty(\Omega; \gamma(H, E_1))$.

Recall the following basic fact (see [115, Theorem 3.17]): a bounded sequence $(x_n^*)_{n\geq 1}$ in E^* where E is a separable Banach space has a weak* convergent subsequence, i.e., there is an $x^* \in E^*$ such that for all $x \in E$, $\langle x, x^*\rangle = \lim_{k\to\infty} \langle x, x_{n_k}^*\rangle$. Moreover, $\|x^*\| \leq \liminf_{k\to\infty} \|x_{n_k}^*\|$.

For every $\omega \in \Omega$, we can consider a canonical extension $T_n(\omega) : \gamma(H, E_0) \to \gamma(H, E_1)$ defined by $(T_n(\omega)R)h = T_n(\omega)(Rh)$, and this extension satisfies $\|T_n(\omega)\| \leq LC$. For every $R \in \gamma(H, E_0)$, the bounded sequence $(T_nR)_{n\geq 1}$ in $L^\infty(\Omega; \gamma(H, E_1))$ contains a weak* convergent subsequence. In particular, this holds for every element R_i with $i \geq 1$. By a diagonal argument we can find a subsequence $(n_k)_{k\geq 1}$ and elements $(z_i)_{i\geq 1}$ in $L^\infty(\Omega; \gamma(H, E_1))$ such that for all $i \geq 1$, $\lim_{k\to\infty} T_{n_k} R_i = z_i$ in the weak*-topology of $L^\infty(\Omega; \gamma(H, E_1))$. Let $\gamma_0(H, E_0) = \text{sp}\{R_1, R_2, \ldots\}$. Define the operator $T : \gamma_0(H, E_0) \to L^\infty(\Omega; \gamma(H, E_1))$ by

$$T\Big(\sum_{i=1}^n a_i R_i\Big) = \sum_{i=1}^n a_i z_i.$$

For each $R \in \gamma_0(H, E)$ one has $\lim_{k\to\infty} T_{n_k} R = TR$ in the weak*-topology of $L^\infty(\Omega; \gamma(H, E_1))$ and therefore,

$$\|TR\|_{L^\infty(\Omega;\gamma(H,E_1))} \leq \liminf_{k\to\infty} \|T_{n_k}R\|_{L^\infty(\Omega;\gamma(H,E_1))} \leq LC\|R\|.$$

It follows that T has a continuous extension $T : \gamma(H, E_0) \to L^\infty(\Omega; \gamma(H, E_1))$. Moreover, an approximation argument shows that for all $R \in \gamma(H, E_0)$, $TR = \lim_{k\to\infty} T_{n_k} R$ in the weak*-topology. We show that for all $\xi \in L^p(\Omega; \gamma(H, E_0))$ and all simple functions $\eta : \Omega \to \gamma(H, E_1)^*$ one has

$$\mathbb{E}\langle T\xi, \eta\rangle = \lim_{k\to\infty} \mathbb{E}\langle T_{n_k}\xi, \eta\rangle. \tag{2.3.6}$$

To prove this, note that if ξ is a simple function as well, then by linearity it suffices to prove (2.3.6) for $\xi = \mathbf{1}_A R$ with $R \in \gamma(H, E_0)$ and $A \in \mathscr{F}$. In that case one has

$$\mathbb{E}\langle \eta, T\xi\rangle = \mathbb{E}\langle \mathbf{1}_A\eta, TR\rangle = \lim_{k\to\infty} \mathbb{E}\langle \mathbf{1}_A\eta, T_{n_k}R\rangle = \lim_{k\to\infty} \mathbb{E}\langle \eta, T_{n_k}\xi\rangle.$$

for all $\eta \in L^1(\Omega; \gamma(H, E_1^*))$. Now let $\xi \in L^p(\Omega; \gamma(H, E_0))$ and let $\eta : \Omega \to \gamma(H, E)$ be simple function. Let $\varepsilon > 0$ be arbitrary. Choose a simple function $\xi_0 : \Omega \to \gamma(H, E_0)$ such that $\|\xi - \xi_0\|_{L^p(\Omega;\gamma(H,E_0))} \leq \varepsilon$. It follows that

$$\begin{aligned}\limsup_{k\to\infty}\left|\mathbb{E}\langle T\xi,\eta\rangle-\mathbb{E}\langle T_{n_k}\xi,\eta\rangle\right| &\le \limsup_{k\to\infty}\left|\mathbb{E}\langle T\xi_0,\eta\rangle-\mathbb{E}\langle T_{n_k}\xi_0,\eta\rangle\right|+2LC\varepsilon\\ &=2LC\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ was arbitrary (2.3.6) follows.

Taking $\xi = DF$ in (2.3.6) and using (2.3.5) it follows that for all simple functions $\eta : \Omega \to \gamma(H, E_1)^*$ one has

$$\mathbb{E}\langle TDF,\eta\rangle=\lim_{k\to\infty}\mathbb{E}\langle T_{n_k}DF,\eta\rangle=\lim_{k\to\infty}\mathbb{E}\langle \phi'_{n_k}(F)DF,\eta\rangle=\mathbb{E}\langle D\phi(F),\eta\rangle.$$

By a density and Hahn-Banach argument this yields $TDF = D\phi(F)$. Hence we can take $T = T_F$. □

Remark 2.20. The first part of the proof is based on the idea in [24, Proposition 5.2], where the result has been proved for Hilbert spaces E_0 and E_1. It is surprising that this argument can be extended to a Banach space setting. We do not know if the assumption that E_0 has a basis can be avoided. In the final part of the argument in [24, Proposition 5.2] a compactness argument is used to construct an operator $T \in L^\infty(\Omega;\mathscr{L}(E_0,E_1))$ such that $D(\phi(F)) = T(DF)$. There seems to be a difficulty in this proof, and at the moment it remains unclear whether such a T exists. Note that there are subtle (measurability) differences between the latter space and $\mathscr{L}(E_0, L^\infty(\Omega;E_1))$ if E_0 and E_1 are infinite dimensional.

2.4 The divergence operator and the Skorohod integral

Definition 2.21. Let $p\in[1,\infty)$. Let $\mathrm{Dom}_{p,E}(\delta)$ be the set of $\zeta\in L^p(\Omega;\gamma(H,E))$ for which there exists an $F \in L^p(\Omega;E)$ such that

$$\mathbb{E}\,\langle\zeta,DG\rangle_{\mathrm{Tr}}=\mathbb{E}\,\langle F,G\rangle_{E,E^*}\,,\qquad G\in\mathscr{S}\otimes E^*.$$

In that case, F is uniquely determined, and we write $\delta(\zeta) = F$. The operator δ with domain $\mathrm{Dom}_{p,E}(\delta)$ is called the *divergence operator*.

The operator δ is closed and densely defined, which easily follows from the scalar setting (see [53, p. 274] and [95]). For $p \in (1,\infty)$, the operator δ coincides with the adjoint of D acting on $\mathbb{D}^{1,q}(E^*)$ where $\frac{1}{p}+\frac{1}{q}=1$ (see [75]). If there is no danger of confusion, we will also write $\mathrm{Dom}(\delta)$ for $\mathrm{Dom}_{p,E}(\delta)$.

The following identity can be found in [75, Lemma 3.2].

Lemma 2.22. *We have* $\mathscr{S}\otimes\gamma(H,E)\subseteq \mathit{Dom}(\delta)$ *and*

$$\delta(f\otimes R)=\sum_{j\ge1}W(h_j)f\otimes Rh_j-R(Df),\qquad f\in\mathscr{S},\ R\in\gamma(H,E).$$

Here, $(h_j)_{j\ge1}$ *denotes an arbitrary orthonormal basis of* H*.*

An important consequence of Meyer's inequalities and multiplier theorem is the following sufficient condition to be in the domain of δ (see [74, Proposition 6.10]).

Proposition 2.23. *Let E be a* UMD *Banach space and let $1 < p < \infty$. The divergence operator δ is continuous from $\mathbb{D}^{1,p}(\gamma(H,E))$ to $L^p(\Omega;E)$.*

For Hilbert spaces H_1 and H_2, let us denote by I_{H_1,H_2} the isomorphism

$$I_{H_1,H_2} : \gamma(H_1, \gamma(H_2, E)) \to \gamma(H_2, \gamma(H_1, E)), \tag{2.4.1}$$

which is defined by $((I_{H_1,H_2}R)(h_2))(h_1) = (Rh_1)(h_2)$ for $h_1 \in H_1$ and $h_2 \in H_2$. We will write $I_{H_1} = I_{H_1,H_1}$. The following proposition gives a certain commutation relation between D and δ.

Proposition 2.24. *Let E be a* UMD *Banach space. If $u \in \mathbb{D}^{2,p}(\gamma(H,E))$, then $\delta(u) \in \mathbb{D}^{1,p}(E)$ and we have the relation*

$$D(\delta(u)) = u + \delta(I_H(Du)).$$

Proof. First, let $E = \mathbb{R}$, and $u = f(W(h_1), \dots, W(h_n)) \otimes h$, with $h_1, \dots, h_n \in H$ orthonormal and $h \in H$ such that $\|h\| = 1$. We can use Lemma 2.22 to obtain

$$D(\delta(u)) = \Big(\sum_{j=1}^{n} \partial_j f \otimes (h_j \otimes h) \Big) W(h) + f \otimes h - \sum_{j,k=1}^{n} \partial_{jk} f \otimes \langle h, h_j \rangle h_k.$$

Another computation yields

$$\delta(I_H(Du)) = \Big(\sum_{j=1}^{n} \partial_j f \otimes (h_j \otimes h) \Big) W(h) - \sum_{j,k=1}^{n} \partial_{jk} f \otimes \langle h, h_j \rangle h_k.$$

The commutation relation can be extended by linearity. Now let E be a UMD Banach space, $u \in \mathscr{S} \otimes \gamma(H,E)$. The commutation relation holds for $\langle u, x^* \rangle$ for all $x^* \in E^*$, and hence it holds for u. For general $u \in \mathbb{D}^{2,p}(\gamma(H,E))$, the identity follows from Proposition 2.23 and an approximation argument. □

An immediate consequence is that Proposition 2.23 extends to $\mathbb{D}^{k,p}(\gamma(H,E))$ for $k \geq 1$.

Corollary 2.25. *Let E be a* UMD *Banach space, $1 < p < \infty$, and $k \geq 1$. The operator δ is continuous from $\mathbb{D}^{k,p}(\gamma(H,E))$ to $\mathbb{D}^{k-1,p}(E)$.*

Another consequence of 2.23 is that [95, Proposition 1.5.8] extends to the UMD-valued setting.

Proposition 2.26. *Let E be a* UMD *space and let $1 < p < \infty$. For all $u \in \mathbb{D}^{1,p}(\gamma(H,E))$, we have*

$$\|\delta(u)\|_{L^p(\Omega;E)} \leq c_p(\|\mathbb{E}(u)\|_{\gamma(H,E)} + \|Du\|_{L^p(\Omega;\gamma^2(H,E))}).$$

Proof. Since δ is continuous from $\mathbb{D}^{1,p}(\gamma(H,E))$ to $L^p(\Omega;E)$, we have

$$\|\delta(u)\|_{L^p(\Omega;E)} \leq c(\|u\|_{L^p(\Omega;\gamma(H,E))} + \|Du\|_{L^p(\Omega;\gamma^2(H,E))}).$$

By the triangle inequality, we have

$$\|u\|_{L^p(\Omega;\gamma(H,E))} \leq \|\mathbb{E}(u)\|_{\gamma(H,E)} + \|u - \mathbb{E}(u)\|_{L^p(\Omega;\gamma(H,E))}.$$

Now the result follows from Proposition 2.10. □

2.4.1 Independence of *p* and weak characterization

One can formulate the following analogue of Theorem 2.12 for the divergence operator δ.

Theorem 2.27. *Let E be a* UMD *Banach space, $p \in (1,\infty)$ and $k \geq 1$. Let $F \in L^p(\Omega;\gamma^k(H,E))$ be such that for all $x^* \in E^*$, $\langle F, x^*\rangle$ is in $Dom_{1,\mathbb{R}}(\delta^k)$. If there exists a $\xi \in L^p(\Omega;E)$ such that for all $x^* \in E^*$ one has*

$$\delta^k \langle F, x^*\rangle = \langle \xi, x^*\rangle,$$

then $F \in Dom_{p,E}(\delta^k)$ and $\delta^k F = \xi$.

Proof. Since E is a UMD Banach space, $\gamma^k(H,E)$ is as well, and as in the proof of Theorem 2.12 we obtain that $P_{\gamma^k(H,E)}(t)$ is an analytic semigroup on $L^p(\Omega;\gamma^k(H,E))$ and

$$P_{\gamma^k(H,E)}(t)F \in \cap_{j\geq 1}\mathrm{Dom}(L^j_{\gamma(H,E)}) \subset \mathbb{D}^{k,p}(E) \subseteq \mathrm{Dom}_{p,E}(\delta^k)$$

for all $t > 0$, where the last inclusion follows from Corollary 2.25.

By the symmetry of $(P(t))_{t\geq 0}$ and a duality argument, it follows from [74, Lemma 6.2] that $\delta^k(P_{\gamma^k(H,\mathbb{R})}(t)G) = e^{kt}P_{\mathbb{R}}\delta^k G$ for all $G \in \mathrm{Dom}_{1,\mathbb{R}}(\delta)$. Hence for all $x^* \in E^*$,

$$\begin{aligned}\left\langle \delta^k(P_{\gamma^k(H,E)}(t)F), x^*\right\rangle &= \delta^k \left\langle P_{\gamma^k(H,E)}(t)F, x^*\right\rangle = \delta^k(P_{\gamma^k(H,\mathbb{R})}(t)\langle F, x^*\rangle) \\ &= e^{kt}P_{\mathbb{R}}\delta^k(\langle F, x^*\rangle) = e^{kt}P_{\mathbb{R}}\langle \xi, x^*\rangle = \left\langle e^{kt}P_E\xi, x^*\right\rangle.\end{aligned}$$

Therefore, $\delta^k(P_{\gamma^k(H,E)}(t)F) = e^{kt}P_E(t)\xi$.

Now, let $t_n \downarrow 0$ as $n \to \infty$, and set $F_n = P_{\gamma^k(H,E)}(t_n)F$. Then, by the strong continuity of $(P(t))_{t\geq 0}$, we get $F_n \to F$ in $L^p(\Omega;\gamma^k(H,E))$ and $\delta^k F_n \to \xi$ in $L^p(\Omega;E)$. Hence, by closedness of δ^k, we get $F \in \mathrm{Dom}_{p,E}(\delta^k)$ and $\delta^k F = \xi$. □

Remark 2.28. If H is replaced with $L^2(0,T;H)$ and $F : (0,T) \times \Omega \to \mathscr{L}(H,E)$ is adapted, a weak characterization of the stochastic integral was given in [86] without assumptions on the filtration. Theorem 2.27 can be viewed as an extension to the non-adapted setting, but only under the additional assumption that the filtration is generated by W.

2.4.2 Additional results

The next lemma is an integration by parts formula for the divergence operator.

Lemma 2.29 (Integration by parts). *Let E be a Banach space, and $p, q, r \in [1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let $u \in L^p(\Omega; \gamma(H, E))$ and $F \in \mathbb{D}^{1,q}(E^*)$. If $u \in \mathrm{Dom}(\delta)$, then $\langle u, F \rangle \in \mathrm{Dom}_{r,\mathbb{R}}(\delta)$ and*

$$\delta(\langle u, F \rangle) = \langle \delta(u), F \rangle_{E,E^*} - \langle u, DF \rangle_{\mathrm{Tr}}.$$

Proof. Let $G \in \mathscr{S}$. Identifying H with its dual, one obtains

$$\langle DG, \langle u, F \rangle \rangle_H = \langle u, DG \otimes F) \rangle_{\mathrm{Tr}}.$$

With Lemma 2.15, we get

$$\begin{aligned} \mathbb{E} \langle \langle u, F \rangle, DG \rangle_H &= \mathbb{E} \langle u, DG \otimes F) \rangle_{\mathrm{Tr}} \\ &= \mathbb{E} \langle u, D(GF) \rangle_{\mathrm{Tr}} - \mathbb{E} \langle u, GDF \rangle_{\mathrm{Tr}} \\ &= \mathbb{E} \langle \delta(u), GF \rangle_{E,E^*} - \mathbb{E} \langle u, GDF \rangle_{\mathrm{Tr}} \\ &= \mathbb{E}(G \langle \delta(u), F \rangle_{E,E^*}) - \mathbb{E}(G \langle u, DF \rangle_{\mathrm{Tr}}). \end{aligned}$$

Therefore, $\langle u, F \rangle \in \mathrm{Dom}_{r,E}(\delta)$ and the identity follow. Since G was arbitrary, this yields the result by a density argument. □

The next lemma gives a relationship between the operators D, δ and L.

Lemma 2.30. *Let E be a* UMD *Banach space and $p \in (1, \infty)$. If $u \in \mathbb{D}^{2,p}(E)$, then $\delta(Du) = -Lu$.*

Proof. Note that by Meyer's inequalities, we have $u \in \mathrm{Dom}(L)$. If $u \in \mathscr{S} \otimes E$, the claim follows from the scalar case (see [95, Proposition 1.4.3]). The general case follows from an approximation argument and Proposition 2.23 and Meyer's inequalities. □

2.4.3 Preliminaries on the Skorohod integral

In this section we recall the vector-valued Itô integral and its extension to the non-adapted setting.

Assume $H = L^2(0, T; U)$ for some separable Hilbert space U, and some $T > 0$. The family $(W_U(t))_{t \in [0,T]}$ of mappings from U to $L^2(\Omega)$ given by

$$W_U(t)u := W(\mathbf{1}_{[0,t]} \otimes u)$$

is a *U-cylindrical Brownian motion.* For any $t \in [0, T]$, we denote by $\mathscr{F}_t$ the σ-algebra generated by $\{W_U(s)u : \ 0 \le s \le t, \ u \in U\}$. Note that $\mathbb{F} := (\mathscr{F}_t)_{t \in [0,T]}$ is a filtration. Let $\Phi : [0, T] \times \Omega \to \mathscr{L}(U, E)$ be a *finite rank adapted step function*:

$$\Phi(t,\omega) := \sum_{i=1}^{m}\sum_{j=1}^{n} \mathbf{1}_{(t_{i-1},t_i]}(t)\mathbf{1}_{A_{ij}}(\omega)\sum_{k=1}^{l} u_k \otimes x_{ijk},$$

where $A_{ij} \in \mathscr{F}_{t_{i-1}}$ are disjoint for each j, and (u_k) are orthonormal in U. For such processes, *the stochastic integral* $\mathrm{Int}(\Phi) \in L^p(\Omega; E)$ *with respect to* W_U is defined by

$$\mathrm{Int}(\Phi) := \int_0^T \Phi(t)\, dW_U(t) := \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} \mathbf{1}_{A_{ij}}(W(t_i)u_k - W(t_{i-1})u_k) \otimes x_{ijk}.$$

Let $L^p_{\mathbb{F}}(\Omega; \gamma(H,E))$ be the closure of the set of adapted finite rank step functions in $L^p(\Omega; \gamma(H,E))$. Recall the following results (see [86, Theorem 3.5] and [75, Theorem 5.4] respectively).

Theorem 2.31 (Stochastic integral I). *Let E be a* UMD *Banach space and let $1 < p < \infty$. The stochastic integral uniquely extends to a bounded operator*

$$\mathrm{Int} : L^p_{\mathbb{F}}(\Omega; \gamma(H,E)) \to L^p(\Omega; E).$$

In this case the process $(t,\omega) \mapsto \mathrm{Int}(\mathbf{1}_{[0,t]}\Phi)(\omega)$ has a continuous version and for all $\Phi \in L^p_{\mathbb{F}}(\Omega; \gamma(H,E))$ we have the two-sided estimate

$$\|\mathrm{Int}(\Phi)\|_{L^p(\Omega; C([0,T];E))} \eqsim_{p,E} \|\Phi\|_{L^p(\Omega;\gamma(H,E))}.$$

In the above result one does not need that $\mathbb{F}$ is generated by W. If the above norm equivalence holds for all $\Phi \in L^p_{\mathbb{F}}(\Omega; \gamma(H,E))$, then E has the UMD property (see [45]).

Theorem 2.32 (Stochastic integral II). *Let E be a* UMD *space and $1 < p < \infty$. The space $L^p_{\mathbb{F}}(\Omega; \gamma(H,E))$ is contained in the domain of δ and for all $\Phi \in L^p_{\mathbb{F}}(\Omega; \gamma(H,E))$ one has $\delta(\Phi) = \mathrm{Int}(\Phi)$.*

Motivated by the above result, we will write

$$\int_0^T u(t)\, dW_U(t) = \delta(u), \quad u \in \mathrm{Dom}(\delta),$$

and the latter is called the *Skorohod integral* of u.

2.4.4 Stochastic integral processes

In this section we will assume that $H = L^2(0,T;U)$ for some Hilbert space U and some $T > 0$, and we will assume that E is a UMD Banach space. With a slight abuse of notation, we will denote $\mathbf{1}_A : H \to H$, $A \in \mathcal{B}[0,T]$, as the bounded linear operator on $H = L^2(0,T;U)$ defined by

$$(\mathbf{1}_A h)(t) := \mathbf{1}_A(t)h(t),$$

for almost every $t \in [0,T]$.
With another slight abuse of notation, we can in a similar way view $\mathbf{1}_A$ as an operator on $\gamma(H,E)$. Indeed, for $R \in \gamma(H,E)$ we define $(\mathbf{1}_A R)h := R(\mathbf{1}_A h)$. From the ideal property yields that $\mathbf{1}_A$ is then indeed an operator on $\gamma(H,E)$.

When $u \in \mathrm{Dom}(\delta)$, it does not generally hold that $\mathbf{1}_{(s,t]}u \in \mathrm{Dom}(\delta)$. Indeed, already in the case $p = 2$ and $E = \mathbb{R}$ a counterexample can be found in [95] and [109]. Define

$$\mathbb{L}^p(E) := \{u \in \mathrm{Dom}_{p,E}(\delta) : \ \mathbf{1}_{[0,t]}u \in \mathrm{Dom}_{p,E}(\delta) \text{ for all } t \in [0,T]\}.$$

For $u \in \mathbb{L}^p(E)$ we define the process

$$\zeta(t) := \delta(\mathbf{1}_{[0,t]}u) := \int_0^t u(s)\,dW_U(s), \quad t \in [0,T].$$

Note that $\mathbb{D}^{1,p}(\gamma(H,E)) \subset \mathbb{L}^p(E)$. Indeed, by Theorem 2.9, one obtains that if $u \in \mathbb{D}^{1,p}(\gamma(H,E))$, then $\mathbf{1}_{[0,t]}u \in \mathbb{D}^{1,p}(\gamma(H,E))$. The inclusion then follows from Proposition 2.23.

Below we will also need Banach spaces of type 2. Let us recall the definition. Let $p \in [1,2]$ and consider a Rademacher sequence (r_n). The Banach space E has *type* p if there is a constant C_p such that for all finite sequences $x_1,\dots,x_N \in E$,

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n x_n\Big\|^2\Big)^{1/2} \le C_p\Big(\sum_{n=1}^N \|x_n\|^p\Big)^{1/p}.$$

An elementary fact is that every Hilbert space has type 2. From the Kahane-Khintchine inequalities, it follows that every Banach space has type 1. Also, if a Banach space has type p, then it has type p_0 for all $p_0 \in [1,p]$. If $p \in [1,\infty)$, E has type $p_0 \in [1,2]$ and $(A,\mathscr{A},\mu)$ is a measure space, then $L^p(A;E)$ has type $\min\{p,p_0\}$.

Recall from [85, Theorem 11.6] that for type 2 spaces E one has the following embedding

$$L^2(0,T;\gamma(U,E)) \hookrightarrow \gamma(H,E), \tag{2.4.2}$$

where again $H = L^2(0,T;U)$. This embedding and Proposition 2.9 yield the embedding $L^2(0,T;\mathbb{D}^{1,p}(\gamma(U,E))) \hookrightarrow \mathbb{D}^{1,p}(\gamma(H,E))$. We will show that, under extra integrability conditions, $\zeta(t) := \int_0^t u(s)\,dW_U(s)$ has a continuous version.

Theorem 2.33. *Let E be a* UMD *Banach space with type 2, let $2 < p < \infty$ and suppose $u \in L^2(0,T;\mathbb{D}^{1,p}(\gamma(U,E)))$. If the map $r \mapsto D(u(r))$ belongs to $L^p([0,T];L^p(\Omega;\gamma(H,\gamma(U,E))))$, then the integral process $\zeta : [0,T]\times\Omega \to E$ defined by*

$$\zeta(t) = \int_0^t u(s)\,dW_U(s), \quad t \in [0,T],$$

has a version with continuous paths.

Proof. Let $v \in \gamma(H,E)$ be given by $v = \mathbb{E}u$. By Theorem 2.31 the process $Y(t) = \int_0^t v\, dW_U$ has a continuous version. Replacing u by $u - \mathbb{E}u$, from now on we can assume $\|\mathbb{E}(u)\|_{\gamma(H,E)} = 0$. Proposition 2.26 yields

$$\begin{aligned}\mathbb{E}\|\zeta(t)-\zeta(s)\|^p &= \mathbb{E}\|\delta(\mathbf{1}_{[s,t]}u)\|^p \\ &\lesssim_{p,E} \mathbb{E}\|D(\mathbf{1}_{[s,t]}u)\|^p_{\gamma(H,\gamma(H,E))} = \mathbb{E}\|\mathbf{1}_{[s,t]}(I_H(Du))\|^p_{\gamma(H,\gamma(H,E))}.\end{aligned}$$

Here, I_H is the isomorphism given in (2.4.1). Since E has type 2, also $\gamma(H,E)$ has type 2. Hence by (2.4.2)

$$\|F\|_{L^p(\Omega;\gamma(H,\gamma(H,E)))} \lesssim_E \|F\|_{L^p(\Omega;L^2(0,T;\gamma(U,\gamma(H,E))))},$$

for all $F \in L^p(\Omega\times[0,T];\gamma(U,\gamma(H,E))))$. This yields, using Hölder's inequality,

$$\begin{aligned}\|\mathbf{1}_{[s,t]}(I_H(Du))\|^p_{\gamma(H,\gamma(H,E))} &\lesssim_{p,E} \mathbb{E}\Big(\int_s^t \|I_{H,U}(D(u(r)))\|^2_{\gamma(U,\gamma(H,E))}\,dr\Big)^{\frac{p}{2}} \\ &\le |t-s|^{\frac{p}{2}-1}\mathbb{E}\Big(\int_s^t \|I_{H,U}(D(u(r)))\|^p_{\gamma(U,\gamma(H,E))}\,dr\Big) \\ &= |t-s|^{\frac{p}{2}-1}\int_s^t A(r)\,dr,\end{aligned}$$

where $A(r) := \mathbb{E}\|D(u(r))\|^p_{\gamma(H,\gamma(U,E))}$. By Fubini's theorem it follows that for all $\theta \in (0,1/2)$,

$$\begin{aligned}\mathbb{E}\int_0^T\int_0^T \frac{\|\zeta(t)-\zeta(s)\|^p_E}{|t-s|^{\theta p+1}}\,dt\,ds &\lesssim_{p,E} \Big(\int_0^T\int_s^T \frac{1}{|t-s|^{2-p(\frac12-\theta)}}\int_s^t A(r)\,dr\,dt\,ds\Big) \\ &= \int_0^T\int_s^T\int_r^T \frac{A(r)}{|t-s|^{2-p(\frac12-\theta)}}\,dt\,dr\,ds \\ &\lesssim_{p,\theta} \int_0^T\int_s^T \big(|r-s|^{p(\frac12-\theta)-1} + |T-s|^{p(\frac12-\theta)-1}\big)A(r)\,dr\,ds \\ &\lesssim_{p,\theta} \int_0^T \big(r^{p(\frac12-\theta)} + (T-r)^{p(\frac12-\theta)} + T^{p(\frac12-\theta)}\big)A(r)\,dr \\ &\lesssim_{p,\theta} T^{p(\frac12-\theta)}\int_0^T A(r)\,dr = \|D(u(r))\|^p_{L^p(\Omega\times[0,T];\gamma(H,\gamma(U,E)))} < \infty.\end{aligned}$$

Also observe that

$$\mathbb{E}\int_0^T \|\zeta(t)\|^p\,dt \lesssim_{p,E} T^{p(\frac12-\theta)}\|Du\|^p_{L^p(\Omega\times[0,T];\gamma(H,\gamma(U,E)))}.$$

It follows that (see [10, section 2]) $\zeta \in L^p(\Omega;W^{\theta,p}(0,T;E))$. If $\theta \in (1/p,1/2)$, it follows from the Sobolev embedding theorem (see [5, Theorem 4.12]) that $\zeta \in L^p(\Omega;C^{0,\gamma}(0,T;E))$ for all $0 < \lambda \le \theta - \frac1p$ and

$$\|\zeta\|_{L^p(\Omega;C^{0,\lambda}(0,T;E))} \lesssim_{E,p,\lambda,\theta,T} \|Du\|^p_{L^p(\Omega\times[0,T];\gamma(H,\gamma(U,E)))}. \tag{2.4.3}$$

In particular, ζ has a continuous version. $\square$

Corollary 2.34. *Assume the conditions of Theorem 2.33 hold. If additionally* $\mathbb{E}u \in L^p(0,T;\gamma(U,E))$*, then* ζ *has a version in* $L^p(\Omega;C^\lambda([0,T];E))$ *for all* $\lambda \in (0,\frac{1}{2}-\frac{1}{p})$*. Moreover, for every* $\lambda \in (0,\frac{1}{2}-\frac{1}{p})$ *there is a constant* C *independent of* u *such that*

$$\|\zeta\|_{L^p(\Omega;C^{0,\lambda}([0,T];E))} \leq C\|\mathbb{E}u\|_{L^p(0,T;\gamma(U,E))} + C\|Du\|_{L^p(\Omega\times[0,T];\gamma(H,\gamma(U,E)))}.$$

Proof. By the previous proof and in particular (2.4.3) it suffices to estimate the $L^p(\Omega;C^{0,\lambda}([0,T];E))$-norm of η, where $\eta : [0,T]\times\Omega \to E$ is given by $\eta(t) = \int_0^t v\,dW_U$ and $v = \mathbb{E}u$. It follows from Theorem 2.31 and (2.4.2) that for $0 \leq s < t \leq T$ one has

$$\begin{aligned}(\mathbb{E}\|\eta(t)-\eta(s)\|^p)^{1/p} &\eqsim_{E,p} \|\mathbf{1}_{[s,t]}v\|_{\gamma(H,E)} \\ &\leq \|\mathbf{1}_{[s,t]}v\|_{L^2(0,T;\gamma(U,E))} \leq |t-s|^{\frac{1}{2}-\frac{1}{p}}\|v\|_{L^p(0,T;\gamma(U,E))}.\end{aligned}$$

Therefore, as in the proof of Theorem 2.33 one has that $\eta \in L^p(\Omega;W^{\theta,p}(0,T;E))$ for all $0<\theta<1/2$ and for all $\lambda \leq \theta - \frac{1}{p}$ one has

$$\|\eta\|_{L^p(\Omega;C^{0,\lambda}([0,T];E))} \lesssim_{p,\lambda,\theta,T} \|\eta\|_{L^p(\Omega;W^{\theta,p}(0,T;E))} \lesssim_{p,E} \|v\|_{L^p(0,T;\gamma(U,E))}.$$

□

2.5 Itô's formula in the non-adapted setting

In the setting of adapted processes with values in a UMD-Banach space E, a version of Itô's formula has been obtained in [21]. A version for Banach spaces with martingale type 2 was already obtained in [92]. Below in Theorem 2.40 we present a version for the Skorohod integral for UMD spaces with type 2. For Hilbert spaces E the result can be found in [47]. Our proof follows the arguments in the scalar-valued case of Itô's formula from [95, Theorem 3.2.2].

Consider the E-valued stochastic process given by

$$\zeta_t = \zeta_0 + \int_0^t v(s)\,ds + \int_0^t u(s)\,dW_U(s).$$

where ζ_0, u and v are non-adapted, but satisfy certain smoothness assumptions. We prove an Itô formula for $F(\zeta)$, where $F : E \to \mathbb{R}$ is twice continuously Fréchet differentiable with bounded derivatives.

2.5.1 Preliminary results for Itô's formula

Next, we will prove a couple of lemmas that are used in Itô's formula. Let U be a Hilbert space, and set $H = L^2(0,T;U)$.

Lemma 2.35. *Let $\xi \in L^2(\Omega; H)$ and $h \in H$. For $N \in \mathbb{N}$, consider the partition $(t_n^N)_{n=0}^N$ of $(0,T)$, where $t_n^N = \frac{nT}{N}$. Then*

$$\mathbb{E} \sum_{n=1}^{N} \Big\langle \mathbf{1}_{[t_{n-1}^N, t_n^N]} h, \xi \Big\rangle_H^2 \to 0, \qquad N \to \infty.$$

Proof. If $h = \mathbf{1}_{[t_{k-1}^K, t_k^K]} \otimes \varphi$ with $\varphi \in U$, the result can be checked using Hölder's inequality. By linearity it extends to linear combinations of such h. For general $h \in H$ one has that

$$\mathbb{E} \sum_{n=1}^{N} \Big\langle \mathbf{1}_{[t_{n-1}^N, t_n^N]} h, \xi \Big\rangle_H^2 \leq \mathbb{E}\|\xi\|_H^2 \sum_{n=1}^{N} \|\mathbf{1}_{[t_{n-1}^N, t_n^N]} h\|_H^2 = \mathbb{E}\|\xi\|^2 \|h\|_H^2. \tag{2.5.1}$$

Therefore, the case $h \in H$ can be proved by approximation and using (2.5.1). □

Lemma 2.36. *Let (a,b) be an open interval, and consider a partition $(t_n^N)_{n=1}^N$ such that $t_{n+1}^N - t_n^N \to 0$ as $N \to \infty$ for all n. If $u_1, u_2 \in U$, then for all $p \in [1, \infty)$, $\lim_{N\to\infty} \xi_N = (b-a) \langle u_1, u_2 \rangle$ in $L^2(\Omega)$ as $N \to \infty$., where*

$$\xi_N = \sum_{n=1}^{N} ((W_U(t_{n+1}^N) - W_U(t_n^N))u_1)((W(t_{n+1}^N) - W_U(t_n^N))u_2), \quad N \geq 1.$$

Proof. Since $\mathbb{E}\xi_N = (b-a)\langle u_1, u_2\rangle$, it suffices to shows that $\lim_{N\to\infty} \mathbb{E}\xi_N^2 = (b-a)^2 |\langle u_1, u_2 \rangle|^2$. This follows from a straightforward computation. □

The next result will be presented and needed only for dyadic partitions, but actually holds for more general partitions.

Theorem 2.37. *Let U be a separable Hilbert space and E a UMD Banach space with type 2. Set $t_i^n = \frac{iT}{2^n}$ for $n \geq 1$ and $i = 0, 1, \ldots, 2^n$. For each $n \geq 1$ and $i = 0, 1, \ldots, 2^n$, let $\sigma_i^n \in [t_i^n, t_{i+1}^n]$. Let $Z, Z^1, Z^2, \ldots : [0,T] \times \Omega \to \mathscr{L}(E, E^*)$ be processes and assume that*

(i) All processes $Z, Z^1, Z^2, \ldots$ have continuous paths.
(ii) Pointwise on Ω one has $\lim_{n\to\infty} \sup_{t\in[0,T]} \|Z(t) - Z^n(t)\|_{\mathscr{L}(E,E^*)} = 0$.
(iii) There is a $C > 0$ such that for all $t \in [0,T]$ and $\omega \in \Omega$, one has $\|Z^n(t,\omega)\|_{\mathscr{L}(E,E^*)} \leq C$.

Then for $u, v \in L^2(0,T; \mathbb{D}^{1,2}(\gamma(U,E)))$ one has

$$\begin{aligned} &\sum_{i=0}^{2^n-1} \Big\langle \int_{t_i^n}^{t_{i+1}^n} u(s)\, dW_U(s), Z^n(\sigma_i^n) \int_{t_i^n}^{t_{i+1}^n} v(s)\, dW_U(s) \Big\rangle \\ &\qquad \to \int_0^T \langle u(s), Z(s)v(s)\rangle_{\mathrm{Tr}}\, ds \quad \text{ in } L^1(\Omega) \text{ as } n \to \infty. \end{aligned} \tag{2.5.2}$$

Proof. For notational convenience, let $X = L^2(0,T;\mathbb{D}^{1,2}(\gamma(U,E)))$ in the proof below. Fix $n \geq 1$. Let $\Phi_n : X \times X \to L^1(\Omega)$ be given by

$$\Phi_n(u,v) = \sum_{i=0}^{2^n-1} \Big\langle \int_{t_i^n}^{t_{i+1}^n} u(s)\, dW_U(s), Z^n(\sigma_i^n) \int_{t_i^n}^{t_{i+1}^n} v(s)\, dW_U(s) \Big\rangle.$$

Observe that the stochastic integrals are well-defined in $L^2(\Omega;E)$ by Proposition 2.23 and (2.4.2). Let $\Phi : X \times X \to L^1(\Omega)$ be given by

$$\Phi(u,v) = \int_0^T \langle u(s), Z(s)v(s)\rangle_{\mathrm{Tr}}\, ds$$

This is well-defined by Lemma 2.3 and the remarks below it. For proof of (2.5.2) it suffices to show that $\lim_{n\to\infty} \Phi_n(u,v) = \Phi(u,v)$ in $L^r(\Omega)$. For this we proceed in four steps below.

Step 1: Uniform boundedness of the bilinear operator Φ_n.

We first show that there exists an $M \geq 0$ such that for all $u, v \in X$ and for all $n \geq 1$ one has

$$\|\Phi_n(u,v)\|_{L^1(\Omega)} \leq M\|u\|_X\|v\|_X. \tag{2.5.3}$$

One has

$$\|\Phi_n(u,v)\| \leq C \sum_{i=0}^{2^n-1} \Big\| \int_{t_i^n}^{t_{i+1}^n} u(s)\, dW_U(s)\Big\|\Big\| \int_{t_i^n}^{t_{i+1}^n} v(s)\, dW_U(s)\Big\|.$$

Therefore, with $\|\cdot\|_1 = \|\cdot\|_{L^1(\Omega)}$,

$$\begin{aligned}\|\Phi_n(u,v)\|_1 &\leq C\Big(\mathbb{E}\sum_{i=0}^{2^n-1} \Big\| \int_{t_i^n}^{t_{i+1}^n} u(s)\, dW_U(s)\Big\|^2\Big)^{\frac{1}{2}} \\ &\qquad \times \Big(\mathbb{E}\sum_{i=0}^{2^n-1} \Big\| \int_{t_i^n}^{t_{i+1}^n} v(s)\, dW_U(s)\Big\|^2\Big)^{\frac{1}{2}} \\ &\overset{(i)}{\lesssim_E} \sum_{i=0}^{2^n-1} \|\mathbf{1}_{[t_i^n,t_{i+1}^n]}u\|_{\mathbb{D}^{1,2}(\gamma(H,E))} \sum_{i=0}^{2^n-1} \|\mathbf{1}_{[t_i^n,t_{i+1}^n]}v\|_{\mathbb{D}^{1,2}(\gamma(H,E))} \\ &\overset{(ii)}{\lesssim_E} \sum_{i=0}^{2^n-1} \|\mathbf{1}_{[t_i^n,t_{i+1}^n]}u\|_X \sum_{i=0}^{2^n-1} \|\mathbf{1}_{[t_i^n,t_{i+1}^n]}v\|_X = \|u\|_X\|v\|_X.\end{aligned}$$

Here (i) follows from Proposition 2.23, and (ii) follows from (2.4.2) and the fact that E has type 2.

Step 2: Boundedness of the bilinear operator Φ.

As in Step 1, there exists an $M \geq 0$ such that for all $u, v \in X$ one has

$$\|\Phi(u,v)\|_{L^1(\Omega)} \leq M\|u\|_X\|v\|_X. \tag{2.5.4}$$

By Lemma 2.3, one has

$$\begin{aligned}|\langle u(s), Z(s)v(s)\rangle_{\mathrm{Tr}}| &\le \|u(s)\|_{\gamma(U,E)}\|Z(s)v(s)\|_{\gamma(U,E^*)}\\ &\le C\|u(s)\|_{\gamma(U,E)}\|v(s)\|_{\gamma(U,E^*)}.\end{aligned}$$

Hence

$$\|\Phi(u,v)\|_{L^1(\Omega)} \le \int_0^T C\|u(s)\|_{L^2(\Omega;\gamma(U,E))}\|v(s)\|_{L^2(\Omega;\gamma(U,E^*))}\,ds \le C\|u\|_X\|v\|_X.$$

Now (2.5.4) follows.

Step 3: Reduction to simple functions of smooth processes.

Let $(e_n)_{n=1}^\infty$ denote an orthonormal basis for U. Note that the following functions form a dense subset of X.

$$\sum_{j=0}^{2^p-1} \mathbf{1}_{[t_j^p,t_{j+1}^p]} \otimes g_j(W(e_1),\dots,W(e_M)) \otimes (\sum_{l=1}^{L} \psi_{jl}\otimes y_{jl}), \tag{2.5.5}$$

where the g_j's are smooth, $\psi_{jl}\in U$ and $y_{jl}\in E$ for $1\le l\le L$ and $1\le j\le 2^p-1$. Now assume (2.5.2) holds for all functions u and v of the form (2.5.5). We will show that the general case with $u,v\in X$, follows from this by a continuity argument.

Let $u,v\in X$ be arbitrary. Fix $\varepsilon\in(0,1)$. Define $\tilde M=\max\{\|u\|_X,\|v\|_X\}+1$. Choose $\tilde u,\tilde v$ of the form (2.5.5) and such that

$$\|u-\tilde u\|_X<\varepsilon/(M\tilde M), \quad \|v-\tilde v\|_X<\varepsilon/M\tilde M.$$

By (2.5.3) and using the bilinearity of Φ_n and writing $\|\cdot\|_1=\|\cdot\|_{L^1(\Omega)}$ one obtains

$$\begin{aligned}\|\Phi_n(u,v)-\Phi_n(\tilde u,\tilde v)\|_1 &\le \|\Phi_n(u,v-\tilde v)\|_1+\|\Phi_n(u-\tilde u,\tilde v)\|_1\\ &\le M\|u\|_X\|v-\tilde v\|_X+M\|u-\tilde u\|_X\|\tilde v\|_X\le 2\varepsilon.\end{aligned}$$

In a similar way one sees that $\|\Phi(u,v)-\Phi(\tilde u,\tilde v)\|_{L^1(\Omega)}\le 2\varepsilon$. It follows that

$$\begin{aligned}&\|\Phi_n(u,v)-\Phi(u,v)\|_1\\ &\quad\le \|\Phi_n(u,v)-\Phi_n(\tilde u,\tilde v)\|_1+\|\Phi_n(\tilde u,\tilde v)-\Phi(\tilde u,\tilde v)\|_1+\|\Phi(\tilde u,\tilde v)-\Phi_n(u,v)\|_1\\ &\quad\le 4\varepsilon+\|\Phi_n(\tilde u,\tilde v)-\Phi(\tilde u,\tilde v)\|_1.\end{aligned}$$

Therefore, taking the lim sup in the above estimate and using (2.5.2) for $\tilde u$ and $\tilde v$ one obtains that

$$\limsup_{n\to\infty}\|\Phi_n(u,v)-\Phi(u,v)\|_1\le 4\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, it follows that $\lim_{n\to\infty}\Phi_n(u,v)=\Phi(u,v)$ in $L^1(\Omega)$. This yields the result.

Step 4: Convergence for simple functions of smooth processes.

We next prove (2.5.2) for u and v of the form (2.5.5). By linearity it then suffices to consider u and v of the form

$$\begin{aligned}u &= f(W(e_1),\dots,W(e_M))\otimes((\mathbf{1}_{[a,b]}\otimes\varphi)\otimes x_1),\\ v &= g(W(e_1),\dots,W(e_M))\otimes((\mathbf{1}_{[c,d]}\otimes\psi)\otimes x_2),\end{aligned}$$

for some $\varphi,\psi\in U$, $x_1,x_2\in E$ and some dyadic intervals $[a,b]$ and $[c,d]$ and smooth $f,g:\mathbb{R}^M\to\mathbb{R}$. By homogeneity we can assume that $\|\varphi\|_U=\|\psi\|_U=1$. Moreover, we can assume $a=c$ and $b=d$. Indeed, if $(a,b)\cap(c,d)=\varnothing$, then both sides of (2.5.2) vanish for n large enough. If $(a,b)\cap(c,d)\neq\varnothing$, then we can write u and v as a sum of smaller dyadic intervals which are either identical or disjoint. Furthermore, for notational convenience we assume that $[a,b]=[0,T]$.

Let $m\le n$ and for $i=0,1,\dots,n$ let us denote by $t_i^{(m)}$ the point of the m-th partition that is closest to t_i^n from the left. For each n,m and j, let $S_j^{n,m}=\{i: t_i^n\in[t_j^m,t_{j+1}^m)\}$. Then

$$\begin{aligned}&\Big|\sum_{i=0}^{2^n-1}\Big\langle\int_{t_i^n}^{t_{i+1}^n}u\,dW_U, Z^n(\sigma_i^n)\int_{t_i^n}^{t_{i+1}^n}v\,dW_U)\Big\rangle-\int_0^t\langle u(s),Z(s)v(s)\rangle_{\mathrm{Tr}}\,ds\Big|\\
&\le\Big|\sum_{i=0}^{2^n-1}\Big\langle\int_{t_i^n}^{t_{i+1}^n}u\,dW_U,(Z^n(\sigma_i^n)-Z(t_i^{(m)}))\int_{t_i^n}^{t_{i+1}^n}v\,dW_U\Big\rangle\Big|\\
&\quad+\Big|\sum_{j=0}^{2^m-1}\sum_{i\in S_j^{n,m}}\Big\langle\int_{t_i^n}^{t_{i+1}^n}u\,dW_U,Z(t_j^m)\int_{t_i^n}^{t_{i+1}^n}v\,dW_U\Big\rangle\\
&\qquad\qquad-\int_{t_j^m}^{t_{j+1}^m}\big\langle u(s),Z(t_j^m)v(s)\big\rangle_{\mathrm{Tr}}\,ds\Big|\\
&\quad+\Big|\sum_{j=0}^{2^m-1}\int_{t_j^m}^{t_{j+1}^m}\big\langle u(s),Z(t_j^m)v(s)\big\rangle_{\mathrm{Tr}}\,ds-\int_0^t\big\langle u(s),Z(s)v(s)\big\rangle_{\mathrm{Tr}}\,ds\Big|\\
&=a_1+a_2+a_3.\end{aligned}$$

For the term a_3, pointwise in Ω one can estimate

$$\begin{aligned}a_3&=\Big|\sum_{j=0}^{2^m-1}\int_{t_j^m}^{t_{j+1}^m}\big\langle u(s),(Z(t_j^m)-Z(s))v(s)\big\rangle_{\mathrm{Tr}}\,ds\Big|\\
&\le\sum_{j=0}^{2^m-1}\int_{t_j^m}^{t_{j+1}^m}\|u(s)\|_{\gamma(U,E)}\|v(s)\|_{\gamma(U,E)}\|Z(t_j^m)-Z(s)\|_{\mathscr{L}(E,E^*)}\,ds\\
&\le\sup_{|s-r|\le T2^{-m}}\|Z(r))-Z(s)\|_{\mathscr{L}(E,E^*)}\,\|u\|_{L^2(0,T;\gamma(U,E))}\|v\|_{L^2(0,T;\gamma(U,E))},\end{aligned}$$

The latter converges to zero in $L^1(\Omega)$ as $n \to \infty$ and then $m \to \infty$ by the path-continuity of Z.

For a_1 pointwise in Ω one can estimate

$$\begin{aligned} a_1 &\le \sup_{|s-r|\le T2^{-m}} \|Z^n(r) - Z(s)\| \sum_{i=0}^{2^n-1} \|\delta(\mathbf{1}_{[t_i^n,t_{i+1}^n]}u)\|\|\delta(\mathbf{1}_{[t_i^n,t_{i+1}^n]}v)\| \\ &\le \sup_{|s-r|\le T2^{-m}} \|Z^n(r) - Z(s)\|\Big(\sum_{i=0}^{2^n-1} \|\delta(\mathbf{1}_{[t_i^n,t_{i+1}^n]}u)\|^2 + \sum_{i=0}^{2^n-1} \|\delta(\mathbf{1}_{[t_i^n,t_{i+1}^n]}v)\|^2\Big). \end{aligned}$$

Define the uniformly bounded sequence of random variables (z_{nm}) and (z_m) by

$$z_{nm} = \sup_{|s-r|\le T2^{-m}} \|Z^n(r) - Z(s)\|, \qquad z_m = \sup_{|s-r|\le T2^{-m}} \|Z(r) - Z(s)\|.$$

Also let the random variables $(q_n(u))_{n\ge1}$ and $q(u)$ be given by

$$q_n(u) = \sum_{i=0}^{2^n-1} \|\delta(\mathbf{1}_{[t_i^n,t_{i+1}^n]}u)\|^2, \qquad q(u) = \int_0^t \|u(s)\|^2\,ds.$$

We find

$$\begin{aligned} a_1 &\le z_{nm}(q_n(u) + q_n(v)) \\ &\le z_m(q(u) + q(v)) + |z_{nm} - z_m|(q(u) + q(v)) \\ &\qquad + z_{nm}\big(|q_n(u) - q(u)| + |q_n(v) - q(v)|\big) \end{aligned}$$

Since the ranges of u and v are one-dimensional in E we can apply [95, Theorem 3.2.1] to obtain $\lim_{n\to\infty} q_n(u) = q(u)$ in $L^1(\Omega)$ and similarly for v. Clearly, pointwise on Ω, $\lim_{n\to\infty} z_{mn} = z_m$. Letting $n \to \infty$, the dominated convergence theorem gives that

$$\limsup_{n\to\infty} \mathbb{E}a_1 \le \mathbb{E}(z_m(q(u) + q(v))).$$

Now letting $m \to \infty$ and again applying the dominated convergence theorem, we can conclude that $\lim_{m\to\infty} \lim_{n\to\infty} a_1 = 0$ in $L^1(\Omega)$.

We finish the proof once we have shown that $a_2 \to 0$ in $L^1(\Omega)$. For the moment, fix j. Let us calculate the second part of the summand of a_2:

$$\begin{aligned} &\int_{t_j^m}^{t_{j+1}^m} \big\langle u(s), Z(t_j^m)v(s)\big\rangle_{\mathrm{Tr}}\; ds \\ &\quad = \int_{t_j^m}^{t_{j+1}^m} \sum_{k=1}^{\infty} \big\langle f \otimes \langle h_1(s), e_k\rangle\, x_1, Z(t_j^m)(g \otimes \langle h_2(s), e_k\rangle\, x_2)\big\rangle_{E,E^*}\; ds \\ &\quad = fg\,\big\langle x_1, Z(t_j^m)x_2\big\rangle \int_{t_j^m}^{t_{j+1}^m} \langle h_1(s), h_2(s)\rangle\; ds, \end{aligned}$$

where $h_1 := \mathbf{1}_{[0,t]} \otimes \varphi$, $h_2 := \mathbf{1}_{[0,t]} \otimes \psi$. Let us compute the first part of a_2. For every $n \in \mathbb{N}$, consider an orthonormal basis $(\tilde{h}_i)_{i=0}^{\infty}$ such that

$$\tilde{h}_i = \frac{1}{\sqrt{t_{i+1}^n - t_i^n}} \mathbf{1}_{[t_i^n, t_{i+1}^n]} \otimes \varphi, \ i = 1, 2, \ldots, 2^{n+1}.$$

Then by Lemma 2.22 one has

$$\int_{t_i^n}^{t_{i+1}^n} u \, dW_U = \Delta_i^n W \varphi f - \langle \mathbf{1}_{[t_i^n, t_{i+1}^n]} \varphi, Df \rangle_H) x_1.$$

where $\Delta_i^n W = W_U(t_{i+1}^n) - W_U(t_i^n)$. A similar identity holds for the truncated Skorohod integral of v. Therefore, one obtains

$$\begin{aligned}
&\sum_{i \in S_j^{n,m}} \Big\langle \int_{t_i^n}^{t_{i+1}^n} u \, dW_U, Z(t_j^m) \int_{t_i^n}^{t_{i+1}^n} v \, dW_U \Big\rangle \\
&= \sum_{i \in S_j^{n,m}} (\Delta_i^n W \varphi f - \langle \mathbf{1}_{[t_i^n, t_{i+1}^n]} \varphi, Df \rangle_H) \\
&\qquad \times (\Delta_i^n W \psi g - \langle \mathbf{1}_{[t_i^n, t_{i+1}^n]} \psi, Dg \rangle_H) \big\langle x_1, Z(t_j^m) x_2 \big\rangle \\
&=: \sum_{i \in S_j^{n,m}} (A_i - B_i)(C_i - D_i) \big\langle x_1, Z(t_j^m) x_2 \big\rangle
\end{aligned}$$

Thus the convergence would follow if

$$\sum_{j=0}^{2^m - 1} \big\langle x_1, Z(t_j^m) x_2 \big\rangle \Big[\sum_{i \in S_j^{n,m}} (A_i - B_i)(C_i - D_i) - fg \int_{t_j^m}^{t_{j+1}^m} \langle h_1(s), h_2(s) \rangle \ ds \Big]$$

converges to 0 in $L^1(\Omega)$ as $n \to \infty$ and then $m \to \infty$. Pointwise on Ω, the above expression is dominated by

$$C \|x_1\| \|x_2\| \Big| \sum_{i=0}^{2^n - 1} (A_i - B_i)(C_i - D_i) - fg \int_0^t \langle h_1(s), h_2(s) \rangle \ ds \Big|$$

Now it suffices to prove that

$$\mathbb{E} \Big| \sum_{i=0}^{2^n - 1} (A_i - B_i)(C_i - D_i) - fg \int_0^t \langle h_1(s), h_2(s) \rangle \ ds \Big| \to 0, \qquad (2.5.6)$$

as $n \to \infty$. To prove (2.5.6) note that with $\| \cdot \|_1 = \| \cdot \|_{L^1(\Omega)}$ one has

$$
\begin{aligned}
\Big\| \sum_{i=0}^{2^n-1} (A_i - B_i)(C_i - D_i) - fg \int_0^t \langle h_1(s), h_2(s)\rangle \, ds \Big\|_1 \\
\leq \Big\| \sum_{i=0}^{2^n-1} A_i C_i - fg \int_0^t \langle h_1(s), h_2(s)\rangle \, ds \Big\|_1 \\
+ \sum_{i=0}^{2^n-1} \|A_i D_i\|_1 + \|B_i C_i\|_1 + \|B_i D_i\|_1.
\end{aligned}
$$

We will show this $L^1(\Omega)$-convergence by showing the convergence for each of the components separately. First,

$$
\begin{aligned}
\mathbb{E} \sum_{i=0}^{2^n-1} |B_i D_i| \leq \Big(\mathbb{E} \sum_{i=1}^{2^n-1} \Big\langle \mathbf{1}_{[t_i^n, t_{i+1}^n]} h_1, Df \Big\rangle^2 \Big)^{\frac{1}{2}} \\
\Big(\mathbb{E} \sum_{i=0}^{2^n-1} \Big\langle \mathbf{1}_{[t_i^n, t_{i+1}^n]} h_2, Dg \Big\rangle^2 \Big)^{\frac{1}{2}} \to 0,
\end{aligned}
$$

by Lemma 2.35. By the same lemma and the properties of W one sees,

$$
\begin{aligned}
\sum_{i=0}^{2^n-1} \mathbb{E}|A_i D_i| &= \sum_{i=0}^{2^n-1} \mathbb{E}\Big| \Delta_i^n W \varphi f \cdot \Big\langle \mathbf{1}_{[t_i^n, t_{i+1}^n]} h_2, Dg \Big\rangle_H \Big| \\
&\leq \|f\|_\infty \Big(\sum_{i=0}^{2^n-1} \mathbb{E}((\Delta_i^n W)\varphi)^2 \Big)^{\frac{1}{2}} \Big(\sum_{i=0}^{2^n-1} \mathbb{E} \Big\langle \mathbf{1}_{[t_i^n, t_{i+1}^n]} h_2, Dg \Big\rangle_H^2 \Big)^{\frac{1}{2}} \\
&\leq \|f\|_\infty \sqrt{T} \Big(\sum_{i=0}^{2^n-1} \mathbb{E} \Big\langle \mathbf{1}_{[t_i^n, t_{i+1}^n]} h_2, Dg \Big\rangle_H^2 \Big)^{\frac{1}{2}} \to 0,
\end{aligned}
$$

and similarly $\mathbb{E} \sum_{i=1}^{2^n-1} |B_i C_i| \to 0$. By Lemma 2.36 one has

$$
\begin{aligned}
\mathbb{E}\Big| \sum_{i=0}^{2^n-1} A_i C_i - fg \int_0^t \langle h_1(s), h_2(s)\rangle \, ds \Big| \\
\leq \|f\|_\infty \|g\|_\infty \Big(\mathbb{E} \Big(\sum_{i=0}^{2^n-1} (\Delta_i^n W)(\varphi)(\Delta_i^n W)(\psi) - T \langle \varphi, \psi \rangle \Big)^2 \Big)^{\frac{1}{2}} \to 0
\end{aligned}
$$

as $n \to \infty$. Hence (2.5.6) follows. □

Let E be a UMD space with type 2. Consider the following assumptions:

$$
\begin{aligned}
&\zeta_0 \in \mathbb{D}^{1,2}(E), && D\zeta_0 \in L^2(\Omega; L^2(0,T;\gamma(U,E))) \\
&u \in \mathbb{D}^{2,2}(L^2(0,T;\gamma(U,E))), && Du \in L^2(0,T;\mathbb{D}^{1,2}(\gamma(U,\gamma(H,E)))), \quad (2.5.7) \\
&v \in \mathbb{D}^{1,2}(L^2(0,T;E)), && Dv \in L^1(0,T;L^2((0,T)\times\Omega;\gamma(U,E))).
\end{aligned}
$$

Note the following two observations regarding the assumptions:

(1) Clearly, $D\zeta_0$ is in $L^2(\Omega;\gamma(H,E))$ whenever $\zeta_0 \in D^{1,2}(E)$. Note that the assumption (2.5.7) states that $D\zeta_0 \in L^2(\Omega;L^2(0,T;\gamma(U,E)))$. The latter space is smaller due to the type 2 condition. The same applies to Du and Dv.
(2) If E is a Hilbert space, (2.5.7) is equivalent to the assumptions $\zeta_0 \in \mathbb{D}^{1,2}(E)$, $u \in \mathbb{D}^{2,2}(\gamma(H,E))$, and $v \in \mathbb{D}^{1,2}(L^2(0,T;E))$.

Let $\zeta : [0,T]\times\Omega \to E$ defined by

$$\zeta_t = \zeta_0 + \int_0^t v(r)dr + \int_0^t u(r)\, dW_U(r). \tag{2.5.8}$$

Observe for each $t\in[0,T]$, $\zeta(t) \in L^2(\Omega;E)$ is well-defined, by Proposition 2.23. Moreover,

$$\begin{aligned}\sup_{t\in[0,T]} \|\zeta(t)\|_{L^2(\Omega;E)} &\lesssim_E \|\zeta_0\|_{L^2(\Omega;E)} + \|v\|_{L^1(0,T;L^2(\Omega;E))} + \|u\|_{\mathbb{D}^{1,2}(\gamma(H,E))}\\ &\le \|\zeta_0\|_{L^2(\Omega;E)} + \|v\|_{L^1(0,T;L^2(\Omega;E))} + \|u\|_{\mathbb{D}^{1,2}(L^2(0,T;\gamma(U,E)))},\end{aligned}$$

where we used (2.4.2) in the last step. In the next lemma we discuss differentiability properties of ζ.

Lemma 2.38. *Let E be a* UMD *Banach space with type 2. Assume that* (2.5.7) *holds and let ζ be as in* (2.5.8)*. Set $Y = L^2(\Omega;L^2(0,T;\gamma(U,E)))$. Then for each $t\in[0,T]$, $\zeta(t)\in\mathbb{D}^{1,2}(E)$, $D(\zeta(t))\in Y$ and*

$$(D\zeta(t))(s) = (D\zeta_0)(s) + \int_0^t (Dv(r))(s)dr + \mathbf{1}_{[0,t]}(s)u(s) + \int_0^t D(u(r))(s)\, dW_U(r),$$

$$\begin{aligned}\sup_{t\in[0,T]} \|D\zeta(t)\|_Y \lesssim_{p,E} \|D\zeta_0\|_Y + \|Dv\|_{L^1(0,T;Y)} + \|u\|_Y\\ + \|Du\|_{L^2(0,T;\mathbb{D}^{1,2}(\gamma(U,\gamma(H,E))))},\end{aligned}$$

Proof. The fact that $\zeta(t)\in\mathbb{D}^{1,2}(\Omega;E)$ and the first identity follow from Proposition 2.24. The estimate follows from Proposition 2.23. □

Define $D^-\zeta$ as the element in $Y = L^2(\Omega;L^2(0,T;\gamma(U,E)))$ given by

$$(D^-\zeta)(s) = (D\zeta_0)(s) + \int_0^s (Dv(r))(s)dr + \delta(\mathbf{1}_{[0,s]} I_{U,H}((Du)(s))),$$

where $I_{U,H}$ is defined as in (2.4.1). In the scalar case a more general definition of D^- is given in [95, p. 173]. For processes of the form (2.5.8), these definitions coincide (see [95, Proposition 3.1.1]). Observe that the last term in (2.5.1) can be written as $\int_0^s D(u(r))(s)\, dW_U(r)$. By our assumptions, this term is well-defined for almost all $s\in[0,T]$, and by continuity of δ, we obtain

$$\|s \mapsto \delta(\mathbf{1}_{[0,s]} I_{U,H}((Du)(s)))\|_Y \le C\|Du\|_{L^2(0,T;\mathbb{D}^{1,2}(\gamma(H,\gamma(U,E))))}.$$

Therefore, as in Lemma 2.38 one has

$$\|D^-\zeta\|_Y \lesssim_{p,E} \|D\zeta_0\|_Y + \|Dv\|_{L^1(0,T;Y)} + \|Du\|_{L^2(0,T;\mathbb{D}^{1,2}(\gamma(H,\gamma(U,E))))}. \tag{2.5.9}$$

Lemma 2.39. *Let E be a UMD Banach space with type 2. Assume that* (2.5.7) *holds and let ζ be as in* (2.5.8). *Suppose that $Z : \Omega \times [0,T] \to \mathscr{L}(E, E^*)$ is bounded, and has continuous paths. Let $w \in L^2(0,T; L^2(\Omega; \gamma(U,E)))$ and $D^-\zeta$ as in (2.5.1). If we fix $t \in [0,T]$ and set $t_i^n := \frac{it}{2^n}$ for $i = 0, 1, \dots, 2^n$, then*

$$\sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \langle w(s), Z(t_i^n) D(\zeta(t_i^n))(s)\rangle_{\mathrm{Tr}}\, ds \to \int_0^t \big\langle w(s), Z(s)((D^-\zeta)(s))\big\rangle_{\mathrm{Tr}}\, ds,$$

in $L^1(\Omega)$, as $n \to \infty$.

Proof. Let

$$G(\zeta_0, v, u) = \|D\zeta_0\|_Y + \|Dv\|_{L^1(0,T;Y)} + \|Du\|_{L^2(0,T;\mathbb{D}^{1,2}(\gamma(H,\gamma(U,E))))}$$

Let $\|Z\|_\infty = \sup_{s\in[0,T],\omega\in\Omega} \|Z(s,\omega)\|$, $\eta = D\zeta$ and $\xi = D^-\zeta$. By Lemma 2.38, $\langle w, Z(t_i^n)(\eta(t_i^n))(s)\rangle_{\mathrm{Tr}}$ is well-defined a.e. in $(0,T)\times\Omega$. Moreover, one has

$$\begin{aligned}
\mathbb{E} \sum_{i=0}^{2^n-1} & \int_{t_i^n}^{t_{i+1}^n} |\langle w, Z(t_i^n)(\eta(t_i^n))(s)\rangle_{\mathrm{Tr}}|\, ds \\
&\le \sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \mathbb{E}\|w(s)\|_{\gamma(U,E)} \|Z(t_i^n)(\eta(t_i^n))(s)\|_{\gamma(U,E^*)}\, ds \\
&\le C\|Z\|_\infty \|w\|_Y (G(\zeta_0, v, u) + \|u\|_Y).
\end{aligned}$$

Similarly, by (2.5.9), $\langle w(s), Z(s)((D^-\zeta)(s))\rangle_{\mathrm{Tr}}$ is well-defined a.e. and

$$\mathbb{E}\int_0^t |\big\langle w(s), Z(s)((D^-\zeta)(s))\big\rangle_{\mathrm{Tr}}|\, ds \lesssim_{p,E} \|Z\|_\infty \|w\|_Y\, G(\zeta_0, v, u).$$

Now observe that

$$\begin{aligned}
\mathbb{E}\Big| & \sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \langle w(s), Z(t_i^n)(D(\zeta(t_i^n))(s))\rangle_{\mathrm{Tr}}\, ds \\
&\qquad - \int_0^t \big\langle w(s), Z(s)((D^-\zeta)(s))\big\rangle_{\mathrm{Tr}}\, ds\Big| \\
&\le \mathbb{E}\Big| \sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \big\langle w(s), Z(t_i^n)((D(\zeta(t_i^n)))(s) - (D^-\zeta)(s))\big\rangle_{\mathrm{Tr}}\, ds\Big| \\
&\quad + \mathbb{E}\Big| \sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \big\langle w(s), (Z(t_i^n) - Z(s))(D^-\zeta)(s)\big\rangle_{\mathrm{Tr}}\, ds\Big| = T_{1,n} + T_{2,n}
\end{aligned}$$

Let $\theta_n = \sup_{|r-s|<T2^{-n}} \|Z(r) - Z(s)\|_{\mathscr{L}(E,E^*)}$. Note that pointwise in Ω, $\lim_{n\to\infty} \theta_n = 0$ and $\theta_n \le 2\|Z\|_\infty$. For each $n \ge 1$ one has

$$\begin{aligned} T_{2,n} &\le \mathbb{E}\Big(\theta_n \int_0^t \|w(s)\|_{\gamma(U,E)} \, \|(D^-\zeta)(s)\|_{\gamma(U,E)} \, ds\Big) \\ &\le \mathbb{E}(\theta_n \|w\|_{L^2(0,T;\gamma(U,E))} \|D^-\zeta\|_{L^2(0,T;\gamma(U,E))}). \end{aligned}$$

Since $\|w\|_{L^2(0,T;\gamma(H,U))} \|D^-\zeta\|_{L^2(0,T;\gamma(H,U))} \in L^1(\Omega)$, it follows from the dominated convergence theorem and the properties of $(\theta_n)_{n\ge1}$, that $\lim_{n\to\infty} T_{2,n} = 0$.

It follows from Lemma 2.38 that

$$\begin{aligned} T_{1,n} &\le \|Z\|_\infty \mathbb{E} \sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \|w(s)\|_{\gamma(U,E)} \|D(\zeta(t_i^n))(s) - (D^-\zeta)(s)\|_{\gamma(U,E)} \, ds \\ &\le \|Z\|_\infty \|w\|_Y \Big(\sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \mathbb{E}\|D(\zeta(t_i^n))(s) - (D^-\zeta)(s)\|^2_{\gamma(U,E)} \, ds\Big)^{1/2} \\ &\le \|Z\|_\infty \|w\|_Y \Big(\sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \mathbb{E}\Big| \int_{t_i^n}^{s} \|(Dv(r))(s)\|_{\gamma(U,E)} dr\Big|^2 \, ds\Big)^{1/2} \\ &\quad + \|Z\|_\infty \|w\|_Y \Big(\sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \mathbb{E}\|\delta(\mathbf{1}_{[t_i^n,s]}(Du)(s))\|^2_{\gamma(U,E)}\Big)^{1/2} \\ &= S_{1,n} + S_{2,n}. \end{aligned}$$

For $S_{1,n}$ one has

$$S_{1,n} \le \|Z\|_\infty \|w\|_Y \int_0^t \Big(\Big| \int_0^t \mathbf{1}_{|r-s|<T2^{-n}} \mathbb{E}\|(Dv(r))(s)\|^2_{\gamma(U,E)} \, ds\Big)^{1/2} dr.$$

The latter converges to zero by the dominated convergence theorem and the assumption on v. For $S_{2,n}$ one has

$$\begin{aligned} S_{2,n} &\le \|Z\|_\infty \|w\|_Y \Big(\sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \|\mathbf{1}_{B(s,T2^{-n})}(Du)(s)\|^2_{\mathbb{D}^{1,2}(\gamma(H,\gamma(U,E)))} \, ds\Big)^{1/2} \\ &= \|Z\|_\infty \|w\|_Y \Big(\int_0^t \|\mathbf{1}_{B(s,T2^{-n})}(Du)(s)\|^2_{\mathbb{D}^{1,2}(\gamma(H,\gamma(U,E)))} \, ds\Big)^{1/2}. \end{aligned}$$

The latter converges to zero by the dominated convergence theorem and the assumption on u. □

2.5.2 Formulation and proof of Itô's formula

Theorem 2.40 (Itô's formula). *Let E be a* UMD *Banach space with type* 2. *Suppose that the conditions* (2.5.7) *hold and let $\zeta : [0,T] \times \Omega \to E$ be as in*

(2.5.8). *Assume ζ has continuous paths. Let $F : E \to \mathbb{R}$ be a twice continuously Fréchet differentiable function. Suppose that F' and F'' are bounded. Then*

$$\begin{aligned} F(\zeta_t) = F(\zeta_0) &+ \int_0^t F'(\zeta_s)(v(s))\,ds + \delta(\big\langle F'(\zeta), \mathbf{1}_{[0,t]}u\big\rangle) \\ &+ \frac{1}{2}\int_0^t \langle u(s), F''(\zeta_s)(u(s))\rangle_{\mathrm{Tr}}\,ds + \int_0^t \big\langle u(s), F''(\zeta_s)((D^-\zeta)(s))\big\rangle_{\mathrm{Tr}}\,ds. \end{aligned} \tag{2.5.10}$$

Note that the term with D^- is an additional term which is not present in the adapted setting. A similar result in the case that E is a Hilbert space can be found in [47]. Our proof is based on the ideas in [95, Theorem 3.2.2].

Remark 2.41.

1. If F' and F'' are not bounded, one can usually approximate F with a sequence of functions that does satisfy the smoothness and boundedness conditions. In particular, such a procedure works in the important case where $F : E \to \mathbb{R}$, where $F(x) = \|x\|^s$, $s \geq 2$ and E is an L^q-space with $q \geq 2$.
2. If the condition (2.5.7) is strengthened to

$$\begin{aligned} \zeta_0 &\in \mathbb{D}^{1,p}(E), & D\zeta_0 &\in L^p(\Omega; L^2(0,T;\gamma(U,E))) \\ u &\in \mathbb{D}^{1,p}(L^2(0,T;\gamma(U,E))), & Du &\in L^p(0,T;\mathbb{D}^{1,p}(\gamma(H,\gamma(U,E)))), \\ v &\in \mathbb{D}^{1,p}(L^2(0,T;E)), & Dv &\in L^1(0,T;L^p(\Omega;L^2(0,T;\gamma(U,E)))), \end{aligned}$$

   for some $p > 2$, then by Lemma 2.15 and Proposition 2.17 one actually has $\big\langle F'(\zeta), \mathbf{1}_{[0,t]}u\big\rangle) \in \mathbb{D}^{1,p/2}(H)$.
3. Using Theorem 2.27 in the same way as in Proposition 2.17 one could extend the result to functions $F : E \to E_1$, where E_1 another UMD Banach space. However, in that case the traces have to be extended to the vector-valued setting as well.
4. Sufficient conditions for the existence of a continuous version of ζ can be found in Theorem 2.33.

Proof. Set $t_i^n = \frac{it}{2^n}$, $0 \leq i \leq 2^n$. Consider the Taylor expansion of $F(\zeta_t)$ up to the second order

$$\begin{aligned} F(\zeta_t) = F(\zeta_0) &+ \sum_{i=1}^{2^n-1} F'(\zeta(t_i^n))(\zeta(t_{i+1}^n) - \zeta(t_i^n)) \\ &+ \sum_{i=0}^{2^n-1} \frac{1}{2}\Big\langle \zeta(t_{i+1}^n) - \zeta(t_i^n), F''(\overline{\zeta}_i^n)(\zeta(t_{i+1}^n) - \zeta(t_i^n))\Big\rangle_{E,E^*}. \end{aligned}$$

Here, $\overline{\zeta}_i^n$ denotes a random intermediate point on the line between $\zeta(t_i^n)$ and $\zeta(t_{i+1}^n)$. It is well known that this can be done in such a way that $\overline{\zeta}_i^n$ is measurable. Now the proof will be decomposed in several steps.

Step 1: We show that

$$\sum_{i=0}^{2^n-1} \Big\langle \zeta(t_{i+1}^n) - \zeta(t_i^n), F''(\overline{\zeta}_i^n)(\zeta(t_{i+1}^n) - \zeta(t_i^n)) \Big\rangle_{E,E^*} \to \int_0^t \langle u(s), F''(\zeta_s) u_s \rangle_{\mathrm{Tr}} \, ds$$

in $L^1(\Omega)$. Note that the increment $\zeta(t_{i+1}^n) - \zeta(t_i^n)$ equals

$$\zeta(t_{i+1}^n) - \zeta(t_i^n) = \int_{t_i^n}^{t_{i+1}^n} v(s)\, ds + \int_{t_i^n}^{t_{i+1}^n} u(s)\, dW_U(s).$$

Therefore, we can divide

$$\sum_{i=0}^{2^n-1} \Big\langle \zeta(t_{i+1}^n) - \zeta(t_i^n), F''(\overline{\zeta}_i^n)(\zeta(t_{i+1}^n) - \zeta(t_i^n)) \Big\rangle_{E,E^*}$$

into 4 parts. Consider the first piece

$$\sum_{i=0}^{2^n-1} \Big\langle \int_{t_i^n}^{t_{i+1}^n} v(s)\, ds, F''(\overline{\zeta}_i^n)\Big(\int_{t_i^n}^{t_{i+1}^n} v(s)\, ds\Big) \Big\rangle.$$

Pointwise in Ω and for all i, n, one has

$$\begin{aligned}\Big| \sum_{i=0}^{2^n-1} \Big\langle \int_{t_i^n}^{t_{i+1}^n} v(s)\, ds, F''(\overline{\zeta}_i^n)\Big(\int_{t_i^n}^{t_{i+1}^n} v(s)\, ds\Big) \Big\rangle \Big| &\leq \|F''\|_\infty \sum_{i=0}^{2^n-1} \Big\| \int_{t_i^n}^{t_{i+1}^n} v(s)\, ds \Big\|^2 \\ &\leq \|F''\|_\infty T 2^{-n} \|v\|_{L^2(0,T;E)}^2\end{aligned}$$

The latter clearly goes to zero in $L^1(\Omega)$ as $n \to \infty$. Next, both the second and the third part are pointwise dominated by

$$\|F''\|_\infty \sum_{i=0}^{2^n-1} \Big\| \int_{t_i^n}^{t_{i+1}^n} v(s)\ ds \Big\| \Big\| \int_{t_i^n}^{t_{i+1}^n} u(s)\ dW_U(s) \Big\| =: \xi_n$$

We show that $\lim_{n\to\infty} \xi_n = 0$ in $L^1(\Omega)$. Indeed, by Meyer's inequalities one has

$$\begin{aligned}&\mathbb{E} \sum_{i=0}^{2^n-1} \Big\| \int_{t_i^n}^{t_{i+1}^n} v(s)\ ds \Big\| \Big\| \int_{t_i^n}^{t_{i+1}^n} u(s)\ dW_U(s) \Big\| \\ &\qquad \leq \sqrt{t 2^{-n}} \|v\|_{L^2(0,T\times\Omega;E)} \Big(\sum_{i=0}^{2^n-1} \|\mathbf{1}_{[t_i^n, t_{i+1}^n]} u\|_{\mathbb{D}^{1,2}(\gamma(H,E))}^2 \Big)^{\frac{1}{2}}.\end{aligned}$$

Now by the type 2 assumption we have

$$\begin{aligned}\sum_{i=0}^{2^n-1} \|\mathbf{1}_{[t_i^n, t_{i+1}^n]} u\|_{\mathbb{D}^{1,2}(\gamma(H,E))}^2 &\lesssim_E \sum_{i=0}^{2^n-1} \|\mathbf{1}_{[t_i^n, t_{i+1}^n]} u\|_{\mathbb{D}^{1,2}(L^2(0,T;\gamma(U,E)))}^2 \\ &\leq \|u\|_{\mathbb{D}^{1,2}(L^2(0,T;\gamma(U,E)))}^2.\end{aligned}$$

Therefore, we find that $\lim_{n\to\infty} \xi_n = 0$ in $L^1(\Omega)$, from which we see that the second and third term converge to zero in $L^1(\Omega)$.

To finish step 1, observe that $Z = F'' \circ \zeta$ that continuous paths. Moreover, the process $Z^n = F'' \circ \zeta_n$ where ζ_n is the process obtained by letting $\zeta_n(t_i^n) = \overline{\zeta}_i^n$, $i = 0, \dots, 2^n$, and by linear interpolation at the intermediate points. Then by the pathwise continuity of ζ, it is clear that pointwise in Ω, $\lim_{n\to\infty} \sup_{t\in[0,T]} \|Z^n(t) - Z(t)\| = 0$. Hence, by Theorem 2.37 with $\sigma_i^n = t_i^n$,

$$\sum_{i=0}^{2^n-1} \Big\langle \int_{t_i^n}^{t_{i+1}^n} u(s)\, dW_U(s), F''(\overline{\zeta}_i^n) \int_{t_i^n}^{t_{i+1}^n} u(s)\, dW_U(s) \Big\rangle \\ \to \int_0^t \mathrm{Tr}(F''(\zeta_s)(u_s, u_s))\, ds$$

in $L^1(\Omega)$ as $n \to \infty$.

Step 2: One has

$$\Big| \sum_{i=0}^{2^n-1} F'(\zeta(t_i^n)) \Big(\int_{t_i^n}^{t_{i+1}^n} v(s)\, ds \Big) - \int_0^t F'(\zeta_s) v(s)\, ds \Big| \\ \leq \sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} |(F'(\zeta(t_i^n)) - F'(\zeta_s)) v(s)|\, ds \\ \leq \sup_{|s-r|\leq t2^{-n}} \|F'(\zeta_s) - F'(\zeta_r)\| \int_0^t \|v(s)\|\, ds.$$

By the pathwise continuity of ζ and the dominated convergence theorem the latter converges to zero in $L^1(\Omega)$ as $n \to \infty$.

Step 3: As in Lemma 2.38 one can show that $\zeta(t) \in \mathbb{D}^{1,2}(E)$ for each $t \in [0,T]$. Therefore, by Proposition 2.17, we have $F'(\zeta(t_i^n)) \in \mathbb{D}^{1,2}(E^*)$ for all i, n. By Proposition 2.23 one has $u \in \mathrm{Dom}(\delta)$, and with Lemma 2.29, we obtain

$$\sum_{i=0}^{2^n-1} F'(\zeta(t_i^n)) \Big(\int_{t_i^n}^{t_{i+1}^n} u(s)\, dW_U(s) \Big) \\ = \sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \langle u(s), F'(\zeta(t_i^n)) \rangle\, dW_U(s) \\ + \sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \langle u(s), F''(\zeta(t_i^n)) D(\zeta(t_i^n))(s) \rangle_{\mathrm{Tr}}\, ds.$$

By Lemma 2.39 with $Z(s) := F''(\zeta(s))$ one obtains

$$\sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \langle u(s), F''(\zeta(t_i^n))D(\zeta(t_i^n))(s)\rangle_{\mathrm{Tr}}\, ds$$
$$\to \int_0^t \big\langle u(s), F''(\zeta(s))((D^-\zeta)(s))\big\rangle_{\mathrm{Tr}}\, ds$$

in $L^1(\Omega)$, as $n \to \infty$. To finish the proof we need to show that

$$\int_0^t \sum_{i=0}^{2^n-1} \mathbf{1}_{(t_i^n, t_{i+1}^n)}(s)\, \langle u(s), F'(\zeta(t_i^n))\rangle\ dW_U(s) \to \int_0^t \langle u(s), F'(\zeta(s))\rangle\ dW_U(s),$$

in $L^1(\Omega)$ as $n \to \infty$. To prove the latter note that because of the identity in the Taylor development in the beginning of the proof, and the convergence in $L^1(\Omega)$ of all other terms, we know that the lefthand side of the previous formula converges in $L^1(\Omega)$ to some ξ, and it remains to identify its limit. Since δ is a closed operator on $L^1(\Omega; H)$, it suffices to note that

$$\lim_{n\to\infty} \Big\| s \mapsto \sum_{i=0}^{2^n-1} \mathbf{1}_{(t_i^n, t_i^n)}(s)\, \langle u(s), F'(\zeta(t_i^n))\rangle - \langle u(s), F'(\zeta(s))\rangle \Big\|_{L^1(\Omega;H)} = 0.$$

where we used the pathwise continuity of ζ. Therefore we can conclude that $\xi = \int_0^t \langle u(s), F'(\zeta(s))\rangle\ dW_U(s)$ and this completes the proof. □

3

A note on the truncated Skorohod integral process

3.1 Introduction and preliminaries

In [95] it is stated that the truncated version of a Skorohod integrable process need not be Skorohod integrable. A proof of this fact is left as an exercise (Exercise 3.2.1), but is given in [94, p.188]. The opposite statement is given in [38, Proposition 2.6], but personal communications with one of the authors has shown that this needs a suitable interpretation. In this chapter, we give an alternative example of the fact that the truncated process is indeed generally not Skorohod integrable.

First we recall some basic facts and notation from [38], [95]. Consider the Hilbert space $H = L^2(0,1)$ and let $W = \{W(h), h \in H\}$ be an isonormal Gaussian process associated with H. Here, we assume that W is defined on a complete probability space $(\Omega; \mathscr{F}, \mathbb{P})$, where $\mathscr{F}$ is generated by W.

This chapter is based on the paper [109].

3.1.1 The multiple Wiener-Itô integrals

For a function $f : (0,1)^n \to \mathbb{R}$ we define its symmetrization, denoted by $\tilde{f}$, by

$$\tilde{f}(t_1, \dots, t_n) = \frac{1}{n!} \sum_{\sigma} f(t_{\sigma(1)}, \dots, t_{\sigma(n)}), \ (t_1, \dots, t_n) \in (0,1)^n,$$

where σ runs over all permutations of $\{1, \dots, n\}$. We call f symmetric if $f = \tilde{f}$. If $f : (0,1)^{n+1} \to \mathbb{R}$ happens to be symmetric in its first n variables, then

$$\tilde{f}(t_1, \dots, t_{n+1}) = \frac{1}{n+1} \sum_{i=1}^{n+1} f(t_1, \dots, t_{i-1}, t_{n+1}, t_{i+1}, \dots, t_n, t_i),$$
$$(t_1, \dots, t_{n+1}) \in (0,1)^{n+1}.$$

Let $f_n : (0,1)^n \to \mathbb{R}, n \geq 1$ be a symmetric and square integrable random variable. For such a function, we define the multiple Wiener-Itô integral

$$I_n(f_n) := n! \int_0^1 \int_0^{t_n} \dots \int_0^{t_2} f_n(t_1, \dots t_n)\, dW_{t_1} \dots dW_{t_n}.$$

We will call $f : (0,1)^n \to \mathbb{R}$ elementary if it is of the form

$$f(t_1, t_2, \dots, t_n) = \sum_{i_1, \dots, i_n = 1}^{N} a_{i_1 \dots i_m} \mathbf{1}_{A_{i_1} \times \dots \times A_{i_n}}(t_1, \dots, t_n),\ (t_1, \dots, t_n) \in (0,1)^n,$$

where $A_1, \dots, A_n$ are pairwise disjoint, and such that $a_{i_1 \dots i_m} = 0$ whenever at least two of the indices $i_1, \dots, i_m$ are equal. When f and g are elementary, then

$$\mathbb{E}(I_m(f) I_n(g)) = \begin{cases} 0 & \text{if } m \neq n, \\ n! \langle \tilde{f}, \tilde{g} \rangle_{L^2((0,1)^n)} & \text{if } m = n. \end{cases}$$

The following theorem is called the Wiener chaos expansion.

Theorem 3.1 (Theorem 1.1.2, [95]). *Any square integrable random variable $F \in L^2(\Omega)$ can be expanded into a series of multiple stochastic integrals*

$$F = \sum_{n=0}^{\infty} I_n(f_n).$$

Here $I_0(f_0) = f_0 = \mathbb{E}(F)$. Moreover, the functions $f_n \in L^2((0,1)^n)$ are symmetric and uniquely determined by F.

3.1.2 The Malliavin derivative and the divergence operator

Consider the Malliavin derivative operator $D : \mathscr{S} \subset L^2(\Omega) \to L^2(\Omega; H)$, where $\mathscr{S}$ is the class of smooth random variables. The set $\mathbb{D}^{1,2}$ is the closure of $\mathscr{S} \subset L^2(\Omega)$ with respect to the norm

$$\|F\|_{\mathbb{D}^{1,2}} = [\mathbb{E}|F|^2 + \mathbb{E}\|DF\|_H^2]^{1/2}.$$

The adjoint of D is denoted by δ, which is called the Skorohod integral operator for the following reason. If we put $W_t := W(\mathbf{1}_{[0,t]})$, then $(W_t)_{t \in [0,T]}$ becomes a Brownian motion. Let $(\mathscr{F}_t)_{t \in [0,T]}$ be a filtration with respect to the Brownian motion, and let $L_a^2(\Omega \times [0,T])$ be the space of all $(\mathscr{F}_t)$-adapted, square-integrable processes. Then for every $u \in L_a^2(\Omega \times [0,T])$ we have $u \in \mathrm{Dom}(\delta)$ and

$$\int_0^T u(t)\, dW_t = \delta(u).$$

Hence $\delta(u)$ can be viewed as an extension of the Itô-integral, and is called the Skorohod integral. Hence for every $u \in \mathrm{Dom}(\delta)$, we will use the notation as in (3.1.2). Using Theorem 3.1, we get that any process $u \in L^2((0,1) \times \Omega)$ has a Wiener chaos expansion

$$u(t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t)),$$

where for each $n \geq 1$, $f_n \in L^2((0,1)^{n+1})$ is symmetric in the first n variables. For a process $u \in L^2((0,1) \times \Omega)$ to be in the domain of δ, one has the following proposition.

Proposition 3.2 (Identity (1.53), [95]). *Let $u \in L^2((0,1) \times \Omega)$ have the Wiener expansion (3.1.2). Then $u \in Dom(\delta)$ if and only if*

$$\sum_{n=0}^{\infty} (n+1)! \|\tilde{f}_n\|^2_{L^2((0,1)^{n+1})} < \infty.$$

In this case, the series equals $\mathbb{E}(\delta(u)^2)$.

3.2 Results

Theorem 3.3. *There exists a process $u \in Dom(\delta)$ such that $\mathbf{1}_{[0,\frac{1}{2})} u \notin Dom(\delta)$.*

To prove this theorem, we need the following elementary identities

$$\sum_{k=1}^{n} \binom{n}{k} k = n 2^{n-1}, \qquad \sum_{k=1}^{n} \binom{n}{k} k^2 = n(n+1) 2^{n-2}.$$

These can be derived by differentiating $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ with respect to x once and twice.

Proof. For $n \geq 1$, consider the function $f_n \in L^2((0,1)^{n+1})$ given by

$$f_n(t_1, \dots t_n, t_{n+1}) = \frac{1}{n\sqrt{n!}} (\mathbf{1}_{[0,\frac{1}{2}]}(t_{n+1}) - \mathbf{1}_{[\frac{1}{2},1]}(t_{n+1})), \qquad (t_1, \dots, t_{n+1}) \in (0,1)^{n+1}.$$

Observe that f_n is symmetric in the first n variables. Let $u : \Omega \times (0,1) \to \mathbb{R}$ be defined by (3.1.2). We will show that u is in $L^2(\Omega \times (0,1))$. After that, we will show that $u \in \text{Dom}(\delta)$ and $\mathbf{1}_{[0,\frac{1}{2}]} u \notin \text{Dom}(\delta)$.

To show that u is in $L^2(\Omega \times (0,1))$, note that

$$\|u\|^2_{L^2(\Omega \times (0,1))} = \mathbb{E} \int_0^1 \Big| \sum_{n=1}^{\infty} I_n(f_n(\cdot, t)) \Big|^2 dt = \sum_{n=1}^{\infty} n! \int_0^1 \langle \tilde{f}_n, \tilde{f}_n \rangle_{L^2((0,1)^n)} dt$$
$$= \sum_{n=1}^{\infty} n! \int_0^1 \dots \int_0^1 \Big(\frac{1}{n\sqrt{n!}} (\mathbf{1}_{[0,\frac{1}{2}]}(t) - \mathbf{1}_{[\frac{1}{2},1]}(t)) \Big)^2 dt_1 \dots dt_n dt = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Next, in order to show that $u \in \text{Dom}(\delta)$, we use Proposition 3.2. Observe that

$$\tilde{f}_n(t_1,\dots,t_{n+1}) = \frac{1}{n(n+1)\sqrt{n!}} \sum_{i=1}^{n+1} (\mathbf{1}_{[0,\frac{1}{2}]}(t_i) - \mathbf{1}_{[\frac{1}{2},1]}(t_i)).$$

One has

$$\begin{aligned}&\|\tilde{f}_n\|^2_{L^2((0,1)^{n+1})}\\ &\quad= \frac{1}{n^2(n+1)^2 n!} \int_0^1 \cdots \int_0^1 \Big(\sum_{i=1}^{n+1} (\mathbf{1}_{[0,\frac{1}{2}]}(t_i) - \mathbf{1}_{[\frac{1}{2},1]}(t_i))\Big)^2 dt_1 \dots dt_{n+1}.\end{aligned}$$

Now, for every $i = 1,\dots,n+1$, we will split up the integral into a $[0,\frac{1}{2}]$-part and a $[\frac{1}{2},1]$-part, giving us a total of 2^{n+1} parts. Integrating over $[0,\frac{1}{2}]^{n+1}$ gives us

$$\begin{aligned}&\int_0^{1/2} \cdots \int_0^{1/2} (f_n(t_1,\dots,t_{n+1}))^2 \, dt_1 \dots dt_{n+1}\\ &\qquad= \frac{1}{n^2(n+1)^2 n!} \int_0^{1/2} \cdots \int_0^{1/2} \Big(\sum_{i=1}^{n+1} \mathbf{1}_{[0,\frac{1}{2}]}(t_i)\Big)^2 \, dt_1 \dots dt_{n+1}\\ &\qquad= \frac{1}{n^2(n+1)^2 n!} \Big[\frac{1}{2^{n+1}}(n+1)^2\Big].\end{aligned}$$

If we compute the case where all integrals are on $[0,\frac{1}{2}]$, except one interval, then a similar computation yields $\frac{1}{n^2(n+1)^2 n!}(\frac{1}{2^{n+1}}(n-1)^2)$. The case where all integrals are on $[0,\frac{1}{2}]$ except two intervals results in $\frac{1}{n^2(n+1)^2 n!}(\frac{1}{2^{n+1}}(n-3)^2)$, and so forth. In total, we obtain

$$\|\tilde{f}_n\|^2_{L^2((0,1)^{n+1})} = \frac{1}{n^2(n+1)^2 n!} \frac{1}{2^{n+1}} \sum_{i=1}^{n+1} \binom{n+1}{i} (n+1-2i)^2.$$

By (3.2), one has

$$\begin{aligned}\sum_{i=1}^{n+1} \binom{n+1}{i} (n+1-2i)^2 &= \sum_{i=1}^{n+1} \binom{n+1}{i} ((n+1)^2 - 4(n+1)i + 4i^2)\\ &= (n+1)^2 2^{n+1} - 4(n+1)^2 2^n + 4(n+1)(n+2)2^{n-1} = (n+1)2^{n+1}.\end{aligned}$$

From this, we conclude that

$$\sum_{n=1}^{\infty} (n+1)! \|\tilde{f}_n\|^2_{L^2((0,1)^{n+1})} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Hence $u \in \mathrm{Dom}(\delta)$ by Proposition 3.2.

Now consider the process $\mathbf{1}_{[0,\frac{1}{2}]} u : \Omega \times (0,1) \to \mathbb{R}$. Using (3.1.2), we have

$$\mathbf{1}_{[0,\frac{1}{2}]}(t) u(t) = \mathbf{1}_{[0,\frac{1}{2}]}(t) \sum_{n=1}^{\infty} I_n(f_n(\cdot,t)) = \sum_{n=1}^{\infty} I_n(g_n(\cdot,t)),$$

where $g_n = g_n(t_1, \ldots, t_n, t) = \mathbf{1}_{[0,\frac{1}{2}]}(t) f_n(t_1, \ldots, t_n, t)$. Since g_n is symmetric in its first n variables, we see that

$$\tilde{g}_n(t_1, \ldots, t_{n+1}) = \frac{1}{n(n+1)\sqrt{n!}} \sum_{i=1}^{n+1} \mathbf{1}_{[0,\frac{1}{2}]}(t_i).$$

Similar to $\tilde{f}_n$, we get

$$\|\tilde{g}_n\|^2_{L^2((0,1)^{n+1})} = \frac{1}{n^2(n+1)^2 n!} \int_0^1 \cdots \int_0^1 \Big(\sum_{i=1}^{n+1} \mathbf{1}_{[0,\frac{1}{2}]}(t_i) \Big)^2 dt_1 \ldots dt_{n+1}.$$

Similar as before, one can show that

$$\|\tilde{g}_n\|^2_{L^2((0,1)^{n+1})} = \frac{1}{n^2(n+1)^2 n!} \frac{1}{2^{n+1}} \sum_{i=1}^{n+1} \binom{n+1}{i} (n+1-i)^2.$$

Now (3.2) gives

$$\begin{aligned} \sum_{i=0}^{n+1} \binom{n+1}{i} (n+1-i)^2 \\ &= (n+1)^2 2^{n+1} - 2(n+1) \sum_{i=0}^{n+1} \binom{n+1}{i} i + \sum_{i=0}^{n+1} \binom{n+1}{i} i^2 \\ &= (n+1)^2 2^{n+1} - 2(n+1)^2 2^n + (n+1)(n+2) 2^{n-1} \\ &= \frac{1}{4} 2^{n+1} (n+1)(n+2). \end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} (n+1)! \|\tilde{g}_n\|^2_{L^2((0,1)^{n+1})} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{n+2}{n^2} = \infty.$$

By Proposition 3.2, $\mathbf{1}_{[0,\frac{1}{2}]} u$ is not Skorohod integrable. □

Finally, we discuss some additional results. Recall from [95, Proposition 1.3.1], that $\mathbb{D}^{1,2}(L^2(0,1)) \subseteq \mathrm{Dom}(\delta)$. Moreover, from [95, p. 180], one has that if $v \in \mathbb{D}^{1,2}(L^2(0,1))$, then $\mathbf{1}_{[a,b]} v \in \mathbb{D}^{1,2}(L^2(0,1))$. In particular, Theorem 3.3 shows that, from the above proof, the process u satisfies $u \notin \mathbb{D}^{1,2}(L^2(0,1))$. We will now show that $u \in \mathbb{D}^{s,p}(L^2(0,1))$ if and only if $s < 1$. Here, $\mathbb{D}^{s,p}(L^2(0,1))$ is defined as follows. Consider for each $n \geq 1$ the closed subspace $\mathcal{H}_n$ of $L^2(\Omega)$ generated by $\{H_n(W(h)),\ h \in L^2(0,1), \|h\| = 1\}$, where H_n is the nth Hermite polynomial. Let J_n be the projection onto $\mathcal{H}_n$. Define the operator $L : L^2(\Omega) \to L^2(\Omega)$ by

$$LF = \sum_{n=0}^{\infty} -n J_n F,$$

provided this series converges in $L^2(\Omega)$. Let $\mathscr{P}$ be the class of random variables of the form $F = f(W(h_1), \dots, W(h_n))$, where f is a polynomial, and $h_1, \dots, h_n \in H$. For $s \in \mathbb{R}$ and $F \in \mathscr{P}$, we define the following seminorm

$$\|F\|_{s,p} := \|(I-L)^{s/2}F\|_{L^p(\Omega)}, \qquad F \in \mathscr{P}.$$

Note that $(I-L)^{s/2}F = \sum_{n=0}^{\infty}(1+n)^{s/2}J_nF$. Now define $\mathbb{D}^{s,p}(L^2(0,1))$ to be the completion of $\mathscr{P}$ with respect to $\|\cdot\|_{s,p}$. Using orthogonality, one derives that $F \in \mathbb{D}^{s,2}(L^2(0,1))$ if and only if $F \in L^2(\Omega)$ and $\sum_{n=0}^{\infty}(1+n)^s\|J_nF\|^2 < \infty$. Moreover, one has $\mathbb{D}^{s,2}(L^2(0,1)) = \mathbb{D}^{k,2}(L^2(0,1))$, whenever $s = k \in \mathbb{N}$, where the latter is defined using the derivative operator. (See also [95, Remarks: 1].) It follows that u from our counterexample cannot be in the space $\mathbb{D}^{s,2}(L^2(0,1))$ with $s \geq 1$. One actually has the following theorem.

Theorem 3.4. *Let u be the process from Theorem 3.3. Then u and $\mathbf{1}_{[0,\frac{1}{2}]}u$ belong to $\mathbb{D}^{s,2}(L^2(0,1))$ if and only if $s < 1$. Consequently, $\mathbb{D}^{s,2}(L^2(0,1)) \subseteq Dom(\delta)$ if and only if $s \geq 1$.*

Proof. Write $u(t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t))$ and I_n maps into $\mathcal{H}_n$, hence $J_k(u(t)) = I_k(f_k(\cdot,t))$. To obtain $\|u\|_{s,2}$ we use the proof of Theorem 3.3 to find

$$\begin{aligned}\mathbb{E}\Big(\Big\|\sum_{n=0}^{\infty}(1+n)^{s/2}J_nu\Big\|^2_{L^2(0,1)}\Big) &= \mathbb{E}\int_0^1\Big(\sum_{n=0}^{\infty}(1+n)^{s/2}I_n(f_n(\cdot,t))\Big)^2\,dt\\ &= \int_0^1\sum_{n=0}^{\infty}n!(1+n)^s\langle\tilde{f}_n(\cdot,t),\tilde{f}_n(\cdot,t)\rangle_{L^2((0,1)^n)}\,dt = \sum_{n=1}^{\infty}\frac{(1+n)^s}{n^2}.\end{aligned}$$

A similar computation yields

$$\|\mathbf{1}_{[0,\frac{1}{2}]}u\|^2_{s,2} = \int_0^1\sum_{n=0}^{\infty}n!(1+n)^s\langle\tilde{g}_n(\cdot,t),\tilde{g}_n(\cdot,t)\rangle_{L^2((0,1)^n)}\,dt = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(1+n)^s}{n^2}.$$

Hence, $\|u\|_{s,2} < \infty$ and $\|\mathbf{1}_{[0,\frac{1}{2}]}u\|_{s,2} < \infty$ exactly when $s < 1$. From the latter we see $\mathbb{D}^{s,2}(L^2(0,1)) \not\subseteq \mathrm{Dom}(\delta)$ when $s < 1$. Also, $\mathbb{D}^{s,2}(L^2(0,1)) \subset \mathbb{D}^{1,2}(L^2(0,1)) \subset \mathrm{Dom}(\delta)$ for $s \geq 1$. ☐

4

Forward integration, convergence and nonadapted pointwise multipliers

4.1 Introduction

In [117] and [118] Russo and Vallois initiated a theory of stochastic integration via regularization procedures. In later years this was further developed by them and several other authors (see [25, 42, 54, 58, 93, 116, 133], and also the lecture notes [122] and its references). The regularization procedure is connected to the celebrated forward and backward integrals which can be used to integrate with respect to more general processes than semimartingales. Applications arise for instance in the case where the integrator is a fractional Brownian motion. Another feature is that the forward and backward integrals allow to integrate non-adapted processes.

Since the development of the Skorohod integral in [129], integration of non-adapted integrands is used in the theory of SDEs (see [38, 95, 103, 122] and references therein). A basic example where non-adapted integrands naturally occur is when the initial value of an SDE depends on the full paths of the underlying stochastic process (see [22, 83]). In many situations the forward integral is easier to work with than the Skorohod integral as a difficult correction term can often be avoided (see the Itô formula in [120], [38, Theorem 8.12]). The forward integral is used widely in the modeling of insider trading, which was introduced in [14]. Since then, this has been further developed (see [38, Chapter 8] and its references). In particular, in [36, 37, 99] the authors generalized the forward integral to the setting of Lévy processes.

In the infinite dimensional setting several authors have worked on stochastic calculus for the Skorohod integral (see [75, 80, 96, 112] and references therein). However, only few results are available for the forward integral in infinite dimensions. In [34], Di Girolami and Russo present a general set-up for an Itô formula and covariation formulas. In [68] León and Nualart have introduced the forward integral in the operator-valued setting and used it to study stochastic evolution equations in Hilbert spaces with an adapted (unbounded) drift.

In this chapter we study several properties of the forward integral where the integrand is an operator-valued process and the integrator a cylindrical Wiener process. We will prove a new approximation result for the forward integral (see Theorem 4.16 and Corollary 4.18 below). In the one-dimensional setting this result takes the following form:

Theorem 4.1. *Let w be a standard Brownian motion and let g be an adapted and measurable process with almost all paths in $L^p(0,T)$ with $p \geq 2$. Then the pathwise defined process*

$$t \mapsto n \int_0^t g(s)(w(s+\tfrac{1}{n}) - w(s))\, ds, \quad t \in [0,T]$$

converges to the Itô integral process $t \mapsto \int_0^\cdot g\, dw$ in $W^{\alpha,p}(0,T)$ in probability for every $\alpha \in [0, \frac{1}{2})$.

The above result will be a particular case of two more general results on forward integration in UMD Banach spaces. The class of UMD Banach spaces was extensively studied in the work of Burkholder (see [23] and references therein). The UMD property plays an important role in both vector-valued stochastic and harmonic analysis. Stochastic integration and calculus in Banach spaces is naturally limited to the class of UMD Banach space (see [21, 86]). Applications to stochastic evolutions equations have been given in [87] and several works afterwards (see the recent survey [90] for further references).

As an application of the convergence result we derive a new pointwise multiplier result for the forward integral (see Section 4.5). It can be interpreted as an integration by parts formula. The main novelty is that we can multiply adapted Itô integrable processes with a process M which is smooth in time but not necessarily adapted. Moreover, it is allowed to have a non-integrable singularity at $t = T$. This result will be obtained in the operator-valued setting. It is particulary interesting in the study of mild solutions of non-autonomous stochastic evolution equations with adapted drift, where indeed the multiplier has a non-integrable singularity. A well-known obstacle in non-autonomous stochastic evolution equations is that the stochastic convolution term is not well-defined as an Itô integral due to adaptedness problems. In [68] this problem has been investigated using integration by parts for the Skorohod integral. This formula for the Skorohod integrals can be obtained in the case M is constant in time and satisfies certain Malliavin differentiability. In chapter 5 we will use the integration by parts formula to give a new approach to non-autonomous stochastic evolution equations with adapted drift.

This chapter is based on the paper [110].

4.2 Preliminaries

In this chapter we let H be a separable Hilbert space and we fix an orthonormal basis $(h_n)_{n\geq 1}$. The number $T \in (0,\infty)$ will be a fixed time and X is a UMD Banach space. All vector spaces will be assumed to be defined over the real scalar field, but with minor adjustments one can also allow complex scalars. We refer to [23] for details on UMD Banach spaces. The space $(\Omega, \mathscr{F}, \mathbb{P})$ will be a probability space with filtration $(\mathscr{F}_t)_{t\geq 0}$ and expectation is denoted by $\mathbb{E}$. Moreover, we write $L^0(\Omega; X)$ for the strongly measurable functions $\xi : \Omega \to X$ with the topology of given by convergence in probability. In the sequel C will be a constant which may vary from line to line.

4.2.1 Radonifying operators

Let $\mathscr{H}$ be a real separable Hilbert space (below we take $\mathscr{H} = L^2(S; H)$). We refer to [39, Chapter 12] and the survey paper [85] for an overview on γ-radonifying operators and unexplained terminology below. The Banach space of γ-radonifying operators from $\mathscr{H}$ into X will be denoted by $\gamma(\mathscr{H}, X)$. It is a subspace of $\mathscr{L}(\mathscr{H}, X)$. It satisfies the left- and right-ideal property. In particular, for $R \in \gamma(\mathscr{H}, X)$, $U \in \mathscr{L}(X)$ and $T \in \mathscr{L}(\mathscr{H})$, one has $URT \in \gamma(\mathscr{H}, X)$ and

$$\|URT\|_{\gamma(\mathscr{H},X)} \leq \|U\| \, \|R\|_{\gamma(\mathscr{H},X)} \, \|T\|.$$

A simple consequence of the right-ideal property is that every operator $T : \mathscr{H} \to \mathscr{H}$ has an extension to an operator

$$\begin{aligned} \tilde{T} : \gamma(\mathscr{H}, X) &\to \gamma(\mathscr{H}, X), \\ R &\mapsto RT^*, \end{aligned} \tag{4.2.1}$$

and $\|\tilde{T}\| = \|T\|$.

Let (S, Σ, μ) be a σ-finite measure space. A function $G : S \to \mathscr{L}(H, X)$ will be called *H-strongly measurable* if for all $h \in H$, $s \mapsto G(s)h$ is strongly measurable. Moreover, for $p \in (1, \infty)$, G will be called *weakly* $L^p(S; H)$ if for all $x^* \in X^*$, $s \mapsto G(s)^* x^*$ is in $L^p(S; H)$. For $G : S \to \mathscr{L}(H, X)$ which is H-strongly measurable and weakly $L^2(S; H)$ we can define, $R_G : L^2(S; H) \to X$ as the (Pettis) integral operator

$$\langle R_G f, x^* \rangle = \int_S \langle G(s) f(s), x^* \rangle \, d\mu(s), \quad f \in L^2(S; H), \quad x^* \in X^*. \tag{4.2.2}$$

Note that

$$\|R_G f\|_X \leq \|R_G\|_{\gamma(L^2(S;H),X)} \|f\|_{L^2(S;H)}. \tag{4.2.3}$$

We will say $G \in \gamma(S; H, X)$ if $R_G \in \gamma(L^2(S; H), X)$ and write $\|G\|_{\gamma(S;H,X)} = \|R_G\|_{\gamma(L^2(S;H),X)}$. It is well-known that the step functions $G : S \to \mathscr{L}(H, X)$ of finite rank are dense in $\gamma(S; H, X)$.

For many operators $T : L^2(S; H) \to L^2(S; H)$ one has the property that $\tilde{T} R_G = R_F$ for a certain function F. In this case it will be convenient to write $TG = F$.

An easy consequence of the definitions and the ideal property is that $\|G \mathbf{1}_{S_0}\|_{\gamma(S;H,X)} = \|G|_{S_0}\|_{\gamma(S_0;H,X)}$. We will also use the following property.

Example 4.2. For $G \in \gamma(S; H, X)$ and $b \in L^\infty(S)$ one has $bG \in \gamma(S; H, X)$ and

$$\|bG\|_{\gamma(S;H,X)} \leq \|b\|_{L^\infty(S)} \|G\|_{\gamma(S;H,X)}. \tag{4.2.4}$$

This is immediate from the right-ideal property with operator $T_b : L^2(S; H) \to L^2(S; H)$ given by $T_b f = bf$.

Finally we recall that in the special case that X is a Hilbert space, one has

$$\gamma(S; H, X) = L^2(S; \mathrm{S}^2(H, X)), \tag{4.2.5}$$

where $\mathrm{S}^2(H, X)$ denotes the space of Hilbert-Schmidt operators.

Lemma 4.3 (γ-Integration by parts). *Let $M \in W^{1,1}(0, T; \mathscr{L}(X))$. Then for every $f \in \gamma(0, T; X)$ one has $Mf \in \gamma(0, T; X)$ and for all $0 \leq a < b \leq T$,*

$$\int_a^b M(s)f(s)\,ds = M(a)F(a) + \int_a^b M'(s)F(s)\,ds, \tag{4.2.6}$$

where $F(t) = \int_t^b f(s)\,ds$.

Proof. By [64, Example] the family $\{M(t) : t \in [0, T]\}$ is R-bounded by C. Therefore, by the Kalton–Weis γ-multiplier theorem (see [85, Theorem 5.2]), one has that $Mf \in \gamma(0, T; X)$ again and $\|Mf\|_{\gamma(0,T;X)} \leq C\|f\|_{\gamma(0,T;X)}$. One also has

$$\|F(t)\| \leq \|f\|_{\gamma(0,T;X)} \|\mathbf{1}_{(t,b)}\|_{L^2(0,T)} \leq T^{1/2} \|f\|_{\gamma(0,T;X)}$$

and hence

$$\begin{aligned}\int_0^T \|M'(t)F(t)\|\,dt &\leq \|M\|_{W^{1,1}(0,T;\mathscr{L}(X))} \sup_{t\in[0,T]} \|F(t)\| \\ &\leq \|M\|_{W^{1,1}(0,T;\mathscr{L}(X))} T^{1/2} \|f\|_{\gamma(0,T;X)}.\end{aligned}$$

For step functions $f : (0, T) \to X$, the identity (4.2.6) is easy to verify. Now the general case follows from the above estimates and a density argument. □

4.2.2 Integration with respect to a cylindrical Brownian motion

Let $\mathscr{H} = L^2(0, T; H)$, where H is a separable real Hilbert space. For details on stochastic integration in UMD Banach space we refer to [86,90]. The operator $W : \mathscr{H} \to L^2(\Omega)$ will be called a *cylindrical Brownian motion* if for all choices $h \in \mathscr{H}$, Wh is a centered Gaussian random variable and for $h, \tilde{h} \in \mathscr{H}$, $\mathbb{E}(WhW\tilde{h}) = [h, \tilde{h}]$, where $[\cdot, \cdot]$ denotes the inner product on $\mathscr{H}$.

A process $G : (0, T) \times \Omega \to \mathscr{L}(H, X)$ will be called H-strongly adapted if for all $t \in (0, T)$ and $h \in H$, $\omega \mapsto G(t, \omega)h$ is strongly $\mathscr{F}_t$-measurable. If G is H-strongly measurable and adapted, then from the separability of H and [100, Theorem 0.1], one can derive that G has a version which is *H-strongly progressively* *measurable*, i.e. for each $h \in H$, $(t,\omega) \mapsto G(t,\omega)h$ is strongly progressively measurable. This will be used below without further notice.

Recall from [29,86,90] that if X is a UMD space and $G \in L^0(\Omega;\gamma(0,T;H,X))$ adapted, one can define a stochastic integral $I(G) = \int_0^T G\,dW$ in a natural way. We also let $J(G)(t) = \int_0^t G\,dW$, and recall that $J(G)$ has a version with continuous paths. Moreover, for all $p \in (0,\infty)$ the following two-sided estimate for the stochastic integral holds:

$$C^{-1}\|G\|_{L^p(\Omega;\gamma(0,T;H,X))} \leq \|J(G)\|_{L^p(\Omega;C([0,T];X))} \leq C\|G\|_{L^p(\Omega;\gamma(0,T;H,X))} \tag{4.2.7}$$

Remark 4.4. All results below hold under the slightly weaker assumption that the right-hand side of (4.2.7) holds. This includes spaces such as $X = L^1$. For details on such spaces we refer to [26, 29].

4.2.3 Function spaces

For $\alpha \in (0,1)$, $p \in [1,\infty)$ and $a < b$, recall that a function $f : (a,b) \to X$ is said to be in the *Sobolev space* $W^{\alpha,p}(a,b;X)$ if $f \in L^p(a,b;X)$ and

$$[f]_{W^{\alpha,p}(a,b;X)} := \Big(\int_a^b\int_a^b \frac{\|f(t)-f(s)\|^p}{|t-s|^{\alpha p+1}}\,ds\,dt\Big)^{1/p} < \infty.$$

Letting $\|f\|_{W^{\alpha,p}(a,b;X)} = \|f\|_{L^p(a,b;X)} + [f]_{W^{\alpha,p}(a,b;X)}$, this space becomes a Banach space. A function $f : (a,b) \to X$ is said to be in the *Hölder space* $C^\alpha(a,b;X)$ if

$$[f]_{C^\alpha(a,b;X)} = \sup_{a<s<t<b} \frac{\|f(t)-f(s)\|}{|t-s|^\alpha} < \infty.$$

Letting $\|f\|_{C^\alpha(a,b;X)} = \sup_{t\in(0,T)}\|f(t)\|_X + [f]_{W^{\alpha,p}(a,b;X)}$, this space becomes a Banach space. Moreover, every $f \in C^\alpha(a,b;X)$ has a unique extension to a continuous function $f : [a,b] \to X$.

If $0 < \alpha < \beta < 1$, then trivially,

$$C^\alpha(a,b;X) \hookrightarrow W^{\alpha,p}(a,b;X)$$

One of the main results in the theory of fractional Sobolev spaces is the following Sobolev embedding: if $\alpha > \frac{1}{p}$, then

$$W^{\alpha,p}(a,b;X) \hookrightarrow C^{\alpha-\frac{1}{p}}(a,b;X). \tag{4.2.8}$$

Here the embedding means that each $f \in W^{\alpha,p}(a,b;X)$ has a version which is continuous and this function lies in $C^{\alpha-\frac{1}{p}}(a,b;X)$. The embedding (4.2.8) can be found in the literature in the scalar setting and the standard proofs extend to the vector-valued setting. We refer to [70, 14.28 and 14.40] and [35, Theorem 8.2] for detailed proofs.

4.3 Forward integral

Recall that H is a separable real Hilbert space with orthonormal basis $(h_n)_{n\geq 1}$. Let P_n be the projection onto the first n basis coordinates.

Definition 4.5. Let $G : [0,T] \times \Omega \to \mathscr{L}(H,X)$ be H-strongly measurable and weakly in $L^2(0,T;H)$. Define the sequence $(I^-(G,n))_{n=1}^\infty$ by

$$I^-(G,n) = \sum_{k=1}^n n \int_0^T G(s)h_k(W(s+1/n)h_k - W(s)h_k)\, ds,$$

where the integral is defined as in (4.2.2)

The process G is called *forward integrable* if $(I^-(G,n))_{n\geq 1}$ converges in probability. In that case, the limit is called *the forward integral* of G and its limit is denoted by

$$I^-(G) = \int_0^T G\, dW^- = \int_0^T G(s)\, dW^-(s).$$

Note that the above definition does not require any adaptedness properties of G. Unfortunately, it is unclear whether I^- is a closable operator. For the Skorohod integral this is indeed the case (see [95, Section 1.3]).

Observe that if G is forward integrable, then so is $\mathbf{1}_{[0,t]}G$ for $t \in (0,T]$. We write $J^-(G,n)$ for the process given by

$$J^-(G,n)(t) = I^-(G\mathbf{1}_{[0,t]}, n). \tag{4.3.1}$$

Then $J^-(G,n) \in L^0(\Omega; C^{1/2}(0,T;X))$. Indeed, by (4.2.3) we have a.s. for $s < t$,

$$\begin{aligned}\|J^-(G,n)(t) - J^-(G,n)(s)\| &\leq \sum_{k=1}^n n\Big\| \int_s^t G(r)h_k(W(r+1/n)h_k - W(r)h_k)\, dr\Big\| \\ &\leq \sum_{k=1}^n n\|Gh_k\|_{\gamma(0,T;X)}\|r \mapsto \mathbf{1}_{[s,t]}(r)(W(r+1/n)h_k - W(r)h_k)\|_{L^2(0,T)} \\ &\leq 2(t-s)^{1/2}\sum_{k=1}^n n\|Gh_k\|_{\gamma(0,T;X)} \sup_{r\in[0,T+1/n]} |W(r)h_k|,\end{aligned}$$

and hence the result follows.

If for every $t \in [0,T]$, $(J^-(G,n)(t))_{n\geq 1}$ converges in probability, we write $J^-(G)$ for the process given by $J^-(G)(t) = \int_0^t G\, dW^-$. In general it seems to be unclear whether $\mathbf{1}_{[0,t]}G$ is forward integrable when G is forward integrable.

First we show that the forward integral extends the Itô integral of Section 4.2.

Proposition 4.6. *Assume $G \in L^0(\Omega; \gamma(0,T;H,X))$ is adapted.*

(1) *For every* $n \geq 1$*, the process* $G_n := n\mathbf{1}_{[0,\frac{1}{n}]} * (\mathbf{1}_{[0,T]}P_n G)$ *is adapted and in* $L^0(\Omega;\gamma(0,T;H,X))$ *and the following identity holds*

$$I^-(G,n) = \int_0^\infty G_n\,dW = \int_0^{T+\frac{1}{n}} G_n\,dW. \tag{4.3.2}$$

(2) *For every* $t \in [0,T]$*,* $\mathbf{1}_{[0,t]}G$ *is forward integrable and stochastically integrable and*

$$J^-(G)(t) = \int_0^t G\,dW.$$

Motivated by the above result, we will write $J(G)$ for $J^-(G)$ in the adapted case. Recall that $J(G)$ always has a continuous version and we will use this version without further notice. It is unclear to us whether $J^-(G,n) \to J(G)$ in $L^0(\Omega;C([0,T];X))$ for all $G \in L^0(\Omega;\gamma(0,T;H,X))$. In the literature there are several attempts to prove such a result in the setting $H = X = \mathbb{R}$, but we could not follow these arguments. In Theorem 4.16 we will give sufficient conditions on G for convergence in $L^0(\Omega;W^{\alpha,p}([0,T];X))$ and in particular in $L^0(\Omega;C^{\alpha-\frac{1}{p}}([0,T];X))$.

Proof. Choose an H-strongly progressively measurable version of G and extend G as zero on (T,∞). Let the operator S_n on $L^2(\mathbb{R}_+;H)$ be given by $S_n f = n\mathbf{1}_{[0,\frac{1}{n}]} * P_n f$. Then $\|S_n\| \leq 1$ and it extends by (4.2.1) to a contraction on $\gamma(L^2(0,T;H),X)$. By duality and (4.2.2), this extension equals R_{G_n}. It follows that G_n is in $L^0(\Omega;\gamma(\mathbb{R}_+;H,X))$ and for every $t \in \mathbb{R}_+$ and $x^* \in X^*$ one has

$$G_n(t)^*x^* = \int_0^t n\mathbf{1}_{[0,\frac{1}{n}]}(t-s)P_n G(s)^*x^*\,ds$$

and since G is progressive measurable, the latter is $\mathscr{F}_t$-measurable and thus G_n is H-strongly adapted. It follows that G_n is stochastically integrable and by the stochastic Fubini theorem we obtain that for all $x^* \in X^*$,

$$\begin{aligned}
\Big\langle \int_0^{T+\frac{1}{n}} G_n\,dW, x^* \Big\rangle &= \int_0^\infty G_n^* x^*\,dW \\
&= n\int_0^\infty \int_0^T \mathbf{1}_{[0,\frac{1}{n}]}(\sigma-s)P_n G(s)^*x^*\,ds\,dW(\sigma) \\
&= n\int_0^T \int_0^\infty \mathbf{1}_{[0,\frac{1}{n}]}(\sigma-s)P_n G(s)^*x^*\,dW(\sigma)\,ds \\
&= n\sum_{k=1}^n \int_0^T \langle G(s)h_k, x^*\rangle (W(s+1/n)-W(s))h_k\,ds \\
&= \langle I^-(G,n), x^*\rangle.
\end{aligned}$$

By the Hahn-Banach theorem this yields (1).

Next we prove (2). Replacing G by $\mathbf{1}_{[0,t]}G$ it suffices to consider $t = T$. Note that by [86, Proposition 2.4] $G_n \to G$ in $\gamma(\mathbb{R}_+; H, X)$ pointwise on ω. Therefore, with [86, Theorem 5.5] we find that $I^-(G,n) = \int_0^\infty G_n\, dW \to \int_0^\infty G\, dW$ in $L^0(\Omega; X)$ and (2) follows. □

In the following lemma we collect some elementary properties of the forward integral.

Lemma 4.7. *Let X_0, X_1 be Banach spaces and let $F, G : [0,T] \times \Omega \to \mathscr{L}(H, X_0)$ be forward integrable processes.*

(1) *For $\alpha, \beta \in \mathbb{R}$, a.s.*

$$\int_0^T \alpha F + \beta G\, dW^- = \alpha \int_0^T F\, dW^- + \beta \int_0^T G\, dW^-.$$

(2) *If $A : \Omega \to \mathscr{L}(X_0, X_1)$ is such that for every $x \in X_0$, Ax is $\mathscr{F}$-measurable, then AG is forward integrable and a.s.*

$$A \int_0^T G\, dW^- = \int_0^T AG\, dW^-.$$

In particular, for any $x^* \in X_0^*$, $G^* x^*$ is forward integrable, and a.s.

$$\Big\langle \int_0^T G\, dW^-, x^* \Big\rangle = \int_0^T G^* x^*\, dW^-.$$

(3) *If $(A, D(A))$ is a closed linear operator on X_0 such that $G \in D(A)$ a.e., AG is weakly in $L^2(0,T;H)$, H-strongly measurable and adapted and forward integrable, then $\int_0^T G\, dW^-$ is in $D(A)$, AG is forward integrable, and a.s.*

$$A \int_0^T G\, dW^- = \int_0^T AG\, dW^-.$$

The property (3) is a stochastic version of Hille's theorem (see [40, Theorem II.6]). A version for the Itô integral can be found in [27, Lemma 2.8].

Proof. (1) and (2) are straightforward from the definition. To prove (3), note that by Hille's theorem,

$$A \int_0^T (G(s)h_k)(W(s+\tfrac{1}{n}) - W(s))h_k\, ds = \int_0^T A(G(s)h_k)(W(s+\tfrac{1}{n}) - W(s))h_k\, ds.$$

It follows that $A(I^-(G,n)) = I^-(AG,n) \to \int_0^T AG\, dW^-$ in probability. Also, $I^-(G,n) \to \int_0^T G\, dW$ in probability. Hence one can find a set $\Omega_0 \in \mathscr{F}$ with $\mathbb{P}(\Omega_0) = 1$ and a subsequence $(n_k)_{k \geq 1}$ such that for all $\omega \in \Omega_0$, $I^-(G, n_k)(\omega) \to \Big(\int_0^T G\, dW^-\Big)(\omega)$ and $A(I^-(G, n_k)(\omega)) \to \Big(\int_0^T AG\, dW^-\Big)(\omega)$. Now the result follows from the assumption that A is closed. □

Using the forward integral it is easy to deduce local properties of the stochastic integral.

Remark 4.8. From Lemma 4.7 it follows that for a forward integrable process G and a set $B \in \mathscr{F}$, $\mathbf{1}_B G$ is forward integrable and

$$\int_0^T \mathbf{1}_B G \, dW^- = \mathbf{1}_B \int_0^t G \, dW^-(s).$$

In particular, if $G \in L^0(\Omega; \gamma(0,T;H,X))$ is adapted and for all $x^* \in X^*$, $G^*x^* = 0$ on a set $(0,T) \times B$, then a.s.

$$0 = \int_0^t \mathbf{1}_B G^* x^* \, dW^- = \Big\langle \mathbf{1}_B \int_0^t G \, dW^-, x^* \Big\rangle, \quad x^* \in X^*, t \in [0,T].$$

In particular, we deduce that $\int_0^{\cdot} G \, dW^- = 0$ on B a.s.

4.4 Convergence and path regularity

In this section we will give conditions under which for adapted G one has $J^-(G,n) \to J(G)$ in the Sobolev norm. Before we start we introduce a useful class of functions.

Definition 4.9. For $\beta \in [0, \frac{1}{2})$ and $p \in [1, \infty)$, let $V^{\beta,p}(0,T;H,X)$ denote the space of H-strongly measurable $G : (0,T) \to \mathscr{L}(H,X)$ for which for almost all $t \in [0,T]$, $r \mapsto (t-r)^{-\beta} G(r)$ is in $\gamma(0,t;H,X)$ and

$$\|G\|_{V^{\beta,p}(0,T;H,X)} := \Big(\int_0^T \|r \mapsto (t-r)^{-\beta} G(r)\|^p_{\gamma(0,t;H,X)} \, dt \Big)^{1/p} < \infty.$$

The spaces $V^{\beta,p}(0,T;H,X)$ were introduced in [87] in order to study stochastic evolution equations of semilinear type. They also play a major role in [28] and [65], where results on approximation of SPDEs have been derived. Although the spaces $V^{\alpha,p}$ look rather involved at first sight they are quite useful and not too difficult to work with. Many properties of Bochner spaces are inherited by the spaces $V^{\beta,p}(0,T;H,X)$. The main motivation for the weight inside the γ-norm is that it increases the integrability properties of G without leaving the γ-setting.

The following embedding results are straightforward from the definition and (4.2.4)

$$V^{\beta,p_0}(0,T;H,X) \hookrightarrow V^{\beta,p_1}(0,T;H,X) \ \text{ if } 1 \leq p_1 < p_0 < \infty.$$
$$V^{\beta_0,p}(0,T;H,X) \hookrightarrow V^{\beta_1,p}(0,T;H,X) \ \text{ if } 0 \leq \beta_1 < \beta_0 < \frac{1}{2}.$$

The next proposition gives several embedding properties for $V^{\beta,p}(0,T;H,X)$. In particular they give new insights for results in [28], [65] and [87]. Details on (co)type properties of a Banach space can be found in [39, Chapter 11]. Recall that every Hilbert space has type 2, and $X = L^q$ (or $X = W^{s,q}$) has type 2 if and only if $q \in [2, \infty)$. Moreover, for $q < \infty$, L^q has cotype $q \vee 2$.

Proposition 4.10. *Let $p \geq 1$ and $\beta \in [0, \frac{1}{2})$.*

(1) *If $G \in V^{\beta,1}(0,T;H,X)$, then for all $\varepsilon \in (0,T)$ one has $G \in \gamma(0,T-\varepsilon;H,X))$ and*

$$\|G\|_{\gamma((0,T-\varepsilon);H,X)} \leq \frac{T^\beta}{\varepsilon}\|G\|_{V^{\beta,1}(0,T;H,X)}.$$

Moreover, if $\beta > \frac{1}{p}$, then

$$V^{\beta,p}(0,T;H,X) \hookrightarrow \gamma(0,T;H,X).$$

(2) *If X has cotype $p \in [2,\infty)$ and $\beta \in [0, \frac{1}{p})$, then*

$$\gamma(0,T;H,X)) \hookrightarrow V^{\beta,p}(0,T;H,X).$$

(3) *If X has type 2 and $p \in [2,\infty)$, then*

$$L^p(0,T;\gamma(H,X)) \hookrightarrow V^{\beta,p}(0,T;H,X).$$

Under type p assumptions one can show that $V^{\beta,p}(0,T;H,X)$ contains certain fractional Sobolev spaces or Hölder spaces, but we will not go into details on this (see [87, Lemma 3.3] and [65, Lemma 3.8] for some details in this direction).

Note that for $G \in V^{\beta,p}(0,T;H,X)$, the function $u \mapsto \mathbf{1}_{[0,u]}G$ is continuous from $[0,T]$ into $V^{\beta,p}(0,T;H,X)$ (see [87, Section 7]).

Proof. (1): For every $s \in [0,T)$, we can write

$$G(s) = \int_0^T (t-s)^{-\beta}G(s)(T-s)^{-1}(t-s)^\beta \mathbf{1}_{[0,t]}(s)\,dt,$$

it follows that for $\varepsilon \in [0,T]$ one has

$$\|G\|_{\gamma((0,T-\varepsilon);H,X)} \leq \int_0^T \|(t-s)^{-\beta}G(s)(T-s)^{-1}(t-s)^\beta \mathbf{1}_{[0,t]}(s)\|_{\gamma((0,T-\varepsilon);H,X)}\,dt. \tag{4.4.1}$$

If $\varepsilon \in (0,T)$ and $s \in [0,T-\varepsilon)$, then $(T-s)^{-1}(t-s)^\beta \leq \varepsilon^{-1}T^\beta$, and thus by (4.2.4)

$$\|G\|_{\gamma((0,T-\varepsilon);H,X)} \leq \varepsilon^{-1}T^\beta\|G\|_{V^{\beta,1}(0,T;H,X)}.$$

Next assume $\beta > \frac{1}{p}$ and take $\varepsilon = 0$ in (4.4.1). Note that for all $t \in [0,T)$ and $s \in [0,t]$, $(T-s)^{-1}(t-s)^\beta \leq (T-t)^{\beta-1}$. Therefore, by (4.2.4), and Hölder's inequality,

$$\begin{aligned}\|G\|_{\gamma(0,T;H,X)} &\leq \int_0^T \|(t-s)^{-\beta}G(s)\mathbf{1}_{[0,t]}(s)\|_{\gamma(0,t;H,X)}(T-t)^{\beta-1}\,dt \\ &\leq \|G\|_{V^{\beta,p}(0,T;H,X)}\Big(\int_0^T (T-t)^{(\beta-1)p'}\,dt\Big)^{1/p'} \\ &\leq C\|G\|_{V^{\beta,p}(0,T;H,X)}.\end{aligned}$$

(2): Let $G \in \gamma(0,T;H,X)$. Let $\phi_t(r) = \mathbf{1}_{(0,t)}(r)(t-r)^{-\beta}$ and $M_\beta : (0,T) \to \mathscr{L}(X, L^p(0,T;X))$ be given by $M_\beta(t)x = \phi_t x$. Observe that by the γ-Fubini isomorphism (see [86, Proposition 2.6]) and the definition of $V^{\beta,p}$

$$c^{-1}\|G\|_{V^{\beta,p}(0,T;H,X)} \leq \|M_\beta G\|_{\gamma(0,T;H,L^p(0,T;X))} \leq c\|G\|_{V^{\beta,p}(0,T;H,X)} \quad (4.4.2)$$

For $\beta < \frac{1}{p}$ and $t \in (0,T)$, one has

$$\begin{aligned} K :&= \int_0^\infty \sup_{t\in(0,T)} \mu\big(\{r \in (0,t) : \phi_t(r) > s\}\big)^{1/p}\, ds \\ &= \int_0^\infty \sup_{t\in(0,T)} (t \wedge s^{-\frac{1}{\beta}})^{1/p}\, ds = \int_0^\infty T^{\frac{1}{p}} \wedge s^{-\frac{1}{\beta p}}\, ds < \infty. \end{aligned}$$

Therefore, it follows from [50, Lemma 3.1] that $\{M_\beta(t) : t \in (0,T)\}$ is R-bounded by CK, and hence by the Kalton–Weis γ-multiplier theorem (see [85, Theorem 5.2]), we find that

$$\|M_\beta G\|_{\gamma(0,T;H,L^p(0,T;X))} \leq CK\|G\|_{\gamma(0,T;H,X)},$$

where we used the fact that $L^p(0,T;X)$ does not contain a copy of c_0 as it has finite cotype (see [39, page 212 and Theorem 11.12]). Combining the latter estimate with (4.4.2), the required result follows.

(3): From $L^2(0,T;\gamma(H,X)) \hookrightarrow \gamma(0,T;H,X)$ (see [85, Theorem 11.6]) and Young's inequality for convolutions we obtain

$$\begin{aligned} \|G\|^p_{V^{\beta,p}(0,T;H,X)} &= \int_0^T \|r \mapsto (t-r)^{-\beta}G(r)\|^p_{\gamma(0,t;H,X)}\, dt \\ &= C\int_0^T \Big(\int_0^t (t-r)^{-2\beta}\|G(r)\|^2_{\gamma(H,X)}\, dr\Big)^{p/2} dt \\ &\leq C\Big(\int_0^T r^{-2\beta}\, dr\Big)^{p/2} \int_0^T \|G(r)\|^p_{\gamma(H,X)}\, dr \\ &= C'\|G\|^p_{L^p(0,T;\gamma(H,X))}. \end{aligned}$$

□

Example 4.11. Let X be a Hilbert space. In the case that $p = 2$ and $\beta \in [0, \frac{1}{2})$, by (4.2.5) and Fubini's theorem, one has

$$V^{\beta,2}(0,T;H,X) = L^2((0,T), \mu_{\alpha,T}; \mathrm{S}^2(H,X)) \quad (4.4.3)$$

where and $d\mu_{\alpha,T}(r) = (T-r)^{1-2\beta}\, dr$. Moreover, by Proposition 4.10 one has

$$L^p(0,T;\gamma(H,X)) \hookrightarrow V^{\beta,p}(0,T;H,X) \quad \text{for all } p \geq 2 \text{ and } \beta \in [0, \tfrac{1}{2})$$

$$L^2(0,T;\mathrm{S}^2(H,X)) \hookrightarrow V^{\beta,p}(0,T;H,X) \quad \text{for all } p \geq 2 \text{ and } \beta \in [0, \tfrac{1}{p})$$

$$V^{\beta,p}(0,T;H,X) \hookrightarrow L^2(0,T;\mathrm{S}^2(H,X)) \quad \text{for all } p \geq 2 \text{ and } \beta \in (\tfrac{1}{p}, \tfrac{1}{2})$$

Proposition 4.12. *Let $p \in [1,\infty)$ and suppose $0 < \alpha < \beta < \frac{1}{2}$. If $G \in L^0(\Omega; V^{\beta,p}(0,T;H,X))$ is adapted, then $J(G) \in L^0(\Omega; W^{\alpha,p}(0,T;X))$. Furthermore, the following assertions hold:*

(1) *There exists a constant C independent of G such that*

$$\|J(G)\|_{L^p(\Omega;W^{\alpha,p}(0,T;X))} \leq C\|G\|_{L^p(\Omega;V^{\beta,p}(0,T;H,X))}.$$

(2) *For every $n \geq 1$, assume that $G_n \in L^0(\Omega; V^{\beta,p}(0,T;H,X))$ is an adapted process. If $G_n \to G$ in $L^0(\Omega; V^{\beta,p}(0,T;H,X))$, then*

$$J(G_n) \to J(G) \quad \text{in } L^0(\Omega; W^{\alpha,p}(0,T;X)).$$

Remark 4.13. Note that by under the above assumptions by Proposition 4.10 (1), one has $G\mathbf{1}_{[0,t]} \in L^0(\Omega; \gamma(0,T;H,X))$ for all $t \in [0,T)$, and therefore, $J(G)(t)$ is well-defined for every $t \in (0,T)$.

Remark 4.14. If $\frac{1}{p} < \alpha < \frac{1}{2}$, we can use the Sobolev embedding theorem (4.2.8), to replace $W^{\alpha,p}(0,T;X)$ by $C^{\alpha-\frac{1}{p}}(0,T;X)$ in the above result.

Example 4.15. Let X be a Hilbert space. From Example 4.11, we see that by (4.4.3) and Proposition 4.12, for every $G \in L^0(\Omega; L^2((0,T),\mu_{\alpha,T}; \mathrm{S}^2(H,X)))$ adapted, one has $J(G) \in L^0(\Omega; W^{\alpha,p}(0,T;X))$. Note that such a process G is not necessarily in $L^0(\Omega; L^2(0,T;\mathrm{S}^2(H,X)))$. In the case $H = X = \mathbb{R}$ an example is given by $G(t) = (T-t)^{-\frac{1}{2}-\varepsilon}$ with $\varepsilon > 0$.

Indeed, one easily checks that $G \in L^2((0,T),\mu_{\alpha,T})$ if and only if $\varepsilon + \alpha < 1/2$, and in that case $J(G) \in W^{\alpha,p}(0,T;X)$ a.s. However, $G \notin L^2(0,T)$. This singular behavior can only occur at the point $t = T$ as follows from Proposition 4.10 (1).

Proof (Proof of Proposition 4.12). To prove (1), note that for $0 \leq s \leq r < t \leq T$, one has $1 \leq (t-s)^{\beta}(t-r)^{-\beta}$, and hence by (4.2.7) and (4.2.4), we have

$$\begin{aligned}\mathbb{E}\|J(G)(t) - J(G)(s)\|^p &\leq C\mathbb{E}\|G\|^p_{\gamma(s,t;H,X)} \\ &\leq C(t-s)^{\beta p}\mathbb{E}\|r \mapsto (t-r)^{-\beta}G(r)\|^p_{\gamma(s,t;H,X)} \qquad (4.4.4)\\ &\leq C(t-s)^{\beta p}\mathbb{E}\|r \mapsto (t-r)^{-\beta}G(r)\|^p_{\gamma(0,t;H,X)}.\end{aligned}$$

By Fubini's theorem we find that

$$\begin{aligned}\mathbb{E}[J(G)]^p_{W^{\alpha,p}(0,T;X)} &= 2\int_0^T \int_0^t \frac{\mathbb{E}\|J(G)(t) - J(G)(s)\|^p}{(t-s)^{\alpha p+1}}\, ds\, dt \\ &\leq C\mathbb{E}\int_0^T \int_0^t \frac{\|r \mapsto (t-r)^{-\beta}G(r)\|^p_{\gamma(0,t;H,X)}}{(t-s)^{1-(\beta-\alpha)p}}\, ds\, dt \\ &\leq CT^{(\beta-\alpha)p}\mathbb{E}\int_0^T \|r \mapsto (t-r)^{-\beta}G(r)\|^p_{\gamma(0,t;H,X)}\, dt \\ &= CT^{(\beta-\alpha)p}\|G\|^p_{L^p(\Omega;V^{\beta,p}(0,T;H,X))},\end{aligned}$$

where we used $\beta > \alpha$. Taking $s = 0$ in (4.4.4), one also obtains

$$\begin{aligned}\mathbb{E}\|J(G)\|^p_{L^p(0,T;X)} &\leq CT^{\beta p}\mathbb{E}\int_0^T \|r \mapsto (t-r)^{-\beta}G(r)\|^p_{\gamma(0,t;H,X)}\,dt \\ &= T^{\beta p}\|G\|^p_{L^p(\Omega;V^{\beta,p}(0,T;H,X))}.\end{aligned}$$

Combining the estimates yields that $J(G) \in L^p(\Omega; W^{\alpha,p}(0,T;X))$ and (1) holds.

Before we continue to the proof of (2), we claim that for adapted $G \in L^0(\Omega; V^{\beta,p}(0,T;H,X))$, one has $J(G) \in L^0(\Omega; W^{\alpha,p}(0,T;X))$. Indeed, let the stopping time τ be given by

$$\tau_n = \inf\{t \in [0,T] : \|\mathbf{1}_{[0,t]}G\|_{V^{\beta,p}(0,T;H,X)} \geq n\},$$

where we put $\tau_n = T$ if the infimum is taken over the empty set. Then $\mathbf{1}_{[0,\tau_n]}G \in L^p(\Omega; V^{\beta,p}(0,T;H,X))$ and hence $t \mapsto J(G)(t \wedge \tau_n) = J(\mathbf{1}_{[0,\tau_n]}G)(t)$ is in $L^0(\Omega; W^{\alpha,p}(0,T;X))$. Since for almost every $\omega \in \Omega$, we can find an $n \geq 1$ with $\tau_n(\omega) = T$, we find that $J(G) \in W^{\alpha,p}(0,T;X)$ almost surely and the claim follows.

To prove (2) we use another stopping time argument. By linearity we can replace G_n by $G_n - G$ and hence it suffices to consider $G = 0$. Moreover, by a subsequence argument it suffices to consider the case that $G_n \to 0$ in $V^{\beta,p}(0,T;H,X)$ almost surely. For $n \geq 1$ let τ_n be the stopping time given by

$$\tau_n = \inf\{s \in [0,T] : \|\mathbf{1}_{[0,s]}G_n\|_{V^{\beta,p}(0,T;H,X)} \geq 1\},$$

Since $G_n \to 0$ in $V^{\beta,p}(0,T;H,X)$ almost surely, we find that $\lim_{n\to\infty}\mathbb{P}(\tau_n = T) = 1$. Since $\|\mathbf{1}_{[0,\tau_n]}G_n\|_{V^{\beta,p}(0,T;H,X)} \leq 1$, and

$$\|\mathbf{1}_{[0,\tau_n]}G_n\|_{V^{\beta,p}(0,T;H,X)} \leq \|G_n\|_{V^{\beta,p}(0,T;H,X)} \to 0 \text{ a.s.},$$

the dominated convergence theorem gives that $\mathbf{1}_{[0,\tau_n]}G_n \to 0$ in the space $L^p(\Omega; V^{\beta,p}(0,T;H,X))$. In particular, by (1) one has $J(\mathbf{1}_{[0,\tau_n]}G_n) \to 0$ in $L^p(\Omega; W^{\alpha,p}(0,T;X))$. Choose $\varepsilon > 0$ arbitrary. Then using $J(\mathbf{1}_{[0,\tau_n]}G_n)(t) = J(G_n)(t \wedge \tau_n)$ we find that

$$\begin{aligned}\mathbb{P}\big(\|J(G_n)\|_{W^{\alpha,p}(0,T;X)} \geq \varepsilon\big) &\leq \mathbb{P}\big(\|J(G_n)\|_{W^{\alpha,p}(0,T;X)} \geq \varepsilon, \tau_n = T\big) + \mathbb{P}(\tau_n < T) \\ &\leq \mathbb{P}\big(\|J(\mathbf{1}_{[0,\tau_n]}G_n)\|_{W^{\alpha,p}(0,T;X)} \geq \varepsilon\big) + \mathbb{P}(\tau_n < T). \\ &\leq \varepsilon^{-p}\mathbb{E}\|J(\mathbf{1}_{[0,\tau_n]}G_n)\|^p_{W^{\alpha,p}(0,T;X)} + \mathbb{P}(\tau_n < T).\end{aligned}$$

Now the result follows by letting $n \to \infty$. □

The next result is the main result of this chapter and gives convergence of paths of the forward integral in Sobolev norms. With Remark 4.14 one can derive convergence in the Hölder norm as a consequence.

Theorem 4.16. *Let $p \in [1,\infty)$ and $0 < \alpha < \beta < \frac{1}{2}$.*

(1) *If* $G \in L^p(\Omega; V^{\beta,p}(0,T;H,X))$ *is adapted, then*

$$J^-(G,n) \to J(G) \quad in \quad L^p(\Omega; W^{\alpha,p}(0,T;X)).$$

(2) *If* $G \in L^0(\Omega; V^{\beta,p}(0,T;H,X))$ *is adapted, then*

$$J^-(G,n) \to J(G) \quad in \quad L^0(\Omega; W^{\alpha,p}(0,T;X)).$$

From Remark 4.13 we see that $J(G)(t)$ and $J^-(G,n)(t)$ are well-defined for every $t \in [0,T)$.

Recall from (4.3.2) that

$$J^-(G,n)(t) = \int_0^\infty G_n \, dW, \quad \text{where} \quad G_n = n\mathbf{1}_{[0,\frac{1}{n}]} * (\mathbf{1}_{[0,t]} P_n G).$$

Since $G_n \to G$ in $L^0(\Omega; V^{\beta,p}(0,T;H,X)$, at first sight it seems that Proposition 4.12 can be used directly to obtain Theorem 4.16. Unfortunately, Proposition 4.12 does not apply because the process G_n also depends on t, and we need to proceed differently.

Proof. Before proving the assertion we note that if $G \in L^0(\Omega; V^{\beta,p}(0,T;H,X))$, $J(G) \in W^{\alpha,p}(0,T;X)$ a.s. by Proposition 4.12. We claim that $J^-(G,n) \in W^{\alpha,p}(0,T;X)$ a.s. Indeed,

$$\begin{aligned}
&\|J^-(G,n)(t) - J^-(G,n)(s)\| \\
&\quad = \Big\| \sum_{k=1}^n n \int_s^t G(r)h_k (W(r+1/n)h_k - W(r)h_k) \, dr \Big\| \\
&\quad \le \sum_{k=1}^n n \Big\| \int_0^T G(r)h_k \mathbf{1}_{[s,t]}(r)(W(r+1/n)h_k - W(r)h_k) \, dr \Big\| =: \sum_{k=1}^n nJ_k
\end{aligned}$$

By (4.2.3) we find that

$$\begin{aligned}
J_k \le\; & \|r \mapsto (t-r)^{-\beta}\mathbf{1}_{[0,r]}G(r)\|_{\gamma(0,t;H,X)} \\
& \times \|r \mapsto (t-r)^{\beta}(W(r+1/n)h_k - W(r)h_k)\|_{L^2(s,t)}.
\end{aligned}$$

Since the paths of $r \mapsto W(r+1/n)h_k - W(r)h_k$ are continuous, we have

$$\|r \mapsto (t-r)^{\beta}(W(r+1/n)h_k - W(r)h_k)\|_{L^2(s,t)} \le C(W,n)(t-s)^{\beta+\frac{1}{2}}.$$

It follows that

$$\begin{aligned}
[J^-(G,n)]^p_{W^{\alpha,p}(0,T;X)} &= 2\int_0^T \int_0^t \frac{\|J^-(G,n)(t) - J^-(G,n)(s)\|^p}{(t-s)^{\alpha p+1}} \, ds \, dt \\
&\le C_{W,n} \int_0^T \int_0^t \|r \mapsto (t-r)^{-\beta}\mathbf{1}_{[0,r]}G(r)\|^p_{\gamma(0,t;H,X)} (t-s)^{(\beta-\alpha+\frac{1}{2})p-1} \, ds \, dt \\
&\le C_{W,n,\alpha,\beta,p} \|G\|^p_{V^{\beta,p}(0,T;X)}
\end{aligned}$$

Similarly, one sees that $\|J^-(G,n)\|_{L^p(0,T;X)} < \infty$ a.s. and the claim follows.

(1): Observe that by (4.3.2) and (4.2.7),

$$\begin{aligned}&\mathbb{E}[J^-(G,n) - J(G)]^p_{W^{\alpha,p}(0,T;X))}\\&\leq C\mathbb{E}\int_0^T\int_0^T \mathbf{1}_{[0,t]}(s)\frac{\|n\mathbf{1}_{[0,\frac{1}{n}]} * (\mathbf{1}_{[s,t]}P_nG) - \mathbf{1}_{[s,t]}G\|^p_{\gamma(\mathbb{R}_+;H,X)}}{|t-s|^{\alpha p+1}}\,ds\,dt\end{aligned} \tag{4.4.5}$$

By Young's inequality one has $\|n\mathbf{1}_{[0,\frac{1}{n}]} * f\|_{L^2(\mathbb{R};H)} \leq \|f\|_{L^2(\mathbb{R};H)}$ for $f \in L^2(\mathbb{R};H)$. Therefore, by the right-ideal property and (4.2.4) for $0 \leq s \leq t \leq T$,

$$\begin{aligned}\|n\mathbf{1}_{[0,\frac{1}{n}]} * (\mathbf{1}_{[s,t]}P_nG) - \mathbf{1}_{[s,t]}G\|_{\gamma(\mathbb{R}_+;H,X)} &\leq 2\mathbb{E}\|\mathbf{1}_{[s,t]}G\|_{\gamma(s,t;H,X)}\\&\leq 2|t-s|^\beta\|(t-r)^{-\beta}G\|_{\gamma(0,t;H,X)}\end{aligned}$$

Now the latter is integrable on the space $\Omega \times [0,T]^2$ with measure $\mathbf{1}_{[0,t]}(s)(t-s)^{-\alpha p-1}\,ds\,dt\,d\mathbb{P}$, and it dominates the function $\mathbf{1}_{[0,t]}(s)\|n\mathbf{1}_{[0,\frac{1}{n}]} * (\mathbf{1}_{[s,t]}P_nG) - \mathbf{1}_{[s,t]}G\|^p_{\gamma(\mathbb{R}_+;H,X)}$, which depends on $0 \leq s \leq t \leq T$ and $\omega \in \Omega$. Moreover, by [86, Proposition 2.4]

$$\lim_{n\to\infty}\|n\mathbf{1}_{[0,\frac{1}{n}]} * (\mathbf{1}_{[s,t]}P_nG) - \mathbf{1}_{[s,t]}G\|_{\gamma(\mathbb{R}_+;H,X)} = 0$$

for all $0 \leq s \leq t \leq T$ and a.s. on Ω. Therefore, by the dominated convergence theorem, the right-hand side of (4.4.5) tends to zero as $n \to \infty$.

A similar argument yields that $\mathbb{E}\|J^-(G,n) - J(G)\|^p_{L^p(0,T;X)} \to 0$ as $n \to \infty$. This proves (1).

Next we prove (2) using a stopping time argument. Consider an element $G \in L^0(\Omega; V^{\beta,p}(0,T;H,X))$. For each $m \geq 1$ define

$$\tau_m = \inf\{[0,T] : \|\mathbf{1}_{[0,t]}G\|_{V^{\beta,p}(0,T;H,X)} \geq m\},$$

where we let $\tau_m = T$ if the infimum is taken over the empty set. Let $G_m = \mathbf{1}_{[0,\tau_m]}G$. Clearly, $\lim_{m\to\infty}\mathbb{P}(\tau_m = T) = 1$. Observe that almost surely, for all $t \in [0,T]$, $J(G)(\tau_m \wedge t) = J(\mathbf{1}_{[0,\tau_m]}G)(t)$ and $J^-(G,n)(\tau_m \wedge t) = J^-(\mathbf{1}_{[0,\tau_m]}G,n)(t)$. The latter is trivial as $J^-(\cdot,n)$ is defined in a pathwise sense.

Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary and choose m so large that $\mathbb{P}(\tau_m < T) < \delta$. It follows that for all $n \geq 1$,

$$\begin{aligned}&\mathbb{P}\big(\|J(G) - J^-(G,n)\|_{W^{\alpha,p}(0,T;X)} \geq \varepsilon\big)\\&\leq \mathbb{P}\big(\|J(G) - J^-(G,n)\|_{W^{\alpha,p}(0,T;X)} \geq \varepsilon, \tau_m = T\big) + \mathbb{P}(\tau_m < T).\\&\leq \mathbb{P}\big(\|J(\mathbf{1}_{[0,\tau_m]}G) - J^-(\mathbf{1}_{[0,\tau_m]}G,n)\|_{W^{\alpha,p}(0,T;X)} \geq \varepsilon\big) + \delta.\\&\leq \varepsilon^{-p}\mathbb{E}\|J(\mathbf{1}_{[0,\tau_m]}G) - J^-(\mathbf{1}_{[0,\tau_m]}G,n)\|^p_{W^{\alpha,p}(0,T;X)} + \delta.\end{aligned}$$

Since $\mathbf{1}_{[0,\tau_m]}G$ satisfies the conditions of (1) it follows that

$$\limsup_{n\to\infty}\mathbb{P}\big(\|J(G) - J^-(G,n)\|_{W^{\alpha,p}(0,T;X)} \geq \varepsilon\big) \leq \delta.$$

Since $\delta > 0$ was arbitrary, the result follows. □

If the space X is not only a UMD space, but additionally has type 2, one can obtain further conditions for a process to be in the spaces considered in Theorem 4.16. Both results below follow immediately from the embedding of Proposition 4.10 (2), Proposition 4.12 and Theorem 4.16. Similar corollaries can be deduced from Proposition 4.10 (3).

Corollary 4.17. *Assume X has type 2, and let $p \in [2,\infty)$ and $0 < \alpha < \frac{1}{2}$. If $G \in L^0(\Omega; L^p(0,T;\gamma(H,X)))$ is adapted, then $J(G) \in L^0(\Omega; W^{\alpha,p}(0,T;X))$. Furthermore, the following assertions hold:*

(1) *There exists a constant C such that independent of G such that*

$$\|J(G)\|_{L^p(\Omega;W^{\alpha,p}(0,T;X))} \leq C\|G\|_{L^p(\Omega;L^p(0,T;\gamma(H,X)))}.$$

(2) *Assume that for every $n \geq 1$, $G_n \in L^0(\Omega; L^p(0,T;\gamma(H,X)))$ is an adapted process. If $G_n \to G$ in $L^0(\Omega; L^p(0,T;\gamma(H,X)))$, then*

$$J(G_n) \to J(G) \quad \text{in } L^0(\Omega; W^{\alpha,p}(0,T;X)).$$

Corollary 4.18. *Assume X has type 2, and let $p \in [2,\infty)$.*

(1) *If $G \in L^p(\Omega; L^p(0,T;\gamma(H,X)))$ is adapted, then for all $\alpha \in (0,\frac{1}{2})$,*

$$J^-(G,n) \to J(G) \quad \text{in } L^p(\Omega; W^{\alpha,p}(0,T;X)).$$

(2) *If $G \in L^0(\Omega; L^p(0,T;\gamma(H,X)))$ is adapted, then for all $\alpha \in (0,\frac{1}{2})$,*

$$J^-(G,n) \to J(G) \quad \text{in } L^0(\Omega; W^{\alpha,p}(0,T;X)).$$

Again, Remark 4.14 applies to the above results and this will give convergence in the Hölder norm. The above result contains as a special case Theorem 4.1.

4.5 Nonadapted pointwise multipliers

In the next result we give sufficient smoothness conditions on a possibly non-adapted operator-valued process M and an adapted process G, such that MG becomes forward integrable. Moreover we derive a neat integration by parts formula which yields a very useful representation formula for the forward integral.

Theorem 4.19. *Let X and Y be UMD Banach spaces. Assume $p \in (2,\infty)$, $\delta \in [0,3/2)$ and $\beta \in (\frac{1}{p},\frac{1}{2})$ are such that $\beta - \frac{1}{p} - \delta + 1 > 0$. Let $M : [0,T] \times \Omega \to \mathscr{L}(X,Y)$ be such that*

(i) For all $x \in X$, $(t,\omega) \mapsto M(t,\omega)x$ is strongly measurable.
(ii) For almost all $\omega \in \Omega$, $t \mapsto M(t,\omega)$ is continuously differentiable on $[0,T)$ and there exists a constant $\delta \in [0,\frac{3}{2})$ such that for almost all $\omega \in \Omega$, there is a constant $C(\omega) > 0$ such that

$$\|M'(t,\omega)\| \leq C(\omega)(T-t)^{-\delta}, \quad t \in [0,T)$$

Assume $G \in L^0(\Omega; V^{\beta,p}(0,T;X))$ *is adapted and assume* MG *is weakly in* $L^2(0,T;H)$. *Then* MG *is forward integrable,* $s \mapsto M'(s)I(\mathbf{1}_{[s,T]}G) \in L^1(0,T;Y)$ *almost surely and*

$$\int_0^T M(s)G(s)\,dW^-(s) = M(0)I(G) + \int_0^T M'(s)I(\mathbf{1}_{[s,T]}G)\,ds. \tag{4.5.1}$$

Note that we do not assume any adaptedness properties on M.

Proof. By Proposition 4.10, (1) $G \in L^0(\Omega; \gamma(0,T;H,X)$.

Fix $t \in (0,T)$. Let $f_k = nGh_k(W(\cdot + 1/n)h_k - W(\cdot)h_k)$. Note that by (4.2.4) and the path continuity of Wh_k, we have $f_k \in L^0(\Omega; \gamma(0,t;X))$. Let $F_k : [0,t] \times \Omega \to X$ be given by $F_k(s) = \int_s^t f_k(r)\,dr$ and note that

$$\sum_{k=1}^n F_k(s) = I^-(\mathbf{1}_{[s,t]}G, n).$$

Fix $\omega \in \Omega$. By Lemma 4.3 both MG and Mf_k are in $\gamma(0,t;H,Y)$ and

$$\begin{aligned} I^-(M\mathbf{1}_{[0,t]}G, n) &= \sum_{k=1}^n \int_0^t M(s)f_k(s)\,ds \\ &= \sum_{k=1}^n M(0)F_k(0) + \int_0^t M'(s)F_k(s)\,ds \\ &= M(0)I^-(\mathbf{1}_{[0,t]}G, n) + \int_0^t M'(s)I^-(\mathbf{1}_{[s,t]}G, n)\,ds \end{aligned}$$

Now letting $t \uparrow T$, it follows from the observation below (4.3.1) that

$$M(0)I^-(\mathbf{1}_{[0,t]}G, n) \to M(0)I^-(G,n) \ \text{ and } I^-(M\mathbf{1}_{[0,t]}G, n) \to I^-(MG, n).$$

Next we claim that for $t \uparrow T$,

$$\int_0^t M'(s)I^-(\mathbf{1}_{[s,t]}G, n)\,ds \to \int_0^T M'(s)I^-(\mathbf{1}_{[s,T]}G, n)\,ds \tag{4.5.2}$$

Indeed, choose $\alpha \in (\frac{1}{p}, \beta)$ such that $\alpha - \frac{1}{p} - \delta + 1 > 0$. Note that by Theorem 4.16 and (4.2.8), $K := \|J^-(G,n)\|_{C^{\alpha - \frac{1}{p}}(0,T;X)} < \infty$ for almost all $\omega \in \Omega$. The difference of both of the terms in (4.5.2) can be estimated by

$$\begin{aligned} &\int_t^T \|M'(s)I^-(\mathbf{1}_{[s,T]}G, n)\|\,ds + \int_0^t \|M'(s)I^-(\mathbf{1}_{[t,T]}G, n)\|\,ds \\ &\quad \le CK\Big(\int_t^T (T-s)^{-\delta}(T-s)^{\alpha - \frac{1}{p}}\,ds + \int_0^t (T-s)^{-\delta}\,ds\,(T-t)^{\alpha - \frac{1}{p}}\Big) \\ &\quad \le CK\Big((T-t)^{\alpha - \frac{1}{p} - \delta + 1} + [T^{-\delta+1} + (T-t)^{-\delta+1}](T-t)^{\alpha - \frac{1}{p}}\Big), \end{aligned}$$

and the latter goes to zero as $t \uparrow T$.

We can conclude that almost surely for every $n \geq 1$

$$I^-(MG, n) = M(0)I^-(G, n) + \int_0^T M'(s)I^-(\mathbf{1}_{[s,T]}G, n)\, ds. \tag{4.5.3}$$

Hence to prove (4.5.1) we let $n \to \infty$ in the right-hand side of (4.5.3). Obviously, $M(0)I^-(G, n) \to M(0)I(G)$. From Theorem 4.16 and (4.2.8) we find that $\xi_n = [J^-(G, n) - J(G)]_{C^{\alpha-\frac{1}{p}}(0,T;X)} \to 0$ in probability as $n \to \infty$. It follows that

$$\begin{aligned}
\int_0^T &\big\|M'(s)[I^-(\mathbf{1}_{[s,T]}G, n) - I^-(\mathbf{1}_{[s,T]}G)]\big\|\, ds \\
&\leq C \int_0^T (T-s)^{-\delta} \|I^-(\mathbf{1}_{[s,T]}G, n) - I^-(\mathbf{1}_{[s,T]}G)]\|\, ds \\
&\leq C\xi_n \int_0^T (T-s)^{-\delta+\alpha-\frac{1}{p}}\, ds \\
&= C'\xi_n T^{1-\delta+\alpha-\frac{1}{p}}.
\end{aligned}$$

Since the latter converges to zero in probability, it follows that the right-hand side of (4.5.3) converges and hence MG is forward integrable and (4.5.1) holds. □

Remark 4.20. Assume M satisfies (i) and (ii) of Theorem 4.19.

(1) If $\delta \in [0, 1)$, then by Lemma 4.3 one has $MG \in L^0(\Omega; \gamma(0, T; H, Y))$ whenever $G \in L^0(\Omega; \gamma(0, T; H, Y))$. In particular MG is weakly in $L^2(0, T; H)$.

(2) If $0 \leq \delta < \frac{3}{2} - \frac{1}{p}$ and $G \in L^0(\Omega; L^p(0, T; \gamma(H, X)))$, then we have $MG \in L^0(\Omega; L^2(0, T; \gamma(H, Y)))$. Indeed, without loss of generality we can assume $\delta > 1$. It follows that

$$\|M(s)\| \leq C \int_0^t (T-s)^{-\delta}\, ds \leq C\big((T-t)^{1-\delta} + T^{1-\delta}\big).$$

Therefore, by Hölder's inequality with $\frac{1}{q} + \frac{2}{p} = 1$,

$$\begin{aligned}
\|MG\|_{L^2(0,T;\gamma(H,Y))} &\leq C\Big(\int_0^T \big((T-t)^{1-\delta} + T^{1-\delta}\big)^2 \|G(t)\|^2_{\gamma(H,X)}\, dt\Big)^{1/2} \\
&\leq C\|G\|_{L^p(0,T;\gamma(H,X))}.
\end{aligned}$$

From Theorem 4.19, Proposition 4.10 and Remark 4.20 we immediately derive the following:

Corollary 4.21. *Assume X and Y are* UMD *Banach space with type* 2 *and assume M satisfies (i) and (ii) of Theorem 4.19. Assume $p > 2$ and $\delta < \frac{3}{2} - \frac{1}{p}$. If $G \in L^0(\Omega; L^p(0, T; \gamma(H, X)))$ is adapted, then MG is forward integrable, $s \mapsto M'(s)I(\mathbf{1}_{[s,t]}G) \in L^1(0, T; Y)$ almost surely, and* (4.5.1) *holds.*

As an illustration we present a brief indication how the results of this section can be applied to stochastic evolution equations.

Example 4.22. Assume that for each $\omega \in \Omega$, $(A(t,\omega))_{t\in[0,T]}$ is a family of unbounded operators which generates an evolution family $(S(t,s,\omega))_{0\leq s\leq t\leq T,\omega\in\Omega}$ on a Banach space X_0. Assume that $X_1 = D(A(t,\omega))$ does not depend on time and $\omega \in \Omega$, and $A : [0,T]\times\Omega \to \mathscr{L}(X_1, X_0)$ is adapted. In general, $\omega \mapsto S(t,s,\omega)$ will only be $\mathscr{F}_t$-measurable, and hence the stochastic convolution

$$\int_0^t S(t,s)G(s)\,dW(s)$$

does not exist as an Itô integral. In many situations one can check that $\frac{d}{ds}S(t,s) = -S(t,s)A(s)$ satisfies $\left\|\frac{d}{ds}S(t,s,\omega)\right\| \leq C(\omega)(t-s)^{-1}$ (see [4] and [73]). Therefore, Theorems 4.19 and Corollary 4.21 with $M(s) = S(t,s)$ can be used to obtain sufficient conditions for the existence of the forward convolution

$$\begin{aligned} U(t) &:= \int_0^t S(t,s)G(s)\,dW^-(s) \\ &= S(t,0)I(\mathbf{1}_{[0,t]}G) - \int_0^t S(t,s)A(s)I(\mathbf{1}_{[s,t]}G)\;ds. \end{aligned} \tag{4.5.4}$$

In [68] León and Nualart have observed that the forward integral gives a weak solution of the stochastic evolution equation

$$dU = A(t)U(t)\,dt + G(t)\,dW(t), \quad U(0) = 0,$$

and even more general equations. Using (4.5.4) one can obtain a rather complete theory for non-autonomous stochastic evolution equations with random drift. Details can be found in chapter 5.

Part II

Stochastic evolution equations

5

A new approach to stochastic evolution equations with adapted drift

5.1 Introduction

Let E_0 be a Hilbert or Banach space and let H be a separable Hilbert space. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathscr{F}_t)_{t\in[0,T]}$. We study the following stochastic evolution system on E_0.

$$\begin{cases} dU(t) = (A(t)U(t) + F(t, U(t)))\, dt + B(t, U(t))\, dW(t) \\ U(0) \ = u_0. \end{cases} \tag{5.1.1}$$

Here $(A(t,\omega))_{t\in[0,T],\omega\in\Omega}$ is a measurable and adapted family of unbounded operators on E_0. Moreover, F and B are semilinear nonlinearities and W is a cylindrical Brownian motion.

The integrated form of (5.1.1) often leads to problems as in general $A(t)U(t)$ is not well-defined or not integrable with respect to time. In the semigroup approach to (5.1.1) this difficulty does not occur. We refer to the monograph [32] and references therein for details on the semigroup approach to (5.1.1) in the Hilbert space setting. Extensions to the class of Banach spaces with martingale type 2 can be found in [20] in the case A is not depending on time. An extension to the nonautonomous setting (i.e. A depends on time) can be found in [135]. In the semigroup approach to (5.1.1), the mild formulation is particularly useful for fixed point arguments. In the time-dependent setting the mild formulation has the following form:

$$U(t) = S(t,0)u_0 + \int_0^t S(t,s)F(s,U(s))\, ds + \underbrace{\int_0^t S(t,s)B(s,U(s))\, dW(s)}_{\text{well-defined?}} \tag{5.1.2}$$

Here, given $\omega \in \Omega$, $(S(t,s))_{0\le s\le t\le T}$ is the evolution system generated by $(A(t,\omega))_{t\in[0,T]}$. In this case, there is an obstruction in the mild formulation of a solution. The problem is that $\omega \mapsto S(t,s,\omega)$ does not satisfy the right adaptedness properties. In general $\omega \mapsto S(t,s,\omega)$ is only $\mathscr{F}_t$-measurable and not

$\mathscr{F}_s$-measurable (see Example 5.5). Therefore, the stochastic integral in (5.1.2) cannot be defined in the sense of Itô. An easy example can be found in Example 5.5 below. Equations with random generators arise naturally in the case A depends on a stochastic process, e.g. in filtering theory (see [140] and references therein).

There are several different approaches to (5.1.1). In the method of monotone operators (see [62], [102], [108], [113]) the problem (5.1.1) is formulated on a Hilbert space and one can use Galerkin approximation well-posedness questions to reduce the problem to the finite-dimensional setting. In this way no additional difficulty arises when A is dependent on Ω and time. Also in the L^p-approach of Krylov [61] one can allow the coefficient of a second order operator A on $\mathbb{R}^d$ to be dependent on Ω and time in a measurable way. The above mentioned approaches do not use the mild formulation (5.1.2).

Mild formulations can be useful in many type of fixed point arguments. They are also used to study long time behavior (invariant measures) and time regularity. There have been several attempts to extend the mild approach to (5.1.1) to the Ω-dependent setting. A possible method for (5.1.1) using mild formulations is to use stochastic integration for nonadapted integrands and Malliavin calculus. This has been studied in $[6, 7, 68, 69, 97]$. This approach is based on Skorohod integration techniques and it requires certain Malliavin differentiability of the operators $A(t)$ or $S(t, s)$. Let us also mention that in [88] a maximal regularity approach to (5.1.1) with random A has been developed.

In this chapter we will develop a new method for the stochastic evolution equation (5.1.1) with random A. It is based on a new representation formula for stochastic convolution. In order to explain this representation formula, consider

$$\begin{cases} dU(t) = A(t)U(t)\,dt + G\,dW(t), \\ U(0) \ \ = 0, \end{cases} \tag{5.1.3}$$

where G is an adapted and measurable process and A is as before. Our new representation formula for the solution to (5.1.3) is:

$$U(t) = -\int_0^t S(t,s)A(s)I(\mathbf{1}_{(s,t)}G)\,ds + S(t,0)I(\mathbf{1}_{(0,t)}G), \tag{5.1.4}$$

where $I(\mathbf{1}_{(s,t)}G) = \int_s^t G\,dW$. The representation (5.1.4) can basically be obtained using by integration by parts formula for the stochastic convolution. The advantage of the formulation is that it does not require stochastic integration of nonadapted integrands. A difficulty in (5.1.4) is that the norm of the operator-valued kernel $S(t, s)A(s)$ is of order $(t-s)^{-1}$. Fortunately, the Bochner integral in (5.1.4) can still be shown to be convergent as the paths of $t \mapsto I(\mathbf{1}_{(0,t)}G)$ have additional Hölder or Sobolev regularity.

In order to have evolution families with sufficient regularity properties, we will restrict ourselves to the parabolic setting. We will assume that the operators $(A(t))_{t\in[0,T]}$ satisfy the so-called (AT)-conditions which were introduced by Acquistapace and Terreni. This is a combination of a uniform sectoriality condition and a Hölder condition on the resolvents. We will allow Ω-dependent Hölder constants in the latter, which is quite reasonable from the point of view of applications.

This chapter is organized as follows. In Section 5.2 we will discuss the (AT)-conditions, and extend some of their results to the ω-dependent setting. In the Section 5.3 we present a new pathwise regularity result, which will allow to obtain the usual parabolic regularity of the solution to (5.1.3). In Section 5.4 we discuss the new representation formula (5.1.4) and its relations to other solutions concepts such as strong, variational, weak and mild solutions. In Section 5.5 we discuss a general semilinear problem and prove well-posedness with a fixed point argument. For this we first obtain well-posedness under the assumption that the constants in the (AT)-conditions are ω-independent Hölder conditions. After that we localize the Hölder condition and extend the result to the general case. Finally, we illustrate our results with Examples in Section 5.6.

This chapter is based on the paper [111].

5.2 Stochastic evolutions families

Let E_0 be a Banach space. In this section we will be concerned with generation properties of families of unbounded operators. For $t \in [0,T]$ and $\omega \in \Omega$ fixed, we consider a closed and densely defined operator

$$A(t,\omega) : E_0 \supset D(A(t,\omega)) \to E_0$$

For convenience, we sometimes write $A(t)$ and $D(A(t))$ instead of $A(t,\omega)$ and $D(A(t,\omega))$, respectively.

We will only consider the parabolic setting (i.e. the case where each $A(t,\omega)$ generates an analytic semigroup). This is well-documented in the literature (see [9, 73, 104, 130, 131]).

5.2.1 Generation theorem

In this subsection we will consider the conditions introduced by Acquistapace and Terreni [2] (see also [1, 4, 9, 125, 131, 141, 142] and references therein). An important difficulty in our situation is that $A(t,\omega)$ depends on the additional parameter $\omega \in \Omega$.

For $\vartheta \in (\pi/2, \pi)$ we set

$$\Sigma_\vartheta = \{\lambda \in \mathbb{C} : \; |\arg \lambda| < \vartheta\}.$$

On A we will assume a sectoriality condition and a Hölder continuity assumption:

(AT1) There exists a $\vartheta \in (\pi/2, \pi)$ and $M > 0$ such that for every $(t,\omega) \in [0,T] \times \Omega$, one has $\Sigma_\vartheta \cup \{0\} \subset \rho(A(t,\omega))$ and

$$\|R(\lambda, A(t,\omega))\|_{\mathscr{L}(E_0)} \leq \frac{M}{|\lambda|+1}, \quad \lambda \in \Sigma_\vartheta \cup \{0\}.$$

(AT2) There exist $0 < \nu, \mu \leq 1$ with $\mu + \nu > 1$ such that for every $\omega \in \Omega$, there exists a constant $L(\omega) \geq 0$ such that for all $s, t \in [0,T]$ and $\lambda \in \Sigma_\vartheta$,

$$|\lambda|^\nu \|A(t,\omega)R(\lambda, A(t,\omega))(A(t,\omega)^{-1} - A(s,\omega)^{-1})\|_{\mathscr{L}(E_0)} \leq L(\omega)|t-s|^\mu.$$

We would like to point out that it will be important that in Hölder continuity assumption the Hölder constant is allowed to depend on ω. Whenever (AT1) and (AT2) hold, it is said that (AT) holds. The abbreviation (AT) stands for Acquistapace and Terreni.

In the sequel we will not write the dependence on $\omega \in \Omega$ explicitly whenever there is no danger of confusion.

Example 5.1. Assume $E_1 = D(A(t,\omega))$ is constant with uniform estimates in $t \in [0,T]$ and $\omega \in \Omega$. Assume (AT1) holds. If there is a $\mu \in (0,1]$ and a mapping $C : \Omega \to \mathbb{R}_+$ such that

$$\|A(t) - A(s)\|_{\mathscr{L}(E_1,E_0)} \leq C|t-s|^\mu, \quad s,t \in [0,T],$$

then (AT2) holds with $\nu = 1$ and $L = MC$ up to a constant multiplicative factor. The above type of condition is sometimes called the Kato–Tanabe condition (see [104, 130]).

Let $\Delta := \{(s,t) \in [0,T]^2 : \ s \leq t\}$. The following result can be derived by applying [1, Theorem 2.3] pointwise in Ω.

Theorem 5.2. *Assume (AT). There exists a unique map $S : \Delta \times \Omega \to \mathscr{L}(E_0)$ such that*

1. *For all $t \in [0,T]$, $S(t,t) = I$.*
2. *For $r \leq s \leq t$, $S(t,s)S(s,r) = S(t,r)$.*
3. *For every $\omega \in \Omega$, the map $S(\cdot,\omega)$ is strongly continuous.*
4. *There exist a mapping $C : \Omega \to \mathbb{R}_+$ such that for all $s \leq t$, one has $\|S(t,s)\| \leq C$.*
5. *For every $s < t$, one has $\frac{d}{dt}S(t,s) = A(t)S(t,s)$ pointwise in Ω, and there exist a mapping $C : \Omega \to \mathbb{R}_+$ such that*

$$\|A(t)S(t,s)\|_{\mathscr{L}(E_0)} \leq C(t-s)^{-1}.$$

In the above situation we say that $(A(t))_{t\in[0,T]}$ *generates the evolution system/family* $(S(t,s))_{0\leq s\leq t\leq T}$.

5.2.2 Measurability

Throughout this subsection we assume that (AT) holds.

As the domains of $D(A(t,\omega))$ also vary in (t,ω), the most natural way is to formulate the adaptedness assumption for the resolvent as follows:

(H1) For some $\lambda \in \Sigma_\vartheta \cup \{0\}$, $R(\lambda, A(\cdot)) : [0,T] \times \Omega \to \mathscr{L}(E_0)$ is strongly measurable and adapted.

Here we consider measurability and adaptedness in the uniform operator topology. The hypothesis (H1) implies that for all $\lambda \in \Sigma_\vartheta \cup \{0\}$, $R(\lambda, A(\cdot))$ is strongly measurable and adapted. This follows from the fact that the resolvent can be expressed as a uniformly convergent power series (see [41, Proposition IV.1.3]).

Example 5.3. Assume the conditions of Example 5.1 hold. If $A : [0,T] \times \Omega \to \mathscr{L}(E_1, E_0)$ is strongly measurable and adapted, then (H1) holds. Indeed, fix $\omega_0 \in \Omega$. Since $(t,\omega) \mapsto A(t,\omega)A(0,\omega_0)^{-1}$ is strongly measurable and adapted and taking inverses is continuous on the open set of invertible operators, it follows that $(t,\omega) \mapsto A(0,\omega_0)A(t,\omega)^{-1}$ is strongly measurable and adapted. This clearly yields (H1).

Let $r > 0$ and $\eta \in (\pi/2, \vartheta)$, and consider the counterclockwise oriented curve

$$\gamma_{r,\eta} := \{\lambda \in \mathbb{C} : \ |\arg \lambda| = \eta, \ |\lambda| \leq r\} \cup \{\lambda \in \mathbb{C} : \ |\lambda| = r, -\eta \leq \arg \lambda \leq \eta\}.$$

For $s \in [0,T]$, consider the analytic semigroup $(e^{tA(s)})_{t\geq 0}$ defined by

$$e^{tA(s)}x = \begin{cases} \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A(s))x \, d\lambda, & t > 0, \\ x, & t = 0. \end{cases}$$

Proposition 5.4. *The evolution system $S : \Delta \times \Omega \to \mathscr{L}(E_0)$ is strongly measurable in the uniform operator topology. Moreover, for each $t \geq s$, $\omega \mapsto S(t,s,\omega) \in \mathscr{L}(E_0)$ is strongly $\mathscr{F}_t$-measurable in the uniform operator topology.*

In Example 5.5 we will show that the above measurability result cannot be improved in general.

Proof. Fix $0 \leq s < t \leq T$. The evolution system $S(t,s)$ is given in [1], as follows. Let $Q(t,s)$ be given by

$$Q(t,s) = A(t)^2 e^{(t-s)A(t)}(A(t)^{-1} - A(s)^{-1}).$$

Define inductively $Q_n(t,s)$ by

$$Q_1(t,s) = Q(t,s), \quad Q_n(t,s) = \int_s^t Q_{n-1}(t,r)Q(r,s) \, dr.$$

Then the evolution system $S(t,s)$ is given by

$$S(t,s) = e^{(t-s)A(s)} + \int_s^t Z(r,s) \, dr,$$

where

$$
\begin{aligned}
Z(t,s) :=\ & A(t)e^{(t-s)A(t)} - A(s)e^{(t-s)A(s)} \\
& + \sum_{n=1}^{\infty} \int_s^t Q_n(t,r)\Big(A(r)e^{(r-s)A(r)} - A(s)e^{(r-s)A(s)}\Big)\, dr \\
& + \sum_{n=1}^{\infty} \int_s^t (Q_n(t,r) - Q_n(t,s))A(s)e^{(r-s)A(s)}\, dr \\
& + \sum_{n=1}^{\infty} Q_n(t,s)(e^{(t-s)A(s)} - 1).
\end{aligned}
$$

The above series converges in $\mathscr{L}(E_0)$, see [1, Lemma 2.2 (1)].

Step 1: $S(t,s)$ is $\mathscr{F}_t$-measurable. Note that

$$
A(t)^n e^{(t-s)A(t)} = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{(t-s)\lambda} \lambda^n R(\lambda, A(t))\, d\lambda, \qquad n \in \mathbb{N}.
$$

Also, for $n \in \mathbb{N}$, note that $\lambda \mapsto e^{(t-s)\lambda}\lambda^n R(\lambda, A(t))$ is continuous on $\gamma_{r,\eta}$, and hence Riemann integrable. The random variable $e^{(t-s)\lambda}\lambda^n R(\lambda, A(t))$ is $\mathscr{F}_t$-measurable for every $\lambda \in \gamma_{r,\eta}$, hence every Riemann sum and thus every Riemann integral is $\mathscr{F}_t$-measurable. It follows that $A(t)^n e^{(t-s)A(t)}$ (as it is the limit of Riemann integrals) is $\mathscr{F}_t$-measurable. In particular this holds for $n = 2$, and hence $Q(t,s)$ is $\mathscr{F}_t$-measurable as well. On (s,t), the map $r \mapsto Q_{n-1}(t,r)Q(r,s)$ is continuous, by [1, Lemma 2.1]. Thus, by a similar argument as above, $Q_n(t,s)$ is $\mathscr{F}_t$-measurable, for $n \geq 2$. Also $e^{(t-s)A(s)}$ is $\mathscr{F}_t$-measurable. Hence the random variable

$$
\sum_{n=1}^{\infty} Q_n(t,s)(e^{(t-s)A(s)} - 1)
$$

is $\mathscr{F}_t$-measurable.

Clearly $r \mapsto A(s)e^{(r-s)A(s)}$ is continuous. By [1, Lemma 2.1], $r \mapsto Q_n(t,r) - Q_n(t,s)$ is continuous as well. Hence, as before we see that

$$
\sum_{n=1}^{\infty} \int_s^t (Q_n(t,r) - Q_n(t,s))A(s)e^{(r-s)A(s)}\, dr
$$

is $\mathscr{F}_t$-measurable.

The map $g : r \mapsto A(r)e^{(r-s)A(r)} - A(s)e^{(r-s)A(s)}$ for $r \in (s,t)$ is continuous. Indeed,

$$
\begin{aligned}
\|g(q) - g(r)\|_{\mathscr{L}(E_0)} \leq\ & \|A(q)e^{(q-s)A(q)} - A(r)e^{(q-r)A(r)}\|_{\mathscr{L}(E_0)} \\
& + \|A(r)(e^{(q-s)A(r)} - e^{(r-s)A(r)})\|_{\mathscr{L}(E_0)} \\
& + \|A(s)(e^{(r-s)A(s)} - e^{(q-s)A(s)})\|_{\mathscr{L}(E_0)}.
\end{aligned}
$$

Now [3, Lemma 1.10(i)] yields the required continuity of g and its integral will be $\mathscr{F}_t$-measurable again. Combining all terms we deduce that $Z(t,s)$ is $\mathscr{F}_t$-measurable. By [1, Lemma 2.2(ii)] the map $r \mapsto Z(r,s)$ is continuous on (s,t) and therefore, we can now deduce that $S(t,s)$ is $\mathscr{F}_t$-measurable.

Step 2: measurability of the process S. For $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n-1$, consider the triangle

$$D_{k,n} = \{(s,t) \in [0,T]^2 : \tfrac{k}{n} \le t \le \tfrac{k+1}{n},\ \tfrac{k}{n} \le s \le t\}.$$

Let I be the identity operator on E_0. Then for $0 \le s \le t \le T$, define $X_n : \Delta \times \Omega \to \mathscr{L}(E_0)$ by

$$X_n(t,s) := \sum_{k=1}^{n-1} \mathbf{1}_{D_{k,n}}(s,t) I + \sum_{k=0}^{n-2} \sum_{m=k+1}^{n-1} \mathbf{1}_{(\frac{k}{n},\frac{k+1}{n}]\times(\frac{m}{n},\frac{m+1}{n}]}(s,t) S\Big(\frac{m}{n},\frac{k}{n}\Big).$$

Since $S(t,s) : \Omega \to \mathscr{L}(E_0)$ is strongly measurable, by Step 1, it follows that $X_n : \Delta \times \Omega \to \mathscr{L}(E_0)$ is strongly measurable. Moreover, by strong continuity of S, pointwise on $\Delta \times \Omega$, one has $X_n \to S$. Hence $S : \Delta \times \Omega \to \mathscr{L}(E_0)$ is strongly measurable. □

Example 5.5. Let $E_0 = \mathbb{R}$ and let $A : [0,T] \times \Omega \to \mathbb{R}$ be a measurable and adapted process such that $\sup_{t\in[0,T]} |A(t,\omega)| < \infty$. Then A generates the evolution system

$$S(t,s,\omega) = \exp\Big(\int_s^t A(r,\omega)\, dr\Big).$$

Obviously $\omega \mapsto S(t,s,\omega)$ is only $\mathscr{F}_t$-measurable in general.

5.2.3 Pathwise regularity properties of evolution families

Throughout this subsection we assume that (AT) holds. First we recall some facts from interpolation theory. An overview on the subject can be found in [9,73,132].

For $\theta \in (0,1)$ and $1 \le p \le \infty$ the real interpolation space $E^t_{\theta,p} := (E_0, D(A(t)))_{\theta,p}$ is the subspace of all $x \in E_0$ for which

$$\|x\|_{(E_0,D(A(t)))_{\theta,p}} := \Big(\int_0^\infty s^{p(1-\theta)} \|A(t)e^{sA(t)}x\|^p_{E_0} \frac{ds}{s}\Big)^{1/p} < \infty,$$

with the obvious modification if $p = \infty$. Clearly, the space $E^t_{\theta,p}$ and its norm also depends on $\omega \in \Omega$, but this will be omitted from the notation. The space $E^t_{\theta,p}$ with the above norm is a Banach space. For convenience we also let $E^t_{0,p} := (E_0, D(A(t)))_{0,p} = E_0$ and $E^t_{1,p} := (E_0, D(A(t)))_{1,p} = D(A(t))$. By applying $A(t)$ finitely many times on both sides we extend the definition of the spaces $E^t_{\theta,p} := (E_0, D(A(t)))_{\theta,p}$ to all $\theta \ge 0$.

For all $\theta \in [0,\alpha)$

$$E^t_{\alpha,1} \hookrightarrow E^t_{\alpha,p} \hookrightarrow E^t_{\alpha,\infty} \hookrightarrow E^t_{\theta,1} \hookrightarrow E_0. \tag{5.2.1}$$

Here, the embedding constants only depend on the constants in (AT1) and thus are independent of time and Ω.

For $\theta \in (0,1)$, let $(-A(t,\omega))^{-\theta}$ be defined by

$$(-A(t))^{-\theta} = \frac{1}{\Gamma(\theta)} \int_0^\infty s^{\theta-1} e^{sA(t)}\, ds.$$

and let $(-A(t))^{\theta} = ((-A(t))^{-\theta})^{-1}$ with as domain the range of $(-A(t))^{-\theta}$. Endowed with the norm $\|x\|_{D((-A(t))^\theta)} = \|(-A(t))^\theta x\|_{E_0}$, the space $D((-A(t))^\theta)$ becomes a Banach space.

For $\theta \geq 0$, the following continuous embeddings hold:

$$E^t_{\theta,1} \hookrightarrow D(-A(t))^\theta \hookrightarrow E^t_{\theta,\infty}. \tag{5.2.2}$$

and again the embedding constants only depend on the constants in (AT1).

The next result follows from pointwise application of [125, (2.13), (2.15) and Proposition 2.4]. Recall that $\mu, \nu \in (0,1]$ are the smoothness constants from (AT2).

Lemma 5.6. *There exists a mapping $C : \Omega \to \mathbb{R}_+$ such that for all $0 \leq s < t \leq T$, for all $0 \leq \alpha < \beta \leq 1$, $\eta \in (0, \mu+\nu-1)$, $\gamma \in [0,\mu)$, $\theta \in [0,1]$ and $\delta, \lambda \in (0,1)$, the following inequalities hold*

$$\|S(t,s)x\|_{E^t_{\beta,1}} \leq C(t-s)^{\alpha-\beta}\|x\|_{E^s_{\alpha,\infty}}, \quad x \in E^s_{\alpha,\infty}. \tag{5.2.3}$$

$$\|A(t)S(t,s)x\|_{E^t_{\eta,1}} \leq C(t-s)^{-1-\eta}\|x\|, \quad x \in E_0. \tag{5.2.4}$$

$$\|A(t)S(t,s)x\|_{E^t_{\eta,1}} \leq C(t-s)^{-1-\eta+\delta}\|x\|_{E^s_{\delta,\infty}}, \quad x \in E^s_{\delta,\infty}. \tag{5.2.5}$$

$$\|S(t,s)(-A(s))^\gamma x\|_{E_0} \leq C(t-s)^{-\gamma}\|x\|, \quad x \in D((-A(s))^\gamma). \tag{5.2.6}$$

$$\|(-A(t))^\theta S(t,s)(-A(s))^{-\theta}\|_{\mathscr{L}(E_0)} \leq C$$

and $\Delta \ni (t,s) \mapsto (-A(t))^\theta S(t,s)(-A(s))^{-\theta}$ is strongly continuous.

In general C depend on the constants of (AT1) and (AT2). Note that to obtain (5.2.3) one needs to use reiteration in order to obtain the improvement from exponent ∞ to 1. Moreover, (5.2.5) follows from interpolation of (5.2.3) and (5.2.4) and reiteration.

5.2.4 Improved regularity under adjoint conditions

Throughout this section we assume the (AT)-conditions hold. To obtain further pathwise regularity properties we will assume in this section that E_0 is reflexive. Then $(A(t)^*)_{t\in[0,T],\omega\in\Omega}$ is a family of closed densely define operators on E_0^*. Moreover, since $R(\lambda, A(t)^*) = R(\lambda, A(t))^*$, (AT1) holds for this family as well. Furthermore, we will assume that the family of adjoints satisfies (AT2) with constants μ^* and ν^*, throughout the rest of this section.

Under the above assumption on the adjoint family, we know that for every $t \in (0,T]$, the family $(A(t-\tau)^*)_{\tau\in[0,t]}$ satisfies the (AT)-conditions as well, and therefore by Theorem 5.2 it generates an evolution family:

$$(V(t;\tau,s))_{0\le s\le\tau\le T}.$$

Recall from [2, Proposition 2.9], that $S(t,s)^* = V(t;t-s,0)$, and by Theorem 5.2 (5) and the chain rule, for $s<t$

$$\frac{d}{ds}S(t,s)^* = -A(s)^*S(t,s)^*. \tag{5.2.7}$$

Moreover, for all $x\in D(A(s)^*) = D(A(t-(t-s))^*)$ one has $s\mapsto S(t,s)^*x^* = V(t;t-s,0)x^*$ is continuously differentiable on $[0,t]$.

Lemma 5.7. *Under the above conditions one has*

(1) For every $t\in(0,T]$, the mapping $s\mapsto S(t,s)$ belongs to $C^1([0,t);\mathscr{L}(E_0))$, and for all $x\in D(A(s))$ one has $\frac{d}{ds}S(t,s)x = -S(t,s)A(s)x$.
(2) For $t>s$, $0\le\theta<\mu^*+\nu^*-1$ and $x\in D(-A(s)^{1+\theta})$, the evolution operator $S(t,s)$ satisfies

$$\|S(t,s)(-A(s))^{1+\theta}x\|_{E_0}\le C(t-s)^{-1-\theta}\|x\|_{E_0}. \tag{5.2.8}$$

(3) For $t>s$, $0<\gamma<\mu^*+\nu^*-1$, $\delta\in(0,1)$, and $x\in D((-A(s))^{1+\gamma})$

$$\|A(t)^{-\delta}S(t,s)(-A(s))^{1+\gamma}x\|_{E_0}\le C(t-s)^{-1-\gamma+\delta}\|x\|_{E_0}. \tag{5.2.9}$$

In particular, we see that for every $s<t$, the operator $S(t,s)A(s)$ uniquely extends to a bounded operator on E_0 of norm $C(t-s)^{-1}$, which will be denoted by $S(t,s,\omega)A(s,\omega)$ again. As before, the constant C depends on the constants in the (AT)-conditions for A and A^*.

Proof. It follows from (5.2.7) that

$$\frac{d}{ds}S(t,s) = \Big(\frac{d}{ds}S(t,s)^*\Big)^* = (-A(s)^*S(t,s)^*)^*,$$

where we identify E_0 and E_0^{**}. It follows $(-A(s)^*S(t,s)^*)^*\in\mathscr{L}(E_0)$. Hence, for any $x\in D(A(s))$ and every $x^*\in E_0^*$, one has

$$\langle(-A(s)^*S(t,s)^*)^*x,x^*\rangle = -\langle x,A(s)^*S(t,s)^*x^*\rangle = \langle -S(t,s)A(s)x,x^*\rangle.$$

By a Hahn-Banach argument, we obtain $\frac{d}{ds}S(t,s)x = -S(t,s)A(s)x$ for all $x\in D(A(s))$.

By (5.2.3) and (5.2.4) for the adjoint family we find that

$$\begin{aligned}\|(-A(s)^*)^{1+\theta}S(t,s)^*\|_{\mathscr{L}(E_0^*)} &= \|((-A(t-(t-s)))^*)^{1+\theta}V(t;t-s,0)\|_{\mathscr{L}(E_0^*)}\\ &\le C(t-s)^{-1-\theta}.\end{aligned}$$

Let $x\in D((-A(s))^{1+\theta})$ be arbitrary. Then

$$
\begin{aligned}
\|S(t,s)(-A(s))^{1+\theta}x\|_{E_0} &= \sup_{\|x^*\|_{E_0^*}\leq 1} |\langle S(t,s)(-A(s))^{1+\theta}x, x^*\rangle| \\
&= \sup_{\|x^*\|_{E_0^*}\leq 1} |\langle x, (-A(s)^*)^{1+\theta}S(t,s)^*x^*\rangle| \\
&\leq \|x\|_{E_0}\|(-A(s)^*)^{1+\theta}S(t,s)^*\|_{\mathscr{L}(E_0^*)} \\
&\leq C(t-s)^{-1-\theta}\|x\|_{E_0}
\end{aligned}
$$

and (5.2.8) follows.

The estimate (5.2.9) can be derived in a similar way from (5.2.5). □

5.3 Pathwise regularity of convolutions

In this section we will assume the following hypothesis.

(H2) Assume that both $(A(t,\omega))_{t\in[0,T],\omega\in\Omega}$ and $(A(t,\omega)^*)_{t\in[0,T],\omega\in\Omega}$ satisfy the (AT)-conditions.

5.3.1 A class of time independent spaces and interpolation

The following hypothesis is needed to deal with the time-dependent domains in an efficient way.

(H3) There exist $\eta_+ \in (0,1]$ and $\eta_- \in (0,\mu^*+\nu^*-1)$ and two family of interpolation spaces $(\tilde{E}_\eta)_{\eta\in[0,\eta_+)}$ and $(\tilde{E}_\eta)_{\eta\in(-\eta_-,0]}$ such that

(i) For all $\eta_4 \leq \eta_3 \leq 0 \leq \eta_2 \leq \eta_1 < \eta_+$

$$\tilde{E}_{\eta_+} \hookrightarrow \tilde{E}_{\eta_1} \hookrightarrow \tilde{E}_{\eta_2} \hookrightarrow \tilde{E}_0 = E_0 \hookrightarrow \tilde{E}_{\eta_3} \hookrightarrow \tilde{E}_{\eta_4}.$$

(ii) For all $\eta \in [0,\eta_+)$, $E^t_{\eta,1} \hookrightarrow \tilde{E}_\eta \hookrightarrow E_0$, with uniform constants in (t,ω).

(iii) For $\eta \in (0,\eta_-)$, $\tilde{E}_{-\eta}$ is dense in E_0 and for all $x \in E_0$ and $\varepsilon > 0$ one has

$$\|(-A(t))^{-\eta-\varepsilon}x\|_{E_0} \leq C\|x\|_{\tilde{E}_{-\eta}},$$

where C is independent of (t,ω).

If $E_1 = D(A(t))$ is constant one can just take $\tilde{E}_\eta = (E_0,E_1)_{\eta,p}$ for some $p \in [1,\infty)$. Moreover, in particular it follows from (iii) that $(-A(t))^{-\eta-\varepsilon}$ has a unique continuous extension to a bounded operator from E_0 into $\tilde{E}_{-\eta}$. From Remark 5.9 it will become clear why we assume $\eta_- < \mu^*+\nu^*-1$.

Remark 5.8.

1. If $A(t)$ is a differential operator with time dependent boundary conditions, then in general E^t_η will be time dependent as well. In this case one typically takes $\tilde{E}_\eta$ to be the space obtained by real interpolation from E_0 and the space $E_1 \supset D(A(t))$ obtained by leaving out the boundary conditions.

2. Note that it is allowed to choose $\tilde{E}_{-\eta} = E_0$ for all $\eta \in (0, \eta_-)$. However, usually the spaces will be taken certain extrapolation spaces which makes the noise term in a stochastic PDE convergent.

Remark 5.9. Assume hypotheses (H2) and (H3). The following observation will be used throughout the rest of the chapter. Let $\varepsilon > 0$, $a \in (0, \eta_+)$. By (5.2.1), (5.2.2) and (5.2.8) for $s < t$ and $x \in \tilde{E}_{-\theta}$, letting $r = \frac{t+s}{2}$ and taking $\varepsilon > 0$ small enough, we find that for all $\theta \in [0, \eta_-)$,

$$\begin{aligned} \|S(t,s)A(s)x\|_{\tilde{E}_a} &\leq C\|S(t,r)\|_{\mathscr{L}(E_0,E_a^t)}\|S(r,s)(-A(s))^{1+\theta+\varepsilon}(-A(s))^{-\theta-\varepsilon}x\|_{E_0} \\ &\leq C(t-s)^{-a-\varepsilon-1-\theta}\|x\|_{\tilde{E}_{-\theta}}. \end{aligned}$$

Note that here we use $\eta_- < \mu^* + \nu^* - 1$. Similarly, we find that for all $\theta \in (0, \eta_-)$

$$\|S(t,s)x\|_{\tilde{E}_a} \leq C(t-s)^{-a-\varepsilon-\theta}\|x\|_{\tilde{E}_{-\theta}},$$

where in both estimates C depends on Ω.

The next lemma is taken from [135, Lemma 2.3], and this is the place where the assumption that $(\tilde{E}_\eta)_{\eta\in[0,\eta_+]}$ are interpolation spaces, is used.

Lemma 5.10. *Assume (H2) and (H3). Let $\alpha \in (0, \eta_+]$ and $\delta, \gamma > 0$ such that $\gamma + \delta \leq \alpha$. Then there exists a constant $C > 0$ depending on ω, such that*

$$\|S(t,r)x - S(s,r)x\|_{\tilde{E}_\delta} \leq C(t-s)^\gamma\|x\|_{E^r_{\alpha,1}}, \quad 0 \leq r \leq s \leq t \leq T, \quad x \in E^r_\alpha.$$

Moreover, if $x \in E^r_{\alpha,1}$, then $t \mapsto S(t,r)x \in C([r,T]; \tilde{E}_\alpha)$.

In the above lemma C depends on Ω.

5.3.2 Sobolev spaces

Let X be a Banach space. For $\alpha \in (0,1)$, $p \in [1,\infty)$ and $a < b$, recall that a function $f : (a,b) \to X$ is said to be in the *Sobolev space* $W^{\alpha,p}(a,b;X)$ if $f \in L^p(a,b;X)$ and

$$[f]_{W^{\alpha,p}(a,b;X)} := \Big(\int_a^b\int_a^b \frac{\|f(t)-f(s)\|^p}{|t-s|^{\alpha p+1}}\,ds\,dt\Big)^{1/p} < \infty.$$

Letting $\|f\|_{W^{\alpha,p}(a,b;X)} = \|f\|_{L^p(a,b;X)} + [f]_{W^{\alpha,p}(a,b;X)}$, this space becomes a Banach space. By symmetry one can write

$$\int_a^b\int_a^b B(t,s)\,ds\,dt = 2\int_a^b\int_a^t B(t,s)\,ds\,dt = 2\int_a^b\int_s^b B(t,s)\,dt\,ds$$

where $B(t,s) = \frac{\|f(t)-f(s)\|^p}{|t-s|^{\alpha p+1}}$. This will be used often below.

A function $f : (a,b) \to X$ is said to be in the *Hölder space* $C^\alpha(a,b;X)$ if

$$[f]_{C^\alpha(a,b;X)} = \sup_{a<s<t<b} \frac{\|f(t)-f(s)\|}{|t-s|^\alpha} < \infty.$$

Letting $\|f\|_{C^\alpha(a,b;X)} = \sup_{t\in(0,T)} \|f(t)\|_X + [f]_{W^{\alpha,p}(a,b;X)}$, this space becomes a Banach space. Moreover, every $f \in C^\alpha(a,b;X)$ has a unique extension to a continuous function $f : [a,b] \to X$. For $p = \infty$ and a Banach space X, we also write $W^{\alpha,\infty}(0,T;X) = C^\alpha(0,T;X)$.

If $0 < \alpha < \beta < 1$, then trivially,

$$C^\alpha(a,b;X) \hookrightarrow W^{\alpha,p}(a,b;X).$$

One of the main results in the theory of fractional Sobolev spaces is the following Sobolev embedding: if $\alpha > \frac{1}{p}$, then

$$W^{\alpha,p}(a,b;X) \hookrightarrow C^{\alpha-\frac{1}{p}}(a,b;X). \tag{5.3.1}$$

Here the embedding means that each $f \in W^{\alpha,p}(a,b;X)$ has a version which is continuous and this function lies in $C^{\alpha-\frac{1}{p}}(a,b;X)$. The embedding (5.3.1) can be found in the literature in the scalar setting and the standard proofs extend to the vector-valued setting. We refer to [70, 14.28 and 14.40] and [35, Theorem 8.2] for detailed proofs.

5.3.3 Regularity of generalized convolutions

We can now present the first main result of this section. It gives a space-time regularity result for the abstract Cauchy problem:

$$u'(t) = A(t)u(t) + f(t), \quad u(0) = 0. \tag{5.3.2}$$

Recall that the solution is given by the convolution:

$$S * f(t) := \int_0^t S(t,\sigma) f(\sigma)\, d\sigma.$$

The next result extends [135, Proposition 3.2], where a space-time Hölder continuity result has been obtained.

Theorem 5.11. *Assume (AT), (H3). Let $\theta \in [0,\eta_-)$, $p \in [1,\infty)$ and $\delta, \lambda > 0$ such that $\delta + \lambda < \min\{1-\theta, \eta_+\}$. Suppose $f \in L^0(\Omega; L^p(0,T;\tilde{E}_{-\theta}))$. Then the stochastic process $S * f$ is in $L^0(\Omega; W^{\lambda,p}(0,T;\tilde{E}_\delta))$ and satisfies*

$$\|S*f\|_{W^{\lambda,p}(0,T;\tilde{E}_\delta)} \le C\|f\|_{L^p(0,T;\tilde{E}_{-\theta})} \quad a.s.,$$

where C depends on Ω.

Maximal L^p-regularity results for (5.3.2) can be found in [107].

Proof. Let $\varepsilon > 0$ be so small that $\delta + \lambda + \theta + 2\varepsilon < \eta_0$. By (5.2.6) and (H3) we find that

$$\begin{aligned}
&\|S * f(t)\|_{\tilde{E}_\delta} \\
&\leq \int_0^t \|S(t, \tfrac{t+\sigma}{2})\|_{\mathscr{L}(E_0, E^t_{\delta,1})} \|S(\tfrac{t+\sigma}{2}, \sigma)(-A(\sigma))^{\theta+\varepsilon}\|_{\mathscr{L}(E_0)} \|f(\sigma)\|_{\tilde{E}_{-\theta}}\, d\sigma \\
&\leq C \int_0^t (t-\sigma)^{-\delta-\theta-\varepsilon} \|f(\sigma)\|_{\tilde{E}_{-\theta}}\, d\sigma
\end{aligned} \tag{5.3.3}$$

Therefore, Young's inequality yields that

$$\|S * f\|_{L^p(0,T;\tilde{E}_\delta)} \leq C\|f\|_{L^p(0,T;\tilde{E}_{-\theta})}.$$

Next we estimate the seminorm $[S * f]_{W^{\lambda,p}(0,T;E_\delta)}$. For $0 \leq s < t \leq T$, we write

$$\begin{aligned}
\|(S * f)(t) - (S * f)(s)\|_{\tilde{E}_\delta} \leq & \int_0^s \|(S(t,\sigma) - S(s,\sigma))f(\sigma)\|_{\tilde{E}_\delta}\, d\sigma \\
& + \int_s^t \|S(t,\sigma)f(\sigma)\|_{\tilde{E}_\delta}\, d\sigma.
\end{aligned}$$

By Remark 5.9 and Lemma 5.10 we obtain

$$\begin{aligned}
\int_0^s \|(S(t,\sigma) - S(s,\sigma))f(\sigma)\|_{\tilde{E}_\delta}\, d\sigma &\leq C(t-s)^{\lambda+\varepsilon} \int_0^s \|S(s,\sigma)f(\sigma)\|_{E^s_{\delta+\lambda+\varepsilon}}\, d\sigma \\
&\leq C(t-s)^{\lambda+\varepsilon} \int_0^s (s-\sigma)^{-\delta-\lambda-\theta-2\varepsilon} \|f(\sigma)\|_{\tilde{E}_{-\theta}}\, d\sigma
\end{aligned}$$

Now it follows from integration over t and then Young's inequality that

$$\begin{aligned}
&\int_0^T \int_s^T (t-s)^{-1-\lambda p} \Big(\int_0^s \|(S(t,\sigma) - S(s,\sigma))f(\sigma)\|_{\tilde{E}_\delta}\, d\sigma\Big)^p\, dt\, ds \\
&\leq C \int_0^T \int_s^T C(t-s)^{-1+\varepsilon p} \Big(\int_0^s (s-\sigma)^{-\delta-\lambda-\theta-2\varepsilon} \|f(\sigma)\|_{\tilde{E}_{-\theta}}\, d\sigma\Big)^p\, dt\, ds \\
&\leq C \int_0^T \Big(\int_0^s (s-\sigma)^{-\delta-\lambda-\theta-2\varepsilon} \|f(\sigma)\|_{\tilde{E}_{-\theta}}\, d\sigma\Big)^p\, ds \\
&\leq C\|f\|^p_{L^p(0,T;\tilde{E}_{-\theta})}.
\end{aligned}$$

For the other term by (5.2.3) we obtain

$$\int_s^t \|S(t,\sigma)f(\sigma)\|_{\tilde{E}_\delta}\, d\sigma \leq \int_0^t \mathbf{1}_{(s,t)}(\sigma)(t-\sigma)^{-\delta-\theta-\varepsilon} \|f(\sigma)\|_{\tilde{E}_{-\theta}}\, d\sigma$$

Integrating over $s \in (0,t)$ it follows from Minkowski's inequality that

$$\Big(\int_0^t \Big((t-s)^{-\frac{1}{p}-\lambda}\int_0^t \mathbf{1}_{(s,t)}(\sigma)\|S(t,\sigma)f(\sigma)\|_{\tilde{E}_\delta}\,d\sigma\Big)^p\,ds\Big)^{1/p}$$
$$\leq C\Big(\int_0^t \Big(\int_0^t (t-s)^{-\frac{1}{p}-\lambda}\mathbf{1}_{(s,t)}(\sigma)(t-\sigma)^{-(\delta+\theta+\varepsilon)}\|f(\sigma)\|_{\tilde{E}_{-\theta}}\,d\sigma\Big)^p\,ds\Big)^{1/p}$$
$$\leq C\int_0^t \Big(\int_0^t (t-s)^{-1-\lambda p}\mathbf{1}_{(s,t)}(\sigma)(t-\sigma)^{-(\delta+\theta+\varepsilon)p}\|f(\sigma)\|^p_{\tilde{E}_{-\theta}}\,ds\Big)^{1/p}\,d\sigma$$
$$\leq C\int_0^t (t-\sigma)^{-(\delta+\theta+\varepsilon+\lambda)}\|f(\sigma)\|_{\tilde{E}_{-\theta}}\,d\sigma.$$

Taking p-th moments in $t\in(0,T)$, it follows from Young's inequality that

$$\int_0^T\int_0^t \Big((t-s)^{-\frac{1}{p}-\lambda}\int_0^t \mathbf{1}_{(s,t)}(\sigma)\|S(t,\sigma)f(\sigma)\|_{\tilde{E}_\delta}\,d\sigma\Big)^p\,ds\,dt$$
$$\leq C\int_0^T \Big(\int_0^t (t-\sigma)^{-(\delta+\theta+\varepsilon+\lambda)}\|f(\sigma)\|_{\tilde{E}_{-\theta}}\,d\sigma\Big)^p\,dt \leq C\|f\|^p_{L^p(0,T;\tilde{E}_{-\theta})}.$$

Combining the estimates, the result follows. □

The second main result of this section looks artificial, but is a major technical tool to obtain pathwise regularity of the solution to the stochastic Cauchy problem:

$$du = A(t)u(t)\,dt + G\,dW.$$

For details on this we refer to Section 5.4 below.

Recall the convention that for a Banach space X, we put $W^{\alpha,\infty}(0,T;X) = C^\alpha(0,T;X)$.

Theorem 5.12. *Assume (H2) and (H3). Let $p\in(1,\infty]$, $\theta\in[0,\eta_-)$ and $\alpha>\theta$. Let $f\in L^0(\Omega;W^{\alpha,p}(0,T;\tilde{E}_{-\theta}))$. Let $\delta,\lambda>0$ be such that $\delta+\lambda<\min\{\alpha-\theta,\eta_+\}$. The following assertions hold:*

1. *The stochastic process ζ defined by*

$$\zeta(t) := \int_0^t S(t,\sigma)A(\sigma)(f(t)-f(\sigma))\,d\sigma$$

belongs to $L^0(\Omega;W^{\lambda,p}([0,T];\tilde{E}_\delta))$ and

$$\|\zeta\|_{W^{\lambda,p}(0,T;\tilde{E}_\delta)} \leq C\|f\|_{W^{\alpha,p}(0,T;\tilde{E}_{-\theta})} \quad a.s.,$$

where C depends on Ω.

2. *If $\alpha>1/p$, assume additionally that the continuous version of f satisfies $f(0)=0$. Then $\tilde{\zeta}=S(t,0)f(t)$ belongs to $L^0(\Omega;W^{\lambda,p}(0,T;\tilde{E}_\delta))$ and*

$$\|\tilde{\zeta}\|_{W^{\lambda,p}(0,T;\tilde{E}_\delta)} \leq C\|f\|_{W^{\alpha,p}(0,T;\tilde{E}_{-\theta})} \quad a.s.,$$

where C depends on Ω.

Note that in (2) the continuous version of f exists in the case $\alpha > 1/p$ by (5.3.1).

Proof. The proofs in the case $p = \infty$ are much simpler and we focus on the case $p \in (1, \infty)$. Let $\beta = \delta + \theta + \varepsilon$.

(1). Let $\varepsilon > 0$ be so small that $\beta + \lambda + \varepsilon < \alpha$. Write $\Delta_{ts} f = f(t) - f(s)$. First we estimate the $L^p(0, T; \tilde{E}_\delta)$-norm of ζ. Note that by Remark 5.9

$$\|S(t,\sigma)A(\sigma)\Delta_{t\sigma} f\|_{\tilde{E}_\delta} \leq C(t-\sigma)^{-1-\beta}\|\Delta_{t\sigma} f\|_{\tilde{E}_{-\theta}}.$$

Therefore, by Holder's inequality applied with measure $(t-\sigma)^{-1+\varepsilon}\, d\sigma$ we find that

$$\begin{aligned}
\|\zeta\|^p_{L^p(0,T;\tilde{E}_\delta)} &\leq C \int_0^T \Big(\int_0^t \frac{\|\Delta_{t\sigma} f\|_{\tilde{E}_{-\theta}}}{(t-\sigma)^{1+\beta}}\, d\sigma\Big)^p\, dt \\
&\leq C \int_0^T \int_0^t \frac{\|\Delta_{t\sigma} f\|^p_{\tilde{E}_{-\theta}}}{(t-\sigma)^{1+(\beta+\varepsilon-\frac{\varepsilon}{p})p}}\, d\sigma\, dt \\
&\leq C\|f\|_{W^{\alpha,p}(0,T;\tilde{E}_{-\theta})}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\|\zeta(t)-\zeta(s)\|_{\tilde{E}_\delta} \leq &\int_0^s \|(S(t,\sigma)A(\sigma)\Delta_{t\sigma} f - S(s,\sigma)A(\sigma)\Delta_{s\sigma} f)\|_{\tilde{E}_\delta}\, d\sigma \\
&+ \int_s^t \|S(t,\sigma)A(\sigma)\Delta_{t\sigma} f\|_{\tilde{E}_\delta}\, d\sigma = T_1(s,t) + T_2(s,t).
\end{aligned}$$

We estimate the $[\cdot]_{W^{\lambda,p}}$-seminorm of each of the terms separately. For T_2 note that by Remark 5.9,

$$\|S(t,\sigma)A(\sigma)\Delta_{t\sigma} f\|_{\tilde{E}_\delta} \leq C(t-\sigma)^{-\beta-1}\|\Delta_{t\sigma} f\|_{\tilde{E}_{-\theta}} =: g(\sigma,t)$$

Therefore, it follows from the Hardy–Young inequality (see [48, p. 245-246]) that

$$\begin{aligned}
\int_0^t (t-s)^{-\lambda p-1}\Big(&\int_s^t \|S(t,\sigma)A(\sigma)\Delta_{t\sigma} f\|_{\tilde{E}_\delta}\, d\sigma\Big)^p\, ds \\
&\leq \int_0^t (t-s)^{-\lambda p-1}\Big(\int_s^t g(\sigma,t)\, d\sigma\Big)^p\, ds \\
&= \int_0^t (t-s)^{-\lambda p-1}\Big(\int_0^{t-s} g(t-\tau,t)\, d\tau\Big)^p\, ds \\
&= \int_0^t r^{-\lambda p-1}\Big(\int_0^r g(t-\tau,t)\, d\tau\Big)^p\, dr \\
&\leq C\int_0^t r^{p-\lambda p-1} g(t-r,t)^p\, dr \\
&= C\int_0^t \frac{g(s,t)^p}{(t-s)^{-p+\lambda p+1}}\, ds
\end{aligned}$$

Integrating with respect to $t \in (0,T)$ and using the definition of g we find that

$$\begin{aligned}\int_0^T \int_0^t \frac{T_2(s,t)^p}{(t-s)^{\lambda p+1}}\,ds\,dt &\le C \int_0^T \int_0^t \frac{(t-s)^{-(\beta+1)p}\|\Delta_{ts}f\|^p_{\tilde{E}_{-\theta}}}{(t-s)^{-p+\lambda p+1}}\,ds\,dt \\ &\le C \int_0^T \int_0^t \frac{\|\Delta_{ts}f\|^p_{\tilde{E}_{-\theta}}}{(t-s)^{\alpha p+1}} \le C\|f\|^p_{W^{\alpha,p}(0,T;\tilde{E}_\delta)}.\end{aligned}$$

For T_1, we can write

$$\begin{aligned}T_1(s,t) &\le \int_0^s \|S(t,\sigma)A(\sigma)\Delta_{ts}f\|_{\tilde{E}_\delta}\,d\sigma \\ &\quad + \int_0^s \|(S(t,\sigma)-S(s,\sigma))A(\sigma)\Delta_{s\sigma}f\|_{\tilde{E}_\delta}\,d\sigma \\ &= T_{1a}(s,t) + T_{1b}(s,t).\end{aligned}$$

For T_{1a}, by Remark 5.9 we have

$$T_{1a}(s,t) \le C\int_0^s (t-\sigma)^{-1-\beta}\|\Delta_{ts}f\|_{\tilde{E}_{-\theta}}\,d\sigma \le C(t-s)^{-\beta}\|\Delta_{ts}f\|_{\tilde{E}_{-\theta}}$$

It follows that

$$\begin{aligned}\int_0^T \int_0^t \frac{T_{1a}(s,t)^p}{(t-s)^{\lambda p+1}}\,ds\,dt &\le C \int_0^T \int_0^t \frac{\|\Delta_{ts}f\|^p_{\tilde{E}_{-\theta}}}{(t-s)^{(\beta+\lambda)p+1}}\,ds\,dt \\ &\le C\|f\|^p_{W^{\alpha,p}(0,T;\tilde{E}_{-\theta})}.\end{aligned}$$

To estimate T_{1b} let $\gamma = \alpha - \varepsilon - \beta$. By Lemma 5.10 and Remark 5.9,

$$\begin{aligned}&\|(S(t,\sigma)-S(s,\sigma))A(\sigma)\Delta_{s\sigma}f\|_{\tilde{E}_\delta} \le \|(S(t,s)-I)S(s,\sigma)A(\sigma)\Delta_{s\sigma}f\|_{\tilde{E}_\delta} \\ &\quad \le C(t-s)^\gamma \|S(s,\sigma)A(\sigma)^{1+\theta+\varepsilon}\Delta_{s\sigma}f\|_{E^s_{\delta+\gamma,1}} \\ &\quad \le C(t-s)^\gamma \|S(s,\tau)\|_{\mathscr{L}(E_0,E^s_{\delta+\gamma,1})}\|S(\tau,\sigma)A(\sigma)^{1+\theta+\varepsilon}\|_{\mathscr{L}(E_0)}\|\Delta_{s\sigma}f\|_{\tilde{E}_{-\theta}} \\ &\quad \le C(t-s)^\gamma (s-\sigma)^{-1-\gamma-\beta}\|\Delta_{s\sigma}f\|_{\tilde{E}_{-\theta}},\end{aligned}$$

with $\tau = (s-\sigma)/2$. It follows that from Hölder's inequality that

$$\begin{aligned}T_{1b}(s,t)^p &\le C(t-s)^{\gamma p}\Big(\int_0^s (s-\sigma)^{-1-\gamma-\beta}\|\Delta_{s\sigma}f\|_{\tilde{E}_{-\theta}}\,d\sigma\Big)^p \\ &\le C(t-s)^{\gamma p}h(s)^p \int_0^s \frac{\|\Delta_{s\sigma}f\|^p_{\tilde{E}_{-\theta}}}{(s-\sigma)^{\alpha p+1}}\,d\sigma\end{aligned}$$

where by the choice of γ, the function $h(s)$ satisfies

$$h(s) = \Big(\int_0^s \Big[(s-\sigma)^{-1-\gamma-\beta+\alpha+\frac{1}{p}}\Big]^{p'}\,d\sigma\Big)^{1/p'} \le C.$$

Using Fubini's theorem and $\gamma > \lambda$ we can conclude that

$$\int_0^T \int_s^T \frac{T_{1b}(s,t)^p}{(t-s)^{\lambda p+1}}\,dt\,ds \le C \int_0^T \int_0^s \int_s^T (t-s)^{-\lambda p-1+\gamma}\,dt \frac{\|\varDelta_{s\sigma} f\|^p_{\tilde{E}_{-\theta}}}{(s-\sigma)^{\alpha p+1}}\,d\sigma\,ds$$
$$\le C\|f\|^p_{W^{\alpha,p}(0,T;\tilde{E}_{-\theta})}.$$

and this finishes the proof of (1).

To prove (2), we first estimate $[\tilde{\zeta}]_{W^{\lambda,p}(0,T;\tilde{E}_\delta)}$ and write

$$\|\tilde{\zeta}(t) - \tilde{\zeta}(s)\|_{\tilde{E}_\delta} \le \|S(t,0)\varDelta_{ts} f\|_{\tilde{E}_\delta} + \|S(t,0) - S(s,0)f(s)\|_{\tilde{E}_\delta}.$$

By Remark 5.9,

$$\|S(t,0)\varDelta_{ts} f\|_{\tilde{E}_\delta} \le Ct^{-\beta}\|\varDelta_{ts} f\|_{\tilde{E}_{-\theta}} \le C(t-s)^{-\beta}\|\varDelta_{ts} f\|_{\tilde{E}_{-\theta}}.$$

It follows that

$$\int_0^T \int_0^t \frac{\|S(t,0)\varDelta_{ts} f\|^p_{\tilde{E}_\delta}}{(t-s)^{\lambda p+1}}\,ds\,dt \le C\|f\|_{W^{\beta+\lambda}(0,T;\tilde{E}_{-\theta})} \le C\|f\|_{W^{\alpha}(0,T;\tilde{E}_{-\theta})}.$$

For the other term we may assume $\beta + \lambda + \varepsilon > 1/p$ in the case $\alpha > 1/p$. In the case $\alpha \le 1/p$, we can assume $\beta + \lambda + \varepsilon < 1/p$. By [35, Theorem 5.4] we can find an extension of f to a function F in $W^{\alpha,p}(\mathbb{R};\tilde{E}_{-\theta})$ and

$$\|F\|_{W^{\alpha,p}(\mathbb{R};\tilde{E}_{-\theta})} \le C\|f\|_{W^{\alpha,p}(0,T;\tilde{E}_{-\theta})}.$$

Moreover, multiplying F by a suitable smooth cut-off function we can assume that additionally $F = 0$ on $[T+1,\infty)$.

We have by Lemma 5.10 and (5.2.3),

$$\begin{aligned}\|(S(t,0) - S(s,0))f(s)\|_{\tilde{E}_\delta} &= \|(S(t,s) - S(s,s))S(s,0)f(s)\|_{\tilde{E}_\delta}\\ &\le C(t-s)^{\lambda+\varepsilon}\|S(s,0)f(s)\|_{E^s_{\delta+\lambda+\varepsilon}}\\ &\le C(t-s)^{\lambda+\varepsilon}\frac{\|f(s)\|_{\tilde{E}_\theta}}{s^{\beta+\lambda+\varepsilon}}\end{aligned}$$

Therefore, we find

$$\int_0^T \int_s^T \frac{\|S(t,0) - S(s,0)f(s)\|^p_{\tilde{E}_\delta}}{(t-s)^{\lambda p+1}}\,dt\,ds \le C\int_0^T \int_s^T (t-s)^{-1+\varepsilon p}\,dt \frac{\|f(s)\|^p_{\tilde{E}_\theta}}{s^{(\beta+\lambda+\varepsilon)p}}\,ds$$
$$\le C\int_0^T \frac{\|f(s)\|^p_{\tilde{E}_\theta}}{s^{(\beta+\lambda+\varepsilon)p}}\,ds \le C\int_{\mathbb{R}_+} \frac{\|F(s)\|^p_{\tilde{E}_\theta}}{s^{(\beta+\lambda+\varepsilon)p}}\,ds$$

Applying, the fractional Hardy inequality (see [57, Theorem 2]) and elementary estimates we find that the latter is less or equal than

$$
\begin{aligned}
C\|F\|^p_{W^{\beta+\lambda+\varepsilon,p}(\mathbb{R}_+;\tilde{E}_{-\theta})} &\le C\|F\|^p_{W^{\beta+\lambda+\varepsilon,p}(\mathbb{R};\tilde{E}_{-\theta})} \\
&\le C\|f\|^p_{W^{\beta+\lambda+\varepsilon,p}(0,T;\tilde{E}_{-\theta})} \le C\|f\|^p_{W^{\alpha,p}(0,T;\tilde{E}_{-\theta})}.
\end{aligned}
$$

Finally we estimate the L^p-norm of $\tilde{\zeta}$. To do so it follows from Lemma 5.6 that

$$\|\tilde{\zeta}(t)\|_{\tilde{E}_\delta} \le Ct^{-\beta}\|f(t)\|_{\tilde{E}_{-\theta}}.$$

Taking L^p-norms and applying the fractional Hardy inequality as before we obtain

$$\|\tilde{\zeta}\|^p_{L^p(0,T;\tilde{E}_\delta)} \le C\int_0^T t^{-\beta p}\|f(t)\|^p_{\tilde{E}_{-\theta}}\,dt \le C\|f\|^p_{W^{\alpha,p}(0,T;\tilde{E}_{-\theta})}.$$

This completes the proof of (2). □

5.4 Representation formula for stochastic convolutions

In this section we will introduce a new solution formula for equations of the form:

$$\begin{cases} dU(t) = (A(t)U(t)\;dt + G\;dW(t), \\ U(0) \;\;= 0, \end{cases} \tag{5.4.1}$$

Here W is an H-cylindrical Brownian motion and $G : [0,T]\times\Omega \to \mathscr{L}(H, E_0)$ is adapted and strongly measurable. Furthermore, $(A(t))_{t\in[0,T]}$ satisfies the (AT)-conditions as introduced before. At first sight one would expect that the solution to (5.4.1) is given by

$$U(t) = \int_0^t S(t,s)G(s)\,dW(s). \tag{5.4.2}$$

However, in general $s \mapsto S(t,s)$ is only $\mathscr{F}_t$-measurable (see Proposition 5.4 and Example 5.5). Therefore, the stochastic integral does not exist in the Itô sense. We will give another representation formula which provides an alternative to mild solutions to (5.4.1):

$$U(t) = -\int_0^t S(t,s)A(s)I(\mathbf{1}_{(s,t)}G)\,ds + S(t,0)I(\mathbf{1}_{(0,t)}G), \tag{5.4.3}$$

where $I(\mathbf{1}_{(s,t)}G) = \int_s^t G\,dW$. Clearly, there is no adaptedness issue in (5.4.3) as the evolution family is only used in integration with respect to the Lebesgue measure. Moreover, the solution U will be adapted and measurable. A difficulty in the representation formula (5.4.3) is that usually the kernel $S(t,s)A(s)$ has a singularity of order $(t-s)^{-1}$, but we will see below that $I(\mathbf{1}_{(s,t)}G)$ is small enough for s close to t to make the integral in (5.4.3) convergent. Moreover, we will show that the usual parabolic regularity results hold.

In Section 5.4.1 we first repeat some basic results from stochastic integration theory. In Section 5.4.2 we show that in the bounded case this yields the right solution. The space-time regularity of U defined by (5.4.3) is studied in Section 5.4.3. In Section 5.4.4 we show that (5.4.3) leads to the usual weak solutions. Finally in Section 5.4.5 we prove that there is an interpretation of (5.4.2) in terms of forward integration. This will not be used in the rest of the chapter, but provides an interesting connection.

In this section we assume the hypotheses (H1)-(H3) and we impose a further condition on the spaces in (H3).

(H4) The spaces $(\tilde{E}_\eta)_{\eta\in(-\eta_-,\eta_+]}$ from (H3) all have UMD and type 2.

Details on type and UMD can be found in [39] and [23], respectively.

5.4.1 Stochastic integration

Below we briefly repeat a part of the stochastic integration theory in UMD spaces E with type 2. Let H be a separable Hilbert space and let W be a cylindrical Brownian motion on H. For spaces E with UMD one can develop an analogue of Itô's theory of the stochastic integral (see [86]). To be more precise: one can precisely characterize which adapted and strongly measurable $G : [0,T]\times\Omega \to \mathscr{L}(H,E)$ are stochastically integrable. Moreover, two-sided estimates can be obtained. If additionally the space E has type 2, then there exists an easy subspace of stochastically integrable processes. Indeed, every adapted and strongly measurable $G \in L^0(\Omega; L^2(0,T;\gamma(H,E)))$, the stochastic integral process $\left(\int_0^t G\,dW\right)_{t\in[0,T]}$ exists and is pathwise continuous. For convenience we write

$$I(G) = \int_0^T G\,dW \quad\text{and}\quad J(G)(t) = I(\mathbf{1}_{[0,t]}G) \quad t\in[0,T].$$

Moreover, for all $p \in (0,\infty)$, there exists a constant C independent of G such that the following one-sided estimate holds:

$$\|J(G)\|_{L^p(\Omega;C([0,T];E))} \le C\|G\|_{L^p(\Omega;L^2(0,T;\gamma(H,E)))}. \tag{5.4.4}$$

Here, $\gamma(H,E)$ is the space of γ-radonifying operators $R : H \to E$. For details on γ-radonifying operators we refer to [90].

One can deduce Sobolev regularity of the integral process (see [110]).

Proposition 5.13. *Assume E has type 2, and let $p \in [2,\infty)$ and $0 < \alpha < \frac{1}{2}$. If $G \in L^0(\Omega; L^p(0,T;\gamma(H,E)))$ is adapted, then $J(G) \in L^0(\Omega; W^{\alpha,p}(0,T;E))$. Furthermore, there exists a constant C_T such that independent of G such that*

$$\|J(G)\|_{L^p(\Omega;W^{\alpha,p}(0,T;E))} \le C_T\|G\|_{L^p(\Omega;L^p(0,T;\gamma(H,E)))},$$

where $C_T \to 0$ as $T \downarrow 0$. Moreover, if $G_n \in L^0(\Omega; L^p(0,T;\gamma(H,E)))$ is adapted, then

$$G_n \to G \text{ in } L^0(\Omega; L^p(0,T;\gamma(H,E))) \implies J(G_n) \to J(G) \text{ in } L^0(\Omega; W^{\alpha,p}(0,T;E)).$$

By (5.3.1) one can also derive Hölder regularity and convergence in the Hölder norm in the case $\alpha > 1/p$.

5.4.2 Motivation in the bounded case

In this section we will motivate the representation formula (5.4.3) and we will show that the solution U defined by (5.4.3) satisfies the usual space-time regularity results.

Below we will show that in a special case, U, defined by (5.4.3), is a solution to (5.4.1).

Proposition 5.14. *Assume $A \in L^0(\Omega; C([0,T]; \mathscr{L}(E_0))$ and A is adapted. If $G \in L^0(\Omega; L^2(0,T;\gamma(H,E_0)))$ is adapted, then U defined by* (5.4.3) *is adapted and satisfies*

$$U(t) = \int_0^t A(s)U(s)\,ds + \int_0^t G(s)\,dW(s).$$

The above result is only included to show that (5.4.3) leads to the "right" solution. In the case A is bounded, one can construct solutions in a more direct way using stopping time techniques and the Banach fixed point theorem.

Proof. By [104, Theorem 5.2], $(A(t,\omega))_{t\in[0,T]}$ generates a unique continuous evolution family $(S(t,s,\omega))_{0\le s\le t\le T}$ and pointwise in Ω the following identities hold

$$\frac{\partial}{\partial t} S(t,s) = A(t)S(t,s) \quad\text{and}\quad \frac{\partial}{\partial s} S(t,s) = -S(t,s)A(s).$$

Moreover, from the construction in [104, Theorem 5.2] one readily checks that for each $0 \le s \le t \le T$, $\omega \mapsto S(t,s,\omega)$ is $\mathscr{F}_t$-measurable and thus U defined by (5.4.3) is adapted. It follows that

$$\begin{aligned} U(t) &= -\int_0^t S(t,s)A(s)I(\mathbf{1}_{(0,t)}G)\,ds \\ &\qquad + \int_0^t S(t,s)A(s)I(\mathbf{1}_{(0,s)}G)\,ds + S(t,0)I(\mathbf{1}_{(0,t)}G) \\ &= I(\mathbf{1}_{(0,t)}G) + \int_0^t S(t,s)A(s)I(\mathbf{1}_{(0,s)}G)\,ds \end{aligned}$$

Therefore, by Fubini's theorem we obtain

$$\begin{aligned} \int_0^t A(r)U(r)\,dr &= \int_0^t A(r)I(\mathbf{1}_{(0,r)}G)\,dr + \int_0^t \int_0^r A(r)S(r,s)A(s)I(\mathbf{1}_{(0,s)}G)\,ds\,dr \\ &= \int_0^t A(r)I(\mathbf{1}_{(0,r)}G)\,dr + \int_0^t \int_s^t A(r)S(r,s)A(s)I(\mathbf{1}_{(0,s)}G)\,dr\,ds \\ &= \int_0^t S(t,s)A(s)I(\mathbf{1}_{(0,s)}G)\,ds \end{aligned}$$

Combining both identities, the result follows. □

Remark 5.15. In the general case that A is unbounded, the integrals in the above proof might diverge and one needs to argue in a different way. However, if $G(s)$ is in $D((-A(s))^{\beta})$ for $\beta \geq 0$ large enough, and under integrability assumptions in $s \in (0,T)$, one can repeat the above calculation in many situations.

5.4.3 Regularity

As a consequence of the previous results we will now derive a pathwise regularity result for U given by (5.4.3).

Theorem 5.16. *Let $p \in (2,\infty)$ and let $\theta \in [0, \eta_- \wedge \frac{1}{2})$. Let $\delta, \lambda > 0$ be such that $\delta + \lambda < \min\{\frac{1}{2} - \theta, \eta_+\}$. Suppose $G \in L^0(\Omega; L^p(0,T;\gamma(H,\tilde{E}_{-\theta})))$ is adapted. The process U given by* (5.4.3) *is adapted and is in $L^0(\Omega; W^{\lambda,p}(0,T;\tilde{E}_{\delta}))$. Moreover, for every $\alpha \in (\lambda+\delta+\theta, \frac{1}{2})$, there is a mapping $C : \Omega \to \mathbb{R}_+$ which only depends on δ, λ, p and the constants in (H1)-(H4) such that*

$$\|U\|_{W^{\lambda,p}(0,T;\tilde{E}_{\delta})} \leq C\|J(G)\|_{W^{\alpha,p}(0,T;\tilde{E}_{-\theta})}.$$

Recall from Proposition 5.13 that $J(G) \in W^{\alpha,p}(0,T;\tilde{E}_{-\theta})$ a.s.

Proof. Let $\alpha \in (\lambda+\delta+\theta, \frac{1}{2})$. By Proposition 5.13, $J(G)$ belongs to the space $L^0(\Omega; W^{\alpha,p}(0,T;\tilde{E}_{-\theta}))$. Therefore, by Theorem 5.12 we find the required regularity and estimate for the paths of U. The measurability and adaptedness of U follows from Proposition 5.4 and approximation. □

5.4.4 Weak solutions

In this section we assume (H1)-(H4).

Formally, applying a functional $x^* \in E_0^*$ on both sides of (5.4.1) and integration yields

$$\langle U(t), x^*\rangle = \int_0^t \langle U(s), A(s)^*x^*\rangle \, ds + \int_0^t G(s)^*x^* \, dW(s), \qquad (5.4.5)$$

where the last expression only makes sense if $x^* \in D(A(s)^*)$ for almost all $s \in (0,T)$ and $\omega \in \Omega$, and $s \mapsto \langle U(s), A(s)^*x^*\rangle$ is in $L^1(0,T)$ almost surely. In this section we will show that the above identity is satisfied in the special cases where the domains $D(A(t))$ and $D(A(t)^*)$ do not depend on time.

In the case the domains are time independent, it is more natural to use time and Ω-dependent functionals $\varphi : [0,t] \times \Omega \to E_0^*$ to derive a weak formulation of the solution. Here φ will be smooth in space and time, but will not be assumed to be adapted. Formally, applying the product rule to differentiate and then integrate the differentiable function $\langle U(t) - I(\mathbf{1}_{(0,t)}G), \varphi(t)\rangle$, one derives that

$$
\begin{aligned}
&\langle U(t) - I(\mathbf{1}_{(0,t)}G), \varphi(t)\rangle \\
&= \int_0^t \langle A(s)U(s), \varphi(s)\rangle \, ds + \int_0^t \langle U(s) - I(\mathbf{1}_{(0,s)}G), \varphi'(s)\rangle \, ds \\
&= \int_0^t \langle U(s), A(s)^*\varphi(s)\rangle \, ds + \int_0^t \langle U(s), \varphi'(s)\rangle \, ds - \int_0^t \langle I(\mathbf{1}_{(0,s)}G), \varphi'(s)\rangle \, ds.
\end{aligned}
$$

Adding the stochastic integral term to both sides yields

$$
\begin{aligned}
\langle U(t), \varphi(t)\rangle = \int_0^t \langle U(s), A(s)^*\varphi(s)\rangle \, ds + \int_0^t \langle U(s), \varphi'(s)\rangle \, ds \\
- \int_0^t \langle I(\mathbf{1}_{(0,s)}G), \varphi'(s)\rangle \, ds + \langle I(\mathbf{1}_{(0,t)}G), \varphi(t)\rangle.
\end{aligned}
\tag{5.4.6}
$$

Clearly, (5.4.6) reduces to (5.4.5) if $\varphi \equiv x^*$. Below we will show that the representation formula (5.4.3) is equivalent to (5.4.6). Moreover, in the case the domains are constant in time, both are equivalent to (5.4.5). Therefore, this provides the appropriate weak setting to extend the equivalence of Proposition 5.14.

First we define a suitable space of test functions.

Definition 5.17. For $t \in [0, T]$ and $\beta \geq 0$ let $\Gamma_{t,\beta}$ be the subspace of all $\varphi \in L^0(\Omega; C^1(0, t; E_0^*))$ such that

(1) for all $s \in [0, t)$ and $\omega \in \Omega$, $\varphi(s) \in D(((-A(s))^{\beta+1})^*)$ and $\varphi'(s) \in D(((-A(s))^{\beta})^*)$.
(2) the process $s \mapsto A(s)^*\varphi(s)$ is in $L^0(\Omega; C([0, t]; E_0^*))$.
(3) There is a mapping $C : \Omega \to \mathbb{R}_+$ and $\varepsilon > 0$ such that for all $s \in [0, t)$,

$$\|((-A(s))^{1+\beta})^*\varphi(s)\| + \|((-A(s))^{\beta})^*\varphi'(s)\| \leq C(t-s)^{-1+\varepsilon}.$$

Example 5.18. Let $x^* \in D(A(t)^*)$. Then for all $\beta \in [0, \mu^* + \nu^* - 1)$ the process $\varphi : [0, t] \times \Omega \to E_0^*$ defined by $\varphi(s) = S(t, s)x^*$ belongs to $\Gamma_{t,\beta}$. Indeed, first of all $\varphi \in L^0(\Omega; C^1([0, t]; E_0^*))$ (see below (5.2.7)). Moreover, $-A(s)^*\varphi(s) = \varphi'(s) = -A(s)^*S(t, s)x^*$ is continuous, and by the adjoint version of (5.2.9) the latter satisfies

$$
\begin{aligned}
\|((-A(s))^{1+\beta})^*S(t, s)^*x^*\| &\leq \|((-A(s))^{1+\beta})^*S(t, s)^*(A(t)^{-\lambda})^*\|\|((-A(t))^{\lambda})^*x^*\| \\
&\leq C(t-s)^{-1-\beta+\lambda}\|((-A(t))^{\lambda})^*x^*\|,
\end{aligned}
$$

for all $\lambda \in (0, 1)$. The later satisfies the required condition when we take $\lambda \in (\beta, 1)$.

In the next theorem we show the equivalence of the formulas (5.4.3) and (5.4.6). It extends Proposition 5.14 to the unbounded setting.

Theorem 5.19. *Let $p \in (2, \infty)$ and let $G \in L^0(\Omega; L^p(0, T; \gamma(H, \tilde{E}_{-\theta})))$ be adapted.*

1. *If for all $t \in [0,T]$, (5.4.3) holds a.s., then for all $\beta \in (\theta, \eta_-)$ and for all $t \in [0,T]$ and for all $\varphi \in \Gamma_{t,\beta}$ the identity (5.4.6) holds a.s.*
2. *If $U \in L^0(\Omega; L^1(0,T;E_0))$ and there is $\beta \in (\theta, \eta_-)$ such that for all $t \in [0,T]$, for all $\varphi \in \Gamma_{t,\beta}$ the identity (5.4.6) holds a.s., then for all $t \in [0,T]$, U satisfies (5.4.3) a.s.*

We have already seen that (5.4.3) is well-defined. Also all terms in (5.4.6) are well-defined. For instance

$$\begin{aligned}|\langle I(\mathbf{1}_{(0,s)}G), \varphi'(s)\rangle| &= |\langle (-A(s))^{-\beta} I(\mathbf{1}_{(0,s)}G), (-A(s))^{\beta}\varphi'(s)\rangle| \\ &\leq C \sup_{r\in[0,T]} \|I(\mathbf{1}_{(0,r)}G)\|_{E_{-\theta}} (t-s)^{-1-\varepsilon}\end{aligned}$$

and the latter is integrable with respect to $s \in (0,t)$.

Proof. (1): Assume (5.4.3) holds and fix $s \in [0,T]$ for the moment. Let $\beta \in (\theta, \eta_-)$ and choose $\lambda \in (\beta, \eta_-)$. Let $x^* \in D(((-A(s))^{\lambda})^*)$. By (5.2.9),

$$\|(-A(s))^{-\lambda} S(s,r)(-A(r))^{1+\beta}\|_{\mathscr{L}(E_0)} \leq C(s-r)^{-1-\beta+\lambda}.$$

Since also $r \mapsto I(\mathbf{1}_{(0,r)}G)$ is in $L^0(\Omega; L^\infty(0,T;\tilde{E}_{-\theta}))$ it follows that

$$\begin{aligned}&\int_0^s |\langle S(s,r)A(r)I(\mathbf{1}_{(r,s)}G), x^*\rangle|\,dr \\ &= \int_0^s |\langle (-A(s))^{-\lambda} S(s,r)(-A(r))^{1+\beta}(-A(r))^{-\beta} I(\mathbf{1}_{(r,s)}G), ((-A(s))^{\lambda})^* x^*\rangle|\,dr \\ &\leq C_G \int_0^s (s-r)^{-1-\beta+\lambda} \|((-A(s))^{\lambda})^* x^*\|\,dr \\ &\leq C_G' \|((-A(s))^{\lambda})^* x^*\|.\end{aligned}$$

Since $I(\mathbf{1}_{(r,s)}G) = I(\mathbf{1}_{(0,s)}G) - I(\mathbf{1}_{(0,r)}G)$, we can write

$$\begin{aligned}&\int_0^s \langle S(s,r)A(r)I(\mathbf{1}_{(r,s)}G), x^*\rangle\,dr \\ &\quad = \int_0^s \langle S(s,r)A(r)I(\mathbf{1}_{(0,s)}G), x^*\rangle\,dr - \int_0^s \langle S(s,r)A(r)I(\mathbf{1}_{(0,r)}G), x^*\rangle\,dr.\end{aligned}$$

Noting that $\frac{\partial S(s,r)}{\partial r} = -S(s,r)A(r)$, an approximation argument yields that

$$\int_0^s \langle S(s,r)A(r)x, x^*\rangle\,dr = \langle S(s,0)x, x^*\rangle - \langle x, x^*\rangle, \quad x \in \tilde{E}_{-\theta}.$$

Then by (5.4.3) and using the above identities (with $x = I(\mathbf{1}_{(0,s)}G)$) we find that

$$\langle U(s), x^*\rangle = \int_0^s \langle S(s,r)A(r)I(\mathbf{1}_{(0,r)}G), x^*\rangle\,dr + \langle I(\mathbf{1}_{(0,s)}G), x^*\rangle. \quad (5.4.7)$$

Now let $\varphi \in \Gamma_{t,\beta}$. Applying the above with $x^* = A(s)^*\varphi(s)$ and integrating over $s \in (0,t)$ we find that

$$\begin{aligned}
&\int_0^t \langle U(s), A(s)^*\varphi(s)\rangle \, ds - \int_0^t \langle I(\mathbf{1}_{(0,s)}G), A(s)^*\varphi(s)\rangle \, ds \\
&\quad = \int_0^t \int_0^s \langle S(s,r)A(r)I(\mathbf{1}_{(0,r)}G), A(s)^*\varphi(s)\rangle \, dr \, ds \\
&\quad = \int_0^t \int_r^t \langle S(s,r)A(r)I(\mathbf{1}_{(0,r)}G), A(s)^*\varphi(s)\rangle \, ds \, dr.
\end{aligned} \tag{5.4.8}$$

Since $\frac{d}{dt}S(t,s) = A(t)S(t,s)$, with an approximation argument it follows that for all $x \in \tilde{E}_{-\theta}$ and $0 \le r \le t \le T$,

$$\begin{aligned}
\langle S(t,r)A(r)x, \varphi(t)\rangle - \langle x, A(r)^*\varphi(r)\rangle = \int_r^t \langle S(s,r)A(r)x, A(s)^*\varphi(s)\rangle \, ds \\
+ \int_r^t \langle S(s,r)A(r)x, \varphi'(s)\rangle \, ds.
\end{aligned} \tag{5.4.9}$$

Note that the above integrals converge absolutely. Indeed, for all $\varepsilon > 0$ small, one has by (5.2.9) and the assumption on φ that

$$\begin{aligned}
|\langle S(s,r)A(r)x, A(s)^*\varphi(s)\rangle| &= |\langle (-A(s))^{-\lambda} S(s,r)A(r)x, ((-A(s))^{1+\lambda})^*\varphi(s)\rangle| \\
&\le C(s-r)^{-1-\theta-\varepsilon+\lambda}(t-s)^{-1+\varepsilon}.
\end{aligned}$$

The latter is clearly integrable with respect to $s \in (r,t)$ for $\varepsilon > 0$ small enough. The same estimate holds with $A(s)^*\varphi(s)$ replaced by $\varphi'(s)$.

Using (5.4.9) in the identity (5.4.8) we find that

$$\begin{aligned}
\int_0^t \langle U(s), A(s)^*\varphi(s)\rangle \, ds = \int_0^t \langle S(t,r)A(r)I(\mathbf{1}_{(0,r)}G), \varphi(t)\rangle \, dr \\
- \int_0^t \int_r^t \langle S(s,r)A(r)I(\mathbf{1}_{(0,r)}G), \varphi'(s)\rangle \, ds \, dr.
\end{aligned}$$

Therefore, by (5.4.7) applied with $s = t$ and $x^* = \varphi(t)$, and Fubini's theorem we find that

$$\begin{aligned}
\int_0^t \langle U(s), A(s)^*\varphi(s)\rangle \, ds = \langle U(t), \varphi(t)\rangle - \langle I(\mathbf{1}_{(0,t)}G), \varphi(t)\rangle \\
- \int_0^t \langle U(s), \varphi'(s)\rangle \, ds + \int_0^t \langle I(\mathbf{1}_{(0,s)}G), \varphi'(s)\rangle \, ds.
\end{aligned}$$

This implies that U satisfies (5.4.6).

(2): Assume (5.4.6) holds. Fix $t \in [0,T]$ and $x^* \in D(A(t)^*)$. By Example 5.18 the process $\varphi : [0,t] \times \Omega \to E_0^*$ given by $\varphi(s) = S(t,s)^*x^*$ is in $\Gamma_{t,\beta}$ for all $\beta \in [0,\eta_-)$. Applying (5.4.6) and using that $\varphi'(s) = -A(s)^*\varphi(s)$ we find that

$$\langle U(t), x^* \rangle = \int_0^t \langle S(t,s)A(s)I(\mathbf{1}_{(0,s)}G), x^* \rangle \, ds + \langle I(\mathbf{1}_{(0,t)}G), x^* \rangle$$

and as in part (1) of the proof this can be rewritten as

$$\langle U(t), x^* \rangle = -\int_0^t \langle S(t,s)A(s)I(\mathbf{1}_{(s,t)}G), x^* \rangle \, ds + \langle S(t,0)I(\mathbf{1}_{(0,t)}G), x^* \rangle. \tag{5.4.10}$$

The identity (5.4.3) follows from the Hahn-Banach theorem and density of $D(A(t)^*)$ in E_0^*. □

Remark 5.20. If φ in (5.4.6) is not dependent of Ω, then the stochastic Fubini theorem and integration by parts show that (5.4.6) is equivalent with

$$\begin{aligned} \langle U(t), \varphi(t) \rangle = \int_0^t \langle U(s), A(s)^* \varphi(s) \rangle \, ds + \int_0^t \langle U(s), \varphi'(s) \rangle \, ds \\ - \int_0^t G(s)^* \varphi(s) \, dW(s). \end{aligned} \tag{5.4.11}$$

This solution concept coincides with the one in [135] and is usually referred to as a variational solution. Using the forward integral one can obtain (5.4.11) from (5.4.6) for φ depending on Ω in a nonadapted way.

In the next theorem we show the equivalence of the representation formula (5.4.3) and the weak formulation (5.4.5). It extends Proposition 5.14 to the unbounded setting.

Theorem 5.21. *Let $G \in L^0(\Omega; L^p(0,T;\gamma(H,\tilde{E}_{-\theta}))$ be adapted. Let*

$$F = \bigcap_{t\in[0,T],\omega\in\Omega} D((A(t,\omega))^*).$$

(1) Assume $D(A(t)) = D(A(0))$ isomorphically with uniform estimates in $t \in [0,T]$ and $\omega \in \Omega$. If for all $t \in [0,T]$, (5.4.3) *holds a.s., then for all $x^* \in F$, for all $t \in [0,T]$,* (5.4.5) *holds a.s.*

(2) Assume $D(A(t)^*) = D(A(0)^*)$ isomorphically with uniform estimates in $t \in [0,T]$ and $\omega \in \Omega$. If for all $x^* \in F$ and $t \in [0,T]$, (5.4.5) *holds a.s., then for all $t \in [0,T]$,* (5.4.3) *holds almost surely.*

Proof. (1): First consider the case where $G(t) \in D(A(t))$ for all $t \in [0,T]$, and the process $t \mapsto A(t)G(t)$ is adapted in $L^0(\Omega; L^p(0,T;\gamma(H,E_0))$. Let $x^* \in F$ and take $\varphi \equiv x^*$. Unfortunately, φ is not in $\Gamma_{t,0}$. However, due to the extra regularity of G, one can still proceed as in the proof of Theorem 5.19 (1). Indeed, the only modification needed is that (5.4.9) holds for $x^* \in F$ and $x \in E_1$. Moreover, since $\varphi' = 0$, (5.4.5) follows from (5.4.6).

Now let $G \in L^0(\Omega; L^p(0,T;\gamma(H,\tilde{E}_{-\theta}))$ and define an approximation by $G_n(t) = n^2 R(n, A(t))^{-2} G(t)$. Let U_n be given by (5.4.3) with G replaced by G_n. Then by the above, U_n satisfies

$$\langle U_n(t), x^*\rangle = \int_0^t \langle U_n(s), A(s)^*x^*\rangle \, ds + \int_0^t G_n(s)^*x^* \, dW(s). \quad (5.4.12)$$

By the dominated convergence theorem we have almost surely $G_n \to G$ in $L^p(0,T;\gamma(H,\tilde{E}_{-\theta}))$. Therefore, Proposition 5.13 and Theorem 5.16 yield that $U_n \to U$ in $L^0(\Omega;L^p(0,T;E_0))$. Letting $n \to \infty$ in (5.4.12), we obtain (5.4.5).

(2): The strategy of the proof is to show that U satisfies (5.4.6). In order to show this we need to allow the functional $x^* \in F$ to be dependent on $s \in [0,t]$ and $\omega \in \Omega$. In order to do so, fix $t \in [0,T]$, let $f \in C^1([0,t])$ and $x^* \in F$. Let $\varphi = f \otimes x^*$. By integration by parts and (5.4.5) (applied twice) we obtain

$$\begin{aligned}
&\langle U(t), \varphi(t)\rangle - \langle I(\mathbf{1}_{[0,t]}G), \varphi(t)\rangle = \int_0^t \langle U(s), A(s)^*x^*\rangle \, ds \, f(t) \\
&= \int_0^t \langle U(s), A(s)^*\varphi(s)\rangle \, ds + \int_0^t \int_0^s \langle U(r), A(r)^*x^*\rangle \, dr \, f'(s) \, ds \\
&= \int_0^t \langle U(s), A(s)^*\varphi(s)\rangle \, ds + \int_0^t \langle U(s), x^*\rangle \, f'(s) \, ds - \int_0^t \langle I(\mathbf{1}_{[0,s]G}), x^*\rangle \, f'(s) \, ds \\
&= \int_0^t \langle U(s), A(s)^*\varphi(s)\rangle \, ds + \int_0^t \langle U(s), \varphi'(s)\rangle \, ds - \int_0^t \langle I(\mathbf{1}_{[0,s]G}), \varphi'(s)\rangle \, ds.
\end{aligned}$$

This yields (5.4.6) for the special φ as above. By linearity and approximation the identity (5.4.6) can be extended to all $\varphi \in C^1([0,t];E_0^*) \cap C([0,t];F)$. Clearly, the identity extends simple functions $\varphi : \Omega \to C^1([0,t];E_0^*) \cap C([0,t];F)$ and by approximation it extends to any $\varphi \in L^0(\Omega;C^1([0,t];E_0^*) \cap C([0,t];F))$. Now let $x^* \in F$ be arbitrary and let $\varphi(s) = S(t,s)^*x^*$. Then as in the proof of Theorem 5.19 (2) we see that $\varphi \in L^0(\Omega;C^1([0,t];E_0^*) \cap C([0,t];F))$ and (5.4.10) follows and the proof can be finished as before. □

5.4.5 Forward integration and mild solutions

In this section we show how the forward integral can be used to define mild solutions of (5.4.1) and show that they coincide with (5.4.3). The forward integral was developed by Russo and Vallois in [117], [118] and can be used to integrate nonadapted integrands and is based on a regularization procedure. We refer to [122] for a survey on the subject and a collection of references.

For $G \in L^0(\Omega;L^2(0,T;\gamma(H,E_0)))$ define the sequence $(I^-(G,n))_{n=1}^\infty$ by

$$I^-(G,n) = \sum_{k=1}^n n \int_0^T G(s)h_k(W(s+1/n)h_k - W(s)h_k) \, ds.$$

The process G is called *forward integrable* if $(I^-(G,n))_{n\geq 1}$ converges in probability. In that case, the limit is called *the forward integral* of G and its limit is denoted by

$$I^-(G) = \int_0^T G \, dW^- = \int_0^T G(s) \, dW^-(s).$$

This definition is less general than the one in Chapter 4, but will suffice for our purposes here.

In Chapter 4 it has been shown that for UMD Banach spaces the forward integral extends the Itô integral from [86]. In particular, the forward integral as defined above extends the stochastic integral as described in Section 5.4.1.

We will now show that the forward integral can be used to extend the concept of mild solutions to the case where $A(t)$ is random. The proof will be based on a pointwise multiplier result for the forward integral from Chapter 4.

Theorem 5.22. *Assume (H1)-(H4). Let $p \in (2, \infty)$. Let $\theta \in [0, \frac{1}{2} \wedge \eta_-)$. Assume $\delta < \min\{\frac{1}{2} - \theta - \frac{1}{p}, \eta_+\}$. Let $G \in L^0(\Omega; L^p(0, T; \gamma(H, \tilde{E}_{-\theta})))$ be adapted. Then for every $t \in [0, T]$, the process $s \mapsto S(t, s)G(s)$ is forward integrable on $[0, t]$ with values in $\tilde{E}_\delta$, and*

$$U(t) = \int_0^t S(t, s)G(s)\, dW^-(s), \tag{5.4.13}$$

where U is given by (5.4.3).

The above identity is mainly of theoretical interest as it is rather difficult to prove estimates for the forward integral in a direct way. Of course (5.4.3) allows to obtain such estimates. Due to (5.4.13) one could call U a *forward mild solution* to (5.4.1).

As a consequence of Theorems 5.21 and 5.22, there is an equivalence between weak solutions and forward mild solutions. Under different assumptions it was shown in [68, Proposition 5.3] that every forward mild solution is a weak solution.

Proof. Define $M : [0, t] \times \Omega \to \mathscr{L}(\tilde{E}_{-\theta}, \tilde{E}_\delta)$ by $M(s) = S(t, s)$. Let $N : [0, t) \times \Omega \to \mathscr{L}(\tilde{E}_{-\theta}, \tilde{E}_\delta)$ be given by $N(s) = -S(t, s)A(s)$. Then by Lemma 5.7, $M(s) = M(0) + \int_0^s N(r)\, dr$ for $s \in [0, t)$ and thus M is continuously differentiable with derivative N. By Remark 5.9 there is a mapping $C : \Omega \to \mathbb{R}_+$ such that

$$\|N(s)\|_{\mathscr{L}(\tilde{E}_{-\theta}, \tilde{E}_\delta)} \leq C(t - s)^{-1-\delta-\theta}.$$

Now by the non-adapted multiplier result for the forward integral from [110] we find that MG is forward integrable and

$$\begin{aligned}\int_0^t S(t, s)G(s)\, dW^-(s) &= \int_0^t M(s)G(s)\, dW^-(s) \\ &= M(0)I(G) + \int_0^t N(s)I(\mathbf{1}_{[s,t]}G)\, ds = U(t).\end{aligned}$$

□

5.5 Semilinear stochastic evolution equations

In this section we assume Hypotheses (H1)-(H4) are satisfied. We will apply the results of the previous sections to study the following stochastic evolution equation on the Banach space E_0

$$\begin{cases} dU(t) = (A(t)U(t) + F(t, U(t)))\, dt + B(t, U(t))\, dW(t), \\ U(0) \ = u_0, \end{cases} \tag{5.5.1}$$

Here F and G will be suitable nonlinearities of semilinear type. In Section 5.5.1 we will first state the main hypotheses on F and G and define what a pathwise mild solution is. In Section 5.5.2 we will prove that there is a unique pathwise mild solution under the additional assumption that the constants in the (AT)-conditions do not depend on ω. The uniformity condition (H5) will be removed in Section 5.5.3 by localizing the random drift A.

5.5.1 Setting and solution concepts

Recall that the spaces $\tilde{E}_\eta$ were defined in (H3) in Section 5.3. We impose the following assumptions on F and B throughout this section:

(HF) Let $a \in [0, \eta_+)$ and $\theta_F \in [0, \eta_-)$ be such that $a + \theta_F < 1$. For all $x \in \tilde{E}_a$, $(t, \omega) \mapsto F(t, \omega, x) \in \tilde{E}_{-\theta_F}$ is strongly measurable and adapted. Moreover, there exist constants L_F and C_F such that for all $t \in [0, T], \omega \in \Omega, x, y \in \tilde{E}_a$,

$$\|F(t, \omega, x) - F(t, \omega, y)\|_{\tilde{E}_{-\theta_F}} \le L_F \|x - y\|_{\tilde{E}_a},$$
$$\|F(t, \omega, x)\|_{\tilde{E}_{-\theta_F}} \le C_F(1 + \|x\|_{\tilde{E}_a}).$$

(HB) Let $a \in [0, \eta_+)$ and $\theta_B \in [0, \eta_-)$ be such that $a + \theta_B < 1/2$. For all $x \in \tilde{E}_a$, $(t, \omega) \mapsto B(t, \omega, x) \in \gamma(U, \tilde{E}_{-\theta_B})$ is strongly measurable and adapted. Moreover, there exist constants L_B and C_B such that for all $t \in [0, T], \omega \in \Omega, x, y \in \tilde{E}_a$,

$$\|B(t, \omega, x) - B(t, \omega, y)\|_{\gamma(U, \tilde{E}_{-\theta_B})} \le L_B \|x - y\|_{\tilde{E}_a},$$
$$\|B(t, \omega, x)\|_{\gamma(U, \tilde{E}_{-\theta_B})} \le C_B(1 + \|x\|_{\tilde{E}_a}).$$

Let $p \in (2, \infty)$ and consider adapted processes $f \in L^0(\Omega; L^p(0, T; \tilde{E}_{-\theta_F}))$ and $G \in L^0(\Omega; L^p(0, T; \gamma(H, \tilde{E}_{-\theta_B})))$. In the sequel we will write

$$S * f(t) := \int_0^t S(t, s) f(s)\, ds,$$
$$S \diamond G(t) := -\int_0^t S(t, s) A(s) I(\mathbf{1}_{(s,t)} G)\, ds + S(t, 0) I(\mathbf{1}_{(0,t)} G)$$

for the deterministic and stochastic (generalized) convolution.

The integral $\int_0^t S(t, s) A(s) I(\mathbf{1}_{(s,t)} G)\, ds$ was extensively studied in Section 5.4. Recall from Theorem 5.16 that is it well-defined and defines an adapted process in $L^0(\Omega; W^{\lambda,p}(0, T; \tilde{E}_\delta))$ for suitable λ and δ.

Definition 5.23. Let $2 < p < \infty$. An adapted process $U \in L^0(\Omega; L^p(0,T;\tilde{E}_a))$ is called a *pathwise mild solution* of (5.5.1) if for all $t \in [0,T]$, almost surely,

$$U(t) = S(t,0)u_0 + S * F(\cdot, U)(t) + S \diamond B(\cdot, U)(t). \tag{5.5.2}$$

Remark 5.24. A good name for the solution (5.5.2) would be a 'pathwise singular representation of the mild solution'. Pathwise appears in the name, since the representation formula (5.4.3) is defined in a pathwise sense. Also, in defining the representation formula 5.4.3, it is of great importance that the singularity of the kernel $S(t,s)A(s)$ appearing in the formula is canceled by the Hölder continuity of $J(G)$ (see Proposition 5.13). We have chosen to abbreviate this name to 'pathwise mild solution'.

Note that the convolutions in (5.5.2) might only be defined for almost all $t \in [0,T]$. However, if $p \in (2,\infty)$ is large enough, then they are defined in a pointwise sense.

One can extend Proposition 5.14 and Theorems 5.19, 5.21 and 5.22 to the nonlinear setting. Indeed, this follows by taking $G = B(\cdot, U)$ and including the terms F and u_0. The latter two terms do not create any problems despite the randomness of A, because the terms are defined in a pathwise way, and therefore can be treated as in [135]. As a consequence we deduce that (5.5.2) yields the "right" solution of (5.5.1) in many ways (variational, forward mild, weak).

5.5.2 Results under a uniformity condition in Ω

In this section we additionally assume the following uniformity condition.

(H5) The mapping $L : \Omega \to \mathbb{R}_+$ from (AT2) for $A(t)$ and $A(t)^*$ is bounded in Ω.

Under Hypothesis (H5), it is clear from the proofs that most of the constants in Sections 5.2 and 5.3 become uniform in Ω. In Section 5.5.3 we will show how to obtain well-posedness without the condition (H5).

For a Banach space X, we write $B([0,T];X)$ for the strongly measurable functions $f : [0,T] \to X$. For $\delta \in (-1, \eta_+)$ and $p \in (2,\infty)$ let Z^p_δ be the subspace of strongly measurable adapted processes $u : [0,T] \times \Omega \to \tilde{E}_\delta$ for which $\|u\|_{Z^p_\delta} := \sup_{t\in[0,T]} \|u(t)\|_{L^p(\Omega;\tilde{E}_\delta)}$ is finite. Define the operator $L : Z^p_\delta \to Z^p_\delta$ by

$$(L(U))(t) = S(t,0)u_0 + S * F(\cdot, U)(t) + S \diamond B(\cdot, U)(t).$$

In the next lemma we show that L is well-defined and is a strict contraction in a suitable equivalent norm on Z^p_a.

Lemma 5.25. *Assume (H1)–(H5), (HF) and (HB). Let $p \in (2,\infty)$. If the process $t \mapsto S(t,0)u_0$ is in Z^p_a, then L maps Z^p_a into itself and there is an equivalent norm $|\!|\!| \cdot |\!|\!|$ on Z^p_a such that for every $u, v \in Z^p_a$,*

$$|\!|\!| L(u) - L(v) |\!|\!|_{Z^p_a} \leq \frac{1}{2} |\!|\!| u - v |\!|\!|_{Z^p_a}.$$

Moreover, there exists a constant C independent of u_0 such that

$$\|L(u)\|_{Z_a^p} \le C + \|t \mapsto S(t,0)u_0\|_{Z_a^p} + \frac{1}{2}\|u\|_{Z_a^p}. \tag{5.5.3}$$

Proof. Choose $\varepsilon > 0$ so small that $a + \theta_F + \varepsilon < 1$ and $a + \theta_B + \varepsilon < 1/2$. We will first prove several estimates for the individual parts of the mapping L. Conclusions will be derived afterwards.

For $\kappa \ge 0$ arbitrary but fixed for the moment and define an equivalent norm on Z_δ^p by

$$\|u\|_{Z_\delta^p} = \sup_{t\in[0,T]} e^{-\kappa t}\|u(t)\|_{L^p(\Omega;\tilde{E}_\delta)}.$$

We also let

$$\|G\|_{Z_\delta^p(\gamma)} = \sup_{t\in[0,T]} e^{-\kappa t}\|G(t)\|_{L^p(\Omega;\gamma(H,\tilde{E}_\delta))}.$$

Deterministic convolution: Let $f \in Z_{-\theta_F}^p$. By (5.3.3) applied pathwise and (H5) one obtains

$$\|S * f(t)\|_{\tilde{E}_a} \le C\int_0^t (t-\sigma)^{-a-\theta_F-\varepsilon}\|f(\sigma)\|_{\tilde{E}_{-\theta_F}}\,d\sigma, \quad t\in[0,T]$$

where C is independent of ω. In particular, taking $L^p(\Omega)$-norms on both sides we find that

$$\|S * f(t)\|_{L^p(\Omega;\tilde{E}_a)} \le C\int_0^t (t-\sigma)^{-a-\theta_F-\varepsilon}\|f(\sigma)\|_{L^p(\Omega;\tilde{E}_{-\theta_F})}\,d\sigma$$

Using $e^{-\kappa t} = e^{-\kappa(t-\sigma)}e^{-\kappa\sigma}$, it follows that

$$\begin{aligned}\|S * f\|_{Z_a^p} &\le C\|f\|_{Z_{-\theta_F}^p}\sup_{t\in[0,T]}\int_0^t e^{-\kappa(t-\sigma)}(t-\sigma)^{-a-\theta-\varepsilon}\,d\sigma\\ &\le C\phi_1(\kappa)\|f\|_{Z_{-\theta_F}^p},\end{aligned} \tag{5.5.4}$$

where

$$\phi_1(\kappa) = \int_0^\infty e^{-\kappa\sigma}\sigma^{-a-\theta-\varepsilon}\,d\sigma.$$

Clearly, $\lim_{\kappa\to\infty}\phi_1(\kappa) = 0$.

Now let $u, v \in Z_a^p$. By the hypothesis (HF), $F(\cdot,u)$ and $F(\cdot,v)$ are in $Z_{-\theta_F}^p$ and therefore, we find that $S * F(\cdot,u)$ and $S * F(\cdot,v)$ are in Z_a^p again. Moreover, applying (5.5.4) with $f = F(\cdot,u) - F(\cdot,v)$ it follows that

$$\begin{aligned}\|S * F(\cdot,u) - S * F(\cdot,v)\|_{Z_a^p} &\le C\phi_1(\kappa)\|F(\cdot,u) - F(\cdot,v)\|_{Z_{-\theta_F}^p}\,d\sigma\\ &\le C\phi_1(\kappa)L_F\|u-v\|_{Z_a^p}.\end{aligned}$$

Stochastic convolution: Let $G \in Z_{-\theta_B}^p$ be arbitrary. Clearly, we can write $\|S \diamond G(t)\|_{L^p(\Omega;\tilde{E}_a)} \le T_1(t) + T_2(t)$, where

$$T_1(t) := \Big\| \int_0^t S(t,s)A(s)I(\mathbf{1}_{(s,t)}G)\, ds \Big\|_{L^p(\Omega;\tilde{E}_a)},$$

and $T_2(t) = \|S(t,0)I(\mathbf{1}_{(0,t)}G)\|_{L^p(\Omega;\tilde{E}_a)}$. To estimate T_1 note that by Remark 5.9

$$T_1(t) \le C \int_0^t (t-s)^{-1-a-\theta_B-\varepsilon} \|I(\mathbf{1}_{(s,t)}G)\|_{L^p(\Omega;\tilde{E}_{-\theta_B})}\, ds$$

By (H5), C is independent of Ω. By (5.4.4) and Minkowski's inequality we have

$$\begin{aligned}\|I(\mathbf{1}_{(s,t)}G)\|_{L^p(\Omega;\tilde{E}_{-\theta})} &\le C\|G\|_{L^p(\Omega;L^2(s,t;\gamma(H,\tilde{E}_{-\theta_B})))} \\ &\le C\|G\|_{L^2(s,t;L^p(\Omega;\gamma(H,\tilde{E}_{-\theta_B})))} \\ &\le C\Big(\int_s^t e^{2\kappa\sigma}\, d\sigma\Big)^{1/2} \|\!|G|\!\|_{Z^p_{-\theta_B}(\gamma)} \\ &= C\kappa^{-1/2}(e^{2\kappa t} - e^{2\kappa s})^{1/2} \|\!|G|\!\|_{Z^p_{-\theta_B}(\gamma)}\end{aligned} \tag{5.5.5}$$

Therefore, we find that

$$\begin{aligned}&\sup_{t\in[0,T]} e^{-\kappa t} T_1(t) \\ &\quad\le C \sup_{t\in[0,T]} \int_0^t (t-s)^{-1-a-\theta_B-\varepsilon}\, ds \big(\kappa^{-1}(1-e^{-2\kappa(t-s)})\big)^{1/2} \|\!|G|\!\|_{Z^p_{-\theta_B}(\gamma)} \\ &\quad= C \sup_{t\in[0,T]} \int_0^t \sigma^{-1-a-\theta_B-\varepsilon} \kappa^{-1/2} (1-e^{-2\kappa\sigma})^{1/2}\, d\sigma \|\!|G|\!\|_{Z^p_{-\theta_B}(\gamma)} \\ &\quad\le C\phi_2(\kappa) \|\!|G|\!\|_{Z^p_{-\theta_B}(\gamma)},\end{aligned}$$

where ϕ_2 is given by

$$\phi_2(\kappa) = \kappa^{-\frac{1}{2}+a+\theta_B+\varepsilon} \int_0^\infty \sigma^{-1-a-\theta_B-\varepsilon}(1-e^{-2\sigma})^{1/2}\, d\sigma.$$

Since $a+\theta_B+\varepsilon < \frac{1}{2}$, the latter is finite. Moreover, $\lim_{\kappa\to\infty} \phi_2(\kappa) = 0$.

To estimate $T_2(t)$ note that by Remark 5.9, (H5), Proposition 5.13 and (5.5.5) with $s = 0$,

$$\begin{aligned}T_2(t) &\le Ct^{-a-\theta_B-\varepsilon} \|I(\mathbf{1}_{(0,t)}G)\|_{L^p(\Omega;E_{-\theta_B})} \\ &\le Ct^{-a-\theta_B-\varepsilon}\kappa^{-1/2}(e^{2\kappa t}-1)^{1/2} \|\!|G|\!\|_{Z^p_{-\theta_B}(\gamma)}\end{aligned}$$

Therefore, using $\sup_{\sigma\ge 0} \sigma^{-a-\theta_B-\varepsilon}(1-e^{-2\sigma})^{1/2} < \infty$, we find that

$$\sup_{t\in[0,T]} e^{-\kappa t} T_2(t) \le C\kappa^{-\frac{1}{2}+a+\theta_B+\varepsilon} \|\!|G|\!\|_{Z^p_{-\theta_B}(\gamma)}.$$

Combining the estimate for T_1 and T_2 we find that

$$\|S \diamond G\|_{Z_a^p} \leq C\phi_3(\kappa)\|G\|_{Z^p_{-\theta_B}(\gamma)}, \tag{5.5.6}$$

where $\phi_3(\kappa) \to 0$ if $\kappa \to \infty$.

Now let $u, v \in Z_a^p$. By the hypothesis (HB), $B(\cdot, u)$ and $B(\cdot, v)$ are in $Z^p_{-\theta_B}$ and therefore, by the above we find that $S \diamond B(\cdot, u)$ and $S \diamond B(\cdot, v)$ are in Z_a^p again. Moreover, applying (5.5.6) with $G = B(\cdot, u) - B(\cdot, v)$ it follows that

$$\begin{aligned}\|S \diamond B(\cdot, u) - S \diamond B(\cdot, v)\|_{Z_a^p} &\leq C\phi_3(\kappa)\|B(\cdot, u) - B(\cdot, v)\|_{Z^p_{-\theta_B}(\gamma)} \\ &\leq C\phi_3(\kappa)L_B\|u - v\|_{Z_a^p}.\end{aligned}$$

Conclusion. From the above computations, it follows that L is a bounded operator on Z_a^p. Moreover, for all $u, v \in Z_a^p$,

$$\|L(u) - L(v)\|_{Z_a^p} \leq C(L_F\phi_1(\kappa) + L_B\phi_3(\kappa))\|u - v\|_{Z_a^p}.$$

Choosing κ large enough, the result follows. Also, (5.5.3) follows when taking $v \equiv 0$. □

As a consequence we obtain the following result.

Theorem 5.26. *Assume (H1)–(H5), (HF) and (HB). Let $p \in (2, \infty)$. Let $\delta, \lambda > 0$ be such that $a + \delta + \lambda < \min\{\frac{1}{2} - \theta_B, 1 - \theta_F, \eta_+\}$. Assume the process $t \mapsto S(t, 0)u_0$ is in Z_a^p. Then there exists a unique pathwise mild solution $U \in Z_a^p$ of (5.5.1). Moreover, $U - S(t, 0)u_0 \in L^p(\Omega; W^{\lambda,p}(0, T; \tilde{E}_\delta))$ and there is a constant independent of u_0 such that*

$$\|U - S(t, 0)u_0\|_{L^p(\Omega;W^{\lambda,p}(0,T;\tilde{E}_{a+\delta}))} \leq C(1 + \|t \mapsto S(t, 0)u_0\|_{Z_a^p}). \tag{5.5.7}$$

Of course by Sobolev embedding (5.3.1) one can further deduce Hölder regularity of the solution.

Proof. By Lemma 5.25 there exists a unique fixed point $U \in Z_a^p$ of L. Clearly, this implies that U is a pathwise mild solution of (5.5.1). Moreover, from (5.5.3) we deduce that

$$\|U\|_{Z_a^p} \leq 2C + 2\|t \mapsto S(t, 0)u_0\|_{Z_a^p}.$$

Next we prove the regularity assertion. From the previous estimate, Theorem 5.11 and (HF) we see that:

$$\begin{aligned}\|S * F(\cdot, U)\|_{L^p(\Omega;W^{\lambda,p}(0,T;\tilde{E}_{a+\delta}))} &\leq C\|F(\cdot, U)\|_{L^p((0,T)\times\Omega;\tilde{E}_{-\theta_F})} \\ &\leq C(1 + \|U\|_{L^p((0,T)\times\Omega;\tilde{E}_a)}) \\ &\leq C(1 + \|U\|_{Z_a^p}) \leq C(1 + \|t \mapsto S(t, 0)u_0\|_{Z_a^p}).\end{aligned}$$

Similarly, by Theorem 5.16 (with $\alpha \in (a + \delta + \theta_B + \lambda, \frac{1}{2})$), Proposition 5.13 and (HB)

$$\begin{aligned}\|S \diamond B(\cdot, U)\|_{L^p(\Omega;W^{\lambda,p}(0,T;\tilde{E}_{a+\delta}))} &\leq C\|J(B(\cdot, U))\|_{L^p(\Omega;W^{\alpha,p}(0,T;\tilde{E}_{-\theta_B}))} \\ &\leq C\|B(\cdot, U)\|_{L^p(\Omega\times(0,T);\gamma(H,\tilde{E}_{-\theta_B}))} \\ &\leq C(1 + \|U\|_{L^p((0,T)\times\Omega;\tilde{E}_a)}) \\ &\leq C(1 + \|U\|_{Z_a^p}) \leq C(1 + \|t \mapsto S(t, 0)u_0\|_{Z_a^p}).\end{aligned}$$

Now the estimate (5.5.7) follows since $U - S(t, 0)u_0 = S * F(\cdot, U) + S \diamond B(\cdot, U)$. □

One can extend the above existence and uniqueness result to the case where $u_0 : \Omega \to \tilde{E}_a$ is merely $\mathscr{F}_0$-measurable. For that, we will continue with a local uniqueness property that will be used frequently.

Lemma 5.27. *Assume (H1)–(H5), (HF) and (HB). Let $\tilde{A}$ be a second operator satisfying (H1), (H2), (H3) and (H5) with the same spaces $(\tilde{E}_\eta)_{-\eta_0<\eta<\eta_+}$ and let the evolution family generated by $\tilde{A}$ be denoted by $(\tilde{S}(t, s))_{0\leq s\leq t\leq T}$. Let $u_0, \tilde{u}_0 : \Omega \to E^0_{a,1}$ be $\mathscr{F}_0$-measurable and such that $S(t, 0)u_0, \tilde{S}(t, 0)\tilde{u}_0 \in Z^p_a$. Let $\tilde{L}$ be defined as L, but with $(S(t, s))_{0\leq s\leq t\leq T}$ and u_0 replaced by $(\tilde{S}(t, s))_{0\leq s\leq t\leq T}$ and $\tilde{u}_0$ respectively. Let $\Gamma \subset \Omega$ and $\tau : \Gamma \to (0,\infty)$.*

Suppose for almost all $\omega \in \Gamma$ for all $t \in [0, \tau(\omega)]$, $A(t, \omega) = \tilde{A}(t, \omega)$ and $u_0(\omega) = \tilde{u}_0(\omega)$. Let $U, \tilde{U} \in Z^p_a$ be such that

$$\mathbf{1}_\Gamma\mathbf{1}_{[0,\tau]}U = \mathbf{1}_\Gamma\mathbf{1}_{[0,\tau]}L(U), \quad \textit{and} \quad \mathbf{1}_\Gamma\mathbf{1}_{[0,\tau]}\tilde{U} = \mathbf{1}_\Gamma\mathbf{1}_{[0,\tau]}\tilde{L}(\tilde{U}).$$

Then for almost all $\omega \in \Gamma$ and all $t \in [0, \tau(\omega)]$ one has $U(t) = \tilde{U}(t)$.

Proof. First we claim that for all $u \in Z^p_a$ one has

$$\mathbf{1}_\Gamma\mathbf{1}_{[0,\tau]}L(u) = \mathbf{1}_\Gamma\mathbf{1}_{[0,\tau]}L(v) = \mathbf{1}_\Gamma\mathbf{1}_{[0,\tau]}\tilde{L}(v) = \mathbf{1}_\Gamma\mathbf{1}_{[0,\tau]}\tilde{L}(u), \tag{5.5.8}$$

where $v = \mathbf{1}_\Gamma\mathbf{1}_{[0,\tau]}u$. Indeed, by the (pathwise) uniqueness of the evolution family one has almost surely on Γ for all $0 \leq s \leq t \leq \tau$, $S(t, s) = \tilde{S}(t, s)$. Now the identity (5.5.8) can be verified for each of the terms in L and $\tilde{L}$. For instance for the first part of stochastic convolution term one has

$$\begin{aligned}&\mathbf{1}_\Gamma\mathbf{1}_{[0,\tau]}(t)\int_0^t S(t, s)A(s)I(\mathbf{1}_{[s,t]}B(\cdot, u))\,ds \\ &\qquad= \mathbf{1}_\Gamma\mathbf{1}_{[0,\tau]}(t)\int_0^t \mathbf{1}_\Gamma\mathbf{1}_{s\leq t\leq\tau}S(t, s)A(s)I(\mathbf{1}_{[s,t]}B(\cdot, u))\,ds\end{aligned} \tag{5.5.9}$$

Now $\mathbf{1}_\Gamma\mathbf{1}_{s\leq t\leq\tau}S(t, s) = \mathbf{1}_\Gamma\mathbf{1}_{s\leq t\leq\tau}\tilde{S}(t, s)$, so we can replace S by $\tilde{S}$ on the right-hand side of (5.5.9). Moreover, using a property of the forward integral [110, Lemma 3.3] (or the local property of the stochastic integral) one sees

$$\begin{aligned}\mathbf{1}_\Gamma\mathbf{1}_{s\leq t\leq\tau}I(\mathbf{1}_{[s,t]}B(\cdot, u)) &= I^-(\mathbf{1}_\Gamma\mathbf{1}_{s\leq t\leq\tau}\mathbf{1}_{[s,t]}B(\cdot, u)) \\ &= I^-(\mathbf{1}_\Gamma\mathbf{1}_{s\leq t\leq\tau}\mathbf{1}_{[s,t]}B(\cdot, v)) \\ &= \mathbf{1}_\Gamma\mathbf{1}_{[0,\tau]}(s)I(\mathbf{1}_{[s,t]}B(\cdot, v)).\end{aligned}$$

Thus we can replace u by v on the right-hand side of (5.5.9).

We will now show how the statement of the lemma follows. Writing $V = \mathbf{1}_\Gamma \mathbf{1}_{[0,\tau]}U$ and $\tilde{V} = \mathbf{1}_\Gamma \mathbf{1}_{[0,\tau]}\tilde{U}$, it follows from the assumption, (5.5.8) and Lemma 5.25 that

$$\begin{aligned} \|V - \tilde{V}\|_{Z^p_a} &= \|\mathbf{1}_\Gamma \mathbf{1}_{[0,\tau]}(L(U) - \tilde{L}(\tilde{U}))\|_{Z^p_a} \\ &= \|\mathbf{1}_\Gamma \mathbf{1}_{[0,\tau]}(L(V) - L(\tilde{V}))\|_{Z^p_a} \\ &\leq \|L(V) - L(\tilde{V})\|_{Z^p_a} \\ &\leq \frac{1}{2}\|V - \tilde{V}\|_{Z^p_a}, \end{aligned}$$

Therefore, $V = \tilde{V}$ in Z^p_a. Since by Theorem 5.26 and Sobolev embedding, $U - S(\cdot,0)u_0$ and $\tilde{U} - \tilde{S}(\cdot,0)\tilde{u}_0$ have continuous paths, it follows that a.s. for all $t \in [0,T]$, $V(t) = \tilde{V}(t)$. This implies the required result. □

Theorem 5.28. *Assume (H1)–(H5), (HF) and (HB). Let $\delta, \lambda > 0$ be such that $a+\delta+\lambda < \min\{\frac{1}{2} - \theta_B, 1-\theta_F, \eta_+\}$. Assume that $u_0 : \Omega \to E_0$ is $\mathscr{F}_0$-measurable and $u_0 \in E^0_{a,1}$ a.s. Then the following holds:*

1. *There exists a unique adapted pathwise mild solution $U \in L^0(\Omega; C([0,T]; \tilde{E}_a))$ of* (5.5.1). *Moreover, $U - S(t,0)u_0 \in L^0(\Omega; C^\lambda(0,T; \tilde{E}_{a+\delta}))$.*
2. *If additionally, $u_0 \in E^0_{a+\beta,1}$ a.s. with $\beta > 0$ and $\lambda + \delta < \beta$, then $U \in L^0(\Omega; C^\lambda(0,T; \tilde{E}_{a+\delta}))$.*

Note that because of the above result we can also view L as a mapping from the adapted subspace of $L^0(\Omega; C([0,T]; \tilde{E}_a))$ into itself.

Proof. Choose $p \in (2,\infty)$ so large that $a + \delta + \lambda + \frac{1}{p} < \min\{\frac{1}{2} - \theta_B, 1 - \theta_F, \eta_+\}$.

Existence. First observe that $\|u_0\|_{E^0_{a,1}}$ is $\mathscr{F}_0$-measurable. Moreover, by Lemma 5.10, $t \mapsto S(t,0)u_0 \in \tilde{E}_a$ has continuous paths. Define $u_n = u_0 \mathbf{1}_{\{\|u_0\|_{E^0_{a,1}} \leq n\}}$. Then u_n is $\mathscr{F}_0$-measurable, and $t \mapsto S(t,0)u_n \in \tilde{E}_a$ has continuous paths and

$$\mathbb{E} \sup_{t\in[0,T]} \|S(t,0)u_n\|^p_{\tilde{E}_a} < \infty.$$

Hence by Theorem 5.26 the problem (5.5.1) with initial condition u_n admits a unique pathwise mild solution $U_n \in L^p(\Omega \times [0,T]; \tilde{E}_a)$. Moreover, by Theorem 5.26 and (5.3.1) there exists a version of U_n such that $U_n - S(t,0)u_n$ has paths in

$$W^{\lambda+\frac{1}{p},p}([0,T]; \tilde{E}_{a+\delta}) \hookrightarrow C^\lambda([0,T]; \tilde{E}_{a+\delta}).$$

In particular, U_n has paths in $C([0,T]; \tilde{E}_a)$. For $1 \leq m \leq n$, by Lemma 5.27, almost surely on $\{\|u_0\| \leq m\}$, one has $U_n \equiv U_m$. Moreover, almost surely on $\{\|u_0\|_{E^0_{a,1}} \leq m\}$, for all $t \in [0,T]$, $U_n(t) = U_m(t)$ when $n \geq m$. It follows that almost surely for all $t \in [0,T]$, the limit $\lim_{n\to\infty} U_n(t)$ exists in $\tilde{E}_a$. Define $U : \Omega \times [0,T] \to \tilde{E}_a$ by

$$U(t) = \begin{cases} \lim_{n\to\infty} U_n(t) & \text{if the limit exists,} \\ 0 & \text{else.} \end{cases}$$

Then U is strongly measurable and adapted. Moreover, almost surely on the set $\{\|u_0\|_{E^0_{a,1}} \leq n\}$, for all $t \in [0,T]$, $U(t) = U_n(t)$. Hence, almost surely, $U \in C([0,T];\tilde{E}_a)$ and one can check that U is a pathwise mild solution to (5.5.1). By construction of U, there exists a version of $U - S(\cdot,0)u_0$ with paths in $C^\lambda([0,T];\tilde{E}_{a+\delta})$ almost surely. In particular, U has almost all paths in $C([0,T];\tilde{E}_a)$. If additionally $u_0 \in E^0_{a+\beta,1}$ with $\delta+\lambda<\beta$, then by Lemma 5.10, $S(t,0)u_0 \in L^0(\Omega;C^\lambda(0,T;\tilde{E}_{a+\delta}))$.

Uniqueness. Suppose U^1 and U^2 are adapted and in $L^0(\Omega;C([0,T];\tilde{E}_a))$ and are both pathwise mild solutions to (5.5.1). We will show that almost surely $U^1 \equiv U^2$. For each $n \geq 1$ and $i=1,2$ define the stopping times

$$\nu^i_n := \inf\Big\{t\in[0,T] : \|U^i(t)\|_{\tilde{E}_a} \geq n\Big\},$$

where we let $\nu^i_n = T$ if the infimum is taken over the empty set. Let $\tau_n = \nu^1_n \wedge \nu^2_n$ and $U^i_n = U^i \mathbf{1}_{[0,\tau_n]}$. Then $U^i_n \in Z^p_a$ and in a similar way as in Lemma 5.27 one can check that $\mathbf{1}_{[0,\tau_n]}U^i_n = \mathbf{1}_{[0,\tau_n]}L(U^i_n)$ for $i=1,2$. Therefore, from Lemma 5.27, we find that for almost surely for all $t\in[0,T]$, $U^1_n(t) = U^2_n(t)$. In particular, almost surely for almost all $t \leq \tau_n$, one has $U^1(t) = U^2(t)$. If we let $n\to\infty$ we obtain that almost surely, for all $t\in[0,T]$, one has $U^1(t) = U^2(t)$. ☐

5.5.3 Results without uniformity conditions in Ω

In this section we will prove a well-posedness result for (5.5.1) without the uniformity condition (H5). The approach is based on a localization argument. Due to technical reasons we use a slightly different condition than (AT2), which is more restrictive in general, but satisfied in many examples. Details on this condition can be found in [3] and [9, Section IV.2]. This condition is based on the assumption that $D(A(t))$ has constant interpolation spaces $E_{\nu,r} = (E_0, D(A(t)))_{\nu,r}$ for certain $\nu > 0$ and $r \in [2,\infty)$, and the fact that the resolvent is μ-Hölder continuous with values in E_ν with $\mu+\nu>1$. Note that in [9, Section IV.2] more general interpolation spaces are allowed. For convenience we only consider the case of constant real interpolation spaces.

(CIS) Condition (AT1) holds and there are constants $\nu\in(0,1]$ and $r\in[1,\infty]$ such that $E_{\nu,r} := (E_0, D(A(t,\omega)))_{\nu,r}$ is constant in $t\in[0,T]$ and $\omega\in\Omega$ and there is a constant C such that for all $x\in E_{\nu,r}$,

$$c^{-1}\|x\|_{E_{\nu,r}} \leq \|x\|_{(E_0,D(A(t,\omega)))_{\nu,r}} \leq c\|x\|_{E_{\nu,r}}, \quad t\in[0,T], \omega\in\Omega.$$

There is a $\mu\in(0,1]$ with $\mu+\nu>1$ and a mapping $K:\Omega\to\mathbb{R}_+$ such that for all $s,t\in[0,T]$, $\omega\in\Omega$,

$$\|(A(t,\omega)^{-1} - A(s,\omega)^{-1})\|_{\mathscr{L}(E_0,E_{\nu,r})} \leq K(\omega)(t-s)^\mu.$$

We have allowed $\nu = 1$ on purpose. In this way we include the important case where $D(A(t,\omega))$ is constant in time.

Clearly, this condition implies (AT2) with constant $L(\omega) \le CK(\omega)$. Indeed, one has for all $\lambda \in \Sigma_\vartheta$:

$$\begin{aligned}&\|A(t,\omega)R(\lambda, A(t,\omega))(A(t,\omega)^{-1} - A(s,\omega)^{-1})\|_{\mathscr{L}(E_0)}\\ &\le \|A(t,\omega)R(\lambda, A(t,\omega))\|_{\mathscr{L}(E_{\nu,r},E_0)}\|A(t,\omega)^{-1} - A(s,\omega)^{-1}\|_{\mathscr{L}(E_0,E_{\nu,r})}\\ &\le CK(t-s)^\mu \|R(\lambda, A(t,\omega))\|_{\mathscr{L}(E_0,E^t_{1-\nu,r})}\\ &\le CK(t-s)^\mu |\lambda|^{-\nu}.\end{aligned} \tag{5.5.10}$$

We will now replace (H5) by the following hypothesis.

(H5)′ Assume E_0 is separable. Assume $(A(t))_{t\in[0,T]}$ and $(A(t)^*)_{t\in[0,T]}$ satisfy (CIS) with constants $\mu + \nu > 1$ and $\mu^* + \nu^* > 1$.

Unlike (H5), the mapping K is allowed to be dependent on Ω.

We can now prove the main result of this section which holds under the hypotheses (H1)–(H4) and (H5)′, (HF) and (HB).

Theorem 5.29. *Assume (H1)–(H4), (H5)′, (HF) and (HB). Let $\delta, \lambda > 0$ be such that $a + \delta + \lambda < \min\{\frac{1}{2} - \theta_B, 1 - \theta_F, \eta_+\}$. Assume that $u_0 : \Omega \to E_0$ is $\mathscr{F}_0$-measurable and $u_0 \in E^0_a$ a.s. Then the assertions (1) and (2) of Theorem 5.28 hold.*

Unlike in Theorem 5.26 one cannot expect that the pathwise mild solution has any integrability properties in general. This is because of the lack of integrability properties of $S(t,s)$.

Proof. For $\varepsilon \in (0,\mu)$ define $\phi : [0,T] \times \Omega \to \mathbb{R}_+$ by

$$\phi(t) = \sup_{s\in[0,t)} \|A(t)^{-1} - A(s)^{-1}\|_{\mathscr{L}(E_0,E_{\nu,r})}|t-s|^{-\mu+\varepsilon}, \quad \text{if } t > 0,$$

and $\phi(0) = 0$. Define ϕ^* in the same way for the adjoints $(A(t)^*)_{t\in[0,T]}$. It follows from (H5)′ and Lemma 5.35 that ϕ and ϕ^* are pathwise continuous. We claim that ϕ and ϕ^* are adapted. Since E_0 is separable, $\|A(t)^{-1} - A(s)^{-1}\|_{\mathscr{L}(E_0,E_{\nu,r})}$ can be written as a supremum of countably many functions $\|A(t)^{-1}x_n\|_{E_{\nu,r}}$, which are all $\mathscr{F}_t$-measurable by the Pettis measurability theorem. The claim follows.

Define the stoping times $\kappa_n, \kappa_n^* : \Omega \to \mathbb{R}$ by $\kappa_n = \inf\{t \in [0,T] :\ \phi(t) \ge n\}$, $\kappa_n^* = \inf\{t \in [0,T] :\ \phi^*(t) \ge n\}$, and let $\tau_n = \kappa_n \wedge \kappa_n^*$. Consider the stopped process A_n given by $A_n(t,\omega) = A(t \wedge \tau_n(\omega),\omega)$. Then for all for all $s,t \in [0,T]$,

$$\|A_n(t)^{-1} - A_n(s)^{-1}\|_{\mathscr{L}(E_0,E_{\nu,r})} \le n|t-s|^{\mu-\varepsilon}$$

and similarly for $A_n(t)^*$, and it follows from (5.5.10) that A_n and A_n^* satisfy (H5) with $\mu - \varepsilon$ instead of μ, and with $L(\omega) = Cn$. Let $(S_n(t,s))_{0\le s\le t\le T}$ be the evolution family generated by A_n. Since $A_n(t) = A(t)$ for $t \leq \tau_n$, it follows from the uniqueness of the evolution family that $S_n(t,s) = S(t,s)$ for $0 \leq s \leq t \leq \tau_n$.

Existence. Let the initial values $(u_n)_{n\geq 1}$ be as in the proof of Theorem 5.28. It follows from Theorem 5.26 that for each $n \geq 1$ there is a unique adapted pathwise mild solution $U_n \in L^p(\Omega; C([0,T]; \tilde{E}_a))$ of (5.5.1) with A and u_0 replaced by A_n and u_n. Moreover, it also has the regularity properties stated in Theorem 5.28. We will use the paths of $(U_n)_{n\geq 1}$ to build a new process U which solves (5.5.1).

For $v \in Z_a^p$ or $v \in L^0(\Omega; C([0,T]; \tilde{E}_a))$ we write

$$L(v)(t) = S(t,0)u_0 + S * F(\cdot, v)(t) + S \diamond B(\cdot, v)(t)$$
$$L_n(v)(t) = S_n(t,0)u_n + S_n * F(\cdot, v)(t) + S_n \diamond B(\cdot, v)(t).$$

Let $\Gamma_n = \{\|u_0\|_{E_a^0} \leq n\}$. Note that $L_n(U_n) = U_n$ for every n. Fix $m \geq 1$ and let $n \geq m$. Note that on Γ_m, $u_n = u_m$ and on $[0,\tau]$, $A_n = A_m$. By Lemma 5.27 we find that almost surely on the set Γ_m, if $t \leq \tau_m$, $U_n(t) = U_m(t)$. Therefore, we can define

$$U(t) = \begin{cases} \lim_{n\to\infty} U_n(t) & \text{if the limit exists,} \\ 0 & \text{else.} \end{cases}$$

Then U is strongly measurable and adapted. Moreover, almost surely on Γ_m and $t \leq \tau_m$, $U(t) = U_m(t)$. For $\omega \in \Omega$ and $m \geq 1$ large enough, $\tau_m(\omega) = T$. Thus the process U has the same path properties as U_m, which yields the required regularity. One easily checks that U is a pathwise mild solution to (5.5.1).

Uniqueness. Let U^1 and U^2 be adapted pathwise mild solutions in the space $L^0(\Omega; C([0,T]; \tilde{E}_a))$. We will show that $U^1 = U^2$. Let κ_n and κ_n^* be as in the existence proof. Let

$$\nu_n^i := \inf\{t \in [0,T] : \ \|U^i(t)\|_{\tilde{E}_a} \geq n\}, \qquad i = 1,2.$$

Set $\nu_n = \kappa_n \wedge \kappa_n^* \wedge \nu_n^1 \wedge \nu_n^2$. Define U_n^i by $U_n^i(t) = \mathbf{1}_{[0,\nu_n]}(t)U^i(t)$. Then as before one sees that $\mathbf{1}_{[0,\nu_n]}L_n(U_n^i) = U_n^i$. Therefore, from Lemma 5.27 it follows that almost surely, for all $t \in [0,\nu_n]$, $U_n^1 = U_n^2$. The result follows by letting $n \to \infty$. $\square$

5.6 Examples

In this section, we will consider the stochastic partial differential equation from [124, 135]. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathscr{F}_t)_{t\in[0,T]}$. Let $n \in \mathbb{N}$ and let S be a domain in $\mathbb{R}^n$. For $p, q \in [1,\infty]$ and $s \in \mathbb{R}$, let $B_{p,q}^s(S)$ be the Besov space (see [132]).

Example 5.30. Set $S = \mathbb{R}^n$, and consider the stochastic partial differential equation

$$\begin{aligned} du(t,s) &= \big(A(t,s,\omega,D)u(t,s) + f(t,s,u(t,s))\big)\,dt \\ &\qquad + g(t,s,u(t,s))\,dW(t,s),\ t \in (0,T],\ s \in \mathbb{R}^n, \\ u(0,s) &= u_0(s),\ s \in \mathbb{R}^n. \end{aligned} \tag{5.6.1}$$

The drift operator A is assumed to be of the form

$$A(t,s,\omega,D) = \sum_{i,j=1}^{n} D_i(a_{ij}(t,s,\omega)D_j) + a_0(t,s,\omega).$$

We assume that all coefficients are real, and satisfy a.s.

$$\begin{aligned} &a_{ij} \in C^\mu([0,T];C(\mathbb{R}^n),\ a_{ij}(t,\cdot) \in BUC^1(\mathbb{R}^n),\ D_k a_{ij} \in BUC([0,T]\times\mathbb{R}^n), \\ &a_0 \in C^\mu([0,T];L^n(\mathbb{R}^n)) \cap C([0,T];C(\mathbb{R}^n)), \end{aligned}$$

for $i,j,k = 1,\dots,n$, $t \in [0,T]$ and a constant $\mu \in (\frac{1}{2},1]$. All coefficients a_{ij} and a_0 are $\mathscr{P}_T \otimes \mathscr{B}(S)$-measurable, where $\mathscr{P}_T$ is the progressive σ-algebra. Moreover, there exists a constant K such that for all $t \in [0,T]$, $\omega \in \Omega$, $s \in \mathbb{R}^n$, $i,j,k = 1,\dots,n$,

$$|a_{ij}(t,s,\omega)| \le K,\ |D_k a_{ij}(t,s,\omega)| \le K,\ |a_0(t,s,\omega)| \le K.$$

We assume there exists an increasing function $w : (0,\infty) \to (0,\infty)$ such that $\lim_{\varepsilon\downarrow 0} w(\varepsilon) = 0$ and such that for all $t \in [0,T]$, $\omega \in \Omega$, $s,s' \in \mathbb{R}^n$, $i,j = 1,\dots,n$,

$$|a_{ij}(t,\omega,s) - a_{ij}(t,\omega,s')| \le w(|s-s'|).$$

Moreover, we assume that (a_{ij}) is symmetric and that there exists a $\kappa > 0$ such that

$$\kappa^{-1}|\xi|^2 \le a_{ij}(t,s,\omega)\xi_i\xi_j \le \kappa|\xi|^2, \qquad s \in \mathbb{R}^n,\ t \in [0,T],\ \xi \in \mathbb{R}^n. \tag{5.6.2}$$

Let $f,g : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be measurable, adapted and Lipschitz continuous functions with linear growth uniformly in $\Omega \times [0,T] \times \mathbb{R}^n$, i.e., there exist L_f, C_f, L_g, C_g such that for all $t \in [0,T], \omega \in \Omega, s \in \mathbb{R}^n$ and $x,y \in \mathbb{R}$,

$$\begin{aligned} |f(t,\omega,s,x) - f(t,\omega,s,y)| &\le L_f|x-y|, \\ |f(t,\omega,s,x)| &\le C_f(1+|x|), \\ |g(t,\omega,s,x) - g(t,\omega,s,y)| &\le L_g|x-y|, \\ |g(t,\omega,s,x)| &\le C_g(1+|x|). \end{aligned}$$

Let W be an $L^2(\mathbb{R}^n)$-valued Brownian motion with respect to $(\mathscr{F}_t)_{t\in[0,T]}$, with covariance $Q \in \mathscr{L}(L^2(\mathbb{R}^n))$ such that

$$\sqrt{Q} \in \mathscr{L}(L^2(\mathbb{R}^n), L^\infty(\mathbb{R}^n)). \tag{5.6.3}$$

Let $p \ge 2$ and set $E = L^p(\mathbb{R}^n)$. On E, we define the linear operators $A(t,\omega)$ for $t \in [0,T]$, $\omega \in \Omega$, by

$$D(A(t,\omega)) = W^{2,p}(\mathbb{R}^n),$$
$$A(t,\omega)u = A(t,\cdot,\omega,D)u.$$

With integration by parts, one observes that the adjoint $A(t,\omega)^*$ of $A(t,\omega)$ is given by

$$D(A(t,\omega)^*) = W^{2,p'}(\mathbb{R}^n)$$
$$A(t,\omega)^*u = A(t,\cdot,\omega,D)u.$$

The operator $A(t,\omega) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a closed operator. In fact, from [59, Theorem 8.1.1], it follows that there exists a constant c depending only on p, κ, K, w and n, such that

$$c^{-1}\|x\|_{W^{2,p}(\mathbb{R}^n)} \leq \|A(t,\omega)x\|_{L^p(\mathbb{R}^n)} + \|x\|_{L^p(\mathbb{R}^n)} \leq c\|x\|_{W^{2,p}(\mathbb{R}^n)},\ x \in W^{2,p}(\mathbb{R}^n).$$

By [104, Theorem 7.3.6], it follows that $A(t,\omega)$ is the generator of an analytic semigroup. In fact, by a careful check of the proof of [104, Theorem 7.3.6], one can find a sector Σ_ϑ, $\vartheta \in (\pi/2,\pi)$, and a constant M, both independent of t and ω, such that for all $\lambda \in \Sigma_\vartheta$,

$$\|\lambda R(\lambda, A(t,\omega))\| \leq M.$$

By [73, Proposition 2.1.11], changing $A(t,\omega)$ to $A(t,\omega) - \lambda_0$ and f to $f + \lambda_0$ if necessary, it follows that (AT1) holds. Note that the constant C_f may be affected when replacing f with $f + \lambda_0$, but it will remain independent of t, ω, s and x. A proof that also (AT2) holds can be found in [125, Example 2.8]. The operator $A(t,\omega)$ satisfies (CIS) with $\nu = 1$, see [141, Theorem 4.1]. Hence (H2) and (H5)$'$ are satisfied.

Hypothesis (H1) is verified by Example 5.3.

To verify (H3), take $\eta_+ = 1$ and for $\eta \in (0,\eta_+)$, set

$$\tilde{E}_\eta := (L^p(\mathbb{R}^n), W^{2,p}(\mathbb{R}^n))_{\eta,p} = B^{2\eta}_{p,p}(\mathbb{R}^n).$$

We do not need to choose an η_-, see Remark 5.8. Since $B^{2\eta}_{p,p}(\mathbb{R}^n)$ is a UMD space, (H4) holds.

Let $F : [0,T] \times \Omega \times E \to E$ be defined by

$$F(t,\omega,x)(s) = f(t,\omega,s,x(s)).$$

Let $B : [0,T] \times \Omega \times E \to \gamma(L^2(\mathbb{R}^n),E)$ be defined by

$$(B(t,\omega,x)h)(s) = g(t,\omega,s,x(s))(\sqrt{Q}h)(s).$$

By assumption (5.6.3) and [135, Lemma 2.7] it follows that for any $x \in E$,

$$\|x\sqrt{Q}\|_{\gamma(L^2(\mathbb{R}^n),E)} \leq C\|x\|_E.$$

It follows that (HF) and (HB) are satisfied with choices $a = \theta_F = \theta_B = 0$. With the above definitions of A, F and B, problem (5.6.1) can be rewritten as

$$\begin{aligned} du(t) &= (A(t)u(t) + F(t, u(t)))\, dt + B(t, u(t))\, dW(t), \\ u(0) &= u_0. \end{aligned} \tag{5.6.4}$$

Hence, if $\delta, \lambda > 0$ such that $\delta + \lambda < \frac{1}{2}$, then Theorem 5.29 can be applied: there exists a unique adapted pathwise mild solution to (5.6.4) such that $u \in L^0(\Omega; C([0,T]; L^p(\mathbb{R}^n)))$. If additionally $u_0 \in W^{1,p}(\mathbb{R}^n) = \tilde{E}_{1/2}$, then the solution u belongs to the space $L^0(\Omega; C^\lambda(0,T; B^{2\delta}_{p,2}(\mathbb{R}^n)))$. This is summarized in the next theorem.

Theorem 5.31. *Let $p \in (2,\infty)$ and suppose $u_0 : \Omega \to L^p(\mathbb{R}^n)$ is $\mathscr{F}_0$-measurable.*

1. *There exists a unique adapted pathwise mild solution u that belongs to the space $L^0(\Omega; C([0,T]; L^p(\mathbb{R}^n)))$.*
2. *If $u_0 \in W^{1,p}(\mathbb{R}^n)$ a.s., and $\delta, \lambda > 0$ such that $\delta + \lambda < \frac{1}{2}$, then u belongs to $L^0(\Omega; C^\lambda(0,T; B^{2\delta}_{p,p}(\mathbb{R}^n)))$.*

Example 5.32. Let S be a bounded domain in $\mathbb{R}^n$ with C^2-boundary and outer normal vector $n(s)$. Consider the equation

$$\begin{aligned} du(t,s) &= \big(A(t,s,\omega,D)u(t,s) + f(t,s,u(t,s))\big)\, dt \\ &\qquad + g(t,s,u(t,s))\, dW(t,s),\ t \in (0,T],\ s \in S, \\ C(t,s,\omega,D)u(t,s) &= 0,\ t \in (0,T],\ s \in \partial S, \\ u(0,s) &= u_0(s),\ s \in S. \end{aligned} \tag{5.6.5}$$

The drift operator A is of the form

$$A(t,s,\omega,D) = \sum_{i,j=1}^n D_i(a_{ij}(t,s,\omega)D_j) + a_0(t,s,\omega), \tag{5.6.6}$$

$$C(t,s,\omega,D) = \sum_{i,j=1}^n a_{ij}(t,s,\omega)n_i(s)D_j,$$

where D_i stands for the derivative in the i-th coordinate. All coefficients are real and satisfy a.s.

$$\begin{aligned} &a_{ij} \in C^\mu([0,T]; C(\overline{S})),\ a_{ij}(t,\cdot) \in C^1(\overline{S}),\ D_k a_{ij} \in C([0,T] \times \overline{S}), \\ &a_0 \in C^\mu([0,T]; L^n(S)) \cap C([0,T]; C(\overline{S})), \end{aligned} \tag{5.6.7}$$

for $i,j,k = 1,\dots,n$, $t \in [0,T]$ and a constant $\mu \in (\frac{1}{2}, 1]$. All other assumptions from Example 5.30 regarding a_{ij}, a_0, f and g hold in this example as well.

Let $p \geq 2$ and set $E = L^p(S)$. On E, we define the linear operators $A(t,\omega)$ for $t \in [0,T]$, $\omega \in \Omega$, by

$$D(A(t,\omega)) = \{u \in W^{2,p}(S) : \ C(t,s,\omega,D)u = 0, \ s \in \partial S\},$$
$$A(t,\omega)u = A(t,\cdot,\omega,D)u.$$

With integration by parts, one observes that the adjoint $A(t,\omega)^*$ of $A(t,\omega)$ is given by

$$D(A(t,\omega)^*) = \{u \in W^{2,p'}(S) : \ C(t,s,D)u = 0, \ s \in \partial S\},$$
$$A(t,\omega)^* u = A(t,\cdot,\omega,D)u.$$

As in the previous example, $A(t,\omega)$ is a closed operator on $L^p(S)$, see [59, Theorem 8.5.6] and the discussion in [59, Section 9.3]. We have

$$c^{-1}\|x\|_{W^{2,p}(S)} \leq \|A(t,\omega)x\|_{L^p(S)} + \|x\|_{L^p(S)} \leq c\|x\|_{W^{2,p}(S)}, \ x \in D(A(t,\omega)).$$

where the constant c depends only on p, κ, K, w, n and the shape of the domain S. As in the previous example, $A(t,\omega)$ and $A(t,\omega)^*$ both satisfy (AT1).

Next, we will show (CIS). By [8, Theorem 5.2] and [8, (5.25)], it follows that for $\nu < \frac{1}{2} + \frac{1}{2p}$,

$$(E, D(A(t,\omega)))_{\nu,p} = B^{2\nu}_{p,p}(S), \tag{5.6.8}$$

with constants independent of t,ω. For $\nu < \frac{1}{2} + \frac{1}{2p'}$, we obtain the same result for the adjoint $A(t,\omega)^*$. Hence, for $\nu < \frac{1}{2}$, by (5.6.8), (5.2.1), (5.2.2) and [141, Theorem 4.1] we obtain for $\varepsilon > 0$ such that $\nu + \varepsilon < \frac{1}{2}$,

$$\begin{aligned}\|A(t)^{-1} - A(s)^{-1}\|_{\mathscr{L}(E_0, B^{2\nu}_{p,q})} &\leq \|(-A(t))^{\nu+\varepsilon}(A(t)^{-1} - A(s)^{-1})\|_{\mathscr{L}(E_0)} \\ &\leq K(\omega)|t-s|^{\mu}.\end{aligned}$$

A similar estimation holds again for the adjoint. This proves (CIS) and therefore (H5)$'$ and (H2) (see also (5.5.10) and its discussion).

The verification of hypothesis (H1) is technical, and is done in the appendix, see Lemma 5.36.

To verify (H3), take $\eta_+ = \frac{1}{2}$ and $\tilde{E}_\eta := (E, W^{2,p})_{\eta,p}$. Note that in particular, regarding (5.6.8), (H3)(ii) is satisfied. As in the previous example, we do not need to consider η_-. Also (H4) is satisfied by the choice of $\tilde{E}_\eta$. Verification of (HF) and (HB) are done as in Example 5.30. In fact, we can take $a = \theta_F = \theta_B = 0$ again. This means that problem (5.6.5) can be rewritten as a stochastic evolution equation

$$\begin{aligned} du(t) &= (A(t)u(t) + F(t,u(t)))\,dt + B(t,u(t))\,dW(t), \\ u(0) &= u_0. \end{aligned} \tag{5.6.9}$$

Hence, if $\delta, \lambda > 0$ such that $\delta + \lambda < \frac{1}{2}$, then Theorem 5.29 can be applied: there exists a unique adapted pathwise mild solution to (5.6.9) such that $u \in L^0(\Omega; C([0,T]; L^p(S)))$. Moreover, if $\beta < \frac{1}{2}$ such that $\lambda + \delta < \beta$ and if $u_0 \in W^{1,p}(S)$ a.s., then $u \in L^0(\Omega; C^\lambda(0,T; B^{2\delta}_{p,p}))$. Summarized, we have the following result.

Theorem 5.33. *Let $p \in (2,\infty)$ and suppose $u_0 : \Omega \to L^p(S)$ is $\mathscr{F}_0$-measurable.*

1. *There exists a unique adapted pathwise mild solution u that belongs to the space $L^0(\Omega; C([0,T]; L^p(S)))$.*
2. *If $u_0 \in W^{1,p}(S)$ a.s., and $\delta, \lambda > 0$ such that $\delta + \lambda < \frac{1}{2}$, then u belongs to $L^0(\Omega; C^\lambda(0,T; B^{2\delta}_{p,p}(S)))$.*

Remark 5.34.

1. A first order differential term in problems (5.6.1) and (5.6.5) may be included. This term may in fact be included in the function f. To handle such a situation, one needs to consider $a > 0$, $\theta_F > 0$.
2. One can also consider the case of non-trace class noise, e.g. space-time white noise, see for instance [87]. In this situation one needs to take $a > 0$, $\theta_B > 0$. Also in the case of boundary noise or random point masses, one can consider $a > 0, \theta_F > 0$ and $\theta_B > 0$, see [126, 127].

5.7 Appendix: A technical result for Hölder continuous functions

Let X be a Banach space. For a given μ-Hölder function $f : [0,T] \to X$ and $\alpha \in (0,\mu)$ let

$$\phi_{f,\alpha}(t) = \begin{cases} \sup_{s\in[0,t)} \frac{\|f(t)-f(s)\|}{|t-s|^\alpha}, & \text{if } t \in (0,T], \\ 0 & \text{if } t = 0, \end{cases}$$

Lemma 5.35. *Let $f \in C^\mu([0,T]; X)$ with $\mu \in (0,1]$. Then for every $\alpha \in (0,\mu)$, the function $\phi_{f,\alpha}$ is in $C^{\mu-\alpha}([0,T]; X)$.*

Proof. Let $C = [f]_{C^\mu([0,T];X)}$. Let $\alpha \in (0,\mu)$ and write $\phi := \phi_{f,\alpha}$. Let $\varepsilon = \mu - \alpha$. Then $\varepsilon \in (0,\mu/2)$. We will prove that there is a constant B depending on μ and α such that for all $0 \le \tau < t \le T$ one has $|\phi(t) - \phi(\tau)| \le BC(t-\tau)^\varepsilon$. Fix $0 \le \tau < t \le T$.

Since ϕ is increasing we have $\phi(t) \ge \phi(\tau)$. If $\tau = 0$, one can write $|\phi(t) - \phi(0)| \le C \sup_{s\in[0,t)} (t-s)^\varepsilon = Ct^\varepsilon$. Next consider $\tau \ne 0$.

Step 1: Assume that $\phi(t) = \sup\limits_{s\in[\tau,t)} \|f(t) - f(s)\|(t-s)^{-\mu+\varepsilon}$. Then

$$|\phi(t) - \phi(\tau)| \le \phi(t) = \sup_{s\in[\tau,t)} \frac{\|f(t) - f(s)\|}{(t-s)^{\mu-\varepsilon}} \le \sup_{s\in[\tau,t)} C(t-s)^\varepsilon \le C(t-\tau)^\varepsilon.$$

Step 2: Now suppose $\phi(t) = \sup\limits_{s\in[0,\tau)} \|f(t) - f(s)\|(t-s)^{-\mu+\varepsilon}$. Then one has

$$|\phi(t) - \phi(\tau)| \le \sup_{s\in[0,\tau)} \Big| \frac{\|f(t) - f(s)\|}{(t-s)^{\mu-\varepsilon}} - \frac{\|f(\tau) - f(s)\|}{(\tau-s)^{\mu-\varepsilon}} \Big|.$$

With the triangle inequality, we find that $|\phi(t) - \phi(\tau)| \leq$

$$\sup_{s\in[0,\tau)} \frac{\|f(t) - f(\tau)\|}{(t-s)^{\mu-\varepsilon}} + \sup_{s\in[0,\tau)} \|f(\tau) - f(s)\| |(t-s)^{-\mu+\varepsilon} - (\tau-s)^{-\mu+\varepsilon}|$$
$$\leq C \sup_{s\in[0,\tau)} \frac{(t-\tau)^{\mu}}{(t-s)^{\mu-\varepsilon}} + C \sup_{s\in[0,\tau)} (\tau-s)^{\mu}((\tau-s)^{-\mu+\varepsilon} - (t-s)^{-\mu+\varepsilon})$$
$$\leq C(t-\tau)^{\varepsilon} + C \sup_{s\in[0,\tau)} (\tau-s)^{\mu}((\tau-s)^{-\mu+\varepsilon} - (t-s)^{-\mu+\varepsilon}) \tag{5.7.1}$$

We claim that for all $s \in [0, \tau)$

$$(\tau-s)^{\mu}((\tau-s)^{-\mu+\varepsilon} - (t-s)^{-\mu+\varepsilon}) \leq (t-\tau)^{\varepsilon} \tag{5.7.2}$$

In order to show this, let $u = \tau - s$ and $v = t - s$. Then $v - u = t - \tau$ and (5.7.2) is equivalent to

$$u^{\varepsilon} - v^{\varepsilon}\left(\frac{u}{v}\right)^{\mu} \leq (v-u)^{\varepsilon}, \quad 0 < u < v \leq T.$$

Writing $u = xv$ with $x \in (0,1)$ and dividing by v^{ε}, the latter is equivalent to

$$x^{\varepsilon} - x^{\mu} \leq (1-x)^{\varepsilon}, \quad x \in (0,1).$$

For all $x \in [0,1]$ one has $x^{\varepsilon} - x^{\mu} \leq 1 - x^{\mu-\varepsilon}$. Thus it suffices to show that $1 - x^{a} \leq (1-x)^{b}$ where $a = \mu - \varepsilon \in (0,1)$ and $b = \varepsilon \in (0,1)$. However, $1 - x^{a} \leq 1 - x \leq (1-x)^{b}$ for all $x \in [0,1]$ and this prove the required estimate.

We can conclude that the right-hand side of (5.7.1) is less or equal than $2C(t-\tau)^{\varepsilon}$. This completes the proof.

□

5.8 Appendix B: measurability of the resolvent

Lemma 5.36. *The drift operator A from Example 5.32 satisfies condition (H1).*

Proof. We will prove adaptedness of the resolvent. Strong measurability can be done similarly, and will be omitted. Fix $t \in [0,T]$.

Step 1. Reduction to approximation of the coefficients.
Consider, besides the operator A, the operator A' satisfying (5.6.6) but with a'_{ij} and a'_0 instead of a_{ij} and a_0, respectively. We assume that a'_{ij}, a'_0 are functions satisfying (5.6.7). Consider the closed operators $A(t), A'(t) : \Omega \to \mathscr{L}(L^p(S))$. Let p' be the Hölder conjugate of p, let $f \in L^p(S)$ and $g \in L^{p'}(S)$. Set $u := (R(\lambda, A(t)) - R(\lambda, A'(t)))f \in W^{2,p}(S)$, and $v := R(\lambda, A(t)^*)g \in \mathrm{Dom(A(t)^*)}$. By applying [125, (2.40)] with $\nu = 0$ and $A'(t)$ instead of $A(s)$, we obtain

$$\begin{aligned}\langle (R(\lambda, A(t)) - R(\lambda, A'(t)))f, g\rangle &= \int_S u(\lambda - A(t))v \, dx \\ &= \sum_{i,j=1}^n \int_S (a'_{ij}(t,x) - a_{ij}(t,x))(D_j R(\lambda, A'(t))f)(x)(D_i R(\lambda, A(t)^*)g)(x) \, dx \\ &\quad + \int_S (a'_0(t,x) - a_0(t,x))(R(\lambda, A'(t))f)(x)(R(\lambda, A(t)^*)g)(x) \, dx\end{aligned}$$

Still following the lines of [125], it follows that

$$\begin{aligned}&|\langle (R(\lambda, A(t)) - R(\lambda, A'(t)))f, g\rangle| \\ &\quad \leq C \max_{i,j,\omega,x} \{|a_{ij}(t,x) - a'_{ij}(t,x)|, |a_0(t,x) - a'_0(t,x)|\} \|f\|_{L^p(S)} \|g\|_{L^{p'}(S)}.\end{aligned}$$

Hence

$$\|R(\lambda, A(t)) - R(\lambda, A'(t))\|_{\mathscr{L}(L^p(S))} \leq C \max_{i,j,\omega,x} |a_{ij}(t,x) - a'_{ij}(t,x)|.$$

Consequently, if $a'_{ij}(t,x)$ converges to $a_{ij}(t,x)$ uniformly for all i,j, then we have $R(\lambda, A'(t)) \to R(\lambda, A(t))$ in $\mathscr{L}(L^p(S))$.

Step 2. Approximation of the coefficients.
Let us denote the space of all symmetric $n \times n$-matrices by $\mathbb{R}^{n\times n}_{\text{sym}}$, endowed with the operator norm. Consider $a(t)$ as a map $a(t) : \Omega \to C^1(\overline{S}, \mathbb{R}^{n\times n}_{\text{sym}})$. For $i,j = 1,\dots,n$ and $s \in \overline{S}$, define $x^*_{i,j,s} \in C^1(\overline{S}, \mathbb{R}^{n\times n}_{\text{sym}})^*$ by the point evaluation $\langle f, x^*_{i,j,s}\rangle = f(s)_{ij}$. Let Γ be the subset of $C^1(\overline{S}, \mathbb{R}^{n\times n}_{\text{sym}})^*$ defined by

$$\Gamma := \{x^*_{i,j,s} \in C^1(\overline{S}, \mathbb{R}^{n\times n}_{\text{sym}})^* : \ i,j = 1,\dots,n, \ s \in \overline{S}\}.$$

Note that Γ is a set separating the points of $C^1(\overline{S}, \mathbb{R}^{n\times n}_{\text{sym}})$. Since for all $s \in \overline{S}$, $a_{ij}(t,s) : \Omega \to \mathbb{R}$ is $\mathscr{F}_t$-measurable, by assumption, it follows from Pettis's theorem [134, Proposition I.1.10] that $a(t)$ is $\mathscr{F}_t$-measurable. Hence, by [134, Proposition I.1.9], there exists a sequence of mappings $a^k(t) : \Omega \to C^1(\overline{S}, \mathbb{R}^{n\times n}_{\text{sym}})$, such that $a^k(t)$ is countably valued and such that $a^k(t)^{-1}(f) \in \mathscr{F}_t$ for all $f \in C^1(\overline{S}, \mathbb{R}^{n\times n}_{\text{sym}})$, with the property that $a^k(t) \to a(t)$ uniformly in Ω. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that for all $k > N$ and all $\omega \in \Omega$, $\sup_{s\in\overline{S}} \|a(t,s) - a^k(t,s)\|_{\mathbb{R}^{n\times n}_s} < \varepsilon$. Since $a(t,s)$ is invertible, by the uniform ellipticity condition (5.6.2), it follows that $a^k(t,s)$ is invertible whenever k is large enough. In fact, by estimating the norm $\|a^k(t,s)^{-1}\|$, one obtains the following result: there exists a $\delta > 0$ and an $\tilde{N} \in \mathbb{N}$ such that for all $k > N$, $a^k(t,s)$ satisfies (5.6.2) with a constant $\tilde{\kappa}$ such that $\tilde{\kappa} \in [\kappa, \kappa + \delta]$.

Consider the operator A_k defined by (5.6.6) but with a^k_{ij} instead of a_{ij}. Note that A_k satisfies (AT1). Since a^k_{ij} is countably valued, $R(\lambda, A_k(t)) : \Omega \to \mathscr{L}(L^p(S))$ is countably valued as well, and hence $\mathscr{F}_t$-measurable. By step 1, we obtain $R(\lambda, A_k(t)) \to R(\lambda, A(t))$ as $k \to \infty$, uniformly in Ω, and therefore it follows that $R(\lambda, A(t))$ is $\mathscr{F}_t$-measurable.

To prove strong measurability, repeat step 1 but with $A : \Omega \times [0,T] \to \mathscr{L}(L^p(S))$ instead of $A(t) : \Omega \to \mathscr{L}(L^p(S))$. Similarly, in step 2 one considers $a : \Omega \times [0,T] \to C^1(\overline{S}, \mathbb{R}^{n\times n}_{\mathrm{sym}})$ and the σ-algebra $\mathscr{F} \otimes \mathscr{B}([0,T])$.

$\square$

6

Forward mild solutions to stochastic evolution equations with adapted drift

6.1 Introduction

In [68], the authors develop techniques to solve stochastic evolution equations of the form

$$\begin{cases} dU(t) = (A(t)U(t) + F(t, U(t)))\, dt + B(t, U(t))\, dW(t) \\ U(0) \ \ = u_0. \end{cases} \tag{6.1.1}$$

in which the drift $A = A(t, \omega)$ is adapted to the filtration generated by the Brownian motion. If for every $\omega \in \Omega$, the drift A generates an evolution system $S(t, s)_{0 \leq s \leq t \leq T}$, then the latter will only be $\mathscr{F}_t$-measurable. Therefore, the stochastic convolution

$$\int_0^t S(t, s)B(s, u(s))\, dW(s), \tag{6.1.2}$$

appearing in the concept of a mild solution, is not well-defined as an Itô integral. Hence for defining a mild solution to (6.1.1), one has to overcome the restriction that the integrated process should be adapted.

There are by now several theories in which one is able to integrate processes that are not adapted. Two stochastic integral that we will use extensively, are the Skorohod integral and the forward integral. Both of them are generalizations of the Itô integral, in the sense that if one would integrate an adapted process, both integrals coincide with the Itô integral. However, neither one is a generalization of the other. The Skorohod integral was first introduced in [129], and is connected to Malliavin calculus. The forward integral is an example of stochastic integration via regularization [118].

When one considers the notion of mild solution, it seems that the forward integral and not the Skorohod integral is the right choice for the extension of the Itô integral. This is mainly because if one considers (6.1.2) as a forward integral, then a mild solution is always a weak solution (see [68, Proposition 5.3]). And although in the case of the Skorohod integral one can rely on functional analytic methods, the forward integral is sometimes easier to work with. For example, the Itô formula for the forward integral does not have an extra correction term which the Skorohod integral does; compare Itô's formula for the Skorohod integral (2.5.10) with Itô's formula for the forward integral (6.4.5).

The proof of the fact that a mild solution is a weak solution relies on a maximal inequality for forward integration. The authors of [68] prove the latter by first proving a same result for the Skorohod integral, and then by comparing the two stochastic integrals. The proof of the maximal inequality for the Skorohod integral relies on functional analytic techniques and the Itô formula for the Skorohod integral. We present a new proof that does not use the result for the Skorohod integral, and consequently less assumptions on the evolution system are needed. Moreover, we will present the results in the setting of UMD Banach spaces having type 2.

6.2 Preliminaries

Consider a Banach space X and a separable Hilbert space H with an orthonormal basis $(h_n)_{n\geq 1}$. We will assume that all vector spaces are defined over the reals.

6.2.1 Radonifying operators

Let $\mathscr{H}$ be a real separable Hilbert space (below we take $\mathscr{H} = L^2(S;H)$). We refer to [39, Chapter 12] and the survey paper [85] for an overview on γ-radonifying operators. The Banach space of γ-radonifying operators from $\mathscr{H}$ into X will be denoted by $\gamma(\mathscr{H}, X)$ and is a subspace of $\mathscr{L}(\mathscr{H}, X)$. It satisfies the left- and right-ideal property which includes the result that for $R \in \gamma(\mathscr{H}, X)$, $U \in \mathscr{L}(X)$ and $T \in \mathscr{L}(\mathscr{H})$, one has $URT \in \gamma(\mathscr{H}, X)$ and

$$\|URT\|_{\gamma(\mathscr{H},X)} \leq \|U\|\,\|R\|_{\gamma(\mathscr{H},X)}\,\|T\|.$$

Let (S, Σ, μ) be a σ-finite measure space. A function $G : S \to \mathscr{L}(H, X)$ will be called *H-strongly measurable* if for all $h \in H$, $s \mapsto G(s)h$ is strongly measurable. Moreover, for $p \in (1, \infty)$, G will be called *weakly $L^p(S;H)$* if for all $x^* \in X^*$, $s \mapsto G(s)^*x^*$ is in $L^p(S;H)$. For an H-strongly measurable $G : S \to \mathscr{L}(H, X)$ that is weakly $L^2(S;H)$, we define define $R_G : L^2(S;H) \to X$ as the (Pettis) integral operator

$$\langle R_G f, x^*\rangle = \int_S \langle G(s)f(s), x^*\rangle \, d\mu(s), \quad f \in L^2(S;H), \quad x^* \in X^*.$$

Note that

$$\|R_G f\|_X \leq \|R_G\|_{\gamma(L^2(S;H),X)}\|f\|_{L^2(S;H)}.$$

We will say $G \in \gamma(S;H,X)$ if $R_G \in \gamma(L^2(S;H), X)$ and write $\|G\|_{\gamma(S;H,X)} = \|R_G\|_{\gamma(L^2(S;H),X)}$. It is well-known that the step functions $G : S \to \mathscr{L}(H, X)$ of finite rank are dense in $\gamma(S;H,X)$.

Example 6.1. If the Banach space X has type 2, then $L^2(S;\gamma(H,X)) \hookrightarrow \gamma(S;H,X)$. See [85, Theorem 11.6]. For more on type and cotype of Banach spaces, see [39].

Recall that if X is a Hilbert space, then $\gamma(S;H,X) = L^2(S;\mathscr{L}_2(H,X))$, where $\mathscr{L}_2(H,X)$ denotes the space of Hilbert-Schmidt operators.

Finally, we define the *trace-duality pairing* between two elements $R \in \gamma(H,X)$ and $S \in \gamma(H,X^*)$ by

$$\langle R, S\rangle_{\mathrm{Tr}} := \mathrm{tr}(S^*R) = \sum_{k\geq 1}\langle Rh_k, Sh_k\rangle_{X,X^*}.$$

If X is a UMD Banach space, then S^*R is of trace class, and we have

$$|\langle R, S\rangle_{\mathrm{Tr}} \leq \|R\|_{\gamma(H,X)}\|S\|_{\gamma(H,X^*)}.$$

6.2.2 Malliavin calculus in UMD Banach spaces

Let X be a UMD space. Let $T > 0$ be a fixed time and set $\mathscr{H} := L^2(0,T;H)$. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space. Let W be an isonormal Gaussian process on $\mathscr{H}$, i.e., a map $W : \mathscr{H} \to L^2(\Omega)$ such that for all $h \in \mathscr{H}$, Wh is centered Gaussian and such that for all $h, \tilde{h}$, $\mathbb{E}(W(h)W(\tilde{h})) = [h, \tilde{h}]_{\mathscr{H}}$. Let $(\mathscr{F}_t)_{t\in[0,T]}$ be the filtration generated by W.

With this setting, one is able to define the space $\mathbb{D}^{1,p}(X)$ of Malliavin differentiable processes $F \in L^p(\Omega;X)$. This can be done via smooth random processes, and we refer to [75] and [112] for further theory on the spaces $\mathbb{D}^{k,p}(X)$, $k \geq 1$, or to [95] for the Hilbert space case.

Recall from [95] or [74] that the Ornstein-Uhlenbeck semigroup $(P(t))_{t\geq 0}$ defined on $L^2(\Omega)$ is given by

$$P(t) := \sum_{n=0}^{\infty} e^{-nt} J_n,$$

where J_n is the orthogonal projection onto the n-th Wiener chaos $\mathcal{H}_n$. By positivity, for all $t \geq 0$, $p \in [1,\infty)$, $P(t) \otimes I_X$ extends to a contraction on $L^p(\Omega;X)$. The following two results follow from [74, Lemma 6.2], and will be frequently used in the sequel. For every $F \in L^p(\Omega;X)$ and $t > 0$, $P(t)F \in \mathbb{D}^{1,p}(X)$ and

$$\begin{aligned} DP(t)F &= e^{-t}P(t)DF,\\ \delta(P(t)F) &= e^{t}P(t)\delta(F). \end{aligned} \tag{6.2.1}$$

In particular, if $F \in L^p(\Omega;X)$ and $t > 0$, then $P(t)F \in \mathbb{D}^{k,p}(X)$ for all $k \geq 1$.

The following lemma is a Fubini theorem for the Skorohod integral.

Lemma 6.2. *Let G be a σ-finite measure space, and $p \in (1,\infty)$. Let $u \in L^p(\Omega \times G;\gamma(\mathscr{H},X))$. Suppose that for almost every $x \in G$, the process $u(x)$ belongs to the domain of $\delta(X)$. If*

$$\mathbb{E}\int_G \|\delta(u(x))\|_X^p \, dx < \infty$$

for almost every x, *then* $\int_G u(\cdot, x)\, dx$ *belongs to* $\mathrm{Dom}(\delta(\mathrm{X}))$, *and*

$$\int_0^T \int_G u(t,x)\, dx\, dW(t) = \int_G \int_0^T u(t,x)\, dW(t)\, dx.$$

Proof. Let q be the Hölder conjugate of p. It suffices to show that for every $F \in \mathbb{D}^{1,q}(X^*)$, one has

$$\mathbb{E}\Big\langle \int_G u(x)\, dx, DF \Big\rangle_\gamma = \mathbb{E}\Big\langle \int_G \delta(u(x))\, dx, F \Big\rangle_{X,X^*}.$$

A Fubini argument yields

$$\begin{aligned}
\mathbb{E}\Big\langle \int_G u(x)\, dx, DF \Big\rangle_\gamma &= \mathbb{E}\sum_{n\geq 1} \Big\langle \int_G u(x)h_n\, dx, (DF)h_n \Big\rangle_{X,X^*} \\
&= \mathbb{E}\int_G \sum_{n\geq 1} \langle u(x)h_n, (DF)h_n\rangle_{X,X^*}\, dx = \mathbb{E}\int_G \langle u(x), DF\rangle_\gamma\, dx \\
&= \mathbb{E}\int_G \langle \delta(u(x)), F\rangle_{X,X^*}\, dx = \mathbb{E}\Big\langle \int_G \delta(u(x))\, dx, F \Big\rangle_{X,X^*}.
\end{aligned}$$

□

6.2.3 Malliavin calculus in the space of bounded linear operators

Let X, Y be Banach spaces. As discussed in [32], to avoid non-measurability issues, we will say that a function $F : \Omega \to \mathscr{L}(X,Y))$ is X-strongly Bochner integrable if it is X-strongly measurable and if there exists a $\Psi \in \mathscr{L}(X,Y)$ such that for all $x \in X$,

$$\int_\Omega F(\omega)x\, d\mathbb{P} = \Psi(x).$$

Define the space $L^p_s(\Omega; \mathscr{L}(X,Y))$ consisting of X-strongly Bochner integrable random variables $F : \Omega \to \mathscr{L}(X,Y)$ such that $\|F\|_{\mathscr{L}(X,Y)} \in L^p(\Omega)$. We now give the definition of the corresponding strong Malliavin-Sobolev space.

Definition 6.3. Let X, Y be Banach spaces and $F \in L^p_s(\Omega; \mathscr{L}(X,Y))$. We say that $F \in \mathbb{D}^{1,p}_s(\mathscr{L}(X,Y))$ if the following conditions hold:

(1) For every $x \in X$, Fx belongs to $\mathbb{D}^{1,p}(Y)$.
(2) There exists a map $\mathscr{D}F \in L^p_s(\Omega; L(X, \gamma(\mathscr{H}, Y)))$ such that for every $x \in X$ we have a.s. $(\mathscr{D}F)x = D(Fx)$.

Remark 6.4. The definition of $\mathbb{D}^{1,p}_s(\mathscr{L}(X,Y))$ from [68, Definition 2.1] is slightly different. There, the authors only define $\mathbb{D}^{1,p}_s(\mathscr{L}(X,Y))$ for $p=2$, and assume that $\mathscr{D}F \in L^2(0,T\times\Omega;\mathscr{L}(X,\mathscr{L}_2(H,Y)))$, where $\mathscr{L}_2(H,Y)$ is the space of Hilbert-Schmidt operators from H to Y. However, in the Hilbert space setting, one has $\mathscr{L}_2(H;Y)=\gamma(H,Y)$, and $L^2(0,T;\gamma(H,Y))\hookrightarrow\gamma(\mathscr{H},Y)$. Hence every $F\in L^2_s(\Omega;\mathscr{L}(X,Y))$ that satisfies the conditions from [68, Definition 2.1] particularly satisfies the conditions from Definition (6.3).

Remark 6.5. Suppose X is a separable Hilbert space and $F:\Omega\to\gamma(X,Y)$. Then in particular, $F:\Omega\to\mathscr{L}(X,Y)$, but as $\gamma(X,Y)$ is separable, we do not need the above definition. Of course one would like that $\mathscr{D}F=DF$ in a suitable sense. This is indeed the case: First, if $F\in\mathbb{D}^{1,p}(\gamma(X,Y))$, then $DF\in L^p(\Omega;\gamma(\mathscr{H},\gamma(X,Y)))$. Using $\gamma(H,\gamma(X,Y))\simeq\gamma(X,\gamma(H,Y))$, one could interpret DF as an element from $L^p(\Omega;\gamma(X,\gamma(H,Y)))$. A careful check of both definitions leads to the conclusion that $DF=\mathscr{D}F$. Second, if $F\in\mathbb{D}^{1,p}_s(\mathscr{L}(X,Y))$ such that $\mathscr{D}F\in L^p(\Omega;\gamma(X,\gamma(H,Y)))$, then one may conclude that $F\in\mathbb{D}^{1,p}(\gamma(X,Y))$, with again $\mathscr{D}F=DF$.

The following lemma is a generalization of [68, Lemma 2.5]. The proof is similar.

Lemma 6.6 (Product rule). *Let X,Y be Banach spaces and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. If $A\in\mathbb{D}^{1,p}_s(\mathscr{L}(X,Y))$ and $F\in\mathbb{D}^{1,q}(X)$, then $AF\in\mathbb{D}^{1,r}(Y)$ with*

$$D(AF)=(\mathscr{D}A)F+A(DF).$$

The identity should be understood in the following sense: for all $h\in\mathscr{H}$ one has

$$D(AF)h=((\mathscr{D}A)F)(h)+A((DF)h).$$

For any $h\in\mathscr{H}$, we will denote $(\mathscr{D}^hA)F$ for $((\mathscr{D}A)F)(h)$. This is analogous to the operator $D^h:\mathscr{S}\to L^p(\Omega;X)$ that is defined by $D^hF=(DF)h$. One can prove that D^h is a closable operator, and one denotes its domain by $\mathbb{D}^{h,p}(X)$. Concerning these operators, we have the following simple result.

Lemma 6.7. *Let X,Y be Banach spaces, $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, $F\in\mathbb{D}^{1,p}_s(L(X,Y))$ and $G\in L^q(\Omega;X)$. Let $t\in[0,T]$ and suppose that G is $\mathscr{F}_t$-measurable. If $h\in\mathscr{H}$ is such that* $\mathrm{supp}(h)\subset[t,T]$, *then $FG\in\mathbb{D}^{h,r}(Y)$ with*

$$D^h(FG)=(\mathscr{D}^hF)G.$$

Proof. First suppose $G\in\mathbb{D}^{1,q}(X)$. By the product rule, we have $FG\in\mathbb{D}^{h,r}(X)$ and

$$D^h(FG)=(\mathscr{D}^hF)G+FD^hG.$$

From the assumption on the support of h, it follows that $D^hG=0$ ([95, Corollary 1.2.1]). Hence the result follows in this special case.

The general case follows from the closedness of D^h, since one can approximate G by a sequence $(G_n)_{n\geq1}$ of smooth $\mathscr{F}_t$-measurable random variables. ☐

6.3 The forward integral

In this section we assume that the Banach space X is UMD and has type 2. Recall that H is a real separable Hilbert space with orthonormal basis (h_n). Recall that W is an isonormal Gaussian process on $\mathscr{H}$.

Definition 6.8. Let $\phi : \Omega \times [0,T] \to \mathscr{L}(H,X)$ be H-strongly measurable and weakly in $L^2(0,T;H)$. Let $I^-(\phi,n)$ be defined by

$$I^-(\phi,n) = n\sum_{k=1}^{n}\int_0^T \phi(s)h_k(W(s+1/n)h_k - W(s)h_k)\,ds.$$

If the sequence $(I^-(\phi,n))_{n=1}^\infty$ converges in probability, then we say that ϕ is forward integrable. The limit is called the forward integral of ϕ, and denoted by $\int_0^T \phi(s)\,dW^-(s)$.

If ϕ is forward integrable, then we will also say that $\phi \in \mathrm{Dom}(\delta^-)$. If the sequence $(I^-(\phi,n))_{n=1}^\infty$ converges in $L^p(\Omega;X)$, then we will write $\phi \in \mathrm{Dom}(\delta^-_{\mathrm{p}})$.

The forward integral extends the Itô integral, see [110, Proposition 3.2]. One of the advantages of the forward integral over the Skorohod integral, is that one may pull any random operator $A : \Omega \to \mathscr{L}(X,Y)$ out of the integral. That is, when ϕ is forward integrable on X, and $A : \Omega \to \mathscr{L}(X,Y)$, then $A\phi$ is again forward integrable and

$$\int_0^T A\phi\,dW^- = A\int_0^T \phi\,dW^-. \tag{6.3.1}$$

More properties on the forward integral, in particular on convergence of the sequence $I^-(\phi,n)$, can be found in [110].

We will use the following subspace of Malliavin differentiable processes.

Definition 6.9. For $p \in [2,\infty)$, we denote by $\mathbb{M}^{1,p}(X)$ the space of all $F \in \mathbb{D}^{1,p}(L^p(0,T;\gamma(H,X)))$ such that for all $t \in [0,T]$ and $h \in H$, $D(F(t)h) \in L^p(\Omega;\gamma(0,T;H,X))$. We will say that $F \in \mathbb{M}^{1,p}_a(X)$ if $F \in \mathbb{M}^{1,p}(X)$ and it is adapted to the filtration $(\mathscr{F}_t)_{t\in[0,T]}$.

It follows that $\mathbb{M}^{1,p}(X)$ is a subspace of $\mathbb{D}^{1,p}(L^p(0,T;\gamma(H,X)))$, and it is dense in $L^p(\Omega\times[0,T];\gamma(H,X))$. To see the latter, note that the set smooth $L^p(0,T;\gamma(H,X))$-valued random variables are dense in $L^p(\Omega;L^p(0,T;\gamma(H,X)))$. Consider such an smooth random variable ξ that is of the simplest form

$$\xi = f(W(\varphi_1),\dots,W(\varphi_n))\otimes\phi,$$

where $\phi \in L^p(0,T;\gamma(H,X))$. Since the φ_i, $i=1,\dots,m$ belong to $L^2(0,T;H)$, one can approximate φ_i by a sequence $(\varphi_i^m)_{m\geq 1}$ of simple functions. Observe that $W : L^2(0,T;H) \to L^p(\Omega)$ is a continuous linear map, and therefore $W(\varphi_i^m) \to W(\varphi_i)$ in $L^p(\Omega)$. It follows that ξ_m, defined by $\xi_m := f(W(\varphi_1^m),\dots,W(\varphi_n^m))\otimes\phi$, converges to ξ in the norm of $L^p(\Omega; L^p(0,T;\gamma(H,X)))$. Moreover, $\xi_m \in \mathbb{M}^{1,p}(X)$. By linearity, every smooth $L^p(0,T;\gamma(H,X))$-valued random variable can be approximated by a sequence of elements in $\mathbb{M}^{1,p}(X)$. Hence the result follows.

A similar result holds for $\mathbb{M}_a^{1,p}(X)$.

Whenever X is a Hilbert space, then $\mathbb{M}^{1,p}(X) = \mathbb{D}^{1,p}(L^p(0,T;\gamma(H,X)))$. Indeed, let $F \in \mathbb{D}^{1,p}(L^p(0,T;\gamma(H,X)))$. Then for every $t \in [0,T]$ and $h \in H$, one has $F(t)h \in \mathbb{D}^{1,p}(X)$. Now recall that D is a closed operator from $L^p(\Omega;X)$ into $L^p(\Omega;L^2(0,T;\mathscr{L}_2(H,X)))$, where $\mathscr{L}_2(H,X)$ is the space of Hilbert-Schmidt operators from H to X.

It is well-known (see [118, Theorem 2.1]) that in the case $H = X = \mathbb{R}$, if $F \in \mathbb{D}^{1,p}(L^2(0,T))$ and there exists a weak trace-term of DF, then F is forward integrable. Moreover, the forward integral and the Skorohod integral differ by the trace of DF. A similar result holds in the infinite-dimensional case. To prove this, we will first prove the following identity of $I^-(F,n)$. This is essentially an infinite-dimensional generalization of [118, Lemma 2.1]. Let P_n be the projection onto the first n basis coordinates.

Lemma 6.10. *Let $p \in (1,\infty)$, and $F \in \mathbb{M}^{1,p}(X) \cap L^p(\Omega;\gamma(0,T;H,X))$. The convolution $F_n := n\mathbf{1}_{[0,1/n]} * P_n F\mathbf{1}_{[0,T]}$ belongs to $\mathbb{M}^{1,p}(X)$, and we have a.s.*

$$I^-(F,n) = \delta(F_n) + \sum_{k=1}^n \int_0^T n \int_t^{t+1/n} D(F(t)h_k)(s)h_k \, ds \, dt$$

Proof. The first statement follows directly from Young's inequality.

Observe that if $\phi = \mathbf{1}_{[a,b]} \otimes h$ for $0 \le a < b \le T$ and $h \in H$ with $\|h\| = 1$, and $R \in \gamma(0,T;H,X)$, then

$$\langle R, \phi \otimes x^* \rangle_{\mathrm{Tr}} = \langle R\phi, x^* \rangle_{X,X^*}.$$

Also, for any $\phi \in L^2(0,T;H)$, one has

$$\begin{aligned} F_n(\phi) &= \int_0^T \int_0^T n\mathbf{1}_{[0,1/n]}(s-r)F(r)(P_n(\phi(s)))\, dr\, ds \\ &= \int_0^T \int_0^T n\mathbf{1}_{[r,r+1/n]}(s) \sum_{k=1}^n [\phi(s), h_k]_H F(r)h_k \, ds\, dr \\ &= \sum_{k=1}^n \int_0^T F(r)h_k[\phi, n\mathbf{1}_{[r,r+1/n]} \otimes h_k]_{L^2(0,T;H)} \, dr. \end{aligned} \tag{6.3.2}$$

Let $G = g \otimes x^*$ be a smooth X^*-valued random variable, and let $(\phi_n)_{n\ge1}$ be an orthonormal basis for $L^2(0,T;H)$. Then by (6.3.2),

$$\begin{aligned} \mathbb{E}\langle G, \delta(F_n)\rangle &= \mathbb{E} \sum_{m\ge1} \langle F_n\phi_m, (DG)\phi_m\rangle \\ &= \mathbb{E} \sum_{k=1}^n \int_0^T \langle F(r)h_k, x^*\rangle [n\mathbf{1}_{[r,r+1/n]} \otimes h_k, Dg]_{\mathrm{HS}}\, dr. \end{aligned} \tag{6.3.3}$$

Moreover, by the integration by parts formula for the divergence operator, see [112, Lemma 4.9], we obtain a.s.

$$[n\mathbf{1}_{[r,r+1/n]} \otimes h_k, Dg]_{\mathrm{HS}} = \delta(gn\mathbf{1}_{[r,r+1/n]} \otimes h_k) - g\delta(n\mathbf{1}_{[r,r+1/n]} \otimes h_k). \quad (6.3.4)$$

By definition of the divergence operator, for the latter we obtain

$$g\delta(n\mathbf{1}_{[r,r+1/n]} \otimes h_k) = gn(W(r+1/n)h_k - W(r)h_k).$$

Hence

$$\mathbb{E}\sum_{k=1}^{n}\int_0^T \langle F(r)h_k, x^*\rangle gn(W(r+1/n)h_k - W(r)h_k)\,dr = \mathbb{E}\langle I^-(F,n), G\rangle. \quad (6.3.5)$$

For the other part of the right hand side of (6.3.4), we use duality again, and (6.3.2) to obtain

$$\begin{aligned}
&\mathbb{E}\sum_{k=1}^{n}\int_0^T \langle F(r)h_k, x^*\rangle\delta(gn\mathbf{1}_{[r,r+1/n]} \otimes h_k)\,dr \\
&= \mathbb{E}\sum_{k=1}^{n}\int_0^T \langle D(F(r)h_k), (n\mathbf{1}_{[r,r+1/n]} \otimes h_k)\otimes x^*\rangle_{\mathrm{Tr}} g\,dr \\
&= \mathbb{E}\sum_{k=1}^{n}\int_0^T \langle D(F(r)h_k)(n\mathbf{1}_{[r,r+1/n]} \otimes h_k), x^*\rangle g\,dr \\
&= \mathbb{E}\Big\langle \sum_{k=1}^{n}\int_0^T n\int_r^{r+1/n} D(F(r)h_k)(s)h_k\,ds\,dr, G\Big\rangle.
\end{aligned} \quad (6.3.6)$$

Now apply the results of (6.3.5) and (6.3.6) to (6.3.3) to obtain

$$\mathbb{E}\langle G, \delta(F_n)\rangle = \mathbb{E}\langle I^-(F,n), G\rangle - \mathbb{E}\Big\langle \sum_{k=1}^{n}\int_0^T n\int_r^{r+1/n} D(F(r)h_k)(s)h_k\,ds\,dr, G\Big\rangle.$$

Now the result follows from the Hahn-Banach theorem. □

Now we are ready to present an infinite-dimensional version of [118, Theorem 2.1].

Theorem 6.11. *Let* $F \in \mathbb{D}^{1,p}(L^p(0,T;\gamma(H,X)))$.

1. *The convolution* $F_n := n\mathbf{1}_{[0,1/n]} * P_nF\mathbf{1}_{[0,T]}$ *converges to* F *in the space* $\mathbb{D}^{1,p}(L^p(0,T;\gamma(H,X)))$;
2. *Suppose also* $F \in \mathbb{M}^{1,p}(X) \cap L^p(\Omega;\gamma(0,T;H,X))$, *and set*

$$\varPhi_n := \sum_{k=1}^{n}\int_0^T n\int_t^{t+1/n} D(F(t)h_k)(s)h_k\,ds\,dt.$$

If Φ_n is a Cauchy sequence in $L^p(\Omega; X)$ (or in probability), then F is forward integrable, and $I^-(F, n)$ converges to $\delta^-(F)$ in $L^p(\Omega; X)$ (or in probability). In that case we denote the limit by Φ, and the following identity holds

$$\int_0^T F(s)\, dW^-(s) = \int_0^t F(s)\, dW(s) + \Phi. \tag{6.3.7}$$

Proof. The first claim follows from the fact that $F_n \to F$ in $L^p(\Omega; \gamma(0, T; H, X))$, and $DF_n \to DF$ in $L^p(\Omega; \gamma(\mathscr{H}, \gamma(0, T; H, X)))$. Both proofs of these facts are variations of [44, Theorem 8.14].

The second claim follows from the representation given in Lemma 6.10, and from the fact that $\delta : \mathbb{D}^{1,p}(\gamma(0, T; H, X)) \to L^p(\Omega; X)$ is a continuous operator. □

Remark 6.12. Claim (1) can be improved in the following way: let $\Phi_n : [0, T] \to \mathbb{D}^{1,p}(L^p(0, T; \gamma(H, X)))$ be given by the convolution

$$\Phi(t) = n\mathbf{1}_{[0,1/n]} * P_n F \mathbf{1}_{[0,t]},$$

then $\Phi_n(T) = F_n$, and we have $\Phi_n \to \mathbf{1}_{[0,t]} F$ in $L^p(0, T; \mathbb{D}^{1,p}(L^p(0, T; \gamma(H, X))))$.

6.4 Itô's formula for the forward integral

Lemma 6.13. *Let $\xi \in L^2(\Omega; L^2(0, T; H))$ and $\varphi \in L^2(0, T; H)$. We have*

$$\frac{1}{\varepsilon} \int_0^T \Big\langle \mathbf{1}_{[s,s+\varepsilon]} h, \xi \Big\rangle^2_{L^2(0,T;H)} \, ds \to 0,$$

in $L^1(\Omega)$ as $\varepsilon \downarrow 0$.

Proof. Suppose first that $\varphi = \mathbf{1}_{[a,b]} \otimes h$ with $h \in H$. Then with Hölder's inequality and Fubini,

$$\begin{aligned} \frac{1}{\varepsilon} \int_0^T \Big\langle \mathbf{1}_{[s,s+\varepsilon]} \varphi, \xi \Big\rangle^2_H &\le \frac{1}{\varepsilon} \int_{a-\varepsilon}^b \Big(\int_s^{s+\varepsilon} \langle h, \xi(r) \rangle_H \, dr \Big)^2 ds \\ &\le \|h\|^2 \int_{a-\varepsilon}^b \int_s^{s+\varepsilon} \|\xi(r)\|_H^2 \, dr\, ds \le \|h\|^2 \varepsilon \int_0^T \|\xi(r)\|^2 \, dr. \end{aligned}$$

Clearly, the latter converges to 0 in $L^1(\Omega)$ as $\varepsilon \downarrow 0$.

For general $\varphi \in L^2(0, T; H)$, one has

$$\mathbb{E} \frac{1}{\varepsilon} \int_0^T \langle \mathbf{1}_{[s,s+\varepsilon]} \varphi, \xi \rangle^2_{L^2(0,T;H)} \le \mathbb{E} \|\xi\|^2_{L^2(0,T;H)} \|\varphi\|^2_{L^2(0,T;H)}.$$

Therefore, the result follows by an approximation argument. □

Lemma 6.14. *Let (a, b) be an open interval and $h \in H$. We have $\lim_{\varepsilon \downarrow 0} \xi_\varepsilon = (b-a)\|h\|_H^2$ in $L^2(\Omega)$, where*

$$\xi_\varepsilon = \int_a^b \frac{1}{\varepsilon}((W(s+\varepsilon) - W(s))h)^2 \, ds. \tag{6.4.1}$$

Proof. We have $\mathbb{E}\xi_\varepsilon = (b-a)\|h\|_H^2$, and since the right-hand side is independent of Ω, it suffices to show $\mathbb{E}\xi_\varepsilon^2 = (b-a)^2\|h\|_H^4$. To see this, note that one can write

$$\begin{aligned}\mathbb{E}\xi_\varepsilon^2 = \mathbb{E}\int_a^b \int_{(a,b)\setminus[s-\varepsilon,s+\varepsilon]} & \frac{1}{\varepsilon^2}((W(s+\varepsilon) - W(s))h)^2 \\ & \times ((W(r+\varepsilon) - W(r))h)^2 \, dr \, ds \\ + \mathbb{E}\int_a^b \int_{s-\varepsilon}^{s+\varepsilon} & \frac{1}{\varepsilon^2}((W(s+\varepsilon) - W(s))h)^2((W(r+\varepsilon) - W(r))h)^2 \, dr \, ds.\end{aligned}$$

The first part converges to $(b-a)^2\|h\|_H^4$ by the properties of Brownian motion. For the second part, use Hölder's inequality and $\mathbb{E}\gamma^4 = 3q^2$ for any Gaussian variable with variance q, to see that this part converges to 0 as $\varepsilon \downarrow 0$. □

Remark 6.15. Following the lines of the above proof, one notices that respectively the right and left end-point of the integral in equation (6.4.1) can be changed into $a + \varepsilon$ or $b - \varepsilon$, or both.

Proposition 6.16. *Let $Y \in \mathbb{D}^{1,2}(L^2(0, T; H))$ be a smooth process of the form $Y = f(W(h_1), \ldots, W(h_n)) \otimes \mathbf{1}_{[a,b]} \otimes \varphi$, with $f \in C_b^\infty(\mathbb{R}^n)$, $0 \le a < b \le T$, $h_i \in L^2(0, T; H)$ and $h \in H$ with $\|h\|_H = 1$. We have*

$$\frac{1}{\varepsilon}\int_0^T \Big| \int_s^{s+\varepsilon} Y(r) \, dW(r) \Big|^2 ds \to \int_0^T \|Y(s)\|_H^2 \, ds \tag{6.4.2}$$

in $L^1(\Omega)$, as $\varepsilon \downarrow 0$.

Proof. Set $\tilde{\varphi}_1 = \frac{1}{\sqrt{\varepsilon}}\mathbf{1}_{[s,s+\varepsilon]} \otimes h$. Note that $\|\tilde{\varphi}_1\|_{L^2(0,T;H)} = 1$. Consider an orthonormal basis for $\mathscr{H} = L^2(0, T; H)$ that has $\tilde{\varphi}_1$ as an element. Then, by [112, Lemma 4.2], we compute

$$\begin{aligned}\int_s^{s+\varepsilon} Y(r) \, dW(r) = f(W((s+\varepsilon) \wedge b) - W(s \vee a))h \\ - \langle \mathbf{1}_{[s\vee a,(s+\varepsilon)\wedge b]} \otimes h, DF\rangle_{L^2(0,T;H)}.\end{aligned}$$

Split the integral from 0 to T of the left hand side of (6.4.2) into three parts: the first one from $a - \varepsilon$ to a, the second one from a to $b - \varepsilon$ and the third one from $b - \varepsilon$ to b. Then one observes that the first and third integral converge to 0 in $L^1(\Omega)$ as $\varepsilon \downarrow 0$. For the second integral we estimate pointwise

$$
\begin{aligned}
&\Big|\frac{1}{\varepsilon}\int_a^{b-\varepsilon}\Big|\int_s^{s+\varepsilon} Y(r)\,dW(r)\Big|^2\,ds - \int_0^T \|Y(s)\|_H^2\,ds\Big| \\
&\quad\le C\int_a^{b-\varepsilon} f^2\Big(\frac{1}{\varepsilon}\big((W(s+\varepsilon)-W(s))h\big)^2 - \|h\|_H^2\Big)\,ds \\
&\qquad + \frac{C}{\varepsilon}\int_0^T |\langle \mathbf{1}_{[s,s+\varepsilon]}\otimes h, Df\rangle_{L^2(0,T;H)}|^2\,ds
\end{aligned}
$$

The first and second part both converge to 0 in $L^1(\Omega)$ by respectively Lemma 6.14 (with Remark 6.15) and Lemma 6.13, respectively. □

Theorem 6.17. *Let $Y\in\mathbb{D}^{1,2}(L^2(0,T;\gamma(H,X)))$. For $\varepsilon>0$, let $Z_0, Z_\varepsilon:\Omega\times[0,T]\to\mathscr{L}(X,X^*)$ be processes such that:*

1. *all processes $Z_\varepsilon, \varepsilon>0$ and Z_0 have continuous paths,*
2. *Pointwise on ω one has $\lim_{\varepsilon\downarrow 0}\sup_{t\in[0,T]}\|Z_\varepsilon(t)-Z_0(t)\|_{\mathscr{L}(X,X^*)}=0$,*
3. *There exists a constant $C>0$ such that for all $t\in[0,T]$ and all $\omega\in\Omega$ one has $\|Z_\varepsilon(t,\omega)\|_{\mathscr{L}(X,X^*)}\le C$.*

Then for all $t\in[0,T]$ we have

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t\Big\langle\int_s^{s+\varepsilon} Y(r)\,dW(r), Z_\varepsilon(s)\int_s^{s+\varepsilon} Y(r)\,dW(r)\Big\rangle\,ds \\
&\qquad\to\int_0^t\langle Y(s), Z_0(s)Y(s)\rangle\,ds
\end{aligned}
$$

in $L^1(\Omega)$ as $\varepsilon\downarrow 0$.

Proof. This is a continuous version of Theorem 2.37. In fact, following the steps of Theorem 2.37, one can conclude that the first three steps can be copied into this proof. This means that it suffices to consider the case when Y is a smooth process. By linearity, we can assume that

$$Y = f(W(h_1),\dots,W(h_n))\otimes(\mathbf{1}_{[a,b]}\otimes h)\otimes x,$$

where $h\in H$, $x\in E$ and $f\in C_b^\infty(\mathbb{R}^n)$. We estimate

$$
\begin{aligned}
&\Big|\frac{1}{\varepsilon}\int_0^t\Big\langle\int_s^{s+\varepsilon} Y(r)\,dW(r), Z_\varepsilon(s)\int_s^{s+\varepsilon} Y(r)\,dW(r)\Big\rangle\,ds \\
&\qquad\qquad -\int_0^t\langle Y(s), Z_0(s)Y(s)\rangle\,ds\Big| \\
&\le\Big|\frac{1}{\varepsilon}\int_0^t\Big\langle\int_s^{s+\varepsilon} Y(r)\,dW(r), (Z_\varepsilon(s)-Z_0(s))\int_s^{s+\varepsilon} Y(r)\,dW(r)\Big\rangle\,ds\Big| \\
&\quad+\Big|\frac{1}{\varepsilon}\int_0^t\Big\langle\int_s^{s+\varepsilon} Y(r)\,dW(r), Z_0(s)\int_s^{s+\varepsilon} Y(r)\,dW(r)\Big\rangle\,ds \\
&\qquad\qquad -\int_0^t\langle Y(s), Z_0(s)Y(s)\rangle\,ds\Big| \\
&=: a_1 + a_2.
\end{aligned}
$$

One can estimate the expectation of a_1 to obtain

$$\begin{aligned}\mathbb{E}|a_1| \le \mathbb{E} \sup_{\sigma\in[0,T]} &\|Z_\varepsilon(\sigma) - Z_0(\sigma)\|_{\mathscr{L}(X,X^*)} \\ &\times \Big|\frac{1}{\varepsilon}\int_0^t \|\delta(\mathbf{1}_{[s,s+\varepsilon]}Y)\|_X^2\, ds - \int_0^t \|Y(s)\|_{\gamma(H,X)}^2\, ds\Big| \\ &+ \mathbb{E} \sup_{\sigma\in[0,T]} \|Z_\varepsilon(\sigma) - Z_0(\sigma)\|_{\mathscr{L}(X,X^*)} \int_0^t \|Y(s)\|^2\, ds \\ \le C\mathbb{E}&\Big|\frac{1}{\varepsilon}\int_0^t \|\delta(\mathbf{1}_{[s,s+\varepsilon]}Y)\|_X^2\, ds - \int_0^t \|Y(s)\|_{\gamma(H,X)}^2\, ds\Big| \\ &+ \mathbb{E} \sup_{\sigma\in[0,T]} \|Z_\varepsilon(\sigma) - Z_0(\sigma)\|_{\mathscr{L}(X,X^*)} \int_0^t \|Y(s)\|^2\, ds.\end{aligned}$$

The latter converges to 0 by the dominated convergence theorem. The first converges to 0 by Proposition 6.16. Hence $a_1 \to 0$ as $\varepsilon \downarrow 0$ in the space $L^1(\Omega)$.

Finally, for a proof of convergence $\mathbb{E}|a_2| \to 0$ as $\varepsilon \downarrow 0$, we refer to the proof of [112, (5.7)]. □

The following theorem is an Itô formula for the forward integral. See the remark below the theorem for a discussion why we call it an Itô formula. Recall that $(P(t))_{t\ge 0}$ is the Ornstein-Uhlenbeck semigroup on $L^p(\Omega; X)$.

Theorem 6.18. *Let $p \in (2,\infty)$ and $Y \in \mathbb{M}^{1,p}(X) \cap L^p(\Omega; \gamma(0,T;H,X))$. Suppose that there exists an element $D^-Y \in L^p(\Omega \times [0,T]; X)$ such that for all $t \in [0,T]$,*

$$\int_0^t n\sum_{k=1}^n \int_s^{s+1/n} D(Y(s)h_k)(r)h_k)\, dr\, ds \to \int_0^t (D^-Y)(s)\, ds,$$

in $L^p(\Omega; X)$ as $n \to \infty$. If $F : X \to \mathbb{R}$ be a twice continuously differentiable function that is bounded and has bounded derivatives, then:

1. *For all $t \in [0,T]$ one has that $\mathbf{1}_{[0,t]}Y$ is forward integrable. The forward integral process ζ, given by $\zeta(t) := \int_0^t Y(s)\, dW^-(s)$, has continuous paths. Moreover, the following convergence holds:*

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^t \langle \zeta(s+\varepsilon) - \zeta(s), F'(\zeta(s))\rangle\, ds \\ &\qquad \to F(\zeta(t)) - F(\zeta(0)) - \int_0^t \langle Y(s), F''(\zeta(s))Y(s)\rangle\, ds,\end{aligned} \tag{6.4.3}$$

   in $L^1(\Omega)$ as $\varepsilon \downarrow 0$,
2. *For all $\tau \ge 0$ one has*

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_0^t &\langle P(\tau)(\zeta(s+\varepsilon)-\zeta(s)), F'(P(\tau)(\zeta(s)))\rangle\, ds \\
&\to F(P(\tau)\zeta(t)) - F(P(\tau)\zeta(0)) \\
&\quad - e^{-2\tau}\int_0^t \langle P(\tau)Y(s), F''(P(\tau)\zeta(s))(P(\tau)Y(s))\rangle\, ds,
\end{aligned}
\tag{6.4.4}
$$

in $L^1(\Omega)$ *as* $n \to \infty$.

Remark 6.19. Suppose that besides that assumptions in the theorem we have the following:

$$
\begin{aligned}
Du &\in L^p(0,T;\mathbb{D}^{1,p}(\gamma(L^2(0,T;H),\gamma(H,E)))),\\
D^-Y &\in \mathbb{D}^{1,p}(L^p(0,T;E)),\\
D(D^-Y) &\in L^1(0,T;L^p(\Omega;L^p(0,T;\gamma(H,E)))).
\end{aligned}
$$

Then one can prove that an actual Itô formula holds: the process

$$
s \mapsto \langle \mathbf{1}_{[0,t]}(s)Y(s), F'(\zeta(s))
$$

is again forward integrable for every $t \in [0,T]$, and we have

$$
F(\zeta(t)) = F(\zeta(0)) + \int_0^t \langle Y(s), F'(\zeta(s))\rangle\, dW^-(s) + \int_0^t \langle Y(s), F''(\zeta(s))Y(s)\rangle_{\mathrm{Tr}}\, ds
\tag{6.4.5}
$$

In this case, the limit Φ appearing in Theorem 6.11 applied to $\mathbf{1}_{[0,t]}Y$ is the Lebesgue integral $\int_0^t (D^-Y)(s)\, ds$, and one can apply the Itô formula in [112, Theorem 5.7]. Comparing this formula with $\delta^-(s \mapsto \langle \mathbf{1}[0,t](s)Y(s), F'(\zeta(s)))$ yields (6.4.5). A proof of this can be found in [95, Theorem 3.2.7] for the case $H = X = \mathbb{R}$. The infinite-dimensional case is similar.

Remark 6.20. The left hand side of (6.4.4) can be rewritten as a sum of three terms. To prove this, note that by (6.2.1),

$$
P(\tau)(\zeta(s+\varepsilon)-\zeta(s)) = e^{-\tau}\int_s^{s+\varepsilon} P(\tau)Y(r)\, dW(r) + \int_s^{s+\varepsilon} P(\tau)(D^-Y)(r)\, dr.
$$

Since $P(\tau)\zeta(s) \in \mathbb{D}^{1,p}(X)$, by the integration by parts formula for the divergence operator (see [112, Lemma 4.9]) we obtain

$$
\begin{aligned}
\big\langle \delta(\mathbf{1}_{[s,s+\varepsilon]}P_\tau Y), F'(P_\tau\zeta(s))\big\rangle &= \delta(\langle \mathbf{1}_{[s,s+\varepsilon]}P_\tau Y, F'(P_\tau\zeta(s))\rangle) \\
&\quad + \langle \mathbf{1}_{[s,s+\varepsilon]}P_\tau Y, D(F'(P_\tau\zeta(s)))\rangle_{\mathrm{Tr}}.
\end{aligned}
$$

Therefore, we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t \langle P_\tau(\zeta(s+\varepsilon)-\zeta(s)), F'(P_\tau\zeta(s))\rangle \, ds \\
&\quad = \frac{e^{-\tau}}{\varepsilon}\int_0^t \delta\langle \mathbf{1}_{[s,s+\varepsilon]}P_\tau Y, F'(P_\tau\zeta(s))\rangle \, ds \\
&\qquad + \frac{e^{-\tau}}{\varepsilon}\int_0^t \langle \mathbf{1}_{[s,s+\varepsilon]}P_\tau Y, D(F'(P_\tau\zeta(s)))\rangle_{\mathrm{Tr}} \, ds \\
&\qquad + \frac{1}{\varepsilon}\int_0^t \Big\langle \int_s^{s+\varepsilon} P_\tau(D^- Y)(r)\, dr, F'(P_\tau\zeta(s))\Big\rangle \, ds.
\end{aligned}$$

Proof. Proof of (1). The first claim follows directly from Theorem 6.11. The second claim can be found in [112, Theorem 4.13]. To prove (6.4.3), we proceed as follows.

With integration by parts, we have $h(1)-h(0) = h'(0) + \int_0^1 (1-t)h''(t)\,dt$ for any twice continuously differentiable h. For any $a, b \in X$ we can apply this identity to $h = FG$, where $G : [0,1] \to X$ is given by $G(t) = ta + (1-t)b$, and obtain

$$F(b) - F(a) = \langle b-a, F'(a)\rangle + \int_0^1 t\langle b-a, F''(ta+(1-t)b)(b-a)\rangle \, dt. \quad (6.4.6)$$

Since ζ is a continuous process, the process Y_ε given by

$$Y_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t (F(\zeta(s+\varepsilon)) - F(\zeta(s)))\, ds = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} F(\zeta(s))\, ds - \frac{1}{\varepsilon}\int_0^\varepsilon F(\zeta(s))\, ds$$

converges almost surely to the process $t \mapsto F(\zeta(t)) - F(\zeta(0))$. Since F is bounded, one can apply the dominated convergence theorem to conclude that the convergence also holds in $L^p(\Omega; X)$. By (6.4.6), $Y_\varepsilon(t)$ satisfies

$$\begin{aligned}
Y_\varepsilon(t) &= \frac{1}{\varepsilon}\int_0^t \langle \zeta(s+\varepsilon)-\zeta(s), F'(\zeta(s))\rangle \, ds \\
&\quad + \frac{1}{\varepsilon}\int_0^t\int_0^1 r\langle \zeta(s+\varepsilon)-\zeta(s), F''(r\zeta(s)+(1-r)\zeta(s+\varepsilon)) \\
&\qquad\qquad \times (\zeta(s+\varepsilon)-\zeta(s))\rangle \, dr\, ds
\end{aligned} \quad (6.4.7)$$

The second part of the right hand side can be written as

$$\frac{1}{\varepsilon}\int_0^t \langle \zeta(s+\varepsilon)-\zeta(s), Z_\varepsilon(s)(\zeta(s+\varepsilon)-\zeta(s))\rangle \, ds,$$

where

$$Z_\varepsilon(s) = \int_0^1 rF''(r\zeta(s)+(1-r)\zeta(s+\varepsilon))\, dr.$$

If we set $Z_0(s) := \frac{1}{2}F''(\zeta(s))$, then $Z_\varepsilon, \varepsilon > 0$ and Z_0 satisfy the three properties from Theorem 6.17. In particular, we have for all $t \in [0,T]$ and $\omega \in \Omega$, that $\|Z_\varepsilon(t,\omega)\| \leq \frac{1}{2}\|F''\|_\infty$.

By identity (6.3.7) we have

$$\begin{aligned}&\frac{1}{\varepsilon}\langle\zeta(s+\varepsilon)-\zeta(s), Z_\varepsilon(s)(\zeta(s+\varepsilon)-\zeta(s))\rangle\\&=\frac{1}{\varepsilon}\Big\langle\int_s^{s+\varepsilon}Y(r)\,dW(r)+\int_s^{s+\varepsilon}(D^-Y)(r)\,dr,\\&\qquad Z_\varepsilon(s)\Big(\int_s^{s+\varepsilon}Y(r)\,dW(r)+\int_s^{s+\varepsilon}(D^-Y)(r)\,dr\Big)\Big\rangle.\end{aligned}\tag{6.4.8}$$

Theorem 6.17, the assumption on D^-Y and the estimate $\|Z_\varepsilon(t,\omega)\|\le\frac{1}{2}\|F''\|_\infty$ yield that

$$\frac{1}{\varepsilon}\int_0^t\langle\zeta(s+\varepsilon)-\zeta(s), Z_\varepsilon(s)(\zeta(s+\varepsilon)-\zeta(s))\,ds\rangle\to\int_0^t\langle Y(s),F''(\zeta(s))Y(s)\rangle\,ds,$$

in $L^1(\Omega;X)$ as $\varepsilon\downarrow 0$. Combining this with equation (6.4.7) gives

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^t\langle\zeta(s+\varepsilon)-\zeta(s),F'(\zeta(s))\rangle\,ds\\&\quad\to F(\zeta(t))-F(\zeta(0))-\int_0^t\langle Y(s),F''(\zeta(s)Y(s)\rangle\,ds,\end{aligned}$$

in $L^1(\Omega)$ as $\varepsilon\downarrow 0$.

Proof of (3). The proof of (6.4.4), is similar to the proof of (6.4.3), but with $P(\tau)\zeta$ instead of ζ. Note that by identity (6.4.8) and (6.2.1), one obtains

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^t\langle P(\tau)\zeta(s+\varepsilon)-\zeta(s),Z_\varepsilon(s)(P(\tau)\zeta(s+\varepsilon)-\zeta(s))\,ds\rangle\\&\quad\to e^{-2\tau}\int_0^t\langle Y(s),F''(\zeta(s))Y(s)\rangle\,ds\end{aligned}$$

which yields the desired result. □

6.5 The random evolution system

Recall that $\Delta:=\{(s,t)\in[0,T]^2;\ s\le t\}$.

Definition 6.21. A random evolution system is a random family of operators $S:\Delta\times\Omega\to\mathscr{L}(X)$ such that

1. $S:\Delta\times\Omega\to\mathscr{L}(X)$ is strongly measurable;
2. $S(t,s)$ is strongly $\mathscr{F}_t$-measurable for each $t\ge s$;
3. For each ω, the family $\{S(t,s,\omega):\ (t,s)\in\Delta\}$ is an evolution system, that is,
   a) $S(s,s)=I$ and $S(t,r)=S(t,s)S(s,r)$ for any $0\le r\le s\le t\le T$;

b) For all $x \in X$, the mapping $(t,s) \mapsto S(t,s)x$ is continuous from Δ into X.

Let us introduce the following hypotheses on a given random evolution system.

(H1) For each $(t,s) \in \Delta$, we have $S(t,s) \in \mathbb{D}^{1,p}_s(\mathscr{L}(X))$. For all $x \in X$, $(\mathscr{D}S(t,s))x$ belongs to $\gamma(0,T;H,X)$. There exists a map

$$\hat{\mathscr{D}}S(t,s) : \Omega \times [0,T] \to \gamma(H, \mathscr{L}(X))),$$

such that for all $\omega \in \Omega$, $(\hat{\mathscr{D}}S(t,s)(r)h)x = (\mathscr{D}S(t,s)x)(r)h$. We assume that $\|\hat{\mathscr{D}}S(t,s)\| : \Omega \times [0,T] \to \mathbb{R}$ is measurable. Moreover, for all $p > 2$ we have

$$\sup_{t\in[0,T]} \int_0^t \Big[\mathbb{E}\|S(t,s)\|^p_{\mathscr{L}(X)} + \mathbb{E}\Big(\int_0^T \|\hat{\mathscr{D}}S(t,s)(\sigma)\|^2_{\gamma(H,\mathscr{L}(X))}\, d\sigma\Big)^{p/2}\Big]\, ds < \infty. \tag{6.5.1}$$

(H2) There exists a map $D^-S(t,\cdot) : \Omega \times [0,T] \to \mathscr{L}(X, \gamma(H,X))$ such that for all $\omega \in \Omega$ and $s \le t$, $D^-S(t,s) \in \gamma(H, \mathscr{L}(X))$ with the identification from (H1). Moreover, for all $x \in X$ the limit

$$\lim_{\varepsilon \downarrow 0} \mathscr{D}S(t, s-\varepsilon)(s)x = D^-S(t,s)x,$$

exists in $\gamma(H,X)$, and $D^-S(t,\cdot) \in L^2_s(\Omega \times [0,T]; \mathscr{L}(X, \gamma(H,X)))$.

(H3) There is a constant $M > 0$ such that the following estimates hold for almost all $r \le s \le t$:
- (H3a) $\|S(t,s)\|_{\mathscr{L}(X)} \le M$;
- (H3b) $\|\mathscr{D}S(t,r)(s)\|_{\gamma(H,\mathscr{L}(X))} \le M$;
- (H3c) $\|D^-S(t,s)\|_{\gamma(H,\mathscr{L}(X))} \le M$.

Whenever a random evolution system $S(t,s)$ satisfies the above properties $(H1), (H2)$ and $(H3)$, then we say that it satisfies (H). From [68, Remark, p158] it follows that

$$\mathscr{D}S(t,r)(s) = (D^-S(t,s))S(s,r), \tag{6.5.2}$$

whenever $r < s < t$.

Next, let us introduce the following space $\mathscr{S}_a(\gamma(\mathscr{H}, X))$ consisting of smooth and adapted $\gamma(\mathscr{H}, X)$-valued random processes G of the form

$$G = \sum_{k=1}^N \sum_{m=1}^N f_{km}(W(\varphi_1^m), \ldots, W(\varphi_N^m)) \otimes (\mathbf{1}_{(t_m,t_{m+1}]} \otimes R_k),$$

where $f_{km} \in C_b^\infty(\mathbb{R}^N)$, $0 \le t_1 < t_2 < \ldots < t_{N+1} \le T$, $\varphi_k^m \in L^2(0,T) \otimes H$ with $\mathrm{supp}(\varphi_j^m) \subset [0, t_m]$ and $R_k = \sum_{i=1}^{N_k} h_i \otimes x_i \in \gamma(H,X)$. Observe that $\mathbf{1}_{(t_m,t_{m+1}]} \otimes R_k \in \gamma(\mathscr{H}, X)$ by the ideal property, using the fact that the multiplication operator $M_{\mathbf{1}_{(a,b)}} : H \to \mathscr{H}$ given by $M_{\mathbf{1}_{(a,b)}} h = \mathbf{1}_{(a,b)} \otimes h$ is a bounded linear operator. We have that $\mathscr{S}_a(\gamma(\mathscr{H}, X))$ is dense in $L^p_a(\Omega \times [0,T]; \gamma(H,X))$

Let us state the following technical lemma.

Lemma 6.22. *If $A \in \gamma(H, \mathscr{L}(X))$ and $B \in \gamma(H, X)$, then $\sum_{n\geq 1} Ah_n(Bh_n)$ is convergent in X. Moreover, for all $N \in \mathbb{N}$ one has*

$$\Big\| \sum_{n=1}^{N} Ah_n(Bh_n) \Big\|_X \leq \|A\|_{\gamma(H,\mathscr{L}(X))} \|B\|_{\gamma(H,X)}.$$

Proof. Consider the bounded linear operator $T : \mathscr{L}(X) \to \mathscr{L}(X^*)$ defined by $T(\Phi) = \Phi^*$. By the ideal property it follows that $TA \in \gamma(H, \mathscr{L}(X^*))$, with $\|TA\|_{\gamma(H,\mathscr{L}(X^*))} \leq \|A\|_{\gamma(H,\mathscr{L}(X))}$. By the Hahn-Banach Theorem and [85, Proposition 3.19], it follows that

$$\begin{aligned}
\Big\| \sum_{n=1}^{N} Ah_n(Bh_n) \Big\|_X &\leq \sup_{N\geq 1} \Big\| \sum_{n=1}^{N} Ah_n(Bh_n) \Big\|_X \\
&= \sup_{N\geq 1} \sup_{\|x^*\|_{X^*}=1} \Big| \sum_{n=1}^{N} \langle Bh_n, ((TA)h_n)x^* \rangle \Big| \\
&= \sup_{N\geq 1} \sup_{\|x^*\|_{X^*}=1} \Big| \sum_{n=1}^{N} \sum_{k=1}^{N} \mathbb{E}\langle \gamma_n Bh_n, \gamma_k((TA)h_k)^* x^* \rangle \Big| \\
&\leq \|B\|_{\gamma(H,X)} \|A\|_{\gamma(H,\mathscr{L}(X))}.
\end{aligned}$$

□

The next proposition gives the first setup for forward integrability of the process

$$s \mapsto Y(t,s) := S(t,s)G(s)(t-s)^{-\alpha}\mathbf{1}_{[0,t]}(s), \tag{6.5.3}$$

in the space $L^p(\Omega; X)$ with $p > 2$ and $\alpha \in [0, \frac{1}{2})$, $S(t,s)$ a random evolution system satisfying (H) and $G \in \mathscr{S}_a(\gamma(\mathscr{H}, X))$.

Theorem 6.23. *Let $S(t,s)$ be a random evolution system satisfying (H) and $G \in \mathscr{S}_a(\gamma(\mathscr{H}, X))$, let $p > 2$ and $\alpha \in [0, \frac{1}{2})$. Consider the process Y as in (6.5.3). Then we have*

(1) For all $(s,t) \in \Delta$,

$$\begin{aligned}
&\int_0^s \sum_{k=1}^{n} n \int_r^{r+1/n} D(Y(t,r)h_k)(\sigma)(h_k)\, d\sigma\, dr \\
&\qquad \to \int_0^s \sum_{k\geq 1} (t-r)^{-\alpha} D^- S(t,r)h_k G(r)h_k\, dr,
\end{aligned} \tag{6.5.4}$$

as $n \to \infty$ in $L^p(\Omega; X)$;

(2) For all $t \in [0,T]$,

$$\int_0^{\cdot} \sum_{k=1}^{n} n \int_r^{r+1/n} D(Y(t,r)h_k)(\sigma)(h_k)\, d\sigma\, dr \\ \to \int_0^{\cdot} \sum_{k\geq 1} (t-r)^{-\alpha} D^- S(t,r) h_k G(r) h_k\, dr,$$

as $n \to \infty$ in $L^p(\Omega \times [0,t]; X)$.

Remark 6.24. From this theorem, it follows that for all $t \in [0,T]$, the process $s \mapsto Y(t,s)$ satisfies the conditions of Theorem 6.18, with $D^-Y(r) = \sum_{k\geq 1}(t-r)^{-\alpha} D^- S(t,r) h_k G(r) h_k$.

Proof. (1). By Lemma 6.7 we have for any $\phi \in L^2(0,T)\otimes U$ with $\operatorname{supp}(\phi) \subset [s,T]$,

$$D(S(t,s)G(s)h_k)\phi = (\mathscr{D}(S(t,s))G(s)h_k)\phi.$$

Therefore,

$$\begin{aligned} &\int_0^{s} \sum_{k=1}^{n} n \int_r^{r+\frac{1}{n}} D(Y(t,r)h_k)(\sigma)(h_k)\, d\sigma\, dr \\ &= \int_0^{s} \sum_{k=1}^{n} n \int_r^{r+\frac{1}{n}} (t-r)^{-\alpha} (\mathscr{D}S(t,r)(\sigma)h_k) G(r) h_k\, d\sigma\, dr \qquad (6.5.5)\\ &= \int_0^{s+\frac{1}{n}} n \int_{0\vee(\sigma-1/n)}^{s\wedge\sigma} \sum_{k=1}^{n} (\mathscr{D}S(t,r)(\sigma)h_k)\tilde{G}(r) h_k\, dr\, d\sigma, \end{aligned}$$

where $\tilde{G}(r) = (t-r)^{-\alpha}G(r)$. With this notation, the right hand side of (6.5.4) equals

$$\int_0^{\frac{1}{n}} \sum_{k\geq 1} D^- S(t,\sigma) h_k \tilde{G}(\sigma) h_k\, d\sigma + \int_{\frac{1}{n}}^{s} \sum_{k\geq 1} D^- S(t,\sigma) h_k \tilde{G}(\sigma) h_k\, d\sigma.$$

Observe that by Lemma 6.22 and (H3c),

$$\int_0^{\frac{1}{n}} \sum_{k\geq 1} D^- S(t,\sigma) h_k \tilde{G}(\sigma) h_k\, d\sigma \to 0,$$

in $L^p(\Omega, X)$.

We divide the interval $(0, s+\frac{1}{n})$ from the integral in (6.5.5) into three parts: $(0,\frac{1}{n}), (\frac{1}{n}, s)$ and $(s, s+\frac{1}{n})$. We will show that

$$\int_{\frac{1}{n}}^{s} n \int_{\sigma-\frac{1}{n}}^{\sigma} \sum_{k=1}^{n} (\mathscr{D}S(t,r)(\sigma)h_k)\tilde{G}(r) h_k\, dr\, d\sigma - \int_{\frac{1}{n}}^{s} \sum_{k\geq 1} D^- S(t,\sigma) h_k \tilde{G}(\sigma) h_k\, d\sigma \qquad (6.5.6)$$

converges to 0 in $L^p(\Omega; X)$. By the triangle inequality, we estimate

$$\begin{aligned}
&\mathbb{E}\Big\| \int_{\frac{1}{n}}^{s} n \int_{\sigma-\frac{1}{n}}^{\sigma} \sum_{k=1}^{n} (\mathscr{D}S(t,r)(\sigma)h_k)\tilde{G}(r)h_k \, dr\, d\sigma \\
&\qquad - \int_{\frac{1}{n}}^{s} \sum_{k\geq 1} D^- S(t,\sigma)h_k \tilde{G}(\sigma)h_k \, d\sigma \Big\|^p \\
&\leq \mathbb{E}\Big\| \int_{\frac{1}{n}}^{s} n \int_{\sigma-\frac{1}{n}}^{\sigma} \Big[\sum_{k=1}^{n} (\mathscr{D}S(t,r)(\sigma)h_k)\tilde{G}(r)h_k \\
&\qquad - \sum_{k=1}^{n} D^- S(t,\sigma)h_k \tilde{G}(\sigma)h_k \Big] \, dr\, d\sigma \Big\|^p \\
&\quad + \mathbb{E}\Big\| \int_{\frac{1}{n}}^{s} \sum_{k=n+1}^{\infty} D^- S(t,\sigma)h_k \tilde{G}(\sigma)h_k \, d\sigma \Big\|^p =: a_1 + a_2.
\end{aligned}$$

Note that $a_2 \to 0$, by the dominated convergence theorem, using (H3c), and by Lemma 6.22. Moreover, by the triangle inequality and (6.5.2),

$$\begin{aligned}
a_1 &\leq \mathbb{E}\Big\| \int_{\frac{1}{n}}^{s} n \int_{\sigma-\frac{1}{n}}^{\sigma} \sum_{k=1}^{n} \big(D^- S(t,\sigma)[S(\sigma,r) - I]G(r)h_k \big)(h_k) \, dr\, d\sigma \Big\|^p \\
&\quad + \mathbb{E}\Big\| \int_{\frac{1}{n}}^{s} n \int_{\sigma-\frac{1}{n}}^{\sigma} \sum_{k=1}^{n} D^- S(t,\sigma)h_k (\tilde{G}(r) - \tilde{G}(\sigma))(h_k) \, dr\, d\sigma \Big\|^p \\
&=: b_1 + b_2.
\end{aligned}$$

The first term, b_1, can be estimated by Lemma 6.22 and (H3c). Indeed, we obtain

$$b_1 \leq M^p \mathbb{E}\Big(\int_0^s (t-r)^{-\alpha} n \int_{\sigma-\frac{1}{n}}^{\sigma} \|(S(\sigma,r) - I)G(r)\|_{\gamma(H,X)} \, d\sigma\, dr \to 0,$$

by the dominated convergence theorem, where we use strong continuity of the evolution system. Likewise, we estimate b_2:

$$b_2 \leq M^p \mathbb{E}\Big(\int_0^s n \int_{\sigma-\frac{1}{n}}^{\sigma} \|\tilde{G}(r) - \tilde{G}(\sigma)\| \, dr\, d\sigma \Big)^p.$$

Again using the dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\Big(\int_0^s n \int_{\sigma-\frac{1}{n}}^{\sigma} \|\tilde{G}(r) - \tilde{G}(\sigma)\| \, dr\, d\sigma \Big)^p \\
&\qquad = \mathbb{E}\Big(\int_0^s \lim_{n\to\infty} n \int_{\sigma-\frac{1}{n}}^{\sigma} \|\tilde{G}(r) - \tilde{G}(\sigma)\| \, dr\, d\sigma \Big)^p.
\end{aligned}$$

However, for fixed $\sigma \in (0,s)$ such that $\sigma \neq t_i$ for some $i = 1, \dots, N$, there exist an $\tilde{N}$ and an m such that for all $n > \tilde{N}$, $(\sigma - \frac{1}{n}, \sigma) \subset (t_m, t_{m+1}]$. Hence for n large enough, one has for all $r \in (\sigma - \frac{1}{n}, \sigma)$,

$$\tilde{G}(r) - \tilde{G}(\sigma) = ((t-r)^{-\alpha} - (t-\sigma)^{-\alpha})G(\sigma).$$

By continuity of $r \mapsto (t-r)^{-\alpha}$, it follows that $b_2 \to 0$ as $n \to \infty$.

Having showed (6.5.6), the proof is finished once we show the following

$$\begin{aligned} &\int_0^{\frac{1}{n}} n \int_0^{s\wedge\sigma} \sum_{k=1}^n (\mathscr{D}S(t,r)(\sigma)h_k)\tilde{G}(r)h_k \, dr \, d\sigma \to 0, \\ &\int_s^{s+\frac{1}{n}} n \int_{0\vee(\sigma-\frac{1}{n})}^{s} \sum_{k=1}^n (\mathscr{D}S(t,r)(\sigma)h_k)\tilde{G}(r)h_k \, dr \, d\sigma \to 0, \end{aligned} \tag{6.5.7}$$

both in $L^p(\Omega, X)$. We will show the first convergence. The proof of the second is similar.

Observe that (H2c) cannot be used directly: The parameter t may be so small, that $t < \frac{1}{n}$ and thus $\sigma > t$. In such case, (H2c) does not hold. For fixed t, one can always take n large enough, such that $\frac{1}{n} < t$. In that situation, one can simply use Hölders inequality to prove (6.5.7). However, this cannot be used when integrating over t, which is done in (2). Therefore, we will prove (6.5.7) separately, in the case $t < \frac{1}{n}$. We only prove the first convergence. The second is done similarly.

First, note that

$$\begin{aligned} &\int_t^{\frac{1}{n}} n \int_0^{s\wedge\sigma} \sum_{k=1}^n (\mathscr{D}S(t,r)(\sigma)h_k)\tilde{G}(r)h_k \, dr \, d\sigma \\ &\quad = n \sum_{k=1}^n \int_0^s \int_t^{\frac{1}{n}} (\mathscr{D}S(t,r)(\sigma)h_k)\tilde{G}(r)h_k \, d\sigma \, dr = 0, \end{aligned}$$

since $S(t,r)$ is $\mathscr{F}_t$-measurable. Here, we have used that $\mathscr{D}S(t,s)x \in \gamma(0,T;H,X)$. Second, note that using (H2c) and Lemma 6.22 we obtain

$$\begin{aligned} &\mathbb{E}\Big\| \int_0^t n \int_0^{s\wedge\sigma} \sum_{k=1}^n (\mathscr{D}S(t,r)(\sigma)h_k)\tilde{G}(r)h_k \, dr \, d\sigma \Big\|^p \\ &\quad \le CM^p \int_0^t n \int_0^{s\wedge\sigma} (t-r)^{-\alpha} \, dr \, d\sigma \to 0 \end{aligned}$$

as $n \to \infty$.

The proof of (2) follows from the above computations. □

Corollary 6.25. *Let $S(t,s)$ be a random evolution system satisfying (H) and $G \in \mathscr{S}_a(\gamma(\mathscr{H}, X))$, let $p > 2$ and $\alpha \in [0, \frac{1}{2})$. Let Y be as in (6.5.3). For all t, the process $s \mapsto Y(t,s)$ belongs to $\mathrm{Dom}(\delta_\mathrm{p})$ and for all $s \ge t$,*

$$\int_0^s Y(t,r) \, dW^-(r) = \int_0^s Y(t,r) \, dW(r) + \int_0^s \sum_{k\ge 1} (t-r)^{-\alpha} D^- S(t,r)h_k G(r)h_k \, dr.$$

Proof. This follows directly from (6.5.4) and (6.3.7). □

6.6 A maximal inequality for the forward integral

In this section we will assume that the Banach space X is a UMD space with type 2, and X satisfies the following geometric property concerning differentiability of the norm. For $p \in [2, \infty)$, let $n_p : X \to \mathbb{R}$ be defined by $n_p(x) = \|x\|^p$.

(D) For some $q \in [2, \infty)$, the function $n_q : X \to \mathbb{R}$ is twice continuously differentiable, and there is a $C > 0$ such that for all $x, y, z \in X$,

$$|\langle y, n_q'(x)\rangle_{X,X^*}| \le C\|x\|^{q-1}\|y\|,$$
$$|\langle z, n_q''(x)y\rangle_{X,X^*}| \le C\|x\|^{q-2}\|y\|\|z\|.$$

With property (D), one can approximate the $n_p : X \to \mathbb{R}$ by a sequence of bounded C^2-functions F_{km}, given by

$$F_{km}(x) := \Big(\|x\|^q + \frac{1}{k}\Big)^{p/q}\psi_m(\|x\|^q), \qquad (6.6.1)$$

where $\psi_m : \mathbb{R} \to [0, 1]$ is a smooth function such that

$$\psi_m(x) = \begin{cases} 1, \text{ if } |x| \le m; \\ 0, \text{ if } |x| \ge m + 1. \end{cases}$$

Moreover, F_{km} satisfies the same bounds as n_q, uniformly in k, m:

$$\begin{aligned} |\langle y, F_{km}'(x)\rangle_{X,X^*}| &\le C\|x\|^{q-1}\|y\|, \\ |\langle z, F_{km}''(x)y\rangle_{X,X^*}| &\le C\|x\|^{q-2}\|y\|\|z\|. \end{aligned} \qquad (6.6.2)$$

Also, if $u \in L^p(\Omega; X)$ then by the dominated convergence theorem, $\mathbb{E}F_{km}(u) \to \mathbb{E}\|u\|^p$.

The following lemma is a version of the lemma proved in [143].

Lemma 6.26. *Let $\phi : [0, T] \to \mathbb{R}$ be continuous and non-negative, $\psi : [0, T] \to \mathbb{R}$ be Lebesgue measurable and non-negative such that $\int_0^T \psi(s)\, ds < \infty$. Let $\alpha \in (0, 1)$. If*

$$\phi(t) \le \int_0^t (\phi(s))^{1-\alpha}\psi(s)\, ds, \qquad t \in [0, T],$$

then

$$\phi(t) \le \Big(\alpha \int_0^t \psi(s)\, ds\Big)^{\frac{1}{\alpha}}, \qquad t \in [0, T].$$

Proof. The function

$$F_n : \theta \mapsto \Big(\int_0^\theta (\phi(s))^{1-\alpha}\psi(s)\, ds + \frac{1}{n}\Big)^\alpha$$

is absolutely continuous for every $n \ge 1$. hence it is differentiable almost everywhere, and its derivative satisfies

$$\begin{aligned}F_n'(\theta) &= \alpha\Big(\int_0^\theta (\phi(s))^{1-\alpha}\psi(s)\,ds + \frac{1}{n}\Big)^{\alpha-1}\phi(\theta)^{1-\alpha}\psi\theta\\ &\le \alpha\phi(\theta)^{\alpha-1}\phi(\theta)^{1-\alpha}\psi\theta = \alpha\psi(\theta).\end{aligned}$$

If we now integrate both left and right hand side from 0 to t, we get

$$F_n(t) \le \int_0^t \alpha\psi(\theta)\,d\theta.$$

Now let $n\to\infty$ to obtain the result:

$$\int_0^t (\phi(s))^{1-\alpha}\psi(s)\,ds \le \Big(\int_0^t \alpha\psi(\theta)\,d\theta\Big)^{\frac{1}{\alpha}}.$$

□

Theorem 6.27. *Let $S(t,s)$ be a random evolution system satisfying (H), $G\in \mathscr{S}_a(\gamma(\mathscr{H},X))$, $p>2$ and $\alpha\in[0,\frac12)$. For all $t\in[0,T]$ we have*

$$\mathbb{E}\Big\|\int_0^t (t-s)^{-\alpha}S(t,s)G(s)\,dW^-(s)\Big\|_X^p \le C\int_0^t (t-s)^{-2\alpha}\mathbb{E}\|G(s)\|^p_{\gamma(H,X)}\,ds. \tag{6.6.3}$$

Proof. Consider the function F_{km} from (6.6.1). Let $(s,t)\in[0,T]^2$ and set

$$B(t,s) := \int_0^s Y(t,r)\,dW^-(r), \tag{6.6.4}$$

where $Y(t,r) := (t-r)^{-\alpha}S(t,r)G(r)\mathbf{1}_{[0,t]}(r)$ as in (6.5.3). By Remark 6.24, $r\mapsto Y(t,r)$ satisfies the conditions of Theorem 6.18 for any $s\in[0,t]$, and for all $\tau>0$ we obtain

$$\begin{aligned}n\int_0^s \langle P_\tau(B(t,r+1/n)-B(t,r)), F_{km}'(P_\tau B(t,r))\rangle\,dr &\to F_{km}(P_\tau B(t,s))\\ - F_{km}(0) - e^{-2\tau}\int_0^s &\langle P_\tau Y(t,s), (F_{km}''(P_\tau B(t,s)))(P_\tau Y(t,s))\rangle\,dr.\end{aligned} \tag{6.6.5}$$

By Remark 6.20, the left hand side of (6.6.5) equals

$$\begin{aligned}n\int_0^s &\langle P_\tau(B(t,r+1/n)-B(t,r)), F_{km}'(P_\tau B(t,r))\rangle\,dr\\ &= e^{-\tau}n\int_0^s \delta\langle \mathbf{1}_{[r,r+1/n]}P_\tau Y(t,\cdot), F_{km}'(P_\tau B(t,r))\rangle\,dr\\ &\quad + e^{-\tau}n\int_0^s \langle \mathbf{1}_{[r,r+1/n]}P_\tau Y(t,\cdot), D(F_{km}'(P_\tau B(t,r)))\rangle_{\mathrm{Tr}}\,dr\\ &\quad + n\int_0^s \Big\langle \int_r^{r+1/n} P_\tau(D^-Y(t,\cdot))(\sigma)\,d\sigma, F_{km}'(P_\tau B(t,r))\Big\rangle\,dr.\end{aligned}$$

As the expectation of the Skorohod integral is always zero, it follows directly that

$$\mathbb{E}e^{-\tau}n\int_0^s \delta\langle \mathbf{1}_{[r,r+1/n]}P_\tau Y(t,\cdot), F'_{km}(P_\tau B(t,r))\rangle\, dr = 0.$$

This combined with the lemmas and proven in the appendix, Lemma 6.43, Lemma 6.44, we can find a sequence $(\tau_i)_{i=1}^\infty$ such that $\tau_i \downarrow 0$ as $i \to \infty$, and such that for all i,

$$\begin{aligned}
&\mathbb{E}n\int_0^s \langle P_{\tau_i}(B(t,r+1/n) - B(t,r)), F'_{km}(P_{\tau_i}B(t,r))\rangle\, dr \\
&\quad\le C\mathbb{E}\int_0^s \Big(\|B(t,r)\|_X^{p-2}\|Y(t,r)\|_{\gamma(H,X)}\Big\|\int_0^r (t-\sigma)^{-\alpha}S(r,\sigma)G(\sigma)\, dW^-(\sigma)\Big\|_X \\
&\qquad + (t-r)^{-\alpha}\|G(r)\|_{\gamma(H,X)}\|B(t,r)\|_X^{p-1}\Big)\, dr
\end{aligned}$$

Next, we estimate the expectation of the last part of (6.6.5): observe that

$$\begin{aligned}
\mathbb{E}e^{-2\tau}&\int_0^s \langle P_\tau Y(t,r), F''_{km}(P_\tau B(t,r))(P_\tau Y(t,r))\rangle\, dr \\
&\le C\int_0^s \mathbb{E}\|P_\tau Y(t,r)\|^2_{\gamma(H,X)}\|P_\tau B(t,r)\|_X^{p-2}\, dr \\
&\le C\int_0^s (\mathbb{E}\|Y(t,r)\|^p_{\gamma(H,X)})^{\frac{2}{p}}(\mathbb{E}\|B(t,r)\|_X^p)^{\frac{p-2}{p}}\, dr
\end{aligned}$$

Finally, $\mathbb{E}(F_{mk}(0) = (1/k)^{p/q}$. Therefore, combining all of the above computations, we conclude that for every $s \in [0,t]$ and all $i \ge 1$,

$$\begin{aligned}
&\mathbb{E}(F_{mk}P(\tau_i)B(t,s)) \le (1/k)^{p/q} \\
&\quad + C\mathbb{E}\int_0^s \Big(\|B(t,r)\|_X^{p-2}\|Y(t,r)\|_{\gamma(H,X)}\Big\|\int_0^r (t-\sigma)^{-\alpha}S(r,\sigma)G(\sigma)\, dW^-(\sigma)\Big\|_X \\
&\qquad + (t-r)^{-\alpha}\|G(r)\|_{\gamma(H,X)}\|B(t,r)\|_X^{p-1}\Big)\, dr \\
&\qquad + C\int_0^s (\mathbb{E}\|Y(t,r)\|^p_{\gamma(H,X)})^{\frac{2}{p}}(\mathbb{E}\|B(t,r)\|_X^p)^{\frac{p-2}{p}}\, dr
\end{aligned} \tag{6.6.6}$$

Next, note that almost surely,

$$\begin{aligned}
\|B(t,r)\| &\le M\Big\|\int_0^r (t-\sigma)^{-\alpha}S(r,\sigma)G(\sigma)\, dW^-(\sigma)\Big\|_X; \\
\|Y(t,r)\|_{\gamma(H,X)} &\le M(t-r)^{-\alpha}\|G(r)\|_{\gamma(H,X)}.
\end{aligned}$$

Hence we can rewrite estimation (6.6.6) into

$$
\begin{aligned}
&\mathbb{E}(F_{mk}P_{\tau_i}B(t,s)) \le (1/k)^{p/q} \\
&\quad + C\mathbb{E}\int_0^s (t-r)^{-\alpha}\|G(r)\|_{\gamma(H,X)}\Big\|\int_0^r (t-\sigma)^{-\alpha}S(r,\sigma)G(\sigma)\,dW^-(\sigma)\Big\|_X^{p-1}\,dr \\
&\quad + C\int_0^s (t-r)^{-2\alpha}(\mathbb{E}\|G(r)\|^p_{\gamma(H,X)})^{\frac{2}{p}} \\
&\qquad \times \Big(\mathbb{E}\Big\|\int_0^r (t-\sigma)^{-\alpha}S(r,\sigma)G(\sigma)\,dW^-(\sigma)\Big\|_X^p\Big)^{\frac{p-2}{p}}\,dr
\end{aligned}
\tag{6.6.7}
$$

Set $\Phi(t,s) := \int_0^s (t-r)^{-\alpha}S(s,r)G(r)\,dW^-(r)$. Then $B(t,s) = S(t,s)\Phi(t,s)$ and in particular $B(t,t) = \Phi(t,t)$. Letting $s = t$ in (6.6.7), we obtain

$$
\begin{aligned}
\mathbb{E}(F_{mk}P_{\tau_i}\Phi(t,t)) \le (1/k)^{p/q} &+ C\mathbb{E}\int_0^t (t-r)^{-\alpha}\|G(r)\|_{\gamma(H,X)}\|\Phi(t,r)\|_X^{p-1}\,dr \\
&+ C\int_0^t (t-r)^{-2\alpha}(\mathbb{E}\|G(r)\|^p_{\gamma(H,X)})^{\frac{2}{p}}\Big(\mathbb{E}\|\Phi(t,r)\|_X^p\Big)^{\frac{p-2}{p}}\,dr
\end{aligned}
$$

By continuity of F_{mk}, it follows that

$$
\mathbb{E}(F_{mk}P_{\tau_i}\Phi(t,t)) \to \mathbb{E}(F_{mk}\Phi(t,t)), \qquad i\to\infty.
$$

Recall that

$$
\mathbb{E}(F_{mk}\Phi(t,t)) \to \mathbb{E}\|\Phi(t,t)\|_X^p, \qquad m,k\to\infty.
$$

We conclude

$$
\begin{aligned}
\mathbb{E}\|\Phi(t,t)\|_X^p \le C\mathbb{E}&\int_0^t (t-r)^{-\alpha}\|G(r)\|_{\gamma(H,X)}\|\Phi(t,r)\|_X^{p-1}\,dr \\
&+ C\int_0^t (t-r)^{-2\alpha}(\mathbb{E}\|G(r)\|^p_{\gamma(H,X)})^{\frac{2}{p}}\Big(\mathbb{E}\|\Phi(t,r)\|_X^p\Big)^{\frac{p-2}{p}}\,dr
\end{aligned}
$$

With Hölder's inequality and notation $\|\cdot\|_p := \|\cdot\|_{L^p(\Omega;X)}$ and $\|\cdot\|_{p,\gamma} := \|\cdot\|_{L^p(\Omega;\gamma(H,X))}$, we can write

$$
\begin{aligned}
&\mathbb{E}\|\Phi(t,t)\|_X^p \\
&\quad \le C\int_0^t (t-r)^{-\alpha}\|G(r)\|_{p,\gamma}\|\Phi(t,r)\|_p^{p-1} + (t-r)^{-2\alpha}\|G(r)\|^2_{p,\gamma}\|\Phi(t,r)\|_p^{p-2}\,dr \\
&\quad = C\int_0^t (\mathbb{E}\|\Phi(t,r)\|_X^p)^{1-2/p} \\
&\qquad\qquad \times \big[(t-r)^{-\alpha}\|G(r)\|_{p,\gamma}\|\Phi(t,r)\|_p + (t-r)^{-2\alpha}\|G(r)\|^2_{p,\gamma}\big]\,dr.
\end{aligned}
$$

Now use Lemma 6.26 to obtain

$$
\begin{aligned}
\mathbb{E}\|\Phi(t,t)\|_X^p &\le C\Big(\int_0^t (t-r)^{-\alpha}\|G(r)\|_{p,\gamma}\|\Phi(t,r)\|_p + (t-r)^{-2\alpha}\|G(r)\|_{p,\gamma}^2\,dr\Big)^{p/2} \\
&\le C\Big[\Big(\int_0^t (t-r)^{-\alpha}\|G(r)\|_{p,\gamma}\|\Phi(t,r)\|_p\,dr\Big)^{p/2} \\
&\qquad + \Big(\int_0^t (t-r)^{-2\alpha}\|G(r)\|_{p,\gamma}^2\,dr\Big)^{p/2}\Big]
\end{aligned}
$$

Use inequality $ab \le \frac{1}{2}(a^2+b^2)$ to obtain

$$
\begin{aligned}
\mathbb{E}\|\Phi(t,t)\|_X^p &\le C\Big[\Big(\int_0^t \|\Phi(t,r)\|_p^2\,dr\Big)^{p/2} + 2\Big(\int_0^t (t-r)^{-2\alpha}\|G(r)\|_{p,\gamma}^2\,dr\Big)^{p/2}\Big] \\
&\le C\Big(\int_0^t \mathbb{E}\|\Phi(t,r)\|_X^p\,dr + \int_0^t (t-r)^{-2\alpha}\mathbb{E}\|G(r)\|_{\gamma(H,X)}^p\,dr\Big),
\end{aligned}
$$

where in the last line we have used Hölder's inequality with respect to the measure $d\mu = (t-r)^{-2\alpha}\,dr$. Now use Gronwall's inequality to obtain

$$
\mathbb{E}\|\Phi(t,t)\|_X^p \le C\int_0^t (t-r)^{-2\alpha}\mathbb{E}\|G(r)\|_{\gamma(H,X)}^p\,dr
$$

Finally, note that $t\in[0,T]$ was chosen arbitrarily, and note that by construction of Φ we have proved (6.6.3). □

For the maximal inequality in Theorem 6.29 we need the following lemma.

Lemma 6.28. *Let $S(t,s)$ be a random evolution system satisfying (H), $G \in \mathscr{S}_a(\gamma(\mathscr{H},X))$, $p>2$ and $\alpha\in[0,\frac{1}{2})$. For every $t\in[0,T]$ and $r<t$, we have*

$$
s \mapsto (r-s)^{-\alpha}S(t,s)G(s)\mathbf{1}_{[0,r]}(s) \in \mathrm{Dom}(\delta_2),
$$

and

$$
\begin{aligned}
&\int_0^t\int_0^r (t-r)^{\alpha-1}(r-s)^{-\alpha}S(t,s)G(s)\,dW(s)\,dr \\
&\qquad = \int_0^t\int_s^t (t-r)^{\alpha-1}(r-s)^{-\alpha}S(t,s)G(s)\,dr\,dW(s).
\end{aligned}
$$

Proof. The first claim is immediate by hypothesis (H1). By Lemma 6.2, it suffices to show that

$$
\mathbb{E}\int_0^t \Big\|\int_0^r (t-r)^{\alpha-1}(r-s)^{-\alpha}S(t,s)G(s)\,dW(s)\Big\|_X^2\,dr < \infty. \tag{6.6.8}
$$

By continuity of $\delta : \mathbb{D}^{1,p}(\gamma(\mathscr{H},X)) \to L^p(\Omega;X)$, and using that X has type 2, we obtain

$$\mathbb{E}\int_0^t \Big\| \int_0^r (t-r)^{\alpha-1}(r-s)^{-\alpha} S(t,s)G(s)\,dW(s)\Big\|_X^2\,dr$$
$$\leq C\int_0^t (t-r)^{2(\alpha-1)}\mathbb{E}\int_0^r \|S(t,s)(r-s)^{-\alpha}G(s)\|^2_{\mathbb{D}^{1,p}(\gamma(H,X))}\,ds\,dr.$$

We will now show that there exists a constant $C>0$ such that for all $r\in[0,t)$, $\int_0^r \|S(t,s)(r-s)^{-\alpha}G(s)\|^2_{\mathbb{D}^{1,p}(\gamma(H,X))}\,ds \leq C$. First note that

$$\mathbb{E}\int_0^r \|S(t,s)(r-s)^{-\alpha}G(s)\|^2_{\gamma(H,X)}\,ds \leq C^p M^p \mathbb{E}\int_0^r (r-s)^{-2\alpha}\,ds \leq C.$$

Let $q>1$ be so small, that $2\alpha q<1$. Let q' be its Hölder conjugate. Then

$$\mathbb{E}\int_0^r\int_0^T \|\mathscr{D}_\sigma S(t,s)(r-s)^{-\alpha}G(s)\|^2_{\gamma(H,\gamma(H,X))}\,d\sigma\,ds$$
$$\leq C\Big(\int_0^r (r-s)^{-2\alpha q}\,ds\Big)^{1/q}\Big(\int_0^r \Big(\mathbb{E}\int_0^T \|\mathscr{D}S(t,s)\|^2_{\gamma(H,\mathscr{L}(X))}\,d\sigma\Big)^{q'}\,ds\Big)^{\frac{1}{q'}}$$
$$\leq C\Big(\int_0^r \|S(t,s)\|^{q'}_{\mathbb{D}^{1,q'}(\mathscr{L}(X))}\Big)^{\frac{1}{q'}} \leq C\Big(\int_0^t \|S(t,s)\|^{q'}_{\mathbb{D}^{1,q'}(\mathscr{L}(X))}\Big)^{\frac{1}{q'}} < \infty.$$

Finally,

$$\mathbb{E}\int_0^r\int_0^T \|S(t,s)(r-s)^{-2\alpha}D_\sigma G(s)\|^2_{\gamma(H,\gamma(H,X))}\,d\sigma\,ds \leq C$$

by hypothesis (H3a) and the fact that G is bounded. Hence, as we wanted to show,

$$\mathbb{E}\int_0^r \|S(t,s)(r-s)^{-\alpha}G(s)\|^2_{\mathbb{D}^{1,p}(\gamma(H,X))}\,ds \leq C,$$

where we have used the product rule, Lemma 6.6. Now (6.6.8) follows. □

Theorem 6.29. *Let $S(t,s)$ be a random evolution system satisfying (H), and let $G\in\mathscr{S}_a(\gamma(\mathscr{H},X))$. For all $p\in(2,\infty)$, we have the estimate*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\Big\|\int_0^t S(t,s)G(s)\,dW^-(s)\Big\|_X^p\Big) \leq C\mathbb{E}\int_0^T \|G(s)\|^p_{\gamma(H,X)}\,ds. \tag{6.6.9}$$

In particular, the operator $J:\mathscr{S}_a\to L^p(\Omega;C([0,T];X))$ defined by

$$J(G)(t) = \int_0^t S(t,s)G(s)\,dW^-(s) \tag{6.6.10}$$

extends uniquely to a linear bounded operator $J: L^p_a(\Omega\times[0,T];\gamma(H,X)) \to L^p(\Omega;C([0,T];X))$ for which (6.6.9) holds.

Proof. Fix $\alpha \in (1/p, 1/2)$. Apply Theorem 6.27 to obtain that the process $s \mapsto S(t,s)G(s)(r-s)^{-\alpha}\mathbf{1}_{[0,r]}(s)$ belongs to the domain of δ_p^-. Applying the factorization method, we have

$$S(t,s)G(s) = C_\alpha \int_s^t S(t,s)G(s)(t-r)^{\alpha-1}(r-s)^{-\alpha}\,dr,$$

with $C_\alpha = \frac{\sin \pi\alpha}{\pi}$. By Corollary 6.25 and Lemma 6.28, it follows that almost surely

$$\begin{aligned}\int_0^t S(t,s)G(s)\,dW^-(s) &= C_\alpha \int_0^t \int_s^t S(t,s)G(s)(t-r)^{\alpha-1}(r-s)^{-\alpha}\,dr\,dW^-(s)\\ &= C_\alpha \int_0^t \int_0^r S(t,s)G(s)(t-r)^{\alpha-1}(r-s)^{-\alpha}\,dW^-(s)\,dr\\ &= C_\alpha \int_0^t S(t,r)(t-r)^{\alpha-1}\int_0^r S(r,s)G(s)(r-s)^{-\alpha}\,dW^-(s)\,dr.\end{aligned}$$

By choice of α, we have $(\alpha-1)\frac{p}{p-1} > -1$. Therefore, with Hölder's inequality, almost surely

$$\begin{aligned}&\sup_{t\in[0,T]}\Big\|\int_0^t S(t,s)G(s)\,dW^-(s)\Big\|_X\\ &\quad\le \frac{M}{\pi}\sup_{t\in[0,T]}\int_0^t (t-r)^{\alpha-1}\Big\|\int_0^r S(r,s)G(s)(r-s)^{-\alpha}\,dW^-(s)\Big\|_X\,dr\\ &\quad\le \frac{M}{\pi}\sup_{t\in[0,T]}\Big(\int_0^t (t-r)^{(\alpha-1)\frac{p}{p-1}}\,dr\Big)^{\frac{p-1}{p}}\\ &\qquad\times\Big(\int_0^T\Big\|\int_0^r S(r,s)G(s)(r-s)^{-\alpha}\,dW^-(s)\Big\|_X^p\,dr\Big)^{1/p}\\ &\quad\le C\Big(\int_0^T\Big\|\int_0^r S(r,s)G(s)(r-s)^{-\alpha}\,dW^-(s)\Big\|_X^p\,dr\Big)^{1/p}.\end{aligned}$$

By Theorem 6.27, we obtain

$$\begin{aligned}&\mathbb{E}\Big(\sup_{t\in[0,T]}\Big\|\int_0^t S(t,s)G(s)\,dW^-(s)\Big\|_X^p\Big)\\ &\quad\le C\mathbb{E}\int_0^T\Big\|\int_0^r S(r,s)G(s)(r-s)^{-\alpha}\,dW^-(s)\Big\|_X^p\,dr\\ &\quad\le C\int_0^T\int_0^r (r-s)^{-2\alpha}\mathbb{E}\|G(s)\|_{\gamma(H,X)}^p\,ds\,dr\\ &\quad\le C\int_0^T\int_s^T (r-s)^{-2\alpha}\,dr\mathbb{E}\|G(s)\|_{\gamma(H,X)}^p\,ds\\ &\quad\le C\mathbb{E}\int_0^T\|G(s)\|_{\gamma(H,X)}^p\,ds.\end{aligned}$$

This proves estimate (6.6.9). Note that by Theorem 6.18, $t \mapsto S(t,s)G(s)\, dW^-(s)$ has continuous paths. Hence the operator J, given by (6.6.10), maps into $L^p(\Omega; C([0,T];X))$. Since $\mathscr{S}_a(\gamma(\mathscr{H},X))$ is dense in $L^p_a(\Omega \times [0,T]; \gamma(H,X))$, it follows that there exists a unique extension to J, also denoted by J, which maps $L^p_a(\Omega \times [0,T]; \gamma(H,X))$ into $L^p(\Omega; C([0,T];X))$, such that (6.6.9) holds. □

6.7 Stochastic evolution equations

Let E_0 be a UMD Banach space with type 2 that satisfies property (D). Let us consider the following stochastic evolution equation:

$$\begin{cases} dU(t) = (A(t)U(t) + F(t,U(t)))\, dt + B(t,U(t))\, dW(t) \\ U(0) \ = u_0. \end{cases} \tag{6.7.1}$$

We assume that u_0 is an E_0-valued $\mathscr{F}_0$-measurable random variable, and impose the following conditions on A, F and B:

(A.1) For each $t \in [0,T]$ and $\omega \in \Omega$, we have

$$A(t,\omega) : E_0 \supset D(A(t,\omega)) \to E_0,$$

and there exists a Banach space $E_1^* \subset E_0^*$ which is dense in E_0^*, such that $E_1^* \subset D(A(t,\omega)^*) \subset E_0^*$. We assume that for all $x^* \in E_1^*$, $A^*(\cdot)x^* \in L^2(\Omega \times [0,T]; E_0^*)$, and there exists a random evolution system satisfying (H) such that

$$S^*(t,s)A^*(t)x^* = \frac{d}{dt} S^*(t,s)x^*, \qquad x^* \in E_1^*.$$

(A.2) For every $x \in E_0$, the map $(t,\omega) \to F(t,\omega,x) \in E_0$ is strongly measurable and adapted. There exists constants C_F and L_F such that for all $t \in [0,T]$, $\omega \in \Omega$ and $x \in E_0$ one has:

$$\|F(t,\omega,x) - F(t,\omega,y)\|_{E_0} \le L_F \|x-y\|_{E_0},$$
$$\|F(t,\omega,x)\|_{E_0} \le C_F(1 + \|x\|_{E_0}),$$

(A.3) For every $x \in E_0$, the map $(t,\omega) \to B(t,\omega,x) \in \gamma(H,E_0)$ is strongly measurable and adapted. There exists constants C_B and L_B such that for all $t \in [0,T]$, $\omega \in \Omega$ and $x \in E_0$, one has

$$\|B(t,\omega,x) - F(t,\omega,y)\|_{\gamma(H,E_0)} \le L_B \|x-y\|_{E_0},$$
$$\|B(t,\omega,x)\|_{\gamma(H,E_0)} \le C_B(1 + \|x\|_{E_0}),$$

Observe that the hypothesis on the drift $A(t,\omega)$ is different from the hypotheses in Chapter 5. The most important difference is that in this situation, the evolution system needs to be Malliavin differentiable. Also note that assumptions (A.2) and (A.3) coincide with assumptions (HF) and (HB), respectively, in the case $a = \theta_B = \theta_F = 0$.

Definition 6.30. A strongly measurable and adapted process $u : \Omega\times[0,T] \to E_0$ is called a *forward mild solution* to problem (6.7.1) if it belongs to the space $L^p(\Omega; C([0,T]; E_0))$ for some $p > 2$, and if for all $t \in [0,T]$,

$$u(t) = S(t,0)u_0 + \int_0^t S(t,s)F(s,u(s))\,ds + \int_0^t S(t,s)B(s,u(s))\,dW^-(s).$$

Already in Chapter 5 the notion of forward mild solution was introduced, see Theorem 5.22 and the discussion following the theorem. There we have only considered the case where G is independent of u, but one can always take $G = B(\cdot,u(\cdot))$. In the above definition, however, no interpolation spaces are considered. In general, one cannot compare this forward mild solution with the pathwise mild solution, as identity (5.4.3) may not be well defined. However, if $A(t,\omega)$ and $A^*(t,\omega)$ satisfy the conditions of Chapter 5 ((H1) - (H4), (HF), (HB)), then a forward mild solution is always a pathwise mild solution.

Definition 6.31. A strongly measurable and adapted process $u : \Omega\times[0,T] \to E_0$ is called a weak solution to problem (6.7.1) if it belongs to $L^p(\Omega; C([0,T]; E_0))$ for some $p > 2$, and if for all $x^* \in E_1^*, t \in [0,T]$,

$$\begin{aligned}\langle u(t), x^*\rangle_{E_0,E_0^*} = \langle u_0, x^*\rangle_{E_0,E_0^*} &+ \int_0^t \langle A^*(s)x^*, u(s)\rangle_{E_0,E_0^*}\,ds \\ &+ \int_0^t \langle F(s,u(s)), x^*\rangle_{E_0,E_0^*}\,ds \\ &+ \int_0^t \langle B(s,u(s)), x^*\rangle_{E_0,E_0^*}\,dW(s).\end{aligned}$$

Proposition 6.32. *Assume hypotheses (A.1) – (A.3). If u is a mild solution, then it is a weak solution.*

Proof. By the assumptions, we have that $B(\cdot,u(\cdot)) \in L^p_a(\Omega\times[0,T]; \gamma(H,E))$. Consider a sequence $(B_n)_{n\geq 1} \subset \mathscr{S}_a$ such that $B_n \to B$ in $L^p_a(\Omega\times[0,T]; \gamma(H,E))$. Then, by definition of J given in (6.6.10), we have

$$\int_0^t S(t,s)B(s,u(s))\,dW^-(s) = \lim_{n\to\infty} \int_0^t S(t,s)B_n(s)\,dW^-(s).$$

Set

$$u^n(t) = S(t,0)u_0 + \int_0^t S(t,s)F(s,u(s))\,ds + \int_0^t S(t,s)B_n(s)\,dW^-(s),$$

and

$$\begin{aligned}u^{n,m}(t) = S(t,0)u_0 &+ \int_0^t S(t,s)F(s,u(s))\,ds \\ &+ m\sum_{i=1}^m \int_0^t S(t,s)B_n(s)u_i(W(s+1/m) - W(s))u_i\,ds.\end{aligned}$$

A straightforward computation leads to the equality

$$u^{n,m}(t) = S(t,s)u^{n,m}(s) + \int_s^t S(t,r)F(r,u(r))\,dr + m\sum_{i=1}^m \int_s^t S(t,r)B_n(r)u_i(W(r+1/m)-W(r))u_i\,dr.$$

For $x^* \in E_1^*$ and $x \in E_0$, we have

$$\int_\sigma^t \langle x, S^*(r,\sigma)A^*(r)x^*\rangle_{E_0,E_0^*}\,dr = \langle x, S^*(t,\sigma)x^* - x^*\rangle_{E_0,E_0^*}.$$

Hence for $x^* \in E_1^*$ we obtain

$$\begin{aligned} m\sum_{i=1}^m \int_s^t \int_\sigma^t &\langle B_n(\sigma)u_i(W(\sigma+1/n)-W(\sigma))u_i, S^*(r,\sigma)A^*(r)x^*\rangle_{E_0,E_0^*}\,dr\,d\sigma \\ &= m\sum_{i=1}^m \int_s^t \langle B_n(\sigma)u_i(W(\sigma+1/n)-W(\sigma))u_i, S^*(t,\sigma)x^* - x^*\rangle_{E_0,E_0^*}\,d\sigma \\ &= \langle u^{n,m}(t), x^*\rangle - \langle S(t,s)u^{n,m}(s), x^*\rangle - \Big\langle \int_s^t S(t,r)F(r,u(r))\,dr, x^*\Big\rangle \\ &\quad - m\sum_{i=1}^m \int_s^t \langle B_n(r)u_i(W(r+1/n)-W(r))u_i, x^*\rangle\,dr. \end{aligned} \tag{6.7.2}$$

On the other hand, using Fubini's theorem twice, the top term of (6.7.2) equals

$$
\begin{aligned}
& m\sum_{i=1}^{m}\int_s^t\int_\sigma^t\langle B_n(\sigma)u_i(W(\sigma+1/n)-W(\sigma))u_i, S^*(r,\sigma)A^*(r)x^*\rangle_{E_0,E_0^*}\,dr\,d\sigma \\
&= m\sum_{i=1}^{m}\int_s^t\Big\langle\int_s^r S(r,\sigma)B_n(\sigma)u_i(W(\sigma+1/n)-W(\sigma))u_i\,d\sigma, A^*(r)x^*\Big\rangle_{E_0,E_0^*}\,dr \\
&= \int_s^t\Big\langle u^{n,m}(r)-S(r,s)u^{n,m}(s)-\int_s^r S(r,\sigma)F(\sigma,u(\sigma))\,d\sigma, A^*(r)x^*\Big\rangle_{E_0,E_0^*}\,dr \\
&= \int_s^t\langle u^{n,m}(r),A^*(r)x^*\rangle_{E_0,E_0^*}\,dr-\int_s^t\langle S(r,s)u^{n,m}(s),A^*(r)\rangle_{E_0,E_0^*}\,dr \\
&\quad -\int_s^t\Big\langle\int_s^r S(r,\sigma)F(\sigma,u(\sigma))\,d\sigma, A^*(r)x^*\Big\rangle_{E_0,E_0^*}\,dr \\
&= \int_s^t\langle u^{n,m}(r),A^*(r)x^*\rangle_{E_0,E_0^*}\,dr-\langle S(t,s)u^{n,m}(s),x^*\rangle_{E_0,E_0^*} \\
&\quad +\langle u^{n,m}(s),x^*\rangle_{E_0,E_0^*}-\int_s^t\int_\sigma^t\langle S(r,\sigma)F(\sigma,u(\sigma)),A^*(r)x^*\rangle_{E_0,E_0^*}\,dr\,d\sigma \\
&= \int_s^t\langle u^{n,m}(r),A^*(r)x^*\rangle_{E_0,E_0^*}\,dr-\langle S(t,s)u^{n,m}(s),x^*\rangle_{E_0,E_0^*} \\
&\quad +\langle u^{n,m}(s),x^*\rangle_{E_0,E_0^*}-\int_s^t\langle S(t,\sigma)F(\sigma,u(\sigma)),x^*\rangle_{E_0,E_0^*}\,d\sigma \\
&\quad +\int_s^t\langle F(\sigma,u(\sigma)),x^*\rangle_{E_0,E_0^*}\,d\sigma.
\end{aligned} \tag{6.7.3}
$$

If we compare the last term of (6.7.2) with the last term of (6.7.3), we obtain

$$
\begin{aligned}
\langle u^{n,m}(t),x^*\rangle &= \langle u^{n,m}(s),x^*\rangle+\int_s^t\langle u^{n,m}(r),A^*(r)x^*\rangle_{E_0,E_0^*}\,dr \\
&\quad +\int_s^t\langle F(\sigma,u(\sigma)),x^*\rangle_{E_0,E_0^*}\,d\sigma \\
&\quad +m\sum_{i=1}^{m}\int_s^t\langle B_n(r)u_i(W(r+1/n)-W(r))u_i,x^*\rangle\,dr
\end{aligned}
$$

If we write

$$\langle u_t^n,y\rangle=\langle u_t^n-u_t^{n,m},y\rangle+\langle u_t^{n,m},y\rangle,$$

then we obtain

$$
\begin{aligned}
&\langle u_t^n,x^*\rangle-\langle u_s^n,x^*\rangle-\int_s^t\langle A^*(r)x^*,u_r^n\rangle\,dr-\int_s^t\langle F(r,X(r)),x^*\rangle\,dr \\
&= \Big\langle -\sum_{i=1}^{m}\int_0^t S(t,s)B_n(s)h_i(W(s+1/m)-W(s))h_i\,ds
\end{aligned}
$$

$$+\int_0^t S(t,s)B_n(s)\,dW^-(s),x^*\Big\rangle \tag{6.7.4}$$

$$-\Big\langle \int_0^s S(s,r)B_n(r)\,dW^-(r)$$

$$-\sum_{i=1}^m \int_0^s S(s,r)B_n(r)h_i(W(r+1/m)-W(r))h_i\,dr,x^*\Big\rangle \tag{6.7.5}$$

$$-\int_s^t \Big\langle A^*(r)x^*, \int_0^r S(r,\sigma)B_n(\sigma)\,dW^-(\sigma)$$

$$-\sum_{i=1}^m \int_0^r S(r,\sigma)B_n(\sigma)h_i(W(\sigma+1/m)-W(\sigma))h_i\,d\sigma\Big\rangle\,dr \tag{6.7.6}$$

$$+\,m\sum_{i=1}^m \int_s^t \langle B_n(r)h_i(W(r+1/m)-W(r))h_i,x^*\rangle\,dr. \tag{6.7.7}$$

We will show that the right hand side of the above equation converges as $m\to\infty$ to $\Big\langle \int_s^t B_n(r)\,dW(r),x^*\Big\rangle$. Note that the stochastic integral is an Itô integral.

By Corollary 6.25 with $\alpha=0$, we obtain readily that (6.7.4) and (6.7.5) converge to zero almost surely, as $m\to\infty$. Moreover, (6.7.7) converges to $\Big\langle \int_s^t B_n(r)\,dW(r),x^*\Big\rangle$ as $m\to\infty$. With Hölder's inequality and the hypotheses on A^*, we can estimate the L^1-norm of (6.7.6),

$$\begin{aligned}\mathbb{E}\Big|\int_s^t \Big\langle A^*(r)x^*, \int_0^r S(r,\sigma)B_n(\sigma)\,dW^-(\sigma)&\\ -\sum_{i=1}^m \int_0^r S(r,\sigma)B_n(\sigma)h_i(W(\sigma+\tfrac{1}{m})-W(\sigma))h_i\,d\sigma\Big\rangle\,dr\Big|&\\ \le C\mathbb{E}\int_s^t \Big\|\int_0^r S(r,\sigma)B_n(\sigma)\,dW^-(\sigma)&\\ -\sum_{i=1}^m \int_0^r S(r,\sigma)B_n(\sigma)h_i(W(\sigma+\tfrac{1}{m})-W(\sigma))h_i\,d\sigma\Big\|^2\,dr.&\end{aligned}$$

The right hand side converges to zero, by Theorem 6.23 (2) and Remark 6.12. We obtain

$$\langle u_t^n,x^*\rangle = \langle u_s^n,x^*\rangle + \int_s^t \langle A^*(r)x^*,u_r^n\rangle\,dr + \int_s^t \langle F(r,X(r)),x^*\rangle\,dr$$
$$+\Big\langle \int_s^t B_n(r)\,dW(r),x^*\Big\rangle.$$

Again, writing

$$\langle u_t,x^*\rangle = \langle u_t-u_t^n,x^*\rangle + \langle u_t^n,x^*\rangle,$$

we have

$$\begin{aligned}
&\langle u_t, x^*\rangle - \langle u_s, x^*\rangle - \int_s^t \langle u_r, A^*(r)x^*\rangle\, dr - \int_s^t \langle F(r,u(r)), x^*\rangle\, dr \\
&= \Big\langle \int_0^t S(t,s)(B(s,u(s)) - B_n(s,u(s)))\, dW^-(s), x^* \Big\rangle && (6.7.8)\\
&\quad - \Big\langle \int_0^s S(s,r)(B(r,u(r)) - B_n(r,u(r)))\, dW^-(r), x^* \Big\rangle && (6.7.9)\\
&\quad - \int_s^t \Big\langle \int_0^r S(r,\sigma)(B(\sigma,u(\sigma)) - B_n(\sigma,u(\sigma)))\, dW^-(\sigma), && (6.7.10)\\
&\qquad\qquad A^*(r)x^* \Big\rangle\, dr \\
&\quad + \Big\langle \int_s^t B_n(r,u(r))\, dW(r), x^* \Big\rangle.
\end{aligned}$$

By Corollary 6.25, it follows that (6.7.8) and (6.7.9) both converge to zero, almost surely. By the hypotheses on A^*, Hölder's inequality and Theorem 6.29, it follows that (6.7.10) converges to zero in $L^1(\Omega)$. Finally, we also have

$$\Big\langle \int_s^t B_n(r,u(r))\, dW(r), x^* \Big\rangle \to \int_0^t \langle B(s,u(s)), x^*\rangle_{E_0,E_0^*}\, dW(s),$$

as $n \to \infty$. This proves the result. □

The following theorem states existence and uniqueness of mild solutions. Its proof is similar to [68, Theorem 5.4] and is therefore omitted.

Theorem 6.33. *Assume hypotheses (A.1) – (A.3). Then problem* (6.7.1) *has a unique a forward mild solution.*

6.8 Examples

6.8.1 Example 1

Let S be either $\mathbb{R}^n$ or a bounded domain in $\mathbb{R}^n$ with smooth boundary. Consider the second order stochastic partial differential operator

$$A(t,s,\omega) = \sum_{i,j=1}^n a_{ij}(t,s,\omega)\partial_i\partial_j + \sum_{i=1}^n b_i(t,s,\omega)\partial_i + c(t,s,\omega),$$

where the coefficients a_{ij}, b_i and c are all measurable from $\overline{S} \times [0,T] \times \Omega \to \mathbb{R}$. Assume that for each $s \in \overline{S}$, $a_{ij}(s), b_i(s)$ and $c(s)$ are adapted random variables. Assume that (a_{ij}) is symmetric and that there exists a $\kappa > 0$ such that

$$\kappa^{-1}|\xi|^2 \le a_{ij}(t,s,\omega)\xi_i\xi_j \le \kappa|\xi|^2, \qquad s \in \mathbb{R}^n,\ t \in [0,T],\ \xi \in \mathbb{R}^n.$$

Also assume that the coefficients are continuous and uniformly bounded in $\overline{S} \times [0,T]$, and they verify the following Hölder condition: there exists a $K > 0$ and an $\alpha > 0$ such that for all $s, s' \in \overline{S}$, $t, t' \in [0,T]$,

$$\begin{aligned}|a_{ij}(t,s) - a_{ij}(t',s')| &\le K(|s'-s|^\alpha + |t'-t|^{\alpha/2}),\\ |b_i(t,s) - b_i(t,s')| &\le K(|s'-s|^\alpha),\\ |c(t,s) - c(t,s')| &\le K(|s'-s|)^\alpha.\end{aligned}$$

We assume that $a_{ij}(t,\cdot)$ is continuously differentiable with uniformly bounded partial derivatives. Regarding Malliavin differentiability, we assume that for each (t,s), $a_{ij}(t,s), \partial_i a_{ij}(t,s), b_i(t,s)$ and $c(t,s)$ are all in $\mathbb{D}^{1,p}$, $p > 2$. Moreover, the norm of the derivatives

$$\|Da_{ij}(t,s)\|_H, \|D(\partial_i a_{ij}(t,s))\|_H, \|Db_i(t,s)\|_H, \|Dc(t,s)\|_H,$$

regarded as elements in $L^p(\Omega; L^2(0,T))$, are bounded by a nonnegative process $\Phi \in L^p(\Omega; L^2(0,T))$. Finally, we assume that there exists a $C > 0$ such that for all $\omega \in \Omega$, $r \in [0,T]$,

$$\sum_{k=1}^{\infty} \sup_{t,s} \Big(|D_r^{h_k} a_{ij}(t,s)|^2 + |D_r^{e_k}(\partial_i a_{ij}(t,s))|^2 + |D_r^{e_k} b_i(t,s)|^2 + |D_r^{e_k} c(t,s)|^2 \Big) \le C.$$

Consider the stochastic partial differential equation

$$\begin{aligned}du(t,s) &= \big(A(t,s,\omega)u(t,s) + f(t,s,u(t,s))\big)\, dt\\ &\qquad + g(t,s,u(t,s))\, dW(t,s),\ t \in (0,T],\ s \in S,\\ u(0,s) &= u_0(s),\ s \in S.\end{aligned}$$

Here, f, g and W satisfy the same hypotheses as in Example 5.30. As in Example 5.30, the stochastic partial differential equation can be rewritten to a stochastic evolution equation, where $A(t,\omega)$ is a closed linear operator on $L^p(S)$, $p \ge 2$. By [68, Proposition 6.7], there exists a random evolution system $S(t,s)$ verifying (H) and such that (A.1) holds. Also in 5.30 it is proved that (A.2) and (A.3) hold. Hence we obtain the following result.

Theorem 6.34. *Let $q > 2, p \ge 2$ and $u_0 : \Omega \to L^p(S)$ be $\mathscr{F}_0$-measurable. There exists a unique forward mild solution $u \in L^q(\Omega; C([0,T]; L^p(S)))$.*

6.8.2 Example 2

Let E_0, E_1 be Banach spaces such that $E_1 \subset E_0$ is dense. In this example, we consider problem (6.7.1) where the drift $A = A(t,\omega)$ is given by the linear operator

$$A(t,\omega) = \sum_{j=1}^{n} a_j(t,\omega) B_j,$$

where $B_j \in \mathscr{L}(E_1, E_0)$, and a_j are $L^\infty(0,T)$-valued smooth random variables of the form

$$a_j = \sum_{k=1}^{m} f_k(W(h_1^k), \dots, W(h_n^k)) \otimes \varphi_k$$

such that $f_k \in C_b^\infty(\mathbb{R}^n)$ and $h_i^j \in C([0,T]) \otimes H)$ and $\varphi_i \in L^\infty(0,T)$. A special case is the situation $A(t,\omega) = \sum_{i,j=1}^n a_{ij}(t,\omega) D_i D_j$.

We assume that $A : [0,T] \times \Omega \to \mathscr{L}(E_1, E_0)$ is uniformly bounded with bound M, strongly measurable and adapted. Set

$$\mathscr{A} = \overline{\text{co}}\{A(t,\omega); t \in [0,T], \omega \in \Omega\}.$$

Let $\theta \in (\pi/2, \pi)$ and

$$\Sigma_\theta = \{\lambda \in \mathbb{C} : \ |\arg \lambda| < \theta\}.$$

Let us impose the following assumptions on the drift:

(A1) For all $B \in \mathscr{A}$, one has

$$\Sigma_\theta \cup \{0\} \subset \rho(B),$$

and

$$\|R(\lambda, B)\|_{\mathscr{L}(E_0)} \le \frac{M}{|\lambda| + 1},$$

(A2) There exists a constant $c \ge 1$ such that for all $x \in E_1$ and all $B_1, B_2 \in \mathscr{A}$,

$$\|B_1 x\|_{E_0} \le c \|B_2 x\|_{E_0},$$

(A3) There are $\lambda, \mu \in \Sigma_\theta \cup \{0\}$ such that for all $B_1, B_2 \in \mathscr{A}$, one has

$$R(\lambda, B_1) R(\mu, B_2) = R(\mu, B_2) R(\lambda, B_1).$$

The following lemma concerns differentiability of the resolvent.

Lemma 6.35. *Let $\lambda \in \Sigma_\theta \cup \{0\}$. Consider a map $B : [0,T] \to \mathscr{L}(E_1, E_0)$, such that $\|R(\lambda, B(t))\| \le \frac{M}{|\lambda|+1}$ for all $t \in [0,T]$. If B is differentiable in $[0,T]$, then $R(\lambda, B)$ is differentiable in $[0,T]$, and*

$$\frac{d}{dt} R(\lambda, B(t)) = R(\lambda, B(t)) B'(t) R(\lambda, B(t)).$$

Proof. First observe that

$$R(\lambda, B(t+h)) - R(\lambda, B(t)) = R(\lambda, B(t+h))(B(t+h) - B(t)) R(\lambda, B(t)).$$

It follows that $R(\lambda, B(t+h)) \to R(\lambda, B(t))$ as $h \to 0$. Consequently,

$$\lim_{h \to 0} R(\lambda, B(t+h)) \frac{B(t+h) - B(t)}{h} R(\lambda, B(t)) = R(\lambda, B(t)) B'(t) R(\lambda, B(t)).$$

□

For $B \in \mathscr{A}$ and $t > 0$, the operator $e^{tB} : E_0 \to E_0$ is defined by

$$e^{tB} := \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{t\lambda} R(\lambda, B)\, d\lambda,$$

where $\gamma_{r,\eta}$, $r > 0$ and $\eta \in (\pi/2, \theta)$, is the counter-clockwise oriented curve given by

$$\gamma_{r,\eta} := \{\lambda \in \mathbb{C} :\ |\arg \lambda| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} :\ |\lambda| = r,\ -\eta \leq \arg \lambda \leq \eta\}.$$

For $t = 0$ we set $e^{tB} = I$ on E_0.

The operator e^{tB} satisfies the fundamental properties of the exponential function: it is well-known that one has $\frac{d}{dt} e^{tB} = B e^{tB}$ (see [73, Proposition 2.1.1]). Moreover, given $s, t \in \mathbb{R}_+$, one can apply this to the function $f : [0,1] \to \mathscr{L}(E_0)$ given by

$$f(r) = e^{r(tB_1 + sB_2)} e^{(1-r)tB_1} e^{(1-r)sB_2},$$

and obtain $\frac{d}{dr} f(r) = 0$. Here, we have used (A3). It follows that f is constant, and in particular $f(0) = f(1)$. The last observation implies $e^{tB_1 + sB_2} = e^{tB_1} e^{sB_2}$.

By Cauchy's differentiation formula, i.e.,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma_{r,\eta}} \frac{f(\lambda)}{(\lambda - a)^{n+1}}\, d\lambda,$$

applied to the function $f(x) = e^x$ and with $x = tB$, one obtains

$$e^{tB} = f^{(n)}(tB) = \frac{n!}{2\pi i} \int_{\gamma_{r,\eta}} e^{\lambda} R(\lambda, tB)^{n+1}\, d\lambda, \qquad n \in \mathbb{N}.$$

Next, we define $A_{st} := \frac{1}{t-s} \int_s^t A(r)\, dr$ for $0 \leq s < t \leq T$. Note that $A_{st} \in \mathscr{A}$. Set

$$S(t,s) := e^{(t-s)A_{st}},$$

with $S(t,t) = I$. Recall that $\Delta := \{(s,t) \in [0,T]^2;\ s \leq t.\}$.

Lemma 6.36. *The operator $S : \Delta \times \Omega \to \mathscr{L}(E_0)$ is uniformly bounded: there exists an $\tilde{M} > 0$ such that for all $s \leq t$ and $\omega \in \Omega$, $\|S(t,s)\| \leq \tilde{M}$.*

Proof. Consider the counterclockwise oriented curve $\gamma'_{r,\eta}$ in $\mathbb{C}$, where $\gamma'_{r,\eta} = \gamma'_1 \cup \gamma'_2 \cup \gamma'_3$ and

$$\begin{aligned} \gamma'_1 &:= \{\rho e^{-i\eta},\ -\infty \leq \rho \leq r\}, \\ \gamma'_2 &:= \{r e^{i\alpha},\ -\eta \leq \alpha \leq \eta\}, \\ \gamma'_3 &:= \{\rho e^{i\eta},\ r \leq \rho \leq \infty\}. \end{aligned}$$

By Cauchy's theorem, it follows that

$$\frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{(t-s)\lambda} R(\lambda, A_{st})\, d\lambda = \frac{1}{2\pi i} \int_{\gamma'_{r,\eta}} e^{(t-s)\lambda} R(\lambda, A_{st})\, d\lambda.$$

The integral is independent of $r > 0$. Set $r = \frac{1}{t-s}$ and substitute $x = \lambda(t-s)$ to obtain

$$S(t,s) = \frac{1}{2\pi i}\int_{\gamma'_{r,\eta}} e^{(t-s)\lambda} R(\lambda, A_{st})\, d\lambda = \frac{1}{2\pi i}\int_{\gamma'_{1,\eta}} e^{x} R(\tfrac{x}{t-s}, A_{st})\frac{1}{t-s}\, dx. \tag{6.8.1}$$

A straightforward calculation leads to the estimate

$$\|S(t,s)\| \le \frac{1}{\pi}\Big(\int_1^\infty e^{-c\rho}\frac{M}{\rho + (t-s)}\, d\rho + \frac{1}{2}\int_\eta^\eta e^{\cos\alpha}\frac{M}{1+(t-s)}\, d\alpha\Big) =: \tilde{M}$$

□

Proposition 6.37. *The map $S : \Delta\times\Omega \to \mathscr{L}(E_0)$, is a random evolution system as in Definition 6.21. Moreover, $A(t)$ is its generator. In fact, we have*

1. $\frac{d}{dt}S(t,s) = A(t)S(t,s)$, *$t > s$, on E_0*
2. $\frac{d}{ds}S(t,s) = -S(t,s)A(s)$, *$t > s$, on E_1*
3. $A(t)S(t,s) = S(t,s)A(t)$ *and* $A(s)S(t,s) = S(t,s)A(s)$, *on E_1.*

Proof. The facts that $S(x) : \Delta\times\Omega \to E$ is measurable, and that $S(t,s)$ is strongly $\mathscr{F}_t$-measurable, follow from the assumption that A is strongly measurable and adapted.
Note that for $r \le s \le t$ we have $(t-s)A_{st} + (s-r)A_{rs} = (t-r)A_{rs}$. Hence by the properties of the exponential, $S(t,s)S(s,r) = S(t,r)$.
Before we prove strong continuity of $S(t,s)$, we first prove the above three properties. For this, note that the map $f : (s,t) \mapsto (t-s)A_{st}$ is differentiable a.e. for $t > s$, in both variables, and $\frac{d}{dt}f(s,t) = A(t)$, $\frac{d}{ds}f(s,t) = -A(s)$. Again from the properties of the exponential function, it follows that $S(t,s)$ is differentiable a.e., in both parameters. In particular, $\frac{d}{dt}S(t,s) = A(t)S(t,s)$ and $\frac{d}{ds}S(t,s) = -S(t,s)A(s)$. Moreover, by assumption (A3) we obtain $A(t)S(t,s) = S(t,s)A(t)$ and $A(s)S(t,s) = S(t,s)A(s)$.

For the strong continuity, let $x \in E_1$. Fix $s \in [0,T]$ and suppose $t_n \downarrow t$. Then, by Lemma 6.36,

$$\begin{aligned}\|S(t_n,s)x - S(t,s)x\| &= \Big\|\int_t^{t_n} A(r)S(r,t)x\, dr\Big\| = \Big\|\int_t^{t_n} S(r,t)A(r)x\, dr\Big\| \\ &\le \tilde{M}\int_t^{t_n} \|A(r)x\|\, dr \to 0,\end{aligned}$$

as $n\to\infty$. Similarly, as $t_n \uparrow t$, we obtain $S(t_n,s)x \to S(t,s)x$, in E_0. A similar computation shows that if t is fixed and $s_n \to s$, then $S(t,s_n)x \to S(t,s)x$ as $n\to\infty$. Combining both results, we can use the triangle inequality to obtain $\|S(t_n,s_n)x - S(t,s)x\| \to 0$, whenever $(t_n,s_n) \to (t,s) \in \Delta$ for all $x \in E_1$. By density, the same result holds for general $x \in E_0$. □

Remark 6.38. By [56, Theorem 6.22], we can conclude that the adjoint $A^*(t,\omega)$ satisfies (A1) - (A3), but with $\mathscr{A}^*$ instead of $\mathscr{A}$. Following the proof of Proposition 6.37 therefore leads to the conclusion that $\frac{d}{dt}S(t,s)^* = S(t,s)^*A(t)^*$.

Now let us prove the following results on Malliavin differentiability of the resolvent. We start with the bounded case, Lemma (6.39), from which the unbounded case, Theorem 6.40, will follow.

Lemma 6.39. *Suppose $B : \Omega \to \mathscr{L}(E_0)$ is a uniformly bounded random variable. Moreover, assume that $\Sigma_\theta \cup \{0\} \subset \rho(B)$ and assume there exists an $M > 0$ such that for all $\lambda \in \Sigma_\theta \cup \{0\}$ and $\omega \in \Omega$, $\|R(\lambda, B)\| \le M$. If $B \in \mathbb{D}_s^{1,p}(\mathscr{L}(E_0))$ such that $\mathscr{D}B \in L^p(\Omega; \gamma(\mathscr{H}, \mathscr{L}(E_0)))$, then $R(\lambda, B) \in \mathbb{D}_s^{1,p}(\mathscr{L}(E_0))$ for each $\lambda \in \Sigma_\theta \cup \{0\}$ and in that case we have*

$$\mathscr{D}R(\lambda, B) = R(\lambda, B)(\mathscr{D}B)R(\lambda, B). \tag{6.8.2}$$

Proof. Let $\lambda \in \Sigma_\theta \cup \{0\}$ be so big, that $\max_\omega \|B(\omega)\| < |\lambda|$. Then we have

$$R(\lambda, B) = \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{B^n}{\lambda^n}.$$

Define $R_k(\lambda, B) := \frac{1}{\lambda} \sum_{n=0}^{k} \frac{B^n}{\lambda^n}$. By a Banach valued extension of [68, Lemma 2.5] it follows that $R_k(\lambda, B) \in \mathbb{D}_s^{1,p}(\mathscr{L}(E_0))$. In particular, we have

$$\mathscr{D}(B^n) = \sum_{i=0}^{n-1} B^i(\mathscr{D}B)B^{n-1-i}.$$

Fix $x \in E_0$. Note that $R_k(\lambda, B)x \to R(\lambda, B)x$ in $L^p(\Omega; E_0)$, as $k \to \infty$. Moreover, observe that pointwise in ω,

$$\|D(B^n x)\|_{E_0} \le n\|B\|^{n-1}_{\mathscr{L}(E_0)} \|\mathscr{D}B\|_{\gamma(\mathscr{H}, \mathscr{L}(E_0))} \|x\|_{E_0}.$$

It follows that for $k \ge m$,

$$\|D(R_k x) - D(R_m x)\| \le \|x\|_{E_0} \frac{\|\mathscr{D}B\|_{\gamma(\mathscr{H}, \mathscr{L}(E_0))}}{|\lambda|^2} \sum_{n=m}^{k} n \Big(\frac{\|B\|_{\mathscr{L}(E_0)}}{|\lambda|} \Big)^{n-1},$$

which converges to 0 as $k, m \to \infty$. Hence, by closedness of D, it follows that $R(\lambda, B)x \in \mathbb{D}^{1,p}(E_0)$. A similar computation shows that $\mathscr{D}R(\lambda, B) \in L_s^p(\Omega; \mathscr{L}(E_0, \gamma(\mathscr{H}, E_0)))$. Hence $R(\lambda, B) \in \mathbb{D}_s^{1,p}(\mathscr{L}(E_0))$.

Next, by assumption we have $\frac{1}{\|R(\mu,B)\|} > \frac{1}{M}$ for all $\mu \in \Sigma_\theta \cup \{0\}$. Therefore, given $|\lambda| \in \Sigma_\theta \cup \{0\}$ such that $\max_\omega \|B(\omega)\| < |\lambda|$, for any $\mu \in \mathbb{C}$ that is within a ball of radius $\frac{1}{M}$ around λ, one can write

$$R(\mu, B) = \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\lambda, B)^{n+1}.$$

By closedness, one concludes that $R(\mu, B) \in \mathbb{D}_s^{1,p}(\mathscr{L}(E_0)$ for such μ. One can repeat the above strategy to obtain $R(\mu, B) \in \mathbb{D}_s^{1,p}(\mathscr{L}(E_0))$ for all $\mu \in \Sigma_\theta \cup \{0\}$.

Finally, identity (6.8.2) follows from the product rule

$$0 = \mathscr{D}(R(\lambda, B)(\lambda - B)) = (\mathscr{D}R(\lambda, B))(\lambda - B) + R(\lambda, B)(\mathscr{D}(\lambda - B)),$$

by applying $R(\lambda, B)$ on the right side in the above equation. ☐

Theorem 6.40. *Assume (A1) - (A3), and suppose* $B : \Omega \to \mathscr{L}(E_1, E_0)$ *is such that* $B \in \mathscr{A}$ *almost surely. If* $B \in \mathbb{D}^{1,p}_s(\mathscr{L}(E_1, E_0))$ *such that* $\mathscr{D}B \in L^p(\Omega; \gamma(\mathscr{H}, \mathscr{L}(E_1, E_0)))$*, then* $R(\lambda, B) \in \mathbb{D}^{1,p}_s(\mathscr{L}(E_0))$ *for each* $\lambda \in \Sigma_\theta \cup \{0\}$*. Moreover,* (6.8.2) *holds.*

Proof. Consider the Yosida approximation $B_n(\omega) := nB(\omega)R(n, B(\overline{\omega}))$, where $\overline{\omega} \in \Omega$ is kept fixed. We will show that B_n satisfies all properties from Lemma 6.39. First of all, note that $B_n : \Omega \to \mathscr{L}(E_0)$ is uniformly bounded: indeed, we have

$$\begin{aligned}\|B_n(\omega)\| &= \|nB(\omega)R(n, B(\overline{\omega}))\| \leq C\|nB(\overline{\omega})R(n, B(\overline{\omega}))\| \\ &= C\|n^2R(n, B(\overline{\omega})) - nI\| \leq C\Big(\frac{n^2M}{n+1} + n\Big).\end{aligned}$$

Second of all, we have $B_n \in \mathbb{D}^{1,p}_s(\mathscr{L}(E_0))$. Indeed, for any $x \in E_0$ we have $R(n, B(\overline{\omega}))x =: y \in E_1$, and hence $B_nx = nBy$ belongs to $\mathbb{D}^{1,p}(E_0)$. Also, $\mathscr{D}B_n \in L^p_s(\Omega; \mathscr{L}(E_0, \gamma(\mathscr{H}, E_0)))$. Moreover, from $\mathscr{D}B \in L^p(\Omega; \gamma(\mathscr{H}, \mathscr{L}(E_1, E_0)))$ it follows readily that $\mathscr{D}B_n \in L^p(\Omega; \gamma(\mathscr{H}, \mathscr{L}(E_0)))$.

Next, we will show that $\Sigma_\theta \cup \{0\} \subset \rho(B_n)$. For that, observe that

$$\lambda - B_n = (\lambda n - nB - \lambda B(\overline{\omega}))R(n, B(\overline{\omega})). \tag{6.8.3}$$

Hence $\lambda \in \rho(B_n)$ if and only if $\lambda \in \rho(B + \frac{\lambda}{n}B(\overline{\omega}))$. Fix $\lambda \in \Sigma_\theta \cup \{0\}$. The operator $\tilde{B} := \frac{\lambda}{n}B(\overline{\omega})$ is $B-$*bounded*, i.e.,

$$\|\tilde{B}x\| \leq c\tfrac{|\lambda|}{n}\|Bx\|, \qquad x \in E_1$$

where c is the constant from (A2). By [41, Lemma III.2.6], for n large enough, $\Sigma_\theta \cup \{0\} \subset \rho(B + \frac{\lambda}{n}B(\overline{\omega}))$, and the estimate

$$\|R(\mu, B + \tfrac{\lambda}{n}B(\overline{\omega}))\|_{\mathscr{L}(E_0)} \leq \frac{1}{1 - c'\frac{|\lambda|}{n}}\|R(\mu, B)\|_{\mathscr{L}(E_0)}, \qquad \mu \in \Sigma_\theta \cup \{0\},$$

holds. In particular, $\Sigma_\theta \cup \{0\} \subset \rho(B_n)$.

Since the domain of $B + \frac{\lambda}{n}B(\overline{\omega})$ equals E_1, we have

$$\|x\|_{E_1} \leq C(\|x\|_{E_0} + \|(B + \tfrac{\lambda}{n}B(\overline{\omega}))x\|_{E_0}).$$

It follows that for $\mu \in \Sigma_\theta \cup \{0\}$,

$$\|R(\mu, B + \tfrac{\lambda}{n}B(\overline{\omega}))\|_{\mathscr{L}(E_0,E_1)} \leq C(\|R(\mu, B + \tfrac{\lambda}{n}B(\overline{\omega}))\|_{\mathscr{L}(E_0)} + \tilde{C}) \leq C', \tag{6.8.4}$$

where $\tilde{C}, C'$ are constants depending on M. Put $\mu = \lambda \in \Sigma_\theta \cup \{0\}$ and use identity (6.8.3) and inequality (6.8.4) to obtain

$$\begin{aligned}\|R(\lambda, B_n)\|_{\mathscr{L}(E_0)} &= \frac{1}{n}\|(n - B(\overline{\omega}))R(\lambda, B + \tfrac{\lambda,n}{B}(\overline{\omega}))\|_{\mathscr{L}(E_0)} \\ &\leq \frac{C'}{n}\|n - B\|_{\mathscr{L}(E_1,E_0)} \leq C,\end{aligned}$$

with C' independent of n. Consequently, we can apply Lemma 6.39 to B_n. We conclude that for every $\lambda \in \rho(B)$, there exists an $N \in \mathbb{N}$, such that for all $n > N$, $\lambda \in \rho(B_n)$ and $\|R(\lambda, B_n)\| \leq C$.

We will show that $R(\lambda, B_n) \to R(\lambda, B)$ for all $\lambda \in \rho(B)$. Choose $\lambda \in \rho(B)$, $N \in \mathbb{N}$ such that $\|R(\lambda, B_n)\| \leq C$ for all $n \geq N$, and choose $x \in E_1$ arbitrarily. For $n \geq N$, set $y_n := (\lambda - B_n)x$ and $y := (\lambda - B)x$. Then by the triangle inequality,

$$\|R(\lambda, B_n)y - R(\lambda, B)y\| \leq \|R(\lambda, B_n)(y - y_n)\|$$

Observe that $\|y - y_n\|_{E_0} = \|Bx - B_n x\|_{E_0} \to 0$, and hence $\|R(\lambda, B_n)y - R(\lambda, B)y\| \to 0$, as $n \to \infty$. Also, $\{(\lambda - B)x;\ x \in E_1\}$ is dense in E_0, since $\lambda - B$ is surjective. By an approximation argument, it follows that $R(\lambda, B_n) \to R(\lambda, B)$.

Finally, it follows that for every $x \in E_0$, $R(\lambda, B)x \in \mathbb{D}^{1,p}(E_0)$. Indeed, $R(\lambda, B_n)x \to R(\lambda, B)x$ in E_0, where $R(\lambda, B_n)x \in \mathbb{D}^{1,p}(E_0)$ by Lemma 6.39. Moreover, by (6.8.2), one obtains $D(R(\lambda, B_n)x) \to R(\lambda, B)(\mathscr{D}B)R(\lambda, B)x$. The result now follows from the closedness of the operator D. □

Theorem 6.41. *The evolution system $S(t,s)$ satisfies hypothesis (H1) - (H3).*

Proof. Observe that $A(t) \in \mathbb{D}^{1,p}_s(\mathscr{L}(E_1, E_0))$. Also under the assumptions one can apply Hille's theorem to obtain $A_{st} \in \mathbb{D}^{1,p}_s(\mathscr{L}(E_1, E_0))$. Moreover, $\mathscr{D}A_{st} \in L^p(\Omega; \gamma(\mathscr{H}, \mathscr{L}(E_1, E_0)))$. Hence Theorem 6.40 yields $DR(\lambda, A_{st}) = D(A_{st})R(\lambda, A_{st})^2$. A computation similar to (6.8.1), (6.8.1), proves that $\lambda \to e^{(t-s)\lambda}\mathscr{D}R(\lambda, A_{st})$ is Bochner integrable on $\gamma_{r,n}$. Therefore, another application of Hille's theorem yields $S(t,s)$ is Malliavin differentiable, in the strong sense, with

$$\begin{aligned}\mathscr{D}S(t,s) &= \frac{1}{2\pi i}\int_{\gamma_{r,n}} e^{(t-s)\lambda}\mathscr{D}R(\lambda, A_{st})\, d\lambda \\ &= \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{(t-s)\lambda}R(\lambda, A_{st})(\mathscr{D}A_{st})R(\lambda, A_{st})d\lambda\end{aligned} \tag{6.8.5}$$

By construction of $A(r)$, we have $D(A(r))$ and consequently $\mathscr{D}S(t,s)$ belong to $L^p(\Omega; L^2(0,T;\gamma(H, \mathscr{L}(E_0))))$.

Finally, we prove (6.5.1). By Lemma 6.36, it suffices to show

$$\sup_{t\in[0,T]} \mathbb{E}\int_0^t \|\mathscr{D}S(t,s)\|^p_{\gamma(\mathscr{H},\mathscr{L}(E_0))} < \infty.$$

This can be done by estimating the integral in (6.8.5) with straightforward complex integration techniques.

Finally, by construction of $A(t)$, we see that $\mathscr{D}S(t,s)(s)$ is well-defined for all $s \in [0,t]$, and (H2) readily follows. Also (H3) follows by the assumptions on $A(t)$. □

Theorem 6.42. *If F and B satisfy (A.2) and (A.3), respectively, then problem 6.7.1 admits a unique weak solution.*

Proof. By Remark 6.38 and the assumptions on A, (A.1) is satisfied. The result then follows from Proposition 6.32 and 6.33. □

6.9 Appendix: Two technical lemmas

Here, we prove two technical results that are needed in Theorem 6.27. Recall that $(P(t))_{t\geq 0}$ is the Ornstein-Uhlenbeck semigroup on $L^p(\Omega;X)$.

Lemma 6.43. *Let $S(t,s)$ be a random evolution system satisfying (H), and let $G\in\mathscr{S}_a$. Let $p>2$ and $\alpha\in(\frac{1}{p},\frac{1}{2})$. Consider Y, B and F_{km} as in* (6.5.3), (6.6.4) *and* (6.6.1)*, respectively. There exists a sequence $(\tau_i)_{i\geq 1}$ with $\tau_i\downarrow 0$ as $i\to\infty$, such that for all $s\leq t$ and all i,*

$$\begin{aligned}&\mathbb{E}e^{-\tau_i}n\int_0^s\langle \mathbf{1}_{[r,r+1/n]}P_{\tau_i}Y(t,\cdot),D(F'_{km}(P_{\tau_i}B(r,t)))\rangle_{\mathrm{Tr}}\,dr\\&\leq C\mathbb{E}\int_0^s\|B(r,t)\|_X^{p-2}\|Y(t,r)\|_{\gamma(H,X)}\Big\|\int_0^s(t-r)^{-\alpha}S(r,\sigma)G(\sigma)\,dW^-(\sigma)\Big\|_X\,dr.\end{aligned}\tag{6.9.1}$$

Proof. First note that for any $r\in[0,s]$, we can apply the chain rule [112, Proposition 3.8] to rewrite

$$\begin{aligned}&n\langle \mathbf{1}_{[r,r+1/n]}P_\tau Y(t,\cdot),D(F'_{km}(P_\tau B(r,t)))\rangle_{\mathrm{Tr}}\\&\quad=\langle\sqrt{n}\mathbf{1}_{[r,r+1/n]}P_\tau Y(t,\cdot),\sqrt{n}F''_{km}(P_\tau B(r,t))(\mathbf{1}_{[r,r+1/n]}D(P_\tau B(r,t)))\rangle_{\mathrm{Tr}}.\end{aligned}$$

By (6.3.1), we can write $B(s,t)=S(t,s)\int_0^s(t-\sigma)^{-\alpha}S(s,\sigma)G(\sigma)\,dW^-(\sigma)$. It follows that for any $h\in H$, using Lemma 6.7,

$$\begin{aligned}&(\sqrt{n}\mathbf{1}_{[r,r+1/n]}D(P_\tau B(r,t)))h\\&\quad=D\Big(P_\tau S(t,r)\Big(\int_0^r(t-\sigma)^{-\alpha}S(r,\sigma)G(\sigma)\,dW^-(\sigma)\Big)\Big)(\sqrt{n}\mathbf{1}_{[r,r+1/n]}h)\\&\quad=e^{-\tau}P_\tau\mathscr{D}^{\sqrt{n}\mathbf{1}_{[r,r+1/n]}h}S(t,r)\Big(\int_0^r(t-\sigma)^{-\alpha}S(r,\sigma)G(\sigma)\,dW^-(\sigma)\Big).\end{aligned}$$

From (H1) we have that $\sqrt{n}\mathbf{1}_{[s,s+1/n]}D(P(\tau)B(s))$ belongs to $\gamma(0,T;H,X)$, and we may write

$$\begin{aligned}&(\sqrt{n}\mathbf{1}_{[r,r+1/n]}D(P_\tau B(r,t)))h\\&\quad=\int_0^T\sqrt{n}\mathbf{1}_{[r,r+1/n]}(\rho)e^{-\tau}P_\tau\mathscr{D}_\rho S(t,r)\\&\qquad\qquad\times\Big(\int_0^r(t-\sigma)^{-\alpha}S(r,\sigma)G(\sigma)\,dW^-(\sigma)\Big)h(\rho)\,d\rho.\end{aligned}$$

Therefore, with the properties of F_{km}, (6.6.2), we estimate

$$\begin{aligned}
&e^{-\tau}\mathbb{E}n\int_0^s \langle \mathbf{1}_{[r,r+1/n]}P_\tau Y(t,\cdot), D(F'_{km}(P_\tau B(r,t)))\rangle_{\mathrm{Tr}}\, ds\\
&\le C\mathbb{E}\int_0^s \|P_\tau B(r,t)\|_X^{p-2}\|\sqrt{n}\mathbf{1}_{[r,r+1/n]}P_\tau Y(t,\cdot)\|_{\gamma(\mathscr{H},X)}\\
&\quad\times \|\sqrt{n}\mathbf{1}_{[r,r+1/n]}D(P_\tau B(r,t))\|_{\gamma(\mathscr{H},X)}\\
&\le C\mathbb{E}\int_0^s \|P_\tau B(r,t)\|_X^{p-2}\Big(n\int_r^{r+1/n}\|P_\tau Y(t,\sigma)\|^2_{\gamma(H,X)}\,d\sigma\Big)^{1/2}\\
&\quad\times\Big(n\int_r^{r+\frac{1}{n}}\Big\|P_\tau \mathscr{D}_\rho S(t,r)\Big(\int_0^r (t-\sigma)^{-\alpha}S(r,\sigma)G(\sigma)\,dW^-(\sigma)\Big)\Big\|_\gamma^2\,d\rho\Big)^{1/2}\\
&\le C\mathbb{E}\int_0^s \|P_\tau B(r,t)\|_X^{p-2}\|P_\tau Y(t,r)\|_{\gamma(H,X)}\|P_\tau D^- S(t,r)\|_{\gamma(H,\mathscr{L}(X))}\\
&\quad\times\Big\|\int_0^r (t-\sigma)^{-\alpha}S(r,\sigma)G(\sigma)\,dW^-(\sigma)\Big\|_X\,dr
\end{aligned} \tag{6.9.2}$$

Recall that $P(\tau)\zeta\to\zeta$ in $L^q(\Omega;X)$ for all $q\in[1,\infty)$ as $\tau\downarrow 0$. Therefore, we can find a sequence $(\tau_i)_{i=1}^\infty$ with $\tau_i\downarrow 0$ as $i\to\infty$ such that almost surely,

$$\begin{aligned}
&\|P_{\tau_i}B(r,t)\|_X^{p-2}\|P_{\tau_i}Y(t,r)\|_{\gamma(H,X)}\|P_{\tau_i}D^-S(t,r)\|_{\gamma(H,\mathscr{L}(X))}\\
&\qquad\to \|B(r,t)\|_X^{p-2}\|Y(t,r)\|_{\gamma(H,X)}\|D^-S(t,r)\|_{\gamma(H,\mathscr{L}(X))}.
\end{aligned} \tag{6.9.3}$$

Suppose $p_1,p_2,p_3,p_4\in(1,\infty)$ is such that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}=1$. Since $P(\tau)$ is a contraction on $L^q(\Omega;X)$ for all $q\in[1,\infty)$, we have by (6.9.2) and hypothesis (H3),

$$\begin{aligned}
&\mathbb{E}\int_0^s \|P_\tau B(r,t)\|_X^{p-2}\|P_\tau Y(t,r)\|_{\gamma(H,X)}\|P_\tau D^-S(t,r)\|_{\gamma(H,\mathscr{L}(X))}\\
&\qquad\times\Big\|\int_0^r (t-\sigma)^{-\alpha}S(r,\sigma)G(\sigma)\,dW^-(\sigma)\Big\|_X\,dr\\
&\le C\int_0^s (\mathbb{E}\|B(r,t)\|^{(p-2)p_1})^{\frac{1}{p_1}}(\mathbb{E}\|Y(t,r)\|^{p_2})^{\frac{1}{p_2}}\\
&\qquad\times\Big(\mathbb{E}\Big\|\int_0^r (t-\sigma)^{-\alpha}S(r,\sigma)G(\sigma)\,dW^-(\sigma)\Big\|^{p_4}\Big)^{\frac{1}{p_4}}\,dr\\
&\le C\int_0^s (\mathbb{E}\|B(r,t)\|^{(p-2)p_1})^{\frac{1}{p_1}}\\
&\qquad\times\Big(\mathbb{E}\Big\|\int_0^r (t-\sigma)^{-\alpha}S(r,\sigma)G(\sigma)\,dW^-(\sigma)\Big\|^{p_4}\Big)^{\frac{1}{p_4}}(t-r)^{-\alpha}\,dr
\end{aligned} \tag{6.9.4}$$

Note that for $r\in[0,s]$, by Corollary 6.25 one has

$$\begin{aligned}
&\Big(\mathbb{E}\Big\| \int_0^r (t-\sigma)^{-\alpha} S(r,\sigma)G(\sigma)\,dW^-(\sigma)\Big\|^{p_4}\Big)^{\frac{1}{p_4}} \\
&\quad = \Big(\mathbb{E}\Big\| \int_0^r (t-\sigma)^{-\alpha} S(r,\sigma)G(\sigma)\,dW(\sigma)\Big\|^{p_4}\Big)^{\frac{1}{p_4}} \\
&\qquad\qquad + \Big(\mathbb{E}\Big\| \int_0^r (t-\sigma)^{-\alpha} \sum_{k\geq 1} D^- S(r,\sigma)h_k G(\sigma)h_k\,d\sigma\Big\|^{p_4}\Big)^{\frac{1}{p_4}}
\end{aligned} \tag{6.9.5}$$

If we let $q > 1$ be so small that $2q\alpha < 1$ and let q' be its Hölder conjugate, then from Meyer's inequalities,

$$\begin{aligned}
&\mathbb{E}\Big\| \int_0^r (t-\sigma)^{-\alpha} S(r,\sigma)G(\sigma)\,dW(\sigma)\Big\|^{p_4} \\
&\quad \leq \mathbb{E}\Big(\int_0^r (t-\sigma)^{-2\alpha} \|S(r,\sigma)G(\sigma)\|^2_{\gamma(H,X)}\,d\sigma\Big)^{\frac{p_4}{2}} \\
&\qquad + \mathbb{E}\Big(\int_0^r (t-\sigma)^{-2\alpha} \|D(S(r,\sigma)G(\sigma)\|^2_{\gamma(\mathscr{H},\gamma(H,X))}\,d\sigma\Big)^{\frac{p_4}{2}} \\
&\quad \leq C + \mathbb{E}\Big(\Big(\int_0^r (t-\sigma)^{-2q\alpha}\,d\sigma\Big)^{1/q} \Big(\int_0^r \|D(S(r,\sigma)G(r))\|^{2q'}\,d\sigma\Big)^{1/q'}\Big)^{\frac{p_4}{2}} \leq C'
\end{aligned} \tag{6.9.6}$$

Moreover, from Lemma 6.22 we obtain

$$\begin{aligned}
&\mathbb{E}\Big\| \int_0^r (t-\sigma)^{-\alpha} \sum_{k\geq 1} D^- S(r,\sigma)u_k G(\sigma)u_k\,d\sigma\Big\|^{p_4} \\
&\qquad \leq \mathbb{E}\Big(\int_0^r (t-\sigma)^{-\alpha} \|D^- S(r,\sigma)\|_{\gamma(H,\mathscr{L}(X))} \|G(\sigma)\|_{\gamma(H,X)}\,d\sigma\Big)^{p_4} \leq C
\end{aligned} \tag{6.9.7}$$

One can obtain a similar estimate for $(\mathbb{E}\|B(r,t)\|^{(p-2)p_1})^{\frac{1}{p_1}}$, $r \in [0,s]$, yielding boundedness of (6.9.4). Together with observation (6.9.3) we can conclude with the dominated convergence theorem, that along a sequence $(\tau_i)_{i=1}^\infty$ with $\tau_i \downarrow 0$ as $i \to \infty$ we obtain

$$\begin{aligned}
&\lim_{i\to\infty} \mathbb{E} \int_0^s \|P_{\tau_i} B(r,t)\|_X^{p-2} \|P_{\tau_i} Y(t,r)\|_{\gamma(H,X)} \|P_{\tau_i} D^- S(t,r)\|_{\gamma(H,\mathscr{L}(X))} \\
&\qquad\qquad \times \Big\| \int_0^r (t-\sigma)^{-\alpha} S(r,\sigma)G(\sigma)\,dW^-(\sigma)\Big\|_X\,dr \\
&\qquad = \mathbb{E} \int_0^s \|B(r,t)\|_X^{p-2} \|Y(t,r)\|_{\gamma(H,X)} \|D^- S(t,r)\|_{\gamma(H,\mathscr{L}(X))} \\
&\qquad\qquad \times \Big\| \int_0^r (t-\sigma)^{-\alpha} S(r,\sigma)G(\sigma)\,dW^-(\sigma)\Big\|_X\,dr
\end{aligned}$$

Hence there exists a subsequence which we denote again by $(\tau_i)_{i=1}^\infty$, such that for all $i \in \mathbb{N}$, (6.9.1) holds. □

Lemma 6.44. *Let $S(t,s)$ be a random evolution system satisfying (H), and let $G \in \mathscr{S}_a$. Let $p > 2$ and $\alpha \in (\frac{1}{p}, \frac{1}{2})$. Consider Y, B and F_{km} as in* (6.5.3), (6.6.4) *and* (6.6.1)*, respectively. For every sequence $(\tau_i)_{i=1}^{\infty}$ such that $\tau_i \downarrow 0$ as $i \to \infty$, there exists a subsequence, also denoted by $(\tau_i)_{i\geq 1}$, such that for all $s \leq t$ and all i,*

$$\mathbb{E}n \int_0^s \Big\langle \int_r^{r+1/n} P_\tau(D^- Y(t,\cdot))(\sigma)\, d\sigma, F'_{km}(P_\tau B(r,t))\Big\rangle\, dr \\ \leq C\mathbb{E}\int_0^s (t-r)^{-\alpha} \|G(r)\|_{\gamma(H,X)} \|B(r)\|^{p-1}\, dr.$$

Proof. Observe that, almost surely, we have

$$n \int_0^s \Big\langle \int_r^{r+1/n} P_\tau(D^- Y(t,\sigma))\, d\sigma, F'_{km}(P_\tau B(r,t))\Big\rangle\, dr \\ \to \int_0^s \langle P_\tau(D^- Y(t,r)), F'_{km}(P_\tau B(r,t))\, dr,$$

as $n \to \infty$. Estimations similar to (6.9.5), (6.9.6) and (6.9.7) yield boundedness of $\|B(r,t)\|_{L^p(\Omega)}$ for all $r \in [0,t]$. From (6.6.2) and the fact that P_τ is a contraction on $L^p(\Omega; X)$, we obtain

$$\mathbb{E}n \int_0^s \Big\langle \int_r^{r+1/n} P_\tau(D^- Y(t,\sigma))\, d\sigma, F'_{km}(P_\tau B(r,t))\Big\rangle\, dr \\ \leq \int_0^s \Big(\mathbb{E}\Big\| \int_r^{r+1/n} P_\tau(D^- Y(t,\sigma))\, d\sigma\Big\|^p\Big)^{\frac{1}{p}} (\mathbb{E}\|B(r,t)\|^p)^{\frac{1}{p}}\, dr \\ \leq C \int_0^s \Big(\mathbb{E}\Big\| \int_r^{r+1/n} P_\tau(D^- Y(t,\sigma))\, d\sigma\Big\|^p\Big)^{\frac{1}{p}}\, dr.$$

Furthermore, by hypothesis (H3c), and Minkowski's inequality,

$$\Big(\mathbb{E}\Big\| \int_r^{r+1/n} P_\tau(D^- Y(t,\sigma))\, d\sigma\Big\|^p\Big)^{\frac{1}{p}} \leq n \int_r^{r+1/n} (\mathbb{E}\|D^- Y(t,\sigma)\|^p)^{\frac{1}{p}}\, d\sigma \\ \leq Cn \int_r^{r+1/n} (t-\sigma)^{-\alpha} \mathbf{1}_{[0,t]}(\sigma)\, d\sigma = C(n\mathbf{1}_{[-\frac{1}{n},0]} * \varphi_t)(r),$$

where $\varphi_t(\sigma) = \mathbf{1}_{[0,t]}(\sigma)(t-\sigma)^{-\alpha}$. By Young's inequality,

$$C \int_0^s \Big(\mathbb{E}\Big\| \int_r^{r+1/n} P_\tau(D^- Y(t,\sigma))\, d\sigma\Big\|^p\Big)^{\frac{1}{p}}\, dr \\ \leq C\|n\mathbf{1}_{[-1/n,0]} * \varphi_t\|_{L^1} \leq C\|\varphi_t\|_{L^1} \leq C'.$$

By the dominated convergence theorem, one can conclude that

$$\lim_{n\to\infty} \mathbb{E}n \int_0^s \Big\langle \int_r^{r+1/n} P_\tau(D^-Y(t,\sigma))\,d\sigma, F'_{km}(P_\tau B(r,t))\Big\rangle\,dr$$
$$= \mathbb{E}\int_0^s \langle P_\tau(D^-Y(t,r)), F'_{km}(P_\tau B(r,t))\,dr,$$

hence in particular, for n large enough,

$$\begin{aligned}&\mathbb{E}n \int_0^s \Big\langle \int_r^{r+1/n} P_\tau(D^-Y(t,\sigma))\,d\sigma, F'_{km}(P_\tau B(r,t))\Big\rangle\,dr\\ &\quad\leq C\mathbb{E}\int_0^s \|P_\tau(D^-Y(t,r))\|_{\gamma(H,X)}\|P_\tau B(r,t)\|^{p-1}_{\gamma(H,X)}\,dr,\end{aligned} \tag{6.9.8}$$

For every $\xi \in L^q(\Omega;X)$, $q\in[1,\infty)$, we have $P(\tau)\xi \to \xi$ in $L^q(\Omega;X)$ as $\tau\downarrow 0$. Hence, given a sequence $(\tau_i)_{i=1}^\infty$, with $\tau_i\downarrow 0$ as $i\to\infty$, we can find a subsequence, denoted again by $(\tau_i)_{i=1}^\infty$, such that $P(\tau_i)\xi\to\xi$ almost surely. By a similar dominated convergence argument as above, one can find yet another subsequence $(\tau_i)_{i=1}^\infty$ such that for all i,

$$\begin{aligned}&\mathbb{E}\int_0^s \|P_\tau D^-Y(t,r)\|_{\gamma(H,X)}\|P_\tau B(r,t)\|^{p-1}_{\gamma(H,X)}\,ds\\ &\leq C\mathbb{E}\int_0^t (t-r)^{-\alpha}\|D^-S(t,r)\|_{\gamma(H,\mathscr{L}(X))}\|G(r)\|_{\gamma(H,X)}\|B(r)\|^{p-1}\,dr\\ &\leq C\mathbb{E}\int_0^t (t-r)^{-\alpha}\|G(r)\|_{\gamma(H,X)}\|B(r)\|^{p-1}\,dr.\end{aligned}$$

This estimate combined with (6.9.8) yields the desired result. □

Summary

In this thesis we study stochastic evolution equations in Banach spaces. We restrict ourselves to the two following cases. First, we consider equations in which the drift is a closed linear operator that depends on time and is random. Such equations occur as mathematical models in for instance mathematical finance and filtration theory. Second, we restrict ourselves to UMD Banach spaces with type 2. As the theory of Itô stochastic integration is insufficient for studying equations of this general type, we need to have a proper understanding of several extensions to the Itô integral. Two of such extensions that are considered rigorously in this thesis are the Skorohod integral and the forward integral.

In Chapter 2 we study Malliavin calculus, the theory that is the basis for Skorohod integration. The main result in this chapter is Itô's formula for Skorohod integration in Banach spaces. Itô's formula is one of the most important results in stochastic integration theory, and it gives in some cases the possibility to explicitly solve stochastic differential equations.

The Skorohod integral lacks a property that most other integrals, like the Lebesgue integral and the Itô integral, have. This being the property that processes that are integrable on an interval $[a, b]$ are also integrable on any subinterval $[c, d] \subset [a, b]$. Fortunately, there exists a large class of processes, namely the space of Malliavin differentiable processes $\mathbb{D}^{1,2}(L^2(0, T))$, for which this property does hold. In chapter 3 we construct a stochastic process that is Skorohod integrable on $[0, 1]$, but which is not Skorohod integrable on $[0, 1/2]$.

In chapter 5 we study stochastic evolution equations of the form

$$\begin{cases} du(t) = (A(t)u(t) + F(t, u(t)))\, dt + B(t, u(t))\, dW(t), & t \in [0, T], \\ u(0) \;\; = u_0, \end{cases} \tag{SEE}$$

where the drift $A(t)$ is dependent on time and on the probability space, and satisfies the (AT)-conditions by Acquistapace and Terreni. We define a new solution concept, and show that being a solution is equivalent to being a weak solution, forward mild solution or variational solution. Under the extra condition that one of the constants from the (AT)-conditions is uniformly bounded with respect

to the probability variable (this we call condition (UC)), we prove that there exists a unique solution to problem (SEE). Furthermore, we prove that if one of the (AT)-conditions is replaced by a slightly stronger condition, then with a localization argument it is possible to obtain a unique solution to (SEE) without condition (UC).

The stochastic integral appearing in the concept of a mild solution is a forward integral. This integral is defined in Chapter 4. It is a vector-valued extension to the forward integral defined by Russo and Vallois [118]. We prove that also in the vector-valued case, the forward integral is an extension to the Itô integral. Furthermore, we show that the sequence that approximates the forward integral does not only converge in probability pointwise in $[0,T]$, but even in probability in the space $W^{\alpha,p}(0,T;E)$, where $\alpha \in (0,1/2)$, $p \in [2,\infty)$ and where E is a Banach space. This section can be read independently of the rest of Chapter 5, and is therefore also interesting for the reader who is interested in the forward integral and not in problem (SEE).

In Chapter 6 we consider again equations of the form (SEE), but this time without the (AT)-conditions. Here, we assume that the drift $A(t)$ is the generator of a random evolution system $S(t,s)_{0\leq s\leq t\leq T}$ that is Malliavin differentiable. With the help of a relationship between the Skorohod integral and the forward integral, we prove that for adapted smooth processes $\varPhi : \varOmega \times [0,T] \to \gamma(U,E)$, the process $s \mapsto S(t,s)\varPhi(s)\mathbf{1}_{[0,t]}(s)$ is forward integrable. Moreover, we deduce the maximal inequality

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \Big\| \int_0^t S(t,s)\varPhi(s)\, dW^-(s) \Big\|_E^p \Big) \leq C\mathbb{E} \int_0^T \|\varPhi(s)\|_{\gamma(U,E)}^p \, ds, \qquad p > 2.$$

Furthermore, we give the solution concept of a weak solution and prove that this concept is equivalent to a mild solution. Finally, we prove that there exists a unique mild solution to problem (SEE). We apply this to the stochastic partial differential equation from Paragraph 6 in [68] to prove the existence and uniqueness of a weak solution under less assumptions than done in [68].

Samenvatting

In dit proefschrift bestuderen we stochastische evolutievergelijkingen in Banachruimten. We specialiseren ons hierbij op twee gebieden. Ten eerste bekijken we vergelijkingen waarvan de drift-term een gesloten lineaire operator die zowel afhankelijk is van de tijd als van de kansruimte. Vergelijkingen van deze vorm komen voor als wiskundige modellen in bijvoorbeeld financiële wiskunde en filtertheorie. Ten tweede beperken we ons tot UMD Banachruimten met type 2. Daar de Itô-theorie in de meeste gevallen tijdens het bestuderen van dit type vergelijkingen onvoldoende is, is het zaak een goed begrip te hebben van de verschillende uitbreidingen van de Itô-integraal. Twee uitbreidingen die in dit proefschrift nauwkeurig worden beschouwd zijn de Skorohod-integraal en de voorwaartse integraal.

In Hoofdstuk 2 wordt Malliavincalculus bestudeerd; de theorie die de basis is voor Skorohod-integratie. Het hoofdresultaat in dit Hoofdstuk is Itô's formule voor de Skorohod-integraal in UMD Banachruimten. Itô's formule is één van de belangrijkste formules uit de theorie van stochastische differentiaalvergelijkingen, en geeft bijvoorbeeld de mogelijkheid stochastische differentiaalvergelijkingen expliciet op te lossen.

De Skorohod-integraal mist een eigenschap die de meeste andere integralen, waaronder de Lebesgue-integraal en de Itô-integraal, wel hebben. Deze eigenschap zegt dat een proces dat integreerbaar is op een interval $[a, b]$, ook integreerbaar is op een deelinterval $[c, d] \subset [a, b]$. Gelukkig bestaat er een rijke klasse van processen, de ruimte van Malliavindifferentieerbare processen $\mathbb{D}^{1,2}(L^2(0, T))$, waarop deze eigenschap wel geldt. In Hoofdstuk 3 construeren we een stochastisch proces dat Skorohod-integreerbaar is op het interval $[0, 1]$ maar niet op het deelinterval $[0, 1/2]$.

In Hoofdstuk 5 bestuderen we stochastische evolutievergelijkingen van de vorm

$$\begin{cases} du(t) = (A(t)u(t) + F(t, u(t)))\, dt + B(t, u(t))\, dW(t), & t \in [0, T], \\ u(0) \;= u_0. \end{cases} \tag{SEV}$$

De niet-autonome drift $A(t)$, die ook afhangt van de kansparameter, voldoet in dit Hoofdstuk aan de (AT)-condities van Acquistapace en Terreni. We definiëren een nieuw oplossingsconcept en tonen aan dit concept equivalent is aan andere concepten, zoals milde oplossing, variationele oplossing en zwakker oplossing. Onder de extra aanname dat één van de constanten in de (AT)-condities uniform begrensd is als functie van de kansvariabele (deze noemen we de (UC)-aanname), bewijzen we dat probleem (SEV) een unieke oplossing heeft. Bovendien tonen we aan dat indien één van de (AT)-condities wordt vervangen door een ietwat sterkere conditie, het mogelijk is met een lokalisatie-argument een unieke oplossing van (SEV) te construeren zonder aanname (UC).

De stochastische integraal die opduikt in het milde oplossingsbegrip uit Hoofdstuk 5 is een voorwaartse integraal. Deze integraal definiëren we in Hoofdstuk 4, en is een vectorwaardige uitbreiding op de bestaande definitie. We bewijzen daar dat ook in de vectorwaardige context de voorwaartse integraal een uitbreiding is op de Itô-integraal. Bovendien laten we zien dat de definiërende rij processen die convergeert naar de voorwaartse integraal niet alleen in kans convergeert voor alle $t \in [0,T]$, maar zelfs in kans in de ruimte $W^{\alpha,p}(0,T;E)$, waar $\alpha \in (0,1/2)$, $p \in [2,\infty)$ en E een Banachruimte is. Deze sectie is onafhankelijk van de rest van Hoofdstuk 5, en daarmee ook interessant voor de lezer die wel geïnteresseerd is in de voorwaartse integraal maar niet in probleem (SEV).

In Hoofdstuk 6 beschouwen we opnieuw vergelijkingen van de vorm (SEV), maar met andere aannamen dan de (AT)-condities. We nemen hier aan dat de drift $A(t)$ de generator is van een random evolutiesysteem $S(t,s)_{0\le s\le t\le T}$, waarbij dit evolutiesysteem Malliavin-differentieerbaar is. Met behulp van een relatie tussen de Skorohod-integraal en de voorwaartse integraal tonen we met Malliavincalculus aan, dat voor gladde en aangepaste processen $\Phi : \Omega \times [0,T] \to \gamma(U,E)$, het proces $s \mapsto S(t,s)\Phi(s)\mathbf{1}_{[0,t]}(s)$ voorwaarts integreerbaar is. Bovendien hebben we de volgende maximaalafschatting

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \Big\| \int_0^t S(t,s)\Phi(s)\, dW^-(s) \Big\|_E^p\Big) \le C\mathbb{E}\int_0^T \|\Phi(s)\|_{\gamma(U,E)}^p\, ds, \qquad p > 2.$$

Verder geven we het oplossingsconcept van zwakke oplossing, en tonen aan dat dit concept equivalent is aan het concept van milde oplossing. Bovendien bewijzen we dat (SEV) een unieke milde oplossing heeft. We passen dit toe op de stochastische partiële differentiaalvergelijking uit Paragraaf 6 van [68] om het bestaan van een unieke zwakke oplossing te bewijzen, onder minder aannamen dan die in [68].

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Curriculum Vitae

Matthijs Pronk was born in Zoetermeer, on November 9th, 1984. He completed his secondary education in 2003 at Het Erasmus College in Zoetermeer. In the same year he started his studies of Mathematics at the University of Leiden. In April 2009 he obtained his MSc. degree 'cum laude' in the specialization of applied mathematics, under supervision of Dr. S.C. Hille at the University of Leiden. In June 2009 he started his PhD research under the supervision of Prof. dr. J.M.A.M. van Neerven and Dr. M.C. Veraar at the Delft University of Technology. Part of his research was carried out during a half-year stay at the University of Oslo.

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