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On a class of exponential changes of measure for stochastic PDEs

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ABSTRACT

Given a mild solution X to a semilinear stochastic partial differential equation (SPDE), we consider an exponential change of measure based on its infinitesimal generator L , defined in the topology of bounded pointwise convergence. The changed measure \mathbb{P}^h depends on the choice of a function h in the domain of L . In our main result, we derive conditions on h for which the change of measure is of Girsanov-type. The process X under \mathbb{P}^h is then shown to be a mild solution to another SPDE with an extra additive drift-term. We illustrate how different choices of h impact the law of X under \mathbb{P}^h in selected applications. These include the derivation of an infinite-dimensional diffusion bridge as well as the introduction of guided processes for SPDEs, generalizing results known for finite-dimensional diffusion processes to the infinite-dimensional case.

1. Introduction

Consider a semilinear *stochastic partial differential equation (SPDE)* of the form

$$\begin{cases} dX(t) = [AX(t) + F(t, X(t))] dt + \sqrt{Q} dW(t), & t \geq s, \\ X(s) = x. \end{cases} \quad (1)$$

The operator A denotes the generator of a strongly continuous semigroup $(S_t)_{t \geq 0}$ on a Hilbert space H , whereas F denotes a non-linear operator and Q is a symmetric, positive operator on H . The process W is a cylindrical Wiener process on H , defined on a stochastic basis $(\Omega, \mathcal{F}, (F_t)_{t \geq 0}, \mathbb{P})$. We assume that the operators A , F and Q satisfy suitable conditions such that Eq. (1) admits a unique mild solution $X = (X(t, s, x))_{t \geq s}$ for any $s \geq 0$ and $x \in H$. Throughout the article we fix some arbitrary $x_0 \in H$ and simply write $X(t)$ if the SPDE in (1) is assumed to be initialized at $X(0) = x_0$.

For any $m \in \mathbb{N}$, let $C_m(\mathbb{R}_+ \times H)$ be the Banach space of continuous functions $\varphi : \mathbb{R}_+ \times H \rightarrow \mathbb{R}$ such that $\|\varphi\|_m = \sup_{t,x} (1 + \|x\|^m)^{-1} |\varphi(t, x)| < \infty$. The process X is Markovian and defines a transition semigroup

$$(T_t \varphi)(s, x) = \mathbb{E}[\varphi(s + t, X(t + s, s, x))], \quad s, t \geq 0, x \in H,$$

on $C_m(\mathbb{R}_+ \times H)$. It is well-known that the semigroup $(T_t)_{t \geq 0}$ is not strongly continuous with respect to the norm topology on $C_m(\mathbb{R}_+ \times H)$, see e.g. Cerrai [1] and Da Prato [2]. However, it does possess the properties of a strongly continuous semigroup in several weaker ‘modes of convergence’. This has been studied in the framework of \mathcal{K} -convergence in Cerrai [1,3] and Cerrai and Gozzi [4], the mixed topology in Goldys and Kocan [5] and of bp - (bounded pointwise) or π -convergence in Priola [6]. See also Fabbri et al. [7], Appendix

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B for a recent survey. In the respective convergence of choice, one can then define an infinitesimal generator $(L, \text{dom}_m(L))$ of the semigroup $(T_t)_{t \geq 0}$ in the usual way. In this article, we will work within the framework of π -convergence as introduced in Priola [6].

Crucially, the operator $(L, \text{dom}_m(L))$ exhibits the common properties that are characteristic for infinitesimal generators of strongly continuous semigroups. Of particular importance for us is the fact that *Dynkin's formula* holds, i.e. for any $h \in (L, \text{dom}_m(L))$ the process

$$D^h(t) = h(t, X(t)) - \int_0^t Lh(s, X(s)) ds$$

is a \mathbb{P} -martingale. In other words, X solves the *martingale problem* of $(L, \text{dom}_m(L))$ as introduced in Stroock and Varadhan [8]. Furthermore, one can show that for any positive $h \in \text{dom}_m(L)$, the process

$$E^h(t) = \frac{h(t, X(t))}{h(0, x_0)} \exp\left(-\int_0^t \frac{Lh}{h}(s, X(s)) ds\right), \quad t \geq 0,$$

whenever existent, is a positive, continuous local \mathbb{P} -martingale with $\mathbb{E}[E^h(0)] = 1$. If E^h is a true \mathbb{P} -martingale, it defines an *exponential change of measure* \mathbb{P}^h on \mathcal{F} such that for any $t \geq 0$

$$d\mathbb{P}^h_{|_{\mathcal{F}_t}} = E^h(t) d\mathbb{P}_{|_{\mathcal{F}_t}}. \quad (2)$$

The change of measure \mathbb{P}^h is well-known in the literature for Markov processes, see Palmowski and Rolski [9] and references within. If the function h is harmonic, i.e. $Lh = 0$, it is known as *Doob's h -transform*, following its introduction in Doob [10]. In Palmowski and Rolski [9] it was shown that X remains Markovian under \mathbb{P}^h and solves the martingale problem corresponding to a perturbation of L .

In this article we aim to establish conditions on the h -function under which X is not only Markovian under the changed measure, but again the mild solution of another SPDE, differing from Eq. (1) by an additional drift-term dependent on h . This can be viewed as a special case in which \mathbb{P}^h is a *Girsanov-type* change of measure. In this spirit, we show the following as the main result of this paper.

Theorem 1.1 (Informal). *Under suitable assumptions on $h \in \text{dom}_m(L)$, there exists a unique measure \mathbb{P}^h on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ that satisfies (2). Furthermore, the process*

$$W^h(t) = W(t) - \int_0^t \sqrt{Q} D_x \log h(s, X(s)) ds, \quad t \in [0, T],$$

is a cylindrical Wiener process with respect to \mathbb{P}^h . In particular, X under \mathbb{P}^h solves the SPDE

$$dX(t) = [AX(t) + F(t, X(t)) + QD_x \log h(t, X(t))] dt + \sqrt{Q} dW^h(t), \quad t \in [0, T].$$

1.1. Approach and challenges

Let us demonstrate our approach on how to derive Theorem 1.1 in the case that $H = \mathbb{R}^d$. Eq. (1) then describes a *stochastic ordinary differential equation (SODE)*

$$dx(t) = b(t, x(t))dt + \sqrt{q}dw(t), \quad (3)$$

where b is some Lipschitz continuous function, q is a symmetric positive definite matrix and w is an \mathbb{R}^d -valued Wiener process. Let $h \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ be differentiable with bounded derivatives. An application of *Itô's formula* then gives that $h(t, x(t))$ is the semimartingale given by

$$h(t, x(t)) = h(0, x_0) + \int_0^t L_0 h(s, x(s)) ds + \int_0^t \langle \sqrt{q} D_x h(s, x(s)), dw(s) \rangle,$$

where L_0 is the *Kolmogorov operator* associated with Eq. (3), defined via

$$(L_0 h)(t, x) = \partial_t h(t, x) + \langle b(t, x), D_x h(t, x) \rangle + \frac{1}{2} \text{tr}[q D_x^2 h(t, x)].$$

From this, one can conclude that $h \in \text{dom}_m(L)$ with $Lh = L_0 h$ and Dynkin martingale D^h given by

$$D^h(t) = h(0, x_0) + \int_0^t \langle \sqrt{q} D_x h(s, x(s)), dw(s) \rangle.$$

An application of the integration by parts formula for semimartingales then shows that the local martingale E^h equals the *stochastic exponential*

$$E^h(t) = \exp\left(M^h(t) - \frac{1}{2} [M^h]_t\right)$$

of the Itô process $M^h(t) = \int_0^t \langle \sqrt{q} D_x \log h(s, x(s)), dw(s) \rangle$. Therefore, if E^h is a true martingale, the *Girsanov theorem* implies that the process $w^h(t) = w(t) - \int_0^t \sqrt{q} D_x \log h(s, x(s)) ds$ is a Wiener process under \mathbb{P}^h and that x solves the SODE

$$dx(t) = b(t, x(t))dt + q D_x \log h(t, x(t))dt + \sqrt{q} dw^h(t).$$

We face two key challenges when generalizing this approach to the setting of an infinite-dimensional Hilbert space H . Firstly, since A is an unbounded operator on H , we generally cannot expect the SPDE (1) to admit a strong solution. This renders any direct application of Itô's formula infeasible and even for smooth functions h , the process $h(t, X(t))$ is in general not a semimartingale.

We circumvent this issue by approximating X by a sequence of strong solutions X_n that satisfy Eq. (1) when substituting A by its Yosida approximations $(A_n)_n$. Under suitable assumptions on h , one can then approximate the process $h(t, X(t))$ by the sequence of semimartingales $h(t, X_n(t))$ for which Itô's formula is applicable.

Secondly, consider the Kolmogorov operator

$$(L_0\varphi)(t, x) = \partial_t \varphi(t, x) + \langle Ax + F(t, x), D_x \varphi(t, x) \rangle + \frac{1}{2} \text{tr}[QD_x^2 \varphi(t, x)]$$

associated with the SPDE (1). In order for $L_0\varphi$ to be a well-defined and continuous function on $\mathbb{R}_+ \times H$, besides the usual smoothness properties of φ , one requires that there exists a continuous extension of the mapping

$$\mathbb{R}_+ \times \text{dom}(A) \rightarrow \mathbb{R}, (t, x) \mapsto \langle Ax, D_x \varphi(t, x) \rangle.$$

This severely limits the class of functions for which $L_0\varphi$ is a well-defined differential operator and substantial work has been done to construct suitable spaces of test functions for Kolmogorov operators in infinite dimensions, see Da Prato [2] and references within.

In particular, for most of our applications, the h -functions of interest cannot be expected to be in the domain of L_0 . However, in Manca [11,12] it is shown that the space of *exponential test functions* $\mathcal{E}_A(\mathbb{R}_+ \times H)$, defined as the real and imaginary parts of the functions

$$\mathbb{R}_+ \times H \rightarrow \mathbb{R}, (s, x) \mapsto \exp(i(\langle x, a \rangle + cs)), \quad a \in \text{dom}(A^*), c \in \mathbb{R},$$

acts as a core for the infinitesimal generator $(L, \text{dom}_m(L))$ with respect to π -convergence. Therefore, under the weaker assumption that $h \in \text{dom}_m(L)$, we may approximate h with a sequence of suitable test functions $(h_n)_n \subset \mathcal{E}_A(\mathbb{R}_+ \times H)$ for which $L_0 h_n$ remains well-defined.

1.2. Related work and applications

In applications of the exponential change of measure E^h , one chooses a suitable h -function such that X under \mathbb{P}^h exhibits certain desired properties. A well-known application from the finite-dimensional setting is the derivation of *diffusion bridges* that describe the process x conditioned on hitting an endpoint $x(T) = y \in \mathbb{R}^d$. An application of our results lifts this to the infinite-dimensional case, thereby allowing us to derive an equation for the *infinite-dimensional diffusion bridge* (or *SPDE bridge*).

To the best of our knowledge, the existing literature on infinite-dimensional bridges is limited to the case where $F \equiv 0$. This ensures mild solutions are Gaussian processes, which leads to an explicit expression for the SPDE bridge, also called an *Ornstein–Uhlenbeck (OU) bridge* in this case. Simão [13] shows that an infinite system of one-dimensional OU bridges defines an OU bridge on a Hilbert space. In a more general, non-diagonal setting, Goldys and Maslowski [14] derive an equation for the OU bridge and apply it to study basic properties of transition semigroups for semilinear SPDEs. More recently, Di Nunno et al. [15] consider a linear stochastic reaction–diffusion equation on a bounded domain where the process is conditioned on a noisy observation at time T . A general framework for the spatial discretization of these bridge processes is developed.

Our approach via the change of measure E^h is more general. In the specific case that $h(t, x) = p(t, x; T, y)$, with p the transition density of the process with respect to an appropriately chosen reference measure, it gives rise to the infinite-dimensional diffusion bridge that conditions the process to hit y at time T . Our results allow for other choices of h , for example $h(t, x) = \tilde{p}(t, x; T, y)$, where \tilde{p} is the transition density of the SPDE without the nonlinearity. The resulting process is called a *guided process*, analogous to the finite-dimensional setting introduced in Schauer et al. [16]. The guided process is different from the conditioned process, but it mimics properties of that process, though the extra term in the drift ignores the nonlinearity. Contrary to p , the transition density \tilde{p} is explicitly known. Therefore, the SPDE for the guided process can be numerically approximated. Taking into account the likelihood ratio of distribution of the true conditioned process with respect to the guided process, *weighted samples* of the conditioned process are obtained. It is exactly this approach which has been proposed in earlier works in the simpler setting of stochastic ordinary differential equations (see, for instance, Schauer et al. [16], Delyon and Hu [17], Bierkens et al. [18] and applications in Mider et al. [19]). The results in this paper prove the existence of the guided process as the mild solution to a particular SPDE.

Another application that we consider is that of *forcing* the process so that its marginal distribution at time T is fixed to a specified distribution. This extends the results for the finite-dimensional case considered in Baudoux [20].

1.3. Outline

We provide an overview of the needed preliminaries in Section 2. Particular attention will be given to semigroups that are strongly continuous with respect to π -convergence as well as their infinitesimal generators. In Section 3, we present, in a detailed manner, the main result of this paper and its proof. Additionally, a modified version is given for the special case that the change of measure is limited to a finite time horizon. We showcase some applications of the main result in Section 4. These include the derivations of the infinite-dimensional diffusion bridge and the forced process as well as the guided process.

2. Preliminaries

2.1. On the stochastic basis

We assume to be working on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ defined as follows. Let H be some Hilbert space, embedded in another Hilbert space H' such that the embedding $J : H \hookrightarrow H'$ is a Hilbert–Schmidt operator. Note that in particular, JJ^* is a positive definite trace-class operator on H' . Define $\Omega = C(\mathbb{R}_+; H')$ as the space of all continuous functions from \mathbb{R}_+ to H' equipped with the metric

$$d(\omega, \tilde{\omega}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|\omega - \tilde{\omega}\|_n}{1 + \|\omega - \tilde{\omega}\|_n},$$

where $\|\omega - \tilde{\omega}\|_n = \sup_{t \in [0, n]} |\omega(t) - \tilde{\omega}(t)|$. Then (Ω, d) is a Polish space and we denote by \mathcal{F} the Borel σ -algebra of (Ω, d) .

On (Ω, \mathcal{F}) define the canonical process $\eta : \mathbb{R}_+ \times \Omega \rightarrow H'$, $\eta_t(\omega) = \omega(t)$ and let $(\mathcal{F}_t)_{t \geq 0}$ be the right-continuous extension of the natural filtration of η , i.e. $\mathcal{F}_t = \bigcap_{\varepsilon > 0} \sigma(\eta_s : s \leq t + \varepsilon)$ for any $t \geq 0$. By the Kolmogorov extension theorem, there exists a Gaussian measure \mathbb{P} on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ such that, under \mathbb{P} , the canonical process η is a Wiener process on H' with covariance operator JJ^* . In particular, η is a cylindrical Wiener process on H .

Denote by \mathbb{P}_t the restriction of \mathbb{P} onto \mathcal{F}_t for any $t \geq 0$. We will need the following result (see Stroock [21], Lemma 4.2).

Lemma 2.1. *Let $(E(t))_{t \geq 0}$ be a non-negative martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with $\mathbb{E}[E(0)] = 1$. Then there exists a unique measure \mathbb{Q} on \mathcal{F} such that $d\mathbb{Q}_t = E(t)d\mathbb{P}_t$ for all $t \geq 0$.*

Remark 2.2. Note that Lemma 2.1 does not give absolute continuity of \mathbb{Q} with respect to \mathbb{P} . However, under the stronger assumption that E is a uniformly integrable martingale, it can be shown that \mathbb{Q} is absolutely continuous with respect to \mathbb{P} on \mathcal{F} with $d\mathbb{Q} = E_\infty d\mathbb{P}$, where E_∞ is the unique random variable such that $E(t) = \mathbb{E}[E_\infty | \mathcal{F}_t]$ for all $t \geq 0$.

In many applications the martingale E is only defined on some half-open interval $t \in [0, T)$. In that circumstance, the following variation of Lemma 2.1 will be useful.

Lemma 2.3. *Let $(E(t))_{t \in [0, T)}$ be a non-negative martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T)}, \mathbb{P})$ with $\mathbb{E}[E(0)] = 1$. Then there exists a unique measure \mathbb{Q} on \mathcal{F}_T such that $d\mathbb{Q}_t = E(t)d\mathbb{P}_t$ for all $0 \leq t < T$.*

Proof. The proof is similar to that of Stroock [21], Lemma 4.2. For details, see Pieper-Sethmacher et al. [22], Appendix A. \square

2.2. On stochastic evolution equations

The following is a standing assumption on the components involved in Eq. (1).

Assumption 2.4.

- (i) A is the generator of a strongly continuous semigroup $(S_t)_{t \geq 0}$ on H . In particular there exists a $C_S > 0, \omega_S \in \mathbb{R}$ such that $\|S_t\|_{L(H)} \leq C_S \exp(\omega_S t)$ for all $t \geq 0$.
- (ii) W is a cylindrical Wiener process on H .
- (iii) Q is a symmetric, positive and bounded operator on H . Furthermore, the family of operators $(Q_t)_{t \geq 0}$ defined by

$$Q_t = \int_0^t S_s Q S_s^* ds \tag{4}$$

is such that $\sup_t \text{tr}(Q_t) < \infty$.

- (iv) F is such that there exists a constant $C_F > 0$ with

$$\|F(t, x) - F(t, y)\| \leq C_F \|x - y\| \text{ and } \|F(t, x)\| \leq C_F (1 + \|x\|)$$

for all $t \geq 0$ and $x, y \in H$.

Under Assumption 2.4, Eq. (1) admits a unique mild solution $X = (X(t, s, x))_{t \geq s}$ for any initial value $x \in H$, i.e. X is an H -valued, \mathcal{F}_t -adapted process that satisfies

$$X(t, s, x) = S_{t-s}x + \int_s^t S_{t-u} F(u, X(u, s, x)) du + \int_s^t S_{t-u} \sqrt{Q} dW(u), \quad t \geq s. \tag{5}$$

The process X is Markovian and has a \mathbb{P} -almost surely continuous modification. Furthermore, for any $m \geq 1$, there exist some constants $C_m > 0, \gamma_m \geq 0$, also depending on ω_S, C_S, C_F and Q , such that

$$\mathbb{E}[\|X(t, s, x)\|^m] \leq C_m \exp(\gamma_m(t-s))(1 + \|x\|^m). \tag{6}$$

Though bounds similar to (6) are well known results in the literature, this particular bound follows from Goldys and Kocan [5], Proposition 3.1. To abbreviate notation, we fix some arbitrary $x_0 \in H$ and simply write $X(t)$ whenever (1) is assumed to be initialized at $X(0) = x_0$.

Due to the unbounded nature of A , Eq. (1) generally does not admit a strong solution. However, one can approximate its mild solution with strong solutions to a sequence of substitute equations in the following way. Let $(A_n)_{n > \omega_S}$ be the Yosida approximation of A , i.e. $(A_n)_{n > \omega_S}$ is the sequence of bounded, linear operators on H defined as

$$A_n = nAR(n, A),$$

where $R(n, A) = (n - A)^{-1}$ is the resolvent of A . It then holds that $\lim_n A_n x = Ax$ for any $x \in \text{dom}(A)$ and that A_n defines a semigroup $S_t^{(n)}$ such that $\lim_n S_t^{(n)} x = S_t x$ for all $x \in H$ and

$$\|S_t^{(n)}\|_{L(H)} \leq C_S \exp(\omega_n t) \quad (7)$$

with $\omega_n = \frac{\omega_S n}{n - \omega_S}$. Now let X_n be the strong solution to the equation

$$dX_n(t) = [A_n X_n(t) + F(t, X_n(t))] dt + \sqrt{Q} dW(t), \quad X_n(s) = x. \quad (8)$$

It is well-known (see e.g. Da Prato and Zabczyk [23], Proposition 7.4) that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{t \in [s, T]} |X_n(t) - X(t)|^p \right] = 0 \quad (9)$$

for any $T > s, p > 1$ and in particular $X_n \rightarrow X$ in probability as $C([s, T]; H)$ -valued random variables.

In the special case that $F = 0$, we denote the time homogeneous mild solution to Eq. (1) by Z and refer to it as the *Ornstein–Uhlenbeck (OU) process*. The random variables $Z(t, s, x)$ are Gaussian with mean $S_{t-s}x$ and covariance operator Q_{t-s} . Under Assumption 2.4 (iii), $Z(t, s, x)$ converges in distribution to its invariant distribution $\nu \sim \mathcal{N}(0, Q_\infty)$ with

$$Q_\infty = \int_0^\infty S_u Q S_u^* du. \quad (10)$$

2.3. On transition semigroups and their generators on spaces of polynomial growth

Let $(E, |\cdot|_E)$ be a Banach space. For any $m \geq 0$, we let $C_m(\mathbb{R}_+ \times H; E)$ be the space of all continuous functions $\varphi : \mathbb{R}_+ \times H \rightarrow E$ that are bounded in t and of at most polynomial growth of order m in x . The space $C_m(\mathbb{R}_+ \times H; E)$ is a Banach space with norm

$$\|\varphi\|_m = \sup_{(t,x) \in \mathbb{R}_+ \times H} \frac{|\varphi(t, x)|_E}{1 + \|x\|^m}.$$

If $E = \mathbb{R}^d$, equipped with the Euclidean norm, we simply write $C_m(\mathbb{R}_+ \times H)$.

It follows from the inequality (6) that the time-homogeneous space–time process of X , given by $Y(t, (s, x)) = (t + s, X(t + s, s, x))$, defines a transition semigroup $(T_t)_{t \geq 0}$ on $C_m(\mathbb{R}_+ \times H)$ via

$$(T_t \varphi)(s, x) = \mathbb{E}[\varphi(s + t, X(t + s, s, x))], \quad \varphi \in C_m(\mathbb{R}_+ \times H), t \geq 0, x \in H. \quad (11)$$

As noted in the introduction, $(T_t)_{t \geq 0}$ is not strongly continuous with respect to the norm topology on $C_m(\mathbb{R}_+ \times H)$. Instead, one has to turn to the weaker mode of π -convergence.

We say a sequence $(\varphi_n)_n \subset C_m(\mathbb{R}_+ \times H)$ is π -convergent to $\varphi \in C_m(\mathbb{R}_+ \times H)$ and write $\pi\text{-}\lim_n \varphi_n = \varphi$ if

$$\sup_n \|\varphi_n\|_m < \infty \text{ and } \lim_n \varphi_n(t, x) = \varphi(t, x) \text{ for all } (t, x) \in \mathbb{R}_+ \times H.$$

The π -closure of a subset B in $C_m(\mathbb{R}_+ \times H)$ is defined as

$$\overline{B}^\pi = \{\varphi \in C_m(\mathbb{R}_+ \times H) : \exists (\varphi_n)_n \subset B \text{ s.t. } \pi\text{-}\lim_n \varphi_n = \varphi\}.$$

The set B is said to be π -closed if $\overline{B}^\pi = B$ and π -dense if $\overline{B}^\pi = C_m(\mathbb{R}_+ \times H)$. A linear operator $L : \text{dom}(L) \subset C_m(\mathbb{R}_+ \times H) \rightarrow C_m(\mathbb{R}_+ \times H)$ is a π -closed operator if the graph $\{(\varphi, L\varphi) : \varphi \in \text{dom}(L)\}$ is π -closed in $C_m(\mathbb{R}_+ \times H) \times C_m(\mathbb{R}_+ \times H)$. If a subdomain $D \subset \text{dom}(L)$ is such that for any $\varphi \in \text{dom}(L)$ there exists a sequence $(\varphi_n)_n$ in D with

$$\pi\text{-}\lim_n \varphi_n = \varphi \text{ and } \pi\text{-}\lim_n L\varphi_n = L\varphi \quad (12)$$

we call D a π -core for $(L, \text{dom}(L))$.

The following definition of π -semigroups is based on Priola [6]. They are exactly those semigroups that are ‘strongly continuous’ with respect to π -convergence.

Definition 2.5 (π -Semigroup). Let $(T_t)_{t \geq 0}$ be a semigroup of bounded, linear operators on $C_m(\mathbb{R}_+ \times H)$, $m \geq 0$. We say $(T_t)_{t \geq 0}$ is a π -semigroup if the following conditions hold:

(i) There exist some $M \geq 1$ and $\omega \in \mathbb{R}$ such that for all $t \geq 0$

$$\|T_t\| \leq M \exp(\omega t). \quad (13)$$

(ii) For any $t \geq 0$ and any $(\varphi_n)_n \subset C_m(\mathbb{R}_+ \times H)$ such that $\pi\text{-}\lim_n \varphi_n = \varphi \in C_m(\mathbb{R}_+ \times H)$ we have

$$\pi\text{-}\lim_n T_t \varphi_n = T_t \varphi \in C_m(\mathbb{R}_+ \times H). \quad (14)$$

(iii) For any $\varphi \in C_m(\mathbb{R}_+ \times H)$ and $(s, x) \in \mathbb{R}_+ \times H$ fixed, the mapping

$$[0, \infty) \longrightarrow \mathbb{R}, \quad t \mapsto T_t \varphi(s, x) \quad (15)$$

is continuous.

Remark 2.6. Note that contrary to the case of semigroups that are strongly continuous with respect to the norm topology on $C_m(\mathbb{R}_+ \times H)$, condition (i) needs to be assumed as it does not follow from the other conditions.

Remark 2.7. Conditions (i) and (iii) imply that any π -semigroup $(T_t)_{t \geq 0}$ is ‘strongly continuous’ with respect to π -convergence, i.e. if $(t_n)_n$ is a sequence such that $t_n \downarrow 0$ then

$$\pi\text{-}\lim_n T_{t_n} \varphi = \varphi, \quad \varphi \in C_m(\mathbb{R}_+ \times H).$$

We then write $\pi\text{-}\lim_{t \downarrow 0} T_t \varphi = \varphi$.

Lemma 2.8. For any $m \geq 0$, the semigroup $(T_t)_{t \geq 0}$ defined in (11) is a π -semigroup on $C_m(\mathbb{R}_+ \times H)$.

Proof. We show that $(T_t)_{t \geq 0}$ satisfies properties (i) to (iii) in Definition 2.5. Let $\varphi \in C_m(\mathbb{R}_+ \times H)$. Then it holds for any (s, x) that

$$\begin{aligned} |(T_t \varphi)(s, x)| &\leq \mathbb{E}[|\varphi(s+t, X(s+t, s, x))|] \\ &\leq \|\varphi\|_m \mathbb{E}[1 + \|X(s+t, s, x)\|^m] \\ &\leq \|\varphi\|_m (1 + C_m \exp(\gamma_m t) (1 + \|x\|^m)), \end{aligned} \quad (16)$$

where we used the definition of $\|\varphi\|_m$ in the second and the inequality (6) in the third step. It follows that

$$\begin{aligned} \|T_t \varphi\|_m &= \sup_{(s,x) \in \mathbb{R}_+ \times H} \frac{|T_t \varphi(s, x)|}{1 + \|x\|^m} \\ &\leq \|\varphi\|_m \sup_{(s,x) \in \mathbb{R}_+ \times H} \frac{(1 + C_m \exp(\gamma_m t) (1 + \|x\|^m))}{1 + \|x\|^m} \\ &\leq \|\varphi\|_m (1 + C_m) \exp(\gamma_m t) \end{aligned}$$

and thus (13) holds with $M = (1 + C_m)$ and $\omega = \gamma_m$.

To show (ii), let $(\varphi_n)_n$ be a sequence in $C_m(\mathbb{R}_+ \times H)$ such that $\pi\text{-}\lim_n \varphi_n = \varphi$. It then follows by the dominated convergence theorem that

$$\begin{aligned} \lim_n (T_t \varphi_n)(s, x) &= \lim_n \mathbb{E}[\varphi_n(t+s, X(t+s, s, x))] \\ &= \mathbb{E}[\varphi(t+s, X(t+s, s, x))] = (T_t \varphi)(s, x). \end{aligned}$$

Furthermore, from (16) and $\sup_n \|\varphi_n\|_m < \infty$ it follows that $\sup_n \|T_t \varphi_n\|_m < \infty$ and therefore that $\pi\text{-}\lim_n T_t \varphi_n = T_t \varphi$. To show (iii), it suffices to note that continuity of $t \mapsto T_t \varphi(s, x)$ follows from the almost sure continuity of $t \mapsto X(\cdot, s, x)$ and dominated convergence. \square

The *infinitesimal generator* of a π -semigroup is defined in a similar manner as for a strongly continuous semigroup.

Definition 2.9. Let $(T_t)_{t \geq 0}$ be a π -semigroup on $C_m(\mathbb{R}_+ \times H)$. The *infinitesimal generator* L of $(T_t)_{t \geq 0}$ is the operator defined via

$$\begin{cases} \text{dom}_m(L) &= \left\{ \varphi \in C_m(\mathbb{R}_+ \times H) : \exists \psi \in C_m(\mathbb{R}_+ \times H) \text{ s.t. } \pi\text{-}\lim_{t \downarrow 0} \frac{T_t \varphi - \varphi}{t} = \psi \right\} \\ (L\varphi)(s, x) &= \lim_{t \downarrow 0} \frac{(T_t \varphi)(s, x) - \varphi(s, x)}{t}, \quad \varphi \in \text{dom}_m(L), (s, x) \in \mathbb{R}_+ \times H. \end{cases} \quad (17)$$

As the upcoming result shows, the infinitesimal generator of a π -semigroup satisfies the common properties characteristic for generators of strongly continuous semigroups.

Theorem 2.10. Let L be the infinitesimal generator of a π -semigroup $(T_t)_{t \geq 0}$ on $C_m(\mathbb{R}_+ \times H)$. Then:

(i) The domain $\text{dom}_m(L)$ is π -dense in $C_m(\mathbb{R}_+ \times H)$.

- (ii) The operator L is the unique π -closed operator such that for all $\lambda > \omega$ the resolvent $R(\lambda, L) = (\lambda - L)^{-1}$ is a bounded, linear operator on $C_m(\mathbb{R}_+ \times H)$ with

$$R(\lambda, L)\varphi(s, x) = \int_0^\infty \exp(-\lambda t) T_t \varphi(s, x) dt. \quad (18)$$

- (iii) It holds that $T_t(\text{dom}_m(L)) \subset \text{dom}_m(L)$ and for all $\varphi \in \text{dom}_m(L)$ and fixed $(s, x) \in \mathbb{R}_+ \times H$ the function $t \mapsto T_t \varphi(s, x)$ is differentiable with

$$\frac{d}{dt} T_t \varphi(s, x) = L T_t \varphi(s, x) = T_t L \varphi(s, x). \quad (19)$$

Proof. See Priola [6], Propositions 3.2 to 3.6., where these properties have been shown in the context of π -semigroups on $C_b(H)$. \square

The following is essentially a version of Dynkin's formula in the context of π -semigroups.

Lemma 2.11. Let $Y(t) = (t, X(t))$ be the space-time process of $X(t)$. Then for any $\varphi \in \text{dom}(L)$ the process

$$D^\varphi(t) = \varphi(Y(t)) - \int_0^t L\varphi(Y(s)) ds \quad (20)$$

is an F_t -adapted \mathbb{P} -martingale. We call $D^\varphi(t)$ the Dynkin martingale (of $(\varphi, L\varphi)$).

Proof. Clearly $Y(t) = (t, X(t, x))$ is F_t -adapted with initial condition $Y(0) = (0, x_0)$. It holds for any $r \leq t$ that

$$\begin{aligned} \mathbb{E}[D^\varphi(t) \mid \mathcal{F}_r] &= \mathbb{E}\left[\varphi(Y(t)) - \int_0^t L\varphi(Y(u)) du \mid \mathcal{F}_r\right] \\ &= \mathbb{E}[\varphi(Y(t)) \mid \mathcal{F}_r] - \mathbb{E}\left[\int_0^r L\varphi(Y(u)) du \mid \mathcal{F}_r\right] - \mathbb{E}\left[\int_r^t L\varphi(Y(u)) du \mid \mathcal{F}_r\right] \\ &= T_{t-r}\varphi(Y(r)) - \int_0^r L\varphi(Y(u)) du - \int_r^t T_{u-r}L\varphi(Y(r)) du, \end{aligned} \quad (21)$$

where we used the Markov property and F_t -adaptedness of $Y(t)$ in the last step. By Theorem 2.10 (iii) and a substitution we have

$$\begin{aligned} \int_r^t T_{u-r}L\varphi(Y(r)) du &= \int_0^{t-r} T_u L\varphi(Y(r)) du \\ &= T_{t-r}\varphi(Y(r)) - \varphi(Y(r)) \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Plugging this into (21) gives $\mathbb{E}[D^\varphi(t) \mid \mathcal{F}_r] = \varphi(Y(r)) - \int_0^r L\varphi(Y(u)) du$. \square

In general, the abstract definition of the infinitesimal generator given in (17) does not lend itself easily to a closed form expression of $L\varphi$ for $\varphi \in \text{dom}_m(L)$. However, in many cases one can instead construct a suitable π -core for $(L, \text{dom}_m(L))$ on which L acts in a more 'descriptive' manner. It turns out that in this case, L acts as the Kolmogorov operator associated with (1) on the space of exponential test functions $\mathcal{E}_A(\mathbb{R}_+ \times H)$, defined as the span of the real and imaginary parts of the functions

$$\mathbb{R}_+ \times H \rightarrow \mathbb{R}, (s, x) \mapsto \exp(i(\langle x, a \rangle + cs)), \quad a \in \text{dom}(A^*), c \in \mathbb{R}, \quad (22)$$

and $\mathcal{E}_A(\mathbb{R}_+ \times H)$ defines a π -core of $(L, \text{dom}_m(L))$ as the following lemma shows. The π -core property of $\mathcal{E}_A(\mathbb{R}_+ \times H)$ will play a crucial part in the proof of our main result.

Lemma 2.12. For any $m \geq 1$, the space $\mathcal{E}_A(\mathbb{R}_+ \times H)$ is a subset of $\text{dom}_m(L)$ with $L\varphi = L_0\varphi$, where

$$(L_0\varphi)(t, x) = \partial_t \varphi(t, x) + \langle x, A^* D_x \varphi(t, x) \rangle + \langle F(t, x), D_x \varphi(t, x) \rangle + \frac{1}{2} \text{tr} \left(Q D_x^2 \varphi(t, x) \right) \quad (23)$$

for any $\varphi \in \mathcal{E}_A(\mathbb{R}_+ \times H)$. Moreover, $\mathcal{E}_A(\mathbb{R}_+ \times H)$ is a π -core for $(L, \text{dom}_m(L))$, i.e. for any $\varphi \in \text{dom}_m(L)$ there exists a sequence $(\varphi_n)_n \subset \mathcal{E}_A(\mathbb{R}_+ \times H)$,¹ such that

$$\pi\text{-}\lim_n \varphi_n = \varphi \text{ and } \pi\text{-}\lim_n L_0 \varphi_n = L\varphi. \quad (24)$$

Furthermore, if $\varphi \in (L, \text{dom}_m(L))$ is such that $D_x \varphi \in C_m(\mathbb{R}_+ \times H; H)$, the approximating sequence $(\varphi_n)_n$ in (24) can be chosen such that

$$\pi\text{-}\lim_n D_x \varphi_n = D_x \varphi. \quad (25)$$

Proof. This follows from some slight generalizations of the results in Manca [12], where the π -core property of exponential test functions for generators of π -semigroups was studied in the context of time homogeneous SPDEs. For more details, see Appendix A. \square

¹ This approximation relies on multi-indexed sequences, see Appendix A. However, for ease of notation, we may assume that the sequence only has one index.

Remark 2.13. Note that for $\varphi \in \mathcal{E}_A(\mathbb{R}_+ \times H)$, $L\varphi$ is in general not a bounded function anymore. Let for example $\varphi(t, x) = \sin(\langle x, h \rangle)$, $h \in \text{dom}(A^*)$. Then $D_x \varphi = h \cos(\langle x, h \rangle)$ and $D_x^2 \varphi(x) = -h \otimes h \sin(\langle x, h \rangle)$ and thus

$$L\varphi = \cos(\langle x, h \rangle)(\langle x, A^* h \rangle + \langle F(t, x), h \rangle) - \frac{1}{2} \sin(\langle x, h \rangle) \langle \sqrt{Q} h, \sqrt{Q} h \rangle,$$

which is not bounded but of linear growth under the Lipschitz assumption on F . In particular, $\mathcal{E}_A(\mathbb{R}_+ \times H)$ is not a π -core for $(L, \text{dom}_m(L))$ if $m = 0$.

3. Main result

In this section we present the proof of our main result. It establishes conditions on the h -function for which, under the exponential change of measure \mathbb{P}^h , the mild solution X to the SPDE (1) is a mild solution to yet another SPDE with an additional drift term dependent on the h -function of choice.

Our point of departure is that, following Lemma 2.11, for any $h \in \text{dom}_m(L)$, the process

$$D^h(t) = h(t, X(t)) - \int_0^t Lh(s, X(s)) \, ds$$

is a \mathbb{P} -martingale. Additionally, it can be shown that for any positive function $h \in \text{dom}_m(L)$, the process

$$E^h(t) = \frac{h(t, X(t))}{h(0, x_0)} \exp\left(-\int_0^t \frac{Lh}{h}(s, X(s)) \, ds\right), \quad t \geq 0, \quad (26)$$

is a continuous \mathbb{P} -local martingale whenever it exists, see for example Palmowski and Rolski [9], Lemma 3.1. If E^h is a true \mathbb{P} -martingale, it follows from Lemma 2.1 that it defines a unique measure \mathbb{P}^h on \mathcal{F} such that

$$\frac{d\mathbb{P}_t^h}{d\mathbb{P}_t} = E^h(t), \quad t > 0. \quad (27)$$

Throughout this section we fix some arbitrary $m \geq 1$. We will need the following assumptions.

Assumption 3.1. The function $h : \mathbb{R}_+ \times H \rightarrow \mathbb{R}_{>0}$ satisfies:

- (i) $h \in \text{dom}_m(L)$ such that $h^{-1} Lh \in C_m(\mathbb{R}_+ \times H)$.
- (ii) h is Fréchet differentiable in x such that $D_x h \in C_m(\mathbb{R}_+ \times H; H)$.
- (iii) h is such that E^h is a \mathbb{P} -martingale.

The following is this sections main result.

Theorem 3.2. Let h satisfy Assumption 3.1 and let \mathbb{P}^h be the measure defined by (27). Then, for any $T > 0$, the process

$$W^h(t) = W(t) - \int_0^t \sqrt{Q} D_x \log h(s, X(s)) \, ds, \quad t \in [0, T], \quad (28)$$

is a cylindrical Wiener process with respect to \mathbb{P}^h . In particular, X under \mathbb{P}^h is a mild solution to the SPDE

$$dX(t) = [AX(t) + F(t, X(t)) + Q D_x \log h(t, X(t))] \, dt + \sqrt{Q} dW^h(t), \quad t \in [0, T]. \quad (29)$$

The main part of the proof of Theorem 3.2 lies in the following lemma.

Lemma 3.3. Let h satisfy Assumption 3.1(i) and (ii) and let D^h be the corresponding Dynkin martingale. Then

$$D^h(t) = h(0, x_0) + \int_0^t \langle \sqrt{Q} D_x h(s, X(s)), dW(s) \rangle, \quad t \geq 0. \quad (30)$$

Furthermore, let E^h be the process as defined in (26). Then E^h is the stochastic exponential given by

$$E^h(t) = \exp\left(M^h(t) - \frac{1}{2} [M^h]_t\right), \quad t \geq 0, \quad (31)$$

where $M^h(t) = \int_0^t \langle \sqrt{Q} D_x \log h(s, X(s)), dW(s) \rangle$.

Proof. *Step One.* We begin by showing the claim in Eq. (30) for any exponential test function $h \in \mathcal{E}_A(\mathbb{R}_+ \times H)$.

Let $(A_n)_{n > \omega_S}$ be the Yosida approximation of A such that $\lim_n A_n x = Ax$ for all $x \in \text{dom}(A)$ and let X_n be the sequence of strong solutions to Eq. (8). Define the sequence of processes $(D_n^h)_n$ via

$$D_n^h(t) = h(t, X_n(t)) - \int_0^t Lh(s, X_n(s)) \, ds. \quad (32)$$

We first show that $D_n^h(t) \xrightarrow{\mathbb{P}} D^h(t)$ for any $t \geq 0$. It follows from (9) that $h(t, X_n(t)) \xrightarrow{\mathbb{P}} h(t, X(t))$ and $Lh(t, X_n(t)) \xrightarrow{\mathbb{P}} Lh(t, X(t))$. Furthermore, it holds that

$$\begin{aligned} \sup_n \mathbb{E} \left[\left| \int_0^t Lh(s, X_n(s)) ds \right| \right] &\leq \sup_n \mathbb{E} \left[\int_0^t \frac{|Lh(s, X_n(s))|}{1 + \|X_n(s)\|^m} (1 + \|X_n(s)\|^m) ds \right] \\ &\leq \|Lh\|_m \sup_n \int_0^t (1 + \mathbb{E}[\|X_n(s)\|^m]) ds \\ &< \infty. \end{aligned} \quad (33)$$

Here, we used in the last step that, by the bounds in (6) and (7), there exists for any $m \geq 1$ some $\tilde{C}_m > 0$ and $\tilde{\gamma}_m \geq 0$ such that

$$\sup_n \mathbb{E}[\|X_n(s)\|^m] \leq \tilde{C}_m \exp(\tilde{\gamma}_m s). \quad (34)$$

By the dominated convergence theorem it follows from (33) that

$$\int_0^t Lh(s, X_n(s)) ds \xrightarrow{L^1(\mathbb{P})} \int_0^t Lh(s, X(s)) ds$$

and therefore in total we get that $D_n^h(t) \xrightarrow{\mathbb{P}} D^h(t)$.

On the other hand, for any $n > \omega_S$, an application of Itô's lemma gives that

$$dh(t, X_n(t)) = L_n h(t, X_n(t)) dt + \langle \sqrt{Q} D_x h(t, X_n(t)), dW(t) \rangle,$$

where $L_n h(t, x) = \partial_t h(t, x) + \langle x, A_n^* D_x h(t, x) \rangle + \langle F(x), D_x h(t, x) \rangle + \frac{1}{2} \text{tr}[Q D_x^2 h(t, x)]$ is the Kolmogorov operator associated with the approximating SPDE in (8). Plugging this into (32) gives that

$$\begin{aligned} D_n^h(t) &= h(0, x_0) + \int_0^t \langle \sqrt{Q} D_x h(s, X_n(s)), dW(s) \rangle \\ &\quad + \int_0^t [L_n h(s, X_n(s)) - Lh(s, X_n(s))] ds \\ &= h(0, x_0) + \int_0^t \langle \sqrt{Q} D_x h(s, X_n(s)), dW(s) \rangle \\ &\quad + \int_0^t \langle X_n(s), (A_n^* - A^*) D_x h(s, X_n(s)) \rangle ds. \end{aligned} \quad (35)$$

It remains to show that

$$\lim_n \int_0^t \langle \sqrt{Q} D_x h(s, X_n(s)), dW(s) \rangle = \int_0^t \langle \sqrt{Q} D_x h(s, X(s)), dW(s) \rangle \quad (36)$$

and

$$\lim_n \int_0^t \langle X_n(s), (A_n^* - A^*) D_x h(s, X_n(s)) \rangle ds = 0 \quad (37)$$

in probability. Then (30) follows from $D_n^h(t) \xrightarrow{\mathbb{P}} D^h(t)$ and (35)–(37).

To show (36), note that by the Itô isometry and (34) we have

$$\begin{aligned} \sup_n \mathbb{E} \left[\left\| \int_0^t \langle \sqrt{Q} D_x h(s, X_n(s)), dW(s) \rangle \right\|^2 \right] &= \sup_n \int_0^t \mathbb{E} [\| \sqrt{Q} D_x h(s, X_n(s)) \|^2 ds] \\ &\leq \sup_n \|Q\| \|D_x h\|_m^2 \int_0^t \mathbb{E}[(1 + \|X_n(s)\|^m)^2] ds \\ &< \infty. \end{aligned}$$

Since $D_x h(s, X_n(s)) \xrightarrow{\mathbb{P}} D_x h(s, X(s))$, we get (36) by an application of the dominated convergence theorem.

Lastly, for any $h \in \mathcal{E}_A(\mathbb{R}_+ \times H)$, it holds that $D_x h(s, x) = \sum_{i=1}^k z_i g_i(s, x)$ for some $z_i \in \text{dom}(A^*)$ and $g_i \in \mathcal{E}_A(\mathbb{R}_+ \times H)$, $i = 1, \dots, k$. We show (37) for $k = 1$. The case that $k > 1$ follows from linearity. We have

$$\int_0^t \langle X_n(s), (A_n^* - A^*) D_x h(s, X_n(s)) \rangle ds = \int_0^t \langle X_n(s), (A_n^* - A^*) z_1 \rangle g_1(s, X_n(s)) ds.$$

By (9) it follows that $X_n \xrightarrow{\mathbb{P}} X$ in $C([0, t]; H)$ and in particular $\sup_n \sup_{s \in [0, t]} \|X_n(s)\| < \infty$ a.s. Thus (37) follows from $\lim_n (A_n^* - A^*) z_1 = 0$. In total this concludes that (30) holds for any $h \in \mathcal{E}_A(\mathbb{R}_+ \times H)$.

Step Two. We proceed to show (30) for any $h \in \text{dom}_m(L)$ that satisfies Assumption 3.1(i) and (ii). By Lemma 2.12 there exists a sequence $(h_n)_n \in \mathcal{E}_A(\mathbb{R}_+ \times H)$ such that

$$\pi\text{-}\lim_n h_n = h, \quad \pi\text{-}\lim_n D_x h_n = D_x h \quad \text{and} \quad \pi\text{-}\lim_n Lh_n = Lh. \quad (38)$$

Denote by $D^{h_n}(t)$ the Dynkin martingale of (h_n, Lh_n) . Firstly, it follows by dominated convergence, the bound in (6) and the π -convergence in (38) that

$$\begin{aligned} D^{h_n}(t) &= h_n(t, X(t)) - \int_0^t Lh_n(s, X(s)) \, ds \\ &\xrightarrow{\mathbb{P}} h(t, X(t)) - \int_0^t Lh(s, X(s)) \, ds \\ &= D^h(t). \end{aligned} \quad (39)$$

On the other hand, since $(h_n)_n \subset \mathcal{E}_A(\mathbb{R}_+ \times H)$, it follows from Step One that

$$D^{h_n}(t) = h_n(0, x_0) + \int_0^t \langle \sqrt{Q} D_x h_n(s, X(s)), dW(s) \rangle, \quad n \in \mathbb{N}. \quad (40)$$

We already have that $\lim_n h_n(0, x_0) = h(0, x_0)$. It thus remains to show that

$$\int_0^t \langle \sqrt{Q} D_x h_n(s, X(s)), dW(s) \rangle \xrightarrow{\mathbb{P}} \int_0^t \langle \sqrt{Q} D_x h(s, X(s)), dW(s) \rangle. \quad (41)$$

By Itô isometry it suffices to establish that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^t \|\sqrt{Q} D_x h_n(s, X(s))\|^2 \, ds \right] = \mathbb{E} \left[\int_0^t \|\sqrt{Q} D_x h(s, X(s))\|^2 \, ds \right],$$

but this again follows from an application of the dominated convergence theorem, the π -convergence of $\pi\text{-}\lim_n D_x h_n = D_x h$ and the bound in (6). In total, the claim (30) thus follows from (39)–(41).

Step Three. We finish the proof by showing the claim in (31). First note that by Assumption 3.1(i) and the bound in (6), the process

$$t \mapsto \int_0^t \frac{Lh}{h}(s, X(s)) \, ds \quad (42)$$

is finite \mathbb{P} -a.s. and the process E^h in (26) is therefore well-defined with $\mathbb{E}[E^h(0)] = 1$. Furthermore, the mapping in (42) is continuous and of finite variation and thus (see e.g. Palmowski and Rolski [9], Lemma 3.1)

$$\left[h(t, X(t)), \exp \left(- \int_0^t \frac{Lh}{h}(s, X(s)) \, ds \right) \right]_t = 0. \quad (43)$$

By Lemma 2.11, the process $h(t, X(t))$ is a semimartingale and with the integration by parts formula for semimartingales it follows that

$$\begin{aligned} dE^h(t) &= \frac{1}{h(0, x_0)} \left[h(t, X(t)) d \left(\exp \left(- \int_0^t \frac{Lh}{h}(s, X(s)) \, ds \right) \right) \right. \\ &\quad \left. + \exp \left(- \int_0^t \frac{Lh}{h}(s, X(s)) \, ds \right) dh(t, X(t)) \right] \\ &= \frac{1}{h(0, x_0)} \left[- \exp \left(- \int_0^t \frac{Lh}{h}(s, X(s)) \, ds \right) Lh(t, X(t)) dt \right. \\ &\quad \left. + \exp \left(- \int_0^t \frac{Lh}{h}(s, X(s)) \, ds \right) dh(t, X(t)) \right] \\ &= \frac{1}{h(0, x_0)} \exp \left(- \int_0^t \frac{Lh}{h}(s, X(s)) \, ds \right) dD^h(t), \end{aligned} \quad (44)$$

where $D^h(t) = h(t, X(t)) - \int_0^t Lh(s, X(s)) \, ds$ is the Dynkin martingale of (h, Lh) . From step two it follows that $dD^h(t) = \langle \sqrt{Q} D_x h(t, X(t)), dW(t) \rangle$ and plugging this into (44) gives

$$\begin{aligned} dE^h(t) &= \frac{1}{h(0, x_0)} \exp \left(- \int_0^t \frac{Lh}{h}(s, X(s)) \, ds \right) \langle \sqrt{Q} D_x h(t, X(t)), dW(t) \rangle \\ &= E^h(t) \langle \sqrt{Q} D_x \log h(t, X(t)), dW(t) \rangle. \end{aligned} \quad (45)$$

In particular, the process $E^h(t)$ is the stochastic exponential $E^h(t) = \exp \left(M^h(t) - \frac{1}{2} [M^h]_t \right)$ of the Itô process

$$M^h(t) = \int_0^t \langle \sqrt{Q} D_x \log h(s, X(s)), dW(s) \rangle. \quad \square \quad (46)$$

We are now ready to give the remainder of the proof of Theorem 3.2, which is essentially just an application of Girsanov's theorem.

Proof of Theorem 3.2. Let E^h be the process as defined in (26). By Assumption 3.1 (iii), E^h is a \mathbb{P} -martingale and thus, following Lemma 2.1, defines a measure \mathbb{P}^h on F such that $d\mathbb{P}_T^h = E^h(T) d\mathbb{P}_T$ on F_T for any $T > 0$. On the other hand, Lemma 3.3 shows that

E^h is the stochastic exponential of the Itô process

$$M^h(t) = \int_0^t \langle \sqrt{Q} D_x \log h(s, X(s)), dW(s) \rangle. \quad (47)$$

It therefore follows from Girsanov's theorem that

$$W^h(t) = W(t) - \int_0^t \sqrt{Q} D_x \log h(s, X(s)) ds, \quad t \in [0, T], \quad (48)$$

is a cylindrical Wiener process with respect to the measure \mathbb{P}^h . In particular, under \mathbb{P}^h , the process X given in (5) is a mild solution to the SPDE

$$dX(t) = [AX(t) + F(t, X(t)) + QD_x \log h(t, X(t))] dt + \sqrt{Q} dW^h(t), \quad t \in [0, T]. \quad \square$$

3.1. The change of measure on a finite time interval

In certain applications we work with h -functions that are only defined on the half open interval $[0, T)$ for some $T > 0$. We then replace [Assumption 3.1](#) with the following.

Assumption 3.4. The function $h : [0, T) \times H \rightarrow \mathbb{R}_+$ satisfies for any $S < T$:

- (i) $h \in C_m([0, S] \times H)$ such that $Lh(t, x)$ exists for any $[0, S] \times H$ and $h^{-1}Lh \in C_m([0, S] \times H)$.
- (ii) h is Fréchet differentiable in x such that $D_x h \in C_m([0, S] \times H; H)$.
- (iii) h is such that $(E^h(t))_{t \in [0, S]}$ is a \mathbb{P} -martingale.

Under [Assumption 3.4](#) (iii), $(E^h(t))_{t \in [0, T)}$ is a non-negative martingale with $\mathbb{E}[E(0)] = 1$. Following [Lemma 2.3](#), it thus defines a measure \mathbb{P}^h on \mathcal{F}_T such that

$$\frac{d\mathbb{P}_t^h}{d\mathbb{P}_t} = E^h(t), \quad t \in [0, T). \quad (49)$$

We get the following version of [Theorem 3.2](#).

Theorem 3.5. Let h satisfy [Assumption 3.4](#) and let \mathbb{P}^h be the measure defined on \mathcal{F}_T by (49). Then for any $S < T$, the process

$$W^h(t) = W(t) - \int_0^t \sqrt{Q} D_x \log h(s, X(s)) ds, \quad t \in [0, S], \quad (50)$$

is a cylindrical Wiener process with respect to \mathbb{P}^h . In particular, X under \mathbb{P}^h is a mild solution to the SPDE

$$dX(t) = [AX(t) + F(t, X(t)) + QD_x \log h(t, X(t))] dt + \sqrt{Q} dW^h(t), \quad t \in [0, T). \quad (51)$$

Proof. Let $S < T$ be arbitrary but fixed. Let \bar{h} be an extension of h onto $\mathbb{R}_+ \times H$ defined via

$$\bar{h}(t, x) = \begin{cases} h(t, x), & t \leq S, \\ h(S, x), & t > S. \end{cases}$$

Then \bar{h} satisfies [Assumption 3.1](#). Following the proof of [Lemma 3.3](#), we get that

$$E^h(t) = \exp \left(M^h(t) - \frac{1}{2} [M^h]_t \right), \quad t \in [0, S],$$

where $M^h(t) = \int_0^t \langle \sqrt{Q} D_x \log h(s, X(s)), dW(s) \rangle$. The claim then follows from [Assumption 3.4](#) (iii) and an application of Girsanov's theorem. \square

Typically, the martingale property of E^h in [Assumption 3.4](#) (iii) is the most difficult of the three to verify. The following lemma summarizes conditions under which it is satisfied.

Lemma 3.6. Either of the following conditions are sufficient for $(E^h(t))_{t \in [0, S]}$ to be a martingale:

1. In addition to [Assumption 3.4](#)(i), $h^{-1}Lh \in C_0([0, S] \times H)$ is bounded.
2. In addition to [Assumption 3.4](#)(i) and (ii), it holds that

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^S \|\sqrt{Q} D_x \log h(t, X(t))\|^2 dt \right) \right] < \infty.$$

3. In addition to [Assumption 3.4](#)(i) and (ii), the mapping $(t, x) \mapsto \sqrt{Q} D_x \log h(t, x)$ is Lipschitz in x , uniformly in $t \in [0, S]$.

Proof. The first condition has been shown in Palmowski and Rolski [9], Proposition 3.2. The second condition is the Novikov condition, see for example Da Prato and Zabczyk [23], Proposition 10.17. A proof for the last condition can be found in [Appendix B](#). \square

4. Applications

4.1. Conditioned SPDEs

In this subsection we introduce a class of h -functions for which the change of measure \mathbb{P}^h corresponds to conditioning the process $(X(t))_{t \in [0, T]}$ on $X(T)$. Applications of this type of conditioning include for example the infinite-dimensional diffusion bridge.

For the construction of the h -functions in this section, we rely on the transition density of X with respect to some suitable reference measure. Since the state space of X is not Euclidean, the typical choice of the Lebesgue measure is not available to us. However, one can construct a suitable Gaussian reference measure on $(H, B(H))$ as follows.

In addition to [Assumption 2.4](#), let the following hold.

Assumption 4.1 (*Strong Feller Assumption*). The semigroup $(S_t)_{t \geq 0}$ and covariance operators $(Q_t)_{t \geq 0}$ defined in (4) are such that $\text{im}(S_t) \subset \text{im}(Q_t^{1/2})$ for all $t \geq 0$.

Under [Assumption 4.1](#), the Ornstein–Uhlenbeck process Z is strongly Feller and absolutely continuous with respect to its invariant measure $\nu \sim \mathcal{N}(0, Q_\infty)$ with covariance operator Q_∞ as defined in (10).

Furthermore, it follows from the Girsanov theorem that $\mathbb{L}(X(t, s, x)) \sim \mathbb{L}(Z(t, s, x))$ for all $t \geq s$. In particular, we have that $\mathbb{L}(X(t, s, x)) \sim \nu$ and we define by $p(s, x; t, y)$ the transition density

$$p(s, x; t, y) = \frac{d\mathbb{L}(X(t, s, x))}{d\nu}(y), \quad \nu\text{-a.e. } y \in H, \quad (52)$$

of X with respect to ν . From the Markov property of X it follows that $p(s, x; t, y)$ satisfies the *Chapman–Kolmogorov* equation

$$p(s, x; t, y) = \int_H p(s, x; r, z) p(r, z; t, y) \nu(dz) \quad (53)$$

for all $s < r < t$ and for ν -a.e. every $y \in H$. Let us now define the function $h : [0, T] \times H \rightarrow \mathbb{R}_{>0}$ via

$$h(t, x) = \int_H p(t, x; T, y) \mu(dy), \quad (54)$$

for some measure μ on $(H, B(H))$ such that $\int_H p(0, x_0; T, y) \mu(dy) < \infty$. The h -function constructed in (54) satisfies the following.

Lemma 4.2. *The function h is space–time harmonic, i.e. for any $(s, x) \in [0, T] \times H$ it holds that*

$$(T_t h)(s, x) = h(s, x), \quad t < T - s.$$

In particular, $Lh = 0$ and $h(t, X(t))$ is a \mathbb{P} -martingale.

Proof. Let $(s, x) \in [0, T] \times H$ be fixed. Then for any $t < T - s$ we have

$$\begin{aligned} (T_t h)(s, x) &= \mathbb{E}[h(t + s, X(t + s, s, x))] \\ &= \int_H h(t + s, z) p(s, x; t + s, z) \nu(dz) \\ &= \int_H \int_H p(t + s, z; T, y) p(s, x; t + s, z) \nu(dz) \mu(dy) \\ &= \int_H p(s, x; T, y) \mu(dy) = h(s, x), \end{aligned}$$

where we use the definition of h in the third and the Chapman–Kolmogorov equation the fourth line. By definition of L it follows that $Lh = 0$. Furthermore, using the Markov property of $(t, X(t))$ we have for any $s < t < T$ that

$$\mathbb{E}[h(t, X(t)) \mid \mathcal{F}_s] = (T_{t-s} h)(s, X(s)) = h(s, X(s)). \quad \square$$

From [Lemma 4.2](#) it follows that h satisfies [Assumption 3.4](#)(i) and (iii). In particular, $h(t, X(t))/h(0, x_0)$ is a non-negative martingale with mean one and thus defines a unique measure \mathbb{P}^h on \mathcal{F}_T via

$$\frac{d\mathbb{P}_t^h}{d\mathbb{P}_t} = \frac{h(t, X(t))}{h(0, x_0)}, \quad t < T. \quad (55)$$

On the other hand, [Assumption 3.4](#) (ii) is hard to verify in general, as the Fréchet differentiability of h depends on $p(s, x; t, y)$ as well as the choice of μ . We thus keep it as an assumption.

Assumption 4.3. The function h defined in (54) is Fréchet differentiable in x such that $D_x h \in C_m([0, S] \times H; H)$ for any $S < T$.

The following result shows how the measure \mathbb{P}^h changes the law of X .

Proposition 4.4. Let \mathbb{P}^h be the measure defined by (55). It then holds for any bounded and measurable function g and $0 \leq t_1 \leq \dots \leq t_n < T$ that

$$\mathbb{E}^h[g(X(t_1), \dots, X(t_n))] = \int_H \mathbb{E}[g(X(t_1), \dots, X(t_n)) \mid X(T) = y] \xi(dy), \quad (56)$$

where ξ is the measure defined on $(H, \mathcal{B}(H))$ via

$$\xi(dy) = \frac{p(0, x_0; T, y) \mu(dy)}{\int_H p(0, x_0; T, y) \mu(dy)}. \quad (57)$$

Additionally, if Assumption 4.3 is satisfied, X under \mathbb{P}^h satisfies the SPDE

$$dX(t) = [AX(t) + F(t, X(t)) + QD_x \log h(t, X(t))] dt + \sqrt{Q} dW^h(t), \quad t \in [0, T],$$

where W^h is the \mathbb{P}^h -cylindrical Wiener process as defined in (50).

Proof. To show (56) it suffices to show that

$$\mathbb{E}^h[g(X(t))] = \int_H \mathbb{E}[g(X(t)) \mid X(T) = y] \xi(dy)$$

for any $t < T$ and continuous and bounded $g : H \rightarrow \mathbb{R}$. The claim then follows by a standard cylindrical argument, see e.g. Ethier and Kurtz [24], Proposition 4.1.6. Indeed it holds that

$$\begin{aligned} \mathbb{E}^h[g(X(t))] &= \mathbb{E} \left[g(X(t)) \frac{h(t, X(t))}{h(0, x_0)} \right] \\ &= \int_H g(x) \frac{h(t, x)}{h(0, x_0)} p(0, x_0; t, x) v(dx) \\ &= \int_H g(x) \frac{\int_H p(t, x; T, y) \mu(dy)}{\int_H p(0, x_0; T, y) \mu(dy)} p(0, x_0; t, x) v(dx) \\ &= \int_H \left(\int_H g(x) \frac{p(0, x_0; t, x) p(t, x; T, y)}{p(0, x_0; T, y)} v(dx) \right) \frac{p(0, x_0; T, y)}{\int_H p(0, x_0; T, y) \mu(dy)} \mu(dy) \\ &= \int_H \mathbb{E}[g(X(t)) \mid X(T) = y] \xi(dy) \end{aligned}$$

with $\xi(dy) = \frac{p(0, x_0; T, y) \mu(dy)}{\int_H p(0, x_0; T, y) \mu(dy)}$. Here we used the definition of h in the third, Fubini's theorem in the fourth and Bayes' theorem in the last step. This shows the first claim of the proposition. The second is a direct consequence of Theorem 3.5 under Assumption 4.3. \square

The formula

$$\mathbb{E}^h[g(X(t_1), \dots, X(t_n))] = \int_H \mathbb{E}[g(X(t_1), \dots, X(t_n)) \mid X(T) = y] \xi(dy)$$

provides a disintegration of the conditioned process: to draw from it one first generates an endpoint $X(T) = y$ from $\xi(dy)$, followed by drawing the path conditioned on this value of y , see Example 4.5. Different choices of the measure μ in (54) correspond to different kinds of conditioning. We illustrate this in the following examples.

Example 4.5 (The Infinite-Dimensional Diffusion Bridge). Let $y \in H$ be such that $p(t, x; T, y)$ is well-defined.² Set $\mu = \delta_y$ as the Dirac measure

$$\delta_y(A) = \begin{cases} 1, & \text{if } y \in A \\ 0, & \text{else.} \end{cases}$$

It follows that $\xi = \delta_y$ and thus the formula (56) reduces to

$$\mathbb{E}^h[g(X(t_1), \dots, X(t_n))] = \mathbb{E}[g(X(t_1), \dots, X(t_n)) \mid X(T) = y].$$

In other words, X under \mathbb{P}^h is the process $(X(t))_{t \in [0, T]}$ conditioned on hitting the endpoint $X(T) = y$. We refer to this process as the *infinite-dimensional diffusion bridge*. If the transition density $p(t, x; T, y)$ satisfies Assumption 4.3, the infinite-dimensional diffusion bridge is characterized by the bridge equation

$$dX^*(t) = [AX^*(t) + F(t, X^*(t)) + QD_x \log p(t, X^*(t); T, y)] dt + \sqrt{Q} dW(t). \quad (58)$$

This equation corresponds to the well known diffusion bridge equation for the case of a finite-dimensional state space.

² This holds for v -a.e. $y \in H$.

Example 4.6 (Conditioning on a Noisy Observation). Suppose we do not observe the endpoint $X(T) = y$ directly, but instead we observe a sample v from a distribution $q(v | y)v(dv)$ where $q(\cdot | y)$ is some probability density function with respect to v . This corresponds to observing $X(T)$ under noise q . Furthermore, assume q is such that

$$\mu(dy) = q(v | y)v(dy)$$

is a finite measure. It then follows that

$$\xi(dy) = \frac{p(0, x_0; T, y)q(v | y)v(dy)}{\int p(0, x_0; T, y)q(v | y)v(dy)}.$$

This has a nice Bayesian interpretation where the endpoint y gets assigned prior density $\pi(y) = p(0, x_0; T, y)$ and the observation is given by v . The likelihood for this observation is $\ell(y | v) = q(v | y)$ and hence

$$\xi(dy) = \frac{\pi(y)\ell(y | v)v(dy)}{\int \pi(y)\ell(y | v)v(dy)}.$$

This shows that $\xi(dy)$ gives the posterior measure of y , conditional upon observing v . Therefore, using this h -transform, the conditioned process is constructed by first sampling the endpoint y conditional on the observation, followed by sampling the bridge to y .

Example 4.7 (The Forced/tilted Process). Let q be some arbitrary density function with respect to v such that

$$\mu(dy) = \frac{q(y)}{p(0, x_0; T, y)}v(dy)$$

defines a finite measure on $(H, B(H))$. By straightforward substitution we get that

$$h(t, x) = \int_H \frac{p(t, x; T, y)}{p(0, x_0; T, y)}q(y)v(dy)$$

and $\xi(dy) = q(y)v(dy)$. Hence this corresponds to *forcing/tilting* $X(T)$ to have the distribution $q(y)v(dy)$.

4.2. Guided processes

In [Example 4.5](#), setting $h(t, x) = p(t, x; T, y)$, we have derived the infinite-dimensional diffusion bridge equation

$$dX^*(t) = [AX^*(t) + F(t, X^*(t)) + QD_x \log p(t, X^*(t); T, y)]dt + \sqrt{Q}dW(t), \quad t \in [0, T],$$

that characterizes the law of the conditioned process $X(t) | X(T) = y$ for v -a.e. $y \in H$. In many practical applications, one seeks to draw samples from X^* . In general, however, the transition density $p(t, x; T, y)$ is not tractable, rendering a direct simulation of X^* infeasible.

This motivates the following construction. Let $\tilde{p}(t, x; T, y)$ be a tractable density of the mild solution \tilde{X} of another auxiliary SPDE. Then, setting $\tilde{h}(t, x) = \tilde{p}(t, x; T, y)$, one can define a change of measure $\mathbb{P}^{\tilde{h}}$ such that

(i) X under $\mathbb{P}^{\tilde{h}}$ equals in law the mild solution X° to the SPDE

$$dX^\circ(t) = [AX^\circ(t) + F(t, X^\circ(t)) + QD_x \log \tilde{h}(t, X^\circ(t))]dt + \sqrt{Q}dW(t), \quad t \in [0, T].$$

(ii) \mathbb{P}^h and $\mathbb{P}^{\tilde{h}}$ - and therefore, the laws of X^* and X° on $C([0, T]; H)$ - are absolutely continuous with some likelihood ratio Φ .

Samples of the bridge process X^* can then be obtained by drawing proposal samples from the law of X° and accepting or rejecting the proposals based on the likelihood ratio Φ . We now showcase the idea of this construction in more detail.

For this, let \tilde{p} denote the transition density of the OU process Z with respect to v , i.e.

$$\tilde{p}(s, x; t, y) = \frac{d\mathbb{L}(Z(t, s, x))}{dv}(y), \quad v\text{-a.e. } y \in H, s < t.$$

Since Z is a Gaussian process and v a Gaussian measure on $(H, B(H))$, the densities $\tilde{p}(s, x; t, y)$ can be obtained by the Cameron–Martin formula. We shall place an additional assumption on the covariance operators of Z .

Assumption 4.8. The covariance operators $(Q_t)_{t \geq 0}$ as defined in [\(4\)](#) commute.

The following proposition shows that \tilde{p} defines a function \tilde{h} that satisfies [Assumption 3.4](#) and that the SPDE induced by the changed measure $\mathbb{P}^{\tilde{h}}$ remains tractable.

Proposition 4.9. For v -a.e. $y \in H$, the function $\tilde{h}(t, x) = \tilde{p}(t, x; T, y)$ satisfies [Assumption 3.4](#). Moreover,

$$D_x \log \tilde{h}(t, x) = \Gamma_{T-t}^* Q_{T-t}^{-\frac{1}{2}} [y - S_{T-t} x], \quad (59)$$

where $\Gamma_{T-t} = Q_{T-t}^{-\frac{1}{2}} S_{T-t}$ is a bounded operator on H by [Assumption 4.1](#). In particular, there exists a unique change of measure $\mathbb{P}^{\tilde{h}}$ such that X under $\mathbb{P}^{\tilde{h}}$ is a mild solution to the SPDE

$$dX^\circ(t) = \left[AX^\circ(t) + F(t, X^\circ(t)) + Q\Gamma_{T-t}^* Q_{T-t}^{-\frac{1}{2}} [y - S_{T-t} X^\circ(t)] \right] dt + \sqrt{Q} dW^{\tilde{h}}(t). \quad (60)$$

Proof. Let $S < T$ be arbitrary but fixed. We start by showing that [Assumption 3.4](#) (ii) is satisfied, i.e. that $\tilde{h}(t, x)$ is Fréchet differentiable in x on $[0, S] \times H$ with derivative $D_x \tilde{h}(t, x)$ of at most polynomial growth in x . For this, write

$$\tilde{p}(t, x; T, y) = \frac{d\mathbb{L}(Z(T, t, x))}{d\mathbb{L}(Z(T, t, 0))}(y) \frac{d\mathbb{L}(Z(T, t, 0))}{d\nu}(y), \quad \nu\text{-a.e. } y \in H.$$

Since Z is a Gaussian process, the Cameron–Martin formula gives that

$$\begin{aligned} \frac{d\mathbb{L}(Z(T, t, x))}{d\mathbb{L}(Z(T, t, 0))}(y) &= \exp \left(\langle Q_{T-t}^{-\frac{1}{2}} y, Q_{T-t}^{-\frac{1}{2}} S_{T-t} x \rangle - \frac{1}{2} \|Q_{T-t}^{-\frac{1}{2}} S_{T-t} x\|^2 \right) \\ &= \exp \left(\langle \Gamma_{T-t}^* Q_{T-t}^{-\frac{1}{2}} y, x \rangle - \frac{1}{2} \|\Gamma_{T-t} x\|^2 \right), \quad \nu\text{-a.e. } y \in H, \end{aligned}$$

where the mapping $t \mapsto \Gamma_{T-t}^* Q_{T-t}^{-\frac{1}{2}} y$ is continuous for ν -a.e. $y \in H$, see [Appendix C](#) for details. It follows that, for ν -a.e. $y \in H$ fixed, the function $\tilde{h}(t, x) = \tilde{p}(t, x; T, y)$ is well-defined and bounded on $[0, S] \times H$. Moreover, it is Fréchet differentiable in x with derivative

$$D_x \tilde{h}(t, x) = \tilde{h}(t, x) \Gamma_{T-t}^* \left(Q_{T-t}^{-\frac{1}{2}} y - \Gamma_{T-t} x \right).$$

It holds that (S_t) , and (Q_t) , are strongly continuous, from which it follows that $t \mapsto \Gamma_{T-t} x$ is continuous for any fixed $x \in H$. In particular, by the uniform boundedness principle and the continuity of $t \mapsto \Gamma_{T-t}^* Q_{T-t}^{-\frac{1}{2}} y$, we have that

$$\sup_{t \in [0, S]} \left(\|\Gamma_{T-t}\| + \|\Gamma_{T-t}^* Q_{T-t}^{-\frac{1}{2}} y\| \right) < \infty.$$

In total we get that $D_x \tilde{h} \in C_1([0, S] \times H; H)$ since

$$\sup_{t \in [0, S], x \in H} \frac{\|D_x \tilde{h}(t, x)\|}{1 + \|x\|} \leq \sup_{t \in [0, S], x \in H} |\tilde{h}(t, x)| \left(\frac{\|\Gamma_{T-t}^* Q_{T-t}^{-\frac{1}{2}} y\|}{1 + \|x\|} + \frac{\|\Gamma_{T-t}^* \Gamma_{T-t} x\|}{1 + \|x\|} \right) < \infty.$$

We proceed to show that \tilde{h} satisfies [Assumption 3.4](#) (i). By the Fréchet differentiability of \tilde{h} , it follows from Manca [\[12\]](#), Theorem 4.1, that

$$(L\tilde{h})(t, x) = \tilde{L}\tilde{h}(t, x) + \langle F(t, x), D_x \tilde{h}(t, x) \rangle,$$

where \tilde{L} is the infinitesimal generator of the Ornstein–Uhlenbeck π -semigroup. By the same arguments as in [Lemma 4.2](#), one shows that \tilde{h} is harmonic with respect to \tilde{L} , i.e. $\tilde{L}\tilde{h}(t, x) = 0$ for any $t < T$. With the Lipschitz continuity of F , it thus follows that $(L\tilde{h})(t, x) \in C_m([0, S] \times H)$ for any $m \geq 2$ with

$$(L\tilde{h})(t, x) = \langle F(t, x), D_x \tilde{h}(t, x) \rangle.$$

Likewise, we have that $\tilde{h}^{-1} L\tilde{h} \in C_m([0, S] \times H)$ for any $m \geq 2$ since

$$\begin{aligned} (\tilde{h}^{-1} L\tilde{h})(t, x) &= \langle F(t, x), D_x \log \tilde{h}(t, x) \rangle \\ &= \left\langle F(t, x), \Gamma_{T-t}^* \left(Q_{T-t}^{-\frac{1}{2}} y - \Gamma_{T-t} x \right) \right\rangle. \end{aligned}$$

Therefore, \tilde{h} satisfies [Assumption 3.4](#)(i) and $E^{\tilde{h}}$ is a well-defined continuous \mathbb{P} -local martingale given by

$$E^{\tilde{h}}(t) = \frac{\tilde{h}(t, X(t))}{\tilde{h}(0, x_0)} \exp \left(- \int_0^t \left\langle F(s, X(s)), \Gamma_{T-s}^* \left(Q_{T-s}^{-\frac{1}{2}} y - \Gamma_{T-s} X(s) \right) \right\rangle ds \right). \quad (61)$$

It remains to show that $E^{\tilde{h}}$ is a true \mathbb{P} -martingale. However, this follows directly from the uniform Lipschitz continuity of $(t, x) \mapsto \sqrt{Q} D_x \log \tilde{h}(t, x)$ and [Lemma 3.6](#). \square

By construction we immediately get the following.

Corollary 4.10. *The measures \mathbb{P}^h and $\mathbb{P}^{\tilde{h}}$ are absolutely continuous on $\mathcal{F}_t, t < T$, with likelihood ratio*

$$\frac{d\mathbb{P}_t^h}{d\mathbb{P}_t^{\tilde{h}}}(X) = \frac{h(t, X(t))}{\tilde{h}(t, X(t))} \frac{\tilde{h}(0, x_0)}{h(0, x_0)} \exp \left(\int_0^t \left\langle F(s, X(s)), \Gamma_{T-s}^* \left(Q_{T-s}^{-\frac{1}{2}} y - \Gamma_{T-s} X(s) \right) \right\rangle ds \right).$$

Proof. This follows from plugging in the likelihood functions $d\mathbb{P}_t^h = E^h(t) d\mathbb{P}_t$ and $d\mathbb{P}_t^{\tilde{h}} = E^{\tilde{h}}(t) d\mathbb{P}_t$ as given in [\(61\)](#) into $d\mathbb{P}_t^h / d\mathbb{P}_t^{\tilde{h}} = d\mathbb{P}_t^h / d\mathbb{P}_t \cdot d\mathbb{P}_t / d\mathbb{P}_t^{\tilde{h}}$. \square

Remark 4.11. Corollary 4.10 shows the absolute continuity of the measures \mathbb{P}^h and $\mathbb{P}^{\bar{h}}$ on any $\mathcal{F}_t, t < T$. In other words, the measures are equivalent as long as we ‘stay away from the conditioning time T ’. However, in order to draw samples from the bridge process X^* , we require absolute continuity of \mathbb{P}^h and $\mathbb{P}^{\bar{h}}$ on \mathcal{F}_T , i.e. on the complete interval $[0, T]$.

To demonstrate this, note that $\mathbb{P}_t^h \sim \mathbb{P}_t$ for any $t < T$, but under \mathbb{P} the event $\{X(T) = y\}$ has measure zero, implying that samples of X under \mathbb{P} will almost surely not hit the endpoint $X(T) = y$.

In contrast to this, the additional drift term $QD_x \log \tilde{p}(t, x; T, y)$ forces the process X° to hit the conditioning point $X^\circ(T) = y$, resulting in absolute continuity of the laws of process X^* and X° on the complete interval. We postpone the proof of this result, along with numerical illustrations, to an upcoming article, as it is beyond the scope of this paper.

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Appendix

The complete proofs of the results in the appendices can be found in Pieper-Sethmacher et al. [22].

Appendix A

We give more details on the approximation properties of the exponential test functions $\mathcal{E}_A(\mathbb{R}_+ \times H)$ with respect to π -convergence in $C_m(\mathbb{R}_+ \times H)$. For this, it does not suffice to consider single-indexed sequences. Hence, the results stated in this section rely on using k -indexed sequences, i.e. sequences $(\varphi_{n_1, \dots, n_k})_{n_1, \dots, n_k}$ that depend on k indices for some $k \in \mathbb{N}$.

A k -indexed sequence $(\varphi_{n_1, \dots, n_k})_{n_1, \dots, n_k}$ in $C_m(\mathbb{R}_+ \times H)$ is said to be π -convergent to some φ in $C_m(\mathbb{R}_+ \times H)$ if for any $i \in \{2, \dots, k\}$ there exists an $(i-1)$ -indexed sequence $(\varphi_{n_1, \dots, n_{i-1}})_{n_1, \dots, n_{i-1}}$ in $C_m(\mathbb{R}_+ \times H)$ such that

$$\varphi_{n_1, \dots, n_{i-1}} = \pi\text{-}\lim_{n_i} \varphi_{n_1, \dots, n_i}$$

and $\varphi = \pi\text{-}\lim_{n_1} \varphi_{n_1}$. We then write $\pi\text{-}\lim_{n_1, \dots, n_k} \varphi_{n_1, \dots, n_k} = \varphi$.

Lemma A.1. Let $\varphi \in C_0(\mathbb{R}_+ \times H)$ such that $D_x \varphi \in C_0(\mathbb{R}_+ \times H; H)$. Then there exists a three-indexed sequence $(\varphi_{n_1, n_2, n_3})_{n_1, n_2, n_3} \subset \mathcal{E}_A(\mathbb{R}_+ \times H)$ such that

$$\begin{aligned} \pi\text{-}\lim_{n_1, n_2, n_3} \varphi_{n_1, n_2, n_3} &= \varphi, \\ \pi\text{-}\lim_{n_1, n_2, n_3} D_x \varphi_{n_1, n_2, n_3} &= D_x \varphi. \end{aligned} \tag{A.1}$$

Proof. The proof goes as in Manca [11], Proposition 2.7, where a similar result was shown in $C_0(H)$. \square

Lemma A.2. Let $\varphi \in C_m(\mathbb{R}_+ \times H)$ be such that $D_x \varphi \in C_m(\mathbb{R}_+ \times H; H)$. Then there exists a four-indexed sequence $(\varphi_{n_1, n_2, n_3, n_4})_{n_1, n_2, n_3, n_4} \subset \mathcal{E}_A(\mathbb{R}_+ \times H)$ such that

$$\begin{aligned} \pi\text{-}\lim_{n_1, n_2, n_3, n_4} \varphi_{n_1, n_2, n_3, n_4} &= \varphi, \\ \pi\text{-}\lim_{n_1, n_2, n_3, n_4} D_x \varphi_{n_1, n_2, n_3, n_4} &= D_x \varphi. \end{aligned} \tag{A.2}$$

Proof. Let $\varphi \in C_m(\mathbb{R}_+ \times H)$ be such that $D_x \varphi \in C_m(\mathbb{R}_+ \times H; H)$. For any $n_1 \in \mathbb{N}$, define

$$\varphi_{n_1}(t, x) = \frac{n_1 \varphi(t, x)}{n_1 + \|x\|^{2m}}.$$

Then $\lim_{n_1} \varphi_{n_1}(t, x) = \varphi(t, x)$ pointwise and $\sup_{n_1} \|\varphi_{n_1}\|_m \leq \|\varphi\|_m$. Furthermore, one can show that $\lim_{n_1} D_x \varphi_{n_1}(t, x) = \varphi(t, x)$ pointwise with $\sup_{n_1} \|D_x \varphi_{n_1}\|_m \leq (\|\varphi\|_m + \|D_x \varphi\|_{0,m})$ and in particular it holds that

$$\pi\text{-}\lim_{n_1} \varphi_{n_1} = \varphi, \pi\text{-}\lim_{n_1} D_x \varphi_{n_1} = D_x \varphi.$$

Noting that $\varphi_{n_1} \in C_0(\mathbb{R}_+ \times H)$ with bounded derivative $D_x \varphi_{n_1} \in C_0(\mathbb{R}_+ \times H; H)$, the claim follows from [Lemma A.1](#) by approximating φ_{n_1} with a suitable sequence $(\varphi_{n_1, n_2, n_3, n_4})_{n_1, n_2, n_3, n_4} \subset \mathcal{E}_A(\mathbb{R}_+ \times H)$ such that

$$\begin{aligned} \pi\text{-}\lim_{n_1, n_2, n_3, n_4} \varphi_{n_1, n_2, n_3, n_4} &= \varphi_{n_1}, \\ \pi\text{-}\lim_{n_1, n_2, n_3, n_4} D_x \varphi_{n_1, n_2, n_3, n_4} &= D_x \varphi_{n_1}. \quad \square \end{aligned}$$

Lemma A.3 ([Lemma 2.12](#) Above). *For any $m \geq 1$, the space $\mathcal{E}_A(\mathbb{R}_+ \times H)$ is a subset of $\text{dom}_m(L)$ with $L\varphi = L_0\varphi$, where*

$$(L_0\varphi)(t, x) = \partial_t \varphi(t, x) + \langle x, A^* D_x \varphi(t, x) \rangle + \langle F(t, x), D_x \varphi(t, x) \rangle + \frac{1}{2} \text{tr} \left(Q D_x^2 \varphi(t, x) \right) \quad (\text{A.3})$$

for any $\varphi \in \mathcal{E}_A(\mathbb{R}_+ \times H)$. Moreover, $\mathcal{E}_A(\mathbb{R}_+ \times H)$ is a π -core for $(L, \text{dom}_m(L))$, i.e. for any $\varphi \in \text{dom}_m(L)$ there exists a sequence $(\varphi_n)_n \subset \mathcal{E}_A(\mathbb{R}_+ \times H)$ such that

$$\pi\text{-}\lim_n \varphi_n = \varphi \text{ and } \pi\text{-}\lim_n L_0 \varphi_n = L\varphi. \quad (\text{A.4})$$

Furthermore, if $\varphi \in (L, \text{dom}_m(L))$ is such that $D_x \varphi \in C_m(\mathbb{R}_+ \times H; H)$, the approximating sequence $(\varphi_n)_n$ in (A.4) can be chosen such that

$$\pi\text{-}\lim_n D_x \varphi_n = D_x \varphi. \quad (\text{A.5})$$

Proof. Using [Lemma A.2](#), the proof goes just as in Manca [12], Theorem 1.3 after noting that $Y(t, (s, x))$ is the mild solution to the noise-degenerate SPDE

$$\begin{cases} dY(t) = [\tilde{A}Y(t) + \tilde{F}(Y(t))] dt + \sqrt{\tilde{Q}} d\tilde{W}(t), & t \geq 0, \\ Y(0) = (s, x). \end{cases} \quad (\text{A.6})$$

Here $\tilde{A}(s, x) = (0, Ax)$ and $\tilde{F}(s, x) = (1, F(s, x))$ and $\tilde{Q} = (0, Q)$ are defined on the product space $\mathbb{R} \times H$ and it is straightforward to show that \tilde{A} , \tilde{F} and \tilde{Q} satisfy [Assumption 2.4](#). \square

Appendix B

Lemma B.1. *Let $G(s, x)$ be a Lipschitz continuous function in x , uniformly on $[0, T]$, i.e. there exists some constant $C > 0$ such that*

$$\|G(s, x) - G(s, y)\| \leq C \|x - y\|$$

for all $s \in [0, T]$ and $x, y \in H$. Let X be the mild solution to (1). Then the process $E(t)$ defined by

$$E(t) = \exp \left(\int_0^t \langle G(s, X(s)), dW(s) \rangle - \frac{1}{2} \int_0^t \|G(s, X(s))\|^2 ds \right), \quad t \in [0, T],$$

is a \mathbb{P} -martingale.

Proof. This follows from a standard stopping time argument. Defining the stopping times

$$\tau_n(X) = \inf \left\{ t \in [0, T] : \int_0^t \|G(s, X(s))\|^2 ds \geq n \right\} \wedge T,$$

it follows that $\mathbb{P}(\lim_n \tau_n = T) = 1$. Applying Novikov's condition on the processes $E_n(t) = E(t \wedge \tau_n)$ as well as the monotone convergence theorem in n then gives that $\mathbb{E}[E(T)] = 1$. \square

Appendix C

Lemma C.1. *Under [Assumption 4.1](#), for any $S < T$, the operators Γ_{T-t} are uniformly Hilbert–Schmidt on $[0, S]$, i.e.*

$$\sup_{t \in [0, S]} \|\Gamma_{T-t}\|_{HS} < \infty.$$

Proof. From the Strong Feller [Assumption 4.1](#) it follows that $\Gamma_r = Q_r^{-\frac{1}{2}} S_r$ is a bounded linear operator and thus $S_r = Q_r^{\frac{1}{2}} \Gamma_r$ is a Hilbert–Schmidt operator for any $r > 0$. Moreover, it follows from [Assumption 4.1](#) that

$$\text{im}(Q_\infty^{\frac{1}{2}}) = \text{im}(Q_r^{\frac{1}{2}}), \quad r > 0,$$

see Proposition 2 in Chojnowska-Michalik and Goldys [25]. From this one concludes that $Q_r^{-\frac{1}{2}} Q_\infty^{\frac{1}{2}}$ and $Q_\infty^{-\frac{1}{2}} S_r$ are bounded linear operators for all $r > 0$. Now, fix some arbitrary $S < T$. Then for any $t \in [0, S]$ it holds that

$$\Gamma_{T-t} = Q_{T-t}^{-\frac{1}{2}} S_{T-t} = (Q_{T-t}^{-\frac{1}{2}} Q_\infty^{\frac{1}{2}}) (Q_\infty^{-\frac{1}{2}} S_{T-t}) = (Q_{T-t}^{-\frac{1}{2}} Q_\infty^{\frac{1}{2}}) (Q_\infty^{-\frac{1}{2}} S_{T+S-t}) S_{T-S}.$$

Noting that $(Q_{T-t}^{-\frac{1}{2}} Q_{\infty}^{\frac{1}{2}})$ and $(Q_{\infty}^{-\frac{1}{2}} S_{T+S-t})$ are strongly continuous in t , it follows from the uniform boundedness principle that

$$\sup_{t \in [0, S]} \|F_{T-t}\|_{HS} \leq \sup_{t \in [0, S]} \|(Q_{T-t}^{-\frac{1}{2}} Q_{\infty}^{\frac{1}{2}})(Q_{\infty}^{-\frac{1}{2}} S_{T+S-t})\| \|S_{T-S}\|_{HS} < \infty. \quad \square$$

Lemma C.2. For any $S < T$, the random process

$$F_{T-t}^* Q_{T-t}^{-\frac{1}{2}} y = \sum_{j=1}^{\infty} q_{j, T-t}^{-\frac{1}{2}} \langle y, e_j \rangle F_{T-t}^* e_j, \quad t \in [0, S], \quad (\text{C.1})$$

is well-defined as a limit in $L^2(H, \nu; C([0, S]; H))$. Moreover, there exists a measurable space H_S with $\nu(H_S) = 1$ such that the limit exists pointwise for ν -a.e. $y \in H_S$.

Proof. For any $n \in \mathbb{N}$ define the process

$$Y_n(t) = \sum_{j=1}^n q_{j, T-t}^{-\frac{1}{2}} \langle y, e_j \rangle F_{T-t}^* e_j$$

where $(q_{j, T-t}, e_j)_j$ is the eigenbasis of Q_{T-t} . From the strong continuity of $(Q_t)_t$ and $(F_t)_t$ it follows that $Y_n \in C([0, S]; H)$ for any $n \in \mathbb{N}$. Furthermore, it holds that

$$\begin{aligned} \int_H \|Y_n\|_0^2 \nu(dy) &= \int_H \sup_{t \in [0, S]} \left\| \sum_{j=1}^n q_{j, T-t}^{-\frac{1}{2}} \langle y, e_j \rangle F_{T-t}^* e_j \right\|^2 \nu(dy) \\ &= \sup_{t \in [0, S]} \sum_{j=1}^n q_{j, T-t}^{-1} \left(\int_H |\langle y, e_j \rangle|^2 \nu(dy) \right) \|F_{T-t}^* e_j\|^2 \\ &= \sup_{t \in [0, S]} \sum_{j=1}^n q_{j, T-t}^{-1} \langle Q_{\infty} e_j, e_j \rangle \|F_{T-t}^* e_j\|^2 \\ &= \sup_{t \in [0, S]} \sum_{j=1}^n \langle Q_{T-t}^{-\frac{1}{2}} Q_{\infty}^{\frac{1}{2}} e_j, Q_{T-t}^{-\frac{1}{2}} Q_{\infty}^{\frac{1}{2}} e_j \rangle \langle F_{T-t}^* e_j, F_{T-t}^* e_j \rangle \\ &\leq \sup_{t \in [0, S]} \left(\|Q_{T-t}^{-\frac{1}{2}} Q_{\infty}^{\frac{1}{2}}\| \|F_{T-t}^*\|_{HS}^2 \right) < \infty \end{aligned}$$

from which we conclude the convergence of $Y_n \rightarrow F_{T-t}^* Q_{T-t}^{-\frac{1}{2}} y$ in $L^2(H, \nu; C([0, S]; H))$. Now, the second claim follows by an application of the Itô–Nisio theorem. \square

Corollary C.3. There exists a measurable space $H_0 \subset H$ with $\nu(H_0) = 1$ such that

$$[0, T) \rightarrow H, \quad t \mapsto F_{T-t}^* Q_{T-t}^{-\frac{1}{2}} y \quad (\text{C.2})$$

is well-defined and continuous for any $y \in H_0$.

Proof. Set $H_0 = \bigcap_n H_{T-1/n}$ where $H_{T-1/n}$ is the measurable space given by Lemma C.2. \square

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