On consistency of actual restoring stiffness formulations in hydroelastic analysis of marine structures

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1 Introduction

In spite of the fact that ship hydroelasticity has been investigated for many years a consistent formulation of restoring stiffness is still an open question [1]. Basically, there are two approaches to this problem, a pure hydromechanical one, and the other extended to the contribution of the structure. Within the former approach, in the well-known Price and Wu formulation, only basic hydrostatic pressure is considered [2]. Newman formula represents an extension, giving the necessary hydrostatic pressure coefficients [3]. However, neither of those formulations gives the complete restoring stiffness coefficients, not even for the rigid body motions, because the gravity part is missing. Riggs overcame the above shortcoming by specifying new pressure coefficients and adding the gravity term [4]. The next two identical expressions are obtained in different way, i.e. by variational principle and vector calculus, Malenica and Molin [5].

2 Huang and Riggs formulation

A noticeable improvement is done by Huang and Riggs [6], offering a combined hydroelastic and structural formulation of restoring stiffness, Eq. (1) in Table 1, written in the index notation, where h_k^i is the k[th] component of the i[th] natural mode, and σ_{kl} is the stress tensor due to gravity load, $\rho_S g$, and hydrostatic pressure, $\rho g Z$. The complete restoring stiffness is defined as sum of hydrostatic part and geometric stiffness, $k_{ij}^{com} = k_{ij}^F + k_{ij}^G$, where k_{ij}^F results from the external load and k_{ij}^G from the internal stresses. k_{ij}^F is obtained as a change of hydrostatic force due to a small displacement by employing consistent linearization via the directional derivative, [6]. The geometric stiffness matrix, k_{ij}^G, is obviously symmetric, while symmetry of hydrostatic matrix, k_{ij}^F, is proved in [6].

The geometric stiffness matrix, Eq. (1e), can be transformed via integration by parts [6]:

$$k_{ij}^G = k_{ij}^S + k_{ij}^V + k_{ij}^{Vc}, \tag{4}$$

where

$$k_{ij}^S = \iint_S \sigma_{kl} h_m^i h_{m,l}^j n_k \mathrm{d}S, \; k_{ij}^V = -\iiint_V \sigma_{kl,k} h_m^i h_{m,l}^j \mathrm{d}V, \; k_{ij}^{Vc} = -\iiint_V \sigma_{kl} h_m^i h_{m,lk}^j \mathrm{d}V. \tag{5}$$

At the wetted surface, S, and within the structure volume, V, the following boundary and equilibrium conditions have to be satisfied, respectively:

$$\sigma_{kl} n_k = -\rho g Z n_l, \quad \sigma_{k3,k} = \rho_S g, \tag{6}$$

while $\sigma_{k1,k} = \sigma_{k2,k} = 0$. Substituting Eqs. (6) into (5) yields

$$k_{ij}^{SZ} = -\rho g \iint_S Z h_k^i h_{k,l}^j n_l \mathrm{d}S, \; k_{ij}^{VZ} = -g \iiint_V \rho_S h_k^i h_{k,3}^j \mathrm{d}V. \tag{7}$$

Table 1. Actual formulations of modal restoring stiffness*

		Eq. (1)	Eq. (2)	Eq. (3)
Contribution from	Notation	Huang and Riggs [6]	Riggs [7] Senjanovic et al. [8]	Senjanović et al. [9]
a) Pressure	C_{ij}^{p}	$\rho g \iint_S h_k^i h_3^j n_k \mathrm{d}S$ (1a)	$\rho g \iint_S h_k^i h_3^j n_k \mathrm{d}S$ (2a)	$\rho g \iint_S h_k^i h_3^j n_k \mathrm{d}S$ (3a)
b) Normal vector and mode	C_{ij}^{nh}	$\rho g \iint_S Z h_k^i h_{l,l}^j n_k \mathrm{d}S$ (1b)	$\rho g \iint_S Z h_k^i h_{l,l}^j n_k \mathrm{d}S$ (2b)	$\rho g \iint_S Z h_k^i h_{l,l}^j n_k \mathrm{d}S$ (3b)
c) Gravity load	C_{ij}^{m}		$g \iiint_V \rho_S h_k^i k_{3,k}^j \mathrm{d}V$ (2c)	
d) Boundary stress	$-k_{ij}^{S0}$	$-\rho g \iint_S Z h_l^i h_{k,l}^j n_k \mathrm{d}S$ (1d)		$-\rho g \iint_S Z h_l^i h_{k,l}^j n_k \mathrm{d}S$ (3d)
e) Geometric stiffness	k_{ij}^{G}	$\iiint_V \sigma_{kl} h_{m,k}^i h_{m,l}^j \mathrm{d}V$ (1e)		$\iiint_V \sigma_{kl} h_{m,k}^i h_{m,l}^j \mathrm{d}V$ (3e)
f) Strain of wetted surface	$-k_{ij}^{SZ} + k_{ij}^{S0}$			$\rho g \iint_S Z h_l^i \left(h_{l,k}^j + h_{k,l}^j \right) n_k \mathrm{d}S$ (3f)
g) Body strain	$C_{ij}^{m} - k_{ij}^{VZ}$			$g \iiint_V \rho_S h_k^i \left(h_{3,k}^j + h_{k,3}^j \right) \mathrm{d}V$ (3g)

*V-body volume, S-wetted surface, Z-coordinate of wetted surface from the free surface, n_k-component of wetted surface normal vector (directed towards the body).

In this way another formulation of the complete restoring stiffness is obtained, Eq. (7) in [7], which can be specified for rigid body modes. By introducing the zero strain constraint, $h_{k,l} = -h_{l,k}$ and $h_{m,lk} = 0$, Eq. (2) is obtained which, strictly speaking, is only valid for rigid body modes [7].

3 Senjanović et al. formulation

The restoring stiffness of the same form as Eq. (2) is derived in [8] by variational principle, strictly following the definition of stiffness as the relation between force and displacement. After estimation, the energy of involved forces is varied per displacement and mode amplitude. Both rigid body and elastic modes are equally valuated and, as a result, the consistent formulation of restoring stiffness is obtained.

In structural analysis of marine structures conventional stiffness, $\mathbf{K}^0$, is the basic stiffness, while the application of $\mathbf{K}^G$ and $\mathbf{C}$ depends on the analysis concerned, as well as on the type of the structure. If both $\mathbf{K}^G$ and $\mathbf{C}$ are used, then their union has to be determined since they have some terms of equivalent sense as a result of the same external load. Hence, one can write [9]:

$$k_{ij}^{U} = k_{ij}^{G} \cup C_{ij} = k_{ij}^{G} + C_{ij} - k_{ij}^{G} \cap C_{ij}. \tag{8}$$

The terms k_{ij}^{SZ} and k_{ij}^{VZ}, Eq. (7), depend on pressure, $\rho g Z$, and gravity load, $g \rho_S$, as C_{ij}^{nh} and C_{ij}^{m}, Eqs. (2b) and (2c), and therefore the former have to be excluded from the geometric stiffness, k_{ij}^{G}, Eq. (1c). By using the expanded form for C_{ij} one can write:

$$k_{ij}^{U} = C_{ij}^{p} + C_{ij}^{nh} - k_{ij}^{S0} + k_{ij}^{G} + \left(-k_{ij}^{SZ} + k_{ij}^{S0} \right) + \left(C_{ij}^{m} - k_{ij}^{VZ} \right). \tag{9}$$

In the above formula, term k_{ij}^{S0}, Eq. (1d), is added and subtracted in order to achieve constitution of $-k_{ij}^{SZ} + k_{ij}^{S0}$ similar to that of $C_{ij}^{m} - k_{ij}^{VZ}$, Eqs. (3f) and (3g), respectively. It is interesting to point out that these two terms depend on the linear strain, $\left(h_{k,l}+h_{l,k}\right)/2$, while geometric stiffness is function of the non-linear strain, $h_{m,k}h_{m,l}/2$.

4 Illustrative example

By comparing Eqs. (1) and (3), it is obvious that, due to deformation of the structure, the latter has two more terms than the former. For evaluation of their contribution, let us consider vertical vibrations of free pontoon with shear influence on bending included, Fig. 1. The basic formulae read:

$$w=w_b+w_s,\ w_s=-\frac{EI}{GA_s}\frac{\mathrm{d}^2w_b}{\mathrm{d}x^2},\ M=-EI\frac{\mathrm{d}^2w_b}{\mathrm{d}x^2},\ Q=GA_s\frac{\mathrm{d}w_s}{\mathrm{d}x}. \tag{10}$$

$$h_1=-\left(Z-z_N\right)\frac{\mathrm{d}w_b}{\mathrm{d}x},\ h_3=w. \tag{11}$$

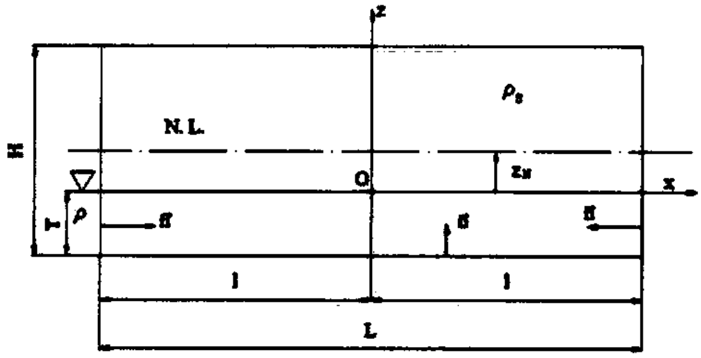

Figure 1. Pontoon particulars

By substituting (11) into (3f), and by using (10) for w_s, the bottom surface integral, where $k=3$ and $Z=-T$, reads

$$-k_{ij}^{SZ}+k_{ij}^{SO}=-\rho gBT\left(T+z_N\right)I_{ij},\ I_{ij}=\int_{-l}^{l}\frac{\mathrm{d}w_b^i}{\mathrm{d}x}\frac{\mathrm{d}w_s^j}{\mathrm{d}x}\mathrm{d}x. \tag{12}$$

Surface integral for the pontoon heads, where $k=1$ and $n_k=\pm1$ for the aft and front, respectively, is zero due to boundary conditions $M=0$ and $Q=0$, Eqs. (10).

Furthermore, by substituting (11) into (3g), for the volume integral, where $-T\leq Z\leq H-T$, one finds

$$C_{ij}^{m}-k_{ij}^{VZ}=-\rho_s gBH\left(\frac{H}{2}-T-z_N\right)I_{ij}. \tag{13}$$

Based on the equilibrium of weight and buoyancy for the homogenous pontoon $\rho_s=\rho T/H$, so that the hydrostatic contribution, Eq. (12), is cancelled with one part of the gravity contribution, Eq. (13). The integral I_{ij}, Eq. (12), can be transformed into the recognizable symmetric form by employing (10) for w_s, integration by parts and applying the boundary condition $M=0$

$$I_{ij}=\frac{EI}{GA_s}\int_{-l}^{l}\frac{\mathrm{d}^2w_b^i}{\mathrm{d}x^2}\frac{\mathrm{d}^2w_b^j}{\mathrm{d}x^2}\mathrm{d}x. \tag{14}$$

Since I_{sj} depends on the shear deflection, w_s, Eq. (12), which is quite small for the first few natural modes usually employed in hydroelastic analysis, the stiffness contribution Eqs. (12) and (13) can be neglected. The other terms of the unified restoring stiffness, Eq. (3), depend on the total deflection w and rotation of cross-section $\mathrm{d}w_b/\mathrm{d}x$, and therefore are dominant.

5 Discussion and conclusion

Three actual formulations of modal restoring stiffness for an elastic body are briefly described and compared. The first, so called complete formulation one, Eq. (1), is related to general marine structures. By employing the rigid body relations, Eq. (1) is reduced to Eq. (2) valid for rigid body modes only. On the other side, Eq. (2) is derived directly without any restriction for elastic modes, so it can be applied for hydroelastic analysis of ship structures, where the contribution of global geometric stiffness is quite small.

The third formulation, Eq. (3), is based on the union of the restoring stiffness, Eq. (2), and geometric stiffness. Compared to Eq. (1), it has two more terms related to the strain of body and wetted surface. Illustrative example of vertical pontoon vibrations shows that contribution of these two terms to global restoring stiffness is quite small for the first few natural modes.

The further investigation should be focused on the influence of the additional terms, Eqs. (3f) and (3g), on the restoring stiffness of 3D FEM models, where substructure vibrations play an important role.

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