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Persistence of periodic traveling waves and Abelian integrals

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Abstract

It is well known that the existence of traveling wave solutions (TWS) for many partial differential equations (PDE) is a consequence of the fact that an associated planar ordinary differential equation (ODE) has certain types of solutions defined for all time. In this paper we address the problem of persistence of TWS of a given PDE under small perturbations. Our main results deal with the situation where the associated ODE has a center and, as a consequence, the original PDE has a continuum of periodic traveling wave solutions. We prove that the TWS that persist are controlled by the zeroes of some Abelian integrals. We apply our results to several famous PDE, like the Ostrovsky, Klein-Gordon, sine-Gordon, Korteweg-de Vries, Rosenau-Hyman, Camassa-Holm, and Boussinesq equations.

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Keywords: Traveling wave; Abelian integral; Melnikov-Poincaré-Pontryagin function; Periodic orbit; Limit cycle; Bifurcation

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1. Introduction

Traveling wave solutions (TWS) are an important class of particular solutions of partial differential equations (PDE). These waves are special solutions which do not change their shape and which propagate at constant speed. They appear in fluid dynamics, chemical kinetics involving reactions, mathematical biology, lattice vibrations in solid state physics, plasma physics and laser theory, optical fibers, etc. In these systems the phenomena of dispersion, dissipation, diffusion, reaction and convection are the fundamental physical common facts. We refer the reader to some interesting sources to know more details about the first appearance of this kind of solutions in the works of Russell (1834), Boussinesq (1877), Korteweg and de Vries (1895), Luther (1906), Fisher (1937), Kolmogorov, Petrovskii and Piskunov (1937), and to find several examples of applications and further motivation to study them: see [12,14–16,18,20,23,26,28] and the references therein.

When studying ordinary differential equations (ODE), especially when they are modeling real world phenomena, it is very important to take into account whether the ODE are *structurally stable*. In a few words this means that if we fix a compact set $\mathcal{K}$ in the phase space it is said that an ODE is structurally stable on $\mathcal{K}$ when any other close enough (in the $\mathcal{C}^1$-topology) differential equation has a conjugated phase portrait. This concept is relevant for applications because it implies that the observed behaviors are qualitatively robust with respect to small changes of the model, see for instance [1,24,27] for more details, in particular concerning the planar case. Recall that the boundary of the sets of structurally stable differential equations is precisely where *bifurcations* (that is, qualitative changes of the phase portraits) may occur.

It is well-known that for many PDE the existence of TWS is established by proving the existence of a particular solution of a planar ordinary differential equation. These particular solutions must be defined for all time and, in the light of the previous definition, can roughly be classified into two categories:

- TWS created by a dynamical behavior that is structurally stable. Examples of this situation are hyperbolic limit cycles or heteroclinic connections where both critical points are hyperbolic and one of them is a node.
- TWS created by a dynamical behavior that is not structurally stable, as for instance continua of periodic orbits, or homoclinic or heteroclinic solutions connecting hyperbolic saddles.

In the first situation, simply take as the set $\mathcal{K}$ a compact neighborhood of the orbit that gives rise to the TWS for a given PDE. Then it can be easily seen that a small enough $\mathcal{C}^1$ perturbation of the original PDE with the same order will still have a TWS. This is so because all the structurally stable phenomena in ODE are robust under $\mathcal{C}^1$-perturbations. The only condition that must be checked is that the $\mathcal{C}^1$-closeness between the two PDE's is translated into a $\mathcal{C}^1$-closeness in $\mathcal{K}$ of the corresponding ODE.

An example corresponding to the first situation is the Fisher-Kolmogorov PDE, $u_t = u_{xx} + u(1-u)$, where the existence of several TWS of front type with different speeds is associated to the existence of a heteroclinic connection between a hyperbolic saddle and a node, see [2,13] and references therein. Therefore, all PDE of the form $u_t = u_{xx} + u(1-u) + \varepsilon g(u, u_x, u_t, \varepsilon)$ for ε small enough have such type of TWS. In fact, the same result holds for many perturbed Fisher-Kolmogorov PDE with a perturbation term of the form $\varepsilon g(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \varepsilon)$. As a second example of the first situation mentioned above, for some PDE of the form $u_t = u_{xx} +$

$h(u)u_x + g(u)$ there are periodic TWS which are associated to the existence of hyperbolic limit cycles, see for instance [8,22] and the references therein.

In this paper we address the second, more delicate, situation. More specifically, we consider several PDE having a *continuum of periodic* TWS associated to a center of a second order ODE associated to the PDE, and we study which conditions have to be imposed on the perturbation of the PDE to be able to ensure that TWS persist and to quantify them.

We split our main results into two theorems, which we state in Section 2 after giving some preliminary definitions and notations. Our first result deals with second order PDE, see Theorem A, and applies to a wide range of equations. Our second result, Theorem B, is more restrictive on the one hand because it only considers some special perturbations, but on the other hand it applies to higher order PDE. In Section 3 we study a particular class of Abelian integrals that will often appear in the analysis of the perturbations in Section 4. For these Abelian integrals our main result is given in Theorem C. Finally, in Section 4 we detail some applications of our results. First, in Section 4.1, we apply Theorem A to perturbations of TWS of second order equations such as the Ostrovsky, Klein-Gordon and sine-Gordon equations. Afterwards, in Section 4.2 we use Theorem B to study perturbations of higher order PDE given by the Korteweg-de Vries, Rosenau-Hyman, Camassa-Holm, and Boussinesq equations.

2. Definitions and main results

Consider m-th order partial differential equations of the form

$$P\Big(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial t}, \frac{\partial^2 u}{\partial t^2}, \dots, \frac{\partial^m u}{\partial x^m}, \frac{\partial^m u}{\partial x^{m-1}\partial t}, \dots, \frac{\partial^m u}{\partial t^m}, \varepsilon\Big) = 0, \tag{1}$$

where $\mathcal{W} \subset \mathbb{R}^{(m+1)(m+2)/2}$ is an open set, $\mathcal{I}$ is an open interval containing 0, $P : \mathcal{W} \times \mathcal{I} \to \mathbb{R}$ is a sufficiently smooth function and ε is a small parameter. Recall that the traveling wave solutions of (1) are particular solutions of the form $u = U(x - ct)$ where $U(s)$ is defined for all $s \in \mathbb{R}$ and satisfies certain boundary conditions at infinity. It is well-known that the existence of such solutions is equivalent to finding solutions defined for all s of the m-th order ordinary differential equation

$$\begin{aligned} &P_c(U, U', U'', \dots, U^{(m)}, \varepsilon) := \\ &\qquad P\big(U, U', -cU', U'', -cU'', c^2U'', \dots, U^{(m)}, -cU^{(m)}, \dots, (-c)^m U^{(m)}, \varepsilon\big) = 0, \end{aligned} \tag{2}$$

satisfying these conditions. Here the prime denotes derivative with respect to s and $P_c : \mathcal{W}_c(\varepsilon) \times I \to \mathbb{R}$, where $\mathcal{W}_c(\varepsilon)$ is an open subset of $\mathbb{R}^{m+1}$.

We will distinguish two cases according to whether (1) is a second order equation ($m = 2$) or a higher order equation ($m > 2$).

Second order equations. Our main result applies to a certain class of perturbed PDE that satisfy three conditions (i)–(iii) that we detail below. Succinctly, it requires the existence of a certain wave speed $c \in \mathbb{R}$ such that: (i) the associated ODE has the form $U'' = f_c(U, U') + \varepsilon g_c(U, U', \varepsilon)$; (ii) after a time reparameterization if necessary the planar system associated with this ODE can be written as a perturbation of a Hamiltonian system; and (iii) this Hamiltonian system has a center, and the Melnikov-Poincaré-Pontryagin function associated with the perturbation has ℓ simple zeroes, see [4, Part II] for further details.

More precisely, we will say that the PDE (1) with $m = 2$ satisfies *Property* $\mathcal{A}$ if there exists $c \in \mathbb{R}$ such that the following three conditions hold:

(i) There exist $\mathcal{C}^1$ functions $f_c : \mathcal{V}_c(\varepsilon) \to \mathbb{R}$ and $g_c : \mathcal{V}_c(\varepsilon) \times \mathcal{I} \subset \mathbb{R}^3 \to \mathbb{R}$, with $\mathcal{V}_c(\varepsilon) \subset \mathbb{R}^2$ and $\mathcal{V}_c(\varepsilon) \times \mathcal{I} \subset \mathbb{R}^3$ open sets, such that, for ε small enough,

$$\{(x, y) \in \mathcal{V}_c(\varepsilon) : z = f_c(x, y) + \varepsilon g_c(x, y, \varepsilon)\} \subset \{(x, y, z) \in \mathcal{W}_c(\varepsilon) : P_c(x, y, z, \varepsilon) = 0\}.$$

Moreover, if $\mathcal{U}_c$ is the limit of the sets $\mathcal{V}_c(\varepsilon)$ when $\varepsilon \to 0$, the only solution of $f_c(x, 0) = 0$ in $\mathcal{U}_c$ is $x = x_c$.

(ii) There exists a $\mathcal{C}^2$ function $H_c : \mathcal{V}_c \subset \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ such that $H_c(x_c, 0) = 0$,

$$\frac{\partial H_c(x, y)}{\partial y} = \frac{y}{s_c(x, y)}, \quad \frac{\partial H_c(x, y)}{\partial x} = -\frac{f_c(x, y)}{s_c(x, y)},$$

for some $\mathcal{C}^1$ function $s_c : \mathcal{V}_c \subset \mathbb{R}^2 \to \mathbb{R}^+$. Notice that s_c is such that

$$\frac{\partial}{\partial x}\left(\frac{y}{s_c(x, y)}\right) + \frac{\partial}{\partial y}\left(\frac{f_c(x, y)}{s_c(x, y)}\right) \equiv 0.$$

(iii) For each $h \in (0, \overline{h}_c)$, where $\overline{h}_c \in \mathbb{R}^+ \cup \{\infty\}$, the set

$$\gamma_c(h) := \{(x, y) \in \mathcal{V}_c : H_c(x, y) = h\}$$

is a closed oval surrounding $(x_c, 0)$ and the function $M_c : (0, \overline{h}_c) \to \mathbb{R}$, defined as the line integral

$$M_c(h) = \int_{\gamma_c(h)} \frac{g_c(x, y, 0)}{s_c(x, y)}\, dx,$$

has $\ell \geq 1$ simple zeroes in $(0, \overline{h}_c)$.

Theorem A. *Assume that the second order PDE*

$$P(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \varepsilon) = 0, \tag{3}$$

satisfies Property $\mathcal{A}$ *for some* $c \in \mathbb{R}$. *Then:*

(a) For $\varepsilon = 0$ *the PDE (3) has a continuum of periodic TWS,* $u = U_h(x - ct)$, *for* h *in an open real interval.*

(b) For ε *small enough it has at least* ℓ *periodic TWS,* $u = U_{h_j}(x - ct, \varepsilon)$, $j = 1, 2, \ldots, \ell$.

Proof of Theorem A. From the discussion at the very beginning of this section, a function $U(s)$ is a TWS for the PDE (1) if it is defined for all time and

$$P_c(U(s), U'(s), U''(s), \varepsilon) = 0, \tag{4}$$

where P_c is defined in (2). By using (i) of Property $\mathcal{A}$ we can write the above expression as

$$U''(s) = f_c(U(s), U'(s)) + \varepsilon g_c(U(s), U'(s), \varepsilon),$$

for some suitable f_c and g_c. In other words, $(x, y) = (U(s), U'(s))$ is a solution of the planar ODE

$$\begin{cases} x' = \dfrac{dx}{ds} = y, \\ y' = \dfrac{dx}{ds} = f_c(x, y) + \varepsilon g_c(x, y, \varepsilon). \end{cases}$$

By item (ii) of Property $\mathcal{A}$ we can parameterize U by a new time, say τ, with $d\tau/ds = s_c(x, y)$, and then $x = U(\tau)$ satisfies the equivalent planar ODE

$$\begin{cases} \dot{x} = \dfrac{dx}{d\tau} = \dfrac{dx}{d\tau}\dfrac{d\tau}{ds} = \dfrac{y}{s_c(x, y)} = \dfrac{\partial H_c(x, y)}{\partial y}, \\ \dot{y} = \dfrac{dy}{d\tau} = \dfrac{dy}{d\tau}\dfrac{d\tau}{ds} = \dfrac{f_c(x, y)}{s_c(x, y)} + \varepsilon \dfrac{g_c(x, y, \varepsilon)}{s_c(x, y)} = -\dfrac{\partial H_c(x, y)}{\partial x} + \varepsilon \dfrac{g_c(x, y, \varepsilon)}{s_c(x, y)}. \end{cases} \tag{5}$$

When $\varepsilon = 0$ the above system is Hamiltonian, and by (i) and (iii) of Property $\mathcal{A}$ the continuum of curves $\gamma_c(h)$ for $0 < h < \overline{h}$ are periodic orbits of system (5) with $\varepsilon = 0$ that surround the center $(x_c, 0)$. The functions $U_h(s, c) = x_h(\tau(s), c)$, where $(x_h(\tau, c), y_h(\tau, c))$ is the parameterization of $\gamma_c(h)$, give rise to the continuum of periodic traveling wave solutions of (3).

When $\varepsilon \neq 0$ is small enough we are in the setting of the perturbations of Hamiltonian systems, [4,9]. Recall that for general perturbed $\mathcal{C}^1$ Hamiltonian systems,

$$\begin{cases} \dot{x} = \dfrac{\partial H(x, y)}{\partial y} + \varepsilon R(x, y, \varepsilon), \\ \dot{y} = -\dfrac{\partial H(x, y)}{\partial x} + \varepsilon S(x, y, \varepsilon), \end{cases} \tag{6}$$

its associated Melnikov-Poincaré-Pontryagin function is

$$M(h) = \int_{\gamma(h)} S(x, y, 0)\, dx - R(x, y, 0)\, dy,$$

where the curves $\gamma(h)$ form a continuum of ovals contained in $\{H(x, y) = h, \text{ for } h \in (h_0, h_1)\}$. Then, it is known that each simple zero $h^* \in (h_0, h_1)$ of M gives rise to a limit cycle of (6) that tends to $\gamma(h^*)$ when $\varepsilon \to 0$. For system (5), $M(h) = M_c(h)$ and so, each simple zero $h_j \in (0, \overline{h}_c)$, $j = 1, 2, \ldots, \ell$ of $M_c(h)$ gives rise to a limit cycle of system (5). Each of these limit cycles correspond to a periodic TWS of (3). □

Higher order equations ($m > 2$). In this situation our approach only works for a particular class of differential equations. Again, fixed $(c, k) \in \mathbb{R}^2$, we will define a property similar to Property $\mathcal{A}$ which will consist of four conditions. The first one, that we will call condition (o), is the most restrictive one and it is totally different to the ones imposed when $m = 2$. It states that the associated ODE can somehow be reduced to a second order equation, or that some of the solutions of the associated ODE are also solutions of a related second order ODE. The rest of the conditions are quite similar to the ones of the planar case.

More precisely, we say that a PDE satisfies *Property* $\mathcal{B}$ if there exist $c, k \in \mathbb{R}$ such that:

(o) There exists a function $Q_c : \mathcal{W}_c \times \mathcal{I} \to \mathbb{R}$, where $\mathcal{W}_c \subset \mathbb{R}^3$ is open and Q_c is sufficiently smooth, such that

$$\frac{d^{m-2}}{ds^{m-2}}\big(Q_c(U, U', U'', \varepsilon)\big) = P_c(U, U', U'', \dots, U^{(m)}, \varepsilon),$$

where P_c is defined in (2) and $U = U(s)$.

(i) There exist two $\mathcal{C}^1$ functions $f_{c,k} : \mathcal{V}_{c,k}(\varepsilon) \to \mathbb{R}$ and $g_{c,k} : \mathcal{V}_{c,k}(\varepsilon) \times \mathcal{I} \to \mathbb{R}$ with $\mathcal{V}_{c,k}(\varepsilon) \subset \mathbb{R}^2$ and $\mathcal{V}_{c,k}(\varepsilon) \times \mathcal{I} \subset \mathbb{R}^3$ open sets, such that, for ε small enough,

$$\{(x, y) \in \mathcal{V}_{c,k}(\varepsilon) \,:\, z = k + f_{c,k}(x, y) + \varepsilon g_{c,k}(x, y, \varepsilon)\} \\ \subset \{(x, y, z) \in \mathcal{W}_{c,k}(\varepsilon) \,:\, Q_c(x, y, z, \varepsilon) = k\}.$$

Moreover, if $\mathcal{U}_{c,k}$ is the limit of the sets $\mathcal{V}_{c,k}(\varepsilon)$ when $\varepsilon \to 0$, the only solution of $f_c(x, 0) = k$ in $\mathcal{V}_{c,k}$ is $x = x_{c,k}$.

(ii) There exists a $\mathcal{C}^2$ function $H_{c,k} : \mathcal{V}_{c,k} \subset \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ such that $H_{c,k}(x_{c,k}, 0) = 0$,

$$\frac{\partial H_{c,k}(x, y)}{\partial y} = \frac{y}{s_{c,k}(x, y)}, \qquad \frac{\partial H_{c,k}(x, y)}{\partial x} = -\frac{f_c(x, y) + k}{s_{c,k}(x, y)},$$

for some $\mathcal{C}^1$ function $s_{c,k} : \mathcal{V}_{c,k} \subset \mathbb{R}^2 \to \mathbb{R}^+$. Notice that the function $s_{c,k}$ is such that

$$\frac{\partial}{\partial x}\left(\frac{y}{s_{c,k}(x, y)}\right) + \frac{\partial}{\partial y}\left(\frac{f_c(x, y) + k}{s_{c,k}(x, y)}\right) \equiv 0.$$

(iii) For each $h \in (0, \overline{h}_{c,k})$, where $\overline{h}_{c,k} \in \mathbb{R}^+ \cup \{\infty\}$, the set

$$\gamma_{c,k}(h) := \{(x, y) \in \mathcal{V}_{c,k} \,:\, H_{c,k}(x, y) = h\}$$

is a closed oval surrounding $(x_{c,k}, 0)$ and the function $M_{c,k} : (0, \overline{h}_{c,k}) \to \mathbb{R}$, defined as the line integral

$$M_{c,k}(h) = \int_{\gamma_{c,k}(h)} \frac{g_{c,k}(x, y, 0)}{s_{c,k}(x, y)}\, dx,$$

has $\ell \geq 1$ simple zeroes in $(0, \overline{h}_{c,k})$.

Theorem B. *Assume that the m-th order PDE (1), with $m > 2$, satisfies Property $\mathcal{B}$, for some $c \in \mathbb{R}$ and $k \in \mathbb{R}$. Then:*

(a) For $\varepsilon = 0$ the PDE (3) has a continuum of periodic TWS, $u = U_{h,k}(x - ct)$ for h in an open real interval.
(b) For ε small enough it has at least ℓ periodic TWS, $u = U_{h_j,k}(x - ct, \varepsilon)$ for $j = 1, 2, \ldots, \ell$.

Proof of Theorem B. From condition (o) of Property $\mathcal{B}$, if we restrict our attention to the solutions of (2) contained in

$$Q_c(U, U', U'', \varepsilon) = k, \tag{7}$$

for the given value of $k \in \mathbb{R}$, we can find some TWS with speed c and associated to this particular value of k. Other values of k give different TWS with the same speed.

Starting from equation (7), instead of equation (4), we can repeat all the steps of the proof of Theorem A, point by point, to get the desired conclusion. □

3. Some particular Abelian integrals

This section is devoted to studying a particular class of Abelian integrals for which we prove a result quantifying their zeros, see Theorem C. We will use this result in the next section when we study the persistence of TWS for several perturbed PDE, which is governed by the number of zeros of integrals of this type.

Proposition 3.1. *Let A, B and D be analytic functions, defined in an open interval $\mathcal{I} \subset \mathbb{R}$ and such that*

$$\begin{aligned} A(x) &= a^2 + O(x - x^*), \qquad B(x) = \frac{(x - x^*)^2}{b^2} + O\big((x - x^*)^3\big), \\ D(x) &= (x - x^*)^{2n} D_0(x) \quad \text{with} \quad D_0(x) = d + O(x - x^*), \end{aligned}$$

for certain $x^* \in \mathcal{I}$, where a, b, c are real constants with $abd \neq 0$ and $n \in \mathbb{N} \cup \{0\}$. Consider the Hamiltonian function $H(x, y) = A(x)y^2 + B(x)$. Then, the following holds:

(a) The Hamiltonian system $\dot{x} = H_y(x, y)$, $\dot{y} = -H_x(x, y)$, has a center at $(x^*, 0)$. We will denote by $\gamma(h)$ the periodic orbits contained in $\{H(x, y) = h\}$, which exist when $h \in (0, \widetilde{h})$ for some $\widetilde{h} \in \mathbb{R}$.
(b) For $h \in (0, \widetilde{h})$ and $p, n \in \mathbb{N}$, define the Abelian integral

$$J_p(h) = \int_{\gamma(h)} D(x) y^p \, dx. \tag{8}$$

Then $J_{2p}(h) \equiv 0$ and

$$J_{2p-1}(h) \sim \frac{2db^{2n+1}}{a^{2p-1}} \frac{(2p-1)!!\,(2n-1)!!\,\pi}{2^{p+n}(p+n)!} h^{p+n} \ \text{at} \ h = 0^+,$$

where $(2k-1)!! = (2k-1)(2k-3)\cdots 3\cdot 1$ *and* $(-1)!! = 1!! = 1$.

Proof. Without loss of generality we will assume that $x^*=0$. To prove (a), notice that the origin is a non-degenerate singular point of the vector field $X=(H_y,-H_x)$ because $\det(\mathrm{D}X(0,0)) = 2A(0)B''(0) = 4a^2/b^2>0$. Moreover, since a singular point of a Hamiltonian system can neither be a focus nor a node, it is a center.

To study the Abelian integral J_p it is convenient to introduce the new variable w as $h=w^2$. Then, by the Weierstrass preparation theorem, see for instance [1,3], in a neighborhood of $(0,0)$ the only solutions of equation $B(x)-w^2 = x^2/b^2 - w^2 + O(x^3)=0$ are

$$x = x^{\pm}(w) = \pm bw + O(w^2),$$

where $x^{\pm}(w)$ are analytic functions at zero. Moreover, in this neighborhood,

$$w^2 - B(x) = (x - x^-(w))(x^+(w)-x)U(x,w), \tag{9}$$

where $U(0,0)=1/b^2$ is also analytic at $(0,0)$. Notice that the points of the oval $\gamma(h)$ satisfy $y=\pm\sqrt{(w^2-B(x))/A(x)}$. When p is even the integral (8) vanishes because of symmetry with respect to $y=0$. Hence

$$J_p(w^2) = \begin{cases} 0, & \text{when } p \text{ is even},\\ 2\displaystyle\int_{x^-(w)}^{x^+(w)} D(x)\left(\frac{w^2-B(x)}{A(x)}\right)^{\frac{p}{2}} dx, & \text{when } p \text{ is odd}.\end{cases}$$

By using (9) we get that

$$\begin{aligned} J_{2p-1}(w^2) =& 2\int_{x^-(w)}^{x^+(w)} D(x)\left(\frac{w^2-B(x)}{A(x)}\right)^{\frac{2p-1}{2}} dx\\ =& 2\int_{x^-(w)}^{x^+(w)} \big((x-x^-(w))(x^+(w)-x)\big)^{\frac{2p-1}{2}} D(x)\left(\frac{U(x,w)}{A(x)}\right)^{\frac{2p-1}{2}} dx\\ =& 2(\Delta(w))^{2p}\int_0^1 \big(z(1-z)\big)^{\frac{2p-1}{2}}\overline{D}(z,w)\left(\frac{\overline{U}(z,w)}{\overline{A}(z,w)}\right)^{\frac{2p-1}{2}} dz, \end{aligned}$$

where in the integral we have introduced the change of variables $z=(x-x^-(w))/\Delta(w)$, being $\Delta(w)=x^+(w)-x^-(w)$, and for any function $E(x,w)$ or $E(x)$, we denote $\overline{E}(z,w) = E\big(\Delta(w)z + x^-(w), w\big)$ or $\overline{E}(z,w) = E(\Delta(w)z+x^-(w))$. In particular,

$$\overline{D}(z,w) = \big(\Delta(w)z + x^-(w)\big)^{2n}\overline{D}_0(z,w) = (\Delta(w))^{2n}\left(z+\frac{x^-(w)}{\Delta(w)}\right)^{2n}\overline{D}_0(z,w),$$

with $\overline{D}_0(0,0)=d$. Hence,

$$J_{2p-1}(w^2)=(\Delta(w))^{2p+2n}\int_0^1 \big(z(1-z)\big)^{\frac{2p-1}{2}} F(z,w)\,dz,$$

where

$$F(z,w)=2\overline{D}_0(z,w)\left(z+\frac{x^-(w)}{\Delta(w)}\right)^{2n}\left(\frac{\overline{U}(z,w)}{\overline{A}(z,w)}\right)^{\frac{2p-1}{2}}.$$

Since $x^{\pm}(w)=\pm bw+O(w^2)$, it holds that $\Delta(w)=2bw+O(w^2)$ and hence $\lim_{w\to 0}\frac{x^-(w)}{\Delta(w)}=-\frac{1}{2}$. Therefore for all $z\in[0,1]$ and w small enough the function $F(z,w)$ is continuous, and as a consequence

$$\begin{aligned}\lim_{w\to 0}\frac{J_{2p-1}(w^2)}{w^{2p+2n}}&=\lim_{w\to 0}\left(\frac{\Delta(w)}{w}\right)^{2p+2n}\int_0^1\big(z(1-z)\big)^{\frac{2p-1}{2}}\lim_{w\to 0}F(z,w)\,dz\\&=(2b)^{2p+2n}\frac{2d}{a^{2p-1}b^{2p-1}}\int_0^1\big(z(1-z)\big)^{\frac{2p-1}{2}}\left(z-\frac{1}{2}\right)^{2n}dz.\end{aligned}$$

Now we claim that

$$K(p,n):=\int_0^1\big(z(1-z)\big)^{\frac{2p-1}{2}}\left(z-\frac{1}{2}\right)^{2n}dz=\frac{(2p-1)!!\,(2n-1)!!}{8^{p+n}(p+n)!}\pi,$$

and we observe that, from this claim, the result follows.

To prove the claim we observe that by using integration by parts, one easily gets that

$$K(p,n)=\frac{2p-1}{2n+1}K(p-1,n+1). \tag{10}$$

Now the claim follows by using induction. First we prove that for any $p\in\mathbb{N}_0$, $K(p,0)$ satisfies the claim. Indeed, if B is the Euler's Beta function, and since $\mathrm{B}(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$, we have

$$K(p,0)=\int_0^1 z^{p-\frac{1}{2}}(1-z)^{p-\frac{1}{2}}dz=\mathrm{B}\left(p+\frac{1}{2},p+\frac{1}{2}\right)=\frac{(\Gamma(p+\frac{1}{2}))^2}{\Gamma(2p+1)}.$$

Since p is an integer number $\Gamma(2p+1)=(2p)!$. On the other hand, it is well known that

$$\Gamma\left(p+\frac{1}{2}\right)=\frac{(2p)!}{4^p\,p!}\sqrt{\pi}=\frac{(2p-1)!!}{2^p}\sqrt{\pi}.$$

Hence,

$$K(p,0)=\frac{(2p-1)!!}{8^p\, p!}\pi$$

as we wanted to prove.

Now we assume that for $n>0$ and for all $p\in\mathbb{N}_0$, $K(p,n)$ satisfies the claim. By using the relation (10), we get that,

$$K(p,n+1)=\frac{2n+1}{2p+1}K(p+1,n)=\frac{(2n+1)\,(2p+1)!!\,(2n-1)!!}{(2p+1)\,8^{p+n+1}(p+n+1)!}\pi=$$
$$=\frac{(2p-1)!!\,(2n+1)!!}{8^{p+n+1}(p+n+1)!}\pi,$$

so the claim follows. □

Before proving the main result of this section, Theorem C, and to motivate one of its hypotheses, we collect some simple observations in the following lemma.

Lemma 3.2. *Let $\gamma(h)\subset\{H(x,y)=h\}$, $h\in(0,\overline{h})=\mathcal{L}$, be a continuum of periodic orbits surrounding a center, corresponding to $h=0$, of the Hamiltonian system associated to a $\mathcal{C}^1$ Hamiltonian function $H(x,y)$ and assume that they have a* clockwise *time parameterization. For each $p,q\in\mathbb{N}\cup\{0\}$, consider the Abelian integral*

$$J_{q,p}(h)=\int\limits_{\gamma(h)} x^q y^p\,dx.$$

The following holds.

(a) When q is even and p is odd then $J_{q,p}(h)>0$ for all $h\in\mathcal{L}$.
(b) When q is even and p is even and $H(x,y)=H(x,-y)$ then $J_{q,p}(h)\equiv 0$ on the whole interval $\mathcal{L}$.
(c) When q is odd and p is odd and $H(-x,y)=H(x,y)$ then $J_{q,p}(h)\equiv 0$ on $\mathcal{L}$.

Proof. Notice that by Green's theorem

$$J_{q,p}(h)=\int\limits_{\gamma(h)} x^q y^p\,dx=\iint\limits_{\mathrm{Int}(\gamma(h))} p x^q y^{p-1}\,dx\,dy,$$

where $\mathrm{Int}(\gamma(\mathrm{h}))$ denotes the interior of the oval. Then, trivially (a) follows. The other two statements are consequence of the symmetries of H and the function $x^q y^{p-1}$. □

The next proposition will be one of the key results to prove Theorem C, which is stated below.

Proposition 3.3. ([6]) *Set $\mathcal{L} \subset \mathbb{R}$ an open real interval and let $F_j : \mathcal{L} \to \mathbb{R}$, $j = 0, 1, \ldots, \ell$, be $\ell + 1$ linearly independent analytic functions. Assume also that one of them, say F_k, $0 \le k \le \ell$, has constant sign on $\mathcal{L}$. Then, there exist real constants d_j, $j = 0, 1, \ldots, \ell$, such that the linear combination $\sum_{j=0}^{\ell} d_j F_j$ has at least ℓ simple zeroes in $\mathcal{L}$.*

Notice that in the next theorem, and due to Lemma 3.2, the monomials of the Abelian integral that we consider are of the form $x^{2q} y^{2p-1}$.

Theorem C. *Let $H(x, y) = A(x)y^2 + B(x)$, with A and B functions satisfying the hypotheses of Proposition 3.1, and denote by $\gamma(h)$, $h \in (0, \overline{h})$, the periodic orbits surrounding the origin of the corresponding Hamiltonian system. For $d_0, d_1, \ldots, d_n \in \mathbb{R}$ and $q_j, p_j \in \mathbb{N}$, $j = 0, 1, \ldots, \ell$ consider the family of Abelian integrals*

$$J(h) = \int_{\gamma(h)} \sum_{j=0}^{\ell} d_j x^{2q_j} y^{2p_j-1} \, dx.$$

If all values $m_j = q_j + p_j$, $j = 0, 1, \ldots, \ell$ are different, there exit values of d_j, $j = 0, 1, \ldots, \ell$, such that the corresponding function $J(h)$ has at least ℓ simple zeroes in $(0, \overline{h})$.

Proof. Notice that

$$J(h) = \sum_{j=0}^{\ell} d_j J_j(h), \quad \text{where} \quad J_j(h) = \int_{\gamma(h)} x^{2q_j} y^{2p_j-1} \, dx.$$

By Proposition 3.1, for each $j = 0, 1, \ldots, \ell$, $J_j(h) = k_j h^{m_j} + o\big(h^{m_j}\big)$ and, by hypothesis, all these m_j are different. This clearly implies that all these $\ell + 1$ functions are linearly independent. Moreover, by item (a) of Lemma 3.2 we know that none of them vanish in $(0, \overline{h})$. Hence we can apply Proposition 3.3 to this set of functions and $\mathcal{L} = (0, \overline{h})$ and the result follows. □

4. Applications

In this section we consider perturbations of several relevant PDE with continua of periodic TWS and prove that the perturbations can be tailored such that a prescribed number of TWS persist in these perturbed PDE. In many examples, for simplicity, we perturb the PDE with an additive term that only contains partial derivatives up to $m - 1$. For more general perturbations, even including terms of order m, most of the results can be adapted.

4.1. Second order PDE

We start with an illustrative toy example for which we give all the details on how a prescribed number of periodic TWS can be obtained.

4.1.1. A toy example

Consider the PDE

$$u + au_{xx} + bu_{xt} + du_{tt} + \varepsilon g(u_x, u_t, \varepsilon) = 0, \tag{11}$$

with g a $\mathcal{C}^1$ function and take c such that $a - bc + dc^2 > 0$. Then equation (2) can be written as

$$U + (a - bc + dc^2)U'' + \varepsilon g(U', -cU', \varepsilon) = 0.$$

We define $C^2 = a - bc + dc^2$ and $g_c(U', \varepsilon) = -g(U', -cU', \varepsilon)/C^2$. Then it is easy to see that this PDE satisfies Property $\mathcal{A}$ with $H_c(x, y) = x^2/(2C^2) + y^2/2$, $s_c(x, y) \equiv 1$ and $(0, \overline{h}) = (0, \infty)$. That is,

$$\begin{cases} \dot{x} = \dfrac{\partial H_c(x, y)}{\partial y} = y, \\ \dot{y} = -\dfrac{\partial H_c(x, y)}{\partial x} + \varepsilon g_c(x, y, \varepsilon) = -\dfrac{x}{C^2} + \varepsilon g_c(y, \varepsilon). \end{cases}$$

Moreover

$$M_c(h) = \int_{\gamma(h)} g_c(y, 0)\, dx,$$

where $\gamma(h)$ is the ellipse $\{x^2/(2C^2) + y^2/2 = h\}$. We parameterize the closed curves $H_c(x, y) = h$ as $(x, y) = (C\sqrt{2h}\cos\theta, \sqrt{2h}\sin\theta)$ for $0 \le \theta \le 2\pi$. Then

$$M_c(h) = -C\sqrt{2h}\int_0^{2\pi} g_c(\sqrt{2h}\sin\theta, 0)\sin\theta\, d\theta.$$

Assume for instance that $g_c(y, 0) = \sum_{j=0}^{N} g_j y^j$ is a polynomial of degree N, and $g_j \in \mathbb{R}$. Then,

$$M_c(h) = -C\sqrt{2h}\sum_{j=0}^{N} g_j(\sqrt{2h})^j\Big(\int_0^{2\pi} \sin^{j+1}\theta\, d\theta\Big).$$

When j is even, by symmetry, the above integrals vanish. Hence

$$M_c(h) = -2Ch\Big(\sum_{i=0}^{[(N-1)/2]} g_{2i+1}2^i I_{2i+2}h^i\Big),$$

where $[\cdot]$ denotes the integer part and $I_{2n} = \int_0^{2\pi} \sin^{2n}\theta\, d\theta > 0$. Removing the factor h, and taking suitable g_{2i+1}, the polynomial $M_c(h)/h$ can be any arbitrary polynomial of degree $[(N-1)/2]$ in h. Hence, by applying Theorem C, for any $\ell \le [(N-1)/2]$, there exist coefficients g_j such that the function $M_c(h)$ has ℓ simple zeros and, therefore, by applying Theorem A, the PDE

(11) has at least ℓ periodic TWS. We remark that the above computations are essentially the same as the ones of the celebrated paper [21] where the authors present the first example of classical polynomial Liénard differential system of degree N with $[(N-1)/2]$ limit cycles.

By doing similar computations we can consider more general perturbations in PDE (11), like for instance

$$u + au_{xx} + bu_{xt} + du_{tt} + \varepsilon(uu_{xx} + g(u, u_x, u_t, \varepsilon)) = 0,$$

and similar results hold.

4.1.2. Reduced Ostrovsky equation

We consider perturbations of the reduced Ostrovsky equation, introduced by L. Ostrovsky in 1978, which is a modification of the Korteweg-de Vries equation that models gravity waves propagating in a rotating background under the influence of the Coriolis force when the high-frequency dispersion is neglected. More concretely, we take

$$(u_t + uu_x)_x - u + \varepsilon g(u, u_x, u_t, \varepsilon) = 0, \tag{12}$$

which satisfies Property $\mathcal{A}$ with $c > 0$, because its associated ODE is

$$(U - c)U'' + (U')^2 - U + \varepsilon g(U, U', -cU', \varepsilon) = 0.$$

Then, taking $g_c(U, U', \varepsilon) = -g(U, U', -cU', \varepsilon)/(U - c)$; $\mathcal{V}_c = \{x < c\}$; $x_c = 0$; and $s_c(x, y) = (x - c)^{-2}$, the system that has to be studied to find TWS is

$$\begin{cases} \dot{x} = \dfrac{\partial H_c(x, y)}{\partial y}, \\ \dot{y} = -\dfrac{\partial H_c(x, y)}{\partial x} + (x - c)^2 \varepsilon g_c(x, y, \varepsilon), \end{cases}$$

with

$$H_c(x, y) = \frac{(x - c)^2 y^2}{2} + \frac{cx^2}{2} - \frac{x^3}{3}.$$

Consider also the Melnikov-Poincaré-Pontryagin function

$$M_c(h) = \int_{\gamma_c(h)} (x - c)^2 g_c(x, y, 0)\, dx,\ h \in (0, c^3/3).$$

As in the toy example, it is not difficult to find a perturbation term g such that the function $M_c(h)$ has several simple zeroes in $(0, c^3/3)$ which, by Theorem C, give rise to periodic TWS of the PDE (12).

4.1.3. Perturbed non-linear Klein-Gordon equation

The Klein-Gordon equation is a wave equation related to the Schrödinger equation, which is used to model spinless relativistic particles. It was introduced in 1926 in parallel by O. Klein, W. Gordon and V. Fock as a tentative to describe the relativistic electron dynamics. In the one-dimensional setting we look at a perturbation of this equation of the form

$$u_{tt} - u_{xx} + \lambda u^p + \varepsilon g(u, u_x, u_t, \varepsilon) = 0,$$

with $\lambda \in \mathbb{R}^+$ and p an *odd* integer. It can readily be seen that it satisfies Property $\mathcal{A}$ and that the system that has to be studied to find TWS is

$$\begin{cases} \dot{x} = \dfrac{\partial H_c(x, y)}{\partial y}, \\ \dot{y} = -\dfrac{\partial H_c(x, y)}{\partial x} + \varepsilon g_c(x, y, \varepsilon), \end{cases}$$

where

$$H_c(x, y) = \frac{C x^{p+1}}{p+1} + \frac{y^2}{2},$$

with $C = \lambda/(c^2 - 1)$, $g_c(x, y, \varepsilon) = -g(x, y, -cy, \varepsilon)/(c^2 - 1)$. The associated Melnikov-Poincaré-Pontryagin function is

$$M_c(h) = \int_{\gamma_c(h)} g_c(x, y, 0)\, dx, \quad h \in (0, \infty).$$

The interested reader can take a look to the papers [5,17] where perturbations of this Hamiltonian system and the zeros of its associated Melnikov-Poincaré-Pontryagin function are studied with two different approaches.

In particular, the zeroes of the above first integral can be studied in a similar way to the toy example considered at the beginning of this section. Notice, however, that when $p \geq 3$, instead of using trigonometric functions to parametrize the invariant closed curves, one can use the generalized polar coordinates introduced by Lyapunov in 1893 in his study of the stability of degenerate critical points, [19]. All the details can be found in [5]. Again, Theorem A guarantees that the zeros of the function $M_c(h)$ correspond with periodic TWS of the Klein-Gordon equation.

4.1.4. Perturbed sine-Gordon equation

The sine-Gordon equation first appeared in 1862 in the context of differential geometry. Specifically in a study by E. Bour on surfaces of constant negative curvature. The equation was rediscovered later by J. Frenkel and T. Kontorova in 1939, in their study of crystal dislocations. The equation is relevant to the community investigating integrable systems because it has soliton solutions. Its perturbation writes as

$$u_{tt} - u_{xx} + \sin u + \varepsilon g(u, u_x, u_t, \varepsilon) = 0.$$

Again, it satisfies Property $\mathcal{A}$ for $c > 1$, and its associated planar system is

$$\begin{cases} \dot{x} = \dfrac{\partial H_c(x,y)}{\partial y}, \\ \dot{y} = -\dfrac{\partial H_c(x,y)}{\partial x} + \varepsilon g_c(x,y,\varepsilon), \end{cases}$$

where

$$H_c(x,y) = C(1-\cos x) + \frac{y^2}{2},$$

with $C = 1/(c^2-1) > 0$ and $g_c(x,y,\varepsilon) = g(x,y,-cy,\varepsilon)/(1-c^2)$. The Melnikov-Poincaré-Pontryagin function is

$$M_c(h) = \int\limits_{\gamma_c(h)} g_c(x,y,0)\,dx, \quad h \in (0,2C).$$

The above type integrals are studied for instance in [11]. There, several condition on g for obtaining many simple zeroes of M_c, and therefore periodic TWS of the considered PDE, are obtained.

4.2. PDE with order greater than 2

In this section we study perturbations of several PDE with order $m > 2$. We start with the following result that helps us to characterize the existence of centers for the unperturbed Hamiltonian systems that will appear.

Lemma 4.1. *Consider a Hamiltonian system of the form*

$$\begin{cases} \dot{x} = \dfrac{\partial H(x,y)}{\partial y} = y\,m(x,y), \\ \dot{y} = -\dfrac{\partial H(x,y)}{\partial x} = f(x,y)\,m(x,y), \end{cases}$$

where $H \in \mathcal{C}^2$, $m(x,y) > 0$ and such that $\frac{\partial}{\partial x}(ym(x,y)) + \frac{\partial}{\partial y}(f(x,y)m(x,y)) \equiv 0$. Then, a singular point of the form $(x_*,0)$ is a center if

$$\frac{\partial}{\partial x} f(x,y)\Big|_{(x_*,0)} < 0. \tag{13}$$

Furthermore, if $m(x,y)$ depends only on x, condition (13) holds, and H is analytic, then the Hamiltonian H satisfies the hypotheses of Proposition 3.1.

Proof. Consider the vector field $X = (H_y, -H_x)$. Since

$$\det(\mathrm{D}X(x_*,0)) = -m^2(x_*,0)\,\frac{\partial}{\partial x}(f(x,y))\Big|_{(x_*,0)},$$

then equation (13) implies that $\det(DX(x_*, 0)) > 0$ and therefore $(x_*, 0)$ is a center (once more, remember that a singular point of a Hamiltonian system cannot be neither a focus nor a node).

If $m(x, y) = m(x)$, then $H(x, y) = y^2 m(x)/2 + B(x)$ for some analytic function B. Since $m(x) > 0$ we can write $m(x) = 2a^2 + O(x - x_*)$ near $x = x_*$. Suppose that condition (13) holds, then $H_{xx}(x_*, 0) = -f_x(x_*, 0)m(x_*) > 0$, and we can write $1/b^2 = B''(x_*) = H_{xx}(x_*, 0)$, obtaining $B(x) = (x - x_*)^2/b^2 + O\big((x - x_*)^3\big)$. So H fulfills the hypotheses of Proposition 3.1. □

Observe that condition (13) is equivalent to the fact that $H_{xx}(x_*, 0) > 0$ and $\det(\mathbf{H}_H(x_*, 0)) > 0$ (where $\mathbf{H}$ is the hessian matrix), which implies that H has a non-degenerate local minimum at $(x_*, 0)$.

4.2.1. Perturbed generalized Korteweg-de Vries equation

We consider a perturbation of a family of PDE which for certain values of the parameters contains the celebrated Korteweg-de Vries and Benjamin-Bona-Mahony equations appearing in several domains of physics (non-linear mechanics, water waves, etc.). More concretely, we consider the family of PDE

$$u_t + au_x + buu_x + duu_t + pu_{xxx} + qu_{xxt} + ru_{xtt} + su_{ttt} + \varepsilon\nabla g(u, u_x, u_t, \varepsilon) \cdot (u_x, u_{xx}, u_{xt}, 0)^t = 0. \quad (14)$$

Notice that the KdV equation corresponds to $\varepsilon = 0$ and $a = d = q = r = s = 0$, $b = -6$ and $p = 1$. The ODE associated to (14) is

$$\left((a - c)U + \frac{b - dc}{2}U^2 + CU'' + \varepsilon g(U, U', -cU', \varepsilon)\right)' = 0,$$

where $C = p - qc + rc^2 - sc^3$. Notice that then, for any function U satisfying previous equation, it holds that there exists $k \in \mathbb{R}$, such that

$$(a - c)U + \frac{b - dc}{2}U^2 + CU'' + \varepsilon g(U, U', -cU', \varepsilon) = k \quad (15)$$

Thus we have to study the equivalent planar system

$$\begin{cases} \dot{x} = y = \dfrac{\partial H_{c,k}(x, y)}{\partial y}, \\ \dot{y} = \alpha_{c,k} + \beta_c x + \gamma_c x^2 + \varepsilon g_c(x, y, \varepsilon) = -\dfrac{\partial H_{c,k}(x, y)}{\partial x} + \varepsilon g_c(x, y, \varepsilon), \end{cases}$$

where

$$H_{c,k}(x, y) = -\alpha_{c,k}x - \frac{\beta_c}{2}x^2 - \frac{\gamma_c}{3}x^3 + \frac{1}{2}y^2,$$

with

$$\alpha_{c,k}=\frac{k}{C},\quad \beta_c=\frac{c-a}{C},\quad \gamma_c=\frac{dc-b}{C},$$

and $g_c(x,y,\varepsilon)=-g(x,y,-cy,\varepsilon)/C$. Hence, using Lemma 4.1, it is not difficult to see that the PDE (14) satisfies Property $\mathcal{B}$ when equation $\alpha_{c,k}+\beta_c x+\gamma_c x^2=0$ has two different real solutions (that correspond to a center and a saddle of the planar system). Then, by Theorem B, the periodic TWS that persist for ε small enough correspond to the simple zeroes of the elliptic integral

$$M_{c,k}(h)=\int\limits_{\gamma_{c,k}(h)} g_c(x,y,0)\,dx$$

in a suitable open interval of energies. This kind of Abelian integrals are studied in detail in the classical paper of Petrov ([25]) and more recently in the Chapter 3 of Part II of the book [4]. Again, it is not difficult to impose conditions on g to get a prescribed number of TWS for (14) for ε small enough and different values of c and k.

4.2.2. *Perturbed Rosenau-Hyman equation*

The Rosenau-Hyman equation is a generalization of the KdV equation. It was introduced in 1993 by P. Rosenau and J.M. Hyman to show the existence of solitary waves with compact support (compactons) in the context of non-linear dispersive equations. We consider the perturbed equation

$$u_t+a(u^n)_x+(u^n)_{xxx}+\varepsilon\nabla g(u,u_x,u_t,\varepsilon)\cdot(u_x,u_{xx},u_{xt},0)^t=0,$$

where $a\in\mathbb{R}$ and $n\in\mathbb{N}$. To find TWS for it we have to study the third order ODE

$$\begin{aligned}&-cU'+a(U^n)'+(U^n)'''+\varepsilon\nabla g(U,U',-cU',\varepsilon)\cdot(U',U'',-cU'',0)^t\\&=\big(-cU+aU^n+(U^n)''+\varepsilon g(U,U',-cU',\varepsilon)\big)'\\&=(-cU+aU^n+n(n-1)U^{n-2}U'+nU^{n-1}U''+\varepsilon g(U,U',-cU',\varepsilon)\big)'=0.\end{aligned}$$

Thus, we need to find solutions of the second order ODE

$$-cU+aU^n+n(n-1)U^{n-2}U'+nU^{n-1}U''+\varepsilon g(U,U',-cU',\varepsilon)=k$$

with $k\in\mathbb{R}$. It writes as the planar system

$$\begin{cases}x'=y,\\[2mm] y'=\dfrac{k+cx-ax^n-n(n-1)x^{n-2}y^2}{nx^{n-1}}+\varepsilon\dfrac{g_c(x,y,\varepsilon)}{nx^{n-1}},\end{cases}$$

where $g_c(x, y, \varepsilon) = -g(x, y, -cy, \varepsilon)$. With the new time τ, where $d\tau/ds = s_{c,k}(x, y)$ and $s_{c,k}(x, y) = x^{2(1-n)}/n$, we get $x = U(\tau)$ satisfies the equivalent planar ODE

$$\begin{cases} \dot{x} = \dfrac{\partial H_{c,k}(x, y)}{\partial y}, \\ \dot{y} = -\dfrac{\partial H_{c,k}(x, y)}{\partial x} + \varepsilon x^{n-1} g_c(x, y, \varepsilon), \end{cases}$$

where

$$H_{c,k}(x, y) = \frac{n}{2} x^{2(n-1)} y^2 - \frac{k}{n} x^n - \frac{c}{n+1} x^{n+1} + \frac{a}{2n} x^{2n}.$$

By Lemma 4.1 (see the comment below its statement), if there exists a singular point $(x_*, 0)$ such that

$$\frac{\partial^2 H_{c,k}}{\partial x^2}(x_*, 0) = x_*^{n-2} \big(a\,(2n-1)\, x_*^n - c\, n\, x_* - k\,(n-1)\big) > 0,$$

then it is a center. Furthermore, since the Hypothesis of Proposition 3.1 is satisfied, we can apply Theorem B and the periodic TWS for the perturbed PDE correspond to simple zeroes of

$$M_{c,k}(h) = \int\limits_{\gamma_{c,k}(h)} x^{n-1} g_c(x, y, 0)\, dx$$

in a suitable interval of the energy. To get examples of perturbations with several simple zeroes we can apply Theorem C.

4.2.3. Camassa-Holm equation and related PDE

The Camassa-Holm equation is a model for the propagation of shallow water waves of moderate amplitude. The horizontal component of the fluid velocity field at a certain depth within the fluid is described by the PDE

$$u_t + (2\kappa + 3u)u_x - 2u_x u_{xx} + u u_{xxx} - u_{xxt} = 0,$$

and the parameter κ is positive. Constantin and Lannes derived in [7] a similar PDE for surface waves also with moderate amplitude in the shallow water regime,

$$u_t + \big(1 + 6u - 6u^2 + 12u^3\big)u_x + 28u_x u_{xx} + 14u u_{xxx} + u_{xxx} - u_{xxt} = 0,$$

see also [10]. Similarly, the Degasperis-Procesi equation

$$u_t + 4u u_x - 3u_x u_{xx} - u u_{xxx} - u_{xxt} = 0,$$

which was derived initially only for its integrability properties, has a similar role in hydrodynamics.

In fact, perturbations of the above equations can be written under the common expression

$$u_t + A'(u)u_x + bu_xu_{xx} + duu_{xxx}$$
$$+ pu_{xxx} + qu_{xxt} + ru_{xtt} + su_{ttt} + \varepsilon\nabla g(u, u_x, u_t, \varepsilon)\cdot(u_x, u_{xx}, u_{xt}, 0)^t = 0,$$

where A is sufficiently smooth and b, d, p, q, r and s are real parameters. Its associated third order ODE is

$$-cU' + A'(U)U' + bU'U'' + dUU''' + CU''' + \varepsilon\nabla g(U, U', -cU', \varepsilon)\cdot(U', U'', -cU'', 0)^t$$
$$= \Big(A_c(U) + b(U')^2/2 + d\big(UU'' - (U')^2/2\big) + CU'' + \varepsilon g(U, U', -cU', \varepsilon)\Big)' = 0$$

where $A_c(U) = A(U) - cU$, with $A_c(0) = 0$, and $C = p - qc + rc^2 - sc^3$. Hence, for any function U satisfying the previous equation, there exists $k \in \mathbb{R}$, such that

$$A_c(U) + \beta(U')^2 + (C + dU)U'' + \varepsilon g(U, U', -cU', \varepsilon) = k,$$

where $\beta = (b - d)/2$. The above equation can be written as the planar system

$$\begin{cases} x' = y, \\ y' = \dfrac{k - A_c(x) - \beta y^2 + \varepsilon g_c(x, y, \varepsilon)}{C + dx}, \end{cases}$$

where $g_c(x, y, \varepsilon) = -g(U, U', -cU', \varepsilon)$. Then, taking $d\tau/ds = s_c(x)$,

$$s_c(x) = \begin{cases} (C + dx)^{-2\beta/d} & \text{when} \quad d \neq 0, \\ e^{-2\beta x/C} & \text{when} \quad d = 0, \end{cases}$$

we get

$$\begin{cases} \dot{x} = \dfrac{\partial H_{c,k}(x, y)}{\partial y}, \\ \dot{y} = -\dfrac{\partial H_{c,k}(x, y)}{\partial x} + \varepsilon \dfrac{g_c(x, y, \varepsilon)}{(C + dx)s_c(x)}, \end{cases}$$

with

$$H_{c,k}(x, y) = \frac{y^2}{2s_c(x)} + \int_0^x \frac{A_c(w) - k}{(C + dw)s_c(w)}\, dw.$$

By Lemma 4.1, any singular point $(x_*, 0)$ such that $H_{xx}(x_*, 0) > 0$, is a center. So by Theorem B, the periodic TWS of the perturbed equations correspond with the simple zeros of

$$M_{c,k}(h) = \int\limits_{\gamma_{c,k}(h)} \frac{g_c(x,y,0)}{(C+dx)s_c(x)}\,dx.$$

Again, for some particular examples, the zeroes of the above type of Abelian integrals can be obtained by using Theorem C. For instance, we observe that this is trivially the case if $g_c(x,y,0) = (C+dx)s_c(x)\left(\sum_{i=0}^{\ell} d_{2i+1}y^{2i+1}\right)$.

4.2.4. Boussinesq-type equations

The Boussinesq equation describes bi-directional surface water waves and reads

$$u_{tt} + uu_{xx} - u_{xx} + (u_x)^2 - u_{xxxx} = 0.$$

Similarly, the modified Boussinesq equation is

$$u_{tt} + uu_{xx} - u_{xx} + (u_x)^2 - u_{xxtt} = 0,$$

and appears in the modeling of non-linear waves in a weakly dispersive medium. We consider the following perturbation of the family of PDE

$$au_{xx} + bu_{xt} + du_{tt} + 2e(uu_{xx} + (u_x)^2) + pu_{xxxx} + qu_{xxxt} + ru_{xxtt} + su_{xttt} + fu_{tttt} + \varepsilon G = 0, \quad (16)$$

where $a, b, d, e, p, q, r, s, f$ are suitable real parameters. We do not detail here the perturbation G, but it is a function of all the partial derivatives of u up to order four, and such that after replacing u by $U(x-ct)$ it holds that there exists a function g_c such that $G = \left(g_c(U, U', \varepsilon)\right)''$. Hence the ODE associated to (16) is

$$CU'' + e(U^2)'' + DU'''' + \varepsilon\left(g_c(U,U',\varepsilon)\right)'' = \left(CU + eU^2 + DU'' + \varepsilon g_c(U,U',\varepsilon)\right)'' = 0,$$

where $C = a - bc + d^2c$, $D = p - qc + rc^2 - sc^3 + fc^4$, and we have used that $(u^2)_{xx} = 2uu_x + 2(u_x)^2$. We are interested in solutions of the above fourth order ODE

$$CU + eU^2 + DU'' + \varepsilon g_c(U,U',\varepsilon) = k, \quad (17)$$

for some $k \in \mathbb{R}$. When $D \neq 0$ we are again under the situation covered by Theorem B. Notice that other solutions would satisfy $CU + eU^2 + DU'' + \varepsilon g_c(U,U',\varepsilon) = k_1 s + k_2$, for some $k_1 \neq 0$, $k_2 \in \mathbb{R}$, but we do not consider them. In fact, from (17) we arrive at the same ODE that appears in the study done in Section 4.2.1 about the perturbed generalized Korteweg-de Vries equation, but with a different notation. Indeed, the above ODE is the same as (15) and it can be studied to get TWS for (16) exactly like in that case.

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