

Monotonicity of entropy of sums of quantum random variables

by

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Abstract

This thesis studies how von Neumann entropy behaves in two closely related discrete quantum settings: a quantum Central Limit Theorem (qCLT) construction and bosonic mode gate cascades used in quantum optics. The qCLT, originally introduced by Cushen and Hudson [CH71], provides a quantum analogue of the classical Central Limit Theorem, which describes how repeated convolution leads to convergence to a Gaussian state. An extension of this analogy would be the monotonic increase of the von Neumann entropy in this setting, as a similar result was proven in the classical case, and monotonicity of the von Neumann entropy would align with thermodynamic intuition.

The first part of the thesis develops the theoretical background of the qCLT, with emphasis on the Stone-von Neumann theorem, which guarantees a unique representation of the canonical commutation relation. Special attention is paid to an interpretation in which the qCLT evolution is seen as a discrete form of thermalisation with the environment. The second part focuses on numerical simulations of the qCLT evolutions, where the resulting entropy behaviour is analysed and compared with theoretical expectations.

In addition to the qCLT evolutions, a mathematically related though physically distinct system is considered in the form of bosonic mode gate cascades. Both fixed and randomly varying unitary gates in the cascade are analysed, and convergence towards the expected limiting states is consistently observed. However, monotonicity of the entropy does not hold in general when the unitary operators representing the gates are chosen randomly, since such unitaries are not restricted to physically structured transformations (unlike beamsplitters, which must satisfy specific physical constraints) and therefore need not systematically increase entropy, with both entropy-increasing and entropy-decreasing effects occurring.

Overall, the results show a clear distinction between convergence and monotonicity. While convergence to equilibrium appears to be a consistent feature of both models, monotonic entropy behaviour depends much more sensitively on the structure of the underlying unitary dynamics. This suggests that physically motivated constraints on the allowed unitaries are essential if one wants to recover entropy monotonicity in discrete quantum evolutions.

Layperson summary

From everyday experiences, we are aware that heat flows from hot to cold areas, and that this process never spontaneously reverses. This behaviour is closely connected to a quantity called entropy, which measures how disordered or mixed a system is; it can be thought of as representing the number of ways a system could be rearranged, without you noticing the difference. The law responsible for heat flowing from higher to lower temperatures, called the second law of thermodynamics, implies that the entropy of a system can only ever increase or potentially remain constant. In what follows, this macroscopic regime will be referred to as the *classical* case. Whether a similar principle also holds in the quantum world is an important open question in physics and forms the central topic of this thesis.

To investigate this question, two different quantum models were studied. In both models, quantum states were repeatedly combined or allowed to interact with their surroundings, after which the behaviour of their entropy was analysed. A quantum state describes a quantum system entirely, encoding everything that can in principle be known about it.

The first approach forms the most direct analogue with the classical case, and it works by repeatedly mixing together quantum states. Though the entropy does approach a well-defined final state, the entropy does not climb steadily to get there. The most likely explanation is that the quantum mixing operations used throughout the simulation do not yet accurately reflect the required physical mechanisms. In other words, more carefully chosen operations may be needed to recover the steady, one-directional (monotonic) increase in entropy as seen in the classical case.

The second approach instead sends a single quantum state through a sequence of interactions with the environment. Upon applying the same interaction at every step, the entropy likewise approaches a fixed value, though again without steady entropy behaviour. When the interactions are chosen randomly at each step, the entropy of the system no longer settles down to any value; interestingly, when the entropy is averaged over many such random processes, the average entropy does stabilise. This reflects the fact that random interactions have an overall tendency to push quantum systems towards scrambled long-term behaviour.

In both methods, the results draw a clear line between two concepts. The systems studied generally settle into predictable long-term behaviour, but this does not mean that entropy increases at every step. As such, the steady increase of quantum entropy appears to require physically motivated constraints on the quantum process, and not just any mathematically valid quantum operation works.

Future work could focus on identifying which physical constraints are needed to bring quantum entropy in line with the thermodynamic intuition that inspired this study, see e.g. [\[BGJ23\]](#).

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1 Introduction

Since its creation in the early twentieth century, quantum mechanics has demanded a level of rigour that sits at the intersection of functional analysis, operator theory and probability. At its foundation lies the Canonical Commutation Relation, relating the momentum operator P and position operator Q through their commutator; $[P, Q] = -i\hbar\mathbf{1}$. For decades, the subtle question on the uniqueness of this relation remained unanswered, until the Stone-von Neumann theorem showed that the mathematical foundation behind the field is, up to unitary equivalence, unique, guaranteeing the physical content of quantum-mechanical theory is independent of its mathematical representation.

A field in which this theorem is commonly used is the theory of bosonic quantum systems [De +15] [DMG14], since these systems are described entirely by their position and momentum operators. It is within this framework that an analogy with classical probability theory can easily be drawn and made precise. In the classical case, the Central Limit Theorem states that repeated convolution of probability distributions will converge toward a Gaussian distribution. A similar theorem was stated and proven by Cushen-Hudson in 1971 [CH71], who established a quantum version of the Central Limit Theorem (qCLT). More precisely, given independent, identically distributed states ρ , the suitably rescaled n -fold tensor product of ρ converges (in an appropriate sense) to the quantum Gaussian state sharing the first and second moments of ρ .

A conjecture accompanying this theorem is that the von Neumann entropy $S(\bar{\rho}_n) = -\text{Tr}(\bar{\rho}_n \log \bar{\rho}_n)$ is monotonically non-decreasing in n , where $\bar{\rho}_n$ denotes the n -th iteration of the qCLT construction. This would further establish a quantum analogue with classical information theory, where entropy monotonicity is a proven result [Art+04]. The fact that this monotonicity holds classically is not trivial, and its proof followed many years after being originally conjectured, underscoring that the quantum analogue should not be expected to follow easily.

The monotonicity of von Neumann entropy in this setting has to do with the second law of thermodynamics; each step of the qCLT construction can be interpreted as a thermalisation process, in which the state under consideration equilibrates with degrees of freedom of the environment. The second law of thermodynamics states that such irreversible processes increase the entropy of the closed system (that is, the state plus the environment), and the monotonicity of $S(\bar{\rho}_n)$ conjectures that the entropy of the state alone is non-decreasing, where the second law by itself does not guarantee it. Convergence to the quantum Gaussian state can likewise be interpreted as the quantum system reaching thermal equilibrium.

The second law of thermodynamics is one of the few principles expected to survive the transition from the classical to the quantum regime, and confirming the entropy monotonicity in the qCLT setting would further provide evidence for this expectation. Failure of this monotonicity would raise the question of whether the finite-dimensional approximations employed in order to carry out computations remain faithful to the underlying, infinite-dimensional theory, or whether they introduce non-physical artefacts.

This project sets out to investigate this process, first analytically by discussing the underlying theory, then computationally, where the qCLT iterates are simulated explicitly and their von Neumann entropies analysed. A physically distinct but mathematically related setting is also considered, which has to do with beamsplitter cascades in quantum optics [Guh08] [Tan+08], and carries a similar thermodynamical interpretation; instead of letting an n -fold tensor product thermalise with an external heat bath, it describes repeated mixing of bosonic modes through a cascade of quantum gates, with the cascade driving

the system toward a more homogeneous equilibrium state.

In both cases, the von Neumann entropy is tracked as a function of n and compared against the entropy of the state it is guaranteed to converge to. The computation results illustrate, among other things, whether monotonicity indeed holds in practice and how rapidly the states $\bar{\rho}_n$ converge to the thermal equilibrium. More broadly, they provide insight into the extent to which finite-dimensional approximations faithfully capture the asymptotic behaviour predicted by the underlying infinite-dimensional theory.

2 Theory

The following section will describe the underlying mathematical principles and theoretical results required to properly understand the construction of the quantum-mechanical Central Limit Theorem (henceforth denoted by qCLT) as given by [CH71], after which monotonicity in von Neumann entropy is discussed. Below follows a brief overview of each subsection.

Subsection 2.1 describes Hilbert spaces, a special type of inner product spaces, and operators acting on them. These spaces play a central role in the description and formalisation of quantum mechanics, and therefore deserve a thorough description. The introduction and derivation of several theorems on Hilbert spaces and their operators is largely adapted from lecture notes by Neerven [Nee22] and lecture notes by Conway [Con19].

A special class of operators - unitary operators - on Hilbert spaces will be specifically highlighted in subsection 2.2. These operators are used throughout many of the theorems presented here and thus we pay special attention to them.

In order to further properly and rigorously construct the qCLT, we must describe a functional calculus given by the theory of spectral measures, as described in subsection 2.3. This theory, again largely adapted from [Nee22], provides a framework for extending certain functions like the matrix exponential to general, possibly unbounded operators on Hilbert spaces satisfying certain properties, and will be extensively used during the construction of the qCLT.

Subsections 2.4 and 2.5 offer a more general way of encoding the Canonical Commutation Relation (CCR), using much of the theory on unitary operators described in earlier sections. Subsection 2.4 describing the connect between unitary operators and physical observables will be largely adapted from lecture notes by Genoud [Gen17]. The motivation and physical interpretation leading to the natural construction of the Weyl relation described in subsection 2.5 is based on lecture notes by Maassen [Maa03].

Next, the Stone-von Neumann Theorem is highlighted and proved in subsection 2.6, where the lecture notes by Maassen [Maa03] were used extensively. This theorem is arguably most important for the construction later described in section 3, and its proof offers insight into usage of many of the topics covered in previous subsections. The section ends with a swift introduction of a generalisation for density matrices in the form of density operators during subsection 2.7, informed by lecture notes by van Neerven [Nee22].

2.1 Hilbert spaces and their operators

The mathematical backbone of quantum mechanics is the theory of Hilbert spaces: wavefunctions are realised as elements of such spaces, and density states as operators acting on them. We therefore begin with a precise account of Hilbert spaces and their operators. Many definitions are adapted from lecture notes by Neerven [Nee22] and lecture notes by Conway [Con19].

2.1.1 Hilbert spaces

Definition 2.1. (Sesquilinearity) Let V, W be vector spaces over a field $\mathbb{K}$, which carries an involution $c \mapsto \bar{c}$. A mapping

$$f : V \times W \rightarrow \mathbb{K}$$

is called *sesquilinear* if it is conjugate-linear in the first variable and linear in the second variable, that is,

$$f(v + v', w) = f(v, w) + f(v', w), \quad f(cv, w) = \bar{c} f(v, w), \quad (1)$$

$$f(v, w + w') = f(v, w) + f(v, w'), \quad f(v, cw) = c f(v, w), \quad (2)$$

for all $c \in \mathbb{K}$, $v, v' \in V$ and $w, w' \in W$.

We will usually take $\mathbb{K} = \mathbb{C}$, with the involution being complex conjugation. Note that an immediate implication of the above sesquilinearity is that for all $v \in V$, $f(v, 0) = 0$.

Definition 2.2. (Inner product spaces) For $\mathcal{H}$ a vector space, we define an *inner product* $\langle \cdot, \cdot \rangle$ on $\mathcal{H} \times \mathcal{H}$ as a sesquilinear mapping to $\mathbb{C}$ with the following properties:

1. $\langle \psi, \psi \rangle \geq 0$ for all $\psi \in \mathcal{H}$, and $\langle \psi, \psi \rangle = 0 \iff \psi = 0$;
2. $\langle \psi, \varphi \rangle = \overline{\langle \varphi, \psi \rangle}$ for all $\psi, \varphi \in \mathcal{H}$.

An *inner product space* is a pair $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, where $\mathcal{H}$ is a vector space and $\langle \cdot, \cdot \rangle$ an inner product on $\mathcal{H} \times \mathcal{H}$.

From this definition of an inner product, a norm may be induced by defining

$$\|\psi\| := \langle \psi, \psi \rangle^{\frac{1}{2}}.$$

Verifying that this indeed defines a norm is a triviality, where the required triangle inequality follows from the Cauchy-Bunyakovsky-Schwarz inequality. This definition allows us to talk about concepts such as convergence under a metric d , where $d(\psi, \varphi) = \|\psi - \varphi\|$. Henceforth it is understood that inner product spaces are given this norm. Under this induced norm, the inner product $\langle \psi, \varphi \rangle$ is jointly continuous as a function of ψ and φ .

Definition 2.3. (Hilbert spaces) A *Hilbert space* $\mathcal{H}$ is a complete inner product space.

Recall that for a metric space to be complete, we require that every Cauchy sequence converges, both notions of convergence defined in terms of the induced metric as described above. This paper will mainly focus on $\mathcal{H} = L^2(\mathbb{R}^n)$ - the equivalence classes of the square-integrable functions $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ - endowed with the standard inner product

$$\langle f, g \rangle := \int_{\mathbb{R}^n} \overline{f(x)} g(x) dx.$$

Examples include $\mathbb{R}^n$, $L^p(\mathbb{R}^n)$ and $\ell^2(\mathbb{N}) = \{(x_n)_{n \in \mathbb{N}} \mid x_n \in \mathbb{C}, \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$.

Definition 2.4. (Orthogonality) Two elements $\psi, \varphi \in \mathcal{H}$ are said to be *orthogonal*, indicated by $\psi \perp \varphi$, if $\langle \psi, \varphi \rangle = 0$. Similarly, two subsets A and B of $\mathcal{H}$ are called orthogonal if $\forall a \in A, \forall b \in B : a \perp b$.

Specifically, we are interested in the definition of orthogonal projections, as will be described now.

Definition 2.5. (Orthogonal projections) Let M be a closed subspace of $\mathcal{H}$. Then for all $\psi \in \mathcal{H}$, there exist unique $\varphi \in M$, $\chi \in M^\perp$ such that $\psi = \varphi + \chi$. We then define the *orthogonal projection* P onto M as the bounded operator defined by $P\psi = \varphi$. For $\psi \in \mathcal{H}$, we will denote the orthogonal projection onto $\text{span}\{\psi\}$ by proj_ψ .

Remark 2.6. For any orthogonal projection $P \in \mathcal{B}(\mathcal{H})$, $\text{Ran}(P)$ is itself a Hilbert space under the inherited inner product.

Definition 2.7. (Orthonormal basis) Let $\{\psi_n\}_{n=0}^\infty \subset \mathcal{H}$ be an orthonormal family. We say that $\{\psi_n\}_{n=0}^\infty$ is an *orthonormal basis* of $\mathcal{H}$ if

$$\text{clspan}\{\psi_n : n \in \mathbb{N}\} = \mathcal{H}.$$

Remark 2.8. If $\{\psi_n\}_{n=0}^\infty$ is an orthonormal basis of $\mathcal{H}$, then every $\psi \in \mathcal{H}$ admits the expansion

$$\psi = \sum_{n=0}^{\infty} \langle \psi_n, \psi \rangle \psi_n,$$

where the series converges in the norm of $\mathcal{H}$.

As is often the case when defining certain characteristics on a set, we would like a way to tell when two such sets are “equal” in important ways. For groups, this is described by group isomorphisms; for topologies, we may describe them by homeomorphisms and for smooth manifolds, we find diffeomorphisms. Similarly, the following definition turns out to be of use.

Definition 2.9. (Isomorphisms) If $\mathcal{H}$ and $\mathcal{G}$ are Hilbert spaces, an *isomorphism* between $\mathcal{H}$ and $\mathcal{G}$ is a function $U : \mathcal{H} \rightarrow \mathcal{G}$ such that for all $\psi, \varphi \in \mathcal{H}$,

$$\langle U\psi, U\varphi \rangle_{\mathcal{G}} = \langle \psi, \varphi \rangle_{\mathcal{H}}.$$

If such an isomorphism between two Hilbert spaces exists, we say they are *isomorphic*.

It is not difficult to check that being isomorphic is an equivalence relation.

2.1.2 Operators on Hilbert spaces

The following subsection is mainly based on lecture notes by Conway titled “A Course on Functional Analysis” [Con19]. Firstly recall that for an operator $A : \mathcal{H} \rightarrow \mathcal{H}$ to be *linear*, we require that, for all $\psi, \varphi \in \mathcal{H}$ and $a, b \in \mathbb{K}$:

$$A(a\psi + b\varphi) = aA(\psi) + bA(\varphi).$$

In this paper, operators on Hilbert spaces are assumed to be linear, unless explicitly stated otherwise.

Definition 2.10. (Bounded operator) A linear operator $L : \mathcal{H} \rightarrow \mathcal{G}$ is *bounded* if there exists a finite constant $C \geq 0$ such that $\forall \psi \in \mathcal{H}$:

$$\|L\psi\|_{\mathcal{G}} \leq C \|\psi\|_{\mathcal{H}}.$$

Note that in the above definition and over the course of this work, $L\psi$ will be written instead of $L(\psi)$ in order to improve visibility. Using the above definition, we will write $\mathcal{B}(\mathcal{H}, \mathcal{G})$ to indicate the set of all bounded, linear operators $A : \mathcal{H} \rightarrow \mathcal{G}$ and we define $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$. Operators in $\mathcal{B}(\mathcal{H})$ may themselves be given a norm as follows, making $\mathcal{B}(\mathcal{H})$ itself into a complete normed vector space (a Banach space).

Definition 2.11. (Operator norm) For $T \in \mathcal{B}(\mathcal{H})$, we define its *operator norm* as follows:

$$\|T\| := \sup_{\|\psi\| \leq 1} \|T\psi\|.$$

Note that this norm matches the infimum of all possible constants $C \geq 0$ as given in definition 2.10. In particular, $T : \mathcal{H} \rightarrow \mathcal{H}$ is bounded if and only if its operator norm is finite.

A very important definition for an operator A is its adjoint. If our Hilbert space $\mathcal{H}$ admits a countable orthonormal basis, then (after choosing such a basis) a bounded linear operator $A : \mathcal{H} \rightarrow \mathcal{H}$ can be

represented as an infinite matrix. In this case, the matrix representation of $A^\dagger$ is precisely the Hermitian of the matrix representation of A . Because we want to preserve generality, we describe the adjoint of an operator using bounded sesquilinear mappings, as will be described below.

Definition 2.12. (Bounded sesquilinear mappings) For $\mathcal{H}, \mathcal{G}$ Hilbert spaces over a field $\mathbb{K}$, a sesquilinear mapping $f : \mathcal{H} \times \mathcal{G} \rightarrow \mathbb{K}$ is *bounded* if there exists some constant $M \geq 0$ such that for all $h \in \mathcal{H}, g \in \mathcal{G}$:

$$|f(h, g)| \leq M \|h\|_{\mathcal{H}} \|g\|_{\mathcal{G}}.$$

It turns out that for any bounded sesquilinear mapping with bound M , we may find *unique* operators $A \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ and $B \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ such that if $\|A\|, \|B\| \leq M$, for all $h \in \mathcal{H}, g \in \mathcal{G}$:

$$f(h, g) = \langle Ah, g \rangle = \langle h, Bg \rangle. \quad (3)$$

Generally, we will consider only one Hilbert space; in other words, we restrict ourselves to the case $\mathcal{H} = \mathcal{G}$. Furthermore, an application of the Cauchy-Bunyakovsky-Schwarz inequality shows every inner product is a bounded sesquilinear mapping. Using this definition, we may rigorously define the *adjoint* of an operator $A \in \mathcal{B}(\mathcal{H})$.

Definition 2.13. (Adjoint of an operator) For $A \in \mathcal{B}(\mathcal{H})$, we call the unique operator $B \in \mathcal{B}(\mathcal{H})$ that satisfies equation (3) the *adjoint* of A and we will denote it by $B = A^\dagger$.

From this definition, many properties true for matrix (Hermitian) adjoints also hold for general linear, bounded operators. A few are listed below for completeness.

Proposition 2.14. 1. *Adjoint reverses products*

$$(AB)^\dagger = B^\dagger A^\dagger, \quad \forall A, B \in \mathcal{B}(\mathcal{H});$$

2. *Adjoint is conjugate-linear*

$$(\lambda A)^\dagger = \bar{\lambda} A^\dagger, \quad \forall \lambda \in \mathbb{C}, A \in \mathcal{B}(\mathcal{H});$$

3. *Adjoint commutes with norm-convergent series*

$$\left(\sum_{n=0}^{\infty} A_n \right)^\dagger = \sum_{n=0}^{\infty} A_n^\dagger, \quad \forall (A_n)_{n \in \mathbb{N}} \subset \mathcal{B}(\mathcal{H}).$$

Definition 2.15. (Symmetric and self-adjoint operators) An operator $A : \mathcal{H} \rightarrow \mathcal{H}$ is said to be *symmetric* if the following holds true: for all $\psi, \varphi \in \text{dom}(A)$, $\langle A\psi, \varphi \rangle = \langle \psi, A\varphi \rangle$. Furthermore, if $\text{dom}(A) = \text{dom}(A^\dagger)$, we will say A is *self-adjoint*.

Clearly, using the above definition, every self-adjoint operator is also symmetric. For bounded operators, they are in fact equivalent. In general however, the converse can be seen to not hold true, see e.g. examples 1.11 and 1.12 of paragraph X.1 in [Con19]; we will mainly set our focus on self-adjoint operators, as they play a central role in the mathematical framework behind quantum mechanics. We end this subsection with the following remark on orthogonal projections.

Remark 2.16. Let P be a bounded operator on $\mathcal{H}$. Then P is an orthogonal projection onto a closed subspace $M \subseteq \mathcal{H}$ in the sense of definition 2.5 if and only if $P^2 = P$ and $P^\dagger = P$.

Proof. ($\Rightarrow$) Suppose P is the orthogonal projection onto M . For $\psi \in \mathcal{H}$, write $\psi = \varphi + \chi$ with $\varphi \in M$ and $\chi \in M^\perp$, so that $P\psi = \varphi$.

$P^2 = P$. Since $P\psi = \varphi \in M$, we have $P^2\psi = P\varphi = \varphi = P\psi$. Hence $P^2 = P$.

$P^\dagger = P$. For $\psi_1, \psi_2 \in \mathcal{H}$, write $\psi_i = \varphi_i + \chi_i$ with $\varphi_i \in M$ and $\chi_i \in M^\perp$. Since $\chi_1 \perp \varphi_2$ and $\varphi_1 \perp \chi_2$, we have

$$\langle \psi_1, P\psi_2 \rangle = \langle \varphi_1 + \chi_1, \varphi_2 \rangle = \langle \varphi_1, \varphi_2 \rangle = \langle \varphi_1, \varphi_2 + \chi_2 \rangle = \langle P\psi_1, \psi_2 \rangle.$$

Hence $P^\dagger = P$.

($\Leftarrow$) Suppose $P^2 = P$ and $P^\dagger = P$. Let $M := \text{ran}(P)$. We verify that P is the orthogonal projection onto M as in definition 2.5. First, M is closed: let $\psi_n \in M$ for every $n \in \mathbb{N}$. if $P\psi_n \xrightarrow{n \rightarrow \infty} \xi$, then $P\xi = P(\lim_{n \rightarrow \infty} P\psi_n) = \lim_{n \rightarrow \infty} P^2\psi_n = \lim_{n \rightarrow \infty} P\psi_n = \xi$, so $\xi \in M$. Next, for any $\psi \in \mathcal{H}$, write $\psi = P\psi + (\mathbf{1} - P)\psi$, where $\mathbf{1}$ denotes the identity operator on $\mathcal{H}$. Then $P\psi \in M$ by definition, and for any $\varphi \in M$, say $\varphi = P\phi$,

$$\langle (\mathbf{1} - P)\psi, \varphi \rangle = \langle \psi - P\psi, P\phi \rangle = \langle \psi, P\phi \rangle - \langle P\psi, P\phi \rangle = \langle P\psi, \phi \rangle - \langle P^2\psi, \phi \rangle = 0,$$

using $P^\dagger = P$ and $P^2 = P$. Hence $(\mathbf{1} - P)\psi \in M^\perp$, so $\psi = P\psi + (\mathbf{1} - P)\psi$ is precisely the decomposition of definition 2.5, and P is the orthogonal projection onto M . $\square$

2.2 Unitary operators

Definition 2.17. (Unitary and normal operators) A bounded, linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ will be called *normal* if

$$T T^\dagger = T^\dagger T.$$

Furthermore, it will be called *unitary* if we have that $T T^\dagger = T^\dagger T = \mathbf{1}$.

Similar to the parallel between operators being self-adjoint and their matrix representation being Hermitian, unitary operators give rise to their matrix representation being unitary. Well-known examples of unitary matrices are rotation matrices R_θ and the family of Pauli spin matrices σ_μ , both acting on the Hilbert space $\mathbb{C}^2$ equipped with the standard Euclidean inner product.

Proposition 2.18. For $U \in \mathcal{B}(\mathcal{H})$, the following statements are equivalent:

1. U is unitary;
2. U is surjective and for all $\psi \in \mathcal{H}$, $\|U\psi\| = \|\psi\|$;
3. U is surjective and for all $\psi, \varphi \in \mathcal{H}$: $\langle U\psi, U\varphi \rangle = \langle \psi, \varphi \rangle$.

Proving the above proposition follows easily by recalling that the adjoint of an operator U satisfies $\langle U\psi, \varphi \rangle = \langle \psi, U^\dagger \varphi \rangle$ for all $\psi, \varphi \in \mathcal{H}$. Notably, the fact that unitary operators preserve the inner product makes them of great interest, as they are a way of transforming wavefunctions ψ while preserving probabilities and norms, ensuring that the physical content of the quantum state remains unchanged under such transformations.

Following the definitions of self-adjoint and unitary operators, we introduce and prove a proposition relating them, which will be used upon making the connection between operators on Hilbert spaces and observables of a quantum system.

Definition 2.19. (Power series of an operator) For $T \in \mathcal{B}(\mathcal{H})$, we define e^{iT} as the natural power series

$$e^{iT} := \sum_{n=0}^{\infty} \frac{i^n}{n!} T^n,$$

where the sum converges in the operator norm. Here, T^n represents n -fold composition of T .

Proposition 2.20. (Exponential of self-adjoint operators) If $T \in \mathcal{B}(\mathcal{H})$ is self-adjoint, then e^{iT} is unitary.

Proof. We have to show that $e^{iT} (e^{iT})^\dagger = (e^{iT})^\dagger e^{iT} = \mathbf{1}$. To this end, we first note that

$$(e^{iT})^\dagger = \left(\sum_{n=0}^{\infty} \frac{i^n}{n!} T^n \right)^\dagger = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} T^n = e^{-iT},$$

where properties 1 and 2 of proposition 2.14 were used. Showing their product equals the identity operator is now a matter of simple and straight-forward calculation.

$$\begin{aligned}
e^{iT} e^{-iT} &= \left(\sum_{n=0}^{\infty} \frac{(iT)^n}{n!} \right) \left(\sum_{m=0}^{\infty} \frac{(-iT)^m}{m!} \right) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(iT)^n}{n!} \frac{(-iT)^m}{m!} \\
&= \sum_{k=0}^{\infty} \left(\sum_{n=0}^k \frac{i^n (-i)^{k-n}}{n! (k-n)!} \right) T^k \\
&= \sum_{k=0}^{\infty} \frac{T^k}{k!} \sum_{n=0}^k \binom{k}{n} i^n (-i)^{k-n} \\
&= \frac{T^0}{0!} + \sum_{k=1}^{\infty} \frac{T^k}{k!} (i - i)^k \\
&= T^0 \\
&= \mathbf{1}.
\end{aligned}$$

Here, the Cauchy product formula was used in the third step. Analogously, it can be shown that $e^{-iT} e^{iT} = \mathbf{1}$, and therefore e^{iT} is unitary. $\square$

Below follows a rule for conjugations of an exponential of a bounded operator, which we state explicitly together with its proof; its generalisation to possibly unbounded operators will be used in subsection 2.5 upon construction of the Weyl relations.

Proposition 2.21. *Let $A \in \mathcal{B}(\mathcal{H})$ be invertible, and let $B \in \mathcal{B}(\mathcal{H})$. Then*

$$Ae^B A^{-1} = e^{ABA^{-1}}.$$

Proof. Since the series for e^B converges absolutely in operator norm, we may conjugate term-by-term:

$$Ae^B A^{-1} = A \left(\sum_{n=0}^{\infty} \frac{B^n}{n!} \right) A^{-1} = \sum_{n=0}^{\infty} \frac{AB^n A^{-1}}{n!} = \sum_{n=0}^{\infty} \frac{(ABA^{-1})^n}{n!} = e^{ABA^{-1}}.$$

$\square$

2.3 Spectral measures

Recall that for normal matrices A - matrices satisfying $A^\dagger A = AA^\dagger$ - we are able to find a spectral decomposition; that is, we are able to write $A = \sum_{i=1}^n \lambda_i \text{proj}_i$, with λ_i being distinct eigenvalues of A and proj_i being orthogonal projections onto the respective eigenspaces.

In quantum mechanics, much of the theory regards operators acting on infinite-dimensional spaces, requiring an extension of this spectral decomposition to more general operators. To this end, we must first describe the underlying theory to complete the analogy with this spectral decomposition for normal matrices. We begin this process by defining projection-valued measures.

Definition 2.22. (Projection-valued measures) We define a *projection-valued measure* $E : \mathcal{B}(\mathbb{C}) \rightarrow \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathbb{C})$ denotes the Borel σ -algebra on $\mathbb{C}$, as a mapping that assigns to every $X \in \mathcal{B}(\mathbb{C})$ an orthogonal projection $E(X)$, and satisfies the following conditions:

1. $E(\mathbb{C}) = \mathbf{1}$;
2. for all $\psi, \varphi \in \mathcal{H}$, we have that the mapping $X \mapsto \langle E(X)\varphi, \psi \rangle$, for $X \in \mathcal{B}(\mathbb{C})$, defines a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$;
3. For any sequence of pairwise disjoint sets (X_n) in $\mathcal{B}(\mathbb{C})$,

$$E\left(\bigcup_n X_n\right)\psi = \sum_n E(X_n)\psi \quad , \forall \psi \in \mathcal{H} ,$$

where the series converges under the $\mathcal{H}$ -norm.

By the second condition, denoting this induced measure for $\psi, \varphi \in \mathcal{H}$ by $E_{\varphi, \psi}$, we have that for all $X \in \mathcal{B}(\mathbb{C})$:

$$\langle E(X)\varphi, \psi \rangle = E_{\varphi, \psi}(X) = \int_{\mathbb{C}} \mathbf{1}_X(\lambda) dE_{\varphi, \psi}(\lambda),$$

where the integral on the right-hand side is to be interpreted as the Lebesgue integral with respect to the complex measure $E_{\varphi, \psi}$. In the special case $\varphi = \psi$, this reduces to integration against the finite positive measure $E_\psi := E_{\psi, \psi}$. When no confusion arises, the dependence on φ and ψ is often suppressed in the notation.

Example 2.23. Let $\mathcal{H} = \mathbb{C}^n$, and let $\{e_1, \dots, e_n\}$ be an orthonormal basis. Fix real numbers $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ corresponding to this basis. For each Borel set $X \subseteq \mathbb{R}$, define $E(X)$ as the orthogonal projection onto $\text{span}\{e_i : \lambda_i \in X\}$. Then $E(X)$ is a projection-valued measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$.

For a vector $\psi \in \mathcal{H}$ written as $\psi = \sum_{i=1}^n c_i e_i$, we compute

$$\langle E(X)\psi, \psi \rangle = \left\langle \sum_{\lambda_i \in X} c_i e_i, \sum_{j=1}^n c_j e_j \right\rangle = \sum_{\lambda_i \in X} |c_i|^2.$$

Hence the induced measure E_ψ is given by

$$E_\psi(X) = \sum_{\lambda_i \in X} |c_i|^2.$$

Recall from linear algebra that normal, finite-dimensional matrices admit a spectral decomposition in the following way.

Theorem 2.24 (Spectral decomposition normal finite-dimensional matrices). *Let $T \in \mathbb{C}^{n \times n}$ be normal, i.e. $TT^\dagger = T^\dagger T$. Then there exists an orthonormal basis of eigenvectors of T , and hence*

$$T = \sum_{i=1}^k \lambda_i \text{proj}_i,$$

where $\lambda_i \in \mathbb{C}$ are the distinct eigenvalues of T , and proj_i are the orthogonal projections onto the corresponding eigenspaces.

Such a representation shows that normal matrices act as multiplication by λ_i on each respective eigenspace. Similarly, using projection-valued measures, such a theorem generalises to bounded, normal operators. We will later also see that a similar theorem holds for unbounded operators under stricter conditions.

Theorem 2.25 (Spectral theorem for bounded normal operators). *Let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator. Then there exists a unique projection-valued measure*

$$E : \mathcal{B}(\mathbb{C}) \rightarrow \mathcal{B}(\mathcal{H})$$

on the Borel σ -algebra $\mathcal{B}(\mathbb{C})$ such that

$$T = \int_{\mathbb{C}} \lambda dE(\lambda),$$

where the integral is understood in the sense that

$$\langle T\varphi, \psi \rangle = \int_{\mathbb{C}} \lambda dE_{\varphi, \psi}(\lambda) \quad \forall \varphi, \psi \in \mathcal{H},$$

and $E_{\varphi, \psi}$ is the complex measure defined by

$$E_{\varphi, \psi}(B) = \langle E(B)\varphi, \psi \rangle, \quad \forall B \in \mathcal{B}(\mathbb{C}).$$

Example 2.26. Let $\mathcal{H} = \mathbb{C}^2$ under the Euclidean inner product and consider the normal, bounded operator

$$T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

Define a map $E : \mathcal{B}(\mathbb{C}) \rightarrow \mathcal{B}(\mathbb{C}^2)$ by

$$E(\{1\}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E(\{i\}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad E(\emptyset) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0} \quad E(\mathbb{C}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}.$$

For an arbitrary Borel set $B \subset \mathbb{C}$, define

$$E(B) := \begin{cases} \mathbf{0} & \text{if } B \cap \{1, i\} = \emptyset, \\ E(\{1\}) & \text{if } 1 \in B, i \notin B, \\ E(\{i\}) & \text{if } i \in B, 1 \notin B, \\ \mathbf{1} & \text{if } \{1, i\} \subset B. \end{cases}$$

Then E is a projection-valued measure. Let $\varphi = (x_1, x_2)$ and $\psi = (y_1, y_2)$ be arbitrary vectors in $\mathbb{C}^2$. Since $E(B)$ depends only on whether 1 and i belong to B , we obtain

$$E_{\varphi, \psi}(B) = x_1 \overline{y_1} \mathbf{1}_B(\{1\}) + x_2 \overline{y_2} \mathbf{1}_B(\{i\}),$$

and hence

$$E_{\varphi,\psi} = x_1 \overline{y_1} \delta_1 + x_2 \overline{y_2} \delta_i,$$

where δ_a denotes the Dirac-measure at $a \in \mathbb{C}$. We now compute the integral

$$\int_{\mathbb{C}} \lambda dE_{\varphi,\psi}(\lambda).$$

By definition of integration with respect to a complex measure,

$$\int_{\mathbb{C}} \lambda dE_{\varphi,\psi}(\lambda) = \int_{\mathbb{C}} \lambda d(x_1 \overline{y_1} \delta_1 + x_2 \overline{y_2} \delta_i).$$

Using linearity of the integral with respect to complex measures, this equals

$$x_1 \overline{y_1} \int_{\mathbb{C}} \lambda d\delta_1(\lambda) + x_2 \overline{y_2} \int_{\mathbb{C}} \lambda d\delta_i(\lambda).$$

Since $\int f d\delta_a = f(a)$ for any measurable f , this becomes

$$x_1 \overline{y_1} \cdot 1 + x_2 \overline{y_2} \cdot i.$$

On the other hand,

$$T\varphi = (x_1, ix_2),$$

so

$$\langle T\varphi, \psi \rangle = x_1 \overline{y_1} + ix_2 \overline{y_2}.$$

Thus,

$$\langle T\varphi, \psi \rangle = \int_{\mathbb{C}} \lambda dE_{\varphi,\psi}(\lambda),$$

as required.

It turns out that a similar theorem remains true for unbounded operators, given additional constraints. Not only do we require for our operator A to be normal (i.e. $AA^\dagger = A^\dagger A$), we will now require it to be self-adjoint; $A = A^\dagger$ and $\text{dom}(A) = \text{dom}(A^\dagger)$.

Theorem 2.27 (The Spectral Theorem). *Every self-adjoint operator A on a Hilbert space $\mathcal{H}$ admits a unique representation*

$$A = \int_{\mathbb{R}} \lambda dE(\lambda),$$

where E is a projection-valued measure on $\mathbb{R}$. Moreover, this correspondence between self-adjoint operators and projection-valued measures is one-to-one.

Note that the integrand has changed from $\mathbb{C}$ to only $\mathbb{R}$; this is analogous to self-adjoint matrices having purely real eigenvalues, which extends to self-adjoint operators in general. Thus, the spectral theorem generalizes the finite-dimensional decomposition of a normal matrix as a finite sum of eigenvalues times projections, replacing the sum by an integral and the discrete family of projections by a projection-valued measure.

Remark 2.28. Let E be a projection-valued measure associated with a self-adjoint operator A as in the spectral theorem. For every bounded function $f : \mathbb{R} \rightarrow \mathbb{C}$ that is measurable with respect to $\mathcal{B}(\mathbb{C})$, we define the operator $f(A)$ by

$$f(A) := \int_{\mathbb{R}} f(\lambda) dE(\lambda),$$

where the integral is understood in the sense of the spectral theorem, i.e.

$$\langle f(A)\varphi, \psi \rangle = \int_{\mathbb{R}} f(\lambda) dE_{\varphi, \psi}(\lambda), \quad \forall \varphi, \psi \in \mathcal{H}.$$

In particular, this defines a functional calculus compatible with the spectral representation of A .

Remark 2.29. The same identity as in proposition 2.21 holds in the setting of the functional calculus for self-adjoint operators. If B is self-adjoint on a Hilbert space and A is unitary, then for any function f measurable with respect to $\mathcal{B}(\mathbb{C})$ we have

$$Af(B)A^{-1} = f(ABA^{-1}),$$

where $f(B)$ is defined via the spectral theorem.

In particular, taking $f(\lambda) = e^{i\lambda}$ yields

$$Ae^{iB}A^{-1} = e^{iABA^{-1}}.$$

Remark 2.30. A fundamental property of the spectral functional calculus is that it is commutative in the following sense: if $f, g : \mathbb{R} \rightarrow \mathbb{C}$ are measurable functions with respect to $\mathcal{B}(\mathbb{C})$ and A is self-adjoint, then the operators $f(A)$ and $g(A)$ commute. In particular,

$$[f(A), g(A)] = 0.$$

This reflects the fact that both operators are defined via integration with respect to the same projection-valued measure associated with A .

2.4 Stone's Theorem

Having established the spectral theorem for unbounded self-adjoint operators and developed the associated functional calculus, we now turn to the study of one-parameter unitary groups. These arise naturally when considering continuous symmetries and dynamical time-reversible evolutions in quantum systems. Stone's theorem provides the corresponding structural characterization in terms of self-adjoint generators, which will serve as the analytic framework for the subsequent quantum central limit construction described in [CH71]. We begin by introducing a one-parameter unitary group.

Definition 2.31 (One-parameter unitary groups). A *one-parameter unitary group* is a mapping $U : \mathbb{R} \rightarrow \mathcal{B}(\mathcal{H})$ such that $U(t) = U_t$ is unitary for all $t \in \mathbb{R}$ and the following hold:

1. $U_0 = \mathbf{1}$;
2. for all $t, s \in \mathbb{R}$, $U_t U_s = U_{t+s}$.

Furthermore, a one-parameter unitary group is said to be *strongly continuous* if, for all $\psi \in \mathcal{H}$, the mapping $t \mapsto U_t \psi$ is continuous with respect to the norm on $\mathcal{H}$.

An immediate consequence of this definition is that such a one-parameter unitary group $(U_t)_{t \in \mathbb{R}}$ satisfies

$$U_{-t} = (U_t)^\dagger = (U_t)^{-1}, \quad \forall t \in \mathbb{R}.$$

Definition 2.32 (Infinitesimal generators). Let $(U_t)_{t \in \mathbb{R}}$ be a strongly continuous one-parameter unitary group. The *infinitesimal generator* of $(U_t)_{t \in \mathbb{R}}$ is the operator G defined on the domain

$$\text{dom}(G) = \left\{ \psi \in \mathcal{H} : \lim_{t \rightarrow 0} \frac{1}{t} (U_t - \mathbf{1}) \psi \text{ exists} \right\}$$

by

$$G\psi = \lim_{t \rightarrow 0} \frac{1}{t} (U_t - \mathbf{1}) \psi.$$

A clear parallel can be seen between the definition of an infinitesimal generator and the derivative of a function at $t = 0$, recalling that we assumed $U_0 = \mathbf{1}$. As an example, the infinitesimal generator G of the one-parameter unitary group given by $(e^{-it} \mathbf{1})_{t \in \mathbb{R}}$ is equal to $-i\mathbf{1}$, as one may easily verify.

The following result is a first step towards the characterization of strongly continuous one-parameter unitary groups, which will be finished by Stone's Theorem.

Theorem 2.33. *Let A be a self-adjoint operator on $\mathcal{H}$ and let E be the projection-valued measure associated to A via the spectral theorem. Then*

$$U_t = e^{itA} = \int_{\mathbb{R}} e^{it\lambda} dE(\lambda), \quad \forall t \in \mathbb{R}, \tag{4}$$

defines a strongly continuous one-parameter unitary group with infinitesimal generator iA . Furthermore, if $\psi \in \text{dom}(A)$, then $U_t \psi \in \text{dom}(A)$ for all $t \in \mathbb{R}$.

Since the proof of this theorem requires a more advanced understanding of spectral measures and the associated functional calculus, we will not present it here. Instead, we illustrate the construction in a concrete example, where all relevant properties can be verified directly.

Example 2.34. Let $\mathcal{H} = L^2(\mathbb{R})$ and define the operator A by

$$(Af)(x) = xf(x), \quad \text{dom}(A) := \{f \in L^2(\mathbb{R}) : xf(x) \in L^2(\mathbb{R})\}.$$

Then A is a self-adjoint operator on $\mathcal{H}$. For each $t \in \mathbb{R}$, define the operator U_t via Stone's Theorem 2.33 by

$$(U_t f)(x) = (e^{itA} f)(x) = e^{itx} f(x), \quad \forall f \in L^2(\mathbb{R}).$$

Proof. We first show that for each $t \in \mathbb{R}$, U_t is unitary. By proposition 2.18, it suffices to show that U_t is norm-preserving and surjective.

To show surjectivity, let $g \in L^2(\mathbb{R})$ and define $f(x) = e^{-itx} g(x)$. Then $f \in L^2(\mathbb{R})$ and

$$(U_t f)(x) = e^{itx} f(x) = e^{itx} e^{-itx} g(x) = g(x),$$

so U_t is surjective.

To show that U_t is norm-preserving, let $f \in L^2(\mathbb{R})$. Then

$$\|U_t f\|^2 = \int_{\mathbb{R}} |e^{itx} f(x)|^2 dx = \int_{\mathbb{R}} |f(x)|^2 dx = \|f\|^2,$$

which implies $\|U_t f\| = \|f\|$, and therefore U_t is unitary. It is trivially seen that $U_0 = \mathbf{1}$ and that for all $t, s \in \mathbb{R}$, $U_t U_s = U_{t+s}$. We thus conclude that $(U_t)_{t \in \mathbb{R}}$ is indeed a one-parameter unitary group.

Secondly, it will be shown that $(U_t)_{t \in \mathbb{R}}$ is strongly continuous. Strong continuity means that for all $t_0 \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$, $\|U_t f - U_{t_0} f\| \rightarrow 0$ as $t \rightarrow t_0$. Using the group property and the fact U_t is norm-preserving,

$$\|U_t f - U_{t_0} f\| = \|U_{t_0}(U_{t-t_0} f - f)\| = \|U_{t-t_0} f - f\|.$$

Hence it suffices to show $\|U_t f - f\| \rightarrow 0$ as $t \rightarrow 0$. Compute

$$\|U_t f - f\|^2 = \int_{\mathbb{R}} |e^{itx} - 1|^2 |f(x)|^2 dx.$$

For each x , $e^{itx} \rightarrow 1$ as $t \rightarrow 0$, and $|e^{itx} - 1|^2 |f(x)|^2 \leq 4|f(x)|^2 \in L^1(\mathbb{R})$. Hence, by the dominated convergence theorem,

$$\|U_t f - f\| \rightarrow 0,$$

so (U_t) is strongly continuous.

We now prove its infinitesimal generator indeed equals iA . For all $f \in L^2(\mathbb{R})$,

$$\frac{U_t f - f}{t}(x) = \frac{e^{itx} - 1}{t} f(x).$$

Since $\frac{e^{itx} - 1}{t} \rightarrow ix$ pointwise as t tends to 0, and if $f \in \text{dom}(A)$ (i.e. $xf(x) \in L^2(\mathbb{R})$), dominated convergence yields

$$\lim_{t \rightarrow 0} \frac{U_t f - f}{t}(x) = ix f(x) = iA f \quad \text{in } L^2(\mathbb{R}).$$

Thus the generator is iA .

Finally, we will show the domain invariance property; that is, if $f \in \text{dom}(A)$, then $U_t f \in \text{dom}(A)$ for all $t \in \mathbb{R}$. If $f \in \text{dom}(A)$, then

$$A(U_t f)(x) = x e^{itx} f(x) = e^{itx} (x f(x)).$$

Since multiplication by e^{itx} preserves being in $L^2(\mathbb{R})$, it follows that $U_t f \in \text{dom}(A)$. □

The previous theorem and example illustrate how strongly continuous one-parameter unitary groups may be induced by self-adjoint operators. Perhaps somewhat surprisingly, it turns out that any strongly

continuous one-parameter unitary group is of the form of equation (4). This result is encapsulated by Stone's Theorem.

Theorem 2.35 (Stone's Theorem). *Let $(U_t)_{t \in \mathbb{R}}$ be a strongly continuous one-parameter unitary group. Then there exists a unique self-adjoint operator A such that*

$$U_t = e^{itA}, \quad \text{for all } t \in \mathbb{R}.$$

Furthermore, if $\psi \in \text{dom}(A)$, then $U_t\psi \in \text{dom}(A)$ for all $t \in \mathbb{R}$.

Note that e^{itA} is to be interpreted by using the Spectral Theorem 2.27, and not by the power series for the exponential function due to the self-adjoint operator A possibly being unbounded. As may be expected and which can be made rigorous using the functional calculus, we have the following equality for every $\psi \in \mathcal{H}$:

$$\left. \frac{d}{dt} e^{itA} \psi \right|_{t=0} = iA\psi.$$

From this, an immediate equation between strongly continuous one-parameter unitary groups $(U_t)_{t \in \mathbb{R}}$ and their associated unique self-adjoint operator A is easily found:

$$A\psi = \frac{1}{i} \left. \frac{d}{dt} U_t \psi \right|_{t=0}. \quad (5)$$

In subsection 2.5, we will make use of the following formula. Its proof can be found in section 5.5 of [Hal15].

Proposition 2.36 (Baker–Campbell–Hausdorff). *Let A and B be operators such that*

$$[A, [A, B]] = [B, [A, B]] = 0.$$

Then

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}.$$

Equivalently,

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B.$$

2.5 The Weyl relation

Over the last few sections, we have developed two fundamental results in the spectral theory of operators, namely Stone's theorem and the spectral theorem. We now turn to the mathematical formulation of quantum mechanics, where these results play a central role in the description of physical observables. Throughout this paragraph, we restrict our attention to the one-dimensional setting, in accordance with the qCLT formulation that will be introduced later in section 3.

A key conclusion from the preceding theory is that an observable can be characterised in three equivalent ways:

1. as a self-adjoint operator on a Hilbert space;
2. as a projection-valued measure;
3. as a strongly continuous one-parameter unitary group.

These three perspectives are interconnected through the spectral theorem and Stone's theorem, and together they provide a unified framework for understanding the structure and dynamics of quantum-mechanical systems. The following section will assume basic undergraduate knowledge of quantum mechanics.

One of the most important observables for a particle on the real line are undoubtedly its *position* and *momentum* observables. By the earlier remark, we may describe these by self-adjoint operators P , respectively Q , which are given by the following equations for $\psi \in L^2(\mathbb{R})$:

$$(Q\psi)(x) = x\psi(x), \quad (P\psi)(x) = -i\hbar \frac{\partial}{\partial x} \psi(x). \quad (6)$$

These self-adjoint operators P and Q will be called the *Schrödinger operators* and they satisfy Heisenberg's *canonical commutation relation* (henceforth denoted by CCR)

$$[P, Q] = PQ - QP = -i\hbar \mathbf{1}. \quad (7)$$

A pair of self-adjoint operators (P, Q) satisfying equation (7) will be called a *canonical pair*.

Although the above formulation of canonical pairs is common and the first one students may run into, it is unsatisfactory as the domains on the left and right side of the equation are not the same. This issue may be prevented by representing these observables using their strongly continuous one-parameter unitary groups. To this end, consider the strongly continuous one-parameter unitary group $(T_t)_{t \in \mathbb{R}}$ given by

$$(T_t\psi)(x) = \psi(x + \hbar t). \quad (8)$$

Its associated self-adjoint operator is readily found using equation (5) to equal P ; similarly, it is found that Q corresponds to the unitary group

$$(S_s\psi)(x) = e^{isx}\psi(x). \quad (9)$$

Weyl proposed to replace equation (7) by a relation between these unitary groups instead, which takes the following form:

$$T_t S_s = e^{i\hbar st} S_s T_t. \quad (10)$$

Using equations (8) and (9), this relation quickly follows by having both sides act on $\psi \in L^2(\mathbb{R})$.

Proof. Let $\psi \in L^2(\mathbb{R})$. Then

$$(T_t S_s \psi)(x) = (S_s \psi)(x + \hbar t) = e^{is(x + \hbar t)} \psi(x + \hbar t).$$

On the other hand,

$$(S_s T_t \psi)(x) = e^{isx} (T_t \psi)(x) = e^{isx} \psi(x + \hbar t).$$

Therefore,

$$T_t S_s = e^{i\hbar st} S_s T_t,$$

which is precisely equation (10). $\square$

It should be remarked that if (P, Q) is a canonical pair, and T_t, S_s their respective strongly continuous one-parameter unitary groups, equation (10) establishes an equivalence with equation (7). This equivalence is shown in Calculation 2.1, and holds even without using the assumed equations (8) and (9).

It was von Neumann's idea to combine the two into a two-parameter family (a Weyl operator), informally defined by

$$W(t, s) := e^{i(sQ + tP)} \quad \forall t, s \in \mathbb{R}, \quad (11)$$

satisfying the *Weyl relation*: for all $s, t, u, v \in \mathbb{R}$,

$$W(t, s) W(u, v) = e^{\frac{i\hbar}{2}(tv - su)} W(t + u, s + v). \quad (12)$$

$\{W(t, s)\}_{t, s \in \mathbb{R}}$ is called a strongly-continuous unitary group, similar to the one-parameter groups given by Stone's Theorem. This relation captures the group property of T_t and S_s together with equation (10). The Weyl relation plays an important role in the Stone-von Neumann Theorem 2.44, which in turn plays a central role in the construction of a quantum-mechanical central limit theorem as described by [CH71]. In order to prove that the Weyl operator of S_s and T_t informally defined by equation (11) indeed satisfies the Weyl relation, we first turn to the following proposition, which yields a more formal definition of Weyl operators.

Proposition 2.37. *Let*

$$W(t, s) := e^{i(sQ + tP)},$$

where Q, P satisfy

$$[P, Q] = -i\hbar \mathbf{1}.$$

Define

$$T_t := e^{itP}, \quad S_s := e^{isQ}.$$

Then

$$W(t, s) = e^{-\frac{i\hbar}{2}st} T_t S_s,$$

and equivalently

$$W(t, s) = e^{\frac{i\hbar}{2}st} S_s T_t.$$

Proof. Set

$$A = isQ, \quad B = itP.$$

Since

$$[A, B] = -i\hbar st \mathbf{1},$$

the Baker–Campbell–Hausdorff formula (proposition 2.36) gives

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B.$$

Hence

$$W(t, s) = e^{\frac{i\hbar}{2}st} S_s T_t = e^{-\frac{i\hbar}{2}st} T_t S_s,$$

where the final equality is established using equation (10). □

Below, the Weyl relation is proven by using proposition 2.37, a relation which will be central in the following subsection 2.6.

Proposition 2.38. *Let*

$$W(t, s) := e^{i(sQ+tP)},$$

where

$$[P, Q] = -i\hbar \mathbf{1}.$$

Then the Weyl relation holds:

$$W(t, s)W(u, v) = e^{\frac{i\hbar}{2}(tv-su)} W(t+u, s+v).$$

Proof.

$$\begin{aligned} W(t, s)W(u, v) &= e^{\frac{i\hbar st}{2}} S_s T_t e^{\frac{i\hbar uv}{2}} S_v T_u \\ &= e^{\frac{i\hbar}{2}(st+uv)} S_s (T_t S_v) T_u \\ &= e^{\frac{i\hbar}{2}(st+uv)} S_s (e^{i\hbar tv} S_v T_t) T_u \\ &= e^{\frac{i\hbar}{2}(st+uv+2tv)} e^{-\frac{i\hbar}{2}(t+u)(s+v)} \left(e^{\frac{i\hbar}{2}(t+u)(s+v)} S_{s+v} T_{t+u} \right) \\ &= e^{\frac{i\hbar}{2}(tv-su)} W(t+u, s+v). \end{aligned}$$

□

Any Weyl operator satisfying this Weyl relation has a rather useful relation with its own unitary, a relation which will be used numerous times in the construction of the next subsection 2.6.

Proposition 2.39. *Let $\{W(t, s)\}_{t,s \in \mathbb{R}}$ be a strongly-continuous unitary group satisfying the Weyl relation*

$$W(t, s)W(u, v) = e^{\frac{i\hbar}{2}(tv-su)} W(t+u, s+v).$$

Then

$$W(t, s)^\dagger = W(-t, -s).$$

Proof. Setting $(u, v) = (-t, -s)$, we obtain

$$W(t, s)W(-t, -s) = e^{\frac{i\hbar}{2}(t(-s)-s(-t))} W(0, 0) = W(0, 0).$$

Similarly,

$$W(-t, -s)W(t, s) = e^{\frac{i\hbar}{2}((-t)s-(-s)t)} W(0, 0) = W(0, 0).$$

Now setting $t = s = u = v = 0$, we obtain

$$W(0, 0)^2 = W(0, 0).$$

As $W(0, 0)$ is unitary and therefore does not equal $\mathbf{0}$, it follows that

$$W(0, 0) = \mathbf{1}.$$

Therefore,

$$W(t, s)W(-t, -s) = W(-t, -s)W(t, s) = \mathbf{1},$$

so that

$$W(t, s)^{-1} = W(-t, -s).$$

Since $W(t, s)$ is unitary,

$$W(t, s)^\dagger = W(t, s)^{-1},$$

and consequently

$$W(t, s)^\dagger = W(-t, -s).$$

□

In this subsection, we have seen an equivalence between the CCR and a relation between their unitary groups given by equation (10), assuming we are considering P, Q as the Schrödinger operators on $L^2(\mathbb{R})$. Proposition 2.38 proves that these standard Schrödinger Weyl operators necessarily satisfy the Weyl relation, as given by equation (12); in the next subsection, we will highlight and subsequently prove a theorem that offers a way of showing the converse of this implication, viz. any Weyl operator satisfying the Weyl relation can be decomposed back to the Weyl operator associated with the Schrödinger canonical pair (P, Q) .

2.6 Stone-von Neumann Theorem

The construction of the quantum-mechanical central limit theorem presented here relies heavily on a fundamental theorem in the theory of Hilbert spaces, namely the Stone-von Neumann Theorem. The derivation presented here closely follows the lecture notes of Maassen [Maa03], with significant elaboration and additional detail.

2.6.1 Kolmogorov's Dilation Theorem

Definition 2.40 (Positive definite kernels). Let S be a set and let K be a *kernel* on S , that is, a function

$$K : S \times S \rightarrow \mathbb{C}.$$

Then K is called *positive definite* if for every $n \in \mathbb{N}$, every choice of points $x_1, \dots, x_n \in S$, and every choice of scalars $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, we have

$$\sum_{i=1}^n \sum_{j=1}^n \lambda_i \overline{\lambda_j} K(x_i, x_j) \geq 0.$$

Positive definite kernels arise naturally in the form $K(x, y) = \langle V(x), V(y) \rangle_{\mathcal{H}}$ associated to a pair of a Hilbert space $\mathcal{H}$ and map $V : S \rightarrow \mathcal{H}$, and two such pairs are considered the same when they are related by a unitary transformation.

Definition 2.41 (Unitary equivalence between pairs). Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces and let $V_i : S \rightarrow \mathcal{H}_i$ for some set S . We say that $(\mathcal{H}_1, V_1)$ and $(\mathcal{H}_2, V_2)$ are *unitarily equivalent* if there exists a unitary operator

$$U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$$

such that

$$UV_1(x) = V_2(x), \quad \forall x \in S.$$

Remark 2.42. If $(\mathcal{H}_1, V_1)$ and $(\mathcal{H}_2, V_2)$ are unitarily equivalent via $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, then in particular

$$\langle V_1(x), V_1(y) \rangle_{\mathcal{H}_1} = \langle UV_1(x), UV_1(y) \rangle_{\mathcal{H}_2} = \langle V_2(x), V_2(y) \rangle_{\mathcal{H}_2}, \quad \forall x, y \in S,$$

since unitary operators preserve inner products. Hence unitarily equivalent pairs induce the same kernel on S . Conversely, as we shall see, this condition is also sufficient: two pairs sharing the same kernel are necessarily unitarily equivalent.

Theorem 2.43 (Kolmogorov's Dilation Theorem). *Let K be a positive definite kernel on a set S . Then, up to unitary equivalence, there exists a unique Hilbert space $\mathcal{H}$ and a map $V : S \rightarrow \mathcal{H}$ such that*

$$\langle V(x), V(y) \rangle_{\mathcal{H}} = K(x, y), \quad \forall x, y \in S, \tag{13}$$

$$\text{cl}(\text{span}\{V(x) : x \in S\}) = \mathcal{H}. \tag{14}$$

Here, V is called an *embedding of S into $\mathcal{H}$* in the sense that it realises S as a set of vectors in a Hilbert space $\mathcal{H}$ whose inner products precisely reproduce the positive definite kernel K .

A map $V : S \rightarrow \mathcal{H}$ satisfying equation (13) is called a (Kolmogorov) *dilation* of K . It is called *minimal* if it furthermore satisfies equation (14), i.e. if the linear span of $V(S)$ is dense in $\mathcal{H}$.

2.6.2 Stone-von Neumann Theorem

Theorem 2.44 (Stone-von Neumann Theorem). *Let $(W(t, s))_{t, s \in \mathbb{R}}$ be a strongly continuous group of unitary operators on a Hilbert space $\mathcal{H}$ satisfying the Weyl relation (12)*

$$W(t, s)W(u, v) = e^{\frac{i\hbar}{2}(tv-su)} W(t+u, s+v), \quad \forall t, s, u, v \in \mathbb{R}.$$

Then $\mathcal{H}$ is unitarily equivalent to $L^2(\mathbb{R}) \otimes \mathcal{G}$ for some Hilbert space $\mathcal{G}$, and under this equivalence $W(t, s)$ corresponds to

$$W_S(t, s) \otimes \mathbf{1},$$

where W_S represents the Schrödinger operators (equation (6)) and thus satisfies

$$W_S(t, s) := e^{\frac{i\hbar}{2}st} T_t S_s = e^{\frac{i\hbar}{2}st} e^{itP} e^{isQ}, \quad \forall t, s \in \mathbb{R},$$

where P, Q are defined on $L^2(\mathbb{R})$ by equation (6).

In other words, given a Hilbert space $\mathcal{H}$ and a Weyl operator $W(t, s)$ satisfying the Weyl relation on $\mathcal{H}$, this theorem guarantees we may find a different Hilbert space $\mathcal{G}$ such that $W(t, s)$ may be represented (by a unitary operator) as acting on $L^2(\mathbb{R}) \otimes \mathcal{G}$, with its unitarily equivalent version $W_S(t, s)$ satisfying (11) and coming from P, Q defined by (6) on $L^2(\mathbb{R})$. The precise implication and its great importance for the eventual construction of the quantum-mechanical central limit theorem [CH71] will be discussed in more detail in section 3.

Proof. Throughout this proof, lengthy calculations are carried out to verify certain properties. In order to maintain the essence of the highlighted proof, these calculations can be found in subsection 3.7.

Let $W : \mathbb{R}^2 \rightarrow \mathcal{B}(\mathcal{H})$ satisfy the Weyl relation (equation (12)). For each integrable function $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ with

$$\int_{\mathbb{R}^2} |f(t, s)| dt ds < \infty,$$

or equivalently $f \in L^1(\mathbb{R}^2)$, informally define a bounded linear operator $A(f)$ on $\mathcal{H}$ by

$$A(f) := \int_{\mathbb{R}} \int_{\mathbb{R}} f(t, s) W(t, s) dt ds.$$

Formally, it is defined as the unique operator satisfying

$$\langle \varphi, A(f)\psi \rangle = \int_{\mathbb{R}^2} f(t, s) \langle \varphi, W(t, s)\psi \rangle dt ds, \quad \forall \varphi, \psi \in \mathcal{H}.$$

We start by noting certain properties of this operator.

$$A(f) + A(g) = A(f + g), \quad A(cf) = cA(f), \quad A(f(t, s))^\dagger = A(f(-t, -s)),$$

and

$$A(f)A(g) = A(f \star_\hbar g), \quad \forall c \in \mathbb{C} \quad \text{and} \quad f, g \in L^1(\mathbb{R}^2),$$

where the twisted convolution is defined by

$$(f \star_\hbar g)(x, y) = \int_{\mathbb{R}^2} f(t, s) g(x - t, y - s) e^{\frac{i\hbar}{2}(ty-sx)} dt ds.$$

Injectivity of $f \mapsto A(f)$ We now proceed to show injectivity of this linear operator $A(f)$, which will be done by proving the following implication:

$$A(f) = \mathbf{0} \implies \left(\mathcal{H} = 0 \quad \text{or} \quad f = 0 \right).$$

Suppose $A(f) = \mathbf{0}$, then for all $a, b \in \mathbb{R}$,

$$\mathbf{0} = W(a, b)^\dagger A(f) W(a, b) = \int_{\mathbb{R}^2} e^{i\hbar(bt-as)} f(t, s) W(t, s) dt ds.$$

In turn, this yields

$$0 = \left\langle \varphi, \left(\int_{\mathbb{R}^2} e^{i\hbar(bt-as)} f(t, s) W(t, s) dt ds \right) \psi \right\rangle \quad \forall \varphi, \psi \in \mathcal{H}.$$

By definition of the integral, we may pass the inner product through the integral, which will give

$$0 = \int_{\mathbb{R}^2} e^{i\hbar(bt-as)} f(t, s) \langle \varphi, W(t, s) \psi \rangle dt ds$$

for all $\varphi, \psi \in \mathcal{H}$ and all $a, b \in \mathbb{R}$.

Define the function $g_{\varphi, \psi} : \mathbb{R}^2 \rightarrow \mathbb{C}$

$$g_{\varphi, \psi}(t, s) := f(t, s) \langle \varphi, W(t, s) \psi \rangle.$$

Then

$$\int_{\mathbb{R}^2} e^{i\hbar(bt-as)} g_{\varphi, \psi}(t, s) dt ds = 0, \quad \forall a, b \in \mathbb{R}.$$

The left-hand side is the Fourier transform of $g_{\varphi, \psi}$ (up to normalisation and sign convention). Since the Fourier transform is injective on $L^1(\mathbb{R}^2)$, it follows that

$$g_{\varphi, \psi}(t, s) = 0 \quad \text{for almost every } (t, s) \in \mathbb{R}^2.$$

Hence

$$f(t, s) \langle \varphi, W(t, s) \psi \rangle = 0 \quad \text{almost everywhere}$$

for all $\varphi, \psi \in \mathcal{H}$.

Suppose $\mathcal{H} \neq \{0\}$. Then we may choose $\varphi, \psi \in \mathcal{H}$ with $\langle \varphi, \psi \rangle \neq 0$. The function $(t, s) \mapsto \langle \varphi, W(t, s) \psi \rangle$ is continuous and equals $\langle \varphi, \psi \rangle \neq 0$ at $(t, s) = (0, 0)$, so it is nonzero on a set of positive measure. Since $f(t, s) \langle \varphi, W(t, s) \psi \rangle = 0$ almost everywhere, we conclude $f = 0$ almost everywhere.

Thus either $\mathcal{H} = \{0\}$ or $f = 0$ almost everywhere, showing that $f \mapsto A(f)$ is an injective operator in the sense that

$$A(f) = A(g) \implies f = g,$$

for non-trivial Hilbert spaces.

Projection $E = A(g)$ The rest of the proof heavily relies on A applied to the following function

$$g(t, s) := \frac{\hbar}{2\pi} \exp\left(-\frac{\hbar}{4}(t^2 + s^2)\right), \quad E := A(g).$$

By the fact that the linear map $f \mapsto A(f)$ is injective, since $g \neq 0$, we have that $E \neq \mathbf{0}$. In Calculation 2.2, we show that

$$EW(a, b)E = \exp\left(-\frac{\hbar}{4}(a^2 + b^2)\right)E, \quad \forall a, b \in \mathbb{R}. \quad (15)$$

We furthermore claim that E is a projection operator, i.e. it satisfies $E^2 = E, E^\dagger = E$.

Indeed,

$$E^\dagger = A(g(t, s))^\dagger = A(g(-t, -s)) = A(g(t, s)) = E,$$

and

$$E^2 = EW(0, 0)E = \exp(0)E = E.$$

Let $\mathcal{G}$ be defined by $\mathcal{G} := \text{Ran}(E)$, which is a Hilbert space by remark 2.6; then for any $\phi \in \mathcal{G}, E\phi = \phi$.

First Kolmogorov dilation Then for all $\phi, \chi \in \mathcal{G}$, we carry out the following calculation:

$$\begin{aligned} \langle W(t, s)\phi, W(u, v)\chi \rangle &= \langle W(t, s)E\phi, W(u, v)E\chi \rangle \\ &= \langle \phi, EW(-t, -s)W(u, v)E\chi \rangle \\ &= e^{-\frac{i\hbar}{2}(tv-su)} \langle \phi, EW(u-t, v-s)E\chi \rangle \\ &= e^{-\frac{i\hbar}{2}(tv-su)} e^{-\frac{\hbar}{4}((u-t)^2 + (v-s)^2)} \langle \phi, E\chi \rangle \\ &= e^{\frac{\hbar}{2}(t+is)(u-iv)} e^{-\frac{\hbar}{4}(t^2+s^2+u^2+v^2)} \langle \phi, \chi \rangle. \end{aligned}$$

Define now

$$V : \mathbb{R}^2 \times \mathcal{G} \rightarrow \mathcal{H}, \quad ((t, s), \phi) \mapsto e^{\frac{\hbar}{4}(t^2+s^2)} W(t, s)\phi. \quad (16)$$

Then V is a Kolmogorov dilation of the kernel

$$K((t, s), \phi; (u, v), \chi) = e^{\frac{\hbar}{2}(t+is)(u-iv)} \langle \phi, \chi \rangle, \quad (17)$$

which follows from multiplying by an exponential factor and comparing to equation (13). By definition of $\mathcal{G}$ and E , it similarly follows that V is in fact a minimal Kolmogorov dilation of K , since it satisfies equation (14).

The rest of the proof now consists of finding another minimal Kolmogorov dilation for our kernel $K : (\mathbb{R}^2 \times \mathcal{G}) \times (\mathbb{R}^2 \times \mathcal{G}) \rightarrow \mathbb{C}$, and then applying Kolmogorov's Dilation Theorem 2.43.

Second Kolmogorov dilation In order to construct another minimal dilation for our kernel K , we will find a dilation $V_S : \mathbb{R}^2 \rightarrow L^2(\mathbb{R})$ in the “standard” case for the Weyl operators, viz. when $W_S(t, s) = e^{\frac{i\hbar}{2}st} S_s T_t$.

In the standard Schrödinger representation,

$$W_S(t, s) = e^{\frac{i\hbar}{2}st} S_s T_t,$$

where

$$(T_t\psi)(x) = \psi(x + \hbar t), \quad (S_s\psi)(x) = e^{isx}\psi(x).$$

Hence

$$(W_S(t, s)\psi)(x) = e^{\frac{i\hbar}{2}st} e^{isx}\psi(x + \hbar t).$$

As before, let

$$g(t, s) = \frac{\hbar}{2\pi} e^{-\frac{\hbar}{4}(t^2+s^2)}.$$

Then it follows by Calculation 2.3 that $E_S := A_S(g)$ is the orthogonal projection onto the one-dimensional subspace

$$\text{Ran}(E_S) = \text{span}\{\Omega\},$$

where

$$\Omega(x) = (\pi\hbar)^{-1/4} e^{-x^2/(2\hbar)}.$$

We now find an explicit expression for a Kolmogorov dilation using the standard Weyl operator. Since

$$\mathcal{G}_S := \text{Ran}(E_S) = \text{span}\{\Omega\},$$

we have (using $\|\Omega\| = 1$) that

$$\langle W_S(t, s)\Omega, W_S(u, v)\Omega \rangle = e^{\frac{\hbar}{2}(t+is)(u-iv)} e^{-\frac{\hbar}{4}(t^2+s^2+u^2+v^2)},$$

following a calculation similar to the one carried out previously.

Define

$$V'_S : \mathbb{R}^2 \rightarrow L^2(\mathbb{R})$$

by

$$V'_S(t, s) := e^{\frac{\hbar}{4}(t^2+s^2)} W_S(t, s)\Omega.$$

Then

$$\begin{aligned} \langle V'_S(t, s), V'_S(u, v) \rangle &= e^{\frac{\hbar}{4}(t^2+s^2)} e^{\frac{\hbar}{4}(u^2+v^2)} \langle W_S(t, s)\Omega, W_S(u, v)\Omega \rangle \\ &= e^{\frac{\hbar}{4}(t^2+s^2)} e^{\frac{\hbar}{4}(u^2+v^2)} e^{\frac{\hbar}{2}(t+is)(u-iv)} e^{-\frac{\hbar}{4}(t^2+s^2+u^2+v^2)} \\ &= e^{\frac{\hbar}{2}(t+is)(u-iv)}, \end{aligned}$$

a calculation similar to the one done before.

Hence V'_S is a Kolmogorov dilation of a different kernel

$$K'((t, s), (u, v)) = e^{\frac{\hbar}{2}(t+is)(u-iv)}.$$

The only difference between V and V'_S is now that their domains differ; V acts on $\mathbb{R}^2 \times \mathcal{G}$, while V'_S , being a projection on a fixed unit vector Ω , acts on $\mathbb{R}^2$ only.

To the end of applying Kolmogorov's dilation theorem, define

$$V_S : \mathbb{R}^2 \otimes \mathcal{G} \rightarrow L^2(\mathbb{R}) \otimes \mathcal{G}, \quad (t, s) \otimes \phi \mapsto e^{\frac{\hbar}{4}(t^2+s^2)} W_S(t, s)\Omega \otimes \phi, \quad (18)$$

which is a minimal Kolmogorov dilation on the kernel K as defined by equation (17), just like the dilation V as defined by equation (16).

Application of Kolmogorov's Dilation Theorem 2.43 Thus Kolmogorov's dilation theorem guarantees a unitary operator

$$U : L^2(\mathbb{R}) \otimes \mathcal{G} \rightarrow \mathcal{H}$$

such that for all $a, b \in \mathbb{R}$ and $\phi \in \mathcal{G}$,

$$UV_S = V,$$

in other words,

$$U\left(e^{\frac{\hbar}{4}(a^2+b^2)}W_S(a,b)\Omega\otimes\phi\right)=e^{\frac{\hbar}{4}(a^2+b^2)}W(a,b)\phi,$$

and the exponential factors cancel.

Validity on the whole of $\mathcal{H}$ We start from

$$U\left(W_S(a,b)\Omega\otimes\phi\right)=W(a,b)\phi, \tag{19}$$

which only holds for $\phi \in \mathcal{G}$. We proceed to show its validity on the whole of $\mathcal{H}$.

Recalling that

$$\mathcal{G} = \text{Ran}(E), \quad E = A(g),$$

let

$$\mathcal{L} := \mathcal{G}^\perp$$

denote the orthogonal complement of $\mathcal{G}$ in $\mathcal{H}$.

We claim that $\mathcal{L}$ is invariant under the Weyl representation. Indeed, since each $W(t, s)$ is unitary and E is a projection commuting with the Weyl operators up to scalar multiples on $\mathcal{G}$ (given by equation (15)), we obtain

$$W(t, s)\mathcal{G} \subseteq \mathcal{G}, \quad \forall t, s \in \mathbb{R}.$$

Taking orthogonal complements and using unitarity of $W(t, s)$ yields

$$W(t, s)\mathcal{L} \subseteq \mathcal{L}, \quad \forall t, s \in \mathbb{R}.$$

Let $W_0(t, s)$ denote the restriction of $W(t, s)$ to $\mathcal{L}$. Define the operator $A_0(f)$ on $\mathcal{L}$ by

$$A_0(f) = \int_{\mathbb{R}^2} f(t, s) W_0(t, s) dt ds,$$

constructed in the same way as $A(f)$ on $\mathcal{H}$.

Let

$$E_0 := A_0(g),$$

the orthogonal projection onto $\mathcal{L}$. In particular,

$$\text{Ran}(E_0) \subseteq \mathcal{L} \quad \text{and} \quad \text{Ran}(E) = \mathcal{G},$$

so that

$$\text{Ran}(E_0) \perp \text{Ran}(E).$$

It follows that E_0 is simultaneously a subprojection of E and orthogonal to E , i.e.

$$\text{Ran}(E_0) \subseteq \text{Ran}(E) \quad \text{and} \quad E_0 \perp E.$$

The only projection satisfying both conditions is the zero operator, hence

$$E_0 = \mathbf{0}.$$

By injectivity of the map $f \mapsto A_0(f)$, this implies

$$g = 0 \quad \text{or} \quad \mathcal{L} = \{0\}.$$

Since $g \neq 0$, we conclude

$$\mathcal{L} = \{0\}.$$

and therefore

$$\mathcal{G} = \mathcal{H},$$

and equation (19) holds on all of $\mathcal{H}$. Finally, we rewrite this equation into one involving only U , $W(a, b)$ and $W_S(a, b)$. Let

$$\mathcal{D} := \text{span}\{W_S(t, s)\Omega \otimes \phi : t, s \in \mathbb{R}, \phi \in \mathcal{H}\},$$

which is dense in $L^2(\mathbb{R}) \otimes \mathcal{H}$.

Fix $a, b \in \mathbb{R}$ and $\psi = W_S(t, s)\Omega \otimes \phi \in \mathcal{D}$. Then

$$\begin{aligned} UW_S(a, b)\psi &= U(W_S(a, b)W_S(t, s)\Omega \otimes \phi) \\ &= e^{\frac{i\hbar}{2}(as-bt)}U(W_S(a+t, b+s)\Omega \otimes \phi) \\ &= e^{\frac{i\hbar}{2}(as-bt)}W(a+t, b+s)\phi, \end{aligned}$$

using the Weyl relations and equation (19).

On the other hand,

$$\begin{aligned} W(a, b)U\psi &= W(a, b)W(t, s)\phi \\ &= e^{\frac{i\hbar}{2}(as-bt)}W(a+t, b+s)\phi. \end{aligned}$$

Hence

$$UW_S(a, b)\psi = W(a, b)U\psi \quad \forall \psi \in \mathcal{D}.$$

By density of $\mathcal{D}$ and boundedness of the operators,

$$W(a, b) = UW_S(a, b)U^{-1}$$

on the whole of $\mathcal{H}$, establishing the correspondence between $W(t, s)$ and $W_S(t, s) \otimes \mathbf{1}$. □

2.7 Density operators

In classical probability theory, the state of a random variable is described by a probability distribution. In quantum mechanics, the corresponding notion is that of a *density operator* or *density state*, which simultaneously captures probabilistic uncertainty and quantum superposition.

Definition 2.45 (Trace-class operators). Let $\mathcal{H}$ be a complex Hilbert space. An operator

$$T : \mathcal{H} \rightarrow \mathcal{H}$$

is called *trace-class* if, for some (and hence every) orthonormal basis $\{e_i\}_{i \in I}$ of $\mathcal{H}$,

$$\sum_{i \in I} \langle e_i, |T| e_i \rangle < \infty,$$

where the operator $|T|$ is defined through the Spectral Theorem 2.27 by

$$|T| := (T^\dagger T)^{1/2}.$$

For trace-class operators, the trace

$$\text{Tr}(T) = \sum_{i \in I} \langle e_i, T e_i \rangle$$

is well-defined and independent of the chosen orthonormal basis.

Remark 2.46. Let ψ be a unit vector and let proj_ψ denote the orthogonal projection onto $\text{span}\{\psi\}$. Then for any trace-class operator A ,

$$\text{Tr}(\text{proj}_\psi A) = \langle \psi, A \psi \rangle.$$

Indeed, if $\{\psi_n\}_{n=0}^\infty$ is any orthonormal basis, then

$$\begin{aligned} \text{Tr}(\text{proj}_\psi A) &= \sum_{n=0}^{\infty} \langle \psi_n, \text{proj}_\psi A \psi_n \rangle \\ &= \sum_{n=0}^{\infty} \langle \psi_n, \langle \psi, A \psi_n \rangle \psi \rangle \\ &= \sum_{n=0}^{\infty} \langle \psi, A \psi_n \rangle \langle \psi_n, \psi \rangle. \end{aligned}$$

Since

$$\psi = \sum_{n=0}^{\infty} \langle \psi_n, \psi \rangle \psi_n,$$

it follows by linearity of the inner product in its second argument and linearity of A that

$$\begin{aligned} \text{Tr}(\text{proj}_\psi A) &= \sum_{n=0}^{\infty} \langle \psi, \langle \psi_n, \psi \rangle A \psi_n \rangle \\ &= \left\langle \psi, A \left(\sum_{n=0}^{\infty} \langle \psi_n, \psi \rangle \psi_n \right) \right\rangle \\ &= \langle \psi, A \psi \rangle, \end{aligned}$$

as required.

Trace-class operators play the role of “integrable” operators on Hilbert spaces and form the natural

setting for quantum states, since expectation values and entropies are expressed through traces.

Definition 2.47 (Density operator). A *density operator*, also called a *density state* on $\mathcal{H}$ is a trace-class operator

$$\rho : \mathcal{H} \rightarrow \mathcal{H}$$

satisfying

$$\rho \geq 0, \quad \text{Tr}(\rho) = 1.$$

Here, the positivity condition $\rho \geq 0$ means that

$$\langle \psi, \rho \psi \rangle \geq 0 \quad \forall \psi \in \mathcal{H}.$$

Positivity guarantees that probabilities and expectation values are nonnegative where appropriate, while the normalization condition ensures that total probability equals one. The following proposition provides a more concrete interpretation of density operators and helps clarify their physical significance.

Proposition 2.48. Let $\rho \in \mathcal{B}(\mathcal{H})$ be a positive trace-class operator. Then there exists an orthonormal basis $\{\phi_i\}_{i \in \mathbb{N}} \subseteq \mathcal{H}$ and nonnegative numbers λ_i such that

$$\rho \psi = \sum_i \lambda_i \langle \phi_i, \psi \rangle \phi_i \quad \forall \psi \in \mathcal{H},$$

where the series converges in norm.

Proof. We first show that ρ is actually self-adjoint. For any $\varphi, \psi \in \mathcal{H}$, positivity gives

$$\langle \varphi + \psi, \rho(\varphi + \psi) \rangle \geq 0 \quad \text{and} \quad \langle \varphi + i\psi, \rho(\varphi + i\psi) \rangle \geq 0,$$

Expanding the first yields

$$\langle \varphi, \rho \varphi \rangle + \langle \varphi, \rho \psi \rangle + \langle \psi, \rho \varphi \rangle + \langle \psi, \rho \psi \rangle \in \mathbb{R},$$

and since $\langle \varphi, \rho \varphi \rangle, \langle \psi, \rho \psi \rangle \in \mathbb{R}$, we conclude that $\langle \varphi, \rho \psi \rangle + \langle \psi, \rho \varphi \rangle \in \mathbb{R}$. Expanding the second similarly gives that $i(\langle \psi, \rho \varphi \rangle - \langle \varphi, \rho \psi \rangle) \in \mathbb{R}$. Together, these two conditions force

$$\langle \varphi, \rho \psi \rangle = \overline{\langle \psi, \rho \varphi \rangle} = \overline{\langle \rho \psi, \varphi \rangle} \quad \forall \varphi, \psi \in \mathcal{H},$$

so ρ is self-adjoint by taking the conjugate of the second equality.

By the Spectral Theorem (2.27), there exists a projection-valued measure E on $\mathbb{R}$ such that

$$\rho = \int_{\mathbb{R}} \lambda dE(\lambda).$$

Since ρ is trace-class, for any orthonormal basis $\{e_i\}$ of $\mathcal{H}$ we have

$$\text{Tr}(\rho) = \sum_i \langle e_i, \rho e_i \rangle = \sum_i \int_{\mathbb{R}} \lambda dE_{e_i}(\lambda) < \infty,$$

where E_{e_i} denotes the positive measure $B \mapsto \langle e_i, E(B)e_i \rangle$. This finiteness forces each measure E_{e_i} to be concentrated on a countable set of points, so that E is a sum of Dirac measures,

$$E = \sum_i E(\{\lambda_i\}) \delta_{\lambda_i},$$

concentrated on a countable set of values $\{\lambda_i\}$. Setting $\text{proj}_i := E(\{\lambda_i\})$, the spectral representation

reduces to

$$\rho\psi = \sum_i \lambda_i \text{proj}_i \psi \quad \forall \psi \in \mathcal{H},$$

with convergence in norm. Since the projections proj_i are mutually orthogonal by Property 3 and their ranges span $\mathcal{H}$ by Property 1 of definition 2.22, choosing an orthonormal basis $\{\phi_i\}_{i \in \mathbb{N}}$ by taking orthonormal bases within each range $\text{Ran}(\text{proj}_i)$ gives

$$\rho\psi = \sum_i \lambda_i \langle \phi_i, \psi \rangle \phi_i \quad \forall \psi \in \mathcal{H}.$$

Finally, positivity gives $\lambda_i = \langle \phi_i, \rho\phi_i \rangle \geq 0$ for all i . □

The numbers λ_i are precisely the eigenvalues of ρ . Since

$$\text{Tr}(\rho) = 1,$$

one furthermore has

$$\sum_i \lambda_i = 1.$$

Equivalently,

$$\rho = \sum_i \lambda_i \text{proj}_{\phi_i},$$

where proj_{ϕ_i} denotes the orthogonal projection onto the one-dimensional subspace spanned by ϕ_i ,

$$\text{proj}_{\phi_i}(\psi) = \langle \phi_i, \psi \rangle \phi_i.$$

The coefficients λ_i may therefore be interpreted as probabilities. In this sense, a density operator describes a statistical ensemble of quantum states.

Recall that observables in quantum mechanics are represented by self-adjoint operators on $\mathcal{H}$. If A is such an observable and the system is in the state ρ , then the expectation value of A in the state ρ is defined by

$$\mathbb{E}_\rho[A] = \text{Tr}(\rho A).$$

In the special case where $\rho = \text{proj}_\psi$ is “pure”, one recovers the familiar expression

$$\mathbb{E}_\rho[A] = \langle \psi, A\psi \rangle.$$

Density operators therefore extend the usual description of quantum states by wave functions, while also allowing for statistical mixtures of different states, making them more suitable for many problems in quantum information theory and quantum statistical mechanics.

2.8 Calculations I

In the following subsection, three calculations postponed throughout section 2 are performed.

Calculation 2.1 shows the equivalence between the CCR and a relation involving the unitary groups T_t and S_s , a relation given by equation (10).

Calculation 2.2 proves equation (15), relating the projection operator E with the Weyl operator $W(a, b)$.

Finally, Calculation 2.3 shows that E_S , the projection operator E in the standard case, is an orthogonal projection onto a one-dimensional subspace $\text{span}\{\Omega\}$.

Calculation 2.1. In this calculation, we show

$$T_t S_s = e^{i\hbar st} S_s T_t \iff [P, Q] = -i\hbar \mathbf{1},$$

where

$$T_t = e^{itP}, \quad S_s = e^{isQ}.$$

Throughout this calculation, we ignore domain issues and work with informal definitions. This is precisely why one typically uses the Weyl relations instead, since they remain formally well-defined and valid in all cases.

($\Leftarrow$) Define

$$Q(t) := T_t Q T_t^{-1}$$

and informally differentiate, where

$$\begin{aligned} \frac{d}{dt} Q(t) &= \frac{d}{dt} (T_t Q T_t^{-1}) \\ &= (iPT_t) Q T_t^{-1} + T_t Q (-iPT_t^{-1}) \\ &= iPT_t Q T_t^{-1} - iT_t Q T_t^{-1} P \\ &= iT_t (PQ - QP) T_t^{-1} \\ &= iT_t [P, Q] T_t^{-1}, \end{aligned}$$

where it was used that $[T_t, P] = 0$ for any $t \in \mathbb{R}$, following remark 2.30. Using $[P, Q] = -i\hbar \mathbf{1}$, we obtain

$$\frac{d}{dt} Q(t) = i(-i\hbar) \mathbf{1} = \hbar \mathbf{1}.$$

Solving the differential equation trivially yields

$$Q(t) = Q + \hbar t \mathbf{1}.$$

Thus we obtain the exact conjugation identity

$$T_t Q T_t^{-1} = Q + \hbar t \mathbf{1}.$$

Now we conjugate the exponential $S_s = e^{isQ}$, following remark 2.29, to get

$$T_t S_s T_t^{-1} = T_t e^{isQ} T_t^{-1} = e^{is(T_t Q T_t^{-1})} = e^{is(Q + \hbar t \mathbf{1})} = e^{isQ} e^{is\hbar t},$$

since Q commutes with the operator $\hbar t \mathbf{1}$. Hence

$$T_t S_s T_t^{-1} = e^{i\hbar s t} S_s,$$

which is rearranged to the desired result.

($\Rightarrow$) To prove the implication from left to right, as noted in the previous remark, we *informally* differentiate and ignore domain issues throughout.

Starting from the Weyl relations

$$e^{isQ} e^{itP} = e^{-i\hbar s t} e^{itP} e^{isQ},$$

differentiate with respect to t and evaluate at $t = 0$. Using

$$\left. \frac{d}{dt} e^{itP} \right|_{t=0} = iP, \quad \left. \frac{d}{dt} e^{-i\hbar s t} \right|_{t=0} = -i\hbar s, \quad \left. e^{itP} \right|_{t=0} = \mathbf{1},$$

we obtain

$$e^{isQ} iP = -i\hbar s e^{isQ} + iP e^{isQ}.$$

Dividing by i yields

$$e^{isQ} P = P e^{isQ} - \hbar s e^{isQ}.$$

Next, differentiate with respect to s and evaluate at $s = 0$. Since

$$\left. \frac{d}{ds} e^{isQ} \right|_{s=0} = iQ,$$

we find

$$iQP = iPQ - \hbar \mathbf{1}.$$

Hence

$$[P, Q] = PQ - QP = -i\hbar \mathbf{1},$$

which is the desired commutation relation.

Calculation 2.2. In this calculation, we show that

$$EW(a, b)E = \exp\left(-\frac{\hbar}{4}(a^2 + b^2)\right)E, \quad \forall a, b \in \mathbb{R},$$

where

$$A(f) := \int_{\mathbb{R}^2} f(t, s) W(t, s) dt ds, \quad g(t, s) := \frac{\hbar}{2\pi} \exp\left(-\frac{\hbar}{4}(t^2 + s^2)\right), \quad E := A(g).$$

We use the following identification:

$$A(\delta_{(a,b)}) = \int_{\mathbb{R}^2} \delta(t - a, s - b) W(t, s) dt ds = W(a, b).$$

Then we compute

$$(g \star_{\hbar} \delta_{(a,b)} \star_{\hbar} g)(x, y).$$

We obtain

$$\begin{aligned}
(g \star_{\hbar} \delta_{(a,b)})(x, y) &= \int_{\mathbb{R}^2} g(t, s) \delta(x - t - a, y - s - b) \exp\left(\frac{i\hbar}{2}(ty - sx)\right) dt ds \\
&= g(x - a, y - b) \exp\left(\frac{i\hbar}{2}(bx - ay)\right) \\
&= \frac{\hbar}{2\pi} \exp\left(-\frac{\hbar}{4}(x - a)^2 + (y - b)^2\right) \exp\left(\frac{i\hbar}{2}(bx - ay)\right).
\end{aligned}$$

Hence

$$\begin{aligned}
(g \star_{\hbar} \delta_{(a,b)} \star_{\hbar} g)(x, y) &= \int_{\mathbb{R}^2} g(x - t, y - s) \frac{\hbar}{2\pi} \exp\left(-\frac{\hbar}{4}(t - a)^2 + (s - b)^2\right) \\
&\quad \cdot \exp\left(\frac{i\hbar}{2}(bt - as)\right) \exp\left(\frac{i\hbar}{2}(ty - sx)\right) dt ds \\
&= \left(\frac{\hbar}{2\pi}\right)^2 \int_{\mathbb{R}^2} \exp\left(-\frac{\hbar}{4}((x - t)^2 + (y - s)^2 + (t - a)^2 + (s - b)^2)\right) \\
&\quad \cdot \exp\left(\frac{i\hbar}{2}(bt - as + ty - sx)\right) dt ds. \\
&= \left(\frac{\hbar}{2\pi}\right)^2 \exp\left(-\frac{\hbar}{4}(x^2 + y^2 + a^2 + b^2)\right) I_t I_s,
\end{aligned}$$

where

$$\begin{aligned}
I_t &= \int_{\mathbb{R}} \exp\left(-\frac{\hbar}{2}t^2 + \frac{\hbar}{2}(x + a + i(b + y))t\right) dt, \\
I_s &= \int_{\mathbb{R}} \exp\left(-\frac{\hbar}{2}s^2 + \frac{\hbar}{2}(y + b - i(a + x))s\right) ds.
\end{aligned}$$

Using

$$\int_{\mathbb{R}} e^{2Ct^2 + Lt} dt = \sqrt{\frac{\pi}{-2C}} \exp\left(-\frac{L^2}{8C}\right),$$

with $C = -\frac{\hbar}{4}$, we obtain

$$\begin{aligned}
I_t &= \sqrt{\frac{2\pi}{\hbar}} \exp\left(\frac{\hbar}{8}(x + a + i(b + y))^2\right), \\
I_s &= \sqrt{\frac{2\pi}{\hbar}} \exp\left(\frac{\hbar}{8}(y + b - i(a + x))^2\right)
\end{aligned}$$

Thus

$$I_t I_s = \frac{2\pi}{\hbar} \exp\left(\frac{\hbar}{8}[(x + a + i(b + y))^2 + (y + b - i(a + x))^2]\right) = \frac{2\pi}{\hbar},$$

where the final equality follows by a direct expansion. Hence

$$(g \star_{\hbar} \delta_{(a,b)} \star_{\hbar} g)(x, y) = \left(\frac{\hbar}{2\pi}\right)^2 \exp\left(-\frac{\hbar}{4}(x^2 + y^2 + a^2 + b^2)\right) I_t I_s = \exp\left(-\frac{\hbar}{4}(a^2 + b^2)\right) g(x, y).$$

Finally, by linearity of the operator $f \mapsto A(f)$ and recalling that $E := A(g)$, we then conclude that

$$EW(a, b)E = \exp\left(-\frac{\hbar}{4}(a^2 + b^2)\right) E \quad \forall a, b \in \mathbb{R},$$

using

$$A(g \star_{\hbar} \delta_{(a,b)} \star_{\hbar} g) = EA(\delta_{(a,b)})E.$$

Calculation 2.3. We consider the Weyl operator in the “standard” case

$$(W_S(t, s)\psi)(x) = e^{\frac{i\hbar}{2}st} e^{isx} \psi(x + \hbar t).$$

As before, let

$$A(f) := \int_{\mathbb{R}^2} f(t, s) W(t, s) dt ds, \quad g(t, s) = \frac{\hbar}{2\pi} e^{-\frac{\hbar}{4}(t^2+s^2)}, \quad E_S := A(g).$$

Define also

$$\Omega(x) = (\pi\hbar)^{-1/4} e^{-x^2/(2\hbar)},$$

and notice that $\|\Omega\| = 1$, i.e. it is a unit vector. Now for all $\psi \in \mathcal{H}$

$$\begin{aligned} (E_S\psi)(x) &= (A(g)\psi)(x) \\ &= \frac{\hbar}{2\pi} \int_{\mathbb{R}^2} e^{-\frac{\hbar}{4}(t^2+s^2)} e^{\frac{i\hbar}{2}st} e^{isx} \psi(x + \hbar t) dt ds. \end{aligned}$$

Integrating first with respect to s , by Fubini’s theorem,

$$\begin{aligned} (E_S\psi)(x) &= \frac{\hbar}{2\pi} \int_{\mathbb{R}} e^{-\frac{\hbar}{4}t^2} \psi(x + \hbar t) \left(\int_{\mathbb{R}} e^{-\frac{\hbar}{4}s^2} e^{is(x+\frac{\hbar t}{2})} ds \right) dt \\ &= \sqrt{\frac{\hbar}{\pi}} \int_{\mathbb{R}} e^{-\frac{\hbar}{4}t^2} e^{-\frac{1}{\hbar}(x+\frac{\hbar t}{2})^2} \psi(x + \hbar t) dt \\ &= \sqrt{\frac{\hbar}{\pi}} e^{-x^2/\hbar} \int_{\mathbb{R}} e^{-xt - \frac{\hbar}{2}t^2} \psi(x + \hbar t) dt \\ &= \frac{1}{\sqrt{\pi\hbar}} e^{-x^2/(2\hbar)} \int_{\mathbb{R}} e^{-y^2/(2\hbar)} \psi(y) dy, \end{aligned}$$

having used the Gaussian Fourier transform

$$\int_{\mathbb{R}} e^{-\frac{\hbar}{4}s^2} e^{i\alpha s} ds = \sqrt{\frac{4\pi}{\hbar}} e^{-\frac{\alpha^2}{\hbar}}.$$

Then

$$\begin{aligned} (E_S\psi)(x) &= \Omega(x) \int_{\mathbb{R}} \overline{\Omega(y)} \psi(y) dy \\ &= \Omega(x) \langle \Omega, \psi \rangle, \end{aligned}$$

hence

$$E_S\psi = \langle \Omega, \psi \rangle \Omega,$$

and therefore

$$E_S = \text{proj}_{\Omega}.$$

3 Construction

In order to discuss monotonicity in entropy of sums of quantum random variables, it must first be clarified what entropy is, and in what sense quantum random variables may be considered monotonically increasing or decreasing. To this end, we first introduce the concept of entropy and remind the reader of the (classical) Central Limit Theorem, henceforth denoted as cCLT, by presenting it in its most commonly taught form.

3.1 Central Limit Theorem and entropy

Theorem 3.1 (Classical Central Limit Theorem (cCLT)). *Let $X_1, X_2, \dots, X_n$ be independent, identically distributed (i.i.d.) random variables with*

$$\mathbb{E}[X_i] = \mu, \quad \text{Var}(X_i) = \sigma^2 < \infty.$$

Define the sample mean

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

Then, as $n \rightarrow \infty$,

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1),$$

where $\xrightarrow{d}$ denotes convergence in distribution. Equivalently,

$$\bar{X}_n \approx \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right) \quad \text{for large } n.$$

The CLT states that the average of many independent random variables tends to follow a normal distribution, regardless of the original distribution of the variables (provided the variance is finite).

Next, we would like to state the classical version of (differential) entropy and some results encompassing it, in order to further our attempts of finding a quantum-mechanical analogue of said results.

Definition 3.2 (Differential entropy). Let X be a continuous random variable on $\mathbb{R}^n$ with density $f_X(x)$. The *differential entropy*, or *Shannon entropy*, of X is defined by

$$h(X) := - \int_{\mathbb{R}^n} f_X(x) \ln f_X(x) dx,$$

whenever the integral exists. Equivalently,

$$h(X) = -\mathbb{E}[\ln f_X(X)].$$

The following proposition establishes a information-theoretic refinement of the cCLT. It was a long-standing open problem, originally posed by Shannon, whether the differential entropy of the normalised sums increases monotonically in n ; this was only resolved in 2004 by Artstein, Ball, Barthe, and Naor [Art+04]. Once monotonicity is established, one sees clearly that convergence in the cCLT is driven by an analogue of the second law of thermodynamics; just as physical entropy increases toward equilibrium, the information-theoretic entropy of the sample mean increases monotonically toward its maximum, attained uniquely by the Gaussian distribution. This proposition will serve as the basis for developing a quantum-mechanical analogue of this monotonicity result.

Proposition 3.3. *Let $X_1, X_2, \dots$ be independent and identically distributed random variables with*

$$\mathbb{E}[X_i] = 0, \quad \text{Var}(X_i) = 1,$$

and define the normalized sums

$$S_n = \frac{X_1 + \dots + X_n}{\sqrt{n}}.$$

Then the sequence of differential entropies is monotone increasing [Art+04]:

$$h(S_1) \leq h(S_2) \leq \dots \leq h(S_n) \leq \dots$$

and moreover [Bar86]

$$\lim_{n \rightarrow \infty} h(S_n) = h(Z),$$

where

$$Z \sim \mathcal{N}(0, 1).$$

Furthermore, since the standard Gaussian distribution has maximal entropy among all random variables with unit variance [PW19]

$$h(S_n) \leq \frac{1}{2} \ln(2\pi e),$$

for every $n \in \mathbb{N}$.

Many components of the above theorem and proposition in the classical case cannot yet easily be “translated” to the quantum case; for instance, we have not yet described “Gaussian distributions” for quantum-mechanical density operators ρ , and we may not simply sum density states as we are able to do with classical random variables.

In order to resolve these issues, the exposition in this section follows the approach of [CH71], with several gaps in the derivations filled in for completeness. This paper describes a quantum-mechanical Central Limit Theorem (qCLT), eventually allowing for an analogous study of the monotonicity of entropy of sums of quantum random variables. We start this construction by an analogue for differential entropy, which we wish to investigate.

Definition 3.4 (von Neumann entropy). Let ρ be a density operator on a Hilbert space $\mathcal{H}$.

The *von Neumann entropy* of ρ is defined by

$$S(\rho) = -\text{Tr}(\rho \ln \rho).$$

If $\{\lambda_i\}$ are the eigenvalues of ρ , then equivalently

$$S(\rho) = -\sum_i \lambda_i \ln \lambda_i.$$

By convention, terms of the form $0 \ln 0$ are taken to be zero.

This definition strongly corresponds to the classical one given by the differential entropy 3.2, keeping in mind the remarks made on density operators and trace-class operators in subsection 2.7.

3.2 Distribution operators

Before applying much of the preceding theory in the construction of “distribution operators” for a canonical pair (P, Q) in the state ρ , we begin by recalling a fact from classical probability theory.

Proposition 3.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X, Y : \Omega \rightarrow \mathbb{R}$ be random variables. Then there exists a probability measure $\mathbb{P}_{X,Y}$ on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$, called the joint distribution of (X, Y) , such that for every measurable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ for which the expectations are well-defined,*

$$\int_{\Omega} f(X, Y) d\mathbb{P} = \int_{\mathbb{R}^2} f(x, y) d\mathbb{P}_{X,Y}(x, y).$$

Moreover, $\mathbb{P}_{X,Y}$ is uniquely determined by X and Y .

Intuitively, a pair of random variables (X, Y) defined on a “general” probability space Ω uniquely bring about a joint probability distribution $\mathbb{P}_{X,Y}$ on an “explicit” probability space $\mathbb{R}^2$. We will now define the “distribution operator” with this analogy in mind, by applying the Stone-von Neumann Theorem 2.44 multidimensionally.

Let (P, Q) be a canonical pair acting on the Hilbert space $\mathcal{H}$. The Weyl operator defined by the unique strongly continuous one-parameter unitary groups (T_t, S_s) corresponding to (P, Q) then satisfies the Weyl relation, after which the Stone-von Neumann Theorem may be applied at $(t, 0)$ and $(0, s)$ to yield the following. There exists some Hilbert space $\mathcal{G}$ and some unitary operator

$$T_{P,Q} : \mathcal{H} \rightarrow L^2(\mathbb{R}) \otimes \mathcal{G}$$

such that for all $t, s \in \mathbb{R}$:

$$\begin{aligned} \exp(itP) &= T_{P,Q}^{-1}(\exp(itP_0) \otimes \mathbf{1}_{\mathcal{G}})T_{P,Q}, \\ \exp(isQ) &= T_{P,Q}^{-1}(\exp(isQ_0) \otimes \mathbf{1}_{\mathcal{G}})T_{P,Q}, \end{aligned}$$

where (P_0, Q_0) is the Schrödinger canonical pair acting on $L^2(\mathbb{R})$ and are thus given by

$$\begin{aligned} P_0\psi(x) &= -i\hbar \frac{\partial\psi(x)}{\partial x}, \\ Q_0\psi(x) &= x\psi(x). \end{aligned}$$

Analogous to the classical case, for each density operator ρ on $\mathcal{H}$ there exists a unique $\rho_{P,Q}$ corresponding to the above decomposition such that for all $A \in \mathcal{B}(L^2(\mathbb{R}))$,

$$\text{Tr}(A\rho_{P,Q}) = \text{Tr}[T_{P,Q}^{-1}(A \otimes \mathbf{1}_{\mathcal{G}})T_{P,Q}\rho].$$

Informally,

$$\rho_{P,Q} = (\mathbf{1} \otimes \text{Tr}_{\mathcal{G}})(T_{P,Q}\rho T_{P,Q}^{-1}).$$

Definition 3.6 (Distribution operator). Let (P, Q) be a canonical pair in state ρ acting on the Hilbert space $\mathcal{H}$. Then the unique density operator $\rho_{P,Q}$ on $L^2(\mathbb{R})$ as described above is called the *distribution operator* for (P, Q) in the state ρ .

The distribution operator will often be called the *reduced density operator*, as is most common in modern literature.

For the purpose of defining different notions of independence, we wish to extend this definition to multiple canonical pairs. As such, let us suppose several canonical pairs $(P_1, Q_1), \dots, (P_n, Q_n)$ such that their Weyl operators informally defined by

$$W_j(t, s) = e^{i(sQ_j + tP_j)}$$

commute for different values of $j \in \{1, \dots, n\}$. The many-dimensional version of the Stone-von Neumann Theorem then states there exists a Hilbert space $\mathcal{H}$ and a unitary operator

$$T_{P_1, Q_1, \dots, P_n, Q_n} : \mathcal{H} \rightarrow \underbrace{L^2(\mathbb{R}) \otimes \dots \otimes L^2(\mathbb{R})}_{n \text{ times}} \otimes \mathcal{G}$$

such that for all $t, s \in \mathbb{R}$ and $j \in \{1, \dots, n\}$:

$$\begin{aligned} \exp(itP_j) &= T_{P_1, Q_1, \dots, P_n, Q_n}^{-1} \left(\underbrace{\mathbf{1}_{L^2(\mathbb{R})} \otimes \dots \otimes \exp(itP_0)_{(j\text{-th position})} \otimes \dots \otimes \mathbf{1}_{L^2(\mathbb{R})}}_{n \text{ times}} \otimes \mathbf{1}_{\mathcal{G}} \right) T_{P_1, Q_1, \dots, P_n, Q_n}, \\ \exp(isQ_j) &= T_{P_1, Q_1, \dots, P_n, Q_n}^{-1} \left(\underbrace{\mathbf{1}_{L^2(\mathbb{R})} \otimes \dots \otimes \exp(isQ_0)_{(j\text{-th position})} \otimes \dots \otimes \mathbf{1}_{L^2(\mathbb{R})}}_{n \text{ times}} \otimes \mathbf{1}_{\mathcal{G}} \right) T_{P_1, Q_1, \dots, P_n, Q_n}, \end{aligned}$$

where (P_0, Q_0) is again the canonical Schrödinger pair acting on $L^2(\mathbb{R})$.

Definition 3.7 (Joint distribution operator). Let $(P_1, Q_1), \dots, (P_n, Q_n)$ be canonical pairs acting on the Hilbert space $\mathcal{H}$, and let ρ be a density operator on $\mathcal{H}$. Then the unique density operator $\rho_{P_1, Q_1, \dots, P_n, Q_n}$ on $L^2(\mathbb{R})$ on $L^2(\mathbb{R})$ satisfying

$$\text{Tr}(A \rho_{P_1, Q_1, \dots, P_n, Q_n}) = \text{Tr}[T_{P_1, Q_1, \dots, P_n, Q_n}^{-1} (A \otimes \mathbf{1}_{\mathcal{G}}) T_{P_1, Q_1, \dots, P_n, Q_n} \rho] \quad (20)$$

for all $A \in \mathcal{B}(L^2(\mathbb{R})^{\otimes n})$ is called the *joint distribution operator* for $(P_1, Q_1), \dots, (P_n, Q_n)$ in the state ρ . Informally,

$$\rho_{P_1, Q_1, \dots, P_n, Q_n} = (\mathbf{1} \otimes \text{Tr}_{\mathcal{G}})(T_{P_1, Q_1, \dots, P_n, Q_n} \rho T_{P_1, Q_1, \dots, P_n, Q_n}^{-1}).$$

Here, and throughout the rest of this paper, $U^{\otimes n}$ denotes the n -fold tensor product of the operator or space U with itself. Often, the distribution operator ρ_{P_j, Q_j} of a canonical pair (P_j, Q_j) will be denoted by ρ when no confusion is caused this way.

3.3 Independence and convergence

The following subsection describes notions of independence and convergence of density operators.

Definition 3.8 (Stochastic independence). Let $(P_1, Q_1), \dots, (P_n, Q_n)$ be canonical pairs acting on the Hilbert space $\mathcal{H}$, and let ρ be a density operator on $\mathcal{H}$. Let $\rho_{P_1, Q_1, \dots, P_n, Q_n}$ be their joint distribution operator, and let (P_j, Q_j) have distribution operator ρ_{P_j, Q_j} . The pairs $(P_1, Q_1), \dots, (P_n, Q_n)$ are said to be *stochastically independent* in the state ρ if

$$\rho_{P_1, Q_1, \dots, P_n, Q_n} = \bigotimes_{j=1}^n \rho_{P_j, Q_j}.$$

The analogy with the classical case becomes clear with the following proposition.

Proposition 3.9. *The canonical pairs $(P_1, Q_1), \dots, (P_n, Q_n)$ are stochastically independent in the state ρ if and only if for all bounded measurable functions $f_1, \dots, f_n$,*

$$\mathbb{E}_\rho \left[\prod_{j=1}^n f_j(P_j, Q_j) \right] = \prod_{j=1}^n \mathbb{E}_\rho[f_j(P_j, Q_j)],$$

where the $f_j(P_j, Q_j)$ are defined through the functional calculus given by remark 2.28.

Proof. For notational ease, define

$$T := T_{P_1, Q_1, \dots, P_n, Q_n}.$$

We first make the following remark: by the multidimensional version of the Stone-von Neumann Theorem,

$$\exp(itP_j) = T^{-1} \left(\mathbf{1} \otimes \dots \otimes \exp(itP_0)_{(j)} \otimes \dots \otimes \mathbf{1} \otimes \mathbf{1}_{\mathcal{G}} \right) T$$

for all $t \in \mathbb{R}$. Since

$$t \mapsto \exp(itP_j)$$

and

$$t \mapsto T^{-1} \left(\mathbf{1} \otimes \dots \otimes \exp(itP_0)_{(j)} \otimes \dots \otimes \mathbf{1} \otimes \mathbf{1}_{\mathcal{G}} \right) T$$

are both strongly continuous one-parameter unitary groups, Stone's Theorem 2.35 implies that their self-adjoint generators are equal. Hence

$$P_j = T^{-1} \left(\mathbf{1} \otimes \dots \otimes (P_0)_{(j)} \otimes \dots \otimes \mathbf{1} \otimes \mathbf{1}_{\mathcal{G}} \right) T.$$

Similarly,

$$Q_j = T^{-1} \left(\mathbf{1} \otimes \dots \otimes (Q_0)_{(j)} \otimes \dots \otimes \mathbf{1} \otimes \mathbf{1}_{\mathcal{G}} \right) T.$$

Let E_{P_j} and E_{Q_j} denote the spectral measures of P_j and Q_j . By unitary equivalence and the uniqueness part of the Spectral Theorem,

$$E_{P_j}(B) = T^{-1} \left(\mathbf{1} \otimes \dots \otimes E_{P_0}(B)_{(j)} \otimes \dots \otimes \mathbf{1} \otimes \mathbf{1}_{\mathcal{G}} \right) T$$

for any $B \in \mathcal{B}(\mathbb{R})$ and similarly for Q_j .

Therefore every bounded operator obtained from the spectral calculus of (P_j, Q_j) out of a bounded, measurable function $f_j : \mathbb{R}^2 \rightarrow \mathbb{C}$ satisfies

$$f_j(P_j, Q_j) = T^{-1} \left(\mathbf{1} \otimes \dots \otimes f_j(P_0, Q_0)_{(j)} \otimes \dots \otimes \mathbf{1} \otimes \mathbf{1}_{\mathcal{G}} \right) T,$$

which in turn leads to

$$\begin{aligned}
\prod_{j=1}^n f_j(P_j, Q_j) &= \prod_{j=1}^n T^{-1} \left(\mathbf{1} \otimes \cdots \otimes f_j(P_0, Q_0)_{(j)} \otimes \cdots \otimes \mathbf{1} \otimes \mathbf{1}_{\mathcal{G}} \right) T \\
&= T^{-1} \left(\prod_{j=1}^n \mathbf{1} \otimes \cdots \otimes f_j(P_0, Q_0)_{(j)} \otimes \cdots \otimes \mathbf{1} \otimes \mathbf{1}_{\mathcal{G}} \right) T \\
&= T^{-1} \left(\bigotimes_{j=1}^n f_j(P_0, Q_0) \otimes \mathbf{1}_{\mathcal{G}} \right) T.
\end{aligned}$$

In order to prove one direction, assume

$$\rho_{P_1, Q_1, \dots, P_n, Q_n} = \bigotimes_{j=1}^n \rho_{P_j, Q_j}.$$

Then using equation (20) with $A = \bigotimes_{j=1}^n f_j(P_0, Q_0)$,

$$\begin{aligned}
\mathbb{E}_\rho \left[\prod_{j=1}^n f_j(P_j, Q_j) \right] &= \text{Tr} \left(\left(\bigotimes_{j=1}^n f_j(P_0, Q_0) \right) \rho_{P_1, Q_1, \dots, P_n, Q_n} \right) \\
&= \text{Tr} \left(\left(\bigotimes_{j=1}^n f_j(P_0, Q_0) \right) \left(\bigotimes_{j=1}^n \rho_{P_j, Q_j} \right) \right) \\
&= \prod_{j=1}^n \text{Tr} (f_j(P_0, Q_0) \rho_{P_j, Q_j}) \\
&= \prod_{j=1}^n \mathbb{E}_\rho [f_j(P_j, Q_j)],
\end{aligned}$$

where it was used that $\text{Tr} \left(\bigotimes_{j=1}^n A_j B_j \right) = \prod_{j=1}^n \text{Tr}(A_j B_j)$.

Conversely, assume

$$\mathbb{E}_\rho \left[\prod_{j=1}^n f_j(P_j, Q_j) \right] = \prod_{j=1}^n \mathbb{E}_\rho [f_j(P_j, Q_j)]$$

for all bounded measurable functions $f_1, \dots, f_n$. Then, by using equation (20),

$$\mathbb{E}_\rho \left[\prod_{j=1}^n f_j(P_j, Q_j) \right] = \text{Tr} \left(\left(\bigotimes_{j=1}^n f_j(P_0, Q_0) \right) \rho_{P_1, Q_1, \dots, P_n, Q_n} \right),$$

which by our assumption equals

$$\begin{aligned}
&= \prod_{j=1}^n \mathbb{E}_\rho [f_j(P_j, Q_j)] \\
&= \prod_{j=1}^n \text{Tr} (f_j(P_0, Q_0) \rho_{P_j, Q_j}) \\
&= \text{Tr} \left(\left(\bigotimes_{j=1}^n f_j(P_0, Q_0) \right) \left(\bigotimes_{j=1}^n \rho_{P_j, Q_j} \right) \right).
\end{aligned}$$

Hence,

$$\mathrm{Tr}(A \rho_{P_1, \dots, Q_n}) = \mathrm{Tr}(A \bigotimes_{j=1}^n \rho_{P_j, Q_j})$$

for all A of the form $A = \bigotimes_{j=1}^n f_j(P_j, Q_j)$, and by linearity and density of such operators in $\mathcal{B}(L^2(\mathbb{R})^{\otimes n})$ we conclude

$$\rho_{P_1, Q_1, \dots, P_n, Q_n} = \bigotimes_{j=1}^n \rho_{P_j, Q_j}. \quad \square$$

Stochastic independence in a state ρ is readily extended to an infinite sequence $\{(P_j, Q_j) : j \in \mathbb{N}\}$ by requiring each finite subset to be stochastically independent following definition 3.8. The final two definitions required to describe the qCLT are stated below, and follow analogously to their classical counterparts.

Definition 3.10 (Identical distribution). Let $\{(P_j, Q_j) : j = 1, \dots, n\}$ be a sequence of canonical pairs on the Hilbert space $\mathcal{H}$. They will said to be *identically distributed in the state ρ* if for all $j = 1, \dots, n$, the distribution operators ρ_{P_j, Q_j} are identical.

Remark 3.11. We note that identical distribution of the canonical pairs (P_j, Q_j) and (P_k, Q_k) does not assert that their density states on $\mathcal{H}$ coincide, but only that their reduced density operators on $L^2(\mathbb{R})$ are identical, i.e. $\rho_{P_j, Q_j} = \rho_{P_k, Q_k}$ for all $j, k \in \mathbb{N}$. The implication for expectations is that for any bounded measurable function f ,

$$\begin{aligned} \mathbb{E}_\rho[f(P_j, Q_j)] &= \mathrm{Tr}\left(T_{P_j, Q_j}^{-1}(f(P_0, Q_0) \otimes \mathbf{1}_{\mathcal{G}})T_{P_j, Q_j} \rho\right) \\ &= \mathrm{Tr}\left(T_{P_k, Q_k}^{-1}(f(P_0, Q_0) \otimes \mathbf{1}_{\mathcal{G}})T_{P_k, Q_k} \rho\right) \\ &= \mathbb{E}_\rho[f(P_k, Q_k)]. \end{aligned}$$

Finally, we speak of a sense of convergence in distribution.

Definition 3.12 (Convergence in distribution). Let $\{(P_j, Q_j) : j \in \mathbb{N}\}$ be a sequence of canonical pairs on the Hilbert space $\mathcal{H}$. Let ρ_0 be a density operator on $L^2(\mathbb{R})$. We shall say the sequence $\{(P_j, Q_j) : j \in \mathbb{N}\}$ *converges in distribution in the state ρ to the limit distribution ρ_0* if for every $A \in \mathcal{B}(L^2(\mathbb{R}))$, we have that

$$\mathrm{Tr}(A \rho_{P_j, Q_j}) \xrightarrow{j \rightarrow \infty} \mathrm{Tr}(A \rho_0).$$

3.4 Gaussian states

In this section, the foundation is laid for describing Gaussian limit distributions to which stochastically independent, identically distributed sequences of canonical pairs will converge in distribution.

Analogous to the classical case, such sequences will converge to the Gaussian distribution with the same covariance as each of the canonical pairs themselves, hence we must first define the very notion of covariance in the quantum-mechanical setting.

Definition 3.13 (Finite covariance matrix). Let (P, Q) be a canonical pair acting on $\mathcal{H}$. We say (P, Q) have *finite covariance matrix in the state ρ* if for all $t, s \in \mathbb{R}$

$$\mathbb{E}_\rho[(tP + sQ)^2] = \text{Tr}(\rho(tP + sQ)^2) < \infty.$$

In that case,

$$\text{Tr}(\rho(tP + sQ)^2) = \Gamma_{11}t^2 + \Gamma_{12}ts + \Gamma_{21}st + \Gamma_{22}s^2,$$

where the coefficients Γ_{ij} are easily seen to be given by

$$\Gamma_{11} = \text{Tr}(\rho P^2), \quad \Gamma_{22} = \text{Tr}(\rho Q^2),$$

and

$$\Gamma_{12} = \Gamma_{21} = \frac{1}{2} \text{Tr}(\rho(PQ + QP)).$$

The matrix

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}$$

is called the covariance matrix of (P, Q) in the state ρ .

Notably, the covariance matrix Γ satisfies the following conditions, which will be used further in the construction and hence will be highlighted and proven here.

Proposition 3.14 (Positivity and uncertainty relation for the covariance matrix). *Let (P, Q) be a canonical pair on a Hilbert space $\mathcal{H}$, and let ρ be a density operator. Assume that (P, Q) possess a finite covariance matrix in the state ρ given by*

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12} & \Gamma_{22} \end{pmatrix},$$

where

$$\Gamma_{11} = \text{Tr}(\rho P^2), \quad \Gamma_{22} = \text{Tr}(\rho Q^2),$$

and

$$\Gamma_{12} = \Gamma_{21} = \frac{1}{2} \text{Tr}(\rho(PQ + QP)).$$

Then:

1. The covariance matrix Γ is positive semidefinite. In particular,

$$\Gamma_{11} \geq 0, \quad \Gamma_{22} \geq 0. \tag{21}$$

- 2.

$$\det(\Gamma) = \Gamma_{11}\Gamma_{22} - \Gamma_{12}^2 \geq \frac{\hbar^2}{4}. \tag{22}$$

Proof. By definition of Γ ,

$$(t, s)\Gamma \begin{pmatrix} t \\ s \end{pmatrix} = \text{Tr}(\rho(tP + sQ)^2) =: q(t, s).$$

By assumption, $q(t, s)$ is well-defined and finite for all $t, s \in \mathbb{R}$, and since it represents the second moment of the observable $tP + sQ$ in the state ρ , it satisfies

$$q(t, s) \geq 0.$$

Hence,

$$(t, s)\Gamma(t, s)^\top \geq 0 \quad \forall t, s \in \mathbb{R},$$

so Γ is positive semidefinite. Taking $(t, s) = (1, 0)$ and $(0, 1)$ yields

$$\Gamma_{11} \geq 0, \quad \Gamma_{22} \geq 0.$$

To prove the determinant inequality, we refer to Calculation 3.1 under subsection 3.7, which uses the Cauchy–Bunyakovsky–Schwarz inequality on the sesquilinear form defined by

$$(X, Y)_\rho := \text{Tr}(\rho X^\dagger Y),$$

ultimately rearranging to the final result

$$\det(\Gamma) \geq \frac{\hbar^2}{4}.$$

□

Conversely, for each real 2x2 matrix Γ satisfying equations (21) and (22) of proposition 3.14, we can construct a state ρ_Γ such that Γ becomes the covariance matrix for a canonical Schrödinger pair in the state ρ_Γ , a process we describe below.

Let

$$c := (\det \Gamma)^{\frac{1}{2}}.$$

Since Γ is a positive-definite matrix, so is its inverse γ , and we can furthermore write

$$\gamma = c^{-1} L^\top L,$$

for L a real matrix with $\det L = 1$. We may define an operator H_Γ by

$$\begin{aligned} H_\Gamma &= \frac{1}{2}(\gamma_{11}P_0^2 + \gamma_{12}P_0Q_0 + \gamma_{21}Q_0P_0 + \gamma_{22}Q_0^2) \\ &= \frac{1}{2} \begin{pmatrix} P_0 & Q_0 \end{pmatrix} \gamma \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix} \\ &= \frac{1}{2} c^{-1} \left(L \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix} \right)^\top L \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix} \\ &= \frac{1}{2} c^{-1} \begin{pmatrix} P'_0 & Q'_0 \end{pmatrix} \begin{pmatrix} P'_0 \\ Q'_0 \end{pmatrix} \\ &= \frac{1}{2} c^{-1} (P_0'^2 + Q_0'^2), \end{aligned}$$

where (P'_0, Q'_0) are defined by

$$\begin{pmatrix} P'_0 \\ Q'_0 \end{pmatrix} = L \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix},$$

and are again a canonical pair:

$$\begin{aligned} [P'_0, Q'_0] &= [L_{11}P_0 + L_{12}Q_0, L_{21}P_0 + L_{22}Q_0] \\ &= (L_{11}L_{22} - L_{12}L_{21})[P_0, Q_0] \\ &= [P_0, Q_0] \\ &= -i\hbar \mathbf{1}. \end{aligned}$$

Recall that H_Γ , being the Hamiltonian of a harmonic oscillator, is a well-defined self-adjoint operator with the discrete spectrum

$$\{\hbar c^{-1} (n + \tfrac{1}{2}), \quad n = 0, 1, 2, \dots\}, \quad (23)$$

with normalised eigenvectors ψ_n on which the actions of P'_0, Q'_0 are

$$\begin{aligned} Q'_0 \psi_n &= \sqrt{\frac{\hbar}{2}} (\sqrt{n+1} \psi_{n+1} + \sqrt{n} \psi_{n-1}), \quad n \geq 1, \\ P'_0 \psi_n &= i\sqrt{\frac{\hbar}{2}} (\sqrt{n+1} \psi_{n+1} - \sqrt{n} \psi_{n-1}), \quad n \geq 1, \\ Q'_0 \psi_0 &= \sqrt{\frac{\hbar}{2}} \psi_1, \\ P'_0 \psi_0 &= i\sqrt{\frac{\hbar}{2}} \psi_1. \end{aligned} \quad (24)$$

In order to now construct ρ_Γ , we must distinguish the two cases of equality and strict inequality of equation (22) of proposition 3.14.

In the case of equality, ρ_Γ is defined as the projection onto the ground state eigenvector of H_Γ :

$$\rho_\Gamma = \text{proj}_{\psi_0}.$$

In the case of inequality, we define

$$\rho_\Gamma = \text{Tr}(e^{-\eta_\Gamma H_\Gamma})^{-1} e^{-\eta_\Gamma H_\Gamma},$$

where η_Γ is defined to be

$$\eta_\Gamma = 2\hbar^{-1}c \operatorname{arccoth}(2\hbar^{-1}c),$$

being a well-defined positive number since we assumed

$$2\hbar^{-1}c = 2\hbar^{-1}(\det \Gamma)^{\frac{1}{2}} > 1.$$

In order to verify the Schrödinger pair in this state indeed admits covariance matrix Γ , first note the following implication, which is explicitly derived in Calculation 3.2. For two canonical pairs (P_0, Q_0) ,

(P'_0, Q'_0) in the same state ρ_Γ ,

$$\begin{pmatrix} P'_0 \\ Q'_0 \end{pmatrix} = L \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix} \implies \text{Cov}' = L \text{Cov} L^\top,$$

where Cov , Cov' are the covariance matrices of (P_0, Q_0) , (P'_0, Q'_0) respectively.

A lengthy derivation, given by Calculation 3.3, uses the equations collectively labelled (24) to show that $\text{Cov}' = cI_2$ in both the equality and inequality cases; hence

$$\text{Cov} = L^{-1} \text{Cov}' (L^\top)^{-1} = L^{-1} (cI_2) (L^\top)^{-1} = c(L^\top L)^{-1} = c(c\gamma)^{-1} = \Gamma$$

in both cases, where we used

$$\gamma = \Gamma^{-1} = c^{-1} L^\top L.$$

3.5 The quantum-mechanical Central Limit Theorem

We now state the qCLT as given by Cushen-Hudson [CH71]. To this end, let $\{(P_n, Q_n) : n \in \mathbb{N}\}$ be a sequence of canonical pairs with commuting Weyl operators. Recall that following Stone's Theorem 2.35, each self-adjoint operator admits a unique strongly continuous one-parameter unitary group. We may in turn define our "sample means" as the unique infinitesimal generators $(\bar{P}_n, \bar{Q}_n)$ defined through

$$e^{it\bar{P}_n} = \prod_{j=1}^n e^{i\frac{t}{\sqrt{n}}P_j}, \quad e^{is\bar{Q}_n} = \prod_{j=1}^n e^{i\frac{s}{\sqrt{n}}Q_j}. \quad (25)$$

Informally,

$$\bar{P}_n = n^{-\frac{1}{2}}(P_1 + \dots + P_n), \quad \bar{Q}_n = n^{-\frac{1}{2}}(Q_1 + \dots + Q_n).$$

Using the definition given by equation (25), it can swiftly be shown that the Weyl operator $\bar{W}_n(t, s)$ corresponding to the pair $(\bar{P}_n, \bar{Q}_n)$ satisfies the Weyl relation, in turn implying that $(\bar{P}_n, \bar{Q}_n)$ is a canonical pair itself. This calculation is performed under Calculation 3.4.

We may now state the qCLT, as it is proven in [CH71].

Theorem 3.15 (Quantum-mechanical Central Limit Theorem). *Let $\{(P_n, Q_n) : n \in \mathbb{N}\}$ be a sequence of stochastically independent, identically distributed (i.i.d.) canonical pairs in state ρ in a Hilbert space $\mathcal{H}$ with finite covariance matrix Γ and for each $j \in \mathbb{N}$,*

$$\mathbb{E}_\rho[P_j] = 0, \quad \mathbb{E}_\rho[Q_j] = 0.$$

Then the sequence of canonical pairs informally defined by

$$\bar{P}_n = n^{-\frac{1}{2}}(P_1 + \dots + P_n), \quad \bar{Q}_n = n^{-\frac{1}{2}}(Q_1 + \dots + Q_n)$$

converges in distribution to the density operator ρ_Γ with covariance matrix Γ .

To end this section, we provide two propositions regarding the density state $\bar{\rho}_n$ that follow from this construction, such that the simulation of entropy for each "sample mean" may be carried out more explicitly.

Proposition 3.16 (Construction of the sample mean state). *Let $\{(P_j, Q_j) : j \in \mathbb{N}\}$ be a sequence of stochastically independent, identically distributed canonical pairs in state ρ on a Hilbert space $\mathcal{H}$. Denote their individual unique reduced density operator by ρ . Then the reduced density operator of the sample mean pair $(\bar{P}_n, \bar{Q}_n)$ is given by*

$$\bar{\rho}_n = (\mathbf{1} \otimes \text{Tr}^{\otimes(n-1)}) (U_n^\dagger \rho^{\otimes n} U_n).$$

Proof. The following construction may be found in greater detail in [Yue76] [Der], and is widely used in the field of quantum information theory.

Since the pairs (P_j, Q_j) are stochastically independent and identically distributed in state ρ , the joint distribution operator of the full n -mode system is

$$\rho^{\otimes n}.$$

The operators $\bar{P}_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n P_j$ and $\bar{Q}_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n Q_j$ are linear combinations of the original canonical pairs. Extend these to a full collection of n canonical pairs $(\bar{P}_n, \bar{Q}_n, \tilde{P}_2, \tilde{Q}_2, \dots, \tilde{P}_n, \tilde{Q}_n)$ by the following

transformation, obtained through a Gram-Schmidt process:

$$\begin{pmatrix} \bar{P}_n \\ \tilde{P}_2 \\ \tilde{P}_3 \\ \vdots \\ \tilde{P}_n \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & \frac{1}{\sqrt{n(n-1)}} & -\frac{n-1}{\sqrt{n(n-1)}} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_n \end{pmatrix}.$$

The transformation works identically on the Q variables. A direct computation shows that the transformation from $\{(P_j, Q_j) : j \in \mathbb{N}\}$ to this extended collection is linear, invertible, and CCR-preserving. By an informal version of the multidimensional Stone-von Neumann Theorem, setting aside domain considerations between P and Q , any such transformation is implemented by a unitary operator on the underlying Hilbert space, yielding a unitary U_n on $\mathcal{H}^{\otimes n}$ satisfying

$$\begin{pmatrix} \bar{P}_n \\ \tilde{P}_2 \\ \vdots \\ \tilde{P}_n \end{pmatrix} = U_n \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{pmatrix} U_n^\dagger,$$

and similarly for the Q variables.

To pass from the level of observables (the Heisenberg picture) to the level of states (the Schrödinger picture), we use that expectation values must be preserved. For any observable $A \mapsto A' := U_n A U_n^\dagger$,

$$\text{Tr}(\rho^{\otimes n} A') = \text{Tr}(\rho U_n A U_n^\dagger) = \text{Tr}(U_n^\dagger \rho^{\otimes n} U_n A),$$

by cyclicity of the trace. Hence conjugating the observables by U_n is equivalently implemented by conjugating the state:

$$\rho^{\otimes n} \mapsto U_n^\dagger \rho^{\otimes n} U_n.$$

The state $U_n^\dagger \rho^{\otimes n} U_n$ lives on the full n -mode system

$$(\bar{P}_n, \bar{Q}_n, \tilde{P}_2, \tilde{Q}_2, \dots, \tilde{P}_n, \tilde{Q}_n).$$

The distribution operator of the sample mean pair $(\bar{P}_n, \bar{Q}_n)$ is finally obtained by tracing out the $n-1$ auxiliary modes, yielding

$$\bar{\rho}_n = (\mathbf{1} \otimes \text{Tr}^{\otimes(n-1)}) (U_n^\dagger \rho^{\otimes n} U_n). \quad (26)$$

□

Using the Stone-von Neumann Theorem, a recursion may also be derived for the U_n allowing for the decomposition of $\mathcal{H}$.

Proposition 3.17 (Recursion of the sample mean unitaries). *The unitaries U_n satisfy the recursion*

$$U_n = (U'_n \otimes \mathbf{1})(\mathbf{1} \otimes U_{n-1}), \quad (27)$$

where U'_n is the unitary on $\mathcal{H}^{\otimes(n-1)}$ implementing the two-mode sample mean transformation, guaranteed via the Stone-von Neumann Theorem.

Proof. We proceed by analysing the two factors in the recursion separately. We suppress domain-theoretic

technicalities and use the Stone–von Neumann theorem in the standard physicist’s sense: all finite-dimensional CCR representations are unitarily equivalent to the Schrödinger representation.

First factor. The operator $\mathbf{1} \otimes U_{n-1}$ acts on $\mathcal{H} \otimes \mathcal{H}^{\otimes(n-1)}$, leaving the first tensor factor unchanged and reorganising the last $n-1$ modes.

By the induction hypothesis, U_{n-1} transforms the canonical operators $(P_1, \dots, P_{n-1})$ and $(Q_1, \dots, Q_{n-1})$ into a new canonical subsystem in which the first mode is the $(n-1)$ -sample mean:

$$\bar{P}_{n-1} = \frac{1}{\sqrt{n-1}} \sum_{j=1}^{n-1} P_j, \quad \bar{Q}_{n-1} = \frac{1}{\sqrt{n-1}} \sum_{j=1}^{n-1} Q_j,$$

together with $n-2$ auxiliary modes.

After applying $\mathbf{1} \otimes U_{n-1}$, the first tensor factor (P_1, Q_1) remains unchanged, while the second factor $\mathcal{H}^{\otimes(n-1)}$ is rewritten in new canonical variables given by

$$(\bar{P}_{n-1}, \bar{Q}_{n-1}, \tilde{P}_2, \tilde{Q}_2, \dots, \tilde{P}_{n-1}, \tilde{Q}_{n-1}).$$

Thus the full system is still $\mathcal{H}^{\otimes n}$, but now understood as a tensor product of the first mode together with a reorganised $(n-1)$ -mode system in which the first mode of that block is the $(n-1)$ -sample mean.

Second factor. We now focus on the two distinguished canonical pairs

$$(P_1, Q_1) \quad \text{and} \quad (\bar{P}_{n-1}, \bar{Q}_{n-1}).$$

Together they form a standard two-mode CCR subsystem. The goal is to replace this pair by a new canonical pair corresponding to the full n -sample mean.

By the Stone–von Neumann theorem, any finite-dimensional representation of the CCR is unitarily equivalent to the Schrödinger representation. Hence there exists a unitary U'_n , acting nontrivially only on the subspace generated by (P_1, Q_1) and $(\bar{P}_{n-1}, \bar{Q}_{n-1})$, such that it transforms these two modes into new canonical variables whose first component is the n -sample mean:

$$\bar{P}_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n P_j, \quad \bar{Q}_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n Q_j.$$

Since the auxiliary modes commute with both (P_1, Q_1) and $(\bar{P}_{n-1}, \bar{Q}_{n-1})$, the unitary U'_n extends trivially to $U'_n \otimes \mathbf{1}$ on the full space.

Composing the two steps,

$$U_n = (U'_n \otimes \mathbf{1})(\mathbf{1} \otimes U_{n-1}),$$

we obtain a unitary that maps the original canonical system on $\mathcal{H}^{\otimes n}$ into one in which the first mode is the n -sample mean pair $(\bar{P}_n, \bar{Q}_n)$. This establishes the recursion. $\square$

3.6 Bosonic mode cascade

In the field of quantum optics, von Neumann entropy has a clear interpretation, as it functions as a quantification for the degree of entanglement in a given density state ρ . For that reason, there are many papers regarding von Neumann entropy [De +15] [DMG14]; more specifically, constructions involving a cascade of bosonic mode beamsplitters are often investigated, and particularly the monotonicity of von Neumann entropy is of interest. Although physically different from the qCLT construction as described in the previous subsection 3.5, it is mathematically closely related, and hence will also be investigated here. The physical interpretation for this case is first given, after which the mathematical foundation is laid.

A reduced density state ρ is established, after which it is tensored with a copy of itself. The combination of these two states is sent through a gate, which is mathematically represented by unitary matrices acting on $\mathcal{H} \otimes \mathcal{H}$, dictating the way the two copies interact. After having gone through the gate, a partial trace is taken, yielding a density state $\bar{\rho}_2$ on $\mathcal{H}$. The process is then repeated, where the tensor product of $\bar{\rho}_2$ and a copy of ρ is taken and consequently sent through a gate. This process, together with its mathematical foundation, is shown in the following proposition.

Proposition 3.18 (Recursive construction of the sample mean state). *Let ρ be the common reduced density operator of the identically distributed canonical pairs, and let $\bar{\rho}_n$ denote the reduced density operator of the “sample mean” canonical pair $(\bar{P}_n, \bar{Q}_n)$. Then*

$$\bar{\rho}_{n+1} = (\mathbf{1} \otimes \text{Tr}) \left(U_n^\dagger (\bar{\rho}_n \otimes \rho) U_n \right), \quad (28)$$

where U_n is the unitary operator on $\mathcal{H} \otimes \mathcal{H}$ provided by Stone-von Neumann implementing the linear transformation

$$\begin{pmatrix} \bar{P}_{n+1} \\ \tilde{P}_{n+1} \end{pmatrix} = M_n \begin{pmatrix} \bar{P}_n \\ P_{n+1} \end{pmatrix}, \quad M_n = \begin{pmatrix} \sqrt{\frac{n}{n+1}} & \sqrt{\frac{1}{n+1}} \\ -\sqrt{\frac{1}{n+1}} & \sqrt{\frac{n}{n+1}} \end{pmatrix},$$

and similarly for the Q -operators.

Proof. The operators $\bar{P}_n = n^{-\frac{1}{2}}(P_1 + \dots + P_n)$ and $\bar{Q}_n = n^{-\frac{1}{2}}(Q_1 + \dots + Q_n)$ satisfy the recursion

$$\bar{P}_{n+1} = \sqrt{\frac{n}{n+1}} \bar{P}_n + \sqrt{\frac{1}{n+1}} P_{n+1}, \quad \bar{Q}_{n+1} = \sqrt{\frac{n}{n+1}} \bar{Q}_n + \sqrt{\frac{1}{n+1}} Q_{n+1}.$$

Define auxiliary modes $\tilde{P}_{n+1}$ and $\tilde{Q}_{n+1}$ as the linear combinations given by the second row of M_n , completing the transformation. Since M_n is orthogonal, a direct computation using orthonormality of its rows shows that the pairs $(\bar{P}_{n+1}, \tilde{P}_{n+1}, \bar{Q}_{n+1}, \tilde{Q}_{n+1})$ again satisfy the CCR. The transformation from $(\bar{P}_n, P_{n+1}, \bar{Q}_n, Q_{n+1})$ to $(\bar{P}_{n+1}, \tilde{P}_{n+1}, \bar{Q}_{n+1}, \tilde{Q}_{n+1})$ is therefore linear, invertible, and preserves the CCR. By an informal application of the Stone-von Neumann Theorem, any such transformation is implemented by a unitary U_n on $\mathcal{H} \otimes \mathcal{H}$ [Der]. Since the $(n+1)$ -mode system is initially in state $\bar{\rho}_n \otimes \rho$, the state transforms as $\bar{\rho}_n \otimes \rho \mapsto U_n^\dagger (\bar{\rho}_n \otimes \rho) U_n$ through the Schrödinger picture. Tracing out the auxiliary mode then yields equation (28). $\square$

3.7 Calculations II

Four calculations relevant to section 3 are presented in this subsection.

Calculation 3.1 proves the determinant inequality (22) for any covariance matrix Γ , with Γ defined by definition 3.13.

Calculation 3.2 proves the implication involving covariance matrix between canonical pairs related by a linear transformation.

Calculation 3.3 explicitly computes the covariance matrix entries for the equilibrium state ρ_Γ in cases of equality and inequality of the determinant inequality (22).

Calculation 3.4 shows the Weyl operator of the sample mean pair $(\bar{P}_n, \bar{Q}_n)$ satisfy the Weyl relation, establishing the sample means are themselves canonical pairs.

Calculation 3.1. This calculation sets out to show $\det(\Gamma) \geq \frac{\hbar^2}{4}$, where Γ is defined by definition 3.13. Consider the positive semidefinite sesquilinear form

$$\langle X, Y \rangle_\rho := \text{Tr}(\rho X^\dagger Y).$$

Applying the Cauchy–Bunyakovsky–Schwarz inequality gives

$$|\text{Tr}(\rho PQ)|^2 \leq \text{Tr}(\rho P^2) \text{Tr}(\rho Q^2) = \Gamma_{11}\Gamma_{22}.$$

Write $z = \text{Tr}(\rho PQ)$. Then

$$|z|^2 = (\text{Re}(z))^2 + (\text{Im}(z))^2,$$

where

$$\text{Re}(z) = \frac{1}{2} \text{Tr}(\rho(PQ + QP)) = \Gamma_{12},$$

and

$$\begin{aligned} \text{Im}(z) &= \frac{1}{2i} (z - \bar{z}) = \frac{1}{2i} \left(\text{Tr}(\rho PQ) - \overline{\text{Tr}(\rho PQ)} \right) \\ &= \frac{1}{2i} \left(\text{Tr}(\rho PQ) - \text{Tr}(Q^\dagger P^\dagger \rho^\dagger) \right) \\ &= \frac{1}{2i} \left(\text{Tr}(\rho PQ) - \text{Tr}(\rho QP) \right), \quad \text{where we used cyclicity of the trace and self-adjointness.} \\ &= \frac{1}{2i} \text{Tr}(\rho(PQ - QP)) \\ &= \frac{1}{2i} \text{Tr}(\rho[P, Q]) \\ &= -\frac{\hbar}{2}, \end{aligned}$$

using $[P, Q] = -i\hbar \mathbf{1}$ and $\text{Tr}(\rho) = 1$.

Combining with the Cauchy–Bunyakovsky–Schwarz bound yields

$$\Gamma_{11}\Gamma_{22} \geq \Gamma_{12}^2 + \frac{\hbar^2}{4},$$

which rearranges to

$$\det(\Gamma) \geq \frac{\hbar^2}{4}.$$

Calculation 3.2. Assume

$$\begin{pmatrix} P'_0 \\ Q'_0 \end{pmatrix} = L \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix}.$$

Below, we first calculate each element of the covariane matrix Cov' associated with (P'_0, Q'_0) using definition 3.13 and verify the relation

$$\text{Cov}' = L\text{Cov}L^\top$$

componentwise.

1. Componentwise calculation of the elements of Cov' .

1a. Computation of Cov'_{11} . Since

$$P'_0 = L_{11}P_0 + L_{12}Q_0,$$

we have

$$\begin{aligned} \text{Cov}'_{11} &= \text{Tr}(\rho(P'_0)^2) \\ &= \text{Tr}(\rho(L_{11}P_0 + L_{12}Q_0)^2) \\ &= L_{11}^2 \text{Tr}(\rho P_0^2) + L_{11}L_{12} \text{Tr}(\rho(P_0Q_0 + Q_0P_0)) + L_{12}^2 \text{Tr}(\rho Q_0^2) \\ &= L_{11}^2 \text{Cov}_{11} + 2L_{11}L_{12} \text{Cov}_{12} + L_{12}^2 \text{Cov}_{22}. \end{aligned} \tag{29}$$

1b. Computation of Cov'_{22} .

Since

$$Q'_0 = L_{21}P_0 + L_{22}Q_0,$$

we have

$$\begin{aligned} \text{Cov}'_{22} &= \text{Tr}(\rho(Q'_0)^2) \\ &= \text{Tr}(\rho(L_{21}P_0 + L_{22}Q_0)^2) \\ &= L_{21}^2 \text{Cov}_{11} + 2L_{21}L_{22} \text{Cov}_{12} + L_{22}^2 \text{Cov}_{22}. \end{aligned} \tag{30}$$

1c. Computation of Cov'_{12} .

Using the definition

$$\text{Cov}'_{12} = \frac{1}{2} \text{Tr}(\rho(P'_0Q'_0 + Q'_0P'_0)),$$

we obtain

$$\begin{aligned} \text{Cov}'_{12} &= \frac{1}{2} \text{Tr}\left(\rho[(L_{11}P_0 + L_{12}Q_0)(L_{21}P_0 + L_{22}Q_0) \right. \\ &\quad \left. + (L_{21}P_0 + L_{22}Q_0)(L_{11}P_0 + L_{12}Q_0)]\right) \\ &= L_{11}L_{21} \text{Cov}_{11} + (L_{11}L_{22} + L_{12}L_{21}) \text{Cov}_{12} + L_{12}L_{22} \text{Cov}_{22}. \end{aligned} \tag{31}$$

1d. Computation of Cov'_{21} .

Since the covariance matrix is symmetric,

$$\text{Cov}'_{21} = \text{Cov}'_{12}.$$

Hence

$$\text{Cov}'_{21} = L_{11}L_{21} \text{Cov}_{11} + (L_{11}L_{22} + L_{12}L_{21}) \text{Cov}_{12} + L_{12}L_{22} \text{Cov}_{22}.$$

2. Verification of the matrix equation relating Cov' and Cov .

Collecting the previous expressions,

$$\text{Cov}' = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

where

$$\begin{aligned} a &= \text{Cov}'_{11}, \\ b &= \text{Cov}'_{21} = \text{Cov}'_{12}, \\ c &= \text{Cov}'_{22}. \end{aligned}$$

a, b and c in terms of Cov are given by equations (29), (31) and (30) respectively. On the other hand,

$$L\text{Cov}L^\top = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} \text{Cov}_{11} & \text{Cov}_{12} \\ \text{Cov}_{12} & \text{Cov}_{22} \end{pmatrix} \begin{pmatrix} L_{11} & L_{21} \\ L_{12} & L_{22} \end{pmatrix},$$

whose entries are exactly those computed above. Therefore

$$\text{Cov}' = L\text{Cov}L^\top.$$

Calculation 3.3. The relevant equations (23), (24) from subsection 3.4 are repeated here for the reader's convenience.

The operator H_Γ has eigenvalue spectrum

$$\{E_n : n = 0, 1, 2, \dots\} = \{\hbar c^{-1} (n + \tfrac{1}{2}) : n = 0, 1, 2, \dots\},$$

with an orthonormal, complete set of eigenvectors ψ_n on which the actions of P'_0, Q'_0 are

$$Q'_0 \psi_n = \sqrt{\frac{\hbar}{2}} (\sqrt{n+1} \psi_{n+1} + \sqrt{n} \psi_{n-1}), \quad n \geq 1,$$

$$P'_0 \psi_n = i\sqrt{\frac{\hbar}{2}} (\sqrt{n+1} \psi_{n+1} - \sqrt{n} \psi_{n-1}), \quad n \geq 1,$$

$$Q'_0 \psi_0 = \sqrt{\frac{\hbar}{2}} \psi_1,$$

$$P'_0 \psi_0 = i\sqrt{\frac{\hbar}{2}} \psi_1.$$

We consider the pair (P'_0, Q'_0) to be in the state ρ_Γ , and prove that

$$\text{Cov}' = c\mathbf{1}$$

in both cases of equality and inequality in the equation

$$c^2 = \det(\Gamma) \geq \frac{\hbar^2}{4}.$$

1. Equality.

In this case

$$\rho_\Gamma = \text{proj}(\psi_0), \quad c = \frac{\hbar}{2}.$$

1a. We first compute Cov'_{11} , where we use remark 2.46,

$$\begin{aligned}
\text{Cov}'_{11} &= \text{Tr}(\rho_{\Gamma} P_0'^2) = \langle \psi_0, P_0'^2 \psi_0 \rangle \\
&= \langle P_0' \psi_0, P_0' \psi_0 \rangle \\
&= \left\langle i \sqrt{\frac{\hbar}{2}} \psi_1, i \sqrt{\frac{\hbar}{2}} \psi_1 \right\rangle \\
&= \frac{\hbar}{2} \langle \psi_1, \psi_1 \rangle \\
&= \frac{\hbar}{2} \\
&= c,
\end{aligned}$$

since $c = \frac{\hbar}{2}$.

1b. Next,

$$\begin{aligned}
\text{Cov}'_{22} &= \text{Tr}(\rho_{\Gamma} Q_0'^2) = \langle \psi_0, Q_0'^2 \psi_0 \rangle \\
&= \langle Q_0' \psi_0, Q_0' \psi_0 \rangle \\
&= \left\langle \sqrt{\frac{\hbar}{2}} \psi_1, \sqrt{\frac{\hbar}{2}} \psi_1 \right\rangle \\
&= \frac{\hbar}{2} \langle \psi_1, \psi_1 \rangle \\
&= \frac{\hbar}{2} \\
&= c,
\end{aligned}$$

again using $c = \frac{\hbar}{2}$.

1c. Finally,

$$\begin{aligned}
\text{Cov}'_{12} &= \text{Cov}'_{21} = \frac{1}{2} \text{Tr}(\rho_{\Gamma} (Q_0' P_0' + P_0' Q_0')) \\
&= \frac{1}{2} \langle \psi_0, (Q_0' P_0' + P_0' Q_0') \psi_0 \rangle.
\end{aligned}$$

Using

$$P_0' \psi_0 = i \sqrt{\frac{\hbar}{2}} \psi_1, \quad Q_0' \psi_0 = \sqrt{\frac{\hbar}{2}} \psi_1,$$

we obtain

$$\begin{aligned}
Q_0' P_0' \psi_0 &= i \frac{\hbar}{2} (\sqrt{2} \psi_2 + \psi_0), \\
P_0' Q_0' \psi_0 &= i \frac{\hbar}{2} (\sqrt{2} \psi_2 - \psi_0).
\end{aligned}$$

Hence

$$(Q_0' P_0' + P_0' Q_0') \psi_0 = i \hbar \sqrt{2} \psi_2.$$

Therefore

$$\begin{aligned}
\text{Cov}'_{12} &= \text{Cov}'_{21} = \frac{1}{2} \langle \psi_0, i \hbar \sqrt{2} \psi_2 \rangle \\
&= \frac{i \hbar \sqrt{2}}{2} \langle \psi_0, \psi_2 \rangle = 0,
\end{aligned}$$

by orthonormality.

Hence

$$\text{Cov}' = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} = c \mathbf{1}.$$

2. Inequality.

In this case

$$\rho_\Gamma = \text{Tr}(e^{-\eta_\Gamma H_\Gamma})^{-1} e^{-\eta_\Gamma H_\Gamma},$$

where η_Γ is defined to be

$$\eta_\Gamma = 2\hbar^{-1}c \operatorname{arccoth}(2\hbar^{-1}c),$$

so that for any $n \geq 0$

$$\rho_\Gamma \psi_n = \text{Tr}(e^{-\eta_\Gamma H_\Gamma})^{-1} e^{-\eta_\Gamma E_n} \psi_n.$$

2.1. We first compute Cov'_{11} .

$$\begin{aligned} \text{Cov}'_{11} &= \text{Tr}(\rho_\Gamma P_0'^2) = \text{Tr}(P_0' \rho_\Gamma P_0') \\ &= \sum_{n=0}^{\infty} \langle \psi_n, P_0' \rho_\Gamma P_0' \psi_n \rangle = \sum_{n=0}^{\infty} \langle P_0' \psi_n, \rho_\Gamma P_0' \psi_n \rangle \\ &= \left\langle i\sqrt{\frac{\hbar}{2}} \psi_1, \rho_\Gamma i\sqrt{\frac{\hbar}{2}} \psi_1 \right\rangle \\ &\quad + \sum_{n=1}^{\infty} \left\langle i\sqrt{\frac{\hbar}{2}} (\sqrt{n+1} \psi_{n+1} - \sqrt{n} \psi_{n-1}), \rho_\Gamma \left(i\sqrt{\frac{\hbar}{2}} (\sqrt{n+1} \psi_{n+1} - \sqrt{n} \psi_{n-1}) \right) \right\rangle \\ &= \frac{\hbar}{2} \langle \psi_1, \text{Tr}(e^{-\eta_\Gamma H_\Gamma})^{-1} e^{-\eta_\Gamma E_1} \psi_1 \rangle \\ &\quad + \frac{\hbar}{2} \sum_{n=1}^{\infty} \langle \sqrt{n+1} \psi_{n+1} - \sqrt{n} \psi_{n-1}, \text{Tr}(e^{-\eta_\Gamma H_\Gamma})^{-1} (e^{-\eta_\Gamma E_{n+1}} \sqrt{n+1} \psi_{n+1} - e^{-\eta_\Gamma E_{n-1}} \sqrt{n} \psi_{n-1}) \rangle \\ &= \frac{\hbar}{2} \text{Tr}(e^{-\eta_\Gamma H_\Gamma})^{-1} e^{-\eta_\Gamma E_1} \\ &\quad + \frac{\hbar}{2} \text{Tr}(e^{-\eta_\Gamma H_\Gamma})^{-1} \sum_{n=1}^{\infty} \left((n+1) e^{-\eta_\Gamma E_{n+1}} + n e^{-\eta_\Gamma E_{n-1}} \right). \end{aligned}$$

For notational ease, define

$$\alpha := 2 \operatorname{arccoth}(2\hbar^{-1}c) > 0, \quad q := e^{-\alpha}.$$

We first note

$$\eta_\Gamma E_n = 2\hbar^{-1}c \operatorname{arccoth}(2\hbar^{-1}c) \hbar c^{-1} (n + \frac{1}{2}) = (n + \frac{1}{2})\alpha$$

and

$$\text{Tr}(e^{-\eta_\Gamma H_\Gamma})^{-1} = \left(\sum_{n=0}^{\infty} \langle \psi_n, e^{-\eta_\Gamma E_n} \psi_n \rangle \right)^{-1} = \left(\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})} \right)^{-1} = q^{-\frac{1}{2}} (1-q),$$

using the geometric sum formula, since $|q| < 1$. Then

$$\begin{aligned}
\text{Cov}'_{11} &= \frac{\hbar}{2} \text{Tr}(e^{-\eta_\Gamma H_\Gamma})^{-1} e^{-\frac{3}{2}\alpha} \\
&\quad + \frac{\hbar}{2} \text{Tr}(e^{-\eta_\Gamma H_\Gamma})^{-1} \sum_{n=1}^{\infty} \left((n+1)e^{-(n+\frac{3}{2})\alpha} + ne^{-(n-\frac{1}{2})\alpha} \right) \\
&= \frac{\hbar}{2} q^{-\frac{1}{2}} (1-q) \left(q^{\frac{3}{2}} + q^{\frac{3}{2}} \sum_{n=1}^{\infty} nq^n + q^{\frac{3}{2}} \sum_{n=1}^{\infty} q^n + q^{-\frac{1}{2}} \sum_{n=1}^{\infty} nq^n \right) \\
&= \frac{\hbar}{2} \frac{1+q}{1-q} = \frac{\hbar}{2} \frac{e^{\frac{\alpha}{2}} + e^{-\frac{\alpha}{2}}}{e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}} = \frac{\hbar}{2} \coth\left(\frac{\alpha}{2}\right) \\
&= \frac{\hbar}{2} 2\hbar^{-1}c \\
&= c,
\end{aligned}$$

where the following sum identities were used

$$\sum_{n=1}^{\infty} q^n = \frac{q}{1-q}, \quad \sum_{n=1}^{\infty} nq^n = \frac{q}{(1-q)^2}.$$

2.2. We proceed similarly when calculating Cov'_{22} .

$$\begin{aligned}
\text{Cov}'_{22} &= \text{Tr}(\rho_\Gamma Q_0'^2) = \text{Tr}(Q_0' \rho_\Gamma Q_0') \\
&= \sum_{n=0}^{\infty} \langle Q_0' \psi_n, \rho_\Gamma Q_0' \psi_n \rangle.
\end{aligned}$$

Using the action of Q_0' on the basis vectors and proceeding exactly as in the computation of Cov'_{11} , one obtains

$$\begin{aligned}
\text{Cov}'_{22} &= \frac{\hbar}{2} \text{Tr}(e^{-\eta_\Gamma H_\Gamma})^{-1} \left(e^{-\eta_\Gamma E_1} + \sum_{n=1}^{\infty} \left((n+1)e^{-\eta_\Gamma E_{n+1}} + ne^{-\eta_\Gamma E_{n-1}} \right) \right) \\
&= \frac{\hbar}{2} q^{-\frac{1}{2}} (1-q) \left(q^{\frac{3}{2}} + q^{\frac{3}{2}} \sum_{n=1}^{\infty} nq^n + q^{\frac{3}{2}} \sum_{n=1}^{\infty} q^n + q^{-\frac{1}{2}} \sum_{n=1}^{\infty} nq^n \right).
\end{aligned}$$

Using

$$\sum_{n=1}^{\infty} q^n = \frac{q}{1-q}, \quad \sum_{n=1}^{\infty} nq^n = \frac{q}{(1-q)^2},$$

it follows that

$$\begin{aligned}
\text{Cov}'_{22} &= \frac{\hbar}{2} \frac{1+q}{1-q} = \frac{\hbar}{2} \coth\left(\frac{\alpha}{2}\right) \\
&= \frac{\hbar}{2} 2\hbar^{-1}c \\
&= c.
\end{aligned}$$

2.3. Finally, we compute Cov'_{12} and Cov'_{21} .

$$\begin{aligned}
Q_0' P_0' \psi_n &= Q_0' \left(i\sqrt{\frac{\hbar}{2}} (\sqrt{n+1} \psi_{n+1} - \sqrt{n} \psi_{n-1}) \right) \\
&= i\frac{\hbar}{2} \left(\sqrt{(n+1)(n+2)} \psi_{n+2} + (n+1) \psi_n - n \psi_n - \sqrt{n(n-1)} \psi_{n-2} \right) \\
&= i\frac{\hbar}{2} \left(\sqrt{(n+1)(n+2)} \psi_{n+2} + \psi_n - \sqrt{n(n-1)} \psi_{n-2} \right),
\end{aligned}$$

for $n \geq 1$. Likewise,

$$\begin{aligned} P'_0 Q'_0 \psi_n &= P'_0 \left(\sqrt{\frac{\hbar}{2}} (\sqrt{n+1} \psi_{n+1} + \sqrt{n} \psi_{n-1}) \right) \\ &= i \frac{\hbar}{2} \left(\sqrt{(n+1)(n+2)} \psi_{n+2} - (n+1) \psi_n + n \psi_n + \sqrt{n(n-1)} \psi_{n-2} \right) \\ &= i \frac{\hbar}{2} \left(\sqrt{(n+1)(n+2)} \psi_{n+2} - \psi_n - \sqrt{n(n-1)} \psi_{n-2} \right). \end{aligned}$$

Adding these expressions yields

$$(Q'_0 P'_0 + P'_0 Q'_0) \psi_n = i \hbar \left(\sqrt{(n+1)(n+2)} \psi_{n+2} - \sqrt{n(n-1)} \psi_{n-2} \right),$$

for $n \geq 1$. For $n = 0$,

$$(Q'_0 P'_0 + P'_0 Q'_0) \psi_0 = i \hbar \sqrt{2} \psi_2.$$

Therefore

$$\begin{aligned} \text{Cov}'_{12} &= \text{Cov}'_{21} = \frac{1}{2} \text{Tr} \left(\rho_\Gamma (Q'_0 P'_0 + P'_0 Q'_0) \right) \\ &= \frac{1}{2} \text{Tr} (e^{-\eta_\Gamma H_\Gamma})^{-1} \left(e^{-\eta_\Gamma E_2} \langle \psi_0, i \hbar \sqrt{2} \psi_2 \rangle \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \langle \psi_n, i \hbar \left(e^{-\eta_\Gamma E_{n+2}} \sqrt{(n+1)(n+2)} \psi_{n+2} - e^{-\eta_\Gamma E_{n-2}} \sqrt{n(n-1)} \psi_{n-2} \right) \rangle \right). \end{aligned}$$

Using the orthonormality of the basis $\{\psi_n\}_{n=0}^\infty$,

$$\langle \psi_0, \psi_2 \rangle = 0, \quad \langle \psi_n, \psi_{n+2} \rangle = 0, \quad \langle \psi_n, \psi_{n-2} \rangle = 0,$$

for all $n \geq 1$. Hence every term in the above sum vanishes and therefore

$$\text{Cov}'_{12} = \text{Cov}'_{21} = 0.$$

Hence

$$\text{Cov}' = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} = c \mathbf{1}.$$

Calculation 3.4. In this calculation, we show the Weyl operator $\overline{W}_n(t, s)$ associated with the pair $(\overline{P}_n, \overline{Q}_n)$ satisfies the Weyl relation, where $(\overline{P}_n, \overline{Q}_n)$ are defined through

$$e^{it\overline{P}_n} = \prod_{j=1}^n e^{it\sqrt{n}P_j}, \quad e^{is\overline{Q}_n} = \prod_{j=1}^n e^{is\sqrt{n}Q_j}.$$

Consider the separate Weyl operators

$$W_j(t, s) = \exp(i(tP_j + sQ_j)),$$

where for each $1 \leq j \leq n$, $W_j(t, s)$ is a strongly continuous unitary group generated (via Stone's Theorem 2.35) by the canonical pair (P_j, Q_j) . These satisfy the Weyl relation

$$W_j(t, s) W_j(t', s') = e^{\frac{i\hbar}{2}(ts' - st')} W_j(t + t', s + s').$$

Since Weyl operators corresponding to distinct indices commute per assumption, it follows by the definition of $(\bar{P}_n, \bar{Q}_n)$ that their associated Weyl operator can be written as

$$\bar{W}_n(t, s) = \exp(i(t\bar{P}_n + s\bar{Q}_n)) = \prod_{j=1}^n W_j\left(\frac{t}{\sqrt{n}}, \frac{s}{\sqrt{n}}\right).$$

We compute

$$\bar{W}_n(t, s) \bar{W}_n(t', s') = \prod_{j=1}^n W_j\left(\frac{t}{\sqrt{n}}, \frac{s}{\sqrt{n}}\right) W_j\left(\frac{t'}{\sqrt{n}}, \frac{s'}{\sqrt{n}}\right).$$

Applying the Weyl relation for each fixed j ,

$$W_j\left(\frac{t}{\sqrt{n}}, \frac{s}{\sqrt{n}}\right) W_j\left(\frac{t'}{\sqrt{n}}, \frac{s'}{\sqrt{n}}\right) = \exp\left(\frac{i\hbar}{2n}(ts' - st')\right) W_j\left(\frac{t+t'}{\sqrt{n}}, \frac{s+s'}{\sqrt{n}}\right).$$

Hence,

$$\bar{W}_n(t, s) \bar{W}_n(t', s') = \prod_{j=1}^n \exp\left(\frac{i\hbar}{2n}(ts' - st')\right) W_j\left(\frac{t+t'}{\sqrt{n}}, \frac{s+s'}{\sqrt{n}}\right).$$

The scalar phase factor combines to give

$$\prod_{j=1}^n \exp\left(\frac{i\hbar}{2n}(ts' - st')\right) = \exp\left(\frac{i\hbar}{2}(ts' - st')\right),$$

and the remaining product recombines as

$$\prod_{j=1}^n W_j\left(\frac{t+t'}{\sqrt{n}}, \frac{s+s'}{\sqrt{n}}\right) = \bar{W}_n(t+t', s+s').$$

Therefore,

$$\bar{W}_n(t, s) \bar{W}_n(t', s') = e^{\frac{i\hbar}{2}(ts' - st')} \bar{W}_n(t+t', s+s'),$$

concluding $(\bar{P}_n, \bar{Q}_n)$ is again a canonical pair.

4 Simulation of the von Neumann entropy

Building upon the framework established in previous sections, this section presents the computational approaches employed to quantify the monotonicity of entropy in the setting of the qCLT. While the theoretical framework is formulated in infinite-dimensional Hilbert spaces, the computational implementation necessarily relies on finite-dimensional approximations. As a result, the computations presented in this section should be viewed as finite-dimensional approximations of the underlying infinite-dimensional theory. For a discrete analogue of the qCLT, we refer to [BGJ23].

First, the computational procedures are introduced and outlined, including the density states under analysis and the manner with which qCLT evolutions are employed. Popular in literature, specifically regarding quantum optics, is the application of the bosonic mode beam splitters [DMG14] [De +15] [Tan+08]. Such a beam splitter enforces the following way of sampling: at each step, the current state is mixed with a fresh copy of the input state ρ_0 via a 50:50 beam splitter unitary, and the output mode is retained while the auxiliary mode is traced out. This yields a recursive procedure of the same structural form as equation (28), though more conditions on the unitary operator are required for this equation to represent that of a beamsplitter gate. As this bosonic mode gate cascade is both physically and mathematically related to the qCLT construction, we perform calculations and draw conclusions on both approaches.

It should be mentioned that the two procedures are distinct in an important sense. In the qCLT as described in theorem 3.15, the von Neumann entropy of the iterates $\bar{\rho}_n$ is guaranteed to converge to that of the Gaussian limit state ρ_Γ . The gate cascade evolution, by contrast, is governed entirely by the unitary operators (representing gates) at every step and the initial state, where we usually consider a cascade consisting of the same, fixed gate. The two procedures are compared in this section, with particular attention to the differing entropy dynamics they induce. A more general and in-depth comparison of the two procedures may be found in [Bec+21].

4.1 Preliminary results

Before showcasing the computational methods employed and the results obtained, we discuss various preliminary results, as they offer ways of verification. It should again be noted that certain results do not translate directly to the finite-dimensional case; they do, however, offer physical intuition as to why certain phenomena in the behaviour of entropy may be observed.

4.1.1 The qCLT construction

The following proposition is given by theorem 5.8 in [Guh08].

Proposition 4.1 (Monotonicity in powers of 2). *Let $n \in \mathbb{N}$ be arbitrary. Following the qCLT construction given by equation (26), it holds true that*

$$S(\bar{\rho}_{2^{n+1}}) \geq S(\bar{\rho}_{2^n}),$$

where $S(\rho) := -\text{Tr}(\rho \ln(\rho))$ is the von Neumann entropy defined by definition 3.4.

The next proposition follows by realising $\bar{\rho}_n$ has its covariance matrix equal to Γ , as shown in Calculation 4.1, and applying property (i) of [WGC06].

Proposition 4.2. *Let ρ be a density state on $\mathcal{H}$ with finite covariance matrix Γ . Let ρ_Γ denote the corresponding Gaussian state, as constructed in subsection 3.4. Then*

$$S(\rho) \leq S(\rho_\Gamma).$$

In particular, for all $n \in \mathbb{N}$,

$$S(\bar{\rho}_n) \leq S(\rho_\Gamma).$$

In finite dimensions furthermore, by continuity of the von Neumann entropy, since $\bar{\rho}_n$ converges in distribution to ρ_Γ ,

$$S(\bar{\rho}_n) \nearrow S(\rho_\Gamma) \quad \text{as } n \rightarrow \infty.$$

By the qCLT [CH71] we are guaranteed that the reduced density states $\bar{\rho}_n$ converge to a Gaussian density operator ρ_Γ , of which a modern result proving this result can be found as theorem 5.10 in [Guh08]. Following proposition 4.2, we know that upon initialising our reduced density operator ρ to be a Gaussian state, the von Neumann entropy of $\bar{\rho}_n$ ought to remain constant throughout the qCLT evolution. In fact, the sample mean reduced density states $\bar{\rho}_n$ themselves stay constant and equal to the Gaussian state, as will be proven in the following proposition. Note that in the discrete case, as opposed to the continuous case described by [CH71], only one discrete Gaussian state is a fixed point of the qCLT evolution, and it takes the form of the maximally mixed state.

Proposition 4.3. *Let the reduced density operator $\rho \in \mathcal{M}_d(\mathbb{C})$ be the maximally mixed state, i.e.*

$$\rho = \frac{1}{d}I_d.$$

Then the reduced density states $\bar{\rho}_n$ generated by the qCLT remain maximally mixed for all $n \in \mathbb{N}$. In particular, their von Neumann entropy is constant and equal to $\ln d$.

Proof. Indeed, let $\rho \in \mathcal{M}_d(\mathbb{C})$ be given by

$$\rho = \frac{1}{d}I_d.$$

It follows that

$$\rho^{\otimes n} = \left(\frac{1}{d}I_d\right)^{\otimes n} = \frac{1}{d^n}I_{d^n}.$$

Hence,

$$U_n^\dagger \rho^{\otimes n} U_n = \frac{1}{d^n} U_n^\dagger I_{d^n} U_n = \frac{1}{d^n} I_{d^n}.$$

Therefore,

$$\begin{aligned} \bar{\rho}_n &= (I_d \otimes \text{Tr}^{\otimes(n-1)}) \left(\frac{1}{d^n} I_{d^n} \right) \\ &= \frac{1}{d^n} (I_d \otimes \text{Tr}^{\otimes(n-1)}) (I_d \otimes I_{d^{n-1}}) \\ &= \frac{1}{d^n} \text{Tr}(I_{d^{n-1}}) I_d \\ &= \frac{d^{n-1}}{d^n} I_d \\ &= \frac{1}{d} I_d. \end{aligned}$$

All d eigenvalues of $\bar{\rho}_n$ are all equal to $1/d$, and thus

$$S(\bar{\rho}_n) = -\text{Tr}(\bar{\rho}_n \ln \bar{\rho}_n) = -\sum_{k=1}^d \frac{1}{d} \ln\left(\frac{1}{d}\right) = \ln d \quad \forall n \in \mathbb{N}. \quad \square$$

4.1.2 The bosonic mode cascade approach

For the cascade case with a fixed gate, we similarly expect the von Neumann entropy to converge to a set value after iterating the process for a sufficient number of times. In contrast with the discrete qCLT construction, which guarantees convergence to a maximally mixed state, the limiting value will be dependent on both the gate (expressed through the fixed unitary operator U) and the initial density state ρ which is being tensored with iteratively. As such, we wish to be able to determine the fixed density state ρ_∞ under a mapping $\mathcal{E}$ given by equation (28)

$$\mathcal{E}(\sigma) := (\mathbf{1} \otimes \text{Tr})(U^\dagger(\sigma \otimes \rho)U), \quad \mathcal{E}(\rho_\infty) = \rho_\infty.$$

Throughout the simulation, it will be assumed $\rho \in \mathcal{M}_2(\mathbb{C})$, the density state of a qubit system. It may easily be seen that the map $\mathcal{E} : \mathcal{M}_2(\mathbb{C}) \rightarrow \mathcal{M}_2(\mathbb{C})$ is a linear mapping, and may thus be represented by a 4x4 matrix T . Determining ρ_∞ therefore becomes a matter of determining the eigenvector of T corresponding to an eigenvalue of 1, which may easily be done computationally.

It should also be noted that, similar to the qCLT case, any combination of gates will admit the maximally mixed state to be a fixed state, as may easily be seen from the following calculation, where ρ has been set to the maximally mixed state.:

$$\begin{aligned} \mathcal{E}\left(\frac{I_d}{d}\right) &= (I_d \otimes \text{Tr}) \left[U^\dagger \left(\frac{I_d}{d} \otimes \frac{I_d}{d} \right) U \right] \\ &= \frac{1}{d^2} (I_d \otimes \text{Tr}) [U^\dagger (I_d \otimes I_d) U] \\ &= \frac{1}{d^2} \cdot \text{Tr}[I_d] \cdot I_d \\ &= \frac{I_d}{d}. \end{aligned} \tag{32}$$

The following proposition is theorem VII.1 in [KS14], and importantly holds only for gates belonging to the class of beamsplitters.

Theorem 4.4 (Quantum entropy power inequality for transmissivity λ). *Let $0 < \lambda < 1$ and let $\mathcal{E}_\lambda$ be the map associated with a beamsplitter of transmissivity λ . Let ρ_X and ρ_Y be arbitrary n -mode states. Then*

$$S(\mathcal{E}_\lambda(\rho_X \otimes \rho_Y)) \geq \lambda S(\rho_X) + (1 - \lambda)S(\rho_Y) \geq \min(S(\rho_X), S(\rho_Y)).$$

In particular, if the gate in equation (28) is a beamsplitter with transmissivity λ ,

$$S(\bar{\rho}_{n+1}) \geq \min(S(\bar{\rho}_n), S(\rho)).$$

Though an important indication for the validity of the monotonicity conjecture, and hence certainly worth mentioning in this work, it will not serve as a means of verification during our computations; this is due to the fact that the gates during the simulations will be chosen randomly, and therefore are highly unlikely to be beamsplitters.

4.2 Entropy computation

This paper limits itself to $\rho \in \mathcal{M}_2(\mathbb{C})$, the density state on a qubit system. As previously stated, the dimensionality of the unitary matrices in the qCLT approach following Cushen-Hudson grows exponentially, since for all $n \in \mathbb{N}$, $U_n \in \mathcal{M}_{2^n}(\mathbb{C})$. As the beamsplitter approach only uses 4x4 matrices, this method is computationally superior and allows for more iterations to be performed; however, convergence is reached at roughly fifty iterations, suggesting that additional computations provide little benefit.

Different classes of states were examined for each approach, and verifications are carried out throughout the analysis. The explicit formulae employed are stated below, and the code utilised can be found in section 6.

1. The qCLT construction.

The qCLT construction involves n -fold tensor products of the same state ρ , and interaction with the environment expressed through unitaries U_n . As we limit ourselves to qubit systems, we consider ρ to be expressed by a 2x2 matrix. Following proposition 3.16, the qCLT evolutions are computed by

$$\bar{\rho}_n = (I_2 \otimes \text{Tr}^{\otimes(n-1)}) (U_n^\dagger \rho^{\otimes n} U_n),$$

where each $U_n \in \mathcal{M}_{2^n}(\mathbb{C})$. The unitary matrices U_n , physically corresponding to entanglement with the environment as in the second law of thermodynamics, are chosen either randomly or through the recursion given by proposition 3.17,

$$U_{n+1} = (U'_{n+1} \otimes I_2)(I_2 \otimes U_n).$$

The measure responsible for generating unitary matrices on $U(2)$ in a random manner is called the Haar measure, on which more information may be found in [Fol16]. Unitary operators generated this way will be called Haar-randomly generated unitary matrices. It should furthermore be noted that in the theoretical construction, U'_n acts on carefully identified degrees of freedom (expressed through modes); in practice, the recursive implementation carries out the recursion purely at the level of tensor-product structure via Kronecker products, neglecting the physical interpretation of which exact modes are being mixed. This latter method is investigated more thoroughly, as it provides a more accurate discrete analogue to the qCLT construction, since it somewhat takes the structure of the Stone-von Neumann Theorem 2.44 into account.

2. The bosonic mode gate cascade approach.

In contrast with the qCLT approach, states in the evolution are computed iteratively, in correspondence with the physical representation of a cascade of quantum gates. A state $\bar{\rho}_n$ is tensored with a copy of the initial state ρ , after which the tensor product is fed through a gate, expressed by a unitary operator. A partial trace is taken over the result, after which $\bar{\rho}_{n+1}$ is obtained and this process is applied iteratively. Mathematically, the reduced densities are given by equation (28),

$$\bar{\rho}_{n+1} = (I_2 \otimes \text{Tr})(U_n^\dagger (\bar{\rho}_n \otimes \rho) U_n) \in \mathcal{M}_2(\mathbb{C}).$$

Notably, convergence of the von Neumann entropy during this process is only expected to occur when U_n , i.e. the gate in the cascade, is fixed and independent of n ; in this case, repeated application of the gate ensures a fixed density state ρ_∞ , while applying different gates at each evolution grants no consistent attractor state. Physically, a fixed unitary U means the same averaging mechanism acts at every step, where convergence to a fixed point is expected, whereas varying U_n offers no notion of equilibrium toward which the system can converge.

4.3 Monotonicity analysis

The results produced by the Python code, to be found in section 6, are analysed in this subsection. Particular attention is given to assessing the accuracy of the computational methods used. To this end, verifications from subsection 4.1 are applied on the respective methods of computation. Alongside these verifications, physical intuition behind the results is briefly given, after which a more rigorous conclusion is drawn in section 5.

4.3.1 The qCLT construction

We first perform analysis on the qCLT method, for a various collection of density matrices. For means of verification, the simulation is first carried out when the initial state equals the maximally mixed state, under the expectation of the entropy remaining constant and equal to $\ln(2)$ following proposition 4.3.

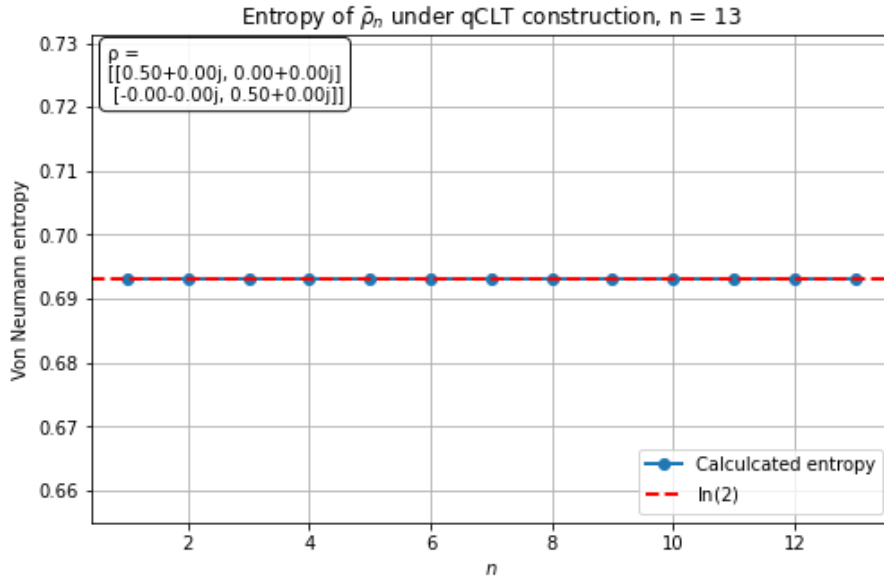


Figure 1: Results of the simulation of the von Neumann entropy for a qCLT evolution of a maximally mixed state. The maximum sample size was chosen to be 13, and ρ was set to $\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$, the maximally mixed state for a qubit system.

From the graph, it is immediately apparent that the entropy remains constant throughout the qCLT evolution. This behaviour is precisely proven in proposition 4.3, providing evidence the computational method employed is consistent with theoretical expectations. It should be noted that the maximally mixed state is a fixed state of the qCLT process, regardless of the unitary matrices used to indicate exchange with the environmental heat bath; physically, this reflects the fact that this state carries no information about the system whatsoever, hence no interaction with an environment can extract any correlations from it or push it further towards disorder.

Next, a state in between the maximally mixed and pure state is chosen, and the qCLT process is carried out for various collections of unitary matrices, where U_n is chosen Haar-randomly for each step.

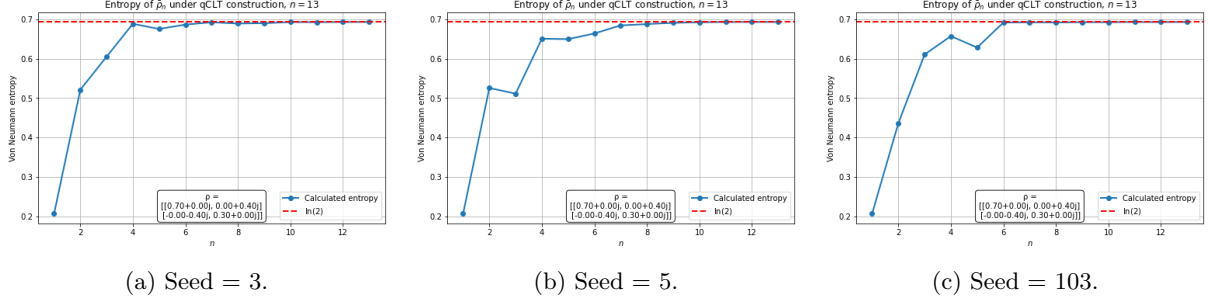


Figure 2: Results of the simulation of the von Neumann entropy for a qCLT evolution, where the unitary matrices are chosen Haar-randomly at each step. The maximum sample size was chosen to be 13 in each case, and the plots shown were simulated using np.random.seed values equal to 3, 5 and 103. In each instance, ρ was set to $\begin{pmatrix} 0.7 & 0.4i \\ -0.4i & 0.3 \end{pmatrix}$.

Convergence to the entropy of the maximally mixed state, the unique fixed point of the qCLT evolution as proven by proposition 4.3, can clearly be seen. Furthermore, monotonicity in powers of two, as stated in proposition 4.1, is similarly satisfied. However, the conjectured monotonicity in entropy over the qCLT evolution is violated, as the absolute difference in entropy between steps in which the entropy decreased is often of the order 10^{-4} , exceeding the order that may be assigned to computational approximation. This may reasonably be attributed to the unitary matrices responsible for the environmental heat bath not forming a direct analogue with the (quantum) thermodynamical theory, as they are chosen Haar-randomly at each step, neglecting the Stone-von Neumann Theorem.

The qCLT evolution is then performed on the same initial state, now including the recursion given by proposition 3.17, in order to incorporate more structure into the way the heat bath interacts with the n -fold tensor product $\rho^{\otimes n}$.

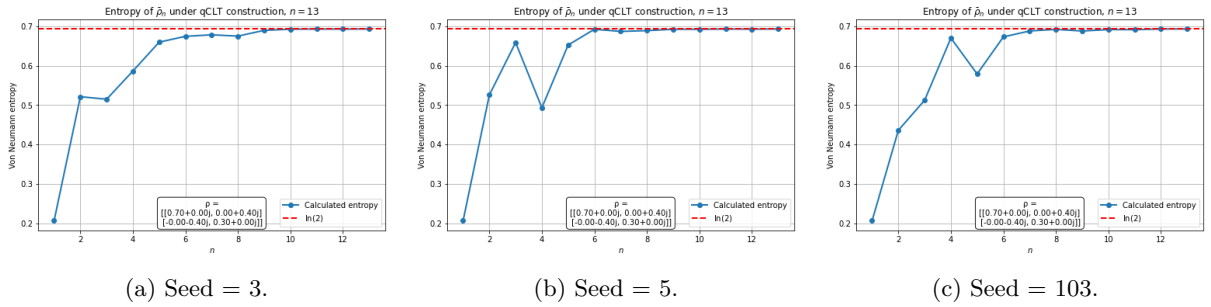


Figure 3: Results of the simulation of the von Neumann entropy for a qCLT evolution, where the unitary matrices are chosen through the recursion given by the infinite-dimensional theory. The maximum sample size was chosen to be 13 in each case, and the plots shown were simulated using np.random.seed values equal to 3, 5 and 103. In each instance, ρ was set to $\begin{pmatrix} 0.7 & 0.4i \\ -0.4i & 0.3 \end{pmatrix}$.

Similar to the case of Haar-random generation, convergence to $\ln(2)$, the entropy of the maximally mixed state, is still observed, and monotonicity in powers of 2 can similarly be seen. Nevertheless, the monotonicity conjecture at every step is clearly breached, which likely is caused by the still present randomness in the unitaries, whereas the theory requires for the unitary matrices to satisfy the strict conditions given by the Stone-von Neumann Theorem.

Finally, a pure state is set as the initial state, and the process is repeated.

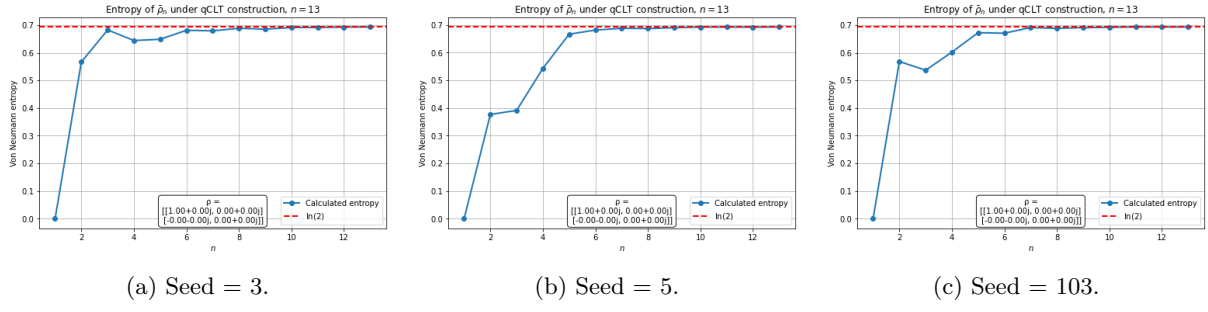


Figure 4: Results of the simulation of the von Neumann entropy for a qCLT evolution, where the unitary matrices are chosen Haar-randomly at each step. The maximum sample size was chosen to be 13 in each case, and the plots shown were simulated using np.random.seed values equal to 3, 5 and 103. In each instance, ρ was set to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

The entropy again converges to that of the maximally mixed state, consistent with proposition 4.3, and monotonicity along powers of two, as stated in proposition 4.1, remains satisfied. Likewise, the conjectured monotonicity of the entropy throughout the qCLT evolution is violated in the same manner as before, with decreases exceeding the scale expected from numerical approximation. No new qualitative phenomena are observed.

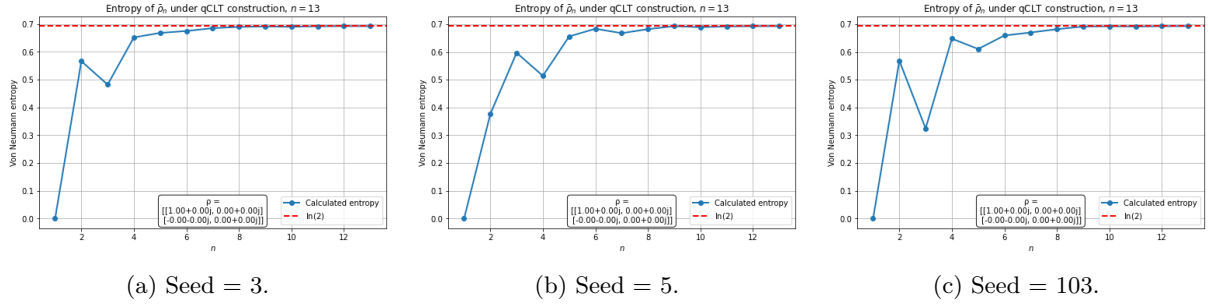


Figure 5: Results of the simulation of the von Neumann entropy for a qCLT evolution, where the unitary matrices are chosen through the recursion given by the infinite-dimensional theory. The maximum sample size was chosen to be 13 in each case, and the plots shown were simulated using np.random.seed values equal to 3, 5 and 103. In each instance, ρ was set to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Upon including the recursion between unitary matrices instead of taking them Haar-randomly, the same qualitative behaviour again persists. Convergence to $\ln(2)$, corresponding to the entropy of the maximally mixed state, is again observed, and monotonicity along powers of two remains intact. The conjectured monotonicity at every step is likewise violated, suggesting that the conclusions are unchanged by the choice of initial condition.

It should be noted the properties observed here are neither seed-dependent nor deviate drastically for other choices of initial states, and deviations from monotonous increase are seen to occur for the vast majority of seeds. Convergence of the von Neumann entropy to the value $\ln(2)$ is also consistently observed.

4.3.2 The bosonic mode cascade approach

As stated previously in subsection 4.2, this approach allows for much more sampling to be done, as all matrices under consideration are 4×4 in size, in contrast with the exponential growth in dimension seen in the qCLT approach. Though this allows for the amount of iterations performed to be extended, the results indicate convergence happens well before fifty iterations, making further computations largely

unnecessary.

Firstly, it is verified that the maximally mixed state remains constant under iteration, as shown in equation (32). As such, we first consider the maximally mixed initial state.

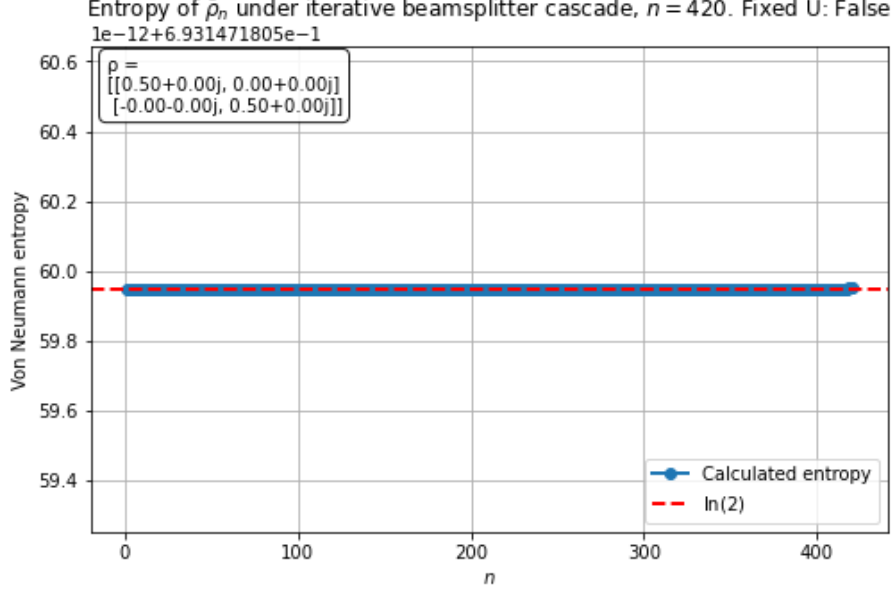
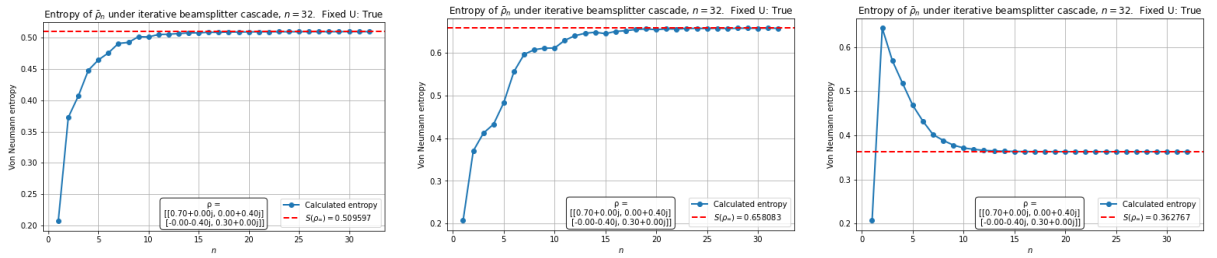


Figure 6: Results of the simulation of the von Neumann entropy for a beamsplitter cascade. ρ was set to $\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$, the maximally mixed state for a qubit system.

Similar to the qCLT case, we observe clear constant behaviour at $\ln(2)$, in accordance with the theoretical results and thereby validating the implementation of the simulation.

In modern literature on quantum optics [Guh08] [DMG14], specifically that on bosonic mode beam-splitters, a cascade of the same, fixed gate is often considered, and therefore special attention is paid to this case here. The fixed 4x4 unitary representing this gate is taken Haar-randomly, and thus does not represent a beamsplitter.



(a) Seed = 3. $S(\rho_\infty) = 0,509597$. (b) Seed = 5. $S(\rho_\infty) = 0,658083$. (c) Seed = 103. $S(\rho_\infty) = 0,362767$.

Figure 7: Results of the simulation of the von Neumann entropy for a beamsplitter cascade, where the gate in each separate instance is chosen Haar-randomly, and kept fixed during the cascade. The maximum sample size was chosen to be 32 in each case, and the plots shown were simulated using np.random.seed values equal to 3, 5 and 103. In each instance, ρ was set to $\begin{pmatrix} 0.7 & 0.4i \\ -0.4i & 0.3 \end{pmatrix}$.

From the above simulations, convergence to the entropy of the fixed density state ρ_∞ is clearly observed. However, due to the random nature of the unitary matrices responsible for representing the gates, monotonous behaviour does not appear, as each instance includes decreasing steps with an absolute dif-

ference in entropy of order 10^{-3} , exceeding what may be attributed to numerical error. Furthermore, it is observed that the entropy of the system under repeated application of a Haar-random unitary may converge in a decreasing manner, rather than the monotonically increasing manner conjectured to hold for beamsplitter gates. In order to represent (discrete) beamsplitter gates, more constraints on the fixed unitary U are to be applied, in which case monotonicity would be guaranteed by proposition 22 of [BGJ23].

Notably, the conclusion of theorem 4.4 seems to hold in each case, since for all $n \in \mathbb{N}$ it can clearly be seen that

$$S(\bar{\rho}_n) \geq S(\rho).$$

This suggests the inequality to be valid for a more broad class of unitary operators, even though the theorem as stated does not cover this case; whether this hints at a deeper and more general entropy principle remains an open question and could be a direction for future work.

A less often investigated case is that of a cascade of different Haar-random gates, i.e. U_n is made dependent on n . For this process, no notion of convergence is expected, since the entirely Haar-random gate cascade enforces no monotonicity on the entropy.

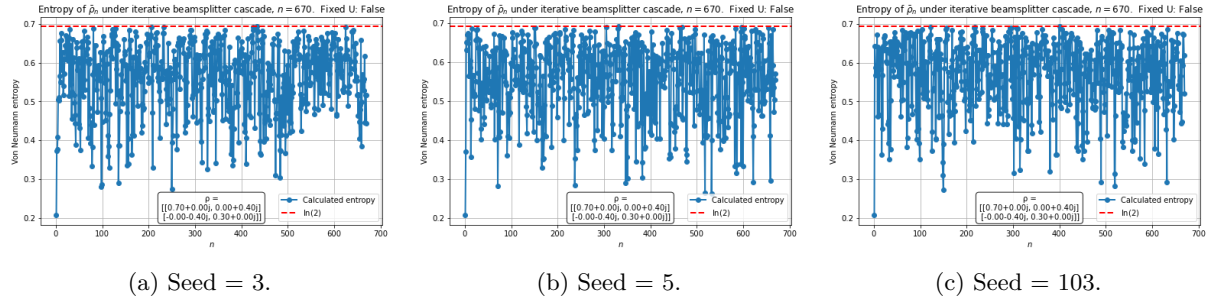


Figure 8: Results of the simulation of the von Neumann entropy for a beamsplitter cascade, where the unitary matrices representing gates are chosen Haar-randomly throughout the entire cascade. The maximum sample size was chosen to be 670 in each case, and the plots shown were simulated using np.random.seed values equal to 3, 5 and 103. In each instance, ρ was set to $\begin{pmatrix} 0.7 & 0.4i \\ -0.4i & 0.3 \end{pmatrix}$.

Corresponding to expectation, no convergence is observed, as the gates through which the sample is sent is altered throughout the cascade. Notably, the entropies remain in the interval $[0, \ln(2)]$, indicating consistency with theory; no entropy may ever become negative or exceed $\ln(2)$.

Since the gates are drawn randomly at each step, we consider an ensemble average over many realisations of the circuit. The averaged entropy of $\bar{\rho}_n$ is not expected to remain constant in general. Instead, it typically rapidly increases from its initial value and then saturates at a steady level. This reflects the fact that random unitary dynamics tend to generate entanglement and drive subsystems toward typical (high-entropy) states, rather than producing a balance between entropy-increasing and entropy-decreasing transformations [Zan01].

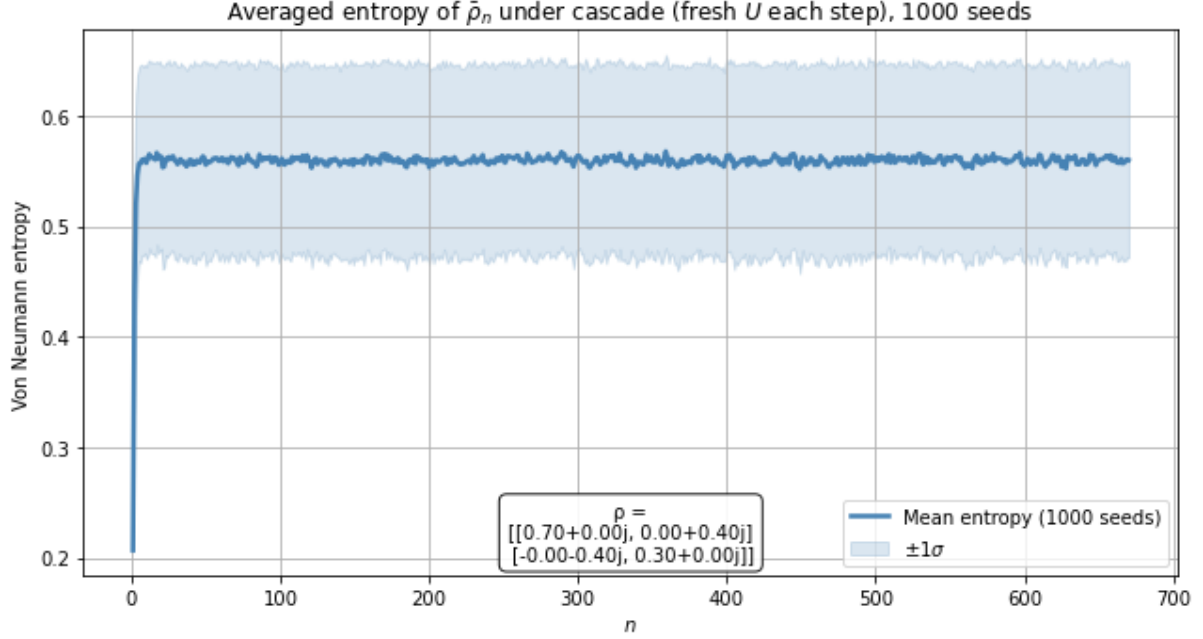
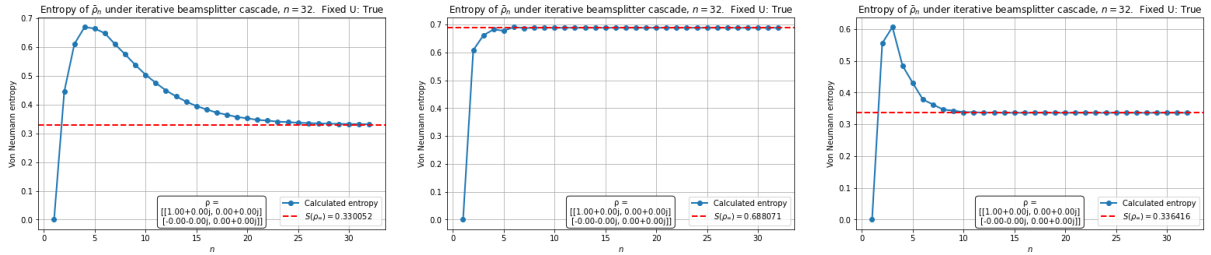


Figure 9: Averaged entropy of $\bar{\rho}_n$, where the average is acquired by taking the sample average over the results from seeds 1 through 1000. ρ was set to $\begin{pmatrix} 0.7 & 0.4i \\ -0.4i & 0.3 \end{pmatrix}$. The mean entropy is $0,559766 \pm 0,088253$.

The mean entropy converges to a set value, dependent on the initial state ρ , as the rapid oscillations in entropy due to the Haar-randomly chosen gates is removed by taking a sample average. After a short transient regime, the system rapidly loses local memory of the initial condition, and the entropy approaches a typical saturated value characteristic of random unitary dynamics. The ensemble average suppresses sample-to-sample fluctuations and reveals this underlying universal behaviour.

A pure state is next set as the initial state, after which the same simulations are performed in order to draw comparisons and identify possible anomalies.



(a) Seed = 3. $S(\rho_\infty) = 0,330052$. (b) Seed = 5. $S(\rho_\infty) = 0,688071$. (c) Seed = 103. $S(\rho_\infty) = 0,336416$.

Figure 10: Results of the simulation of the von Neumann entropy for a beamsplitter cascade, where the gate in each separate instance is chosen Haar-randomly, and kept fixed during the cascade. The maximum sample size was chosen to be 32 in each case, and the plots shown were simulated using np.random.seed values equal to 3, 5 and 103. In each instance, ρ was set to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Entirely analogously to the case where $\rho = \begin{pmatrix} 0.7 & 0.4i \\ -0.4i & 0.3 \end{pmatrix}$, the von Neumann entropy of $\bar{\rho}_n$ converges to $S(\rho_\infty)$, while monotonicity does not occur, as the Haar-random gates do not satisfy the constraints required for a beamsplitter gate.

Lastly, we also repeat the simulation for a random cascade, together with analysis of the averaged entropy of $\bar{\rho}_n$ with the initial state chosen to be pure.

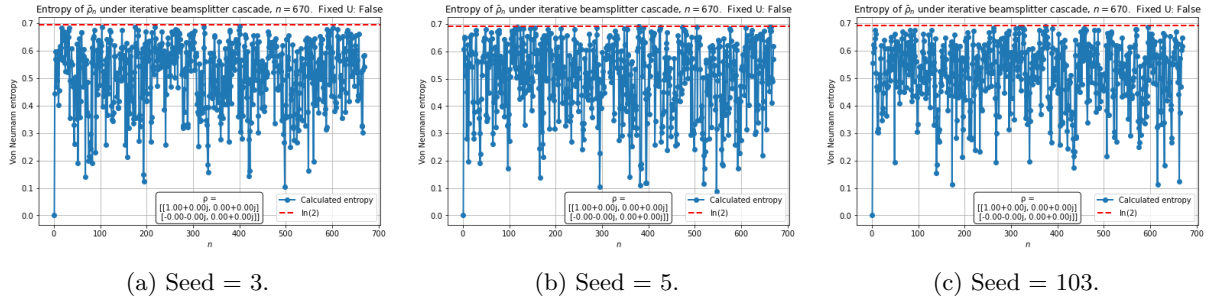


Figure 11: Results of the simulation of the von Neumann entropy for a beamsplitter cascade, where the unitary matrices representing gates are chosen Haar-randomly throughout the entire cascade. The maximum sample size was chosen to be 670 in each case, and the plots shown were simulated using np.random.seed values equal to 3, 5 and 103. In each instance, ρ was set to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

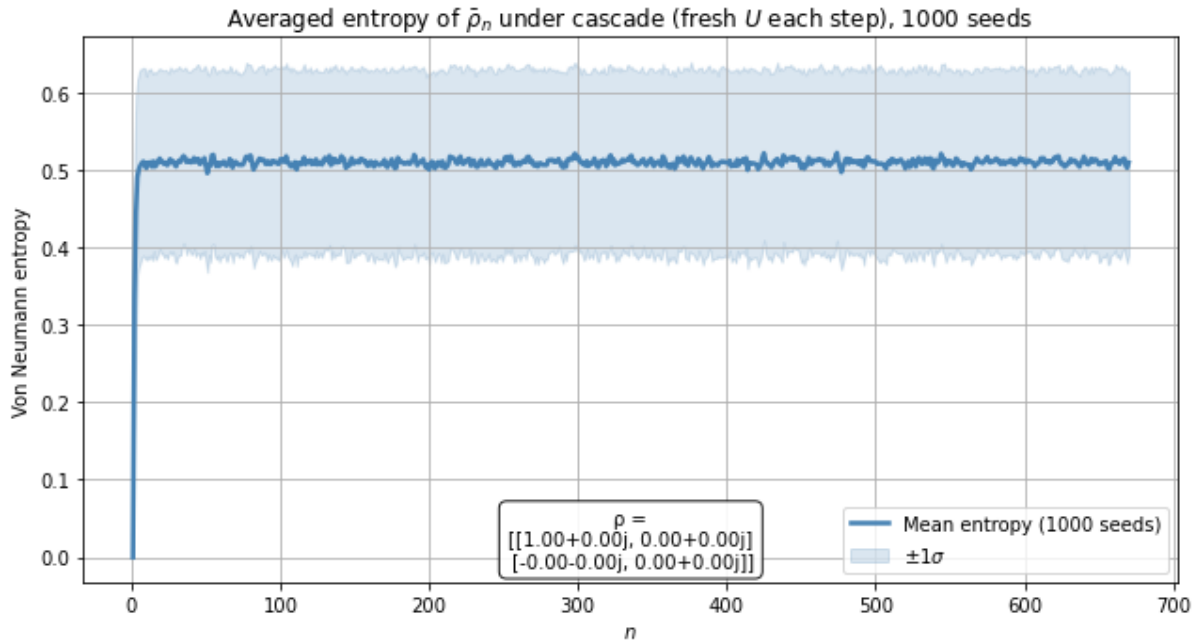


Figure 12: Averaged entropy of $\bar{\rho}_n$, where the average is acquired by taking the sample average over the results from seeds 1 through 1000. ρ was set to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The mean entropy is 0.509126 ± 0.118677 .

One again observes a rapid initial increase followed by convergence to a stable saturated regime, with comparable fluctuation levels across realisations. This agreement across distinct initial states indicates that the observed entropy dynamics is not specific to a particular choice of input state, but rather reflects a generic feature of the underlying unitary evolution generated by the random gate cascade.

Finally, although the numerical results presented in this thesis focus on two representative initial states, the simulations have been carried out for a significantly larger class of initial conditions. As can be verified independently using the code provided in the appendix 6, the qualitative behaviour remains unchanged. This indicates that all of the observed behaviour is robust with respect to the choice of initial state.

4.4 Calculations III

This subsection consists of a calculation performed in section 4, written here in order to avoid interrupting the main discussion with technical details.

Calculation 4.1 shows that a sample mean pair has the same covariance matrix as the i.i.d. pairs it was constructed by, analogous to the classical case.

Calculation 4.1. In this calculation, we show the “sample mean” covariance matrix equals that of each separate, i.i.d. canonical pair (P_j, Q_j) . Recall the informal definitions

$$\bar{P}_n = n^{-\frac{1}{2}}(P_1 + \dots + P_n), \quad \bar{Q}_n = n^{-\frac{1}{2}}(Q_1 + \dots + Q_n),$$

and the covariance matrix entries

$$\Gamma_{11} = \text{Tr}(\rho P^2), \quad \Gamma_{22} = \text{Tr}(\rho Q^2),$$

and

$$\Gamma_{12} = \Gamma_{21} = \frac{1}{2} \text{Tr}(\rho(PQ + QP)).$$

Assume each i.i.d. canonical pair (P_j, Q_j) has covariance matrix Γ , and assume that $\mathbb{E}_\rho[P_j] = \mathbb{E}_\rho[Q_j] = 0$. We calculate $\bar{\Gamma}_{11}, \bar{\Gamma}_{22}, \bar{\Gamma}_{12}$ and $\bar{\Gamma}_{21}$ explicitly.

1. Calculation of $\bar{\Gamma}_{11}, \bar{\Gamma}_{22}$.

We carry out the calculation for the (1,1)-entry of the covariance matrix; the remaining entries follow similarly. Expanding $\bar{P}_n^2$ explicitly,

$$\bar{P}_n^2 = \left(\frac{1}{\sqrt{n}} \sum_{j=1}^n P_j \right)^2 = \frac{1}{n} \sum_{j=1}^n P_j^2 + \frac{1}{n} \sum_{\substack{j,k=1 \\ j \neq k}}^n P_j P_k,$$

so that

$$\mathbb{E}_\rho[\bar{P}_n^2] = \frac{1}{n} \sum_{j=1}^n \mathbb{E}_\rho[P_j^2] + \frac{1}{n} \sum_{\substack{j,k=1 \\ j \neq k}}^n \mathbb{E}_\rho[P_j P_k].$$

For the cross terms, fix $j \neq k$ and apply the stochastic independence of the canonical pairs: taking $f_j(P_j, Q_j) = P_j$ and $f_k(P_k, Q_k) = P_k$ as bounded measurable functions and applying proposition 3.9,

$$\mathbb{E}_\rho[P_j P_k] = \mathbb{E}_\rho[f_j(P_j, Q_j) f_k(P_k, Q_k)] = \mathbb{E}_\rho[P_j] \mathbb{E}_\rho[P_k] = 0,$$

where the last equality uses the assumption that the pairs are centered. Hence the double sum vanishes entirely, and since the pairs are identically distributed, remark 3.11 yields

$$\mathbb{E}_\rho[\bar{P}_n^2] = \frac{1}{n} \sum_{j=1}^n \mathbb{E}_\rho[P_j^2] = \frac{1}{n} \cdot n \cdot \mathbb{E}_\rho[P^2] = \mathbb{E}_\rho[P^2] = \Gamma_{11}.$$

The identical argument applies to $\mathbb{E}_\rho[\bar{Q}_n^2]$.

2. Calculation of $\bar{\Gamma}_{12}, \bar{\Gamma}_{21}$.

For the $(1, 2)$ -entry, we expand the symmetrised product explicitly,

$$\overline{P}_n \overline{Q}_n + \overline{Q}_n \overline{P}_n = \frac{1}{n} \sum_{j=1}^n (P_j Q_j + Q_j P_j) + \frac{1}{n} \sum_{\substack{j,k=1 \\ j \neq k}}^n (P_j Q_k + Q_k P_j),$$

so that

$$\mathbb{E}_\rho[\overline{P}_n \overline{Q}_n + \overline{Q}_n \overline{P}_n] = \frac{1}{n} \sum_{j=1}^n \mathbb{E}_\rho[P_j Q_j + Q_j P_j] + \frac{1}{n} \sum_{\substack{j,k=1 \\ j \neq k}}^n \mathbb{E}_\rho[P_j Q_k + Q_k P_j].$$

For the cross terms, fix $j \neq k$ and apply stochastic independence, taking $f_j(P_j, Q_j) = P_j$ and $f_k(P_k, Q_k) = Q_k$ as bounded measurable functions and applying proposition 3.9,

$$\mathbb{E}_\rho[P_j Q_k] = \mathbb{E}_\rho[f_j(P_j, Q_j) f_k(P_k, Q_k)] = \mathbb{E}_\rho[P_j] \mathbb{E}_\rho[Q_k] = 0,$$

and identically $\mathbb{E}_\rho[Q_k P_j] = 0$, where both last equalities use centredness. Hence the double sum vanishes, and since the pairs are identically distributed, remark 3.11 yields

$$\frac{1}{2} \mathbb{E}_\rho[\overline{P}_n \overline{Q}_n + \overline{Q}_n \overline{P}_n] = \frac{1}{2n} \sum_{j=1}^n \mathbb{E}_\rho[P_j Q_j + Q_j P_j] = \frac{1}{2n} \cdot n \cdot \mathbb{E}_\rho[PQ + QP] = \frac{1}{2} \mathbb{E}_\rho[PQ + QP] = \Gamma_{12} = \Gamma_{21}.$$

Since each expectation of the “sample mean” pair equals that of the separate canonical pairs, we conclude its covariance matrix equals Γ .

5 Conclusion and Discussion

This thesis studied how the von Neumann entropy behaves under two related, discrete quantum evolutions; a qCLT evolution, described by [CH71], and a bosonic mode gate cascade, often used in the field of quantum optics.

For the qCLT model, the discrete numerical results are consistent with the expected structure, as the system converges to the maximally mixed state, resulting in the von Neumann entropy to converge to the entropy of the maximally mixed state. This forms a finite-dimensional analogue of the theoretical results developed in the preliminary sections of the thesis, where the core properties of the qCLT were established. However, the conjectured monotonicity of the entropy throughout the qCLT evolution was not observed in practice, as clear fluctuations beyond numerical precision remained visible, suggesting that the use of Haar-random unitaries does not capture the conditions imposed by the Stone-von Neumann framework.

In the gate cascade setting, the structure of the chosen unitaries largely influences the behaviour of the entropy. When the cascade consists of a fixed gate, the system converges to a fixed point dependent on the repeated gate and on the initial state, though again no monotonic entropy behaviour is observed. When the cascade is allowed to include different gates, each chosen randomly according to the Haar measure, convergence is no longer expected nor observed, although the entropy remains bounded by physically meaningful limits. Notably, when averaging over many random realisations in the second case, the sample mean entropy stabilises at a state-dependent value, which reflects the balancing effect of Haar-random sampling driving subsystems toward typical, high-entropy states.

Overall, the results show a clear distinction between convergence and monotonicity. Convergence repeatedly appears as a feature of both models, whereas monotonicity is much more sensitive to the precise structure of the underlying unitary operators involved. In particular, the absence of monotonicity in the qCLT framework may be attributed to the fact that the unitary operators used during the simulation do not sufficiently satisfy the structural constraints implied by the Stone-von Neumann framework.

Future research could aim towards formulating a meaningful discrete analogue of the Stone-von Neumann framework, which would allow for an appropriate choice of unitary operators in the qCLT framework in finite-dimensional settings. Such a formulation could potentially restore monotonicity properties of the von Neumann entropy in the qCLT evolution. In a similar spirit, it would be of interest to further research and implement a discrete theory of bosonic mode gate cascades, see e.g. [BGJ23].

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6 Appendix

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import unitary_group
import time

# =====
# METHOD SELECTION
# =====

Method = 'Beamsplitter' # <-- SET TO 'qCLT' OR 'Beamsplitter'

# Additional options (only relevant for their respective methods)
max_n      = 40 # for qCLT: max number of qubits; for Beamsplitter: number of
               iterations
FIXED_U    = True # Beamsplitter only: True = reuse one Haar-random unitary,
               False = resample each step
Iteration  = True # qCLT only: True = using the iteration between unitaries,
               False = Haar-random each step

# =====
# AVERAGING OPTIONS
# Averaging is only applicable to the Beamsplitter method with FIXED_U = False.
# Each seed produces an independent trajectory (fresh unitaries drawn at every
step).
# =====

Averaging   = False # If True, average over multiple seeds (Beamsplitter +
FIXED_U=False only)
num_seeds    = 1000 # Number of independent seeds to average over
seed_start   = 1     # First seed value; subsequent seeds are seed_start+1,
               seed_start+2, ...
plot_individual = False # If True, also plot each individual seed's trajectory
               (faint lines)

# =====
# BASIC LINEAR ALGEBRA
# =====

def dagger(A):
    return np.conjugate(A.T)

def is_density_matrix(rho, tol=1e-10):
    hermitian = np.allclose(rho, dagger(rho), atol=tol)
    trace_ok  = np.isclose(np.trace(rho), 1.0, atol=tol)
    eigvals   = np.linalg.eigvalsh(rho)
    positive  = np.all(eigvals >= -tol)
```

```

    return hermitian and trace_ok and positive

# =====
# VON NEUMANN ENTROPY
# =====

def von_neumann_entropy(rho):
    eigvals = np.linalg.eigvalsh(rho)
    eigvals = np.real_if_close(eigvals)
    eigvals = eigvals[eigvals > 1e-15]
    entropy = -np.sum(eigvals * np.log(eigvals))
    return np.real_if_close(entropy)

# =====
# TENSOR PRODUCT
# =====

def kron_n(rho, n):
    result = rho
    for _ in range(n - 1):
        result = np.kron(result, rho)
    return result

# =====
# PARTIAL TRACE
# =====

def partial_trace(rho, subsystems_to_trace_out, dims):
    n = len(dims)
    tensor = rho.reshape([*dims, *dims])

    for i in sorted(subsystems_to_trace_out, reverse=True):
        tensor = np.trace(tensor, axis1=i, axis2=i + n)
        dims = dims[:i] + dims[i + 1:]
        n -= 1

    d = int(np.prod(dims))
    return tensor.reshape(d, d)

# =====
# RANDOM UNITARY
# =====

def random_unitary(dim):
    return unitary_group.rvs(dim)

```

```

# =====
# METHOD A      qCLT
# =====

def build_recursive_unitaries(max_n):
    U_dict = {}
    U_dict[2] = random_unitary(4)

    for n in range(3, max_n + 1):
        dim_prev = 2 ** (n - 1)
        print(
            f"Building U_{n}|previous dimension={dim_prev}|"
            f"matrix size={2**n}x{2**n}"
        )
        step_start = time.time()

        U_tilde = random_unitary(dim_prev)
        identity = np.eye(2)
        left = np.kron(U_tilde, identity)
        right = np.kron(identity, U_dict[n - 1])
        U_dict[n] = left @ right

        print(f"Finished U_{n} in {time.time() - step_start:.2f}s")

    return U_dict

def construct_rho_bar_n(rho, U_n, n):
    rho_n = kron_n(rho, n)
    evolved = U_n @ rho_n @ dagger(U_n)
    dims = [2] * n
    subsystems_to_trace_out = list(range(1, n))
    return partial_trace(evolved, subsystems_to_trace_out=
        subsystems_to_trace_out, dims=dims)

def run_qCLT(rho, max_n):
    if Iteration:
        print("\nBuilding recursive unitaries...")
        U_dict = build_recursive_unitaries(max_n)
    else:
        U_dict = {}
        for n in range(2, max_n + 1):
            print(
                f"Building U_{n}|previous dimension={2**(n-1)}|"
                f"matrix size={2**n}x{2**n}"
            )
            step_start = time.time()
            U_dict[n] = random_unitary(2**n)
            print(f"Finished U_{n} in {time.time() - step_start:.2f}s")

```

```

entropy_values = [von_neumann_entropy(rho)]
print(f"\nS(rho)={entropy_values[0]:.6f}")

for n in range(2, max_n + 1):
    print("\n" + "=" * 60)
    print(f"Constructing rho_bar_{n}")
    print("=" * 60)

    rho_bar_n = construct_rho_bar_n(rho, U_dict[n], n)
    S_n = von_neumann_entropy(rho_bar_n)

    if S_n < entropy_values[-1] - 1e-12:
        print(f"DECREASE BETWEEN STEPS {n-1} and {n}")

    entropy_values.append(S_n)
    print(f"rho_bar_{n} shape={rho_bar_n.shape}")
    print(f"S(rho_bar_{n})={S_n:.6f}")

return entropy_values

# =====
# METHOD B      Beamsplitter (iterative cascade)
# =====

def iterate_rho_bar(rho_bar, rho, U=None):
    if U is None:
        U = random_unitary(4)

    joint = np.kron(rho_bar, rho)
    evolved = dagger(U) @ joint @ U
    return partial_trace(evolved, subsystems_to_trace_out=[1], dims=[2, 2])

def run_beamsplitter(rho, max_n, FIXED_U, verbose=True):
    """
    Runs the beamsplitter/iterative cascade method and returns
    a list of entropy values for n=1, 2, ..., max_n.
    verbose is turned off when looped over multiple seeds in order to prevent
    repeated output:)
    """

    if FIXED_U:
        U_fixed = random_unitary(4)
        if verbose:
            print("\nUsing a single fixed Haar-random unitary for all steps.")
    else:
        U_fixed = None
        if verbose:
            print("\nSampling a fresh Haar-random unitary at each step.")

```

```

entropy_values = [von_neumann_entropy(rho)]
if verbose:
    print(f"\nS(rho)={entropy_values[0]:.6f}")

rho_bar = rho.copy()

for n in range(1, max_n):
    if verbose:
        print("\n" + "=" * 60)
        print(f"Iterating: rho_bar_{n}->rho_bar_{n+1}")
        print("=" * 60)

    step_start = time.time()
    rho_bar = iterate_rho_bar(rho_bar, rho, U=U_fixed)
    elapsed = time.time() - step_start

    if verbose and not is_density_matrix(rho_bar):
        print(f"WARNING: rho_bar_{n+1} is not a valid density matrix!")

    S_n = von_neumann_entropy(rho_bar)

    if verbose and S_n < entropy_values[-1] - 1e-12:
        print(f"DECREASE at step_{n}->{n+1}")

    entropy_values.append(S_n)

    if verbose:
        print(f"rho_bar_{n+1}_shape={rho_bar.shape}")
        print(f"S(rho_bar_{n+1})={S_n:.6f} [{elapsed:.3f}s]")

return entropy_values, U_fixed

# =====
# AVERAGING (Beamsplitter + FIXED_U=False only)
# =====

def run_beamsplitter_averaged(rho, max_n, num_seeds, seed_start):
    """
    Runs the beamsplitter method (FIXED_U=False) independently for each seed
    in [seed_start, seed_start+num_seeds) and returns:
    all_entropies: array of shape (num_seeds, max_n)
    mean_entropy: array of shape (max_n,)
    std_entropy: array of shape (max_n,)
    """
    all_entropies = []

    for i in range(num_seeds):
        seed = seed_start + i
        np.random.seed(seed)
        print(f"\n{'='*60}")

```

```

        print(f"Averaging run {i+1}/{num_seeds} (seed={seed})")
        print(f"{'='*60}")

        entropies, _ = run_beamsplitter(rho, max_n, FIXED_U=False, verbose=False)
        all_entropies.append(entropies)
        print(f"Done. Final S={entropies[-1]:.6f}")

    all_entropies = np.array(all_entropies)          # shape: (num_seeds, max_n)
    mean_entropy = np.mean(all_entropies, axis=0)   # shape: (max_n,)
    std_entropy = np.std(all_entropies, axis=0)     # shape: (max_n,)

    return all_entropies, mean_entropy, std_entropy

# =====
# MAIN
# =====

if __name__ == "__main__":
    SEED = 20
    np.random.seed(SEED)

    # -----
    # Initial single-qubit density matrix
    # -----
    rho = np.array([
        [0.6, 0.0j],
        [-0.0j, 0.4]
    ], dtype=complex)

    print("Initial rho:")
    print(rho)

    if is_density_matrix(rho):
        print("Valid density matrix")
    else:
        print("Invalid density matrix")
        raise SystemExit

    # -----
    # Dispatch to selected method
    # -----
    if Method == 'qCLT':
        entropy_values = run_qCLT(rho, max_n)
        title = (
            r"Entropy of  $\bar{\rho}_n$  under qCLT construction,  $n =$ "
            + str(max_n)
        )

```

```

elif Method == 'Beamsplitter':

    if Averaging and not FIXED_U:
        # -----
        # Averaging branch: independent seeds, FIXED_U=False
        # -----
        all_entropies, mean_entropy, std_entropy =
            run_beamsplitter_averaged(
                rho, max_n, num_seeds, seed_start
            )
        x_vals = list(range(1, max_n + 1))

        plt.figure(figsize=(9, 5))

        if plot_individual:
            for i, entropies in enumerate(all_entropies):
                plt.plot(
                    x_vals, entropies,
                    color='steelblue', alpha=0.25, linewidth=0.8,
                    label='Individual runs' if i == 0 else '_nolegend_'
                )

        plt.plot(
            x_vals, mean_entropy,
            color='steelblue', linewidth=2.5, label=f'Mean entropy ({
                num_seeds} seeds)'
        )
        plt.fill_between(
            x_vals,
            mean_entropy - std_entropy,
            mean_entropy + std_entropy,
            color='steelblue', alpha=0.20, label=r'$\pm 1 \sigma$'
        )

        rho_text = (
            "\n"
            f"[{rho[0,0]:.2f}, {rho[0,1]:.2f}]\n"
            f"[{rho[1,0]:.2f}, {rho[1,1]:.2f}]"
        )
        plt.gca().text(
            0.5, 0.02, rho_text,
            transform=plt.gca().transAxes,
            fontsize=10,
            horizontalalignment='center',
            verticalalignment='bottom',
            bbox=dict(boxstyle="round,pad=0.4", edgecolor="black",
                    facecolor="white", alpha=0.9)
        )

        plt.xlabel(r"$n$")
        plt.ylabel("Von Neumann entropy")

```

```

plt.title(
    r"Averaged entropy of  $\bar{\rho}_n$  under cascade (fresh  $U$  each step), "
    + f"{num_seeds} seeds"
)
plt.grid(True)
plt.legend(loc="lower right")
plt.tight_layout()
plt.show()

# Print summary
print("\n" + "=" * 60)
print("AVERAGING SUMMARY")
print("=" * 60)
print(f"Final mean S = {mean_entropy[-1]:.6f}")
print(f"Final std S = {std_entropy[-1]:.6f}")

raise SystemExit # skip the shared plotting block below

else:
    if Averaging and FIXED_U:
        print("\nWARNING: Averaging=True has no effect when FIXED_U=
            True"
              "(the fixed unitary fully determines the trajectory). "
              "Running single-seed beamsplitter instead.\n")

    entropy_values, U_used = run_beamsplitter(rho, max_n, FIXED_U)
    title = (
        r"Entropy of  $\bar{\rho}_n$  under iterative beamsplitter cascade,  $n =$  "
        + str(max_n)
        + f". Fixed U: {FIXED_U}"
    )

else:
    raise ValueError(f"Unknown Method '{Method}'. Choose 'qCLT' or 'Beamsplitter'.")

# -----
# Shared plotting block (single-run paths)
# -----
if Method == "Beamsplitter" and FIXED_U:
    print("\n" + "=" * 60)
    print("Fixed 4x4 unitary used throughout the simulation:")
    print("=" * 60)
    print(U_used)

x_vals = list(range(1, len(entropy_values) + 1))

plt.figure(figsize=(8, 5))

```

```

plt.plot(x_vals, entropy_values, marker='o', linewidth=2, label="Calculated
        entropy")

if Method == 'Beamsplitter' and FIXED_U:
    basis = [
        np.array([[1, 0], [0, 0]], dtype=complex),
        np.array([[0, 1], [0, 0]], dtype=complex),
        np.array([[0, 0], [1, 0]], dtype=complex),
        np.array([[0, 0], [0, 1]], dtype=complex)
    ]

    T = np.zeros((4, 4), dtype=complex)

    for j, E in enumerate(basis):
        F = iterate_rho_bar(E, rho, U_used)
        T[:, j] = F.reshape(-1)

    eigvals, eigvecs = np.linalg.eig(T)
    idx = np.argmin(np.abs(eigvals - 1))
    rho_inf = eigvecs[:, idx].reshape(2, 2)
    rho_inf /= np.trace(rho_inf)
    rho_inf = (rho_inf + rho_inf.conj().T) / 2
    S_inf = von_neumann_entropy(rho_inf)

    plt.axhline(
        y=S_inf, color='red', linestyle='--', linewidth=2,
        label=r'$S(\rho_{\infty})={S_{inf}:.6f}$'
    )
else:
    plt.axhline(
        y=np.log(2), color='red', linestyle='--', linewidth=2,
        label=r'$\ln(2)$'
    )

rho_text = (
    "\n"
    f"[[{rho[0,0]:.2f},{rho[0,1]:.2f}]\n"
    f"[{rho[1,0]:.2f},{rho[1,1]:.2f}]]\n"
)

plt.gca().text(
    0.5, 0.02, rho_text,
    transform=plt.gca().transAxes,
    fontsize=10,
    horizontalalignment='center',
    verticalalignment='bottom',
    bbox=dict(boxstyle="round,pad=0.4", edgecolor="black",
        facecolor="white", alpha=0.9)
)

plt.xlabel(r"$n$")
plt.ylabel(r"Von Neumann entropy")

```

```
plt.title(title)
plt.grid(True)
plt.legend(loc="lower_right")
plt.show()
```