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A micro-mechanics perspective to the invariant-based approach to stiffness

A. Arteiro[a,*], L.F. Pereira[b], M.A. Bessa[c], C. Furtado[a], P.P. Camanho[a,b]

[a] DEMec, Faculdade de Engenharia, Universidade do Porto, Rua Dr. Roberto Frias, s/n, 4200-465, Porto, Portugal
[b] INEGI, Instituto de Ciência e Inovação em Engenharia Mecânica e Engenharia Industrial, Rua Dr. Roberto Frias, 400, 4200-465, Porto, Portugal
[c] Faculty of Mechanical, Maritime and Materials Engineering, Delft University of Technology, Mekelweg 2, 2628, CD, Delft, the Netherlands

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ABSTRACT

To simplify the analysis and characterisation of composite laminates, an invariant-based approach to stiffness that takes the trace of the plane stress stiffness matrix as a material property was recently proposed. In the present work, a study based on micro-mechanical models brings new insight to this invariant-based approach. The Rule of Mixtures and the Halpin-Tsai models are used to establish the relations between the fibre volume fraction, the fibre/matrix stiffness ratios, and the trace-normalised engineering constants of unidirectional laminae and multidirectional laminates. For sufficiently high longitudinal fibre/matrix stiffness ratios and for fibre volume fractions between 50% and 70%, typical of advanced CFRPs, the variation of the trace-normalised longitudinal Young's modulus is within 6% for unidirectional laminae and within 1% for multidirectional laminates, supporting the definition of an invariant-based approach to stiffness based on a **Master Ply** concept and laminate factors derived thereof, defining clearly a domain of applicability of the invariant theory and confirming the empirical observations of the past.

1. Introduction

Despite the growing use of advanced polymer composite laminates in structural applications, especially in the aerospace industry, the experimental programmes currently required to generate the mechanical properties and design allowables for these materials are too costly and time consuming. With the aim to simplify testing, design and understanding of composites in general, Tsai and Melo [1] proposed an invariant-based approach to stiffness that takes the trace of the plane stress stiffness matrix as a material property.

The invariant-based approach proposed by Tsai and Melo [1] relies on the normalisation of the stiffness components using the trace of the plane stress stiffness matrix, which is invariant with respect to the stacking sequence:

$$\mathrm{Tr}(\mathbf{Q}) = \mathrm{Tr}(\mathbf{A}) = \mathrm{Tr}(\mathbf{D}) \tag{1}$$

where $\mathrm{Tr}(\bullet)$ stands for the trace of $(\bullet)$. $\mathbf{Q}$ is the plane stress stiffness matrix that, in the material coordinate system 1–2 (Fig. 1), can be written as:

$$\mathbf{Q} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & 2Q_{66} \end{bmatrix} = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & 2G_{12} \end{bmatrix} \tag{2}$$

with E_1 and E_2 the longitudinal and transverse Young's moduli, ν_{12} and ν_{21} the major and minor Poisson's ratios, where:

$$\nu_{21} = \nu_{12}\frac{E_2}{E_1} \tag{3}$$

and G_{12} the shear modulus of the composite material. $\mathbf{A}$ and $\mathbf{D}$ are respectively the normalised in-plane and flexural laminate stiffness matrices, which can be written in the laminate coordinate system x-y as:

$$\mathbf{A} = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \mathbf{Q}_\theta \,\mathrm{d}z\,, \quad \mathbf{D} = \frac{12}{h^3}\int_{-\frac{h}{2}}^{\frac{h}{2}} \mathbf{Q}_\theta \, z^2 \,\mathrm{d}z \tag{4}$$

where $\mathbf{Q}_\theta$ is the plane stress stiffness matrix in the laminate coordinate system x-y and h is the laminate thickness.

Tsai and Melo [1] observed that the stiffness components normalised by trace:

$$Q_{11}^* = \frac{Q_{11}}{\mathrm{Tr}(\mathbf{Q})},\ Q_{22}^* = \frac{Q_{22}}{\mathrm{Tr}(\mathbf{Q})},\ Q_{12}^* = \frac{Q_{12}}{\mathrm{Tr}(\mathbf{Q})},\ Q_{66}^* = \frac{Q_{66}}{\mathrm{Tr}(\mathbf{Q})} \tag{5}$$

of several CFRP systems are approximately the same. In particular, the trace-normalised longitudinal stiffness component Q_{11}^* had a coefficient of variation of only 1.5% [1]. Using Q_{11}^*, Q_{22}^*, Q_{12}^* and Q_{66}^*, trace-normalised engineering constants can be determined from the components

* Corresponding author.
E-mail address: aarteiro@fe.up.pt (A. Arteiro).

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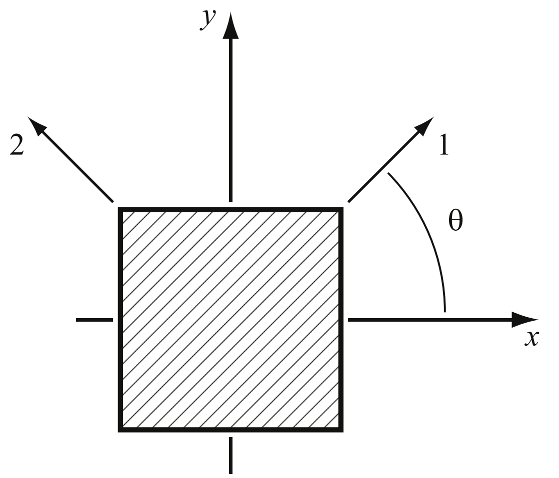

Fig. 1. Material symmetry axes 1–2 and laminate coordinate system *x*-*y*.

Table 1
Master ply and examples of laminate factors obtained from the **Master ply** using CLPT (after Tsai and Melo [2]). The elastic properties of any laminate can be obtained multiplying the trace-normalised laminate factors by the value of trace.

Master Ply	E_1^*	E_2^*	G_{12}^*	ν_{12}
[0]	0.880	0.052	0.031	0.320
Laminate factors	E_x^*	E_y^*	G_{xy}^*	ν_{xy}
[0/90]	0.468	0.468	0.031	0.036
[0/±45/90]	0.336	0.336	0.129	0.308
[0_7/±45/90]	0.662	0.175	0.070	0.310
[0_5/±45_2/90]	0.518	0.208	0.109	0.423
[0_2/±45/90]	0.445	0.289	0.109	0.308
[0/±45_4/90]	0.217	0.217	0.187	0.552
[0/±45]	0.370	0.155	0.161	0.734
[0/±45/0]	0.499	0.141	0.129	0.701
[0/±30]	0.510	0.074	0.129	1.220
[0/±30/0]	0.611	0.072	0.104	1.079
[±12.5]	0.764	0.053	0.066	0.913

of the trace-normalised compliance matrix (in engineering notation) $\mathbf{S}^* = (\mathbf{Q}^*)^{-1}$:

$$E_1^* = \frac{1}{S_{11}^*},\ E_2^* = \frac{1}{S_{22}^*},\ G_{12}^* = \frac{1}{S_{66}^*},\ \nu_{12} = -\frac{S_{12}^*}{S_{11}^*} \tag{6}$$

Using the median values of the trace-normalised engineering constants of several CFRPs, Tsai and Melo [1] proposed a **Master Ply** (Table 1) that defines a universal relation between the orthotropic in-plane engineering constants E_1, E_2, G_{12} and ν_{12} and the trace Tr(**Q**) of any CFRP:

$$E_1^* = \frac{E_1}{\mathrm{Tr}(\mathbf{Q})},\ E_2^* = \frac{E_2}{\mathrm{Tr}(\mathbf{Q})},\ G_{12}^* = \frac{G_{12}}{\mathrm{Tr}(\mathbf{Q})},\ \nu_{12} \tag{7}$$

Using the **Master Ply** E_1^*, E_2^*, and ν_{12} (Table 1), the trace-normalised laminate engineering constants E_x^*, E_y^*, G_{xy}^* and ν_{xy}, or *laminate factors*:

$$E_x^* = \frac{E_x}{\mathrm{Tr}(\mathbf{A})},\ E_y^* = \frac{E_y}{\mathrm{Tr}(\mathbf{A})},\ G_{xy}^* = \frac{G_{xy}}{\mathrm{Tr}(\mathbf{A})},\ \nu_{xy} \tag{8}$$

where Tr(**A**) = Tr(**Q**), can be determined for virtually any laminate using the Classical Laminated Plate Theory (CLPT) (Sect. 3.2). Table 1 shows the **Master Ply** proposed by Tsai and Melo [2] and examples of laminate factors for several lay-ups. This concept allows the determination of the elastic properties E_x, E_y, G_{xy} and ν_{xy} of virtually any CFRP laminate using the laminate factors E_x^*, E_y^*, G_{xy}^* and ν_{xy} obtained from the **Master Ply** and CLPT (e.g. Table 1), and one independent elastic property only — the trace Tr(**Q**) = Tr(**A**) of the plane stress stiffness matrix of the CFRP:

$$E_x = E_x^* \cdot \mathrm{Tr}(\mathbf{A}),\ E_y = E_y^* \cdot \mathrm{Tr}(\mathbf{A}),\ G_{xy} = G_{xy}^* \cdot \mathrm{Tr}(\mathbf{A}),\ \nu_{xy} \tag{9}$$

To determine the value of trace (Tr(**Q**) = Tr(**A**)), the plane stress stiffness matrices do not need to be determined. Instead, it is sufficient to know the longitudinal Young's modulus of a 0° unidirectional laminate, E_1, or the longitudinal Young's modulus of a multidirectional laminate, E_x, and to divide it by the corresponding trace-normalised laminate factor, E_1^* or E_x^* (e.g. Table 1), whose coefficients of variation are within 4% for CFRPs [2]:

$$\mathrm{Tr}(\mathbf{Q}) = \frac{E_1}{E_1^*} \qquad (\textbf{Master Ply}),\ \text{or} \tag{10a}$$

$$\mathrm{Tr}(\mathbf{Q}) = \mathrm{Tr}(\mathbf{A}) = \frac{E_x}{E_x^*} \qquad (\text{laminate factor}) \tag{10b}$$

Hence, this **Master Ply** concept can simplify laminate stiffness characterisation considerably by reducing the number of tests required to fully characterise the orthotropic in-plane elastic properties of unidirectional composites from 3 tests at the ply level (including the highly non-linear in-plane shear test on a ±45° coupon, which adds complexity and cost to the characterisation programme due to off-axis lamination and associated scrap material) to 1 test at the ply level (to obtain E_1), or 1 test at the laminate level (to obtain E_x — for example, of a simple cross-ply laminate [3]). It is interesting to note that, with 1 test at the laminate level, lamination effects are intrinsically accounted for, potentially replacing additional tests at the laminate level sometimes required to account for these effects (e.g. aerospace industry).

Although the reliability of this approximate approach to determine the elastic properties of engineering laminates has been thoroughly validated in the past [1–4], its empirical nature has been regarded as an obstacle to its acceptance and adoption by the academic and industrial communities. In the present work, a study based on micro-mechanical models is presented, which brings a new perspective to the invariant-based approach to stiffness proposed by Tsai and Melo [1].

2. Effective ply elastic properties

The Rule of Mixtures (RoM), being the simplest micro-mechanics model, provides reasonable predictions for the effective longitudinal stiffness of a unidirectional ply, E_1, by assuming that the longitudinal strain in the fibres and matrix is uniform. This is equivalent to a parallel model, which can be written as:

$$E_1 = ((\gamma_1 - 1)V_f + 1)E_m \tag{11}$$

where V_f is the fibre volume fraction and $\gamma_1 = E_{1f}/E_m$ is the ratio between the longitudinal Young's modulus of the fibres E_{1f}, assumed to be transversely isotropic, and the Young's modulus of the matrix E_m, assumed to be isotropic. The same assumption (uniform longitudinal strain in the fibres and matrix) also leads to reasonable estimations of the effective Poisson's ratio ν_{12}:

$$\nu_{12} = \left(\left(\frac{\nu_{12f}}{\nu_m} - 1\right)V_f + 1\right)\nu_m \tag{12}$$

where ν_{12f} is the Poisson's ratio of the fibres and ν_m the Poisson's ratio of the isotropic matrix.

However, simple RoM cannot provide reasonable predictions for the effective transverse and shear elastic properties. Alternative micro-mechanical models, often based on empirical parameters, are required to obtain better estimates of these properties. One of these models is the Halpin-Tsai model [5–7], which gives the following estimates for the effective transverse Young's modulus, E_2, and shear modulus, G_{12}, respectively:

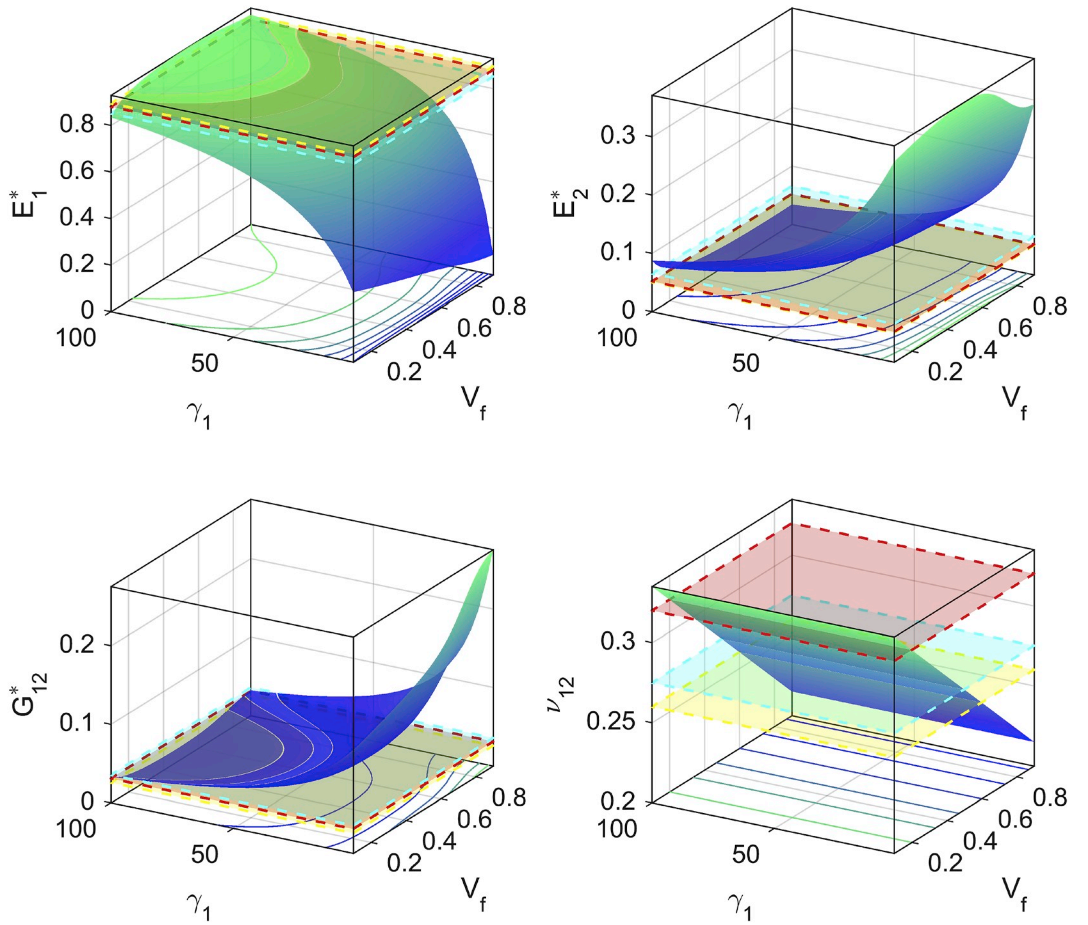

Fig. 2. Effective trace-normalised ply elastic moduli E_1^*, E_2^* and G_{12}^* and effective ply Poisson's ratio ν_{12} (**Master Ply**) as a function of the fibre volume fraction V_f and of the ratio between the longitudinal Young's modulus of the fibres and the Young's modulus of the matrix γ_1, with $\gamma_2 = 4$, $\gamma_{12} = 10$, $\nu_{12f} = 0.2$ and $\nu_m = 0.35$. The dashed lines are, respectively, the **Master Ply** values from Table 1 (red), the median values of the predictions for $0.5 < V_f < 0.7$ and $\gamma_1 \gtrsim 50$ (yellow), and the median values of the predictions for $0.1 < V_f < 0.9$ and $1 < \gamma_1 < 100$ (cyan). (For interpretation of the references to colour in this figure legend, the reader is referred to the Web version of this article.)

$$E_2 = \frac{1 + \zeta_2 \eta_2 V_f}{1 - \eta_2 V_f} E_m \,, \quad \eta_2 = \frac{\gamma_2 - 1}{\gamma_2 + \zeta_2} \tag{13}$$

$$G_{12} = \frac{1 + \zeta_{12} \eta_{12} V_f}{1 - \eta_{12} V_f} G_m \,, \quad \eta_{12} = \frac{\gamma_{12} - 1}{\gamma_{12} + \zeta_{12}} \tag{14}$$

where $\gamma_2 = E_{2f}/E_m$ is the ratio between the transverse Young's modulus of the fibres (E_{2f}) and the Young's modulus of the matrix, $\gamma_{12} = G_{12f}/G_m$ is the ratio between the shear modulus of the fibres (G_{12f}) and the shear modulus of the matrix (G_m):

$$G_m = \frac{E_m}{2(1 + \nu_m)} \tag{15}$$

and ζ_2 and ζ_{12} are empirical parameters that account for the reinforcement geometry under transverse and shear loading, respectively. For simplicity, and based on finite element analyses [8] and curve fitting of experimental data [9], in the following it will be assumed that $\zeta_2 = 1.5$ and $\zeta_{12} = 1 + 40V_f^{10}$, results that are valid for at least $0.20 \lesssim V_f \lesssim 0.55$ and $V_f \lesssim 0.70$, respectively.

3. Trace-normalised stiffness components

3.1. Master Ply

Using Eq. (15) in Eq. (14), the effective elastic moduli in Eqs. (11), (13) and (14) can be written as functions of the fibre volume fraction V_f, of the ratios between the elastic moduli of the fibres and matrix γ_i ($i = 1,2,12$), and of the Young's modulus of the matrix E_m. The effective elastic moduli and the effective Poisson's ratio (Eq. (12)) can be used to determine the components of the effective stiffness matrix **Q** using Eq. (2). These components can then be used to determine the trace of the effective stiffness matrix, $\mathrm{Tr}(\mathbf{Q}) = Q_{11} + Q_{22} + 2Q_{66}$, which is used to normalise the effective elastic moduli E_1, E_2 and G_{12}. The resulting trace-normalised elastic moduli E_1^*, E_2^* and (Eq. (7)) are now only functions of the fibre volume fraction V_f, of the ratios between the elastic moduli of the fibres and matrix γ_i, and of the Possion's ratios ν_{12f} and ν_m.

Fig. 2 shows the effect of the fibre volume fraction ($0.1 < V_f < 0.9$) and of the ratio between the longitudinal Young's modulus of the fibres and the Young's modulus of the matrix ($1 < \gamma_1 < 100$) on the effective trace-normalised ply elastic moduli E_1^*, E_2^* and and on the effective ply Poisson's ratio ν_{12}. Based on the standard properties of CFRPs [10], $\gamma_2 = 4$, $\gamma_{12} = 10$, $\nu_{12f} = 0.2$ and $\nu_m = 0.35$ were used.

For fibre volume fractions in the range $0.5 < V_f < 0.7$ and longitudinal fibre/matrix stiffness ratios $\gamma_1 \gtrsim 50$, typical of advanced CFRPs [10], the effective trace-normalised ply elastic moduli tend to a constant value in the range of the **Master Ply** values reported by Tsai and Melo [2] (see Table 1), shown as red dashed lines in Fig. 2. The effective Poisson's ratio ν_{12}, however, although independent of γ_1, as would be expected from Eq. (12), shows a linear variation with V_f, and it is usually below the **Master Ply** value reported by Tsai and Melo [2] for fibre volume fractions in the range $0.5 < V_f < 0.7$. Nevertheless, the results presented in Fig. 2 support that, in the case of CFRPs (or other

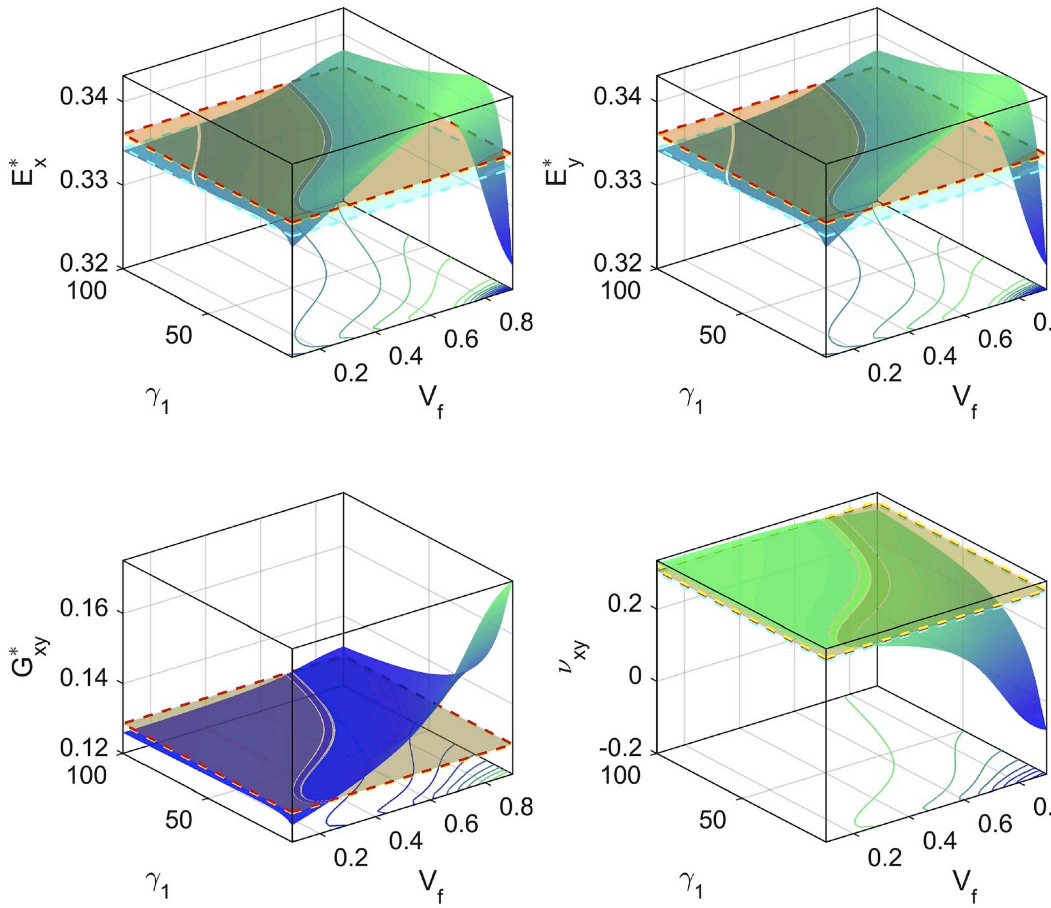

Fig. 3. Trace-normalised laminate factors E_x^*, E_y^*, G_{xy}^* and ν_{xy} of a [0/±45/90] lay-up as a function of the fibre volume fraction V_f and of the ratio between the longitudinal Young's modulus of the fibres and the Young's modulus of the matrix γ_1, with $\gamma_2 = 4$, $\gamma_{12} = 10$, $\nu_{12f} = 0.2$ and $\nu_m = 0.35$. The dashed lines are, respectively, the values of the laminate factors in Table 1 (red), the values calculated using the median values of the predictions for $0.5 < V_f < 0.7$ and $\gamma_1 \gtrsim 50$ (yellow), and the values calculated using the median values of the predictions for $0.1 < V_f < 0.9$ and $1 < \gamma_1 < 100$ (cyan). (For interpretation of the references to colour in this figure legend, the reader is referred to the Web version of this article.)

FRPs with $\gamma_1 \gtrsim 50$), it is possible to define the stiffness of composite laminae based on an invariant of the stiffness matrix using a **Master Ply** concept, reducing the number of independent stiffness properties to one.

For lower values of the longitudinal fibre/matrix stiffness ratio (e.g. in the range $\gamma_1 \approx 22$–24 as in GFRPs [10]), higher variations on the effective trace-normalised ply elastic moduli E_1^*, E_2^* and can be observed. These variations become even more important when the reinforcing fibres are assumed isotropic, i.e. $E_{2f} = E_{1f}$ and $G_{12f} = E_{1f}/(2(1 + \nu_{12f}))$, as it is usually the case in glass fibres.[1] In addition, in this case ($\gamma_1 \lesssim 25$ and isotropic fibres), the fibre volume fraction has a greater effect, not only on E_1^*, but more importantly on E_2^* and . This explains why the **Master Ply** concept or an invariant-based approach to stiffness for GFRPs does not exist.

3.2. Laminate factors

Tsai and Melo [2] showed that another benefit of the **Master Ply** concept comes from the ability to define laminate factors for virtually any lay-up. Consequently, also the stiffness of composite laminates can be defined using just an invariant of the stiffness matrix (independent of lay-up and coordinate system — see Eq. (1)) and the corresponding **Master Ply**-based laminate factors.

Using the calculated effective trace-normalised ply elastic moduli E_1^*, E_2^* and and effective ply Poisson's ratio ν_{12} to determine the trace-normalised plane stress stiffness matrix $\mathbf{Q}^*$ (in engineering notation):

$$\mathbf{Q}^* = \begin{bmatrix} \frac{E_1^*}{1-\nu_{12}\nu_{21}} & \frac{\nu_{12}E_2^*}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{21}E_1^*}{1-\nu_{12}\nu_{21}} & \frac{E_2^*}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12}^* \end{bmatrix} \tag{16}$$

the effective laminate factors E_x^*, E_y^*, G_{xy}^* and ν_{xy} can be derived using the CLPT:

$$E_x^* = \frac{1}{a_{11}^*}\,, \quad E_y^* = \frac{1}{a_{22}^*}\,, \quad G_{xy}^* = \frac{1}{a_{66}^*}\,, \quad \nu_{xy} = -\frac{a_{12}^*}{a_{11}^*} \tag{17}$$

where the normalised compliance matrix of the **Master Ply**, $\mathbf{a}^*$, is obtained as:

$$\mathbf{a}^* = (\mathbf{A}^*)^{-1}\,, \quad \text{with } \mathbf{A}^* = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \mathbf{Q}_\theta^*\,\mathrm{d}z \tag{18}$$

after transformation of $\mathbf{Q}^*$ to the laminate coordinate system x-y (Fig. 1) — $\mathbf{Q}_\theta^*$. Figs. 3–5 show the effect of the fibre volume fraction ($0.1 < V_f < 0.9$) and of the ratio between the longitudinal Young's modulus of the fibres and the Young's modulus of the matrix

[1] The effective trace-normalised ply elastic moduli E_1^*, E_2^* and G_{12}^* and effective ply Poisson's ratio ν_{12} as a function of the fibre volume fraction V_f and of the ratio between the longitudinal Young's modulus of the fibres and the Young's modulus of the matrix γ_1 for an isotropic fibre are provided as Supplementary Data on the online version of this paper.

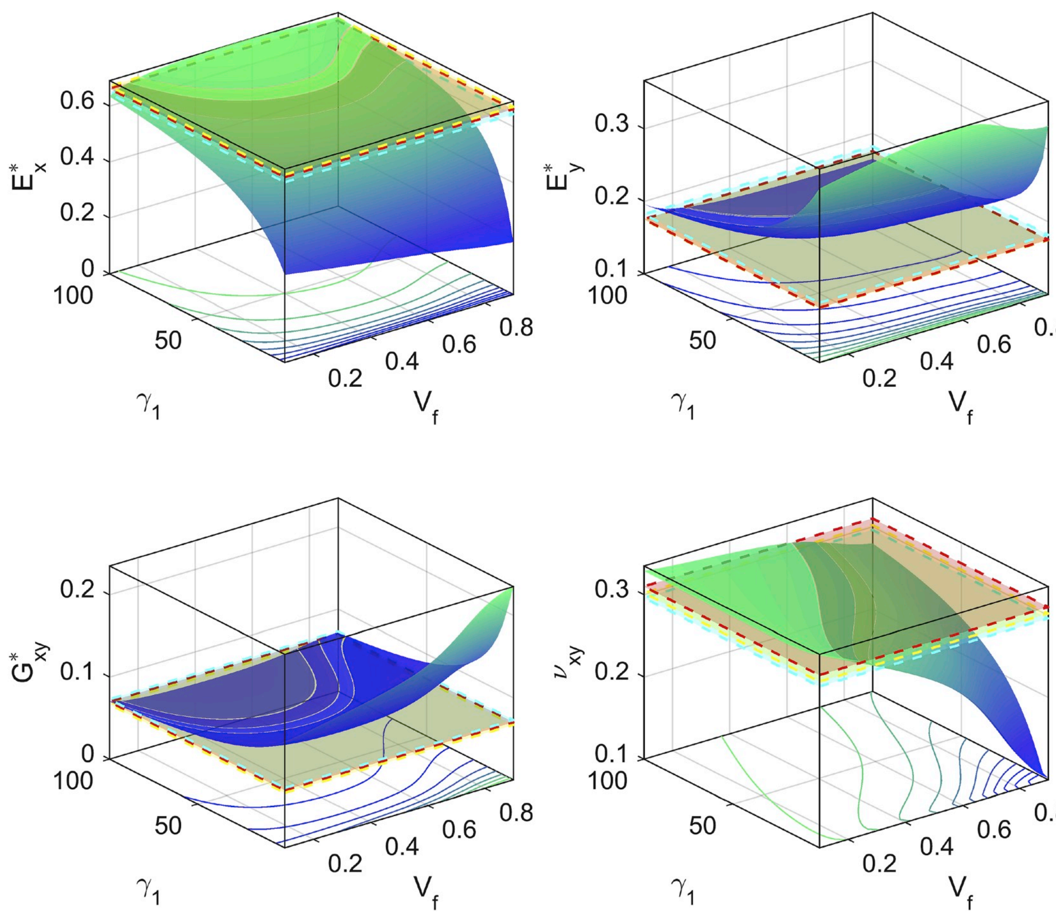

Fig. 4. Trace-normalised laminate factors E_x^*, E_y^*, G_{xy}^* and ν_{xy} of a $[0_7/\pm45/90]$ lay-up as a function of the fibre volume fraction V_f and of the ratio between the longitudinal Young's modulus of the fibres and the Young's modulus of the matrix γ_1, with $\gamma_2 = 4$, $\gamma_{12} = 10$, $\nu_{12f} = 0.2$ and $\nu_m = 0.35$. The dashed lines are, respectively, the values of the laminate factors in Table 1 (red), the values calculated using the median values of the predictions for $0.5 < V_f < 0.7$ and $\gamma_1 \gtrsim 50$ (yellow), and the values calculated using the median values of the predictions for $0.1 < V_f < 0.9$ and $1 < \gamma_1 < 100$ (cyan). (For interpretation of the references to colour in this figure legend, the reader is referred to the Web version of this article.)

($1 < \gamma_1 < 100$) on the trace-normalised laminate factors for three laminates from Table 1 (similar analyses for the remaining laminates in Table 1 are provided as Supplementary Data on the online version of this paper). Again, it is clear that, for the fibre volume fractions V_f and fibre/matrix stiffness ratios γ_1 typical of advanced CFRPs ($0.5 < V_f < 0.7$ and $\gamma_1 \gtrsim 50$), the effective trace-normalised laminate properties tend to constant values close to the laminate factors determined using the **Master Ply** concept [2] (red dashed lines in Figs. 3–5). It is interesting to note that, in this case, also the effective laminate Poisson's ratios ν_{xy} tend to constant values for the typical fibre volume fractions and fibre/matrix stiffness ratios of advanced CFRPs, close to the **Master Ply**-based values in Table 1 (red dashed lines in Figs. 3–5).

4. Analysis verification

To validate the previous results, four FRP systems, two CFRPs and two GFRPs, will be analysed based on the experimental data available from Ref. [10] (Table 2). Using the elastic properties of the constituents, the predictions of the effective trace-normalised elastic properties for each of the four unidirectional laminae in Table 2 were obtained using the micro-mechanical models presented in Sect. 2, for fibre volume fractions in the range $0.1 < V_f < 0.9$. These predictions are shown in Figs. 6 and 7, together with the **Master Ply** values from Table 1 (dashed horizontal lines) and the experimentally determined trace-normalised elastic properties for the unidirectional laminae in Table 2 (open circles). As can be observed, the RoM and Halpin-Tsai models provide reasonable predictions of the elastic properties of all unidirectional laminae, here normalised by the trace of the plane stress stiffness matrix.

For the carbon/epoxy laminae (Fig. 6), the effective trace-normalised stiffness properties for fibre volume fractions in the range $0.5 < V_f < 0.7$ (typical of advanced CFRPs) are within the **Master Ply** values from Table 1 (the difference for E_1^* is lower than 2%), confirming the observations from Sect. 3.1. To further elucidate about the applicability of the **Master Ply** concept for general CFRPs, Fig. 8 shows the effect of γ_1 on the predicted effective trace-normalised elastic properties of AS4/3501-6, obtained for variations of the longitudinal Young's modulus of the fibres (E_{1f}) within 50% (the effect of the remaining fibre/matrix stiffness ratios on AS4/3501-6 and on T300/BSL914C are provided as Supplementary Data on the online version of this paper). As can be observed, the variation of the longitudinal Young's modulus of the fibres leads to variations of the trace-normalised stiffness properties within the range of the **Master Ply** values from Table 1, especially for E_1^*.

In addition, the effective trace-normalised stiffness properties of fibreglass/epoxy laminae (Fig. 7) are not only substantially different from the **Master Ply** values from Table 1, as would be expected (since the latter were obtained specifically from CFRPs), but also the variation of the trace-normalised stiffness properties is more pronounced for the usually wide range of fibre volume fractions observed on GFRPs (the effect of the fibre/matrix stiffness ratios on Gevetex/LY556 and Silenka/MY750 are provided as Supplementary Data on the online version of this paper). This observation again supports the difficulties of defining a **Master Ply** concept or an invariant-based approach to stiffness for GFRPs.

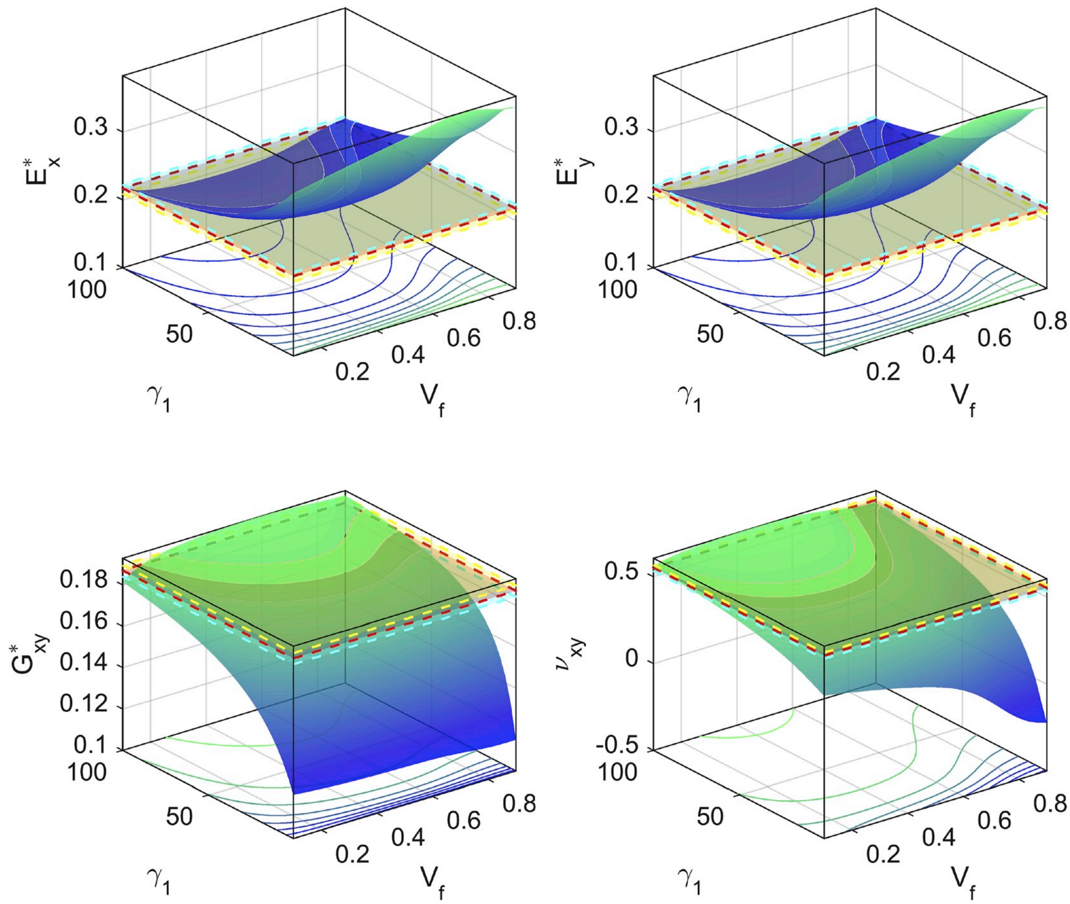

Fig. 5. Trace-normalised laminate factors E_x^*, E_y^*, G_{xy}^* and ν_{xy} of a $[0/\pm45_4/90]$ lay-up as a function of the fibre volume fraction V_f and of the ratio between the longitudinal Young's modulus of the fibres and the Young's modulus of the matrix γ_1, with $\gamma_2 = 4$, $\gamma_{12} = 10$, $\nu_{12f} = 0.2$ and $\nu_m = 0.35$. The dashed lines are, respectively, the values of the laminate factors in Table 1 (red), the values calculated using the median values of the predictions for $0.5 < V_f < 0.7$ and $\gamma_1 \gtrsim 50$ (yellow), and the values calculated using the median values of the predictions for $0.1 < V_f < 0.9$ and $1 < \gamma_1 < 100$ (cyan). (For interpretation of the references to colour in this figure legend, the reader is referred to the Web version of this article.)

Table 2
Elastic properties of four unidirectional laminae, their fibres and their matrices [10].

Material system	E_1 (GPa)	E_2 (GPa)	G_{12} (GPa)	ν_{12} (−)	V_f (−)
AS4/3501-6	126	11	6.6	0.28	0.60
T300/BSL914C	138	11	5.5	0.28	0.60
Gevetex/LY556	53.48	17.7	5.83	0.278	0.62
Silenka/MY750	45.6	16.2	5.83	0.278	0.60

Fibres	E_{1f} (GPa)	E_{2f} (GPa)	G_{12f} (GPa)	ν_{12f} (−)
AS4	225	15	15	0.2
T300	230	15	15	0.2
E-glass 21xK43 Gevetex	80	80	33.33	0.2
Silenka E-Glass 1200tex	74	74	30.8	0.2

Matrices	E_m (GPa)	ν_m (−)
3501–6	4.2	0.34
BSL914C	4.0	0.35
LY556/HT907/DY063	3.35	0.35
MY750/HY917/DY063	3.35	0.35

5. Discussion

Based on the micro-mechanical models presented herein, it is observed that, for $\gamma_1 \gtrsim 50$ and $0.5 < V_f < 0.7$, the variations of the effective E_1^* are within 6% (Fig. 2), and the variations of the effective E_x^* are within 1% (e.g. Fig. 3), supporting the definition of an invariant-based approach to stiffness for CFRPs (or other FRPs with similar fibre/matrix stiffness ratios) using a **Master Ply** concept and laminate factors derived thereof. In fact, a **Master Ply** can be defined based on the results from Sect. 3.1 by taking the median values of the effective trace-normalised elastic properties for $\gamma_1 \gtrsim 50$ and $0.5 < V_f < 0.7$, for example. This is shown as yellow dashed lines in Figs. 2 and 9. The difference to the **Master Ply** values from Table 1 is very small, except for the Poisson's ratio ν_{12}, as already pointed out in Sect. 3.1. Similarly, it is also possible to define a **Master Ply** using the entire dataset shown, for instance, in Fig. 2, obtained using the micro-mechanical models from Sect. 2. This is shown as cyan dashed lines (Fig. 2). Interestingly, also in this case, the obtained values do not differ substantially from the **Master Ply** values from Table 1, except for the Poisson's ratio ν_{12}[2]. It is, however, noted, that these values, obtained from either a reduced or the entire dataset, provide reasonable approximations of the trace-normalised elastic properties only for $\gamma_1 \gtrsim 50$. This observation highlights the difficulties to define an invariant-based approach to stiffness for FRPs with low fibre/matrix stiffness ratios, such as GFRPs with $\gamma_1 \approx 22$–24.

Using the calculated **Master Ply** values from the reduced and from the entire micro-mechanics datasets, it is also possible to generate

[2] Based on these results, a new **Master Ply** value of the Poisson's ratio ν_{12} in the range of 0.260–0.275 would possibly provide a better approximation of the ply Poisson's ratio for advanced CFRPs instead of the proposed 0.320 (Table 1) [2].

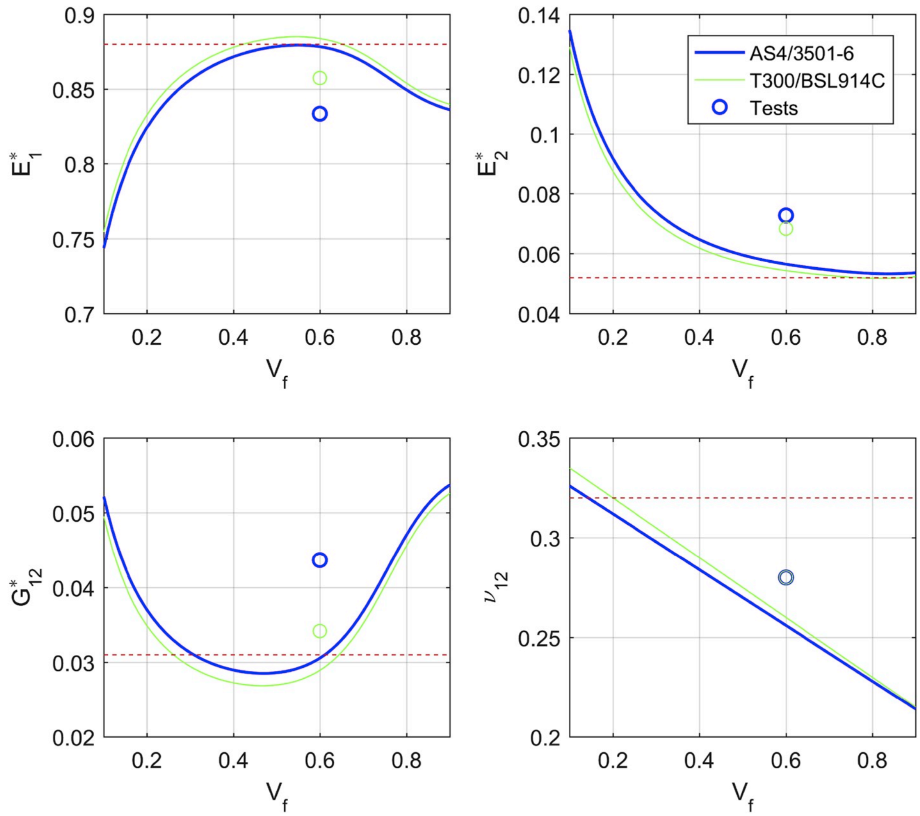

Fig. 6. Effective trace-normalised ply elastic moduli E_1^*, E_2^* and G_{12}^* and effective ply Poisson's ratio ν_{12} for two carbon/epoxy systems as a function of fibre volume fraction V_f (full lines). Comparison with the **Master Ply** values from Table 1 (dashed horizontal lines) and experimental data from Table 2 (open circles).

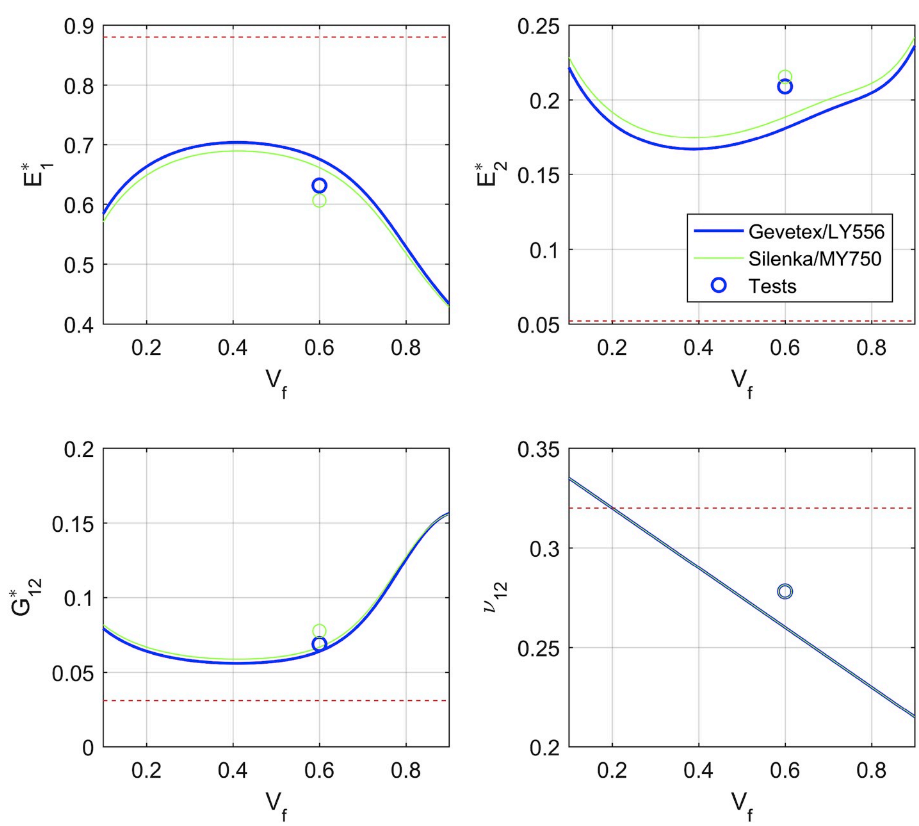

Fig. 7. Effective trace-normalised ply elastic moduli E_1^*, E_2^* and G_{12}^* and effective ply Poisson's ratio ν_{12} for two fibreglass/epoxy systems as a function of fibre volume fraction V_f (full lines). Comparison with the experimental data from Table 2 (open circles). For reference, the **Master Ply** values from Table 1 are also shown (dashed horizontal lines).

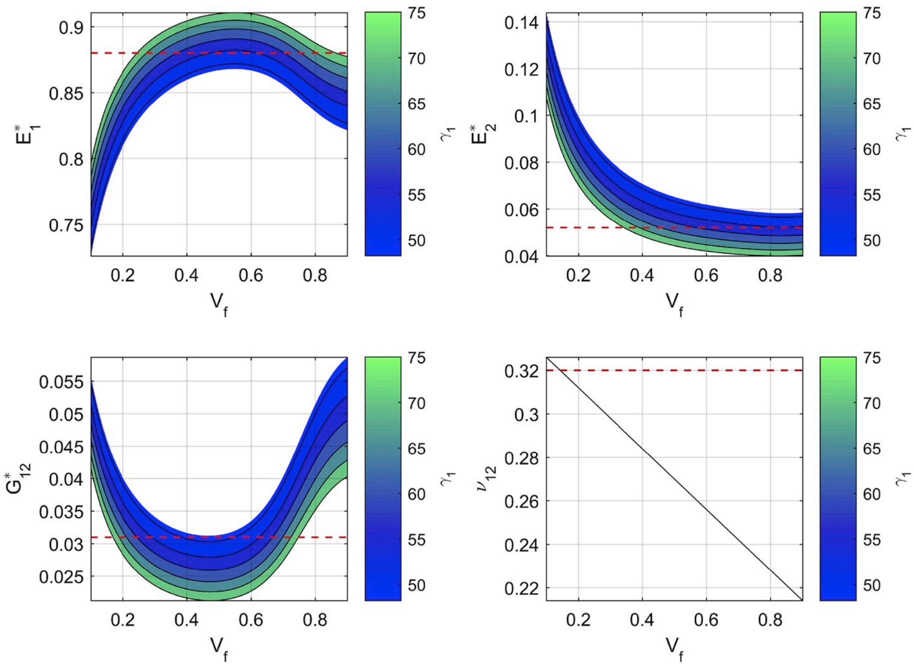

Fig. 8. Effect of γ_1 on the effective trace-normalised ply elastic moduli E_1^*, E_2^* and G_{12}^* and effective ply Poisson's ratio ν_{12} for AS4/3501-6 carbon/epoxy as a function of fibre volume fraction V_f. Comparison with the **Master Ply** values from Table 1 (dashed horizontal lines).

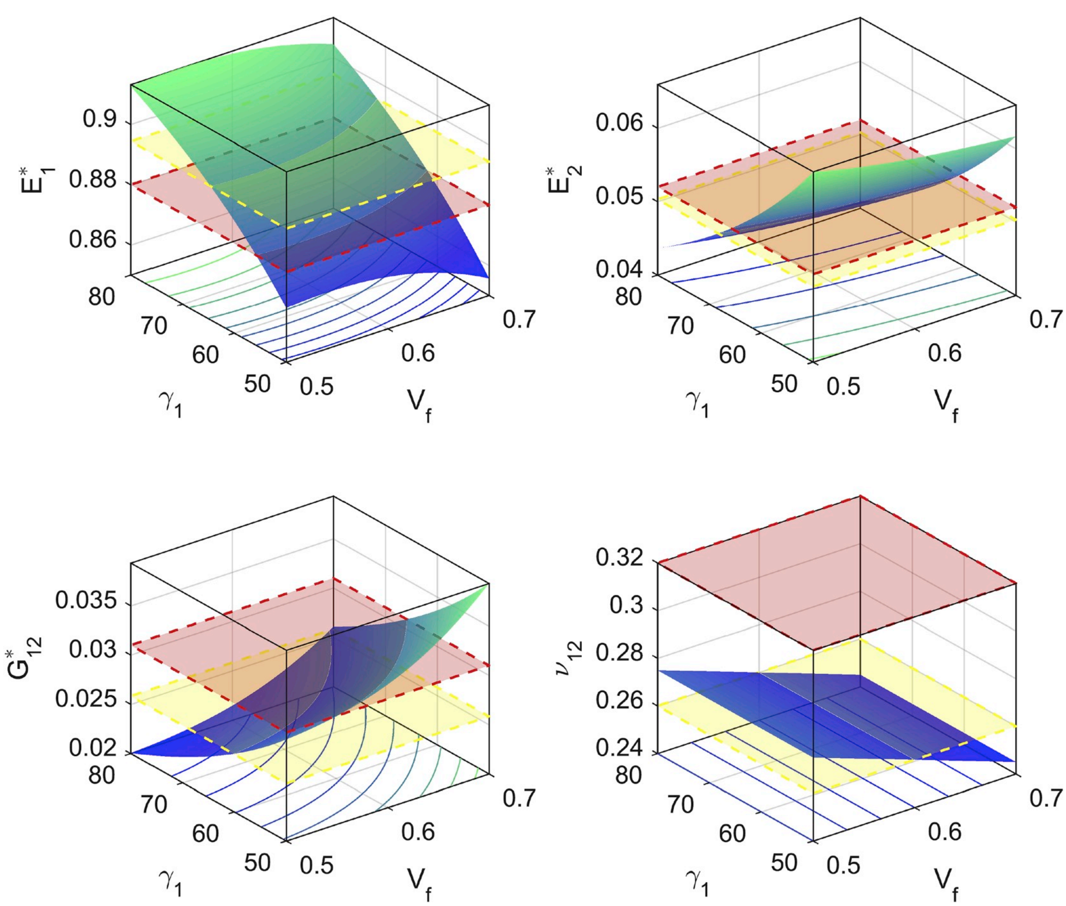

Fig. 9. Effective trace-normalised ply elastic moduli E_1^*, E_2^* and G_{12}^* and effective ply Poisson's ratio ν_{12} (**Master Ply**) for fibre volume fractions $0.5 < V_f < 0.7$ and fibre/matrix stiffness ratios $\gamma_1 \gtrsim 50$, with $\gamma_2 = 4$, $\gamma_{12} = 10$, $\nu_{12f} = 0.2$ and $\nu_m = 0.35$. The red dashed lines are the **Master Ply** values from Table 1, and the yellow dashed lines are the median values of the predictions for $0.5 < V_f < 0.7$ and $\gamma_1 \gtrsim 50$. (For interpretation of the references to colour in this figure legend, the reader is referred to the Web version of this article.)

laminate factors for the lay-ups presented in Table 1. These laminate factors are shown, respectively, as yellow and cyan dashed lines in Figs. 3–5. Independently of the lay-up, reasonable approximations of the trace-normalised elastic properties are obtained for $\gamma_1 \gtrsim 50$, typical, for example, of advanced CFRPs. Moreover, the difference between the calculated laminate factors based on the micro-mechanical data and the laminate factors from Table 1 is remarkably small. This is attributed to the predominance of E_1^* in the determination of the laminate factors, diluting the wider variation of the remaining trace-normalised ply elastic properties [2]. This observation shows clearly that an invariant-based approach to stiffness relying on a **Master Ply** concept is a viable solution to not only simplify the analysis of FRPs with stiff fibres (e.g. CFRPs), as proposed by Tsai and Melo [1], but also reduce the number of material properties required to characterise these materials.

6. Conclusions

With the aim of understanding the origin and domain of validity of the invariant-based approach to stiffness proposed by Tsai and Melo [1], micro-mechanical models were used to obtain the relation between the fibre volume fraction, the fibre/matrix stiffness ratios, and the trace-normalised engineering constants of unidirectional laminae and multidirectional laminates. Based on the RoM and Halpin-Tsai models, it was shown that for longitudinal fibre/matrix stiffness ratios $\gamma_1 \gtrsim 50$ and for fibre volume fractions in the range $0.5 < V_f < 0.7$, the variation of the trace-normalised longitudinal Young's modulus is within 6% for unidirectional laminae (E_1^*) and within 1% for multidirectional laminates (E_x^*). This observation supports the definition of an invariant-based approach to stiffness based on a **Master Ply** concept and laminate factors derived thereof for FRPs whose fibres are substantially stiffer than the matrix material.

For lower fibre/matrix stiffness ratios, small variations on the elastic properties of the constituents lead to higher variations on the trace-normalised elastic properties. These results confirm the difficulties to define a **Master Ply** for FRPs with fibres of lower stiffness, including GFRPs (with $\gamma_1 \approx 22$–24).

It is important to note that, for the first time, the empirical observations from Tsai and Melo [1] have been confirmed by micro-mechanical models, clearly defining the domain of application of the invariant-based approach to stiffness for FRPs based on the properties of the composite constituents. The applicability of the **Master Ply** proposed by Tsai and Melo [2] based on the median values of several CFRPs is verified for virtually any advanced carbon fibre-reinforced composite (typically with $\gamma_1 \gtrsim 50$ and $0.5 < V_f < 0.7$), especially when estimating laminate elastic properties based on laminate factors obtained using the **Master Ply** concept.

Because the Master Ply concept is based on an averaging procedure of normalised stiffness properties, for some, less abundant composites, such as high modulus CFRPs, the discrepancies to the experimental values can be up to 6% [4]. Although an estimate within 6% is reasonable enough for most applications, the present work indicates that this difference can be reduced (for high strength CFRPs this difference is around 1% [4]) by considering a **Master Ply** based on the averaging of normalised stiffness properties of high modulus CFRPs. Thus, the definition of the range of applicability of the invariant-based approach to stiffness is of critical importance to enable the use of this approach by the industry.

It is also important to stress that the adoption of the invariant-based approach to stiffness allows the definition of the stiffness of composite laminae and laminates based solely on an invariant of the stiffness matrix — the trace — reducing the number of independent stiffness properties to one. This not only reduces the number and complexity of the tests required to characterise advanced FRPs [3], but it also makes preliminary material and laminate scrutiny easier and more cost effective, laminate design and optimisation faster [11,12], and generation of design allowables more efficient [13], improving the competitiveness of composite materials in applications such as in aerospace and automotive. Although the use of very simple models such as the RoM and Halpin-Tsai models gives a reasonably good estimation of the elastic constants, they require characterisation of the constituent materials (fibres and matrix), which is complex (e.g. fibre testing requires a great amount of work due to preparation and repetition of the tests to tackle the intrinsic variability in the properties, while polymer characterisation requires the preparation and adoption of non-conventional manufacturing procedures that do not resemble completely the composite manufacturing conditions) and cannot usually be done by the composite end-users for characterisation and certification purposes. Hence, unless the constituent properties are available, the RoM and Halpin-Tsai models cannot be applied. In addition, lamination effects and variations in the manufacturing process (from user to user) cannot be easily considered based solely on the constituent properties. The **Master Ply** concept allows considering these effects with a minimal number of independent properties and test configurations.

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Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.compscitech.2019.04.002.

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