Analyzing the effect of Coriolis forces on shallow water waves using Lie theory

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Abstract

The Korteweg-de Vries equation is a partial differential equation that can be used to describe water waves. A few years back, a modified version of this equation was derived, to take coriolis forces into account. Before that, the Korteweg-de Vries equation had also been found in the realm of Lie theory, as an equation of motion on the Virasoro-Bott group.

This paper aimed to find a relation between the two equation, through studying the equations themselves, as well as through the Lie algebras in which they also appear. This way, two similar transformations from solutions of one equation, to solutions of the other have been found. Both of these come down to a vertical shift, though different ones. Another transformation was also found for the equation, but no similar transformation between the Lie algebras was found.

Laymen summary

There exists a large variety of mathematical models for water waves. One such model is an equation called the Korteweg-de Vries (KdV) Equation, which is used to model waves that keep their shape as they travel.

This paper explores how this classic wave equation changes when we take into account a force called the Coriolis effect—an effect caused by the Earth's rotation. The Coriolis effect slightly alters how water moves, which changes the way we need to model the waves.

An altered version of the wave equation, called the geophysical KdV (gKdV) equation, includes this rotational force. The paper examines how the two versions of the equation are related. One key finding is that the solutions to both equations are fairly similar. In fact, one can retrieve a solution for one of them, by shifting a solution of the other vertically. The paper goes further by diving into advanced math called Lie theory, which connects these wave equations to movement through abstract spaces (called Lie groups),. This theory treats the equations like paths on the Lie groups, which helps us understand the deeper structure of the equations.

Both the KdV and the gKdV equation have a corresponding Lie group. Their Lie groups are very similar, and a transformation between the two exists. This transformation also gives rise to a vertical shift, although this vertical shift is different.

Introduction

The Korteweg-de Vries (KdV) equation, a non-linear partial differential equation, has, since its first derivation by Joseph Valentin Boussinesq in 1877[1], and its later rediscovering by Diederik Korteweg and Gustav de Vries, who intended it to describe the surface waves of shallow water streams such as rivers, been extensively studied. These days, it has more applications than just that, seeing use in other areas of science, such as solid state physics, as well.

Another place where the KdV equation shows up is in the domain of Lie theory, where it is the solution to a type of equation of motion called the Euler-Arnold equation, for the Virasoro-Bott group.

More recently, a variant on this equation has been derived by Geyer and Quirchmayr.[2] This variant, the geophysical Korteweg-de Vries (gKdV) equation is derived to take the virtual force due to the Coriolis effect into account. This equation does not deviate all too much from the regular KdV equation, which begs the question: How does the inclusion of the Coriolis effect in the gKdV equation affect the behavior and solutions of the classical KdV equation? More specifically, to what extent do the solutions of the gKdV equation differ from those of the KdV equation, and how can the two be related? Furthermore, is this gKdV equation also to be found in the realm of Lie theory, and how is it related to the KdV equation in this area?

As such, the intent of this paper is to compare the two, and see in what way the two are connected. To do this, the following will be done:

-We will first describe the derivation of both equations, and compare them to each other.

-We will then move into the framework of Lie theory, where the two are connected as well.

-We will then look at known solutions of the KdV equation, and check if and how these translate to the gKdV equation.

We will compare these different possible transformations, and see how they relate, if they relate at all and, if they do relate, how.

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Chapter 1

Derivation

In this first chapter, we will derive the Korteweg-de Vries equation itself, as well as its geophysical variant. We will first derive the Euler equations, the general equation for inviscid fluid flow, and set the boundary conditions we will use. We will then rescale the variables in the equations, to remove their dimensions. Afterwards, we will change our coordinate system, to move with the wave, the so-called far field coordinates. We will then use asymptotic expansion, to retrieve the KdV equation.

After that, we will do the same steps again, but inserting the Coriolis effect into the Euler equations, which will give us the geophysical KdV equation. Finally, we will see what possible connection or transformation might exist between solutions of the two.

The following derivation of the KdV equation has been adapted from [3]:

1.1 Derivation of the Euler Equations

Since the Korteweg-de Vries equation is an equation describing (inviscid) water waves, one can start to derive it from the Euler equations. Those will be derived in this section. This will be done by studying the flow of both the water itself, as well as its impulse flow, in and out of a control volume.

Firstly, the change in mass of water is determined by the flow of water in and out of a control volume. For the entire volume, this can be described as:

$$\int_V \partial_t \rho d^3x = -\oint_{\partial V} \rho \mathbf{v} d^2x = -\int_V \nabla \cdot (\rho \mathbf{v}) d^3x$$

Here, $\mathbf{u} = (u, v, w)$ is the fluid velocity, and ρ the density of water. When putting both terms on the left-hand side, and putting them in the same integral, we get:

$$\int_V (\partial_t \rho + \nabla \cdot (\rho \mathbf{v})) d^3x = 0$$

Since this should be true for any control volume, we get the following (continuity) equation:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$$

For an incompressible fluid, the density will remain constant. Then, the continuity equation will become:

$$\nabla \cdot \mathbf{v} = 0$$

From this point onward, we will make use of the following notation, which is called the material derivative:

$$\frac{D}{Dt} = \partial_t + (\mathbf{v} \cdot \nabla)$$

The change in momentum of a liquid is determined by the force working on it, and the momentum of the in- and outflow of it. The force on the liquid is due to viscosity, pressure, and outside forces. However, we will not take viscosity into account when modeling fluids with the Euler equation, as the KdV and Euler equations do not take this into account. Suppose P is the pressure gradient in the fluid, and $\mathbf{F}$ the external force per unit mass. Then, the force working on the fluid in the container is:

$$\int_V \rho \mathbf{F} d^3x - \oint_{\partial V} P\mathbf{n} d^3x = \int_V (\rho \mathbf{F} - \nabla P) d^3x$$

The change due to flow is given by:

$$-\oint_{\partial V} \rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n}) d^2x = -\int_V (\rho\mathbf{v}(\nabla\cdot\mathbf{v}) + \mathbf{v}(\mathbf{v}\cdot\nabla)\rho + \rho(\mathbf{v}\cdot\nabla)\mathbf{v}) d^3x$$

The first term on the right hand side is dependent on the divergence of velocity for incompressible flow, and is, as a consequence, zero.

Finally, the change in momentum is given by the time derivative of the total momentum in the control volume:

$$\frac{d}{dt}\int_V \rho\mathbf{v} d^3x = \int_V (\mathbf{v}\partial_t\rho + \rho\partial_t\mathbf{v}) d^3x$$

As stated above, this change is equal to the change due to flow, together with the change due to force, so

$$\int_V (\mathbf{v}\partial_t\rho + \rho\partial_t\mathbf{v}) d^3x = -\int_V (\mathbf{v}(\mathbf{v}\cdot\nabla)\rho + \rho(\mathbf{v}\cdot\nabla)\mathbf{v}) d^3x + \int_V (\rho\mathbf{F} - \nabla P) d^3x$$

Since we have constant density, all terms which differentiate it, will be zero. If we then put the remaining terms on the same side, and under the same integral, the equation becomes as follows:

$$\int_V (\rho\partial_t\mathbf{v} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \rho\mathbf{F} - \nabla P) d^3x = 0$$

The two left most terms, together, become the material derivative of velocity, times the density. Since the equation should hold true for arbitrary control volumes we will gain the following equation:

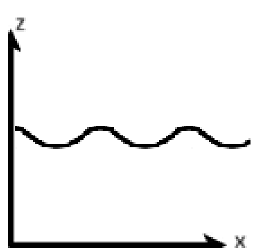
$$\rho\frac{D\mathbf{v}}{Dt} + \rho\mathbf{F} - \nabla P = 0$$

Dividing all terms by the density, and separating the velocity term from the force terms, we get the Euler equations:

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla P + \mathbf{F} \tag{1.1}$$

1.2 Setting the Stage

We shall henceforth use a two-dimensional, cartesian coordinate system, with coordinates (x, z), with x the horizontal axis, oriented to the direction of wave propagation, and z vertical axis, oriented opposite the direction of gravity.



For our purposes, the only external force working on the water will be gravity (though this will change in section 1.9, with the inclusion of the Coriolis effect). This force, in these coordinates, equates to $\mathbf{F} = (0, -g)$, with g the gravitational acceleration.

The pressure here, P, will be described as follows:

$$P = P_a + \rho g(h_0 - z) + \tilde{P}$$

where P_a is the atmospheric pressure, $\rho g(h_0 - z)$ is the hydrostatic pressure due to gravity[4], and $\tilde{P}$ the deviation from this pressure. Here, only the terms $-\rho g z$ and $\tilde{P}$ will be non-constant, with the former only being dependent on z. When put in the Euler equations, we see the following happens:

$$-\frac{1}{\rho}\nabla P + \mathbf{F} = -\frac{1}{\rho}[+\partial_x \tilde{P}, -\rho g + \partial_z \tilde{P}] + [0, -g] = -\frac{1}{\rho}\nabla \tilde{P}$$

This implies that the Euler equations become:

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla \tilde{P} \tag{1.2}$$

Lastly, in these coordinates we will use the notation $\mathbf{v} = [u, w]$ for velocity.

1.3 Boundary Conditions

As boundary conditions for the equations, we shall consider both the border the water shares with the bottom, as well as with the air. The bottom boundary, we will assume is constant, in time and space. The condition on the water surface is what the KdV equation will eventually describe.

Let $z = h$ be the height profile of the surface of the body of water. Here, we use $h = h_0 + \alpha\eta(\mathbf{x}, t)$, where h_0 would be the unperturbed height of the water surface, and $\alpha\eta$ the disturbance due to waves, with α their amplitude. Then:

$$\frac{D}{Dt}(0) = \frac{D}{Dt}(z - h) = (\partial_t + u\partial_x + w\partial_z)(z - h)$$

z is time- and x independent, so we gain the following:

$$(\partial_t + u\partial_x + w\partial_z)(-h) + w = 0 \Rightarrow w = (\partial_t + u\partial_x + w\partial_z)(h)$$

Since the height of the surface is only dependent on x and t, this is equal to the following boundary condition:

$$w = \partial_t h + u\partial_x h \tag{1.3}$$

Similarly, the z-directional velocity at the bottom of the water is as follows:

$$w = \partial_t b + u\partial_x b = 0 \tag{1.4}$$

We will assume a flat, unchanging bottom surface, $b = 0$, which is why the z-directional velocity there is zero as well.

Lastly, we will assume that the pressure deviation $\tilde{P}$ on the surface is constant, save for a term depending on the surface tension. However, this surface tension term will, after the Non-dimensionalization in the next section, be linearly dependent on the inverse of the weber number, W^{-1}. This is equal to $W^{-1} = \frac{\Gamma}{\rho g h_0^2}$, where Γ is the surface tension, which, for water, is equal to 0.07 N/m.[5] For a body of water 1 meter deep, the inverse weber number becomes of the order 10^5. So, we can say, with reasonable certainty that the surface tension term is negligible.

1.4 Non-dimensionalization

Next, in order to be able to easily make use of the approximations which the KdV equation relies on, we will first rescale the various variables in the equation by way of some constant values related to said coordinates, in order to remove the dimensions of the variables. In this section, we shall refer to the non-dimensionalized variables with a hat, e.g. $\hat{z}$ for the non-dimensionalized z.

The first rescaling will be to scale the z-axis with the undisturbed water height, as seen in the definition of the surface boundary condition, as follows: $z = h_0\hat{z}$. Next, we shall scale the x-axis with the typical wavelength λ we expect the waves to have: $x = \lambda\hat{x}$.

Lastly, using a characteristic speed of the wave, $\sqrt{gh_0}$, and the typical wavelength, we will rescale the time as follows: $t = \frac{\lambda}{\sqrt{gh_0}}\hat{t}$

These changes will have the following effects on the flow velocity:

$$u = \lambda \cdot \frac{\sqrt{gh_0}}{\lambda}\hat{u} = \sqrt{gh_0}\hat{u}, \quad w = h_0\frac{\sqrt{gh_0}}{\lambda}\hat{w}$$

where the time rescaling and respectively the x- and z-directional space rescaling have been used.

Next, the partial derivative of the various dimensions are scaled by the reciprocal of their respective scaling factors:

$$\frac{\partial}{\partial x} = \frac{1}{\lambda}\frac{\partial}{\partial \hat{x}}, \quad \frac{\partial}{\partial z} = \frac{1}{h_0}\frac{\partial}{\partial \hat{z}}, \quad \frac{\partial}{\partial t} = \frac{\sqrt{gh_0}}{\lambda}\frac{\partial}{\partial \hat{t}}$$

Combining these all together for the material derivative, we get the following rescaling:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z} = \frac{\sqrt{gh_0}}{\lambda}\frac{\partial}{\partial \hat{t}} + \sqrt{gh_0}\hat{u}\cdot\frac{1}{\lambda}\frac{\partial}{\partial \hat{x}} + h_0\frac{\sqrt{gh_0}}{\lambda}\hat{w}\cdot\frac{1}{h_0}\frac{\partial}{\partial \hat{z}} = \frac{\sqrt{gh_0}}{\lambda}\frac{D}{D\hat{t}}$$

Then, when we check the material derivatives of u and w, we get the following:

$$\frac{Du}{Dt} = \frac{\sqrt{gh_0}}{\lambda}\sqrt{gh_0}\frac{D\hat{u}}{D\hat{t}}, \quad \frac{Dw}{Dt} = \frac{\sqrt{gh_0}}{\lambda}h_0\frac{\sqrt{gh_0}}{\lambda}\frac{D\hat{w}}{D\hat{t}}$$

We will then non-dimensionalize the pressure deviation as follows:

$$\tilde{P} = \rho h_0 g\hat{p}$$

For the pressure gradient, we then get, in horizontal direction:

$$-\frac{1}{\rho}\partial_x P = -\partial_x(\tilde{P}) = -gh_0\partial_x\hat{p} = -gh_0\frac{1}{\lambda}\partial_{\hat{x}}\hat{p}$$

and, in vertical direction:

$$-\frac{1}{\rho}\partial_z P + g = -\partial_z(\tilde{P}) = -gh_0\partial_z\hat{p} = -gh_0\frac{1}{h_0}\partial_{\hat{z}}\hat{p} = -g\partial_{\hat{z}}\hat{p}$$

We can see that, in vertical direction,the pressure gradient is scaled the same way as the material derivative of velocity. However, for the vertical direction, the material derivative of velocity is scaled differently: $g(h_0/\lambda)^2$ compared to just g. When we put these together, we find the new Euler equations:

$$\frac{D\hat{u}}{D\hat{t}} = -\partial_{\hat{x}}\hat{p}$$

$$\delta^2\frac{D\hat{w}}{D\hat{t}} = -\partial_{\hat{z}}\hat{p}$$

Here, $\delta = \frac{h_0}{\lambda}$ is the shallowness parameter.

The scalings of the boundary conditions are as follows:

$$h = h_0\hat{h}, \quad \hat{h} = 1 + \frac{\alpha}{h_0}\eta(x_1, t) = 1 + \epsilon\eta$$

Here, $\epsilon = \frac{\alpha}{h_0}$ is the amplitude parameter. This will result to the following boundary condition for the horizontal velocity: At $z = \hat{h}$:

$$\hat{w} = \epsilon\left(\partial_t\eta + u\partial_x\eta\right)$$

Finally, after non-dimensionalization, the boundary condition for pressure, as given above will become:

$$p_z = \epsilon\eta - \epsilon\left[\frac{\Gamma}{\rho g\lambda^2}\right]\left(\frac{\eta_{xx}}{(1+\epsilon\eta_x^2)^{1/2}}\right)$$

However, as stated in section 1.3, this second term due to surface tension will be negligible in size, compared to the first term. It will therefore be neglected in the KdV equation. Now, since w and p are proportional to ϵ, it is reasonable to rescale the pressure and velocity (including u, as to keep mass conservation) with ϵ:

$$\hat{w} \mapsto \epsilon\hat{w}, \quad \hat{p} \mapsto \epsilon\hat{p}, \quad \hat{u} \mapsto \epsilon\hat{u}$$

This will result to the following change to the material derivative:

$$\frac{D}{D\hat{t}} = \partial_{\hat{t}} + \epsilon\left(\hat{u}\partial_{\hat{x}} + \hat{w}\partial_{\hat{z}}\right)$$

Note:

From this point onwards, we shall only use the non-dimensionalized values, and, as such, will use the normal symbols, e.g. t or w, for the non-dimensionalized values, instead of adding a hat on top.

To summarize, the new dimensionless Euler equations are:

$$\begin{gathered}\partial_t u + \epsilon\left(u\partial_x u + w\partial_z u\right) = -\partial_x p, \quad \delta^2(\partial_t w + \epsilon(u\partial_x w + w\partial_z w)) = -\partial_z p \\ \partial_x u + \partial_z w = 0\end{gathered} \tag{1.5}$$

and the dimensionless boundary conditions have become:

$$\begin{gathered}w = \epsilon\left(\partial_t\eta + u\partial_x\eta\right), \quad p_z = \eta \quad \text{at } z = 1 + \epsilon\eta \\ w = 0 \quad \text{at } z = 0\end{gathered} \tag{1.6}$$

1.5 Assumptions

Now that we have rewritten the equations into this form, we can implement the following approximation conditions for the KdV equation.

. We shall use $\delta^2 = \mathcal{O}(\epsilon)$. This is called the 'shallow water approximation', which is the domain in which the KdV is usable. It describes bodies of water, where the depth is very low, compared to the wavelengths of the waves that are described. It will reduce the number of these parameters by one, which will make the asymptotic expansion that is to come easier.

We can then rescale $x \mapsto \frac{\delta}{\epsilon^{1/2}}x$ and $t \mapsto \frac{\delta}{\epsilon^{1/2}}t$. Furthermore, to keep consistency in the material derivative and in the continuity equation, we should rescale $w \mapsto \frac{\epsilon^{1/2}}{\delta}w$.

This results to the following changes to 1.5 and 1.6:

$$\begin{gathered}\partial_t u + \epsilon(u\partial_x u + w\partial_z u) = -\partial_x p, \quad \epsilon(\partial_t w + \epsilon(u\partial_x w + w\partial_z w)) = -\partial_z p, \quad \partial_x u + \partial_z w = 0 \\ p = \eta, \quad w = \partial_t\eta + \epsilon u\partial_x\eta \quad \text{on} \quad z = h = 1 + \epsilon\eta \\ w = 0 \quad \text{on} \quad z = 0\end{gathered} \tag{1.7}$$

1.6 $\epsilon \to 0$ approximation

We will now see what these new Euler equations give us, for very small ϵ. For the approximation $\epsilon \to 0$, the equations will result in the following:

$$\partial_t u = -\partial_x p, \quad \partial_z p = 0, \quad \partial_z w = -\partial_x u$$

$$w = \partial_t \eta = \partial_t p, \quad p = \eta \qquad \text{on } z = 1$$

Since p is independent of z, and $p = \eta$ on $z = 1$, $p = \eta$ everywhere. u must also be independent of z as a consequence, so $w = -z\partial_x u$, which implies, at $z = 1$, $-\partial_x u = \partial_t \eta$.

Then, we have the equations:

$$\partial_t \eta = -\partial_x u, \quad \partial_t u = -\partial_x \eta$$

So:

$$\partial_t^2 \eta = -\partial_t \partial_x u = \partial_x^2 \eta$$

This leads to the standard one-dimensional wave equation:

$$\partial_t^2 \eta - \partial_x^2 \eta = 0$$

This has solutions of the form $\eta = f(x \pm t)$ for some function f.

1.7 Far field Variables

The results in the previous section seem to imply that the exists knowledge to be gained from studying the left and right moving waves it gives. To study the development of the right-moving wave which this equation describes, the following coordinate transformations ought to be done: $\xi = x - t$, $\tau = \epsilon t$, the first to move along with the wave, the second to change the timescale to a slower one. From these transformations, which give $\partial_t = -\partial_\xi + \epsilon\partial_\tau$, and $\partial_x = \partial_\xi$ we get the following changes to equations 1.7:

$$-\partial_\xi u + \epsilon(\partial_\tau u + u\partial_\xi u + w\partial_z u) = -\partial_\xi p, \quad \epsilon(-\partial_\xi w + \epsilon(\partial_\tau w + u\partial_\xi w + w\partial_z w)) = -\partial_z p, \quad \partial_\xi u + \partial_z w = 0$$

$$p = \eta, \quad w = -\partial_\xi \eta + \epsilon(\partial_\tau \eta + u\partial_\xi \eta) \quad \text{on} \quad z = 1 + \epsilon\eta$$

$$w = 0 \quad \text{on} \quad z = 0$$

1.8 Asymptotic Expansion

The last step to deriving the KdV equation is to use asymptotic expansion:

$$\eta(x,t) \sim \sum_{n=0}^{\infty} \epsilon^n \eta_n(x,t)$$

and

$$p(x,z,t) \sim \sum_{n=0}^{\infty} \epsilon^n p(x,z,t), \quad u(x,z,t) \sim \sum_{n=0}^{\infty} \epsilon^n u(x,z,t), \quad w(x,z,t) \sim \sum_{n=0}^{\infty} \epsilon^n w(x,z,t)$$

This is, as might be guessed from its form, a method to approximate the function for small values of ϵ. For the derivation of the KdV equation we will only use the leading (zeroth) order term, and the first order term. Every term afterwards will not be taken into consideration. When put in the equations, we will get the following:

$$-\partial_\xi u_0 + \epsilon(-\partial_\xi u_1 + \partial_\tau u_0 + u_0\partial_\xi u_0 + w_0\partial_z u_0) = -\partial_\xi p_0 + \epsilon(-\partial_\xi p_1) + \mathcal{O}(\epsilon^2)$$

$$-\epsilon\partial_z w_0 = -\partial_z p_0 - \epsilon\partial_z p_1 + \mathcal{O}(\epsilon^2), \quad \partial_\xi u_0 + \partial_z w_0 + \epsilon(\partial_\xi u_1 + \partial_z w_1) + \mathcal{O}(\epsilon^2) = 0$$

The boundary conditions become:

$$p_0 + \epsilon p_1 = \eta_0 + \epsilon\eta_1 + \mathcal{O}(\epsilon^2), \quad w_0 + \epsilon w_1 = -\eta_{0\xi} + \epsilon(-\partial_\xi \eta_1 + \partial_\tau \eta_0 + u_0 \partial_\xi \eta_0) + \mathcal{O}(\epsilon^2) \quad \text{at } z = 1 + \epsilon\eta$$

$$w_0 + \epsilon w_1 = 0 \quad \text{at } z = 0$$

The $+\epsilon\eta$ term in the height of the water surface boundary condition will cause issues when determining the equations for the first order of ϵ. To get rid of it we will Taylor expand p and w around $z_0 = 1 + \epsilon\eta$, giving:

$$p_0 + \epsilon\eta_0 \partial_z p_0 + \epsilon p_1 = \eta_0 + \epsilon\eta_1 + \mathcal{O}(\epsilon^2)$$

$$w_0 + \epsilon\eta_0 \partial_z w_0 + \epsilon w_1 = -\partial_\xi \eta_{0\xi} + \epsilon(-\partial_\xi \eta_1 + \partial_\tau \eta_0 + u_0 \partial_\xi \eta_0) + \mathcal{O}(\epsilon^2) \quad \text{on } z = 1$$

If these are put into the system of equations, the leading order, $n = 0$, will give us:

$$\partial_\xi u_0 = \partial_\xi p_0, \quad \partial_z p_0 = 0, \quad \partial_z w_0 = -\partial_\xi u_0, \quad w_0 = -\partial_\xi \eta_0 \text{ and } p_0 = \eta_0 \text{ on } z = 1$$

Clearly, since p_0 is equal to η_0 on $z = 1$, and independent of z, they must be equal in general. This leads to:

$$u_0 = p_0 = \eta_0, \quad w_0 = -z\partial_\xi \eta_0$$

This is not enough to give us an equation for η_0, so we must look further. The first order, $n = 1$, will give us:

$$-\partial_\xi u_1 + \partial_\tau u_0 + u_0 \partial_\xi u_0 + w_0 \partial_z u_0 = -\partial_\xi p_1, \quad \partial_\xi w_0 = \partial_z p_1, \quad \partial_\xi u_1 = -\partial_z w_1$$

$$\eta_0 \partial_z p_0 + p_1 = \eta_1, \quad \eta_0 \partial_z w_0 + w_1 = -\partial_\xi \eta_1 + \partial_\tau \eta_0 + u_0 \partial_\xi \eta_0 \quad \text{on } z = 1$$

$$w_1 = 0 \quad \text{on } z = 0$$

Since p_0 is independent of z, the pressure boundary condition becomes $p_1 = \eta_1$ on $z = 1$. When plugging in the value for w_0 into the second Euler equation above, we get the following expression for the first order pressure term:

$$\partial_z p_1 = -z\partial_\xi^2 \eta_0, \quad p_1 = \eta_1 \text{ at } z = 1 \Rightarrow p_1 = \eta_1 + (1/2 - 1/2z^2)\partial_\xi^2 \eta_0$$

Then, its partial derivative, with respect to ξ, becomes:

$$\partial_\xi p_1 = \partial_\xi \eta_1 + (1/2 - 1/2z^2)\partial_\xi^3 \eta_0$$

Turning our attention now to the first Euler equation, as well as the continuity equation, we can see that:

$$\partial_z w_1 = -\partial_\tau \eta_0 - \eta_0 \partial_\xi \eta_0 - w_0 \partial_\xi u_0 - \partial_\xi p_1$$

We know that the partial derivative with respect to z of u_0 equals zero, so, combined with the value found for p_1, this equation becomes:

$$\partial_z w_1 = -\partial_\tau \eta_0 - \eta_0 \partial_\xi \eta_0 - \partial_\xi \eta_1 - \partial_\xi^3 \eta_0 (1/2 - 1/2z^2)$$

Integrating this with respect to z, and evaluating at $z = 1$, we get:

$$w_1 = -\partial_\tau \eta_0 - \partial_\xi \eta_1 - 1/3\partial_\xi^3 \eta_0 - \eta_0 \partial_\xi \eta_0$$

We already what w_1 is from the boundary conditions. When plugging in the values for the leading order terms for w and u, we get:

$$w_1 = -\partial_\xi \eta_1 + \partial_\tau \eta_0 + 2\eta_0 \partial_\xi \eta_0$$

So, when putting these two together, we get:

$$-\partial_\xi \eta_1 + \partial_\tau \eta_0 + 2\eta_0 \partial_\xi \eta_0 = -\partial_\tau \eta_0 - \partial_\xi \eta_1 - 1/3\partial_\xi^3 \eta_0 - \eta_0 \partial_\xi \eta_0$$

Rewriting this, we get the KdV equation:

$$2\partial_\tau \eta_0 + 3\eta_0 \partial_\xi \eta_0 + 1/3\partial_\xi^3 \eta_0 = 0$$

This ends the derivation of the regular KdV equation.

1.9 Adding Coriolis Effect

The Coriolis effect is a consequence of defining a coordinate system on a rotating body. The effect defines a 'force' on a moving object on said body as described by the following equation: $\mathbf{F} = 2m(\omega \times \mathbf{v})$ perpendicular to both the velocity of the object, as the axis of rotation. The cross product here will make describing a one dimensional flow equation rather hard, since it will add a component perpendicular to the flow in all cases where this direction isn't parallel to the axis of rotation. Therefore, we will only use it to describe water waves around, and parallel to, the equator. The following derivation is taken from [2].

To add the Coriolis effect, another force needs to be added to the Euler equation in the beginning: $\mathbf{F}_c = 2(0, \omega, 0) \times (u, 0, w) = (2\omega w, 0, -2\omega u)$. This ω term will then be non-dimensionalized by rescaling it: $\omega \mapsto \frac{\sqrt{gh_0}}{h_0}\omega$, and then further rescaled by $\omega = \epsilon\omega_0$. Since the ω and ϵ seem to have a similar magnitude, it is reasonable to perform this rescaling.[2] Inserting this in the equations found in the assumptions section, we get:

$$\partial_t u + \epsilon(u\partial_x u + w\partial_z u + 2\omega_0 w) = -\partial_x p, \quad \epsilon(\partial_t w + \epsilon(u\partial_x w + w\partial_z w - 2\omega_0 u)) = -\partial_z p, \quad \partial_x u + \partial_z w = 0$$

$$p = \eta, \quad w = \partial_t \eta + \epsilon u \partial_x \eta \quad \text{on} \quad z = h = 1 + \epsilon\eta$$

$$w = 0 \quad \text{on} \quad z = 0$$

After changing to far-field coordinates, we get:

$$-\partial_\xi u + \epsilon(\partial u_\tau + u\partial_\xi u + w\partial_z u + 2\omega_0 w) = -\partial_\xi pi, \quad \epsilon(-\partial_\xi w + \epsilon(\partial_\tau w + u\partial_\xi w + w\partial_z w - 2\omega_0 u)) = -\partial_z p, \quad \partial_\xi u + \partial_z w = 0$$

$$p = \eta, \quad w = -\partial_\xi \eta + \epsilon(\partial_\tau \eta + u\partial_\xi \eta) \quad \text{on} \quad z = 1 + \epsilon\eta$$

$$w = 0 \quad \text{on} \quad z = 0$$

The descriptions of the leading order terms remain the same:

$$u_0 = p_0 = \eta_0, \quad w_0 = -z\partial_\xi \eta_0$$

The first order terms become:

$$-\partial_\xi u_1 + \partial_\tau u_0 + u_0\partial_\xi u_0 + w_0\partial_z u_0 + 2\omega_0 w_0 = -\partial_\xi p_1, \quad \partial_\xi w_0 = \partial_z p_1, \quad \partial_\xi u_1 = -\partial_z w_1$$

$$\eta_0\partial_z p_0 + p_1 = \eta_1, \quad \eta_0\partial_z w_0 + w_1 = -\partial_\xi \eta_1 + \partial_\tau \eta_0 + u_0\partial_\xi \eta_0 \quad \text{on } z = 1$$

$$w_1 = 0 \quad \text{on } z = 0$$

The only change to $\partial_z w_1$ is:

$$\partial_z w_1 = -\partial_\xi u_1 = -\partial_\tau \eta_0 - \eta_0\partial_\xi \eta_0 - \partial_\xi \eta_1 - \partial_\xi^3 \eta_0 (1/2 - 1/2z^2) + 2\omega_0\partial_\xi \eta_0$$

As done above, we integrate it with respect to z, and evaluate it at $z = 1$:

$$\partial_z w_1 = -\partial_\tau \eta_0 - \eta_0\partial_\xi \eta_0 - \partial_\xi \eta_1 - 1/3\partial_\xi^3 \eta_0 + 2\omega_0\partial_\xi \eta_0$$

When we put this together with the value for w_1 at the boundary, we get the following:

$$-\partial_\tau \eta_0 - \eta_0\partial_\xi \eta_0 - \partial_\xi \eta_1 - 1/3\partial_\xi^3 \eta_0 + 2\omega_0\partial_\xi \eta_0 = -\partial_\xi \eta_1 + \partial_\tau \eta_0 + 2\eta_0\partial_\xi \eta_0$$

which results in the final equation becoming a geophysical KdV equation:

$$2\partial_\tau \eta_0 - 2\omega_0\partial_\xi \eta_0 + 3\eta_0\partial_\xi \eta_0 + 1/3\partial_\xi^3 \eta_0 = 0$$

1.10 Transformation Between gKdV and KdV

Now that the derivations are done, we shall look to how these two could possibly relate.

If we were to assume some function η is a solution to the geophysical Korteweg-de Vries equation, then, if we were to consider the function $\tilde{\eta} = \eta - 2/3\omega_0$, the following can be seen:

$$2\partial_\tau \eta - 2\omega_0 \partial_\xi \eta + 3\eta_0 \partial_\xi \eta + 1/3\partial_\xi^3 \eta = 0$$

implies

$$2\partial_\tau \eta + (3\eta - 2\omega_0)\partial_\xi \eta + 1/3\partial_\xi^3 \eta = 0$$

implies

$$2\partial_\tau \tilde{\eta} + 3\tilde{\eta}\partial_\xi \tilde{\eta} + 1/3\partial_\xi^3 \tilde{\eta} = 0$$

since ω_0 is a constant. So, if η is a solution for the gKdV equation, then $\tilde{\eta} = \eta - 2/3\omega_0$ is a solution to the KdV equation.

Conversely, consider a function η, which is a solution to the KdV equation. Then, let $\tilde{\eta} = \eta + 2/3\omega_0$. Then:

$$2\partial_\tau \eta \partial_\xi \eta + 3\eta_0 \partial_\xi \eta + 1/3\partial_\xi^3 \eta = 0$$

implies

$$2\partial_\tau \eta + (3\eta + 2\omega_0 - 2\omega_0)\partial_\xi \eta + 1/3\partial_\xi^3 \eta = 0$$

implies

$$2\partial_\tau \tilde{\eta} - 2\omega_0 \partial_\xi \tilde{\eta} + 3\tilde{\eta}\partial_\xi \tilde{\eta} + 1/3\partial_\xi^3 \tilde{\eta} = 0$$

So, for a KdV solution η, $\eta + 2/3\omega_0$ is a solution for the gKdV equation.

In other words, the solution of the one equation, is a vertically shifted version of the solution of the other, with KdV$\rightarrow$gKdV shifting upwards by $2/3\omega_0$, and gKdV$\rightarrow$KdV shifting downwards by $2/3\omega_0$. There is a small problem here, which is related to the boundary conditions for ξ used with the Korteweg-de Vries equations when modeling waves. These assume that the waves go to zero, as ξ goes to (positive or negative) infinity. This will be addressed in chapter 3.

Chapter 2

gKdV and Lie Groups

In the first part, the Korteweg-de Vries equation, as well as its geophysical counterpart, have been described, and a transformation between their solutions has been described. In this chapter, these equations will be viewed in a different light. We will first define Lie groups, Lie algebras, and state the Euler-Arnold equation. This last equation is what will lay the link between these Lie Groups, and the KdV equation. We will then show how this Euler-Arnold equation works, with a relatively simple example. After that, we will use it to show which Lie Algebra corresponds to the KdV equation, and how this relates to the algebra, which corresponds to the gKdV equation.

2.1 Lie Groups

We should start with defining what a Lie group actually is. A Lie group is a smooth manifold which is also a group, which has a smooth operation. This definition has three components, all of which require more elaboration. Firstly, what is a smooth manifold?

A smooth manifold is a type of space which locally 'behaves' like a regular Euclidean space. This allows us to define a sense of differentiability on said space. To define this properly, we have to define a myriad different concepts.

Definition: A Hausdorff topological space is a space which satisfies the following property: for any two points, there exist two open sets, each of which includes one of those points, such that the sets are disjoint.[6]

This property might look irrelevant, but it is necessary for a space to have limits converge to just one point.

Definition: A homeomorphism $\phi : X \to Y$ is a bijective map, such that it, as well as its inverse, is continuous.[6]

Definition: Suppose we have a space M, with a subspace U, such that there exists a homeomorphism from $\phi : U \to V \subseteq \mathbb{R}^n$, with V an open subset, then we call ϕ a chart from M to R^n.[7]

With this, we can now compare continuous functions on our space, with continuous functions on a Euclidean space. We can also define a sense of differentiability on the subset U, by calling a function f smooth, if $\phi \circ f \circ \phi^{-1} : \mathbb{R}^n \supseteq V \to V$ is smooth.

To define smoothness on the whole space, we need it to be covered by subsets U, with charts to some space R^n. We do need to take into account that, for two of these subsets U_a, U_b which overlap, composing their charts in the following form: $\phi_a \circ \phi_b^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ should be smooth for the areas where the domains of these two intersect. This type of map is called a transition function.[7] Were this not the case, a function f could be smooth when using one chart, and not smooth when using the other, at the same points. Then smoothness would not be well defined. The smoothness of the transfer function implies that, if f is smooth when using one of the charts, it is also smooth using the other one.

Definition: A smooth atlas is a collection of subsets of a space which cover said space, together with charts of these subsets, such that their transition functions are smooth.[7]

This gives us all components to define smooth manifolds:

Definition: A smooth manifold M is a Hausdorff topological space with a smooth atlas. Its dimension is equivalent to the dimension of the Euclidean space which its charts map to, R^n.[7]

An example of this type of space is the sphere. Here, we can use stereographic projection to map all but one point on the sphere, namely, the pole from which we project, to the space R^2. We can then do the same from the pole on the other side of the sphere.

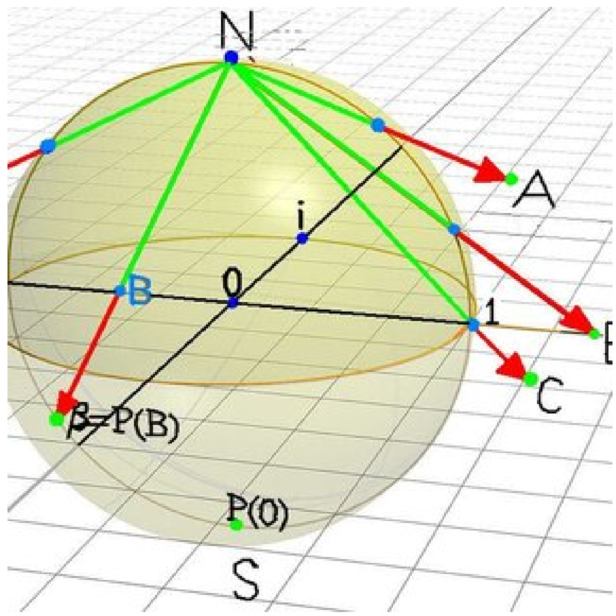


Figure 2.1: An illustration of the stereographic projection, using the north pole (0,0,1) as reference point.

Now that we have an idea of what a smooth manifold is, we can move on to the second aspect of Lie groups, that being, groups.

Definition[8]: A set G, together with operation $\circ$ is a group, if the following properties of $\circ$ are satisfied:

1): (Closure of the operation)For any two elements x, y of G, $x \circ y \in G$.

2): (Associativity) For any three elements $x, y, z \in G$, $(x \circ y) \circ z = x \circ (y \circ z)$

3): (Identity elements) There exists an element $e \in G$ such that, for all x, $e \circ x = x \circ e = x$

4): (Inverse) For all $x \in G$, there exists an element $a \in G$, such that $x \circ a = a \circ x = e$, where e is the identity element. We can say $a = x^{-1}$.

This makes the operation on G act like the familiar arithmetic operations such as addition and multiplication. The only property which is not required for G to be a group, is commutativity.

An example of a non-commutative group is the group of all invertible matrices of same dimension, with the matrix multiplication operation. This satisfies the properties mentioned above, but is not commutative. For the rest of this paper, we will describe this operation as a kind of multiplication, though it should not be confused for regular multiplication.

A Lie group is a smooth manifold, and a group. It has, however, one more property which must be mentioned, which is that the operation on the Lie group is smooth. That is, for any elements $a, x \in G$

where G is the Lie group, the map $(a, x) \mapsto a \circ x$ is smooth. With this, we can define lie groups the following way:

Definition: A Lie group is a group, which is also a smooth manifold, for which multiplication is a smooth map.[9]

An example of a Lie group is, again, the set of all invertible matrices of same dimension.[9]

There is one thing left to mention regarding the definition of Lie groups, and that is that the definition above is only valid for Lie groups of finite dimensions, since their charts all take their values in some Euclidean space, which are of finite dimension.

We can extend the definition of smooth manifolds to include spaces which have charts taking their values from a different vector space, like a Banach space, or, as will be used later this paper, a Fréchet space. This will lead to a loss of some properties, but we will not use these in this paper.

A Fréchet space is defined as a complete, hausdorff, locally convex, metrizable space.[10] This means the following properties are true for a Fréchet space:

1): (Complete) Every cauchy sequence in the space converges.[11]

2): (locally convex) Any neighborhood of any point in the space, contains a convex subset.[10]

3): (metrizable) There exists a metric, which yields the topology of the space.[12]

In essence, a Fréchet space is a weaker kind of Banach space, where no norm needs to exist. This becomes important in section 2.6, when the diffeomorphism group of the unit circle is used. This space is not a Banach space, but it is a Fréchet space.[10]

The smoothness of a Lie group's multiplication operation gives us a tool to map smoothly from one point on the Lie group, to its identity element, by simply multiplying with its inverse element. This will be very useful moving on.

If we want to define velocity, or a derivative on the manifold in general, we can describe this by using tangent vector spaces on each of the points on the manifold. To illustrate this, we can look at the sphere again. Here, we can associate an infinitesimal change in position with the tangent plane on the position.

The tangent space on the identity element of a Lie group is called the Lie algebra.[9] This Lie algebra has, besides an addition operation, also a multiplication operation, the so-called Lie bracket. This Lie bracket is determined by the multiplication operation of the Lie Group, but always has the following properties:

Given a Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra, the following properties hold:[10]

1):(Bilinearity) For any $X, Y, Z \in \mathfrak{g}$, and any scalars a, b, $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ and also: $[X, aY + bZ] = a[X, Y] + b[X, Z]$

2):(Skew-symmetry) For any $X, Y \in \mathfrak{g}$: $[X, Y] = -[Y, X]$

3):(Jacobi Identity) For any $X, Y, Z \in \mathfrak{g}$, $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

Some more things of note:
Most Lie brackets used in this paper will be of the form $[X, Y] = XY - YX$, but this is not the only form which a Lie bracket may take.
Any Lie group has a Lie algebra, and any finite dimensional Lie algebra has a Lie group. However, an infinite dimensional Lie algebra does not always correspond to a Lie group.[10]

2.2 Adjoint and Coadjoint Representation

To use the Euler-Arnold equation, one must also be familiar with the adjoint and coadjoint representations of Lie algebras.

The adjoint representation of a Lie algebra $\mathfrak{g}$ is a method to describe elements of a Lie algebra, as linear transformations of the algebra itself, i.e.: $\text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g})$, where $\text{End}(\mathfrak{g})$ is the group of endomorphisms on $\mathfrak{g}$.[10] Here, an endomorphism is simply a map from an algebra, to itself, which preserves the operations on the algebra. This representation of an element X, acting on an element Y, is given by the Lie bracket: $\mathrm{ad}_X(Y) = [X, Y]$.

The coadjoint representation of a Lie algebra $\mathfrak{g}$ is very similar to the adjoint representation, but instead of the elements being transformations of the algebra itself, they are transformations of the dual space of the algebra, i.e.: $\mathrm{ad}^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$. This representation is given by the following relation: $\langle \mathrm{ad}^*_X(\xi), Y\rangle = -\langle \xi, \mathrm{ad}_X(Y)\rangle$, with $X, Y \in \mathfrak{g}$, as before, and $\xi \in \mathfrak{g}^*$.[10]

This second representation will be used in the Euler-Arnold equation.

2.3 Central Extensions

A tool which we will use quite extensively in the latter parts of this chapter is the central extension. A central extension of a Lie algebra is a way of creating a bigger Lie algebra from it by 'adding' a vector space to it. A central extension by a vector space n gives a new Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \bigoplus n$, with the Lie bracket of the algebra being $[(X, v), (Y, w)] = [X, Y] \bigoplus \omega(X, Y)$, where $v, w \in n$.

The map $\omega : \mathfrak{g} \times \mathfrak{g} \to n$ must be a bilinear map, which is skew-symmetric, and satisfies the identity

$$\omega([X, Y], Z) + \omega([Z, X], Y) + \omega([Y, Z], X) = 0 \tag{2.1}$$

These properties are to preserve the requirements for the bracket to be a Lie bracket, and thus, for the algebra to be a Lie algebra. Such a map is called a 2-cocycle. It's quite easy to see from these requirements that two cocycles added together is a new cocycle, given their bilinearity.[10]

One thing we can observe is that the vector space elements don't have any influence on the Lie bracket.

A special type of 2-cocycle is the 2-coboundary. A 2-coboundary $d_\alpha : \mathfrak{g} \times \mathfrak{g} \to n$ of the form $d_\alpha(X, Y) = \alpha([X, Y])$, where $\alpha : \mathfrak{g} \to n$ is a linear map. These 2-coboundaries are special, because when adding them to another cocycle, the Lie algebra you get is isomorphic to the Lie algebra you would have gotten without adding the coboundary. This will be important in section 2.8.

Isomorphism:

Suppose $\mathfrak{g}_\omega = \mathfrak{g} \oplus n$ is a central extension of $\mathfrak{g}$ with 2-cocycle ω, and $\mathfrak{g}_{\omega + d_\alpha} = \mathfrak{g} \oplus n$ one with 2-cocycle $\omega + d_\alpha$, where d_α is a coboundary, then these two algebras are isomorphic.

Proof:

Let $\phi_\alpha : \mathfrak{g} \oplus n \to \mathfrak{g} \oplus n$ be as follows: $\phi_\alpha((X, v)) = (X, v + \alpha(X))$. Let $[(X, v), (Y, w)]_\omega = [X, Y] \oplus \omega(X, Y)$ and $[(X, v), (Y, w)]_{\omega+d_\alpha} = [X, Y] \oplus (\omega(X, Y) + d_\alpha(X, Y))$

To show:

1): $\phi_\alpha(aX + bY) = a\phi_\alpha(X) + b\phi_\alpha(Y)$, for all $X, Y \in \mathfrak{g} \oplus n$ and a, b scalars.

2): ϕ_α is bijective.

3): $\phi_\alpha([X, Y]_\omega) = [\phi_\alpha(X), \phi_\alpha(Y)]_{\omega+d_\alpha}$

1): $a(X, w) = (aX, aw)$ and $(X, v) + (Y, w) = (X + Y, v + w)$ Then $\phi_\alpha(a(X, v) + b(Y, w)) = \phi_\alpha((aX + bY, av + bw)) = (aX + bY, av + bw + \alpha(aX + bY)) = (aX, a(v + \alpha(X))) + (bY, b(w + \alpha(Y))) = a\phi_\alpha((X, v)) + b\phi_\alpha((Y, w))$

2): Let (Y, w) be some arbitrary element in $\mathfrak{g} \oplus n$, then there should exist an element (X, w) in $\mathfrak{g} \oplus n$, such that $\phi_\alpha((X, v)) = (Y, w)$. This is true, if $X = Y$, and $w = v + \alpha(X)$. So, $(X, v) = (Y, w - \alpha(Y))$ satisfies these conditions. So ϕ_α is surjective.

Suppose there is a different element (Z, u) that maps to (Y, w), then we know that $Z = Y = X$ and $u + \alpha(Z) = w = v + \alpha(X)$, so $u = v$. So there exists only one element that maps to (Y, w), so ϕ_α is injective, and as such, bijective.

3):$\phi_\alpha([(X, v), (Y, w)]_\omega) = \phi_\alpha(([X, Y], \omega(X, Y))) = ([X, Y], \omega(X, Y) + \alpha(X, Y)) = [\phi_\alpha(X), \phi_\alpha(Y)]_{\omega+d_\alpha}$

So, the map ϕ_α is an isomorphism. □

2.4 Euler-Arnold

Now, the final thing to address before we can formulate the Euler-Arnold equation, is how to define distance on the Lie Group. The metric one uses in this case is called a Riemannian metric.

A Riemannian metric is essentially a choice of inner product for the tangent spaces of each point in the manifold. This inner product can vary from tangent space to tangent space, however.

For the Euler-Arnold equation, we need metrics which are left-invariant. This means that, for any r, x, y, the metric g satisfies $g(rx, ry) = g(x, y)$. With this property, we can pull back vectors from the tangent spaces of the point g to the Lie algebra via left-hand multiplication. Then, for example, if we wanted to determine the inner product of vector v in the tangent space of point g with itself, we could simply use the fact that $(v, v)_g = (g^{-1}v, g^{-1}v)_e$, and use the inner product at the Lie algebra. This inner product at the Lie algebra can be defined by $(v, w) = \langle Av, w\rangle$, where $A : \mathfrak{g} \to \mathfrak{g}^*$ is an invertible map between an algebra and its dual, and $\langle\cdot,\cdot\rangle$ is the pairing between them.

The Euler-Arnold Equation:

Finally, we have the tools to discuss the Euler-Arnold equation. The Euler-Arnold equation is essentially the equation of motion on the Lie Group. Specifically, it is a rewritten version of the principle of least action, with lagrangian $L(g, v) = 1/2(v, v)_g$, where $(\cdot,\cdot)_g$ is the inner product at g, given by the metric, as described above, of the tangent vectors of g. This can be pulled back to by way of left-hand multiplication, and can be rewritten as a differential equation for some element of the dual space of the Lie algebra, specifically, the element $m(t) = A(g^{-1}(t)g'(t))$, with $g(t)$ being the path of least action. For the left-invariant metric on a group generated by an inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$, the Euler-Arnold equation assumes the form

$$m'(t) = -\mathrm{ad}^*_{A^{-1}m(t)}m(t) \tag{2.2}$$

on the dual space $\mathfrak{g}^*$. The proof for this can be found in the book by Khesin and Wendt[10].

2.5 Lie group SO(3)

To familiarize ourselves with the Euler-Arnold equation, let's first use it with a finite dimensional Lie group in mind. The derivation of the Euler top equations of motion below has been adapted from [10].

The Special Orthogonal group (SO(n)) is the group describing rotational transformations in n dimensions. Then, SO(3) is the group of all 3 dimensional rotations. This group shall lead to a system of differential equations for rotating rigid bodies.

The group SO(3) is a three-dimensional smooth manifold. As a consequence, it is a Lie group. Its Lie algebra, $\mathfrak{so}(3)$, can be shown to be the vector space of skew-symmetric matrices, with the Lie bracket: $[A, B] = AB - BA$. It can be shown that this algebra is isomorphic with the algebra $\mathbb{R}^3$, using the cross product for the Lie bracket: $\mathrm{ad}_X(Y) = [X, Y] = X \times Y$.

In this new representation, we can quickly see that its dual algebra is just $\mathbb{R}^3$ as well. Since the pairing in this case is the same as the dot product between two vectors, the coadjoint representation of the Lie algebra is given by:

$$\langle \mathrm{ad}^*_X(\xi), Y\rangle = -\langle \xi, \mathrm{ad}_X(Y)\rangle = -\xi \cdot (X \times Y) = (X \times \xi) \cdot Y$$

So, $\mathrm{ad}^*_X(\xi) = X \times \xi$.

Given $(X, Y)_e = \langle IX, Y\rangle$, with $I : \mathfrak{so}(3) \to \mathfrak{so}^*(3)$ being the inertia tensor, in this case $I = \mathrm{diag}(I_1, I_2, I_3)$, we find the following Euler-Arnold equation:

$$m'(t) = -\mathrm{ad}^*_{I^{-1}m(t)}(m(t)) = -(I_1^{-1}m_1, I_2^{-1}m_2, I_3^{-1}m_3) \times (m_1, m_2, m_3)$$

From this, we can retrieve

$$m_1' = (I_3 - I_2)m_2m_3$$

$$m_2' = (I_1 - I_3)m_3m_1$$
$$m_3' = (I_2 - I_1)m_1m_2$$

which are the equations of motion for the Euler top.[10]

As seen here, the process of deriving the Euler-Arnold equation for a manifold mostly comes down to determining the coadjoint action.

2.6 Diffeomorphisms of the Circle

From this point forward, only infinite-dimensional Lie groups and algebras are discussed. However, the Euler-Arnold equation still works for the Frechét Lie groups used from this point forward.

The derivation of the KdV equation below has been adapted from [10].

The first Lie group to be studied is the diffeomorphism group of the circle. A diffeomorphism is a bijective map between two manifold, which is smooth, and has a smooth inverse. It can be seen as a sort of smooth homeomorphism. In this case, it maps from the circle to itself. We denote this by $G = \text{diff}(S^1)$. Its Lie algebra is given by the algebra of smooth vector fields on the circle, $\mathfrak{g} = \text{Vect}(S^1)$.[10]

The concept of a smooth vector field requires a bit more elaboration. A vector field is essentially a collection of vectors from the tangent spaces of the manifold, where each point has exactly one vector in the field. A smooth vector field has these vectors transition smoothly from point to point. The vector fields can be described as $f(\theta)\partial_\theta$. Here, $f : [-\pi, \pi] \to \mathbb{R}$ is a smooth, 2π-periodic function (to preserve continuity over the circle), which any order derivatives are also 2π-periodic (to preserve smoothness). This gives us the following Lie bracket:

$$[f(\theta)\partial_\theta, g(\theta)\partial_\theta] = f(\theta)\partial_\theta g(\theta)\partial_\theta - g(\theta)\partial_\theta f(\theta)\partial_\theta = (f(\theta)g'(\theta) - g(\theta)f'(\theta))\partial_\theta$$

We note that the Lie algebra $\text{Vect}(S^1)$ has the smooth dual space $\{u(\theta)(d\theta)^2\}$[10], the space of quadratic differentials, with the following pairing:

$$\langle u(\theta)(d\theta)^2, f(\theta)\partial_\theta\rangle = \int_{S^1} f(\theta)u(\theta)d\theta$$

We specify the smooth dual space, since any element which differs from an element in the smooth dual at countable points, would still be in the dual of the algebra. These functions, however, have no impact on the inner product, since their non-equal points would have a measure of zero. To determine the coadjoint representation, we use the following arbitrary elements: $X = f(\theta)\partial_\theta$, $Y = g(\theta)\partial_\theta$, and $\xi = u(\theta)(d\theta)^2$. Putting them into the coadjoint representation relation, we get:

$$\langle \text{ad}^*_X(\xi), Y\rangle = -\langle\xi, [X, Y]\rangle = -\int_{S^1} u(\theta)(f(\theta)g'(\theta) - f'(\theta)g(\theta))d\theta$$

Rewriting this, we find two parts, one where Y ($g(\theta)$) remains untouched (the left part), and one where it is differentiated:

$$\langle \text{ad}^*_X(\xi), Y\rangle = \int_{S^1} f'(\theta)u(\theta)g(\theta)d\theta - \int_{S^1} f(\theta)u(\theta)g'(\theta)d\theta$$

We want to separate the Y part from the rest, as we did with the $SO(3)$ case, so we can find the coadjoint action. Using integration by parts on the right integral, we find:

$$\int_{S^1} f'(\theta)u(\theta)g(\theta)d\theta + \int_{S^1} (f(\theta)u(\theta))'g(\theta)d\theta = \int_{S^1} (2f'(\theta)u(\theta) + f(\theta)u'(\theta))g(\theta)d\theta$$

Then, we can see that the coadjoint action is given by $\text{ad}^*_{f(\theta)\partial_\theta}(u(\theta)(d\theta)^2) = (2f'(\theta)u(\theta) + f(\theta)u'(\theta))(d\theta)^2$

Using the identity mapping as inertia tensor $I(f(\theta)\partial_\theta) = f(\theta)(d\theta)^2$, we get the following Euler-Arnold equation:

$$\partial_t m(\theta, t)(d\theta)^2 = -\text{ad}^*_{m(\theta.t)\partial_\theta}(m(\theta, t)(d\theta)^2) = -3m(\theta, t)\partial_\theta m(\theta, t)(d\theta)^2$$

Then,

$$\partial_t m(\theta, t) + 3m(\theta, t)\partial_\theta m(\theta, t) = 0$$

which is the (inviscid) burgers' equation, after some rescaling of m.

2.7 Virasoro Algebra and the Korteweg-de Vries Equation

We use a central extension on the diffeomorphism group to change this burger's equation to the desired Korteweg-de Vries equation. It goes as follows:

The Virasoro Algebra is the Lie algebra of the Virasoro-Bott Group, which is a central extension of the diffeomorphism group from the section above. For the Lie algebra, the central extension is given by the following 2-cocycle:[10]

$$\omega(f(\theta)\partial_\theta, g(\theta)\partial_\theta) = \int_{S^1} f'(\theta)g''(\theta)d\theta$$

ω is clearly bilinear, as well as skew-symmetric, as can be shown by using integration by parts. It also satisfies the Jacobi Identity. Using integration by parts (twice) to get it in the right form for our purposes, as we did above, we find

$$\int_{S^1} f'''(\theta)g(\theta)d\theta$$

So, the Virasoro algebra, $\mathfrak{vir}$, is the vector space $\mathrm{Vect}(S^1) \oplus \mathbb{R}$, with Lie bracket $\mathrm{ad}_{(X,a)}((Y,b)) = ([X,Y], \omega(X,Y))$, where $a, b \in \mathbb{R}$.

The dual space of this algebra has, as might be expected, the elements $\xi = (u(\theta)(d\theta)^2, c)$, with $c \in \mathbb{R}$ and the pairing between the two is given by

$$\langle (u(\theta)(d\theta)^2, c), (f(\theta)\partial_\theta, a)\rangle = \int_{S^1} f(\theta)u(\theta)d\theta + a \cdot c$$

The coadjoint representation is then given by:

$$\langle \mathrm{ad}^*_X(\xi), Y\rangle = -\langle \xi, \mathrm{ad}_X(Y)\rangle = -\langle u(\theta)(d\theta)^2, [f(\theta)\partial_\theta, g(\theta)\partial_\theta]\rangle + c \cdot \omega(f(\theta)\partial_\theta, g(\theta)\partial_\theta)$$

We know both the left term, which is the same as the original algebra, and the right term, which is a rescaled cocycle. Adding them together, we get the following:

$$\langle \mathrm{ad}^*_X(\xi), Y\rangle = \int_{S^1} (2f'(\theta)u(\theta) + f(\theta)u'(\theta) + c \cdot f'''(\theta))g(\theta)d\theta$$

and, the coadjoint representation becomes

$$\mathrm{ad}^*_{(f(\theta)\partial_\theta, a)}((u(\theta)(d\theta)^2), c) = ((2f'(\theta)u(\theta) + f(\theta)u'(\theta) + cf'''(\theta))(d\theta)^2, 0)$$

Then, using the same inertia tensor as previous section, we find the new Euler-Arnold equation:

$$\partial_t(m(\theta,t)(d\theta)^2, c) = -\mathrm{ad}^*_{(m(\theta)\partial_\theta, c)}((m(\theta)(d\theta)^2, c)) = ((-3m(\theta)\partial_\theta m(\theta,t) - c\partial^3_\theta m(\theta,t))(d\theta)^2, 0)$$

which gives us

$$\partial_t m + 3m\partial_\theta m + c\partial^3_\theta m = 0$$

and

$$c' = 0$$

which is the KdV equation.

Here ends the derivation from the book. If we use $c = 1/6$, and $m = 1/2\eta$, we get it in the same form as in chapter 1.

2.8 A coboundary to add Coriolis

In order to turn this into the gKdV-equation, one can add a coboundary to the cocycle.

If we use the linear map

$$\alpha(f(\theta)\partial_\theta) = a\int_{S^1} f(\theta)d\theta$$

we find the new cocycle

$$\omega(X,Y) = \int_{S^1} f'''(\theta)g(\theta)d\theta + a\int_{S^1}(f(\theta)g'(\theta) - f'(\theta)g(\theta))d\theta$$

Using integration by parts to remove the unwanted differentiation of g, we get:

$$\omega(X,Y) = \int_{S^1} f'''(\theta)g(\theta)d\theta - 2a\int_{S^1} f'(\theta)g(\theta)d\theta$$

Then, we see that the coadjoint action of this Lie algebra is just the previous one, with one term added to it:

$$\mathrm{ad}^*_{(f(\theta)\partial_\theta,a)}((u(\theta)(d\theta)^2),c) = ((2f'(\theta)u(\theta) + f(\theta)u'(\theta) + cf'''(\theta) - 2caf'(\theta))(d\theta)^2, 0)$$

and, as a consequence, the Euler-Arnold equation becomes:

$$\partial_t(m(\theta,t)(d\theta)^2,c) = -\mathrm{ad}^*_{(m(\theta,t)\partial_\theta,c)}((m(\theta,t)(d\theta)^2,c)) = ((-3m(\theta)\partial_\theta m(\theta,t) - \partial^3_\theta cm(\theta,t) + 2ca\partial_\theta m(\theta,t))(d\theta)^2, 0)$$

Which brings us to the following two equations:

$$\partial_t m + 3m\partial_\theta m - 2ca\partial_\theta m + c\partial^3_\theta m = 0$$

with c, a constants.

Let $c = 1/6$, $\eta = m/2$, and $a = 3\omega_0$. Then, the equation becomes the gKdV equation, as seen in the part before:

$$2\partial_t\eta + 3\eta\partial_\theta\eta - 2\omega_0\partial_\theta\eta + \frac{1}{3}\partial^3_\theta\eta = 0$$

Note that the algebra here is isomorphic to the Virasoro algebra itself. The question is, if this algebra isomorphism has some relation to the change of coordinates, which lets us transform between the two equations, as seen in the first chapter.

2.9 Non-Constant Term a

Suppose that, instead of a being constant, our coboundary would be:

$$\alpha(f(\theta)\partial_\theta) = \int_{S^1} a(\theta)f(\theta)d\theta$$

Then, the new cocycle would have another term, due to the integration by parts:

$$\omega(X,Y) = \int_{S^1} f'''(\theta)g(\theta)d\theta - 2\int_{S^1} a(\theta)f'(\theta)g(\theta))d\theta - \int_{S^1} a'(\theta)f(\theta)g(\theta)d\theta$$

Then, the coadjoint action would be:

$$\mathrm{ad}^*_{(f(\theta)\partial_\theta,a)}((u(\theta)(d\theta)^2),c) = ((2f'(\theta)u(\theta) + f(\theta)u'(\theta) + cf'''(\theta) - 2ca(\theta)f'(\theta) - 2ca'(\theta)f(\theta))(d\theta)^2, 0)$$

So, the new equation would be:

$$\partial_t m + 3m\partial_\theta m - 2ca\partial_\theta m + c\partial^3_\theta m - 2cfa' = 0$$

Note that this is neither a non-constantly shifted version of the Korteweg-de Vries equation, nor what a non-constant ω_0 would turn the gKdV equation into.

Instead, a non-constant ω_0 would simply result in the same gKdV equation as a constant omega:

$$2\partial_t\eta + 3\eta\partial_\theta\eta - 2\omega_0(\theta)\partial_\theta\eta + \frac{1}{3}\partial^3_\theta\eta = 0$$

For a non-constant shift by ω_0, one would get:

$$2\partial_t\eta + 3\eta\partial_\theta\eta - 2\omega_0\partial_\theta\eta - 2\eta\partial_\theta\omega_0 + 2\omega_0\partial_\theta\omega_0 + \frac{1}{3}\partial_\theta^3\eta - \frac{2}{9}\partial_\theta^3\omega_0 = 0$$

Clearly, neither of these are the same equation. This isn't too worrying, when considering physical systems, since the angular momentum of the reference frame isn't dependent on the position, but it does show that neither of the transformations, both the one which follows from the coordinate change found in 1.10, as the isomorphism between the two algebras, work for non-constant a like it does for the constant case.

2.10 Relation Between the Two Results

To see if any relation might be afoot, we can look at the isomorphism between the two: $\phi_\alpha((f(\theta)\partial_\theta, c)) = (f(\theta)\partial_\theta, c + \int_{S^1} f(\theta)d\theta)$. We could begin with questioning whether this isomorphism on the Lie algebra, has a dual on its dual space. Suppose it does: $\langle\phi_\alpha^*(\xi), X\rangle = \langle\xi, \phi_\alpha(X)\rangle$

Then, if we write this out, we get

$$\langle\xi, \phi_\alpha(X)\rangle = \int_{S^1} u(\theta)f(\theta)d\theta + a(c + \alpha\int_{S^1} f(\theta)d\theta) = \int_{S^1}(u(\theta) + \alpha a)d\theta + ac$$

So: $\phi_\alpha^*((u(\theta)(d\theta)^2, a)) = ((u(\theta) + \alpha a)(d\theta)^2, a)$

We can see that, in the case of constant a, the algebra isomorphism's dual is just a shift up in the dual.

We must then make note that this isomorphism does not preserve the metric:

$$\langle((u(\theta)+\alpha a)(d\theta)^2, a), ((f(\theta)+\alpha c)\partial_\theta, c)\rangle = \int_{S^1}(f(\theta)+\alpha c)(u(\theta)+\alpha a)d\theta + ac \neq \langle(u(\theta)(d\theta)^2, a), (f(\theta)\partial_\theta, c)\rangle$$

Then, since the Euler-Arnold equation depends on the metric given map $A : \mathfrak{g} \to \mathfrak{g}^*$, we cannot simply plug the isomorphism into the Euler-Arnold equation, since, due to the metric being different, this would give us another equation. Therefore, we need to find the correct metric via the pullback of the isomorphism:

$$(\xi, \zeta)_{new} = (\phi^{*-1}(\xi), \phi^{*-1}(\zeta))_{old}$$

Here, we will use $\xi = (m(\theta)d\theta^2, a)$ and $\zeta = (n(\theta)d\theta^2, b)$ The inverse of ϕ is given by $\phi^{*-1}(m(\theta)d\theta^2, a) = ((m(\theta) - a)d\theta^2, a)$ So, the new metric will be:

$$(X, Y) = \int_{S^1}(m(\theta) - \alpha a)(n(\theta) - \alpha b)d\theta + ab$$

All that has to be done now, is to rewrite this in the form of $\langle X, AY\rangle$.

Rewriting this metric, we get:

$$\int_{S^1}(m(\theta) - \alpha a)n(\theta)d\theta + ab - \alpha b\int_{S^1}(m(\theta) - \alpha a)d\theta$$

This equates to:

$$\langle((m(\theta) - \alpha a)d\theta^2, a - \alpha(m(\theta) - \alpha a), (n(\theta)d\theta^2, b)\rangle = \langle A^{-1}\xi, \theta\rangle$$

So, we can see that, when we pull the new space back to the old one, the metric is given by the following operator:

$$A^{-1}(\xi) = (m(\theta) - \alpha a)d\theta^2, a - \alpha(m(\theta) - \alpha a))$$

Note that, because of the addition and subtraction, due to the isomorphism and the metric, we get the following term for Euler-Arnold:

$$A^{-1}\phi_\alpha^*((u(\theta)(d\theta)^2, a)) = (u(\theta), a - \alpha(m(\theta)))$$

Now, suppose we have a function If we then put this all together, into the Euler-Arnold equation, we get the following:

$$-\mathrm{ad}^*_{A^{-1}\phi^*(m)}\phi^*(m) = (-2(m+\alpha a)\partial_\theta m - m\partial_\theta m - a\partial_\theta^3 m)$$

This does not result in the Euler-Arnold equation we expect, since it returns the wrong sign for the coriolis term. Interestingly, were we to use the inverse isomorphism, where $-\alpha$ is used instead of α, the Euler-Arnold equation would return the right equation. This is, because the approach above is wrong.

In the approach above, one asks for a value of m, in the original dual space, such that that the image of m satisfies the regular KdV equation, instead of the gKdV equation. And, indeed, if we want to gain a function n in the new dual space, such that $\phi^{*-1}(n)$ satisfies the KdV equation, we get the gKdV equation.

So, if some m in the dual space satisfies the KdV equation, then its image in the new dual space satisfies the gKdV equation.

We must observe, however, that, even though this isomorphism changes the functions in the same way as the transformation from chapter 1, that being a constant vertical shift, it does so differently. If we use the values used in section 2.8 which gave us the original gKdV equation, $a = 1/6$, $\alpha = 3\omega_0$, $\eta = 1/2m$, then we get a vertical shift ω_0, instead of $2/3\omega_0$. This isn't too cataclysmic a discovery, since we aren't doing the same thing as in chapter 1, so these don't contradict each other. It does, however, confirm that they are not the same transformation. They are, however, linearly related, since one is always 3/2 times larger than the other. Whether this is coincidental, or hints to some underlying relation, remains to be discovered.

Chapter 3

Solutions of the KdV and gKdV Equation

Now, we will look at solutions to the KdV equation, and how the Coriolis effect will affect them. Firstly, we will only look to traveling wave solutions, in other words, equations of the form $f(\xi - c\tau) = f(\chi)$. Here, we must note that c isn't a regular velocity value, since it is dimensionless. We must also take note of the fact that these aren't the only type of solutions to the Korteweg-de Vries equation. If we rewrite the KdV equation with respect to this new, only variable, $\partial_\xi = \frac{d}{d\chi}$ and $\partial_\tau = -c\frac{d}{d\chi}$, which will result in the following rewritten gKdV equation:

$$-2(c+\omega_0)\eta' + 3\eta\eta' + 1/3\eta''' = 0$$

Note that this would become the regular KdV equation at $\omega_0 = 0$. If we integrate this, we can remove the third order derivative, and get the following equation:

$$-2(c+\omega_0)\eta + 3/2\eta^2 + 1/3\eta'' = 1/3A$$

With A some integration constant. These terms can't be integrated as easily as the ones above, due to the η terms on the right hand side. However, if we multiply this equation by η', the equation becomes

$$-2(c+\omega_0)\eta\eta' + 3/2\eta^2\eta' + 1/3\eta'\eta'' = 1/3A\eta'$$

This can be integrated again, which results in the following equation:

$$-(c+\omega_0)\eta^2 + 1/2\eta^3 + 1/6(\eta')^2 = 1/3A\eta + 1/3B$$

with B another integration constant. Rewriting this, we get the following equation:

$$1/2(\eta')^2 = -3/2\eta^3 + 3(c+\omega_0)\eta^2 + A\eta + B$$

We will use this form of the equation for the derivation of the solutions in the rest of this chapter.

3.1 Solitary Waves

A known solution of the KdV equation is that of the solitary wave. A solitary wave is a singular wave, instead of a periodic wave front. An example would be a tsunami. Such a soliton solution is of the form $a\text{sech}^2(b\chi)$. If we put this into the equation, we get the following:

$$1/2(-2ab\tanh(b\chi)\text{sech}^2(b\chi))^2 = -3/2a^3\text{sech}^6(b\chi) + 3a^2(c+\omega_0)\text{sech}^4(b\chi) + Aa\text{sech}^2(b\chi) + B$$

The left-hand term here is equal to the following:

$$2a^2b^2\text{sech}^6(b\chi)(\cosh^2(b\chi)-1)) = 2a^2b^2(\text{sech}^4(b\chi)-\text{sech}^6(b\chi))$$

Note that, for this form of solution to be able to exist, the quadratic and constant term on the right-hand side should be zero for any χ. Then, for A=B=0, we get the following requirements for a and b, for the function to satisfy the KdV equation:

$$2a^2b^2 = 3a^2(c+\omega_0), \quad 2a^2b^2 = 3/2a^3$$

From this, we get the values of a and b, which give a solution to the gKdV equation:

$$b = \sqrt{3/2(c+\omega_0)}, \quad a = 2(c+\omega_0)$$

This gives us the following solution to the gKdV equation:

$$\eta_1(\xi,\tau) = 2(c+\omega_0)\text{sech}^2(\sqrt{3/2(c+\omega_0)}(\xi - c\tau)) \tag{3.1}$$

This solution was also found in the original paper on the gKdV equation.[2]

We have also seen, in section 1.10, that, for any solution of the regular KdV equation, a translation upwards also give a solution to the KdV equation. Since equation 3.1 is a solution to the KdV equation if $\omega_0 = 0$, we can construct a different solitary wave solution for the gKdV equation:

$$\eta_2(\xi,\tau) = 2/3\omega_0 + 2c\cdot\text{sech}^2(\sqrt{3/2c}(\xi - c\tau)) \tag{3.2}$$

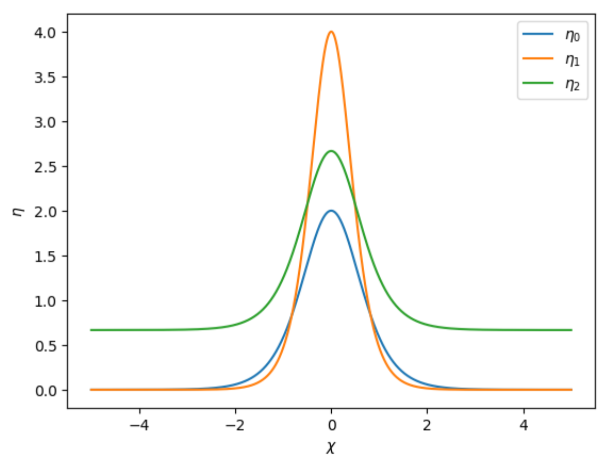


Figure 3.1: Two solutions to the gKdV equation, compared to the solution to the KdV equation, from which they were constructed. Here, $\omega_0 = c = 1$

The effect of the shift is not hard to distinguish, and the other solution narrows and heightens the wave.

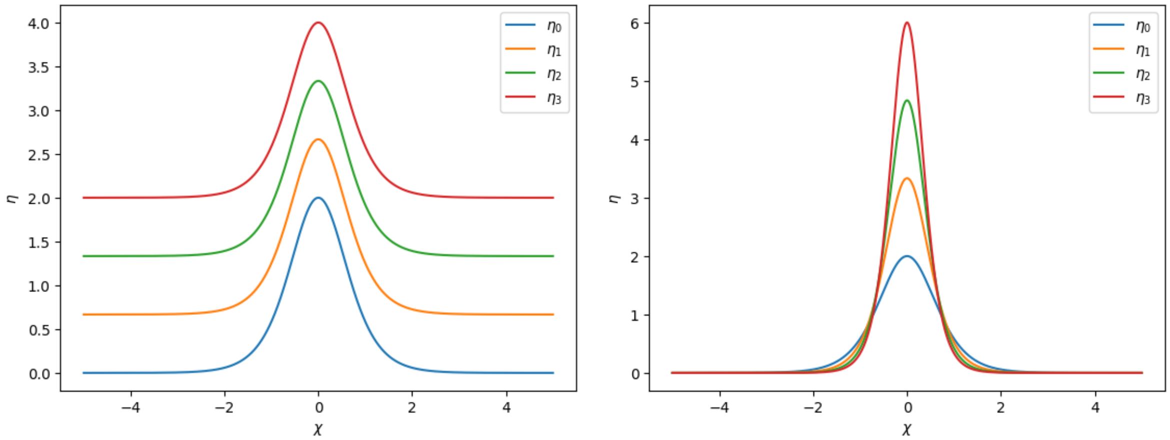


Figure 3.2: Here, the changes as a result of different values for ω_0 are visualized. On the right, the changes to the shape of the function, due to the Coriolis effect. Here, ω_0 is increased by steps of $2/3$. On the left, the upwards shift due to Coriolis, with ω_0 increasing in steps of 1. In both cases, $c = 1$.

There is one more thing of note, which was hinted at at the end of chapter 1. That being, the boundary conditions of ξ. When describing waves with the KdV equation, we have the function go to zero, as ξ goes to infinity. If this property needs to be satisfied for the geophysical variant, simply shifting a physical solution of the regular KdV equation will not yield a physical solution for the gKdV. However, a non-physical solution of the regular equation might still yield a physical solution to the geophysical equation, when shifted.

For example, the equation

$$2(c + \omega_0)\text{sech}^2(\sqrt{3/2(c + \omega_0)}(\xi - c\tau)) - 2/3\omega_0$$

is a solution to the KdV equation, though one not satisfying the regular boundary conditions. If shifted upwards, we get the previously discovered solution of the gKdV equation, which does satisfy the boundary conditions.

3.2 Jacobi Elliptic Functions

A more general form of this type of solutions is that of cnoidal waves, which are of the form:

$$\eta(\chi) = \gamma + \alpha \text{cn}^2(\beta\chi, m)$$

Here, cn is one of the Jacobi elliptic functions.[13] These functions are the equivalent to trigonometric functions, but for ellipses instead of circles. The function $\text{cn}(x, m)$ is the equivalent of cosine. Here, m describes the eccentricity of the ellipse it would describe, and is, for an ellipse, between 0 and 1. At zero, such an ellipse would be a circle, and at one, it seizes to be an ellipse, and instead is a parabola. We see then, that, for $0 \leq m < 1$, the function $\text{cn}(x, m)$ is periodic. The case for $m = 1$ is unique, and will be discussed in the next section.

α, β, γ are constants yet to be determined. Putting this in

$$1/2(\eta')^2 = -3/2\eta^3 + 3(c + \omega_0)\eta^2 + A\eta + B$$

then, on the left hand, we get:

$$1/2(-2\alpha\beta \text{dn}(\beta\chi, m)\text{sn}(\beta\chi, m)\text{cn}(\beta\chi, m))^2 = 2\alpha^2\beta^2\text{dn}^2(\beta\chi, m)\text{sn}^2(\beta\chi, m)\text{cn}^2(\beta\chi, m)$$

Using the following identities:[13]

$$\text{cn}^2(\beta\chi, m) = 1 - \text{sn}^2(\beta\chi, m), \quad \text{dn}^2(\beta\chi, m) = 1 - m \cdot \text{sn}^2(\beta\chi, m)$$

and $y = sn^2(\beta\chi, m)$, we get the following:

$$2\alpha^2\beta^2 y(1-my)(1-y)$$

Since $\eta(\chi) = \gamma + \alpha \text{cn}^2(\beta\chi, m) = \gamma + \alpha(1-y)$, we have $y = 1 - \frac{\eta-\gamma}{\alpha}$. If we plug this in, we get the following left-hand side of the equation:

$$-2\alpha^2\beta^2 m(\frac{1}{m} - y)(1-y)(0-y) = -2\frac{\beta^2 m}{\alpha}(\eta - (\alpha - \frac{\alpha}{m}))(\eta-\gamma)(\eta - (\gamma+\alpha))$$

This implies that, since α, β, γ, and m are real, so must be the roots of the right-hand side. We shall rewrite the polynomial on the right-hand side in terms of its roots:

$$-2\frac{\beta^2 m}{\alpha}(\eta - (\alpha - \frac{\alpha}{m} + \gamma))(\eta-\gamma)(\eta-(\gamma+\alpha)) = -3/2(\eta - r_1)(\eta - r_2)(\eta - r_3)$$

Then, we can write down the following equations, which have to be satisfied:

$$-2\frac{\beta^2 m}{\alpha} = 3/2, \quad r_1 = \gamma + \alpha, \quad \gamma = r_2, \quad r_3 = \alpha + \gamma - \frac{\alpha}{m}$$

Here, which roots are equal to which values isn't relevant, and the order here has been chosen for reasons which will be revealed later. Then, from this, we find the following values:

$$\gamma = r_2, \quad \alpha = r_1 - r_2, \quad \beta = \sqrt{\frac{3}{4}\frac{\alpha}{m}}$$

Furthermore, we note that:

$$\frac{\alpha}{m} = \eta_1 - \eta_3$$

And, as such:

$$m = \frac{r_1 - r_2}{r_1 - r_3}, \quad \beta = \sqrt{\frac{3}{4}(r_1 - r_3)}$$

Here, we see that, since we only use real values, $r_1 \geq r_3$. We also know that, since $0 \leq m \leq 1$, $r_1 - r_2 \leq r_1 - r_3$, which implies that $r_3 \leq r_2 \leq r_1$ (hence the chosen order). We must also make note that $r_3 < r_1$, to prevent division by zero. This is not all too important, since if they were the same, amplitude of the wave would be zero, since r_1 This gives us the following solution for the gKdV equation:

$$\eta(\xi, \tau) = r_2 + (r_1 - r_2)\text{cn}^2(\sqrt{\frac{3}{4}(r_1 - r_3)}(\xi - c\tau), \frac{r_1 - r_2}{r_1 - r_3})$$

where r_1, r_2, r_3 are the roots of the polynomial

$$-3/2\eta^3 + 3(c + \omega_0)\eta^2 + A\eta + B \tag{3.3}$$

which satisfy $r_3 \leq r_2 \leq r_1$, with only real roots.

Now, if we want our solutions to satisfy the boundary conditions mentioned in the previous section, we note that the second root r_2 needs to be equal to zero. This gives the following simplification to the equation

$$\eta(\xi, \tau) = r_1 \text{cn}^2(\sqrt{\frac{3}{4}(r_1 - r_3)}(\xi - c\tau), \frac{r_1}{r_1 - r_3})$$

For $m \neq 1$, we get periodic waves, where m determines the shape of the waves. That is, the larger m, the more prolonged the troughs, and the sharper the waves.

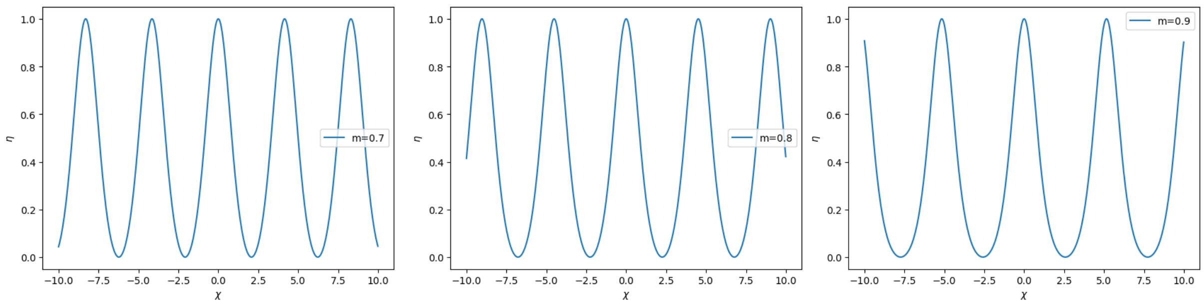


Figure 3.3: Here are three cnoidal waves shown, to illustrate the changes due to differing m. As can be seen, the higher the m, the more prolonged the troughs, and the sharper the waves.

Note that, due to the requirement for one of the roots to be zero, the constant term of polynomial 3.3, B should be zero as well.

3.2.1 Solitary Waves from the Elliptic Function

If we look back to the solitary wave solutions of the gKdV equation, and try to calculate the elliptic function solutions for the values of A and B, we find that these solutions are the same as the solitary solutions.

Morphed version:

To demonstrate this, let $A = B = 0$. Then polynomial 3.3 becomes:

$$-3/2\eta^3 + 3(c+\omega_0)\eta^2 = -3/2(\eta - 2(c+\omega_0))(\eta - 0)^2$$

The roots of this polynomial, when put in the right order, are:

$$r_1 = 2(c+\omega_0), \quad r_2 = r_3 = 0$$

Then:

$$\gamma = 0, \quad \alpha = 2(c+\omega_0), \quad \beta = \sqrt{\frac{3}{2}(c+\omega_0)}, \quad m = 1$$

For $m = 1$, $\text{cn}(x, 1) = \text{sech}(x)$, so the solution becomes:

$$\eta(\xi, \tau) = 2(c+\omega_0)\text{sech}^2(\sqrt{3/2(c+\omega_0)}(\xi - c\tau))$$

which is, as expected, exactly the same solution as 3.1.

Shifted version:

For the shifted version, we shall take polynomial 3.3 to be

$$-3/2(\eta - (2c + 2/3\omega_0))(\eta - 2/3\omega_0)(\eta - 2/3\omega_0)$$

This polynomial is specifically chosen to result in the shifted form, and is possible to be written in the original form. Then,

$$A = 4\omega_0 c + 2/3\omega_0^2, \quad B = -4/3\omega_0^2 c - 4/9\omega_0^3$$

The roots of this polynomial are:

$$r_1 = 2c + 2/3\omega_0, \quad r_2 = r_3 = 2/3\omega_0$$

This gives values:

$$\gamma = 2/3\omega_0, \quad \alpha = 2c, \quad \beta = \sqrt{\frac{3}{2}c}, \quad m = 1$$

This gives solution:

$$\eta(\xi, \tau) = 2/3\omega_0 + 2c \cdot \mathrm{sech}^2(\sqrt{3/2c}(\xi - c\tau))$$

In general, $r_2 = r_3$ gives a (possibly shifted) soliton solution, since this gives $m = 1$. If $r_2 = r_3 = 0$, then the solution will be a non-shifted soliton. However, the only one which satisfies this, is the one we know already. If $r_1 = r_2$, then the solution becomes a constant.

One final thing of note is that the only differing constraint the gKdV equation has when compared to the KdV equation, is in the quadratic term in the polynomial. To see the effect, we use that this quadratic term is equal to the following:

$$3(c + \omega_0) = 3/2(r_1 + r_2 + r_3)$$

Besides the requirement that these roots are real, this is the only constraint the roots have, and herein also lies the difference between the cnoidal wave solutions which satisfy the gKdV equation, and those which satisfy the KdV equation. That is, the only difference between polynomial 3.3 for the KdV and the gKdV equations is the value of ω_0 in the quadratic term, which is zero for the regular Korteweg-de Vries equation, and non-zero for the geophysical Korteweg-de Vries equation.

This gives us another transformation between the KdV equation, and the gKdV equation. That being, the shifting of the wave velocity coefficient everywhere, except at the velocity itself, since $\chi = \xi - c\tau$ remains unchanged.

It must be noted that this new transformation is dependent on the fact that a solution would be periodic. This does not have to be the case, as stated previously in the introduction of this chapter. This lack of generality suggests that, unlike the vertical shift transformation, this transformation might not have a similar Lie isomorphism, as the other has. If a more general transformation, which the velocity shift is a specific form of, exists, there might be a Lie isomorphism of the same form, but that is assuming that there is a relation between the isomorphism found in chapter 2, and the transformation found in chapter 1.

Chapter 4

Conclusions

We know now that, both through direct means, and through studying the Lie algebras connected to the two equations, we can find transformations from solutions to the Korteweg-de Vries equation to solutions to the geophysical Korteweg-de Vries equation, and the other way around. Not only that, but both of these are just a vertical shift of these solutions. However, these transformations shift said solutions by different amounts. Where the simple transformation from KdV to gKdV shifts the solutions upwards by $2/3\omega_0$, the transformation through the Lie algebra isomorphism shifts it by a full ω_0. This means that these two transformations, however similar and possibly even related to each other, are not the same.

This isn't totally unexpected, since the two transformations do not do the same thing. The Lie isomorphism transformation simply states that, if a function is a solution to the gKdV equation, in the dual space where the KdV equation is the Euler-Arnold equation, its image, in the space where the gKdV equation is the Euler-Arnold equation, is a solution to the KdV equation. However, whether their similar transformations are a coincidence, or have some deeper cause, remains to be seen.

For the third transformation, which is caused by a change in the velocity term for periodic functions, no existing variant in the Lie theory domain has been found. This also is not that surprising, since that transformation is reliant on the solution being periodic, which is by no means necessary for a function to solve the KdV equation. This restriction was taken into account nowhere in the Lie algebra. If a more general form of this transformation exists, which is doubtful, perhaps a version for the Lie algebra can be found.

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