Model of a Proposed Superconducting Phase Slip Oscillator: A Method for Obtaining Few-Photon Nonlinearities

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(Received 29 January 2010; published 17 February 2011)

We theoretically investigate a driven oscillator with the superconducting inductance subject to quantum phase slips (QPS). We find uncommon nonlinearities in the proposed device: they oscillate as a function of the number of photons N with a local period of the order of $\sqrt{N}$. We prove that such nonlinearities result in multiple metastable states encompassing few photons and study oscillatory dependence of various responses of the oscillator. Such nonlinearities enable new possibilities for quantum manipulation of photon states and very sensitive measurements to confirm the coherence of phase slips.

DOI: 10.1103/PhysRevLett.106.077004 PACS numbers: 74.78.Na, 74.25.Gz, 74.81.Fa

A phase slip process in a superconducting wire is a topological fluctuation of the superconducting order parameter whereby it reaches zero at a certain time moment and at a certain point of the wire [1]. Such a process results in a $\pm 2\pi$ change of the superconducting phase difference between the ends of the wire; this produces a voltage pulse. Incoherent thermally activated phase slips were shown to be responsible for residual resistance of the wire slightly below critical temperature [2,3]. At lower temperatures and in thinner wires phase slips are quantum fluctuations. Although resistance measurements indicate the quantum nature of the phase slips [4,5], they cannot prove a possible quantum coherence of phase slips. Coherent phase slips are not manifested as countable incoherent events detected in the course of resistance measurements. A set of other nanodevices [6,7] have been proposed to verify the coherence experimentally. To facilitate this verification was the initial motivation of our research.

Nonlinear effects on driven oscillations are important in many fields of physics, ranging from applied mechanics to optics. They are instrumental for quantum applications [8,9] and in superconducting resonators [10,11], where the nonlinearities enabled measurement of several quanta in the resonant cavity. However, a limitation is that the nonlinearities in almost all physical systems are featureless (polynomial) functions of the average number of photons N in the oscillator, typically N^2 [12–15]. In this Letter we show that the QPSs induce very special nonlinearities in the oscillator. They have a particular oscillatory dependence on the number of photons [16] with a local period $\propto \sqrt{N}$ and are manifesting already at several photons. We show that this brings up new possibilities for quantum manipulation of photons states in the resonantly driven oscillator. An important functionality of the proposed device, the phase slip oscillator, is an unambiguous verification of existence of coherent QPS by two signatures: (i) periodic gate-voltage dependence of oscillator response at small QPS amplitudes, (ii) multiple stable points at bigger QPS amplitudes as opposite to two stable points commonly found in nonlinear driven oscillators. As compared with previous proposals to verify the coherence experimentally [6,7], the device is sensitive to five order of magnitude smaller amplitudes.

The inductance L of the wire brings about the inductive energy scale $E_L = \Phi_0^2/2L$, where $\Phi_0 = \pi\hbar/e$ is the flux quantum with $\hbar$ the Planck constant and e the electron charge. It is usually assumed that experimental observation of coherent QPS requires the phase slip amplitude E_S to be comparable with E_L [7]. The QPS amplitude E_S depends exponentially on the wire parameters, so its value can hardly be predicted and it may be small. This is why it is important to be able to detect arbitrary small values of E_S. Our idea is to use a driven oscillator. We prove that in this case the detectable values of E_S are only limited by damping of the oscillator $E_S \approx \hbar\Gamma \ll \hbar\omega_0$. There is an outburst of activity in applying superconducting oscillators for quantum manipulation purposes [17–19]. The inductance of such an oscillator may be either a thin superconducting wire [11,20] or a chain of Josephson junctions [21,22]. The multijunction chains also exhibit QPS and for our purposes are very similar to a wire. Typical experimental values for the main frequency and dissipation rate are $\omega_0 \simeq 10^{10}$ Hz and $\Gamma \simeq 10^5$.

This brings us to the system under consideration: the phase slip oscillator [see Fig. 1(a) and the equivalent circuit in Fig. 1(b)]. For simplicity, we neglect the effects of the capacitance distribution along the wire attributing all the capacitance C to the "island." Without QPS, the system is a linear LC one-mode oscillator. The ac component of the gate electrode excites the oscillator while the dc component induces constant charge $q = CV_g$ to the island. The oscillator is subject to small damping characterized by the energy loss rate $\Gamma \ll \omega_0$, $\omega_0 = 1/\sqrt{LC}$. Without QPS, the dynamics of the oscillator can be described by ϕ—the superconducting phase difference dropping along the wire; ϕ can take any value not being restricted to the interval $(-\pi, \pi)$. The dynamics are entirely linear with the inductive energy given by $E_L(\phi/2\pi)^2$. Consequently,

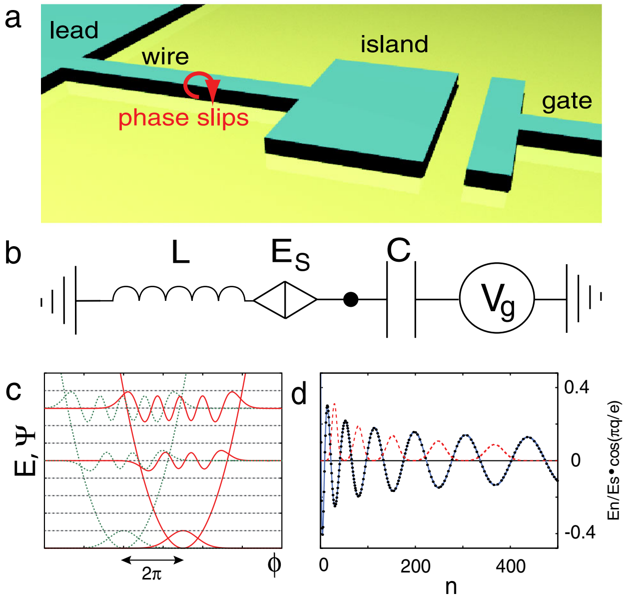

FIG. 1 (color online). (a) Phase-slip oscillator. A thin superconducting wire connects a lead and an island. The nearby gate electrode induces charge to the island. The wire is subject to QPS. (b) The inductance L of the wire and the capacitance C of the island form an oscillator that can be excited with the gate voltage. The crossed diamond represents QPS of amplitude E_S. (c) Energy potential and wave functions for $n = 0$; 5; 8 for two harmonic oscillators shifted by 2π with respect with each other. The QPS induced correction $E(n)$ is proportional to the overlap integral of the wave functions shifted. Wave function oscillations in ϕ give rise to the oscillations of the overlap in n. (d) The QPS corrections to the levels of the oscillator for $\gamma = 0.92$. Exact values (dots) are fitted with Eq. (3) (curve). The photon distribution in several coherent states is plotted to compare its width and the local period of oscillations (dashed).

in the absence of QPS, the charge q on the island does not affect the dynamics. The QPS shifting the phase by $\pm 2\pi$ can be described by a Hamiltonian acting on the wave function of the system $\Psi(\phi)$ [7]:

$$\hat{H}_S\Psi(\phi) = \frac{E_S}{2}\sum_{\pm} e^{\pm i\pi q/e}\Psi(\phi \pm 2\pi). \qquad (1)$$

The effect of weak QPS ($E_S \ll \hbar\omega_0$) on the resonantly driven oscillator originates from the shifts $E_n = \langle n|H_S|n\rangle$ to the otherwise equidistant levels of the oscillator, labeled by the integer n (photon number). We immediately see from Eq. (1) that $E_n \propto \cos(\pi q/e)$, so the charge induced affects the quantum interference of QPS with opposite shifts. Any effect of QPS is thus periodic in gate voltage with a period $2e/C$. The experimental observation of such dependence unambiguously identifies the quantum coherence of phase slips. As mentioned, we are more interested in the oscillatory dependence on the number of photons n. One envisages the origin of such a dependence from the fact that the energy shifts E_n are proportional to overlaps of the oscillator wave functions shifted by $\pm 2\pi$ with respect to each other [see Fig. 1(c)]. The wave functions oscillate in the space of variable ϕ with a typical local period of $\Delta\phi \propto 1/\sqrt{n}$. These oscillations are converted into oscillatory dependence of the overlaps $\int d\phi \Psi^*(\phi)\Psi(\phi \pm 2\pi)$ on the photon number [see Fig. 1(c)].

The important parameter $\gamma = (2G_Q Z/\pi)^{-1/2}$ (where $G_Q \equiv e^2/\pi\hbar$ is the conductance quantum) measures the effective impedance of the oscillator $Z = \sqrt{L/C}$, and defines the quantum fluctuations of phase $\propto (4\pi^2\gamma^2)^{-1}$. Commonly, electrical resonators have $\gamma \gg 1$. However, superconducting wires provide significant kinetic inductance which may make $\gamma \approx 1$ [21,22]. In this letter, we concentrate on experimentally accessible range $0.3 < \gamma < 3$.

We expect that QPS have a measurable effect when the oscillator is resonantly driven. Therefore we include the driving force $2V_{ac}(t) = \tilde{V}\exp(i(\omega_0 - \omega)t) + \text{H.c.}$, with the detuning $|\omega| \ll \omega_0$. It is convenient to normalize the driving force such that it enters the Hamiltonian in a combination $\hbar F(\hat{b} + \hat{b}^\dagger)/2$, where $\hat{b}$ and $\hat{b}^\dagger$ are the boson annihilation and creation operators, defined as: $b = \frac{\gamma}{2\pi}\phi + i\frac{\pi}{\gamma}Q$, and $b^\dagger$ is obtained by conjugation. The force is then $F = (e\gamma/\pi\hbar)\tilde{V}$ and the Hamiltonian of the driven oscillator reads

$$\hat{H} = \hbar\omega_0 b^\dagger b + \text{Re}\{\hbar F e^{i(\omega_0-\omega)t}\}\frac{b^\dagger + b}{2} + \hat{H}_S,$$

where the QPS term H_S is given by Eq. (1).

We implement the rotating-wave approximation to arrive to the equation for density matrix $\hat{\rho}$ (valid at H_R, $\hbar\Gamma \ll \hbar\omega_0$):

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H}_R, \hat{\rho}] + \Gamma\Big(b\hat{\rho}b^\dagger - \frac{1}{2}(b^\dagger b\hat{\rho} + \hat{\rho}b^\dagger b)\Big), \qquad (2)$$

where the terms including the dissipation Γ in Eq. (2) are of conventional form [23] (assuming $k_B T \ll \hbar\omega_0$) and

$$\hat{H}_R = E(b^\dagger b) + \frac{\hbar F b^\dagger + \hbar F^* b}{2} + \hbar\omega b^\dagger b.$$

The energy shifts induced by the QPS, $E(n) = \langle n|\hat{H}_S|n\rangle$, are expressed as the first order correction with respect to E_S through the hypergeometric function ${}_1F_1$:

$$E_n = 2E_S\cos(\pi q/e)\exp(-\gamma^2/2)\,{}_1F_1[-n, 1, \gamma^2].$$

We have found that if the parameter γ lies within the interval [0.3, 3], the shifts at any $n \neq 0$ can be sufficiently well approximated by the large-n asymptotic:

$$E_n = 2E_S\cos(\pi q/e)\frac{\cos(2\gamma\sqrt{n} - \frac{\pi}{4})}{\sqrt{\pi\gamma}n^{1/4}}. \qquad (3)$$

This simple formula emphasizes the central point of this Letter: it shows that the phase slips add very unusual nonlinearities to the resonantly driven oscillator. The local period of oscillations reads $\Delta n = 2\pi\sqrt{n}/\gamma$. At $\gamma \simeq 1$ it is of the order of the "width" $\langle n^2\rangle = \sqrt{n}$ of a coherent state of the oscillator corresponding to the average number of bosons n. For larger γ, the QPS shift E_n will make more oscillations at the scale of the coherent state width. This suppresses the effect of QPS at large γ.

In the absence of QPS, a driving force F brings the oscillator into a coherent state with the amplitude given by [23]

$$\lambda \equiv \langle b \rangle = -i \frac{F}{\frac{\Gamma}{2} + i\omega}. \qquad (4)$$

A straightforward but involved perturbation theory (see supplemental material [24]) gives the first order correction ($\propto E_S$) to this amplitude, valid for any γ and $k_B T$,

$$\delta\lambda = \delta\langle b \rangle = -2E_S \cos(\pi q/e)\mathcal{E}(\gamma)\frac{\gamma\lambda^2}{|\lambda|F} J_1(2\gamma|\lambda|), \qquad (5)$$

with $\mathcal{E}(\gamma) = \exp(-\frac{\gamma^2}{2}\operatorname{cotanh}\frac{\omega_0}{2T})$. The last factor ($J_1$ being the first order Bessel function) incorporates oscillations corresponding to the oscillatory behavior of the energy shifts. The exponential factor $\mathcal{E}(\gamma)$ is best understood as the effect of averaging of these oscillations over the width of the coherent state. The first order correction is exponentially suppressed at high temperature and $\gamma \gg 1$. While not all the corrections vanish at $\gamma \gg 1$, we prefer to work at $\gamma \simeq 1$, where the exponent is $\simeq 1$. We will also assume $k_B T \ll \hbar\omega_0$.

In the linear regime, $F \to 0$, the correction amounts to the frequency shift $\hbar\omega \to \hbar\omega - 2E_S\gamma^2\cos(\pi q/e)\mathcal{E}(\gamma)$. The correction becomes noticeable if it is of the order of the line width, $E_S \simeq \hbar\Gamma$ [25], and can be revealed owing to the oscillatory dependence on gate voltage.

However, the applicability of the linear regime is limited to almost no photons excited, $\lambda \ll 1$. At larger driving, the correction slowly decays with increasing $N \equiv |\lambda|^2$. At $N \gg 1$, the correction becomes significant if $E_S \gtrsim \hbar \max(\omega, \Gamma)N^{3/4}$. It is interesting to note that the oscillatory correction enhances the dependence on the detuning ω. This is why the correction becomes significant at much smaller E_S, $E_S \sim \hbar \max(\omega, \Gamma)N^{1/4}$, if one concentrates on the derivative of the amplitude with respect to detuning, $\partial\lambda/\partial\omega$. We illustrate the scale of the correction and its oscillatory dependence on λ in Fig. 2 and refer to supplemental material for details of these estimations [24].

Let us go beyond perturbation theory, to the regime where the QPS correction becomes large, leading to qualitatively different physics. We present a comprehensive semiclassical analysis that captures the essence of the full quantum solution.

In the semiclassical approximation we replace n by a continuous variable $N \approx \langle n \rangle$. The nonlinearities modify the detuning ω in Eq. (4), $\omega \to \omega + dE(N)/\hbar dN$, where $E(N)$ is defined by Eq. (3) at $\gamma \approx 1$. Squaring Eq. (4) yields a self-consistency equation for N at given F and ω [23]:

$$N = \frac{F^2}{(\frac{\Gamma}{2})^2 + (\frac{1}{\hbar}\frac{dE}{dN} + \omega)^2}. \qquad (6)$$

That suffices to make implicit plots $N(F, \omega)$. For common nonlinearities dE/dN is $\propto N$. This gives either a single solution for $N(\omega)$ or three solutions corresponding to two metastable states. The oscillatory dependence on N changes this drastically. To elucidate, we plot $N(\omega)$ in Fig. 3 at fixed $F = 15\Gamma$. At negligible E_S, $N(\omega)$ is a Lorentzian. QPS corrections shift the curve horizontally, the magnitude of the shift oscillating with a local period $\simeq \sqrt{N}$. At sufficiently large E_S this results in an impressive characteristic "corkscrew" shape. At any given ω within the oscillator line width one finds a multitude of states that differ in N. About half of these states are stable. We stress the tunability of this QPS oscillator: small changes of the driving force, detuning, or charge induced change the number of stable states, thereby enabling easy manipulation of N.

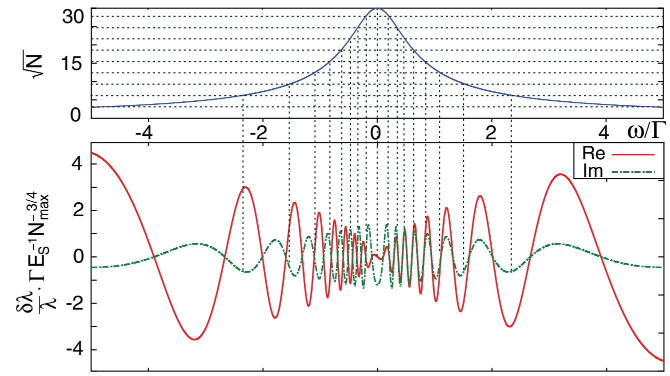

FIG. 2 (color online). QPS induced correction $\delta\lambda$ of a driven oscillation versus detuning at $F = 15\Gamma$. Real and imaginary parts are shown. Upper pane: $\sqrt{N}$ versus detuning at the same force. It illustrates the oscillation period $\Delta(\sqrt{N}) = \pi/\gamma$.

Generally, one expects fluctuation-induced switching between the available stable states. The semiclassical analysis does not account for that. Nor does it prove if a given metastable solution corresponds to a pure quantum state. It is also not clear if the semiclassical prediction for the metastable solutions works for the states with few photons. To understand this, we have performed numerical simulations using the full quantum equation Eq. (2) for density matrix. For illustration, we set $\omega = 0$ and E_S to a moderate value of $6\hbar\Gamma$. We initialize the density matrix to vacuum, $|0\rangle\langle 0|$, and compute its time-dependence while making a

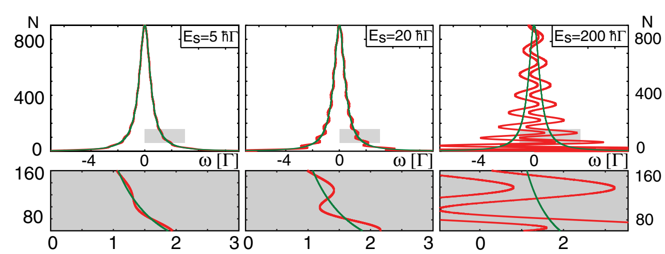

FIG. 3 (color online). Multiple stability in QPS oscillator. We plot the number of photons N versus detuning ω at $F = 15\Gamma$ as predicted by semiclassical Eq. (6). The lower panels represent insets of the shaded areas in the upper panel. For $E_S = 0$ (no QPS), $N(\omega)$ is a Lorentzian, plotted in all panes with a thinner line. For three values of QPS amplitude, there are increasing deviations from the Lorentzian. There are multiple intervals of ω with two ($E_S = 20\hbar\Gamma$) and multiple ($E_S = 200\hbar\Gamma$) stable configurations.

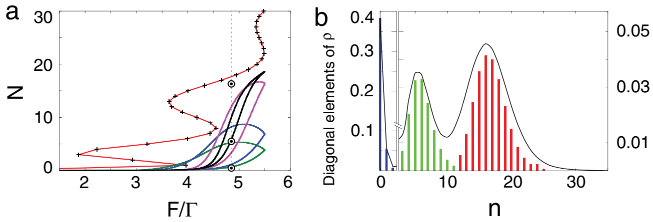

FIG. 4 (color online). (a) Hysteresis in QPS oscillator. Crossed curve: semiclassical prediction for $N(F)$ at $\omega = 0$. The solid curves give the result of slow sweep of F from 0 to 5.5Γ and back, where the sweep duration takes values 10^3, 2×10^3, 10^4, and $4 \times 10^4 \Gamma^{-1}$ from lowest to uppermost curve. The pronounced hysteresis indicates exponentially long lifetimes of the metastable states. Black circles indicate the pure states that contribute to the equilibrium density matrix at $F = 4.85\Gamma$. (b) Pure states. We show diagonal elements of the equilibrium density matrix at $F = 4.85\Gamma$. The pulses of different shades of gray show contributions to diagonal elements from three pure states of highest probability: a "dark state" with predominantly 0 photons and two coherentlike states centered around 5.5 and 16.3 photons.

linear sweep of F from 0 to 5.5Γ and back [Fig. 4(a)]. Plotting $\langle n \rangle$ versus F for different sweep durations T gives a series of curves with evident hysteresis [see Fig. 4(a)]. Generally, one expects the relaxation time of the density matrix to be of the order of $1/\Gamma$. Remarkably, a noticeable hysteresis persists even at time intervals $4 \times 10^4 \Gamma^{-1}$. This clearly indicates an exponentially long lifetime of the metastable states even for a few photons. From semiclassics we expect up to 3 metastable states in this force interval. We hypothesize that the oscillator spends most of the time in one of such states while rare switching between these states result in equilibration of the probabilities to be in these states. Such equilibration occurs at the time scale corresponding to the slowest switching rate. To prove this illustratively, we have computed the equilibrium density matrix at $F = 4.85\Gamma$ and expanded it into a sum of contributing pure states [see Fig. 4(b)]. We have found that the density matrix is mainly contributed by three pure states: one "dark" state $\approx |0\rangle$ and two coherentlike state centered around 5.5 and 16.3 photons, respectively, with probabilities 0.46, 0.25, and 0.15. The remaining probability corresponds to "excited" states that have nodes at positions of the coherentlike states centered at 5.6, 16.3. The relaxation time that characterizes the slow switching is $300\Gamma^{-1}$ at this value of F. About 4000 photons are absorbed and emitted in the oscillator during this time interval; this proves the extraordinary robustness of the states involved.

To conclude, we have investigated the effect of nonlinearities produced in a superconducting resonator by coherent phase slips. These nonlinearities are very distinct from those previously known owing to the oscillatory dependence on number of photons with a local period $\simeq \sqrt{n}$. We have demonstrated that at semiclassical level the nonlinearities result in a multitude of metastable states; this is specific for the oscillatory nonlinearity presented. The position and number of these metastable states can easily be tuned by changing the driving force. At quantum level, we have demonstrated that there is a single quantum state corresponding to the semiclassical metastable states. These states are robust, their switching time is exponentially long, although they encompass only a few photons. These features of the *phase slip oscillator* make it useful for a wide range of applications, such as ultrasensitive measurements, quantum manipulation and naturally, an unambiguous experimental verification of coherent phase slips.

We are grateful to J. E. Mooij, members of his team and to A. Ustinov for useful discussions. This work is supported by the "Stichting voor Fundamenteel Onderzoek der Materie (FOM)" and the "Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO)."

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[25] The phase slip rate quantified in M. Sahu *et al.*, Nature Phys. **5**, 503 (2009)) corresponds to $E_S \simeq 10^5$ to 10^7 Hz. For realistic $\Gamma = 10^5$ Hz this range of phase slip amplitudes can be explored with the proposed QPS oscillator.