# A moment extension of Lions' method for SPDEs

J.P.C. Hoogendijk

Technische Universiteit Delft

# A moment extension of Lions' method for SPDEs

by

J.P.C. Hoogendijk

to obtain the degree of Master of Science
at the Delft University of Technology,
to be defended publicly on Monday August 30th, 2021 at 10:00 AM.

| | | |
|---|---|---|
| Student number: | 4583736 | |
| Project duration: | November 24, 2020 – August 30, 2021 | |
| Thesis committee: | Prof. dr. ir. M. C. Veraar, | TU Delft, supervisor |
| | Dr. M. V. Gnann, | TU Delft, supervisor |
| | Dr. ir. R. C. Kraaij, | TU Delft |

*This thesis is confidential and cannot be made public until August 23, 2021.*



TU Delft

# Abstract

This master's thesis introduces a new $p$-dependent coercivity condition through which $L^p(\Omega; L^2([0,T]; X))$ estimates can be obtained for a large class of SPDEs in the variational framework. Using these estimates, we obtain existence and uniqueness results by using a Galerkin approximation argument. The framework that is built is applied to many SPDEs such as stochastic heat equations with Dirichlet and Neumann boundary conditions, Burger's equation and Navier-Stokes in 2D. Furthermore, we obtain known results for systems of SPDEs and higher order SPDEs using our unifying coercivity condition. We also obtain first steps towards a theory of higher order regularity of stochastic heat equations.

# Preface

This thesis would not have existed without the help of some people, for which I owe them my gratitude. In particular, I would like to thank my supervisors Manuel Gnann and Mark Veraar. Thanks for all your suggestions and all the time you have taken to help me. As has been said by many people before me, the pandemic has displaced all of us, but I do not feel that the process has lacked because of this. I would also like to thank everyone at the analysis department, students and faculty members, for the nice atmosphere, the seminars and the subjects that have been taught. I would also like to thank Richard Kraaij for taking part in my thesis committee and introducing me to the interesting world of large deviations. Last but not least, I would like to thank my family and friends for keeping the process of writing a thesis somewhat bearable. I would have for sure gone mad without you.

*Jochem Hoogendijk*
*Katwijk, August 2021*

# Contents

# 1 Introduction

In his *Théorie Analytique des Probabilités* [26], Pierre-Simon Laplace remarks

> It is remarkable that a science which began with the consideration of games of chance should have become the most important object of human knowledge.

This quote remains more true than ever before, possibly beyond all of Laplace's expectations. However, probability theory has not always taken this important place in mathematics. Starting with the conception of classical mechanics in the 17th century by Isaac Newton, models of nature through Newton's laws were governed by deterministic equations. This approach was expanded by people such as Lagrange and Hamilton through their own respective formulations of mechanics, with still no room for probability in the equations. In the 19th century, this slowly started to change. Beginning with Ludwig Boltzmann's statistical physics, Boltzmann showed that macroscopic thermodynamic variables could be derived from a probabilistic examination of the microscopic variables, which was revolutionary and controversial at the time. Boltzmann's discoveries were especially remarkable if one considers that scientists had no notion of atoms and the microscopic world during Boltzmann's time! One success of statistical physics was given by Albert Einstein and Paul Langevin. They managed to give an explanation of the Brownian motion of pollen grains by doing a probablistic, microscopic examination of the system, describing one of the first stochastic differential equations (SDEs) in history [25, 27]. Proceeding into the early 20th century, classical mechanics was superseded by quantum mechanics, which gives a probabilistic description of objects on small scales. It became clearer and clearer that probability theory had a fundamental place in nature. [1]

Since Laplace's remark, probability theory and analysis have not stood still either. Both have gotten rigorous fundamentals, starting with analysis in the 19th century by Augstin-Louis Cauchy and Karl Weierstrass. For probality theory, the modern axiomatic foundation was layed by Andrei Kolmogorov only in 1933. This allowed for a rigourous study of many new concepts, such as Markov processes, martingales and Wiener processes. Kyoshi Itô introduced a theory of stochastic integration [17, 18], through which a rigorous mathematical interpretation to the SDE of Einstein and Langevin could be given. As a generalization of

[1] Of course, a probabilistic view of nature has also led to some controversialism in philosphy. For different interpretations of probability theory, see [16]

SDEs, one can also consider stochastic partial differential equations (SPDEs), which will be the main topic of this master's thesis. As the name suggests, these are a natural extension of partial differential equations with some stochastic dependence included in the equation. For example, a natural extension of the heat equation $\partial_t u = \Delta u$ on $\mathbb{R}^d$ is the stochastic heat equation

$$\partial_t u = \Delta u + \xi$$

on $\mathbb{R}^d$ ,where $\xi$ is some random signal in space and time. Just as SDEs turned up in the study of Brownian motion by Einstein and Langevin, SPDEs turn up naturally in many other physical models, such as the $\Phi_3^4$ model in quantum field theory [32] and stochastic fluid models [1, 6]. Other areas where they turn up are filtering theory [21, 24], reaction diffusion equations, neurophysiology and finance [10, 21], amongst others. From a more mathematical point of view, SPDEs bring their own interesting problems. The most common approach to study PDEs and SPDEs is to reformulate them as infinite dimensional ordinary differential equations (ODEs) or stochastic differential equations (SDEs). However, it is well-known that not all results from finite dimensions carry over to infinite dimensions. A famous theorem in the study of finite dimensional SDEs is Itô's lemma, which can be used to prove existence and uniqueness of solutions of SDEs and very useful theorems such as the Burkholder-Davis-Gundy (BDG) inequalities (useful for energy estimates) and Girsanov's theorem (often used for a change of measure in mathematical finance). However, Itô's lemma does not carry over to the infinite dimensional setting for all SPDEs. Therefore, it is not possible to simply generalize finite dimensional results to infinite dimensions. This prompts a different approach, which leads to interesting mathematics of itself and makes SPDEs definitely worth studying.

## 1.1. Exposition and background

In this thesis, we will introduce a new coercivity condition to obtain existence and uniqueness of a large class of SPDEs. In particular, we are interested in $L^p(\Omega; L^2([0,T]; X))$ results, where $\Omega$ is a probability space, $p \geq 2$, $[0,T]$ is a time interval and $X$ is a Hilbert space (take for instance the Sobolev space $W_0^{2,k}(\mathscr{D})$ on a domain $\mathscr{D}$). The main motivation to study this problem comes from the theory of stochastic maximal regularity. To illustrate this problem, we first introduce maximal regularity in the deteriministic setting. In this setting, we can consider the inhomogeneous Cauchy problem

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) + Au(t) = f(t), \quad t \in [0,T], \quad u(0) = u_0, \tag{1.1}$$

in $L^p([0,T], X)$ for $p \in (1,\infty)$ and $X$ a Banach space. We say that a closed and densely defined operator $A : \mathrm{D}(A) \subset X \to X$ has maximal $L^p$-regularity if for each $f \in L^p([0,T]; X)$ there exists a unique $u \in W^{1,p}([0,T]; X) \cap L^p([0,T]; D(A))$ satisfying the above equation a.e. in $[0,T]$ with $u_0 = 0$. We can also derive the following:

$$\|u\|_{L^p([0,T];X)} + \|Au\|_{L^p([0,T];X)} \leq C\|f\|_{L^p([0,T];X)}. \tag{1.2}$$

It is well-known that $A$ has maximal regularity if and only if $-A$ generates an analytic semigroup [33]. Now, maximal regularity can be used to obtain a wide range of well-posedness results for nonlinear PDEs using fixed-point arguments or others. In particular, the above estimate is very useful in proving this. A long standing aim in the study of SPDEs has been to obtain a theory of stochastic maximal regularity. In this theory, we search for existence and uniqueness of solutions of SPDEs in a class $L^p(\Omega; L^q([0,T]; X)$ where $p, q \geq 2$, $\Omega$ is a probability space, $[0,T]$ is a time-interval and $X$ is a Banach space. However, a theory of stochastic maximal regularity only exists for certain subcases of $p, q$ and certain choices of $X$. The aim of this thesis is to introduce a new coercivity condition that leads to existence and uniqueness in $L^p(\Omega; L^2([0,T]; X))$ for a large class of SPDEs. This can be applied to a wide range of equations, such as stochastic heat equations with both Dirichlet and Neumann boundary conditions, as well as Burgers' equation and the Navier-Stokes equations in 2D. Furthermore, we are able to recover results from the literature for systems of SPDEs and higher order SPDEs by using our theory.

In order to study SPDEs, we will reformulate them as infinite dimensional stochastic evolution equations of the following form:

$$\mathrm{d}u_t = A(t, u_t)\,\mathrm{d}t + B(t, u_t)\,\mathrm{d}W_t, \quad u(0) = u_0. \tag{1.3}$$

There are three main approaches that have tackled these type of equations over the past 70 years. The most studied approach is probably the semigroup approach. In this approach, the operator $A$ is assumed to be the generator of some semigroup and $B$ a bounded operator satisfying a Lipschitz condition. Using contraction mapping arguments, both existence and uniqueness of solutions can be obtained. Another approach is the martingale approach. This approach generalizes the notion of weak solutions of SDEs to SPDEs. As it is easy to confuse the notion of weak solution to SDEs with weak solutions to PDEs, this approach is called the martingale problem approach by some authors [10]. Both approaches will not be discussed in the sequel, but it is good to be aware of their limitations. In particular, it is standard to assume that the operator $A$ is the generator of a $C_0$ semigroup. However, many SPDEs do not satisfy this assumption. For example, certain SPDEs arising in control theory are not included in the semigroup or martingale approaches, see [23].

The third approach is the variational approach, which will be the only important approach for this thesis. The variational approach for SPDEs was initiated by Alain Bensoussan in 1971 [2] [3], based on Lions' approach for PDEs. In this approach, a coercivity condition on the operator $A$ is assumed, together with $B = I$. Using time discretization methods, existence and uniqueness of solutions is proved. Bensoussan's work was later generalized in 1975 by Étienne Pardoux [31], who proved existence and uniqueness results for monotone operators in both the parabolic and hyperbolic setting based on Galerkin approximations. Influential work by Krylov and Rozovskii in 1981 [23] loosens restrictions on the operators $A$ and $B$ made by Pardoux, but still uses the Galerkin approximation idea. More recently, Wei Liu and Michael Röckner [28] have generalized this framework even further by using locally monotone operators instead of monotone operators. This allows for new equations to be studied, such as the stochastic Burgers equation. However, the assumptions in their paper still allow limited growth on $B$ and the estimates for their solutions are only restricted to $L^2$ estimates in $\Omega$. Work published by Zdzislaw Brzeźniak, Wei Liu and Jiahui Zhu [4] expands their framework by considering a slightly better growth condition on $B$ and Lévy noise. The latest point in this development is a paper published by David Šiška and Neelima Varshney [38], in which $L^{rp}(\Omega; L^2([0,T]; X))$ estimates are obtained, where $r \in (0,1)$. This paper will be the starting point of this thesis. We will use a slightly more general coercivity condition and improve the argument given by Šiška and Varshney to actually obtain

$L^p(\Omega; L^2([0,T]; X))$ estimates, which we will describe now. In order to study (1.3) in the variational approach, we need to introduce the Gelfand triple setting. This means that we have a triple of spaces $(V, H, V^*)$, where $V$ is a separable Banach space, $H$ is a Hilbert space, $V \subseteq H$ densely and $V^*$ is the Banach space dual of $V$. We will also need an extra separable Hilbert space $U$ to make sense of the stochastic part of the equation. Then, we consider $A$ as a map $A : V \to V^*$ and $B$ as a map $B : V \to L_2(U, H)$, where $L_2(U, H)$ is the space of Hilbert-Schmidt operators. We also assume that $A$ and $B$ satisfy local monotonicity bounds, as well as certain indivdual bounds, details of which can be found in chapter 4 of the thesis. The most important assumption is the following: for all $v \in V, v \neq 0$, $t \in [0,T]$ a.s.

$$2\langle A(t,v), v\rangle + \|B(t,v)\|^2_{L_2(U,H)} + (p-2)\frac{\|B(t,v)^* v\|^2_U}{\|v\|^2_H} \leq -\theta\|v\|^\alpha_V + f_t + K_c\|v\|^2_H \tag{1.4}$$

where $p \geq 2$, $\theta > 0$, $\alpha > 1$, $K_c \geq 0$ and $f \in L^{\frac{p}{2}}(\Omega; L^1([0,T];\mathbb{R}))$ nonnegative. With these assumptions in hand we can state the most important result from this thesis. Given the above assumptions, suppose a solution exists to (1.3). Then, there exists a constant $C \geq 0$ depending on $\theta, \alpha$ and $p$ such that

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}\|u_t\|^p_H + \mathbb{E}\int_0^T \|u_t\|^{p-2}_H\|u_t\|^\alpha_V \mathrm{d}t + \mathbb{E}\left(\int_0^T \|u_t\|^\alpha_V \mathrm{d}t\right)^{\frac{p}{2}}\\ &\leq Ce^{CT}\left(\mathbb{E}\|u_0\|^p_H + \mathbb{E}\left(\int_0^T f_t \mathrm{d}t\right)^{\frac{p}{2}}\right).\end{aligned} \tag{1.5}$$

Moreover, these estimates are optimal. With optimal, we mean that it is not possible to obtain better results such as higher moments based on the assumptions we used to prove this estimate in the first place. These a priori estimates will then be used to prove existence and uniqueness for (1.3) using the method of Galerkin approximations. This means that we project (1.3) to finite dimensions and use the finite dimensional theory to obtain a sequence of approximate solutions. It is then shown that this sequence converges and that its limit is indeed a solution of (1.3). It can be shown that this framework applies to a wide range of equations. The equations we work out in this thesis are stochastic heat equations with both Dirichlet and Neumann boundary conditions, as well as Burgers' equation and Navier-Stokes in 2D. Furthermore, we are able to recover results from the literature for systems of SPDEs and higher order SPDEs by using our theory.

We also study the higher order space regularity of stochastic heat equations on smooth domains in the $L^p(\Omega;\ldots)$ setting. First results in this direction were obtained by Krylov for $p = 2$ [20], though he needs weighted Sobolev spaces to do this, unlike the deterministic setting [14]. We aim to produce the same results for $p \geq 2$. In this thesis, we have made the first steps in this direction by proving higher order space regularity for $p \geq 2$ on $\mathbb{R}^d_+$ for stochastic heat equations with constant coefficients. A large part in proving this is taken up by the framework built in this thesis.

## 1.2. Reading guide

Now that we have established the exposition of this thesis, we will provide a brief overview of the structure of this thesis and how different chapters relate to each other.

- *Chapter 2 (Preliminaries):* The intended audience for whom this chapter has been written are mathematics students with some knowledge of PDEs, functional analysis and stochastic integration. The chapter mostly contains statements of important theorems and lemmas that will be used later. References for proofs are also provided. Only some theorems and lemmas were proven by the author himself.

- *Chapter 3 (A coercivity condition for higher order moments):* The main problem of the thesis is treated in this chapter. We build a framework based on the new coercivity assumption by first proving a priori estimates and then proving existence and uniqueness. Optimality of the framework is also proven in this chapter.

- *Chapter 4 (Examples of coercive SPDEs):* In this chapter we introduce a variety of SPDEs to which our framework applies. The examples here are certainly non-exhaustive, but might provide a good insight in what the framework is capable of. For example, we show that the stochastic heat equation with Dirichlet boundary conditions reduces to the known setting using our framework. We are also able to derive previously obtained results in the literature for systems of SPDEs and higher order SPDEs.

- *Chapter 5 (Higher order regularity):* This chapter proves higher order space regularity of stochastic heat equations with constant coefficients.

# 2
# Preliminaries

The most essential knowledge that is needed to understand the results in this thesis is treated in this chapter. The topics included are ordered in three sections: functional analysis, PDE theory and stochastic integration theory. Most results can also be found in the literature and are referenced accordingly, while some other, less standard results are taylored for this thesis.

## 2.1. Some functional analysis

### 2.1.1. Bochner integration and Bochner spaces

Suppose one is given a function $f : E \to X$, where $E$ and $X$ are both Banach spaces. In many situations in mathematics, one is interested in integrating such a function for a variety of reasons. The easiest case one could think of, and for which integration was introduced historically, is determining the area under a curve. In this case, $X = \mathbb{R}$ and integration can be made rigorous in both the Riemann sense and the Lebesgue sense. To motivate integration for infinite dimensional Banach spaces $F$, we inspect the heat equation

$$\tfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x).$$

If we surpress the spatial variable $x$ from notation, we can see this PDE as a function valued ODE, where $u(t,\cdot)$ takes values in some function space $X$. Informally, one would then solve this PDE by finding a function $u(t,\cdot)$ that satisfies the following integrated form of the PDE:

$$u(T,\cdot) - u(0,\cdot) = \int_0^T \Delta u(t,\cdot)\mathrm{d}t.$$

Therefore, to solve this PDE, one needs to be able to make sense of the integral on the RHS. This can be done using the Bochner integral, which allows integration of functions $f : E \to X$, where both $E$ and $X$ are Banach spaces. The Bochner integral generalizes the Lebesgue integral, and accordingly one can also introduce Bochner spaces as a generalization of the classical Lebesgue spaces.

For ease of exposition and since all Banach spaces in this thesis will be separable, we only treat the special case where the space $X$ is separable. The exposition in this section is adapted from [10]. The interested reader can find the case of non-separable Banach spaces in [29].

To introduce the Bochner integral, we have to recall some basic measure theoretic notions. We consider a measurable space to be a pair $(E,\mathscr{E})$, where $E$ is a set and $\mathscr{E}$ is a $\sigma$-algebra. If $E$ is a metric space, one particular choice of $\sigma$-algebra is the Borel $\sigma$-algebra. This is the $\sigma$-algebra generated by the open sets of $E$. A measurable space can be turned into a measure space by equipping it with a measure $\mu:\mathscr{E}\to[0,\infty]$. Now, consider two measurable spaces $(E,\mathscr{E})$ and $(X,\mathscr{X})$. A mapping $f:E\to X$ is said to be measurable if for every $A\in\mathscr{X}$, $\{X\in A\}=\{a\in E: X(a)\in A\}\in\mathscr{E}$.

We call a function $f:E\to X$ simple if it can be expressed as $f=\sum_{i=1}^n a_i 1_{A_i}$ where $a_i\in X$ and $A_i\in\mathscr{E}$. To construct the Bochner integral, we first define integration for simple functions and then use an approximating sequence. Therefore, we need the following approximation lemma

**Lemma 2.1.** *Let $E$ and $X$ be separable Banach spaces with norms $\|\cdot\|_E$, $\|\cdot\|_X$. Consider a measurable map $f:E\to X$. Then there exists a sequence $\{f_n\}$ of simple $X$-valued functions such that for every $a\in E$, $\|f_n(a)-f(a)\|_X\to 0$ monotically.*

*Proof.* [See lemma 1.3, p. 16 in [10]] Let $E_0=\{e_k\}_{k\in\mathbb{N}}$ be a countably dense subset of $E$. For $m\in\mathbb{N}$ and $a\in E$, define the following:

$$\begin{aligned}\rho_m(a)&=\min_{k\in\{1,\ldots,m\}}\|f(a)-e_k\|_X\\ k_m(a)&=\min\{k\le m:\rho_m(a)=\|f(a)-e_k\|_X\}\\ f_m(a)&=e_{k_m(a)}\end{aligned}$$

Then, every $f_m$ is a simple function, since $f_m(E)\subseteq\{e_1,\ldots,e_m\}$. Since $E_0$ is dense in $E$, $\rho_m(a)$ is monotonically decreasing to 0 in $m$ for all $a\in E$. It is clear to see that $\rho_m(a)=\rho(f(a),f_m(a))$. □

We are now in a position to define the Bochner integral. Suppose $f:E\to X$ is a simple function on a measure space $(E,\mathscr{E},\mu)$ of the form

$$f=\sum_{i=1}^N x_i 1_{A_i},\quad A_i\in\mathscr{E},\quad x_i\in E,\quad N\in\mathbb{N}.$$

We set

$$\int_E f\,\mathrm{d}\mu=\sum_{i=1}^N x_i\mu(A_i) \tag{2.1}$$

This integral does not depend on the representation of $f$, as can be checked similarly for Lebesgue integration by using refinement. It is also straightforward to prove that the Bochner integral for simple functions is additive and linear. The following triangle inequality-like property also holds:

$$\left\|\int_E f\,\mathrm{d}\mu\right\|_X\le\int_E\|f\|_X\,\mathrm{d}\mu. \tag{2.2}$$

Next, we want to extend this integral to all measurable functions $f: E \to X$. First note that $\|f\|_X$ is also a measurable function, since $\|\cdot\|_X$ is a continuous function. We call $f$ Bochner integrable if

$$\int_E \|f\|_X \, \mathrm{d}\mu < \infty. \tag{2.3}$$

Note that the above integral is a Lebesgue integral, since $\|f\|_X$ is a real function. Using lemma 2.1, we obtain an approximating sequence of simple functions $\{f_m\}_{m\in\mathbb{N}}$ such that $\{\|f(a) - f_m(a)\|_X\}$ decreases to 0 for all $a \in E$. It follows that $\{\int_E f_m \, \mathrm{d}\mu\}_{m\in\mathbb{N}}$ is a Cauchy sequence, since

$$\begin{aligned} &\left\| \int_E f_m \, \mathrm{d}\mu - \int_E f_n \, \mathrm{d}\mu \right\|_X \\ &\leq \int_E \|f - f_m\|_X \, \mathrm{d}\mu + \int_E \|f - f_n\|_X \, \mathrm{d}\mu \end{aligned} \tag{2.4}$$

and the RHS decreases to 0 as $m, n \to \infty$. Since $X$ is a Banach space, we can therefore define

$$\int_E f \, \mathrm{d}\mu = \lim_{m\to\infty} \int_E f_m \, \mathrm{d}\mu. \tag{2.5}$$

We call the quantity on the RHS the Bochner integral of $f$ against $\mu$. It is routine to check that the integral does not depend on the approximating sequence $\{f_m\}_{m\in\mathbb{N}}$. For a bounded linear operator $T$, it is clear that the operator $T$ and the integral can be interchanged. However, we can prove the same for a certain class of closed operators, which is the content of the next proposition.

**Proposition 2.1.** *Suppose $f: E \to X$ is Bochner integrable and $T$ a closed linear operator with domain $D(T) \subseteq X$ and values in some Banach space $Y$. Suppose $f$ takes values in $D(T)$ a.e. and that $Tf: E \to Y$ is Bochner integrable. Then, $\int_E Tf \mathrm{d}\mu \in D(T)$ and*

$$T \int_E f \mathrm{d}\mu = \int_E Tf \mathrm{d}\mu.$$

*Proof.* See [29]. □

### 2.1.2. Hilbert-Schmidt operators

In order to build the stochastic integral in Hilbert spaces, we need to introduce the concept of Hilbert-Schmidt operators. To preserve some type of Itô isometry, Hilbert-Schmidt operators form the natural arena for stochastic integration in Hilbert spaces. The material treated in this section is entirely standard and also treated in [28]. Let $(U, (\cdot,\cdot)_U)$ and $(H, (\cdot,\cdot)_H)$ be two separable Hilbert spaces. We denote the space of bounded linear operators between $U$ and $H$ by $L(U, H)$.

**Definition 2.1.** An element $T \in L(U, H)$ is said to be a *nuclear operator* if there exists a sequence $(a_j)_{j\in\mathbb{N}}$ in $H$ and a sequence $(b_j)_{j\in\mathbb{N}}$ in $U$ such that

$$Tx = \sum_{j=1}^{\infty} a_j (b_j, x)_U, \qquad \text{for all } x \in U$$

and

$$\sum_{j=1}^{\infty} \|a_j\|_H \|b_j\|_U < \infty.$$

The space of all nuclear operators is denoted by $L_1(U,H)$. If $U = H$ and $T \in L_1(U,U)$ is nonnegative and symmetric, then $T$ is called *trace class.*

**Definition 2.2.** A bounded linear operator $T : U \to H$ is called a *Hilbert-Schmidt operator* if

$$\sum_{k\in\mathbb{N}} \|Te_k\|_H^2 < \infty,$$

where $e_k, k \in \mathbb{N}$, is an orthonormal basis of $U$.

The space of all Hilbert-Schmidt operators is denoted by $L_2(U,H)$. We can define a norm on $L_2(U,H)$ by setting

$$\|T\|^2_{L_2(U,H)} = \sum_{k=1}^{\infty} \|Te_k\|_H^2. \tag{2.6}$$

The following properties of Hilbert-Schmidt operators turn out to be crucial later for stochastic integration in Hilbert spaces:

**Proposition 2.2.** *Let $T \in L_2(U,H)$. Then,*

1. *the definition of Hilbert-Schmidt operators and the number $\|T\|^2_{L_2(U,H)}$ are independent of the basis on $U$.*
2. $\|T\|_{L_2(U,H)} = \|T^*\|_{L_2(H,U)}$.
3. $\|T\|_{L(U,H)} \le \|T\|_{L_2(U,H)}$.
4. *Let $(G,(\cdot,\cdot)_G)$ be another separable Hilbert space, $S_1 \in L(H,G)$, $S_2 \in L(G,U)$ and $T \in L_2(U,H)$. Then, $S_1T \in L_2(U,G)$ and $TS_2 \in L_2(G,H)$ with estimates*

$$\|S_1T\|_{L_2(U,G)} \le \|S_1\|_{L(H,G)}\|T\|_{L_2(U,H)},$$

$$\|TS_2\|_{L_2(G,H)} \le \|T\|_{L_2(U,H)}\|S_2\|_{L(G,U)}.$$

*Proof.* The proof is taken from [28, Remark B.0.6, p. 217]. We prove items 1 and 2 at the same time.

1. Let $\{e_k\}_{k\in\mathbb{N}}$, $\{f_k\}_{k\in\mathbb{N}}$ be two orthonormal basises of $U$ and $\{g_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $H$. We use Parseval's identity to obtain:

$$\sum_{k=1}^{\infty} \|Te_k\|_H^2 = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \left|\langle Te_k, g_k\rangle\right|^2 = \sum_{j=1}^{\infty} \|T^*g_j\|_U^2.$$

   We can now apply the same trick to obtain:

$$\sum_{j=1}^{\infty} \|T^*g_j\|_U^2 = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \left|\langle Tf_k, g_k\rangle\right|^2 = \sum_{k=1}^{\infty} \|Tf_k\|_H^2.$$

   In the course of the proof we have also obtained that $\|T\|_{L_2(U,H)} = \|T^*\|_{L_2(H,U)}$

2. See item 1

3. Let $x \in U$ and $\{f_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $H$. We can then use item 2 and Parseval's identity to obtain:

$$\|Tx\|_H^2 = \sum_{k=1}^{\infty} \left|\langle Tx, f_k\rangle\right|^2 \le \|x\|_U^2 \sum_{k=1}^{\infty} \|T^* f_k\|_U^2 = \|T\|_{L_2(U,H)}^2 \|x\|_U^2.$$

By definition of the operator norm, we obtain $\|T\|_{L(U,H)} \le \|T\|_{L_2(U,H)}$.

4. Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $U$. Then,

$$\sum_{k=1}^{\infty} \|S_1 Te_k\|_G^2 \le \|S_1\|_{L(H,G)}^2 \|T\|_{L_2(U,H)}^2.$$

Using item 2 and the property $(TS_2)^* = S_2^* T$ it follows that

$$\begin{aligned} \|TS_2\|_{L_2(G,H)} &= \|(TS_2)^*\|_{L_2(H,G)} \\ &= \|S_2^* T^*\|_{L_2(G,H)} \\ &\le \|S_2\|_{L(G,U)} \|T\|_{L_2(U,H)}. \end{aligned}$$ □

We end this section by showing that the space $L_2(U,H)$ can be turned into a Hilbert space.

**Proposition 2.3.** *Let $S, T \in L_2(U,H)$ and let $\{e_k\}_{k\in\mathbb{N}}$, be an orthonormal basis of $U$. If we define*

$$\langle T, S\rangle_{L_2} = \sum_{k=1}^{\infty} \langle Se_k, Te_k\rangle \tag{2.7}$$

*we obtain that $(L_2(U,H), \langle\cdot,\cdot\rangle_{L_2})$ is a separable Hilbert space. Furthermore, if $\{f_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $H$ we get that $f_j \otimes e_k = f_j\langle e_k, \cdot\rangle_U$, where $j, k \in \mathbb{N}$, is an orthonormal basis of $L_2(U,H)$*

*Proof.* See [28, Proposition B.0.7, p. 218]. □

### 2.1.3. Weak convergence

As outlined in the exposition, we are interested in treating the variational approach for SPDEs. In this approach, we build a sequence of approximate solutions of which we extract a convergent subsequence in some way. One way to do this is by using weak compactness arguments. The basics to make such an argument are recalled in this section. We will take the easy route and avoid any reference to the theory of locally convex spaces, following Evans [14]. For a more detailed treatise of weak topologies, weak convergence and locally convex spaces, see [9, 35]. In this subsection we let $X$ be a real Banach space.

**Definition 2.3.** A sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq X$ is said to converge weakly to $x \in X$, denoted by

$$x_k \rightharpoonup x,$$

if

$$\langle x_k, x^*\rangle \to \langle x, x^*\rangle$$

for each bounded linear functional $x^* \in X^*$.

The following proposition is a simplified version of the famous Banach-Alaoglu theorem. The proof is taken from [36].

**Proposition 2.4.** *Let $X$ be a separable normed space and let $(x_n^*)_{n\geq 1}$ be a bounded sequence in the dual space $X^*$. Then there exists a subsequence $(x_{n_k}^*)_{k\geq 1}$ and an $x^* \in X^*$ such that*

$$\lim_{k\to\infty} \langle x, x_{n_k}^* \rangle = \langle x, x^* \rangle \quad \textit{for all } x \in X. \tag{2.8}$$

*Proof.* Let $(x_j)_{j\geq 1}$ be a countable set whose linear span $X_0$ is dense in $X$ (which is possible by separability). By a diagonal argument we find a subsequence $(x_{n_k}^*)_{k\geq 1}$ such that the limit $\phi(x_j) := \lim_{k\to\infty}\langle x_j, x_{n_k}^*\rangle$ exists for all $j \geq 1$. Then, the limit $\phi(x) := \lim_{k\to\infty}\langle x, x_{n_k}^*\rangle$ exists for all $x \in X_0$, which follows from linearity. It is easy to see that $x \mapsto \phi(x)$ is a bounded linear map from $X_0 \to \mathbb{R}$. Since $X_0$ is dense in $X$, we can obtain a unique bounded extension of $\phi$ of the same norm, which we also denote by $\phi$. Since the operators $\langle \cdot, x_{n_k}^*\rangle$ are uniformly bounded, we obtain that $\phi(x) = \lim_{k\to\infty}\langle x, x_{n_k}^*\rangle$ for all $x \in X$. Therefore, $x^* := \phi$ has the required properties. □

We can obtain a very useful corollary from this proposition. For example, one can identify a candidate limit from having a bounded sequence of certain approximate solutions to a PDE, using this corollary.

**Corollary 2.1.** *Let $X$ be a separable reflexive Banach space and $(x_n)_{n\geq 1}$ a bounded sequence on $X$. Then, there exists a subsequence $(x_{n_k})_{k\geq 1}$ and an $x \in X$ such that*

$$\lim_{k\to\infty} \langle x_{n_k}, x^* \rangle = \langle x, x^* \rangle \quad \textit{for all } x^* \in X^*. \tag{2.9}$$

*Proof.* We first note $X^*$ is also separable, since $X$ is reflexive and separable. This allows us to apply Proposition 2.4 with $X^*$ as our normed space and $(x_n)_{n\geq 1}$ as our bounded sequence, which is possible by reflexivity. We immediately obtain a subsequence $(x_{n_k})_{k\geq 1}$ and an $x \in X$ such that

$$\lim_{k\to\infty} \langle x_{n_k}, x^* \rangle = \langle x, x^* \rangle \quad \text{for all } x^* \in X^*. \tag{2.10}$$

This finishes the proof. □

We proceed to prove one extra lemma related to weakly convergent sequences, which will be useful in proving existence of the SPDEs we consider. Recall that for any weakly convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ we have the following bound:

$$\|x\|_X \leq \liminf \|x_n\|_X. \tag{2.11}$$

We can generalize the above bound in a special way that is needed for one of the results in this thesis. Let $\Omega$ be a probability space and $H$ be a separable Hilbert space. Then we have the following:

**Lemma 2.2.** *Suppose a sequence $(u^{(n)})_{n\geq 1}$ converges weakly to $u$ in $L^2(\Omega; L^2([0,T]; H))$. Then, for any weight $w : [0,T] \times \Omega \to [0,\infty)$ with $w \leq 1$, we have the following inequality:*

$$\mathbb{E}\int_0^T \|u_t\|_H^2 w_t \mathrm{d}t \leq \liminf \mathbb{E}\int_0^T \|u_t^{(n)}\|_H^2 w_t \mathrm{d}t \tag{2.12}$$

*Proof of Lemma 2.2.* Let $v \in L^2(\Omega; L^2([0,T]; H))$. By definition of weak convergence, we then have:

$$\lim_{n\to\infty} \langle u^{(n)}, v\rangle = \langle u, v\rangle.$$

Now, consider the following:

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\int_0^T (u_t^{(n)}, v_t)_H w_t \mathrm{d}t &= \lim_{n\to\infty} \mathbb{E}\int_0^T (u_t^{(n)}, w_t v_t)_H \mathrm{d}t \\ &= \mathbb{E}\int_0^T (u_t, w_t v_t)_H \mathrm{d}t\end{aligned} \tag{2.13}$$

where the last line follows since $wv \in L^2(\Omega; L^2([0,T]; H))$. Now, we can consider the LHS and RHS as duality pairings between the spaces $L^2(\Omega; L^2([0,T]; w; H))$ and $L^2(\Omega; L^2([0,T]; w; H))$. Therefore,

$$\mathbb{E}\int_0^T (u_t, v_t)_H w_t \mathrm{d}t \leq \liminf \| u^{(n)} \|_{L^2(\Omega; L^2([0,T]; w; H))} \| v \|_{L^2(\Omega; L^2([0,T]; H))} \tag{2.14}$$

We can therefore conclude that the lemma holds. □

## 2.2. Stochastc integration theory

The main motivation to study stochastic integration theory is to study stochastic differential equations. One could consider the following ODE:

$$\frac{\mathrm{d}u_t}{\mathrm{d}t} = f(t, u_t) + \xi(t), u(0) = u_0, \tag{2.15}$$

where $\xi(t)$ is some kind of forcing signal. In many situations in nature, it is reasonable to assume that $\xi$ is a random signal. For instance, the Langevin equation (also called the Ornstein-Uhlenbeck process by mathematicians)

$$m\frac{\mathrm{d}^2 x_t}{\mathrm{d}t^2} = -6\pi\mu a \frac{\mathrm{d}x}{\mathrm{d}t} + \xi(t)$$

describes the motion of a particle of radius $a$ suspended in a liquid with viscosity $\mu$ with Gaussian white noise $\xi(t)$, see [25] (translated version [27]). Using the theory of stochastic integration introduced by Kyoshi Itô, we can give rigorous meaning to the above equation [18]. For other motivations to study stochastic differential equations and stochastic integration theory, see for instance the introduction of [30].

Of course, we can consider stochastic forcings for PDEs as well. This leads to the study of SPDEs, which we briefly touched upon in the exposition. To make sense of the SPDEs that we are going to study, we need a theory of stochastic integration in Hilbert spaces. Suppose one is given an SPDE of the form

$$\frac{\partial u_t}{\partial t} = \Delta u_t + \xi(t, x),$$

where $\xi(t, x)$ is a white noise in space and time (this is a stochastich heat equation). The aim is then to reformulate this SPDE as a function-space valued ODE, just as in the deterministic setting. The function spaces we consider in this master's thesis are almost always Hilbert spaces. We can give meaning to the white noise[1] by using stochastic integration theory in Hilbert spaces.

### 2.2.1. Stochastic integration in $\mathbb{R}^d$

For a primer on stochastic integration theory in $\mathbb{R}^d$ we refer to [19]. We also refer to the short callback in [28].

### 2.2.2. Gaussian measure theory and infinite dimensional Wiener processes

The goal of this subsection is to introduce enough theory so that we can build the stochastic integral for Hilbert spaces in the next section. We start of by recalling the definition of an $\mathbb{R}$-valued Wiener process (also called Brownian motion sometimes).

**Definition 2.4.** A stochastic process $\{W_t\}_{t\geq 0}$ is called a *Wiener process* whenever the following four properties hold:

1. $W(0) = 0$ a.s.

2. $W(t) - W(s) \sim \mathcal{N}(0, t-s)$ for all $t \geq s \geq 0$.

[1] Strictly speaking the spatial dimension should be of dimension 1. In higher dimensions, the white noise can only be interpreted as a distribution.

3. For any finite collection of time points $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, the collection $W_{t_n} - W_{t_{n-1}}$, $W_{t_{n-1}} - W_{t_{n-2}}, \ldots, W_{t_1}$ is independent.

4. The paths $t \mapsto W_t$ are continuous a.s.

It is not straightforward how one should extend the second property. One standard characterization of the normal distribution is that it has Gaussian density. However, this does not work in infinite dimensions, since there exists no infinite dimensional Lebesgue measure. Therefore, we will introduce infinite dimensional Wiener processes using a different approach, namely Gaussian measures. To this end, let $(U, (\cdot,\cdot))$ be a separable Hilbert space.

**Definition 2.5.** A probability measure $\mu$ on $(U, (\cdot,\cdot))$ is said to be Gaussian whenever the measure $\mu \circ l^{-1}$ has Gaussian density for all $l \in U^*$, that is,

$$\mu(l \in A) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_A e^{-\frac{(x-m)^2}{2\sigma^2}} \mathrm{d}x \qquad \text{for all } A \in \mathscr{B}(\mathbb{R})$$

where $m \in \mathbb{R}$ and $\sigma > 0$, and $\mathscr{B}(\mathbb{R})$ denotes the Borel $\sigma$-algebra on $\mathbb{R}$.

If one takes $U = \mathbb{R}^d$, where $d \geq 1$, and $X$ is an $\mathbb{R}^d$-valued Gaussian random variable, we can find that the above definition also holds for $X$. Indeed, for any $a \in \mathbb{R}^d$, $a \cdot X$ is normally distributed with mean zero. Since any linear functional in $U^*$ can be identified with some $a \in \mathbb{R}^d$, we obtain the above definition. This gives a hint that the new definition is indeed the 'right' one.

It turns out that Gaussian measures can be characterized by a mean $m \in U$ and a nonnegative, symmetrice, finite trace, covariance matrix $Q \in L(U)$, just as in the real case. We have the following theorem:

**Theorem 2.1.** *A measure $\mu$ on a separable Hilbert space $(U, (\cdot,\cdot))$ is Gaussian if and only if*

$$\widehat{\mu}(u) = \int_U e^{i(u,v)} \mathrm{d}\mu = e^{i(m,u) - \frac{1}{2}(Qu,u)}, \quad u \in U,$$

*where $m \in U$, $Q \in L(U)$ is nonnegative, symmetric and with finite trace.*

*Proof.* See [28, Theorem 2.1.2, p. 10]. □

From now on, we will denote a Gaussian measure $\mu$ as $N(m,Q)$ where $m$ and $Q$ are specified using the above theorem. However, this theorem does not show us if we can form any Gaussian measure given arbitrary $m \in U$ and $Q \in L(U)$. It turns out that this is also possible, which will be helpful to define the infinite dimensional Wiener process.

**Proposition 2.5.** *Let $(U, (\cdot,\cdot))$ be a separable Hilbert space, $m \in U$ and $Q \in L(U)$ be nonnegative, symmetric with tr $Q < \infty$. Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $U$, consisting of the eigenvectors of $Q$ with corresponding eigenvalues $\lambda_k$, where $k \in \mathbb{N}$. Then, a $U$-valued random variable $X$ is Gaussian with $\mathbb{P} \circ X^{-1} = N(m,Q)$ if and only if*

$$X = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k e_k + m$$

*where $\{\beta_k\}_{k\in\mathbb{N}}$ are independent real-valued standard normal random variables. Furthermore, the above series converges in $L^2(\Omega; U)$.*

This leads to the following corollary:

**Corollary 2.2.** *Let $Q$ be a nonnegative and symmetric operator in $L(U)$ with finite trace and let $m \in U$. Then there exists a Gaussian measure $\mu = N(m, Q)$ on $(U, (\cdot,\cdot))$.*

*Proof of proposition 2.5.* See [28, Proposition 2.1.6, p. 14]. □

This allows us to introduce the definition of the infinite dimensional Wiener process on a Hilbert space.

**Definition 2.6.** Let $(U, (\cdot,\cdot))$ be a separable Hilbert space. A $U$-valued stochastic process $\{W(t)\}_{t\in[0,T]}$ is called a $Q$-Wiener process if:

1. $W(0) = 0$ a.s.
2. $W(t) - W(s) \sim N(0, (t-s)Q)$ for all $t \geq s \geq 0$
3. For any finite collection of time points $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, the collection $W_{t_n} - W_{t_{n-1}}$, $W_{t_{n-1}} - W_{t_{n-2}}, \dots, W_{t_1}$ is independent.
4. The paths $t \mapsto W_t$ are continuous a.s.

The existence of such a process can be derived from the real case. The interested reader is again referred to [10, 28]. Ideally, we would like to set $Q = I$, but this is not possible in the current setting, since the identity operator does not have finite trace. If we go back to Definition 2.5, this means that it is not possible to seek convergence for

$$\sum_{k=1}^{\infty} \beta_k(t) e_k$$

in $L^2(\Omega; L^2([0,T]; U))$. Using Hilbert-Schmidt embedding, we can let the above series converge in a different space, and therefore define the infinite dimensional version of the Wiener process for traceless $Q$. This type of Wiener process is called the *cylindrical* Wiener process. We will only consider setting $Q = I$, though it is also possible to take other traceless $Q$. It is also clear that the identity operator $I$ is nonnegative and symmetric. Then we have the following proposition:

**Proposition 2.6.** *Let $(U, (\cdot,\cdot))$ be a separable Hilbert space. Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $U$ and $\{\beta_k\}_{k\in\mathbb{N}}$ be a family of independent, real-valued Brownian motions. Let $(U_1, (\cdot,\cdot))$ be a further separable Hilbert space such that $U$ is Hilbert-Schmidt embedded in $U_1$, i.e. $U \subseteq U_1$ and $J : U \to U_1$ is Hilbert-Schmidt. Last but not least, define $Q_1 = JJ^*$, which is nonnegative and symmetric with finite trace and $Q_1 \in L(U_1)$. Then, the series*

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) J e_k, \quad t \in [0,T], \tag{2.16}$$

*converges in $L^2(\Omega; L^\infty([0,T]; U_1))$ and defines a $Q_1$-Wiener process on $U_1$. Moreover, we have $Q_1^{\frac{1}{2}}(U_1) = J(U)$ and for all $u \in U$,*

$$\|u\|_U = \|Q_1^{\frac{1}{2}} J u\|_{U_1} = \|Ju\|_{Q_1^{\frac{1}{2}}(U_1)}.$$

*Proof.* See [28, Proposition 2.5.2, p. 50]. □

**Remark 2.1.** One can also introduce Hilbert space-valued Wiener processes through isonormal processes. See for instance [37].

### 2.2.3. Stochastic integration in Hilbert spaces

Having made sense of the Wiener process in separable Hilbert spaces, we are in the position to introduce the stochastic integral. We will do this through 4 steps, which mimick the construction in the real case:

**Step 1:** Defining the stochastic integral for a certain class of elementary $L(U,H)$-valued processes.

**Step 2:** Prove an Itô isometry and use the isometry to extend the integral to a larger class of integrands.

**Step 3:** Identify the extension.

**Step 4:** Extend the stochastic integral through localization.

The construction we give follows [28]. Throughout this section, $H$ and $U$ are assumed to be separable Hilbert spaces.
**Step 1:**
Let $\phi : \mathbb{R}_+ \times \Omega \to L(U,H)$ be an adapted elementary process of finite rank. That is, $\phi$ is a linear combination of processes of the form

$$1_{(s,t]\times F}(u \otimes h)$$

where $0 \le s < t$, $F \in \mathscr{F}_s$, $u \in U$, $h \in H$ and $u \otimes h$ is a linear operator $u \otimes h : U \to H$ such that $(u \otimes h)u' = (u,u')h$ for all $u' \in U$. We can then define the stochastic integral against a cylindrical Wiener process $W$ by setting

$$\int_0^\infty 1_{(s,t]\times F}(u \otimes h)\mathrm{d}W := 1_F(W(t) - W(s), u) \otimes h$$

and extend this to all adapted elentary processes of finite rank by linearity.

**Step 2:**
Let $\phi$ be an adapted elementary process of finite rank. We can write $\phi$ as

$$\phi = \sum_{n=1}^{N} 1_{(t_{n-1},t_n]} \sum_{m=1}^{M} 1_{F_{mn}} \sum_{j=1}^{k} u_j \otimes h_{jmn}$$

where $(u_j)_{j=1}^k$ is an orthonormal system of $U$, for each $1 \le n \le N$, the sets $F_{mn}$, $1 \le m \le M$, are disjoint and belong to $\mathscr{F}_{t_{n-1}}$. Finally, $h_{jmn} \in H$. We see that

$$\int_0^\infty \phi\,\mathrm{d}W = \sum_{n,m,j=1}^{N,M,k} 1_{F_{mn}}(W(t_n) - W(t_n), u_j) \otimes h_{jmn}. \tag{2.17}$$

We use the inner product structure of the Hilbert space $H$ to obtain the following isometry:

$$\begin{aligned}\mathbb{E}\left\|\int_0^\infty \phi\,\mathrm{d}W\right\|_H^2 &= \mathbb{E}\left\|\sum_{n,m,j=1}^{N,M,k} 1_{F_{mn}}(W(t_n) - W(t_{n-1}), u_j) \otimes h_{jmn}\right\|_H^2 \\ &\overset{(*)}{=} \sum_{n,m,j=1}^{N,M,k} (t_n - t_{n-1})\mathbb{E}\left[1_{F_{mn}}\|h_{jmn}\|_H^2\right] \\ &= \mathbb{E}\int_0^\infty \|\phi_t\|_{L_2(U,H)}^2 \mathrm{d}t.\end{aligned} \tag{2.18}$$

In the $(*)$ step, we use that $W$ is a Wiener process, orthonormality of the collection $(u_j)_{j=1}^k$ and adaptedness of $\phi$. For details, see [28].

**Step 3:**

Let $\mathscr{E}$ be the class of all adapted finite rank elementary processes. Then, the above isometry can be used to extend the stochastic integral to the closure of $\mathscr{E}$, which turns out to be all progressively measurable $\phi : \mathbb{R}_+ \times \Omega \to L_2(U,H)$ such that $\mathbb{E}\int_0^\infty \|\phi_t\|^2_{L_2(U,H)} \mathrm{d}t < \infty$. We will not prove this here, but refer to [28].

**Step 4:**

The above stochastic integral can also be extended to integrands $\phi : \mathbb{R}_+ \times \Omega \to L_2(U,H)$ for which $\phi$ is progressively measurable and $\mathbb{P}\left(\int_0^\infty \|\phi_t\|^2_{L_2(U,H)} \mathrm{d}t < \infty\right)$, again see [28].

We will also state Burkholder's inequality, arguably the most important inequality of this master's thesis:

**Theorem 2.2.** *Let $(U,(\cdot,\cdot))$ and $(H,(\cdot,\cdot))$ be separable Hilbert spaces. Then, for any progressively measurable $\phi : \mathbb{R}_+ \times \Omega \to L_2(U,H)$ and $0 < p < \infty$, there exists a constant $C$ only depending on $p$ such that*

$$\mathbb{E}\sup_{t\geq 0}\left\|\int_0^t \phi \mathrm{d}W\right\|_H^p \leq C\|\phi\|^p_{L^p(\Omega;L^2(\mathbb{R}_+;L_2(U,H)))} \tag{2.19}$$

*Proof.* See [12]. □

### 2.2.4. Itô formulas for the square norm and $p$-norm

A cornerstone of much of the study of stochastic differential equations is Itô's lemma. For example, Itô's lemma allows us to verify solutions of stochastic differential equations and prove certain important theorems such as the martingale representation theorem and the Burkholder-Davis-Gundy inequalities. Seeing that we are interested in obtaining solutions of SPDEs, it is therefore desirable to produce a similar formula in an infinite-dimensional setting. We give such a result for the square norm and subsequentially for the $p$-th norm where $p \geq 2$. The results in the next chapters crucially hinge on the Itô formulas presented here.

In the sequel we assume that $(V,H,V^*)$ is a Gelfand triple. That is, $V$ is a reflexive Banach space with its dual $V^*$ and $H$ is a Hilbert space such that $V$ is densely embedded in $H$ (from which it follows that $H^*$ is densely embedded in $V^*$). We have the following two theorems:

**Theorem 2.3.** *Let $\alpha \in (1,\infty)$, $X_0 \in L^2(\Omega;\mathscr{F}_0,\mathbb{P};H)$ and $Y \in L^{\frac{\alpha}{\alpha-1}}([0,T]\times\Omega;\mathrm{d}t\otimes\mathbb{P};V^*)$, $Z \in L^2([0,T]\times\Omega;\mathrm{d}t\otimes\mathbb{P};L_2(U,H))$, both progressively measurable. Define the following continuous $V^*$-valued process*

$$X(t) = X_0 + \int_0^t Y(s)\mathrm{d}s + \int_0^t Z(s)\mathrm{d}W(s), \quad t \in [0,T].$$

*If for its $\mathrm{d}t\otimes\mathbb{P}$-equivalence class $\hat{X}$ we have $\hat{X} \in L^\alpha([0,T]\times\Omega;\mathrm{d}t\otimes\mathbb{P};V)$ and if $\mathbb{E}(\|X(t)\|_H^2) < \infty$ for $\mathrm{d}t$-a.e. $t \in [0,T]$ then $X$ is a continuous $H$-valued $\mathscr{F}_t$-adapted process and the following*

*Itô-formula holds for the square of its $H$-norm $\mathbb{P}$-a.s.:*

$$\begin{aligned}\|X(t)\|_H^2 = \|X_0\|_H^2 + \int_0^t \Big(2\langle Y(s), \bar{X}(s)\rangle_V + \|Z(s)\|^2_{L_2(U,H)}\Big)\,\mathrm{d}s \\ + 2\int_0^t (X(s), Z(s)\cdot)\,\mathrm{d}W(s) \quad \textit{for all } t\in[0,T],\end{aligned} \tag{2.20}$$

*where $\bar{X}$ is a $V$-valued progressively measurable version of $X$.*

*Proof.* See [28, Theorem 4.2.5, p. 91] or [34, Theorem 3.2, p. 73]. □

We can generalize the above theorem to any $p \in [2,\infty)$

**Theorem 2.4.** *Let $p \in [2,\infty)$, $\alpha \in (1,\infty)$, $X_0 \in L^p(\Omega;\mathscr{F}_0,\mathbb{P};H)$ and $Y \in L^{\frac{\alpha}{\alpha-1}}([0,T]\times\Omega;\mathrm{d}t\otimes\mathbb{P};V^*)$, $Z\in L^2([0,T]\times\Omega;\mathrm{d}t\otimes\mathbb{P};L_2(U,H))$, both progressively measurable. Define the following continuous $V^*$-valued process*

$$X(t) = X_0 + \int_0^t Y(s)\mathrm{d}s + \int_0^t Z(s)\mathrm{d}W(s), \quad t\in[0,T].$$

*If for its $\mathrm{d}t\otimes P$-equivalence class $\hat{X}$ we have $\hat{X}\in L^\alpha([0,T]\times\Omega;\mathrm{d}t\otimes\mathbb{P};V)$ and if $\mathbb{E}(\|X(t)\|_H^2)<\infty$ for $\mathrm{d}t$-a.e. $t\in[0,T]$, then $X$ is a continuous $H$-valued $\mathscr{F}_t$-adapted process and the following Itô-formula holds $\mathbb{P}$-a.s.:*

$$\begin{aligned}\|X(t)\|_H^p = \|X_0\|_H^p + p\int_0^t \|X(s)\|_H^{p-2}(X(s),Z(s)\cdot)\mathrm{d}W_s \\ + \frac{p(p-2)}{2}\int_0^t \|X(s)\|_H^{p-4}\|Z(s)^*X(s)\|_U^2\mathrm{d}s \\ + \frac{p}{2}\int_0^t \|X(s)\|_H^{p-2}\Big(2\langle Y(s),\bar{X}(s)\rangle + \|Z(s)\|^2_{L_2(U,H)}\Big)\mathrm{d}s \quad \textit{for all } t\in[0,T],\end{aligned} \tag{2.21}$$

*where $\bar{X}$ is a $V$-valued progressively measurable version of $X$.*

*Proof.* Since $X_0\in L^p(\Omega;\mathscr{F}_0;H)\subseteq L^2(\Omega;\mathscr{F}_0;H)$ for any $p\in[2,\infty)$, we can apply Theorem 2.3. Therefore, we obtain continuity and $\mathscr{F}_t$-adaptedness for $X$ as an $H$-valued process. We also obtain the formula stated in Theorem 2.3:

$$\begin{aligned}\|X(t)\|_H^2 = \|X_0\|_H^2 + \int_0^t \Big(2\langle Y(s), \bar{X}(s)\rangle_V + \|Z(s)\|^2_{L_2(U,H)}\Big)\,\mathrm{d}s \\ + 2\int_0^t (X(s), Z(s)\cdot)\,\mathrm{d}W(s), \quad \text{for all } t\in[0,T],\end{aligned} \tag{2.22}$$

Now, observe that this formula implies that $\|X(t)\|_H^2$ is a real-valued local semi-martingale. We can therefore apply the real-valued version of Itô's lemma [19, theorem 3.3, p. 149]. Since $f(x)=x^{\frac{p}{2}}$ is not $C^2$ for $p\in(2,4)$, we use an approximating sequence of $C^2$ functions $f_\varepsilon(x)=\left(x^2+\varepsilon\right)^{\frac{p}{4}}$, where $\varepsilon>0$. It is clear that the theorem holds for $p\in[4,\infty)$. For exposition, we first calculate the relevant derivatives of $f_\varepsilon$. These are

$$f_\varepsilon'(x) = \tfrac{p}{2}\left(x^2+\varepsilon\right)^{\frac{p}{4}-1}x \tag{2.23}$$

and

$$f_\varepsilon''(x) = \tfrac{p}{2}\left(x^2+\varepsilon\right)^{\frac{p}{4}-1} + p\left(\tfrac{p}{4}-1\right)\left(x^2+\varepsilon\right)^{\frac{p}{4}-2}x^2. \tag{2.24}$$

Using these derivatives, we can apply Itô's lemma. This results in:

$$\begin{aligned}
&\left(\|X(t)\|_H^4+\varepsilon\right)^{\frac{p}{4}}\\
&= \left(\|X_0\|_H^4+\varepsilon\right)^{\frac{p}{4}} + \underbrace{p\int_0^t \left(\|X(s)\|_H^4+\varepsilon\right)^{\frac{p}{4}-1}\|X(s)\|_H^2(X(s),Z(s)\cdot)\mathrm{d}W_s}_{\boxed{\mathrm{A}}}\\
&\quad+\underbrace{\frac{p}{2}\int_0^t \left(\|X(s)\|_H^4+\varepsilon\right)^{\frac{p}{4}-1}\|X(s)\|_H^2\left(2\langle Y(s),\bar{X}(s)\rangle+\|Z(s)\|_{L_2(U,H)}^2\right)\mathrm{d}s}_{\boxed{\mathrm{B}}}\\
&\quad+\underbrace{2\int_0^t \left[p\left(\frac{p}{4}-1\right)\left(\|X(s)\|_H^4+\varepsilon\right)^{\frac{p}{4}-2}\|X(s)\|_H^4+\frac{p}{2}\left(\|X(s)\|_H^4+\varepsilon\right)^{\frac{p}{4}-1}\right]\|Z(s)^*X(s)\|_U^2\mathrm{d}s}_{\boxed{\mathrm{C}}}
\end{aligned} \tag{2.25}$$

We will take the limit in probability by taking $\varepsilon \to 0$ on both sides. By considering subsequences, we obtain the formula stated in the theorem. Therefore, we inspect terms $\boxed{A}$, $\boxed{B}$ and $\boxed{C}$ individually.

For term $\boxed{A}$, we use the Itô isometry. Therefore, it suffices to show the following:

$$\mathbb{E}\int_0^t \left[\left(\|X(s)\|_H^4+\varepsilon\right)^{\frac{p}{4}-1}\|X(s)\|_H^2-\|X(s)\|_H^{p-2}\right]^2\|Z(s)^*X(s)\|_U^2\mathrm{d}s \to 0 \qquad \text{as } \varepsilon\to 0 \tag{2.26}$$

We will show this by using the dominated convergence theorem. Taking $\varepsilon > 0$ small, we have the following bound:

$$\begin{aligned}
&\left(\|X(s)\|_H^4+\varepsilon\right)^{\frac{p-4}{4}}\|X(s)\|_H^2-\|X(s)\|_H^{p-2}\\
&\le \left(\|X(s)\|_H^4+\varepsilon\right)^{\frac{p-2}{4}}-\|X(s)\|_H^{p-2}\\
&\le C\left(\|X(s)\|_H^{p-2}+\varepsilon^{\frac{p-2}{4}}\right)\\
&\le C\left(\|X(s)\|_H^{p-2}+M\right),
\end{aligned} \tag{2.27}$$

where the third line follows since $p\in[2,\infty)$. Since $X$ is continuous in $H$, the above multiplied by $\|Z^*X\|_U^2$, is integrable in $t$ a.s. Therefore, we can apply the DCT to obtain the limit (2.26). Therefore, we have shown the following:

$$\begin{aligned}
\mathbb{P}\text{-}\lim_{\varepsilon\to 0}\boxed{\mathrm{A}} &= \mathbb{P}\text{-}\lim_{\varepsilon\to 0}\int_0^t \left(\|X(s)\|_H^4+\varepsilon\right)^{\frac{p}{4}-1}\|X(s)\|_H^2(X(s),Z(s)\cdot)\mathrm{d}W_s\\
&= \int_0^t \|X(s)\|_H^{p-2}(X(s),Z(s)\cdot)\mathrm{d}W_s.
\end{aligned} \tag{2.28}$$

The limits need to be interpreted as limits in probability. We continue with term $\boxed{B}$. The integrand in $\boxed{B}$ can be dominated in a similar way as in (2.27). This results in:

$$\begin{aligned}&\left(\|X(s)\|_H^4+\varepsilon\right)^{\frac{p-4}{4}}\|X(s)\|_H^2\left(2\langle Y(s),\bar{X}(s)\rangle+\|Z(s)\|_{L_2(U,H)}^2\right)\\&\leq C(\|X(s)\|_H^{p-2}+M)\left(2\|Y(s)\|_{V^*}\|\bar{X}(s)\|_V+\|Z(s)\|_{L_2(U,H)}^2\right)\end{aligned}\tag{2.29}$$

Therefore, if we pick the RHS as dominating function for the DCT, we only need to check integrability. Indeed,

$$\begin{aligned}&\int_0^t C(\|X(s)\|_H^{p-2}+M)(2\|Y(s)\|_{V^*}\|\bar{X}(s)\|_V+\|Z(s)\|_{L_2(U,H)}^2)\mathrm{d}s\\&\leq C(\sup_{t\in[0,T]}\|X(t)\|_H^{p-2}+M)\int_0^t 2\|Y(s)\|_{V^*}\|\bar{X}(s)\|_V+\|Z(s)\|_{L_2(U,H)}^2\mathrm{d}s\\&\leq C(\sup_{t\in[0,T]}\|X(t)\|_H^{p-2}+M)\Bigg(2\left(\int_0^t\|Y(s)\|_{V^*}^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\alpha-1}{\alpha}}\left(\int_0^t\|\bar{X}(s)\|_V^{\alpha}\mathrm{d}s\right)^{\frac{1}{\alpha}}\\&\quad+\int_0^t\|Z(s)\|_{L_2(U,H)}^2\mathrm{d}s\Bigg).\end{aligned}\tag{2.30}$$

Since $X$ is continuous, $Y\in L^{\frac{\alpha}{\alpha-1}}(\Omega\times[0,T];V^*)$, $X\in L^{\alpha}(\Omega\times[0,T];V)$ and $Z\in L^2(\Omega\times[0,T];L_2(U,H))$, all above quantities are finite $\mathbb{P}$-a.s. Therefore, we can apply the DCT for term $\boxed{B}$. This results in:

$$\begin{aligned}\mathbb{P}\text{-}\lim_{\varepsilon\to0}\boxed{\mathrm{B}}&=\mathbb{P}\text{-}\lim_{\varepsilon\to0}\int_0^t\left(\|X(s)\|_H^4+\varepsilon\right)^{\frac{p-4}{4}}\|X(s)\|_H^2\left(2\langle Y(s),\bar{X}(s)\rangle+\|Z(s)\|_{L_2(U,H)}^2\right)\\&=\int_0^t\|X(s)\|_H^{p-2}\left(2\langle Y(s),\bar{X}(s)\rangle+\|Z(s)\|_{L_2(U,H)}^2\right)\mathrm{d}s.\end{aligned}\tag{2.31}$$

Here, the limits should also be interpreted as limits in probability. We are only left to treat term $\boxed{C}$. We will do this again by using the DCT. First note by the Cauchy-Schwarz inequality and some other elementary inequalities, that for all $\varepsilon>0$ small enough, there exists $C\geq0$ such that:

$$\begin{aligned}&\left|p\left(\frac{p}{4}-1\right)\left(\|X(s)\|_H^4+\varepsilon\right)^{\frac{p}{4}-2}\|X(s)\|_H^4+\frac{p}{2}\left(\|X(s)\|_H^4+\varepsilon\right)^{\frac{p}{4}-1}\right|\|Z(s)^*X(s)\|_U^2\\&\leq p\left(\frac{p-2}{4}\right)\left(\|X(s)\|_H^4+\varepsilon\right)^{\frac{p}{4}-1}\|Z(s)^*X(s)\|_U^2\\&\leq p\left(\frac{p-2}{4}\right)\left(\|X(s)\|_H^4+\varepsilon\right)^{\frac{p-2}{4}}\|Z(s)\|_{L_2(U,H)}^2\\&\leq Cp\left(\frac{p-2}{4}\right)(\|X(s)\|_H^{p-2}+M)\|Z(s)\|_{L_2(U,H)}^2,\end{aligned}\tag{2.32}$$

where the last line follows since $p>2$. By similar arguments for terms $\boxed{A}$ and $\boxed{B}$, the last quantity is integrable. Therefore, we can apply the DCT to conclude:

$$\begin{aligned}
&\mathbb{P}\text{-}\lim_{\varepsilon\to 0}\boxed{\text{C}}\\
&=\mathbb{P}\text{-}\lim_{\varepsilon\to 0}\int_0^t\left[p\left(\frac{p}{4}-1\right)\left(\|X(s)\|_H^4+\varepsilon\right)^{\frac{p}{4}-2}\|X(s)\|_H^4+\frac{p}{2}\left(\|X(s)\|_H^4+\varepsilon\right)^{\frac{p}{4}-1}\right]\|Z(s)^*X(s)\|_U^2\mathrm{d}s\\
&=p\left(\frac{p-2}{4}\right)\int_0^t\|X(s)\|_H^{p-4}\|Z(s)^*X(s)\|_U^2\mathrm{d}s
\end{aligned} \tag{2.33}$$

Again, the limits should be interpreted as limits in probabilty. We can combine all limits and take a subsequence to obtain the Itô formula stated in the theorem, which concludes the proof. □

**Remark 2.2.** It turns out that Theorem 2.3 can be generalized even further than Theorem 2.4. See for example [22].

## 2.3. Standard notions from PDE theory

In this section, we will state some definitions and notation used for elements of PDE theory used in this master's thesis. We will

### 2.3.1. Sobolev spaces

We first state the definition of a Sobolev space on some set $\mathscr{D} \subseteq \mathbb{R}^d$.

**Definition 2.7.** The Sobolev space $W^{k,p}(\mathscr{D})$ is defined as the space of all $u \in L^1_{\text{loc}}(\mathscr{D})$ such that for each multiindex $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\mathscr{D})$.

We can turn the Sobolev spaces into normed spaces, identifying functions that are equal a.e., by defining

$$\|u\|_{W^{k,p}(\mathscr{D})} := \left( \sum_{|\alpha| \leq k} \int_{\mathscr{D}} \left| D^\alpha u \right|^p \mathrm{d}x \right)^{\frac{1}{p}}, \qquad \text{for } 1 \leq p < \infty. \tag{2.34}$$

For $p = \infty$, we define

$$\|u\|_{W^{k,\infty}(\mathscr{D})} := \sum_{|\alpha| \leq k} \operatorname{ess\,sup}_{\mathscr{D}} \left| D^\alpha u \right|. \tag{2.35}$$

It is a standard exercise to prove that the Sobolev spaces are Banach spaces. However, unlike the $L^p$ spaces, $C_c^\infty$ functions are not always dense in the Sobolev spaces. Therefore, we make the following separate definition.

**Definition 2.8.** We let $W_0^{k,p}(\mathscr{D})$ be the closure of $C_c^\infty(\mathscr{D})$ in $W^{k,p}(\mathscr{D})$ under the above norms.

**Remark 2.3.** The Sobolev spaces for which $p = 2$ take a special place in the theory of PDE, because they are Hilbert spaces. They are usually denoted as $H^k(\mathscr{D}) = W^{k,2}(\mathscr{D})$. Accordingly, the closure of $C_c^\infty(\mathscr{D})$ in $W^{k,2}(\mathscr{D})$ is denoted by $H_0^k(\mathscr{D})$.

**Remark 2.4.** We will denote the dual of $H_0^k(\mathscr{D})$ by $H^{-k}(\mathscr{D})$.

# 3

# A coercivity condition for higher order moments

In this chapter we will introduce the new $p$-dependent coercivity condition that will allow us to obtain $L^p(\Omega; L^2([0,T]; X))$ estimates for solutions of SPDEs in the respective class we study. It must be mentioned that this condition is only a slight generalization of a relatively new coercivity condition introduced in a recent paper by David Šiška and Neelima Varshney [38]. In turn, they extended earlier work by Liu and Röckner, a more extensive summary of which can be found in the introduction.The main novelty of this thesis over the work of Šiška is two-fold: We slightly generalize the coercivity condition as well as obtaining endpoints for $L^p(\Omega; L^2([0,T]; X))$ estimates. Slightly generalizing the $p$-dependent coercivity condition allows us to distinguish between $p$-dependent and $p$-independent examples such as the stochastic heat equation and the $p$-Laplacian. Furthermore, Šiška and Neelima only obtained almost $p$-th moments, i.e. estimates of type

$$\mathbb{E} \sup_{t \in [0,T]} \|u\|_H^{rp}, \tag{3.1}$$

where $r \in (0,1)$, whereas our methods allow for $r = 1$. This is because we avoid the use of Lenglart's inequality.

This chapter is structured as follows. We first state the setting and assumptions we use for the SPDEs we study. We then proceed to prove our main workhorse of this thesis, which are a priori estimates on the solution of the SPDEs in our class. Using these a priori estimates, we prove existence and uniqueness using Galerkin approximations. We proceed by extending our framework to additive equations, which will be especially useful when we consider examples. Last but not least, we prove that the results that we have obtained are optimal in the sense that it is not possible to obtain higher moments based on the assumptions we use.

## 3.1. Setting, assumptions and a solution definition

This section will first illustrate the assumptions from which we will build our theory, and define variational solutions for SPDEs. Afterwards, we will give some comments on these assumptions.

**Assumptions 3.1.** *Let $(U,(\cdot,\cdot)_U)$, $(H,(\cdot,\cdot)_H)$ be separable Hilbert spaces and $(V,\|\cdot\|_V)$ a reflexive Banach space embedded continuously and densely in $H$. We denote the dual of $V$ by $V^*$ and the duality pairing by $\langle\cdot,\cdot\rangle$. We can then consider the Gelfand triple $(V,H,V^*)$ where all embeddings are dense and continuous.*

*Now, consider the SPDE*

$$\mathrm{d}u_t = A(t,u_t)\mathrm{d}t + B(t,u_t)\mathrm{d}W_t, \tag{3.2}$$

*where $A$ is a nonlinear operator*

$$A : [0,T]\times\Omega\times V \to V^*$$

*and $B$ a nonlinear operator*

$$B : [0,T]\times\Omega\times V \to L_2(U,H).$$

*Both operators are assumed to be progressively measurable. To shorten notation, we will denote $A(t,u_t)$ as $A_t(u_t)$ from now on, in which we also surpress the dependence on $\omega\in\Omega$. The same remark holds for $B(t,u_t)$. We consider $(W_t)_{t\in[0,T]}$ to be a $U$-valued cylindrical Wiener process, as defined in the previous chapter, proposition 2.6. Furthermore, it is assumed that there exist constants $\alpha>1,\beta\geq 0,p_0\geq\beta+2,\theta>0,K_c,K_A,K_B,K_\alpha\geq 0$ and a nonnegative function $f\in L^{\frac{p_0}{2}}(\Omega;L^1([0,T];\mathbb{R}))$ such that the following five conditions hold for $t\in[0,T]$ a.s.*

*(H1) (Hemicontinuity) For all $u,v,w\in V$, $\omega\in\Omega$, the map*

$$\lambda\mapsto\langle A_t(u+\lambda v,\omega),w\rangle_V$$

*is continuous.*

*(H2) (Local weak monotonicity) For all $u,v\in V$,*

$$\begin{aligned}&2\langle A_t(u)-A_t(v),u-v\rangle_V+\|B_t(u)-B_t(v)\|^2_{L_2(U,H)}\\&\leq K(1+\|v\|_V^\alpha)(1+\|v\|_H^\beta)\|u-v\|_H^2.\end{aligned}$$

*(H3) (Coercivity) For all $v\in V$, $v\neq 0$,*

$$2\langle A_t(v),v\rangle_V+\|B_t(v)\|^2_{L_2(U,H)}+(p_0-2)\frac{\|B_t(v)^*v\|_U^2}{\|v\|_H^2}\leq-\theta\|v\|_V^\alpha+f_t+K_c\|v\|_H^2.$$

*(H4) (Boundedness 1) For all $v\in V$,*

$$\|A_t(v)\|_{V^*}^{\frac{\alpha}{\alpha-1}}\leq K_A(f_t+\|v\|_V^\alpha)(1+\|v\|_H^\beta).$$

*(H5) (Boundedness 2) For all $v\in V$,*

$$\|B_t(v)\|^2_{L_2(U,H)}\leq\left(f_t+K_B\|v\|_H^2+K_\alpha\|v\|_V^\alpha\right).$$

**Remark 3.1.** Most conditions are fairly standard and appear in previous works that treat the variational approach to SPDEs [31] [23] [28]. The coercivity condition we use is slightly more general than the one in Šiška and Neelima's paper, which is as follows:

$$2\langle A_t(v), v\rangle + (p_0 - 1)\|B_t(v)\|^2_{L_2(U,H)} \le -\theta\|v\|^\alpha_V + f_t + K_c\|v\|^2_H. \tag{3.3}$$

Whenever Šiška and Neelima's condition holds, we can derive our coercivity condition H3 using Cauchy-Schwarz and the operator adjoint isometry for Hilbert-Schmidt operators. We have also introduced an extra condition, H5, which bounds the norm $\|B_t(v)\|^2_{L_2(U,H)}$. In most other works [28][23], this bound is only mentioned as an extra assumption in the main theorems. We also use a slightly more general bound, taking inspiration from recent work by Brzezniak, Liu and Zhang [4]. Last but not least, we mention that the constant $\beta$ allows us to incorporate some polynomial growth in the operator $A$.

**Remark 3.2.** In most cases, $\alpha = 2$, $\beta = 0$, $K_B = 0$ and $f = 0$. This is for example the case in the stochastic heat equation treated in the next chapter.

Having laid out the assumptions from which we will work, we need to define what it means to be a solution to equation (3.2).

**Definition 3.1.** Given the assumptions in assumptions 3.1, let $u_0 \in L^{p_0}(\Omega; H)$ be the initial condition to (3.2). An adapted, continuous $H$-valued process $u$ is called a solution of the stochastic evolution equation (3.2) if:

1. $\mathrm{d}t \times \mathbb{P}$ almost everywhere $u \in V$ and
$$\int_0^T \|u_t\|^\alpha_V \mathrm{d}t < \infty \quad \text{a.s.}$$
2. For every $t \in [0,T]$, $\mathbb{P}$-a.s.,
$$u_t = u_0 + \int_0^t A_s(u_s)\mathrm{d}s + \int_0^t B_s(u_s)\mathrm{d}W_s,$$

where the $u_s$ on the RHS is taken to be a progressively measurable version to ensure the stochastic integral exists.

## 3.2. A priori estimates on the solution

We first present a priori estimates (which we will also call energy estimates occasionally) on the solution of SPDE (3.2) under assumptions H3, H4 and H5. These are estimates under the solution that SPDE (3.2) indeed has a solution. Eventhough this might seem superfluous, a priori estimates will later help us to prove existence and uniqueness for SPDE (3.2). The a priori estimates are summarized in the following theorem:

**Theorem 3.1.** *Suppose $u$ is a solution of equation* (3.2) *with initial condition $u_0 \in L^{p_0}(\Omega; H)$ and assumptions H3, H4 and H5 hold. Then, for all $p \in [2, p_0]$, there exists a constant $C$ depending on $\theta$, $\alpha$, and $p$ such that*

$$\begin{aligned} &\mathbb{E}\sup_{t\in[0,T]}\|u_t\|^p_H + \mathbb{E}\int_0^T \|u_t\|^{p-2}_H\|u_t\|^\alpha_V \mathrm{d}t + \mathbb{E}\left(\int_0^T \|u_t\|^\alpha_V \mathrm{d}t\right)^{\frac{p}{2}} \\ &\le Ce^{CT}\left(\mathbb{E}\|u_0\|^p_H + \mathbb{E}\left(\int_0^T f_t \mathrm{d}t\right)^{\frac{p}{2}}\right). \end{aligned} \tag{3.4}$$

If we assume that $K_B = K_C = 0$ in H3 and H5, we can make energy estimates with constants that do not on $p$. This will be commented on in the remark after the next proof.

*Proof of Theorem 1.* The above estimate can be proven by proving the inequality for each term on the LHS individually. Before we do this, we first lay out some general estimates that will be useful for all three terms. To this end, suppose that $u$ is a solution of equation (3.2). We introduce a sequence of stopping times to apply the function $\|\cdot\|_H^p$ to $u$ by using theorem 2.4. Fix $S \in (0,T]$and consider the following sequence of stopping times:

$$\tau_n = \inf\{t \in [0,S] : \|u_t\|_H > n\} \wedge \inf\{t \in [0,S] : \int_0^t \|u_s\|_V^\alpha ds \geq n\} \wedge S, \tag{3.5}$$

where $n \in \mathbb{N}$. It is clear that $\tau_n \to S$ a.s. as $n \to \infty$. Applying theorem 2.4, this results in the following formula for $\|u_{t\wedge\tau_n}\|_H^p$:

$$\begin{aligned}\|u_{t\wedge\tau_n}\|_H^p = \|u_0\|_H^p &+ p\int_0^{t\wedge\tau_n} \|u_s\|_H^{p-2}(u_s, B_s(u_s)\cdot)\mathrm{d}W_s \\ &+ \frac{p(p-2)}{2}\int_0^{t\wedge\tau_n} \|u_s\|_H^{p-4}\|B_s^*(u_s)u_s\|_U^2 \mathrm{d}s \\ &+ \frac{p}{2}\int_0^{t\wedge\tau_n} \|u_s\|_H^{p-2}\left(2\langle A_s(u_s), u_s\rangle + \|B_s(u_s)\|_{L_2(U,H)}^2\right)\mathrm{d}s,\end{aligned} \tag{3.6}$$

where the dot in $(u_s, B_s(u_s)\cdot)$ is short notation for a linear operator. We combine the last two deterministic integrals, which results in:

$$\begin{aligned}\|u_{t\wedge\tau_n}\|_H^p = \|u_0\|_H^p &+ p\int_0^{t\wedge\tau_n} \|u_s\|_H^{p-2}(u_s, B_s(u_s)\cdot)\mathrm{d}W_s \\ &+ \frac{p}{2}\int_0^{t\wedge\tau_n} \|u_s\|_H^{p-2}\Big(2\langle A_s(u_s), u_s\rangle + \|B_s(u_s)\|_{L_2(U,H)}^2 \\ &+ (p-2)\frac{\|B_s(u_s)^* u_s\|_U^2}{\|u_s\|_H^2}\Big)\mathrm{d}s.\end{aligned} \tag{3.7}$$

The last expression on the RHS can be simplified by invoking the coercivity assumption H3, giving:

$$\begin{aligned}\|u_{t\wedge\tau_n}\|_H^p + \theta\frac{p}{2}\int_0^{t\wedge\tau_n}\|u_s\|_V^\alpha \mathrm{d}s \leq \|u_0\|_H^p &+ p\int_0^{t\wedge\tau_n} \|u_s\|_H^{p-2}(u_s, B_s(u_s)\cdot)\mathrm{d}W_s \\ &+ \frac{p}{2}\int_0^{t\wedge\tau_n} \|u_s\|_H^{p-2}\left(f_s + K_c\|u_s\|_H^2 - \theta\|u_s\|_V^\alpha\right)\mathrm{d}s.\end{aligned} \tag{3.8}$$

This inequality will be our starting point to derive different inequalities which ultimately lead to our result. Taking expectations on both sides of (3.8) and using that the stochastic integral is a martingale results in:

$$\begin{aligned}\mathbb{E}\|u_{t\wedge\tau_n}\|_H^p + \theta\frac{p}{2}\mathbb{E}\int_0^{t\wedge\tau_n}\|u_s\|_H^{p-2}\|u_s\|_V^\alpha \mathrm{d}s \leq \mathbb{E}\|u_0\|_H^p &+ \frac{p}{2}\mathbb{E}\int_0^{t\wedge\tau_n}\|u_s\|_H^{p-2}f_s\mathrm{d}s \\ &+ \frac{p}{2}K_c\mathbb{E}\int_0^{t\wedge\tau_n}\|u_s\|_H^p\mathrm{d}s.\end{aligned} \tag{3.9}$$

This estimate will be re-used at different times throughout the proof. At this point, we specialize to indivual terms of the inequality in the theorem.

*Step 1: Supremum term*
We proceed by making an estimate on the quantity

$$\mathbb{E}\sup_{t\in[0,S]}\|u_t\|_H^p. \tag{3.10}$$

By re-using the Ito formula obtained in (3.8) for $\mathbb{E}\sup\|u_{t\wedge\tau_n}\|_H^p$, we obtain the following estimate:

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,S]}\|u_{t\wedge\tau_n}\|_H^p \le \mathbb{E}\|u_0\|_H^p &+ p\mathbb{E}\sup_{t\in[0,S]}\int_0^{t\wedge\tau_n}\|u_s\|_H^{p-2}(u_s, B_s(u_s)\cdot)\mathrm{d}W_s \\ &+ \frac{p}{2}\mathbb{E}\sup_{t\in[0,S]}\int_0^{t\wedge\tau_n}\|u_s\|_H^{p-2}\left(f_s + K_c\|u_s\|_H^2\right)\mathrm{d}s.\end{aligned} \tag{3.11}$$

Note that we can remove the supremum on the last term as the integrand is nonnegative. Therefore, we are left with

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,S]}\|u_{t\wedge\tau_n}\|_H^p \le \mathbb{E}\|u_0\|_H^p &+ p\mathbb{E}\sup_{t\in[0,S]}\int_0^{t\wedge\tau_n}\|u_s\|_H^{p-2}(u_s, B_s(u_s)\cdot)\mathrm{d}W_s \\ &+ \frac{p}{2}\mathbb{E}\int_0^{\tau_n}\|u_s\|_H^{p-2}f_s\mathrm{d}s \\ &+ \frac{p}{2}K_c\mathbb{E}\int_0^{\tau_n}\|u_s\|_H^p\mathrm{d}s.\end{aligned} \tag{3.12}$$

Note that the last two integrals range from 0 to $\tau_n$ as $S\wedge\tau_n = \tau_n$ by definition. We estimate the stochastic integral first, which we will recombine with estimate (3.12). Let $\varepsilon_1 > 0$. We estimate the stochastic integral term by doing a combination of the Burkholder-Davis-Grundy inequality, Hölder, the Peter-Paul inequality and invoking H5:

$$\begin{aligned}&p\mathbb{E}\sup_{t\in[0,S]}\int_0^{t\wedge\tau_n}\|u_s\|_H^{p-2}(u_s, B_s(u_s)\cdot)\mathrm{d}W_s \\ &\overset{(BDG)}{\le} 2\sqrt{2}p\mathbb{E}\left(\int_0^{\tau_n}\|u_s\|_H^{2p-2}\|B_s(u_s)\|_{L_2(U,H)}^2\mathrm{d}s\right)^{\frac{1}{2}} \\ &\le\quad p2\sqrt{2}\mathbb{E}\left(\sup_{t\in[0,S]}\|u_{t\wedge\tau_n}\|_H^p\int_0^{\tau_n}\|u_s\|_H^{p-2}\|B_s(u_s)\|_{L_2(U,H)}^2\mathrm{d}s\right)^{\frac{1}{2}} \\ &\overset{\text{(Hölder)}}{\le} p2\sqrt{2}\left(\mathbb{E}\sup_{t\in[0,S]}\|u_{t\wedge\tau_n}\|_H^p\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^{\tau_n}\|u_s\|_H^{p-2}\|B_s(u_s)\|_{L_2(U,H)}^2\mathrm{d}s\right)^{\frac{1}{2}} \\ &\overset{\text{(PP)}}{\le}\quad p\sqrt{2}\varepsilon_1\mathbb{E}\sup_{t\in[0,S]}\|u_{t\wedge\tau_n}\|_H^p + \frac{p}{\varepsilon_1}\sqrt{2}\mathbb{E}\int_0^{\tau_n}\|u_s\|_H^{p-2}\|B_s(u_s)\|_{L_2(U,H)}^2\mathrm{d}s \\ &\overset{\text{(H5)}}{\le}\quad p\sqrt{2}\varepsilon_1\mathbb{E}\sup_{t\in[0,S]}\|u_{t\wedge\tau_n}\|_H^p \\ &\quad+\quad\sqrt{2}\frac{p}{\varepsilon_1}\left(\mathbb{E}\int_0^{\tau_n}\|u_s\|_H^{p-2}(f_s + K_B\|u_s\|_H^2 + K_\alpha\|u_s\|_V^\alpha)\mathrm{d}s\right).\end{aligned} \tag{3.13}$$

We recombine this estimate with (3.12), taking the supremum term to the LHS. This results in

$$
\begin{aligned}
&(1-p\sqrt{2}\varepsilon_1)\mathbb{E}\sup_{t\in[0,S]}\|u_{t\wedge\tau_n}\|_H^p\\
&\leq \mathbb{E}\|u_0\|_H^p + p\left(\frac{\sqrt{2}}{\varepsilon_1}+\frac{1}{2}\right)\mathbb{E}\int_0^{\tau_n}\|u_s\|_H^{p-2}f_s\mathrm{d}s + p\left(\frac{\sqrt{2}K_B}{\varepsilon_1}+\frac{K_c}{2}\right)\mathbb{E}\int_0^{\tau_n}\|u_s\|_H^p\mathrm{d}s\\
&\quad + pK_\alpha\frac{\sqrt{2}}{\varepsilon_1}\mathbb{E}\int_0^{\tau_n}\|u_s\|_H^{p-2}\|u_s\|_V^\alpha\mathrm{d}s. \qquad (3.14)
\end{aligned}
$$

The last term on the RHS can be estimated by re-using estimate (3.9). This results in

$$
\begin{aligned}
\mathbb{E}\int_0^{\tau_n}\|u_s\|_H^{p-2}\|u_s\|_V^\alpha\mathrm{d}s \leq \frac{2}{p\theta}\mathbb{E}\|u_0\|_H^p + \frac{1}{\theta}\mathbb{E}\int_0^{\tau_n}\|u_s\|_H^{p-2}f_s\mathrm{d}s&\\
+\frac{K_c}{\theta}\mathbb{E}\int_0^{\tau_n}\|u_s\|_H^p\mathrm{d}s.& \qquad (3.15)
\end{aligned}
$$

Recombining with estimate (3.14) gives

$$
\begin{aligned}
&(1-p\sqrt{2}\varepsilon_1)\mathbb{E}\sup_{t\in[0,S]}\|u_{t\wedge\tau_n}\|_H^p \qquad (3.16)\\
&\leq \left(1+K_\alpha\frac{2\sqrt{2}}{\varepsilon_1\theta}\right)\mathbb{E}\|u_0\|_H^p + p\left(\frac{\sqrt{2}}{\varepsilon_1}+\frac{1}{2}+K_\alpha\frac{\sqrt{2}}{\theta\varepsilon_1}\right)\underbrace{\mathbb{E}\int_0^{\tau_n}\|u_s\|_H^{p-2}f_s\mathrm{d}s}_{\boxed{\mathrm{A}}}\\
&\quad + p\left(\frac{\sqrt{2}K_B}{\varepsilon_1}+\frac{K_c}{2}+\frac{\sqrt{2}K_\alpha K_c}{\theta\varepsilon_1}\right)\underbrace{\mathbb{E}\int_0^{\tau_n}\|u_s\|_H^p\mathrm{d}s}_{\boxed{\mathrm{B}}}. \qquad (3.17)
\end{aligned}
$$

It remains to estimate items $\boxed{\mathrm{A}}$ and $\boxed{\mathrm{B}}$. We first estimate item $\boxed{\mathrm{A}}$, whereas term $\boxed{\mathrm{B}}$ will be estimated later using Gronwall's inequality. To this end, let $\varepsilon_2 > 0$. We use Hölder's inequality and Young's inequality to obtain

$$
\begin{aligned}
\mathbb{E}\int_0^{\tau_n}\|u_s\|_H^{p-2}f_s\mathrm{d}s &\leq \mathbb{E}\sup_{t\in[0,S]}\|u_{t\wedge\tau_n}\|_H^{p-2}\int_0^{\tau_n}f_s\mathrm{d}s\\
&\overset{\text{(Hölder)}}{\leq} \left(\mathbb{E}\sup_{t\in[0,S]}\|u_t\|_H^p\right)^{\frac{p}{p-2}}\left(\mathbb{E}\left(\int_0^{\tau_n}f_s\mathrm{d}s\right)^{\frac{p}{2}}\right)^{\frac{2}{p}} \qquad (3.18)\\
&\overset{\text{(Young)}}{\leq} \frac{p-2}{p}\varepsilon_2\mathbb{E}\sup_{t\in[0,S]}\|u_{t\wedge\tau_n}\|_H^p + \frac{2}{p\varepsilon_2^{\frac{p-2}{2}}}\mathbb{E}\left(\int_0^{\tau_n}f_s\mathrm{d}s\right)^{\frac{p}{2}}.
\end{aligned}
$$

Filling in this estimate on the RHS of (3.16), taking the supremum term to the LHS again, we finally obtain:

$$
\begin{aligned}
&\left(1-p\sqrt{2}\varepsilon_1-(p-2)\varepsilon_2\left(\frac{\sqrt{2}}{\varepsilon_1}+\frac{1}{2}+K_\alpha\frac{\sqrt{2}}{\theta\varepsilon_1}\right)\right)\mathbb{E}\sup_{t\in[0,S]}\|u_{t\wedge\tau_n}\|_H^p\\
&\leq\left(1+K_\alpha\frac{2\sqrt{2}}{\varepsilon_1\theta}\right)\mathbb{E}\|u_0\|_H^p\\
&\quad+\frac{2}{\varepsilon_2^{\frac{p-2}{p}}}\left(\frac{\sqrt{2}}{\varepsilon_1}+\frac{1}{2}+K_\alpha\frac{\sqrt{2}}{\theta\varepsilon_1}\right)\mathbb{E}\left(\int_0^{\tau_n}f_s\mathrm{d}s\right)^{\frac{p}{2}}\\
&\quad+p\left(\frac{\sqrt{2}K_B}{\varepsilon_1}+\frac{K_c}{2}+\frac{\sqrt{2}K_\alpha K_c}{\theta\varepsilon_1}\right)\mathbb{E}\int_0^{\tau_n}\|u_s\|_H^p\mathrm{d}s
\end{aligned}
$$

We can simplify the above expression in such way that we can apply Gronwall's inequality. We do this by choosing $\varepsilon_1$ and $\varepsilon_2$ such that the bracket term on the LHS is positive. Denoting the resulting constant by $C$ (depending on $K_c, K_A, K_B, K_\alpha$, $p$ and $\theta$), we obtain

$$\mathbb{E}\sup_{t\in[0,S]}\|u_{t\wedge\tau_n}\|_H^p\leq C\left(\mathbb{E}\|u_0\|_H^p+\mathbb{E}\left(\int_0^{\tau_n}f_s\mathrm{d}s\right)^{\frac{p}{2}}+\mathbb{E}\int_0^{\tau_n}\|u_s\|_H^p\right) \tag{3.19}$$

As $S$ was arbitrarily chosen in $(0,T]$ and the constant $C$ does not depend on $S$, we can apply Gronwall's inequality as a function of S. We find:

$$\mathbb{E}\sup_{t\in[0,T]}\|u_{t\wedge\tau_n(T)}\|_H^p\leq Ce^{CT}\left(\mathbb{E}\|u_0\|_H^p+\mathbb{E}\left(\int_0^{\tau_n(T)}f_s\mathrm{d}s\right)^{\frac{p}{2}}\right) \tag{3.20}$$

We can remove the stopping times by overestimating the RHS and then using Fatou's lemma on the LHS. This leaves us with:

$$\mathbb{E}\sup_{t\in[0,T]}\|u_t\|_H^p\leq Ce^{CT}\left(\mathbb{E}\|u_0\|_H^p+\mathbb{E}\left(\int_0^{T}f_s\mathrm{d}s\right)^{\frac{p}{2}}\right) \tag{3.21}$$

Therefore, up to some constants, we have derived the first part of the inequality stated in the theorem.

*Step 2: Mixed norm*

The second part of the inequality in the theorem follows by re-using equation (3.9) at time $t=T$. We therefore also need to reintroduce the stopping time $\tau_n$ used before. Specifically, we reintroduce $\tau_n$ where we take $S=T$:

$$\tau_n=\inf\{t\in[0,T]:\|u_t\|_H>n\}\wedge\inf\{t\in[0,T]:\int_0^t\|u_s\|_V^\alpha ds\geq n\}\wedge T. \tag{3.22}$$

We can now re-use equation (3.9). Leaving out one of the terms on the LHS, this results in the following estimate:

$$\theta\frac{p}{2}\mathbb{E}\int_0^{t\wedge\tau_n}\|u_s\|_H^{p-2}\|u_s\|_V^\alpha\mathrm{d}s\leq\mathbb{E}\|u_0\|_H^p+\frac{p}{2}K\mathbb{E}\int_0^T\|u_s\|_H^{p-2}f_s\mathrm{d}s+\frac{p}{2}K\mathbb{E}\int_0^T\|u_s\|_H^p\mathrm{d}s \tag{3.23}$$

The second term on the RHS is estimated as in equation (3.18). This results in:

$$\mathbb{E}\int_0^T \|u_s\|_H^{p-2} f_s \mathrm{d}s \le \frac{p-2}{p}\mathbb{E}\sup_{t\in[0,T]}\|u_t\|_H^p + \frac{2}{p}\mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}} \tag{3.24}$$

Using the estimate derived in (3.21), we obtain:

$$\mathbb{E}\int_0^T \|u_s\|_H^{p-2} f_s\mathrm{d}s \le \frac{p-2}{p}Ce^{CT}\mathbb{E}\|u_0\|_H^p + \left(\frac{2}{p}+\frac{p-2}{p}Ce^{CT}\right)\mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}} \tag{3.25}$$

We now proceed to estimate the third term on the RHS of equation (3.23). We again use (3.21) to get:

$$\begin{aligned}\mathbb{E}\int_0^T \|u_s\|_H^p\mathrm{d}s &\le T\mathbb{E}\sup_{t\in[0,T]}\|u_t\|_H^p\\ &\le TCe^{CT}\left(\mathbb{E}\|u_0\|_H^p + \mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}}\right)\\ &\le C'e^{C'T}\left(\mathbb{E}\|u_0\|_H^p + \mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}}\right).\end{aligned} \tag{3.26}$$

Recombining estimates (3.25) and (3.26) in equation (3.23), we finally obtain:

$$\mathbb{E}\int_0^{t\wedge\tau_n}\|u_s\|_H^{p-2}\|u_s\|_V^\alpha\mathrm{d}s \le C''e^{C''T}\left(\mathbb{E}\|u_0\|_H^p + \mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}}\right). \tag{3.27}$$

Applying Fatou's lemma on the LHS and setting $t = T$, we obtain the estimate

$$\mathbb{E}\int_0^T\|u_s\|_H^{p-2}\|u_s\|_V^\alpha\mathrm{d}s \le C''e^{C''T}\left(\mathbb{E}\|u_0\|_H^p + \mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}}\right). \tag{3.28}$$

*Step 3: V-norm*
There is only a third part of the inequality stated in the theorem that we have to prove. We are interested in the following quantity:

$$\mathbb{E}\Big(\int_0^T\|u_s\|_V^\alpha\mathrm{d}s\Big)^{\frac{p}{2}} \tag{3.29}$$

In order to estimate this quantity, we have to apply Itô's formula for $\|u_t\|_H^2$ as given in [28], Theorem 4.2.5. This results in:

$$\|u_t\|_H^2 = \|u_0\|_H^2 + \int_0^t\Big(2\langle A_s(u_s),u_s\rangle + \|B_s(u_s)\|_{L_2(U,H)}^2\Big)\mathrm{d}s + 2\int_0^t(u_s,B_s(u_s)\cdot)\mathrm{d}W_s \tag{3.30}$$

Apply the coercivity assumption H3 to the second term to get:

$$\begin{aligned}\|u_t\|_H^2 &+ \int_0^t\Big((p_0-2)\frac{\|B_s(u_s)^*u_s\|_U^2}{\|u_s\|_H^2} + \theta\|u_s\|_V^\alpha\Big)\mathrm{d}s\\ &\le \|u_0\|_H^2 + K_A\int_0^t\big(f_t + \|u_t\|_H^2\big)\mathrm{d}t + 2\int_0^t(u_s,B_s(u_s)\cdot)\mathrm{d}W_s\end{aligned} \tag{3.31}$$

In particular,

$$\theta\int_0^t \|u_s\|_V^\alpha \mathrm{d}s \le \|u_0\|_H^2 + K_A\int_0^t \left(f_t+\|u_t\|_H^2\right)\mathrm{d}t + 2\int_0^t (u_s, B_s(u_s)\cdot)\mathrm{d}W_s \tag{3.32}$$

We introduce the stopping time $\tau_n$ again:

$$\tau_n = \inf\{t\in[0,S] : \|u_t\|_H > n\} \wedge \inf\{t\in[0,S] : \int_0^t \|u_s\|_V^\alpha > n\}\wedge T \tag{3.33}$$

Applying the function $|\cdot|^{\frac{p}{2}}$ to both sides and taking expectations at time $t\wedge\tau_n$, we obtain the following estimate:

$$\begin{aligned}
&\theta^{\frac{p}{2}}\mathbb{E}\left(\int_0^{t\wedge\tau_n}\|u_s\|_V^\alpha\mathrm{d}s\right)^{\frac{p}{2}}\\
&\le C\left(\mathbb{E}\|u_0\|_H^p + \mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}} + K\left(\mathbb{E}\int_0^{t\wedge\tau_n}\|u_s\|_H^2\mathrm{d}s\right)^{\frac{p}{2}} + 2^{\frac{p}{2}}\mathbb{E}\left|\int_0^{t\wedge\tau_n}(u_s,B_s(u_s)\cdot)\mathrm{d}W_s\right|^{\frac{p}{2}}\right)
\end{aligned} \tag{3.34}$$

We are only left to estimate the third term on the RHS. Applying the BDG inequality, Hölder's inequality and Young's inequality, we obtain:

$$\begin{aligned}
\mathbb{E}\left|\int_0^{t\wedge\tau_n}(u_s,B_s(u_s)\cdot)\mathrm{d}W_s\right|^{\frac{p}{2}} &\le \mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t(u_s,B_s(u_s)\cdot)\mathrm{d}W_s\right|^{\frac{p}{2}}\\
&\le C_p\mathbb{E}\left(\int_0^T\|u_s\|_H^2\|B_s(u_s)\|_{L_2(U,H)}^2\mathrm{d}s\right)^{\frac{p}{4}}\\
&\le C_p\mathbb{E}\left(\sup_{t\in[0,T]}\|u_t\|_H^2\int_0^T\|B_s(u_s)\|_{L_2(U,H)}^2\mathrm{d}s\right)^{\frac{p}{4}}\\
&\le C_p\left(\mathbb{E}\sup_{t\in[0,T]}\|u_t\|_H^p\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\int_0^T\|B_s(u_s)\|_{L_2(U,H)}^2\mathrm{d}s\right)^{\frac{p}{2}}\right)^{\frac{1}{2}}\\
&\overset{\text{Yng.}}{\le} C_p\frac{1}{2\varepsilon}\mathbb{E}\sup_{t\in[0,T]}\|u_t\|_H^p + C_p\frac{\varepsilon}{2}\mathbb{E}\left(\int_0^T\|B_s(u_s)\|_{L_2(U,H)}^2\mathrm{d}s\right)^{\frac{p}{2}}
\end{aligned} \tag{3.35}$$

We use H5 to estimate the second term on the RHS. This results in:

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T\|B_s(u_s)\|_{L_2(U,H)}^2\mathrm{d}s\right)^{\frac{p}{2}}\\
&\le\mathbb{E}\left(\int_0^T f_s+\|u_s\|_H^2+\|u_s\|_V^\alpha\mathrm{d}s\right)^{\frac{p}{2}}\\
&\le C\mathbb{E}\left(\int_0^T f_s\mathrm{d}s\right)^{\frac{p}{2}} + C\mathbb{E}\left(\int_0^T\|u_s\|_H^2\mathrm{d}s\right)^{\frac{p}{2}} + C\mathbb{E}\left(\int_0^T\|u_s\|_V^\alpha\mathrm{d}s\right)^{\frac{p}{2}}\\
&\le C\mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}} + CT^{\frac{p}{2}}\mathbb{E}\sup_{t\in[0,T]}\|u_t\|_H^p + C\mathbb{E}\left(\int_0^T\|u_s\|_V^\alpha\mathrm{d}s\right)^{\frac{p}{2}}
\end{aligned} \tag{3.36}$$

Recombine this estimate with inequality (3.35) to get:

$$
\begin{aligned}
&\mathbb{E}\left|\int_0^t (u_s, B_s(u_s)\cdot)\mathrm{d}W_s\right|^{\frac{p}{2}} \\
&\leq \left(C_p\frac{1}{2\varepsilon} + CC_pT^{\frac{p}{2}}\frac{\varepsilon}{2}\right)\mathbb{E}\sup_{t\in[0,T]}\|u_t\|_H^p + CC_p\frac{\varepsilon}{2}\mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}} + CC_p\frac{\varepsilon}{2}\mathbb{E}\left(\int_0^T \|u_s\|_V^\alpha\mathrm{d}s\right)^{\frac{p}{2}} \\
&\overset{(3.21)}{\leq} C_\varepsilon\mathbb{E}\|u_0\|_H^p + C'_\varepsilon\mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}} + CC_p\frac{\varepsilon}{2}\mathbb{E}\left(\int_0^T \|u_s\|_V^\alpha\mathrm{d}s\right)^{\frac{p}{2}}
\end{aligned}
\tag{3.37}
$$

We can finally combine this with the estimate (3.34) to obtain:

$$
\begin{aligned}
&\theta^{\frac{p}{2}}\mathbb{E}\left(\int_0^{t\wedge\tau_n} \|u_s\|_V^\alpha\mathrm{d}s\right)^{\frac{p}{2}} \\
&\leq \left(C+C'_\varepsilon\right)\mathbb{E}\|u_0\|_H^p + (C+C_\varepsilon)\mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}} + CC_p\frac{\varepsilon}{2}\mathbb{E}\left(\int_0^T \|u_s\|_V^\alpha\mathrm{d}s\right)^{\frac{p}{2}}
\end{aligned}
\tag{3.38}
$$

Choosing time $t = T$, we obtain:

$$
\left(\theta^{\frac{p}{2}} - CC_p\frac{\varepsilon}{2}\right)\mathbb{E}\left(\int_0^T \|u_s\|_V^\alpha\mathrm{d}s\right)^{\frac{p}{2}} \leq (C+C_\varepsilon)\mathbb{E}\|u_0\|_H^p + (C+C'_\varepsilon)\mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}} \tag{3.39}
$$

Choosing $\varepsilon > 0$ small enough, we finally obtain the estimate we are interested in:

$$
\mathbb{E}\left(\int_0^T \|u_s\|_V^\alpha\mathrm{d}s\right)^{\frac{p}{2}} \leq C''e^{C''T}\left(\mathbb{E}\|u_0\|_H^p + \mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}}\right) \tag{3.40}
$$

To conclude the proof we combine the three main estimates (3.21), (3.23) and (3.40) □

**Remark 3.3.** When $K_B = K_c = 0$ in assumptions H1-H5, it is possible to make energy estimates where some of the constants in the estimate do not depend on $p$. Specifically, we can prove the following estimates:

$$
\begin{aligned}
&\left(\mathbb{E}\sup_{t\in[0,T]}\|u_t\|_H^p\right)^{\frac{1}{p}} \\
&\leq C\left[\|u_0\|_{L^p(\Omega;H)} + \|f\|^2_{L^{\frac{p}{2}}(\Omega;L^1([0,T];\mathbb{R}))}\right],
\end{aligned}
\tag{3.41}
$$

where $C$ depends on $\theta, \alpha, K_A$ and $K_\alpha$, but not on p!. We also get:

$$
\left(\mathbb{E}\left(\int_0^T \|u_s\|_V^\alpha\mathrm{d}s\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \leq C'\left[\|u_0\|_{L^p(\Omega;H)} + \|f\|^2_{L^{\frac{p}{2}}(\Omega;L^1([0,T];\mathbb{R}))}\right], \tag{3.42}
$$

where $C'$ depends on $\theta, \alpha, K_A$ and $K_\alpha$ and $p$. In this case, the constant $C'$ behaves as $C'_p \sim \sqrt{p}$. The first estimate allows one to obtain $L^\infty$ estimates, while the second estimate is still useful to obtain tail estimates. Applications of this remark can be found in the next chapter when we treat the Burgers' equation and the stochastic Navier-Stokes equation in 2D.

*Proof of Remark 3.3.* We prove the inequality for the two terms on the LHS individually.

*Step 1: Supremum norm* We can do exactly the same steps in the previous proof to obtain inequality (3.19), where $\varepsilon_1, \varepsilon_2 > 0$:

$$\begin{aligned}
&\left(1 - p\sqrt{2}\varepsilon_1 - (p-2)\varepsilon_2\left(\frac{\sqrt{2}}{\varepsilon_1} + \frac{1}{2} + K_\alpha \frac{\sqrt{2}}{\theta\varepsilon_1}\right)\right)\mathbb{E}\sup_{t\in[0,S]}\|u_{t\wedge\tau_n}\|_H^p \\
&\leq \left(1 + K_\alpha \frac{2\sqrt{2}}{\varepsilon_1\theta}\right)\mathbb{E}\|u_0\|_H^p \\
&\quad + \frac{2}{\varepsilon_2^{\frac{p-2}{p}}}\left(\frac{\sqrt{2}}{\varepsilon_1} + \frac{1}{2} + K_\alpha\frac{\sqrt{2}}{\theta\varepsilon_1}\right)\mathbb{E}\left(\int_0^{\tau_n} f_s \mathrm{d}s\right)^{\frac{p}{2}}.
\end{aligned} \tag{3.43}$$

We can choose

$$\varepsilon_1 = \frac{1}{2\sqrt{2}p}, \qquad \varepsilon_2 = \frac{1}{2(p-2)(8p+1+K_\alpha\frac{8p}{\theta})}$$

which results in the constant on the LHS being equal to $\frac{1}{4}$. We also note that

$$\frac{1}{\varepsilon_2} \leq 16p^2(1+K_\alpha\frac{1}{\theta}) + p^2 + 1 \leq Ap^2 + 1,$$

where $A$ is a constant depending on $K_\alpha$ and $\theta$. Therefore, we get:

$$\begin{aligned}
&\frac{1}{4}\mathbb{E}\sup_{t\in[0,S]}\|u_{t\wedge\tau_n}\|_H^p \\
&\leq \left(1 + K_\alpha\frac{8p}{\theta}\right)\mathbb{E}\|u_0\|_H^p + (Ap^2+1)^{\frac{p-2}{p}}\mathbb{E}\left(\int_0^{\tau_n} f_s\mathrm{d}s\right)^{\frac{p}{2}}
\end{aligned} \tag{3.44}$$

We take $1/p$ powers on both sides to obtain:

$$\begin{aligned}
&\left(\mathbb{E}\sup_{t\in[0,T]}\|u_t\|_H^p\right)^{\frac{1}{p}} \\
&\leq 4^{\frac{1}{p}}\left[\left(1 + K_\alpha\frac{8p}{\theta}\right)^{\frac{1}{p}}\|u_0\|_{L^p(\Omega;H)} + \left(Ap^2+1\right)^{\frac{p-2}{p^2}}\|f\|^2_{L^{\frac{p}{2}}(\Omega;L^1([0,T];\mathbb{R}))}\right].
\end{aligned} \tag{3.45}$$

Both constants can be bounded uniformly in $p$. Therefore, we obtain the estimate we were looking for:

$$\left(\mathbb{E}\sup_{t\in[0,T]}\|u_t\|_H^p\right)^{\frac{1}{p}} \leq C\left[\|u_0\|_{L^p(\Omega;H)} + \|f\|^2_{L^{\frac{p}{2}}(\Omega;L^1([0,T];\mathbb{R}))}\right], \tag{3.46}$$

where $C$ depends on $\theta$ and $K_\alpha$.

*Step 2: V-norm* To prove the inequality for the V term, we re-use equation (3.34). We therefore obtain:

$$\begin{aligned}&\mathbb{E}\left(\int_0^{t\wedge\tau_n}\|u_s\|_V^\alpha \mathrm{d}s\right)^{\frac{p}{2}}\\&\leq \frac{2^{\frac{p}{2}}}{\theta^{\frac{p}{2}}}\left(\mathbb{E}\|u_0\|_H^p+\mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}}+2^{\frac{p}{2}}\underbrace{\mathbb{E}\left|\int_0^{t\wedge\tau_n}(u_s,B_s(u_s)\cdot)\mathrm{d}W_s\right|^{\frac{p}{2}}}_{\boxed{A}}\right).\end{aligned}\tag{3.47}$$

We estimate term $\boxed{A}$ as in (3.35) using the BDG inequality. This results in

$$\boxed{A}\leq\left(\frac{p}{2}\right)^{\frac{p}{4}}\frac{1}{\varepsilon}\mathbb{E}\sup_{t\in[0,T]}\|u_t\|_H^p+\left(\frac{p}{2}\right)^{\frac{p}{4}}\varepsilon\mathbb{E}\left(\int_0^{\tau_n}\|B_s(u_s)\|^2_{L_2(U,H)}\mathrm{d}s\right)^{\frac{p}{2}},\tag{3.48}$$

where $\varepsilon>0$ is chosen retrospectively. We continue to estimate the second term on the RHS. Using assumption H5, we obtain

$$\mathbb{E}\left(\int_0^{\tau_n}\|B_s(u_s)\|^2_{L_2(U,H)}\mathrm{d}s\right)^{\frac{p}{2}}\leq 2^{\frac{p}{2}}\mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}}+2^{\frac{p}{2}}K_\alpha^{\frac{p}{2}}\mathbb{E}\left(\int_0^{\tau_n}\|u_s\|_V^\alpha\mathrm{d}s\right)^{\frac{p}{2}}.\tag{3.49}$$

Recombine all terms with inequality (3.47) to obtain

$$\begin{aligned}&\mathbb{E}\left(\int_0^{t\wedge\tau_n}\|u_s\|_V^\alpha \mathrm{d}s\right)^{\frac{p}{2}}\\&\leq \frac{2^{\frac{p}{2}}}{\theta^{\frac{p}{2}}}\left(\mathbb{E}\|u_0\|_H^p+\left(1+2^p\left(\frac{p}{2}\right)^{\frac{p}{4}}\varepsilon\right)\mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}}\right.\\&\quad\left.+2^{\frac{p}{2}}\sqrt{\frac{p}{2}}\frac{1}{\varepsilon}\mathbb{E}\sup_{t\in[0,T]}\|u_t\|_H^p+2^pK_\alpha^{\frac{p}{2}}\left(\frac{p}{2}\right)^{\frac{p}{4}}\varepsilon\mathbb{E}\left(\int_0^{\tau_n}\|u_s\|_V^\alpha\right)^{\frac{p}{2}}\right).\end{aligned}\tag{3.50}$$

Set time $t=T$. We can take the last term to the LHS to obtain

$$\begin{aligned}&\left(1-\frac{2^{\frac{p}{2}}}{\theta^{\frac{p}{2}}}2^pK_\alpha^{\frac{p}{2}}\left(\frac{p}{2}\right)^{\frac{p}{4}}\varepsilon\right)\mathbb{E}\left(\int_0^{\tau_n}\|u_s\|_V^\alpha\mathrm{d}s\right)^{\frac{p}{2}}\\&\leq\frac{2^{\frac{p}{2}}}{\theta^{\frac{p}{2}}}\left(\mathbb{E}\|u_0\|_H^p+\left(1+2^p\left(\frac{p}{2}\right)^{\frac{p}{4}}\varepsilon\right)\mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}}+2^{\frac{p}{2}}\left(\frac{p}{2}\right)^{\frac{p}{4}}\frac{1}{\varepsilon}\mathbb{E}\sup_{t\in[0,T]}\|u_t\|_H^p\right).\end{aligned}\tag{3.51}$$

We choose

$$\varepsilon=\frac{\theta^{\frac{p}{2}}}{2^{p+1}K_\alpha^{\frac{p}{2}}2^{\frac{p}{2}}\left(\frac{p}{2}\right)^{\frac{p}{4}}}.$$

This leads to the constant on the LHS being equal to $\frac{1}{2}$. The following is obtained:

$$\begin{aligned}&\frac{1}{2}\mathbb{E}\left(\int_0^{\tau_n}\|u_s\|_V^\alpha\mathrm{d}s\right)^{\frac{p}{2}}\\&\leq\frac{2^{\frac{p}{2}}}{\theta^{\frac{p}{2}}}\left(\mathbb{E}\|u_0\|_H^p+\left(1+\frac{\theta^{\frac{p}{2}}}{K_\alpha^{\frac{p}{2}}2^{\frac{p}{2}+1}}\right)\mathbb{E}\Big(\int_0^T f_s\mathrm{d}s\Big)^{\frac{p}{2}}+\left(\frac{p}{2}\right)^{\frac{p}{2}}\frac{2^{2p+1}K_\alpha^{\frac{p}{2}}}{\theta^{\frac{p}{2}}}\mathbb{E}\sup_{t\in[0,T]}\|u_t\|_H^p\right).\end{aligned}\tag{3.52}$$

We can take $1/p$ powers on both sides and re-use the estimate in step 1, to obtain

$$\begin{aligned}&\left(\mathbb{E}\left(\int_0^{\tau_n}\|u_s\|_V^\alpha \mathrm{d}s\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\\ &\leq 2^{\frac{1}{p}}\sqrt{\frac{2}{\theta}}\left(\|u_0\|_{L^p(\Omega;H)}+\left(1+\frac{\sqrt{\theta}}{\sqrt{K_\alpha}2^{\frac{p+2}{2p}}}\right)\|f\|^2_{L^{\frac{p}{2}}(\Omega;L^1([0,T];\mathbb{R}))}\right.\\ &\quad\left.+\sqrt{\frac{p}{2}}\frac{2^{\frac{2p+1}{p}}\sqrt{K_\alpha}}{\sqrt{\theta}}\mathbb{E}\sup_{t\in[0,T]}\|u_t\|_H^p\right)\end{aligned} \tag{3.53}$$

All constants can be bounded by a constant growing as $\sqrt{p}$. We can also re-use the estimate in step 1 to estimate the supremum term on the RHS. To summarize, we obtain the following bound, where the constant $C$ depends on $\theta, \alpha, K_A$ and $K_\alpha$ and $\sqrt{p}$:

$$\left(\mathbb{E}\left(\int_0^{\tau_n}\|u_s\|_V^\alpha \mathrm{d}s\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \leq C\left(\|u_0\|_{L^p(\Omega;H)}+\|f\|^2_{L^{\frac{p}{2}}(\Omega;L^1([0,T];\mathbb{R}))}\right) \tag{3.54}$$

Using Fatou's lemma on the LHS to remove the stopping time finishes the proof. □

## 3.3. Existence and uniqueness result for coercive SPDEs

The a priori estimates derived in the previous section can be used to prove existence and uniqueness results by using a Galerkin approximation. This will be done in the next theorem:

**Theorem 3.2.** *If assumptions H1 to H5 hold and* $u_0 \in L^p(\Omega; H)$ *with* $p \in [2, p_0]$*, then the stochastic evolution equation* (3.2) *has a unique solution u and the following estimate holds for all* $q \in [2, p]$*:*

$$\begin{aligned} &\mathbb{E} \sup_{t\in[0,T]} \|u_t\|_H^q + \mathbb{E}\int_0^T \|u_t\|_H^{q-2}\|u_t\|_V^\alpha \mathrm{d}t + \mathbb{E}\left(\int_0^T \|u_t\|_V^\alpha \mathrm{d}t\right)^{\frac{q}{2}} \\ &\leq Ce^{CT}\left(\mathbb{E}\|u_0\|_H^q + \mathbb{E}\left(\int_0^T f_t \mathrm{d}t\right)^{\frac{q}{2}}\right). \end{aligned} \tag{3.55}$$

*Proof of Theorem 3.2 (Existence).* We start the proof with some constructive preliminaries in order to project the SPDE to a finite dimensional subspace. This allows us to perform the Galerkin approximation later.

*Step 1: Existence and uniqueness of projected SPDE*
Let

$$\{e_1, e_2, \ldots\} \subseteq V$$

be an orthonormal basis of $H$. This is indeed possible, since we can apply the Gram-Schmidt procedure to any countably dense subset of $V$ in $H$. Define $H_n = \text{span}\{e_1, \ldots, e_n\}$ for every $n \in \mathbb{N}$ and the operator $P_n : V^* \to H_n$ by

$$P_n y = \sum_{i=1}^n \langle y, e_i\rangle_V e_i$$

for $y \in V^*$. Similarly, let

$$\{g_1, g_2, \ldots\} \subseteq U$$

be an orthonormal basis of $U$ and define the operator $\tilde{P}_n : U \to \text{span}\{g_1, \ldots, g_n\}$ by

$$\widetilde{P}_n u = \sum_{i=1}^n (u, g_i)_U g_i.$$

Using this operator, define

$$W^n(t) = \tilde{P}_n W(t).$$

This allows us to consider the following projected SPDE (which is actually an SDE now), where we suggestively write $u_t^n$ for the solution process:

$$\mathrm{d}u_t^n = P_n A_t(u_t^n)\mathrm{d}t + P_n B_t(u_t^n)\widetilde{P}_n \mathrm{d}W_t, \qquad u_0^n = P_n u_0. \tag{3.56}$$

For notational convenience, we will denote $A_t(u_t^n)$ and $B_t(u_t^n)$ by $a_t^n$ and $b_t^n$ respectively from now on. Furthermore, we claim that the quantities appearing in equation (3.56) satisfy the premises of Theorem (3.1). This will allow us to place a bound on several quantities involving $u_t^n$, independent of $n \in \mathbb{N}$. To this end, we take the following Gelfand triple:

$$(H_n, \|\cdot\|_V) \subseteq (H_n, \|\cdot\|_H) \subseteq (H_n, \|\cdot\|_{V^*}).$$

The embedding constants in this triple are independent of $n \in \mathbb{N}$, since the norms used are associated with the Gelfand triple $(V, H, V^*)$. This will be useful for obtaining uniform bounds independent of $n$. To obtain these, we only need to check whether a solution of equation (3.56) exists and assumptions H1 to H5 are satisfied.

Before we do this, we first take note of one identity that will simplify these checks. For $u \in V, v \in H_n$,

$$\langle P_n A_t(u), v\rangle_V = \langle P_n A_t(u), v\rangle_H = \langle A_t(u), v\rangle_V \tag{3.57}$$

which can be found by using the definition of $P_n$.

We use Theorem 3.1.1 from [28, Theorem 3.1.1, p. 56] to show that a solution exists for equation (3.56). First note that $\widetilde{P}_n \mathrm{d}W_t$ turns equation (3.56) into a finite dimensional SDE. Indeed, $\mathrm{d}W^n(t) = \widetilde{P}_n \mathrm{d}W(t)$ and by definition of $W(t)$, the projection $\widetilde{P}_n$ creates a finite dimensional Wiener process, when applied to $W(t)$. Let $u, v \in H_n$ such that $\|u\|_{H_n}, \|v\|_{H_n} \leq R$ where $R \in [0,\infty)$ and $t \in [0, T]$. We check the local weak monotonicity and weak coercivity conditions. For the local weak coercivity condition, we get:

$$\begin{aligned}
&2\langle P_n A_t(u) - P_n A_t(v), u - v\rangle_H + \|P_n B_t(u) - P_n B_t(v)\|^2_{L_2(\mathrm{span}(g_1,\dots,g_n),H_n)} \\
&\leq 2\langle A_t(u) - A_t(v), u - v\rangle_V + \|B_t(u) - B_t(v)\|^2_{L_2(U,H)} \\
&\leq K(1 + \|v\|_V^\alpha)(1 + \|v\|_H^\beta)\|u - v\|_H^2 \\
&\leq K(1 + R^\alpha)(1 + R^\beta)\|u - v\|_H^2
\end{aligned}$$

For the weak coercivity condition, we get:

$$\begin{aligned}
\langle P_n A_t(v), v\rangle_H + \|P_n B_t(v)\|^2_{L_2(\mathrm{span}(g_1,\dots,g_n),H_n)} &\leq 2\langle A_t(v), v\rangle_V + \|B_t(v)\|^2_{L_2(U,H)} \\
&\leq f_t + K\|v\|_H^2 \\
&\leq \max\{f_t, K\}(1 + \|v\|_H^2)
\end{aligned}$$

where the second step follows by assumption H3 for equation (3.2). We conclude that equation (3.56) has a unique solution.

Next, we show that assumptions H3, H4 and H5 hold for equation (3.56). To this end, let $v \in H_n$ be arbitrary. Then,

$$\begin{aligned}
&2\langle P_n A_t(v), v\rangle_V + \|P_n B_t(v)\widetilde{P}_n\|^2_{L_2(U,H_n)} + (p_0 - 2)\frac{\|(P_n B_t(v)\widetilde{P}_n)^* v\|_U^2}{\|v\|_H^2} \\
&\overset{(3.57)}{\leq} 2\langle A_t(v), v\rangle_V + \|P_n B_t(v)\widetilde{P}_n\|^2_{L_2(U,H_n)} + (p_0 - 2)\frac{\|(P_n B_t(v)\widetilde{P}_n)^* v\|_U^2}{\|v\|_H^2} \\
&\leq 2\langle A_t(v), v\rangle_V + \|B_t(v)\|^2_{L_2(U,H)} + (p_0 - 2)\frac{\|B_t(v)^* v\|_U^2}{\|v\|_H^2} \\
&\overset{(\mathrm{H3})}{\leq} -\theta\|v\|_V^\alpha + K\|v\|_H^2 + f_t
\end{aligned} \tag{3.58}$$

Now, let $u, v \in H_n$ be arbitrary. We determine the operator norm $\|A_t(v)\|_{V^*}$ for $v \in H_n$, using equation (3.57):

$$|\langle P_n A_t(u), v\rangle_V| = |\langle A_t(u), v\rangle_V| \leq \|A_t(u)\|_{V^*}\|v\|_{H_n} \tag{3.59}$$

We conclude that $\|P_n A_t(u)\|_{V^*} \leq \|A_t(u)\|_{V^*}$, which implies that

$$\|P_n A_t(u)\|_V^{\frac{\alpha}{\alpha-1}} \leq \|A_t(u)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \leq (f_t + K\|u\|_V^\alpha)(1+\|u\|_H^\beta) \tag{3.60}$$

The last assumption H5 follows by a similar argument, since projections are contractions. We can consequently apply Theorem 1 and conclude the following:

For all $p \in [2, p_0]$ and for all $n \in \mathbb{N}$, there exists a constant $C$ independent of $n$ such that the following bound holds for all $n$:

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}\|u_t^n\|_H^p + \mathbb{E}\int_0^T \|u_t^n\|_H^{p-2}\|u_t^n\|_V^\alpha \mathrm{d}t + \mathbb{E}\left(\int_0^T \|u_t\|_V^\alpha \mathrm{d}t\right)^{\frac{p}{2}}\\ &\leq C\left(\mathbb{E}\|u_0^n\|_H^p + \mathbb{E}\left(\int_0^T f_s \mathrm{d}s\right)^{\frac{p}{2}}\right)\end{aligned} \tag{3.61}$$

The above bound is relevant, since it will allow us to build a solution using the Galerkin approximation method.

*Step 2: Weak approximation of solutions*

We now proceed to apply weak compactness arguments to obtain a weak approximation of a candidate solution to the SPDE (3.2). To obtain these, one needs uniform bounds on a sequence of approximations. In this case, we aim for a candidate solution that is the weak limit of the sequence $(u_n)_{n\in\mathbb{N}}$. Having obtained this candidate solution, we need to show that it is actually a solution. We do this by extracting weakly convergent subsequences of $a_t^n := A_t(u^n)$ and $b_t^n := B_t(u^n)$ in some space. Specifically, we look for uniform bounds on

$$\mathbb{E}\int_0^T \|a_t^n\|_{H_n}^{\frac{\alpha}{\alpha-1}} dt, \qquad \mathbb{E}\int_0^T \|b_t^n\|_{L_2(U,H)}^2 dt$$

We start with $a_t^n$. Using H4, H5 and $p_0 \geq \beta + 2$, we obtain:

$$\begin{aligned}\mathbb{E}\int_0^T \|a_s^n\|_{V^*}^{\frac{\alpha}{\alpha-1}}\mathrm{d}s &\leq \mathbb{E}\int_0^T \left(f_s + K\|u_s^n\|_V^\alpha\right)\left(1+\|u_s^n\|_H^\beta\right)\mathrm{d}s\\ &\leq \mathbb{E}\int_0^T (f_s + K\|u_s^n\|_V^\alpha)(1+\|u_s^n\|_H^\beta)^{\frac{p_0-2}{\beta}}\mathrm{d}s\\ &\leq 2^{\frac{p_0-2}{\beta}-1}\mathbb{E}\int_0^T (f_s + K\|u_s^n\|_V^\alpha)(1+\|u_s^n\|_H^{p_0-2})\mathrm{d}s\\ &\leq C_{\beta,p_0}\mathbb{E}\int_0^T f_s + f_s\|u_s^n\|_H^{p_0-2} + K\|u_s^n\|_V^\alpha + K\|u_s^n\|_V^\alpha\|u_s^n\|_H^{p_0-2}\mathrm{d}s\end{aligned} \tag{3.62}$$

One of the terms in equation (3.62) can be estimated using Hölder's inequality, giving:

$$
\begin{aligned}
\mathbb{E}\int_0^T f_s\|u_s^n\|_H^{p_0-2}\mathrm{d}s &\leq \mathbb{E}\sup_{t\in[0,T]}\|u_t^n\|_H^{p_0-2}\int_0^T f_s\mathrm{d}s \\
&\leq \left(\mathbb{E}\sup_{t\in[0T]}\|u_t^n\|_H^{p_0}\right)^{\frac{p_0-2}{p_0}}\left(\mathbb{E}\left(\int_0^T f_s\mathrm{d}s\right)^{\frac{p_0}{2}}\right)^{\frac{2}{p_0}} \\
&\leq \frac{p_0-2}{p_0}\mathbb{E}\sup_{t\in[0,T]}\|u_t^n\|_H^{p_0}+\frac{2}{p_0}\mathbb{E}\left(\int_0^T f_s\mathrm{d}s\right)^{\frac{p_0}{2}}.
\end{aligned}
$$

We can estimate equation (3.62) by combining the above estimate with equation (3.61) where we take $p=p_0$ and $p=2$. This results in:

$$
\begin{aligned}
&\mathbb{E}\int_0^T \|a_s^n\|_{V^*}^{\frac{\alpha}{\alpha-1}}\mathrm{d}s \\
&\leq C_{\beta,p_0}\mathbb{E}\int_0^T f_s ds + C_{\beta,p_0}\frac{p_0-2}{p_0}\mathbb{E}\sup_{t\in[0,T]}\|u_t^n\|_H^{p_0}+C_{\beta,p_0}\frac{2}{p_0}\mathbb{E}\left(\int_0^T f_s\mathrm{d}s\right)^{\frac{p_0}{2}} \\
&\quad + C_{\beta,p_0}KC\mathbb{E}\|u_0\|_H^2 + C_{\beta,p_0}KC\mathbb{E}\int_0^T f_s\mathrm{d}s + C_{\beta,p_0}KC\mathbb{E}\|u_0\|_H^{p_0} \\
&\quad + C_{\beta,p_0}KC\mathbb{E}\left(\int_0^T f_s\mathrm{d}s\right)^{\frac{p_0}{2}} \\
&\leq C_{\beta,p_0}(1+KC)\mathbb{E}\int_0^T f_s\mathrm{d}s + C_{\beta,p_0}\left(C\frac{p_0-2}{p_0}+\frac{2}{p_0}+KC\right)\mathbb{E}\left(\int_0^T f_s\mathrm{d}s\right)^{\frac{p_0}{2}} \\
&\quad + C_{\beta,p_0}KC\mathbb{E}\|u_0\|_H^2 + C_{\beta,p_0}\left(KC+C\frac{p_0-2}{p_0}\right)\mathbb{E}\|u_0\|_H^{p_0}.
\end{aligned} \tag{3.63}
$$

We conclude that the term on the LHS of the inequality is therefore uniformly bounded in $n$. We do similar estimates for the $b^n$ term, using the growth condition H5:

$$
\mathbb{E}\int_0^T \|b_s^n\|_{L_2(U,H)}^2\mathrm{d}s \tag{3.64}
$$

$$
\leq K\mathbb{E}\int_0^T f_s + \|u_s^n\|_H^2 + \|u_s^n\|_V^\alpha\mathrm{d}s \tag{3.65}
$$

$$
\leq K\mathbb{E}\int_0^T f_s\mathrm{d}s + TK\mathbb{E}\sup_{t\in[0,T]}\|u_t^n\|_H^2 + K\mathbb{E}\int_0^T \|u_s\|_V^\alpha\mathrm{d}s \tag{3.66}
$$

$$
\leq K\mathbb{E}\int_0^T f_s\mathrm{d}s + KC(1+T)\mathbb{E}\|u_0\|_H^2 + KC(1+T)\mathbb{E}\int_0^T f_s\mathrm{d}s. \tag{3.67}
$$

$$
\tag{3.68}
$$

We conclude that the LHS is also uniformly bounded in $n$. These bounds consequently allow us to extract weakly convergent subsequences in the following reflexive Banach spaces:

$$
L^2(\Omega;L^2([0,T];H)),\quad L^\alpha([0,T]\times\Omega;V),\quad L^{\frac{\alpha}{\alpha-1}}([0,T]\times\Omega;V^*),\quad L^2([0,T]\times\Omega;L_2(U,H)).
$$

One consequence of reflexivity is that unit balls (and multiples of it) of these spaces are weakly compact, which follows by the Banach-Alaoglu theorem. This implies that any bounded sequence in one of the four spaces must always have a weakly convergent subsequence. We have shown earlier in expressions (3.61), (3.63) and (3.64), that the sequences $(u_n)_{n\in\mathbb{N}}, (a^n)_{n\in\mathbb{N}}, (b^n)_{n\in\mathbb{N}}$ all are uniformly bounded in $n$, respectively. This implies that they have weakly convergent subsequences. We summarize this as follows. There exist $\bar{u}$, $a$ and $b$ with $\bar{u} \in L^2(\Omega; L^2([0,T]; H))$, $\bar{u} \in L^\alpha([0,T] \times \Omega; V)$, $a \in L^{\frac{\alpha}{\alpha-1}}([0,T] \times \Omega; V^*)$ and $b \in L^2([0,T] \times \Omega; L_2(U,H)$, and a subsequence $(n_k)_{k\in\mathbb{N}}$ (by taking further subsequences if needed) such that:

1. $u^{(n_k)} \rightharpoonup \bar{u}$ in $L^2(\Omega; L^2([0,T]; H))$.
2. $u^{(n_k)} \rightharpoonup \bar{u}$ in $L^\alpha([0,T] \times \Omega; V)$.
3. $a^{(n_k)} \rightharpoonup a$ in $L^{\frac{\alpha}{\alpha-1}}([0,T] \times \Omega; V^*)$.
4. $b^{(n_k)} \rightharpoonup b$ in $L^2([0,T] \times \Omega; L_2(U,H))$.

We further note, by boundedness of the stochastic integral, that

$$\int_0^{\cdot} P_{n_k} b_s^{n_k} dW^{n_k}(s) \rightharpoonup \int_0^{\cdot} Z(s) dW(s)$$

in $\mathcal{M}_T^2(H)$. Here, $\mathcal{M}_T^2(H)$ is the space of square integrable continuous martingales taking values in $H$ with norm

$$\|v\|_{\mathcal{M}_T^2(H)} = \left(\mathbb{E} \sup_{t\in[0,T]} \|v_t\|_H^2\right)^{\frac{1}{2}}$$

for $v \in \mathcal{M}_T^2(H)$.

We use the last two limits to define a candidate solution

$$u_t = u_0 + \int_0^t a_s \mathrm{d}s + \int_0^t b_s \mathrm{d}W_s. \tag{3.69}$$

*Step 3: Showing that the candidate solution is a solution*

In the previous step 2, we defined the candidate solution $u$. We are only left to show that this is an actual solution to the SPDE (3.2). To do this, we first show that $\bar{u} = u$ almost everywhere. To this end, let $\eta \in L^\infty(\Omega \times [0,T]; \mathbb{R})$, $\phi \in \bigcup_{n\geq 1} H_n$. By weak convergence and equation (3.57), we get:

$$
\begin{aligned}
&\mathbb{E}\int_0^T \langle \bar{u}_t, \eta_t \phi\rangle_V \mathrm{d}t \\
&= \lim_{k\to\infty} \mathbb{E}\int_0^T \left\langle u_t^{(n_k)}, \eta_t\phi \right\rangle_V \mathrm{d}t \\
&= \lim_{k\to\infty} \mathbb{E}\int_0^T \left\langle u_0^{(n_k)} + \int_0^t P_{n_k} a_s^{n_k} \mathrm{d}s + \int_0^t P_{n_k} b_s^{n_k} \mathrm{d}W_s^{(n_k)}, \eta_t\phi \right\rangle \mathrm{d}t \\
&= \mathbb{E}\int_0^T \left\langle u_0, \eta_t\phi \right\rangle \mathrm{d}t + \lim_{k\to\infty} \mathbb{E}\int_0^T \left\langle \int_0^t b_s^{n_k} \mathrm{d}s, \eta_t\phi \right\rangle \mathrm{d}t \\
&\quad + \lim_{k\to\infty} \mathbb{E}\int_0^T \left\langle \int_0^t b_s^{n_k} \mathrm{d}W_s^{(n_k)}, \eta_t\phi \right\rangle \mathrm{d}t \\
&= \mathbb{E}\int_0^T \left\langle u_0, \eta_t\phi \right\rangle \mathrm{d}t + \mathbb{E}\int_0^T \left\langle \int_0^t a_s \mathrm{d}s, \eta_t\phi \right\rangle \mathrm{d}t \\
&\quad + \mathbb{E}\int_0^T \left\langle \int_0^t b_s \mathrm{d}W_s, \eta_t\phi \right\rangle \mathrm{d}t \\
&= \mathbb{E}\int_0^T \langle u_t, \eta_t\phi\rangle_V \mathrm{d}t
\end{aligned}
\tag{3.70}
$$

We find that $u = \bar{u}$ a.e., since linear combinations of $\eta\phi$ are dense in $L^\alpha(\Omega \times [0,T];V)$. By [28, Theorem 4.2.5, p. 91] (see also theorem 2.3), it follows that $u_t$ is continuous in $H$ a.s. To show that $u$ solves the SPDE, we are only left to show that $a = A(u)$ and $b = B(u)$ almost everywhere. We introduce the space $\Psi$ to help us in this task.

For any $v \in V$, define $\rho(v) = K(1 + \|v\|_V^\alpha)(1 + \|v\|_H^\beta)$. Let $\Psi$ be the collection of $V$-valued $\mathscr{F}_t$-adapted processes $\psi$ satisfying

$$\int_0^T \rho(\psi_t)\mathrm{d}t < \infty \qquad \text{a.s.}$$

Now take $\psi \in \Psi$ arbitrarily. Let $\psi \in L^\alpha([0,T]\times\Omega;V) \cap \Psi \cap L^2(\Omega; L^\infty([0,T];H))$. We use Itô's lemma, to obtain an expression for the following two similar quantities:

$$e^{-\int_0^t \rho(\psi_s)\mathrm{d}s}\|u_t^{(n_k)}\|_H^2 \qquad e^{-\int_0^t \rho(\psi_s)\mathrm{d}s}\|u_t\|_H^2 \tag{3.71}$$

We first note that [28, Theorem 4.2.5, p. 91] (see also theorem 2.3) implies an Itô type formula for both $\|u_t^{(n_k)}\|_H^2$ and $\|u_t\|_H^2$, since both $u^{(n_k)}$ and $u$ satisfy the hypotheses of that theorem. The resulting Itô formulas are:

$$
\begin{aligned}
\|u_t^{(n_k)}\|_H^2 = \|u_0^{(n_k)}\|_H^2 &+ \int_0^t 2\langle a_s^{n_k}, u_s^{(n_k)}\rangle_V + \|b_s^{n_k}\|_{L_2(U,H)}^2 \mathrm{d}s \\
&+ \int_0^t (u_s^{(n_k)}, b_s^{n_k}\cdot)_H \mathrm{d}W_s
\end{aligned}
\tag{3.72}
$$

and

$$\|u_t\|_H^2 = \|u_0\|_H^2 + \int_0^t 2\langle a_s, u_s\rangle_V + \|b_s\|_{L_2(U,H)}^2 \mathrm{d}s + \int_0^t (u_s, b_s\cdot)_H \mathrm{d}W_s. \tag{3.73}$$

We apply Itô's lemma again, using the function $f(x,y) = e^{-x}y$. We apply this to the pair $(X_t, Y_t)$ where $X_t = \int_0^t \rho(\psi_s)\mathrm{d}s$ and $M_t$ is a real-valued semimartingale of the form

$$M_t = M_0 + \int_0^t F(s)\mathrm{d}s + \int_0^t (M_s, G(s)\cdot)\mathrm{d}W_s.$$

We assume that $F, G$ are such that $\mathbb{E}\sup M_t$ is finite and $G \in L^2([0,T]\times\Omega; L_2(U,H))$. This results in:

$$\begin{aligned} e^{-\int_0^t \rho(\psi_s)\mathrm{d}s}\|M_t\|_H^2 = \|M_0\|_H^2 + \int_0^t e^{-\int_0^s \rho(\psi_r)\mathrm{d}r}(M_s, G(s)\cdot)\mathrm{d}W_s \\ - \int_0^t \rho(\psi_s)e^{-\int_0^s \rho(\psi_r)\mathrm{d}r}\|M_s\|_H^2\mathrm{d}s + \int_0^t e^{-\int_0^s \rho(\psi_r)\mathrm{d}r}F(s)\mathrm{d}s \end{aligned} \tag{3.74}$$

Since we have assumed that $G \in L^2([0,T]\times\Omega : L_2(U,H))$ and $\mathbb{E}\sup M_t$ are both finite, the stochastic integrals are martingales. On taking expectations, this results in:

$$\begin{aligned} &\mathbb{E}\left(e^{-\int_0^t \rho(\psi_s)\mathrm{d}s}\|M_t\|_H^2\right) - \mathbb{E}\|M_0\|_H^2 \\ &= \mathbb{E}\int_0^t e^{-\int_0^s \rho(\psi_r)dr}F(s)\mathrm{d}s - \mathbb{E}\int_0^t \rho(\psi_s)e^{-\int_0^s \rho(\psi_r)\mathrm{d}r}\|M_s\|_H^2\mathrm{d}s \end{aligned} \tag{3.75}$$

Choosing $M_t = \|u_t^{(n_k)}\|_H^2$, we obtain:

$$\begin{aligned} &\mathbb{E}\left(e^{-\int_0^t \rho(\psi_s)\mathrm{d}s}\|u_t^{(n_k)}\|_H^2\right) - \mathbb{E}\|u_0^{(n_k)}\|_H^2 \\ &= \mathbb{E}\int_0^t e^{-\int_0^s \rho(\psi_r)\mathrm{d}r}\left(2\langle a_s^{n_k}, u_s^{(n_k)}\rangle + \|b_s^{n_k}\|_{L_2(U,H)}^2 - \rho(\psi_s)\|u_s^{(n_k)}\|_H^2\right)\mathrm{d}s \end{aligned} \tag{3.76}$$

Similarly, we can choose $M_t = \|u_t\|_H^2$ to get:

$$\begin{aligned} &\mathbb{E}\left(e^{-\int_0^t \rho(\psi_s)\mathrm{d}s}\|u_t\|_H^2\right) - \mathbb{E}\|u_0\|_H^2 \\ &= \mathbb{E}\int_0^t e^{-\int_0^s \rho(\psi_r)\mathrm{d}r}\left(2\langle a_s, u_s\rangle + \|b_s\|_{L_2(U,H)}^2 - \rho(\psi_s)\|u_s\|_H^2\right)\mathrm{d}s \end{aligned} \tag{3.77}$$

Starting from equation (3.76), we work our way towards invoking the local monotonicity assumption H2.

$$\begin{aligned} &\mathbb{E}\left(e^{-\int_0^t \rho(\psi_s)\mathrm{d}s}\|u_t^{(n_k)}\|_H^2\right) - \mathbb{E}\|u_0^{(n_k)}\|_H^2 \\ &= \mathbb{E}\left[\int_0^t e^{-\int_0^s \rho(\psi_r)\mathrm{d}r}\left(2\langle a_s^{n_k}, u_s^{(n_k)}\rangle + \|b_s^{n_k}\|_{L_2(U,H)}^2 - \rho(\psi_s)\|u_s^{(n_k)}\|_H^2\right)\mathrm{d}s\right] \\ &= \mathbb{E}\Big[\int_0^t e^{-\int_0^t \rho(\psi_s)\mathrm{d}s}\Big(2\langle a_s^{n_k} - A_s(\psi_s), u_s^{(n_k)} - \psi_s\rangle + 2\langle A_s(\psi_s), u_s^{(n_k)}\rangle \\ &\quad + 2\langle a_s^{n_k} - A_s(\psi_s), \psi_s\rangle + \|b_s^{n_k} - B_s(\psi_s)\|_{L_2(U,H)}^2 - \|B_s(\psi_s)\|_{L_2(U,H)}^2 \\ &\quad + 2(b_s^{n_k}, B_s(\psi_s))_{L_2(U,H)} - \rho(\psi_s)\left[\|u_s^{(n_k)} - \psi_s\|_H^2 - \|\psi\|_H^2 + 2(u_s^{(n_k)}, \psi_s)_H\right]\Big)\mathrm{d}s\Big] \end{aligned} \tag{3.78}$$

Applying H2, this results in:

$$
\begin{aligned}
&\mathbb{E}\left(e^{-\int_0^t \rho(\psi_s)\mathrm{d}s}\|u_t^{(n_k)}\|_H^2\right) - \mathbb{E}\|u_0^{(n_k)}\|_H^2 \\
&\leq \mathbb{E}\Big[\int_0^t e^{-\int_0^s \rho(\psi_r)\mathrm{d}r}\Big(2\langle A_s(\psi_s), u_s^{(n_k)}\rangle + 2\langle a_s^{n_k} - A_s(\psi_s), \psi_s\rangle \\
&\quad - \|B_s(\psi)\|_{L_2(U,H)}^2 + 2(b_s^{n_k}, B_s(\psi_s))_{L_2(U,H)} - \rho(\psi_s)\Big[2(u_s^{(n_k)}, \psi_s) - \|\psi\|_H^2\Big]\Big)\mathrm{d}s\Big]
\end{aligned}
\tag{3.79}
$$

In order to proceed we use lemma 2.2, which results in the following inequality by the weak convergence of $u^{(n_k)} \rightharpoonup u$ in $L^2(\Omega; L^2([0,T]; H))$:

$$
\mathbb{E}\int_0^T e^{-\int_0^t \rho(\psi_s)\mathrm{d}s}\|u_t\|_H^2 \mathrm{d}t \leq \liminf_{k\to\infty}\mathbb{E}\left[\int_0^T e^{-\int_0^t \rho(\psi_s)\mathrm{d}s}\|u_t^{(n_k)}\|_H^2 \mathrm{d}t\right] \tag{3.80}
$$

Using this observation, we can proceed from inequality (3.79). Observe that for all $k \in \mathbb{N}$, $\|u_0^{(n_k)}\|_H^2 \leq \|u_0\|_H^2$. Therefore, it follows that $-\|u_0\|_H^2 \leq \liminf(-\|u_0^{(n_k)}\|_H^2)$. Invoking weak convergence and inequality (3.79),

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \left(e^{-\int_0^t \rho(\psi_s)\mathrm{d}s}\|u_t\|_H^2 - \|u_0\|_H^2\right)\mathrm{d}t\right] \\
&\leq \liminf_{k\to\infty}\mathbb{E}\left[\int_0^T \left(e^{-\int_0^t \rho(\psi_s)\mathrm{d}s}\|u_t^{(n_k)}\|_H^2 - \|u_0^{(n_k)}\|_H^2\right)\mathrm{d}t\right] \\
&\leq \mathbb{E}\Big[\int_0^T \int_0^t e^{-\int_0^s \rho(\psi_r)\mathrm{d}r}\Big(2\langle A_s(\psi_s), u_s\rangle + 2\langle Y_s - A_s(\psi_s), \psi_s\rangle \\
&\quad - \|B_s(\psi_s)\|_{L_2(U,H)}^2 + 2(Z(s), B_s(\psi))_{L_2(U,H)} - \rho(\psi_s)\Big[2(u_s, \psi_s) - \|\psi_s\|_H^2\Big]\Big)\mathrm{d}s\mathrm{d}t\Big]
\end{aligned}
\tag{3.81}
$$

We know a different expression for the LHS of the above inequality by using equation (3.77). Filling this in and simplifying both LHS and RHS of inequality (3.81), we obtain:

$$
\begin{aligned}
\mathbb{E}\Big[\int_0^T \int_0^t e^{-\int_0^t \rho(\psi_r)\mathrm{d}r}\Big(2\langle a_s - A_s(\psi_s), u_s - \psi_s\rangle + \|B_s(\psi_s) - b_s\|_{L_2(U,H)}^2 \\
- \rho(\psi_s)\|u_s - \psi_s\|_H^2\Big)\mathrm{d}s\mathrm{d}t\Big] \leq 0
\end{aligned}
\tag{3.82}
$$

Take $\psi = u$. We immediately observe that this implies that $B(u) = b$ in $L^2([0,T]\times\Omega; L_2(U,H))$. To show that $A(u) = a$, let $\eta \in L^\infty([0,T]\times\Omega;\mathbb{R})$, $\phi \in V$, $\varepsilon \in (0,1)$ and let $\psi = u - \varepsilon\eta\phi$. This results in:

$$
\begin{aligned}
\mathbb{E}\Big[\int_0^T \int_0^t e^{-\int_0^s \rho(u_r - \varepsilon\eta_r\phi)\mathrm{d}r}(2\langle a_s - A_s(u_s - \varepsilon\eta_s\phi), \eta_s\phi\rangle \\
- \varepsilon\rho(u_s - \varepsilon\eta_s\phi)\|\eta_s\phi\|_H^2)\mathrm{d}s\mathrm{d}t\Big] \leq 0
\end{aligned}
\tag{3.83}
$$

Using the dominated convergence theorem, sending $\varepsilon \to 0$, and H1, we obtain that

$$
\mathbb{E}\left[\int_0^T \int_0^t e^{-\int_0^s \rho(u_r)\mathrm{d}r} 2\eta_s\langle a_s - A_s(u_s), \phi\rangle \mathrm{d}s\mathrm{d}t\right] \leq 0 \tag{3.84}
$$

We conclude from this that

$$\int_0^t e^{-\int_0^s \rho(u_r)dr} 2\eta_s \langle a_s - A_s(u_s), \phi \rangle \mathrm{d}s = 0 \qquad \mathrm{d}t \times \mathbb{P} \text{ a.e}$$

Since this holds for almost all $t \in [0,T]$, $\eta$ and $\phi$ were arbitrary, we obtain that $A(u) = a$ in $L^{\frac{\alpha}{\alpha-1}}([0,T] \times \Omega; V^*)$. □

*Proof of Theorem 2 (Uniqueness).* Consider $u_t$ and $v_t$ to be two solutions of equation (3.2). This implies that

$$u_t - v_t = \int_0^t A_s(u_s) - A_s(v_s)\mathrm{d}s + \int_0^t B_s(u_s) - B_s(v_s)\mathrm{d}W_s \tag{3.85}$$

almost surely for all $t \in [0,T]$. We aim to use Itô's lemma in a similar way as the previous proof. We first introduce the following sequence of stopping times:

$$\sigma_n = \inf\{t \in [0,T] : \|u_t\|_H > n\} \wedge \inf\{t \in [0,T] : \|v_t\|_H > n\} \wedge T.$$

We use this stopping time to note that $\|u_t - v_t\|_H^2$ is also semimartingale, by using Itô's lemma [28, Theorem 4.2.5, p. 91] (or see theorem 2.3), which gives the following Itô formula:

$$\begin{aligned}\|u_{t\wedge\sigma_n} - v_{t\wedge\sigma_n}\|_H^2 = &\int_0^{t\wedge\sigma_n} 2\langle A_s(u_s) - A_s(v_s), u_s - v_s\rangle_V + \|B_s(u_s) - B_s(v_s)\|_{L_2(U,H)}^2 \mathrm{d}s \\ &+ \int_0^{t\wedge\sigma_n} (u_s - v_s, (B_s(u_s) - B_s(v_s))\cdot)_H \mathrm{d}W_s\end{aligned} \tag{3.86}$$

Since the above satisfies the assumptions for equation (3.74), we use $M_t = \|u_t - v_t\|_H^2$. This results in the following formula:

$$\begin{aligned}e^{-\int_0^{t\wedge\sigma_n}\rho(v_s)ds}\|u_{t\wedge\sigma_n} - v_{t\wedge\sigma_n}\|_H^2 = &\int_0^{t\wedge\sigma_n} e^{-\int_0^s \rho(v_r)dr}\Big(2\langle A_s(u_s) - A_s(v_s), u_s - v_s\rangle_V \\ &+ \|B_s(u_s) - B_s(v_s)\|_{L_2(U,H)}^2 - \rho(v_s)\|u_s - v_s\|_H^2\Big)\mathrm{d}s \\ &+ \int_0^{t\wedge\sigma_n} e^{-\int_0^s \rho(v_r)dr}(u_s - v_s, (B_s(u_s) - B_s(v_s))\cdot)_H \mathrm{d}W_s\end{aligned} \tag{3.87}$$

Invoke assumption H2 to obtain:

$$e^{-\int_0^{t\wedge\sigma_n}\rho(v_s)ds}\|u_t - v_t\|_H^2 \leq \int_0^{t\wedge\sigma_n} e^{-\int_0^s \rho(v_r)\mathrm{d}r}(u_s - v_s, (B_s(u_s) - B_s(v_s))\cdot)_H \mathrm{d}W_s \tag{3.88}$$

The stopping time turns the stochastic integral term into a martingale. Taking expectations on both sides in the above equation, we obtain that:

$$\mathbb{E}\left(e^{-\int_0^{t\wedge\sigma_n}\rho(v_s)\mathrm{d}s}\|u_{t\wedge\sigma_n} - v_{t\wedge\sigma_n}\|_H^2\right) = 0. \tag{3.89}$$

We apply Fatou's lemma to conclude that $\|u_t - v_t\|_H^2 = 0$ almost surely for every $t \in [0,T]$. Invoking continuity of $u - v$ in $H$, we find that $u$ and $v$ are indistinguishable, i.e.

$$\mathbb{P}\left(\sup_{t\in[0,T]}\|u_t - v_t\|_H = 0\right) = 1$$

□

## 3.4. Extension to additive equations

We can generalize the framework that has been built to include certain additive equations. That is, we can treat equations of the form

$$\mathrm{d}u_t = (A_t(u_t) + f_t)\mathrm{d}t + (B_t(u_t) + g_t)\mathrm{d}W_t, \tag{3.90}$$

where certain assumptions on $f$ and $g$ are made. We will first state assumptions underlying treatment of this equation and then state the results based on our theory.

**Assumptions 3.2.** *Consider equation* (3.90)*, a given Gelfand triple* $(V,H,V^*)$ *and a cylindrical Wiener process* $W$*, taking values in a separable Hilbert space* $U$*. We assume the operators* $A$ *and* $B$ *satisfy the assumptions H1 to H5, given constants* $\alpha > 1$*,* $\beta \geq 0$*,* $p_0 \geq \beta + 2$*,* $\theta > 0$*,* $K \geq 0$ *and a nonnegative function* $h \in L^{\frac{p_0}{2}}(\Omega; L^1([0,T];\mathbb{R}))$*. We also take*

$$f \in L^{\frac{p_0\alpha}{2(\alpha-1)}}(\Omega; L^{\frac{\alpha}{\alpha-1}}([0,T];V^*)), \quad g \in L^{p_0}(\Omega; L^2([0,T]; L_2(U,H))).$$

**Theorem 3.3.** *Given assumptions 3.2, define* $\widetilde{A} = A + f$ *and* $\widetilde{B} = B + g$*. Then assumptions H1 to H5 hold again for* $\widetilde{A}$ *and* $\widetilde{B}$ *with* $\tilde{\alpha} = \alpha$*,* $\tilde{\beta} = \beta$*,* $\tilde{p}_0 = p_0$*, but different* $\tilde{\theta} > 0$ *and function*

$$h_t + \|f_t\|_{V^*}^{\frac{\alpha}{\alpha-1}} + \|g_t\|_{L_2(U,H)}^2.$$

*Furthermore, under assumptions 3.2 and for any initial condition* $u_0 \in L^p(\Omega;H)$ *with* $p \in [2,p_0]$*, there exists a unique solution to equation* (3.90) *with corresponding energy inequality for all* $q \in [2,p]$ *and some constant* $C$ *depending on* $\theta, \alpha$ *and* $q$*:*

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\|u_t\|_H^q + \mathbb{E}\int_0^T \|u_t\|_H^{q-2}\|u_t\|_V^\alpha \mathrm{d}t + \mathbb{E}\left(\int_0^T \|u_t\|_V^\alpha\right)^{\frac{q}{2}} \\
&\leq Ce^{CT}\left(\mathbb{E}\|u_0\|_H^q + \mathbb{E}\left(\int_0^T h_t \mathrm{d}t\right)^{\frac{q}{2}} + \mathbb{E}\left(\int_0^T \|f_t\|_{V^*}^{\frac{\alpha}{\alpha-1}}\mathrm{d}t\right)^{\frac{q}{2}} + \mathbb{E}\left(\int_0^T \|g_t\|_{L_2(U,H)}^2 \mathrm{d}t\right)^{\frac{q}{2}}\right)
\end{aligned} \tag{3.91}$$

*Proof of Theorem 3.3.* We only show H3, the other assumptions are straightforward. The second result follows by invoking theorem 3.2 with operators $\widetilde{A}$ and $\widetilde{B}$. To this end, let $v \in V$, $v \neq 0$ and $t \in [0,T]$. We use Young's inequality, H3 and H5 for $A$ and $B$ to derive: We first split the additive terms and use Cauchy-Schwarz to obtain:

$$\begin{aligned}
&2\langle A_t(v) + f_t, v\rangle_V + \|B_t(v) + g_t\|_{L_2(U,H)}^2 + (p_0-2)\frac{\|(B_t(v)+g_t)^* v\|_U^2}{\|v\|_H^2} \\
&\leq 2\langle A_t(v), v\rangle_V + 2\langle f_t, v\rangle_V + \|B_t(v)\|_{L_2(U,H)}^2 + 2(B_t(v), g_t)_{L_2(U,H)} \\
&\quad + \|g_t\|_{L_2(U,H)}^2 + (p_0-2)\frac{\|(B_t(v))^* v\|_U^2}{\|v\|_H^2} + 2(p_0-2)\frac{(B_t(v)^* v, g_t^* v)_U}{\|v\|_H^2} + (p_0-2)\|g_t\|_{L_2(U,H)}^2.
\end{aligned} \tag{3.92}$$

We can apply H3 for operators $A$ and $B$ and Cauchy-Schwarz to derive:

$$\begin{aligned}
&2\langle A_t(v)+f_t,v\rangle_V+\|B_t(v)+g_t\|^2_{L_2(U,H)}+(p_0-2)\frac{\|(B_t(v)+g_t)^*v\|_U^2}{\|v\|_H^2}\\
&\overset{(H3)}{\le} h_t+K\|v\|_H^2+2\langle f_t,v\rangle_V+2(B_t(v),g_t)_{L_2(U,H)}+(p_0-1)\|g_t\|^2_{L_2(U,H)}\\
&\quad+2(p_0-2)\frac{(B_t(v)^*v,g_t^*v)_U}{\|v\|_H^2}-\theta\|v\|_V^\alpha\\
&\overset{(C-S)}{\le} h_t+K\|v\|_2+2\|f_t\|_{V^*}\|v\|_V+2(p_0-1)\|B_t(v)\|_{L_2(U,H)}\|g_t\|_{L_2(U,H)}+(p_0-1)\|g_t\|^2_{L_2(U,H)}\\
&\quad-\theta\|v\|_V^\alpha.
\end{aligned} \tag{3.93}$$

Next, we make use of Young's inequality, where $\varepsilon>0$ will be chosen later. We also use H5 to estimate $\|B_t(v)\|^2_{L_2(U,H)}$. This results in:

$$\begin{aligned}
&2\langle A_t(v)+f_t,v\rangle_V+\|B_t(v)+g_t\|^2_{L_2(U,H)}+(p_0-2)\frac{\|(B_t(v)+g_t)^*v\|_U^2}{\|v\|_H^2}\\
&\le h_t+K\|v\|_H^2+C_\varepsilon\frac{2(\alpha-1)}{\alpha}\|f_t\|_{V^*}^{\frac{\alpha}{\alpha-1}}+\varepsilon\|B_t(v)\|^2_{L_2(U,H)}+C'_\varepsilon\|g_t\|^2_{L_2(U,H)}\\
&\quad+\left(-\theta+\varepsilon\frac{2}{\alpha}\right)\|v\|_V^\alpha\\
&\overset{(H5)}{\le} h_t+K\|v\|_H^2+C_\varepsilon\frac{2(\alpha-1)}{\alpha}\|f_t\|_{V^*}^{\frac{\alpha}{\alpha-1}}+\varepsilon K(h_t+\|v\|_H^2+\|v\|_V^\alpha)\\
&\quad+C'_\varepsilon\|g_t\|^2_{L_2(U,H)}+\left(-\theta+\varepsilon\frac{2}{\alpha}\right)\|v\|_V^\alpha\\
&\le(\varepsilon K+1)h_t+C_\varepsilon\frac{2(\alpha-1)}{\alpha}\|f_t\|_{V^*}^{\frac{\alpha}{\alpha-1}}+C'_\varepsilon\|g_t\|^2_{L_2(U,H)}+K(1+\varepsilon)\|v\|_H^2\\
&\quad+\left(-\theta+\varepsilon\frac{2}{\alpha}+\varepsilon K\right)\|v\|_V^\alpha
\end{aligned} \tag{3.94}$$

By choosing $\varepsilon>0$ small, we have shown H3 for $\tilde{A}$ and $\tilde{B}$. □

## 3.5. Optimality of results within this framework

This section will show that the results obtained in Theorem 3.90 are optimal. By optimal, we mean that it is not possible to obtain better results, such as higher moments, based on assumptions 3.1. We prove this by using a rather academic example given by Brzezniak and Veraar in [7]. The exposition of this section is based on the same arguments as given in [38]. Consider the equation

$$\mathrm{d}u_t=\Delta u_t\mathrm{d}t+2\gamma(-\Delta)^{\frac{1}{2}}u_t\mathrm{d}W_t \tag{3.95}$$

on the torus $\mathbb{T}$, with $\gamma\in\mathbb{R}$, a $\mathscr{F}_0$-measurable initial condition $u_0$ and $W$ a real-valued Wiener process. It follows from [7] that equation (3.95) is well-posed in $L^{p_0}(\Omega;L^2(\mathbb{T}))$ as long as $2\gamma^2(p_0-1)<1$. Using our framework, we will derive that equation (3.95) indeed has a solution with values in $L^{p_0}([0,T]\times\Omega;L^2(\mathbb{T}))$ whenever $2\gamma^2(p_0-1)<1$, corresponding to the condition derived in [7]. After this, we show that it is not possible to even obtain existence whenever $2\gamma^2(p_0-1)>1$.

**Assumptions 3.3.** *Let $\gamma^2 \in (0, \frac{1}{2})$ and $u_0 \in L^2(\Omega; L^2(\mathbb{T}))$. Consider equation* (3.95)*, where $W_t$ is a real-valued Wiener process. We use the following Gelfand triple in this example:*

$$W^{1,2}(\mathbb{T}) \subseteq L^2(\mathbb{T}) \subseteq W^{-1,2}(\mathbb{T}).$$

*Let $\mathscr{F}$ be the Fourier transform. We define the operator $(-\Delta)^{\frac{1}{2}} : W^{1,2}(\mathbb{T}) \to L^2(\mathbb{T})$ to be:*

$$(-\Delta)^{\frac{1}{2}} u = \mathscr{F}^{-1}((|k|(\mathscr{F}u)(k))_{k\in\mathbb{Z}}) \tag{3.96}$$

*Subsequently, we let the operators $A : W^{1,2}(\mathbb{T}) \to W^{-1,2}(\mathbb{T})$ and $B : W^{1,2}(\mathbb{T}) \to L^2(\mathbb{T})$ be defined by $A(u) = \Delta u$ and $B(u) = 2\gamma(-\Delta)^{\frac{1}{2}}$.*

**Theorem 3.4.** *Let $p_0 \in [2,\infty)$ be such that $2\gamma^2(p_0-1) < 1$. Then, equation* (3.95) *has a unique solution $u \in L^p((0,T)\times\Omega; L^2(\mathbb{T}))$ for all $p \in [2, p_0]$. Furthermore, we have an energy estimate of the form:*

$$\mathbb{E} \sup_{t\in[0,T]} \|u_t\|^p_{L^2(\mathbb{T})} \leq Ce^{CT}\mathbb{E}\|u_0\|^p_{L^2(\mathbb{T})} \tag{3.97}$$

*Proof.* To apply our theory, we first show assumptions H1 to H5 and then apply Theorem 3.2. We will only show H2 and H3. For H2, let $u, v \in W^{1,2}(\mathbb{T})$. Using Plancherel's theorem, we obtain:

$$\begin{aligned}
&2\langle A(u) - A(v), u - v\rangle + \|B(u) - B(v)\|^2_{L^2(\mathbb{T})} \\
&\leq (-2 + 4\gamma^2) \sum_{k=1}^{\infty} k^2 |(\mathscr{F}u)(k) - (\mathscr{F}v)(k)|^2 \\
&\leq 0
\end{aligned} \tag{3.98}$$

Now we check the coercivity condition H3. Let $v \in W^{1,2}(\mathbb{T})$. We have the following sequence of inequalities by using that $\|T\| = \|T^*\|$ for Hilbert-Schmidt operators in Hilbert-Schmidt norm and boundedness of $(-\Delta)^{\frac{1}{2}}$:

$$\begin{aligned}
&2\langle A(v), v\rangle + \|B(v)\|^2_{W^{1,2}(\mathbb{T})} + \frac{|B(v)^* v|^2}{\|v\|_{L^2(\mathbb{T})}} \\
&\leq (4\gamma^2(p_0 - 1) - 2) \sum_{k=1}^{\infty} k^2 |(\mathscr{F}u)(k)|^2 \\
&\leq 0
\end{aligned} \tag{3.99}$$

where part of the inequality follows by Plancherel's theorem. We see that there exists a $\theta > 0$ such that the coercivity condition holds as long as $2\gamma^2(p_0 - 1) < 1$. Boundedness of H5 is also straightforward from the above calculations. Application of theorem 3.2 then yields the main result. □

We now proceed to show that we don't get well-posedness in $L^p((0,T)\times\Omega; L^2(\mathbb{T}))$ as soon as $2\gamma^2(p_0 - 1) > 1$. This shows that our results are sharp.

**Theorem 3.5.** *Let $p_0 \in [2,\infty)$ be such that $2\gamma^2(p_0-1) > 1$. Then, equation* (3.95) *has a unique solution $u \in L^p((0,\tau)\times\Omega; L^2(\mathbb{T}))$ for all $p \in [2, p_0]$, where $\tau = (2\gamma^2(p_0 - 1) - 1)^{-1}$. Moreover,*

$$\limsup_{t\uparrow\tau} \|u(t)\|_{L^p(\Omega; L^2(\mathbb{T}))} = \infty \tag{3.100}$$

*Proof.* The existence and uniqueness follows from [7], since the equation doesn't fit into our framework under the above assumption. By taking Fourier transforms of equation (3.95), we have the following equation of SDEs:

$$\begin{cases} \mathrm{d}v_n(t) = -n^2 v_n(t)\mathrm{d}t + 2\gamma|n|v_n(t)\mathrm{d}W(t) \\ v_n(0) = a_n \end{cases} \tag{3.101}$$

where $a_n = e^{-n^2}$ for all $n \in \mathbb{Z}\setminus\{0\}$ and $a_0 = 0$. Then, one can check by Itô's formula that the above has solution

$$v_n(t) = e^{-t(n^2+2\gamma^2 n^2)} e^{2\gamma|n|W(t)} a_n \tag{3.102}$$

for $n \in \mathbb{Z}\setminus\{0\}$. If (3.95) has a solution, it must then be of form $u(t,x) = \sum v_n(t)e^{inx}$. Taking $L^2$ norms, we compute the following:

$$\begin{aligned} \|u(t)\|^2_{L^2(\mathbb{T})} &= \sum_{n\in\mathbb{Z}\setminus\{0\}} |v_n(t)|^2 \\ &= \sum_{n\in\mathbb{Z}\setminus\{0\}} e^{-2t(n^2+2\gamma^2 n^2)} e^{4\gamma|n|W(t)} e{-2n^2} \\ &= \sum_{n\in\mathbb{Z}\setminus\{0\}} e^{-2n^2(t+1+2t\gamma^2)} e^{4\gamma|n|W(t)}. \end{aligned} \tag{3.103}$$

As an intermediate step, we note the following sequence of equalities, which can be verified easily:

$$\begin{aligned} -2n^2(t+1+2t\gamma^2) + 4\gamma|n|W(t) &= -2(t+1+2t\gamma^2)\left[|n| - \frac{\gamma W(t)}{(t+1+2t\gamma^2)}\right]^2 + \frac{2\gamma^2 W(t)^2}{(t+1+2t\gamma^2)} \\ &= -2f(t)(|n|-g(t))^2 + 2h(t), \end{aligned} \tag{3.104}$$

where we define

$$f(t) = t+1+2t\gamma^2, \quad g(t) = \frac{\gamma W(t)}{(t+1+2t\gamma^2)}, \quad h(t) = \frac{\gamma^2 W(t)^2}{(t+1+2t\gamma^2)} \tag{3.105}$$

Returning to equation (3.103), we see that:

$$\begin{aligned} \|u(t)\|^2_{L^2(\mathbb{T})} &= e^{2h(t)} \sum_{n\in\mathbb{N}\setminus\{0\}} e^{-2f(t)(|n|-g(t))^2} \\ &= 2e^{2h(t)} \sum_{n\geq 1} e^{-2f(t)(n-g(t))^2} \\ &\geq 2e^{2h(t)} e^{-2f(t)}. \end{aligned} \tag{3.106}$$

Taking powers and expectation, we find:

$$\begin{aligned} \mathbb{E}\|u(t)\|^p_{L^2(\mathbb{T})} &= 2^{\frac{p}{2}} e^{-pf(t)} \int_\Omega e^{ph(t)} \mathrm{d}\mathbb{P} \\ &= 2^{\frac{p}{2}+1} e^{-pf(t)} \int_{W(t)>0} e^{ph(t)} \mathrm{d}\mathbb{P} \\ &= 2^{\frac{p}{2}+1} e^{-pf(t)} \int_{W(1)>0} e^{\frac{p\gamma^2 W(1)^2}{(1+t^{-1}+2\gamma^2)}} \mathrm{d}\mathbb{P} \end{aligned} \tag{3.107}$$

The last integral becomes unbounded as $t \to \tau$. □

# 4

# Examples of coercive SPDEs

We will give a fair amount of examples to which our framework applies. We will show that we recover the previously known coercivity condition for existence and uniqueness of the stochastic heat equation with Dirichlet boundary conditions. Accordingly, we will also treat the stochastic heat equation with Neumann boundary conditions, but we need a different coercivity condition for existence of higher order moments. We proceed by investigating the stochastic Burgers' equation and Navier-Stokes equations in 2D, for which we show that the solution is bounded almost surely in $C([0,T];L^2)$.

## 4.1. The stochastic heat equation with Dirichlet boundary conditions

The first equation we consider is a stochastic heat equation with stochastic forcing and Dirichlet boundary conditions on a $C^1$ domain $\mathscr{D} \subseteq \mathbb{R}^d$. This leads us to consider the following equation:

$$\mathrm{d}u_t = \left(\sum_{i,j=1}^{d} \partial_i(a^{ij}\partial_j u_t) + f_t\right)\mathrm{d}t + \sum_{k=1}^{\infty}\left(\sum_{i=1}^{d} b^{ik}\partial_i u_t + g_t^k\right)\mathrm{d}W_t^k. \tag{4.1}$$

We will first make some assumptions on the coefficients of the equation and the initial condition, and then state the results derived from our framework with proof. It will turn out that the $p$ dependent term in the coercivity condition vanishes. Therefore, the solution will admit moment estimates of all orders $p \geq 2$, only limited by integrability of the additive noise and the initial condition.

**Assumptions 4.1.** *Consider equation* (4.1) *on a $d$-dimensional $C^1$ domain $\mathscr{D}$ with Dirichlet boundary conditions and initial condition $u_0 \in L^{p_0}(\Omega; H)$ where $p_0 \in [2,\infty)$. We reformulate this equation into an equation of the form*

$$\mathrm{d}u_t = A_t(u_t)\,\mathrm{d}t + \sum_{k=1}^{\infty} B_t^k(u_t)\,\mathrm{d}W_t^k. \tag{4.2}$$

*To do this, we use the following Gelfand triple for this example:*

$$H_0^1(\mathscr{D}) \subseteq L^2(\mathscr{D}) \subseteq H^{-1}(\mathscr{D}).$$

*Here, $H^{-1}(\mathscr{D})$ is considered to be the dual of $H_0^1(\mathscr{D})$. We set the deterministic part of the equation to be an operator $A_t : H_0^1(\mathscr{D}) \to H^{-1}(\mathscr{D})$ defined as:*

$$\langle A_t(u), v\rangle = -\sum \int_{\mathscr{D}} a^{ij}(t,x)\partial_i u \partial_j v \, \mathrm{d}x + \langle f_t, v\rangle \qquad \text{for } u, v \in H_0^1(\mathscr{D}). \tag{4.3}$$

*The coefficients $a^{ij}$ depend on $x, \omega$ and $t$, and are bounded almost surely. We also require that $f \in L^{p_0}(\Omega; L^2([0,T]; H^{-1}(\mathscr{D})))$. We set the operators $B_t^k : H_0^1(\mathscr{D}) \to L^2(\mathscr{D})$ to be defined as:*

$$B_t^k(v) = \sum_{i=1}^d b^{ik}\partial_i v + g_t^k \qquad \text{for } v \in H_0^1(\mathscr{D}). \tag{4.4}$$

*We assume the coefficients $b^{ik}$ depend on $x$, $\omega$ and $t$, are bounded almost surely and are smooth for all $i, k \in \mathbb{N}$. An additional requirement is that $g^k \in L^{p_0}(\Omega; L^2([0,T]; L^2(\mathscr{D})))$. We also assume a uniform ellipticity condition holds for the coefficients $a^{ij}$ and $b^{ik}$. Define*

$$\sigma^{ij} = \sum_k b^{ik} b^{jk} \tag{4.5}$$

*Then, we assume the following uniform ellipticity condition:*

$$\sum_{i,j=1}^d \left(2a^{ij} - \sigma^{ij}\right)\xi_i\xi_j \geq \theta|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^d, \tag{4.6}$$

*where $\theta > 0$.*

**Proposition 4.1.** *Suppose the assumptions in 4.1 hold, $u_0 \in L^{p_0}(\Omega; L^2(\mathscr{D}))$, where $p_0$ is the same as in 4.1. Then, a unique solution $u$ of equation* (4.1) *exists and the following estimate holds for all $p \in [2, p_0]$:*

$$\begin{aligned} &\mathbb{E}\sup_{t\in[0,T]} \|u_t\|^p_{L^2(\mathscr{D})} + \mathbb{E}\int_0^T \|u_t\|^{p-2}_{L^2(\mathscr{D})}\|u_t\|^2_{H_0^1(\mathscr{D})}\mathrm{d}t + \mathbb{E}\left(\int_0^T \|u_t\|^2_{H_0^1(\mathscr{D})}\mathrm{d}t\right)^{\frac{p}{2}} \\ &\leq Ce^{CT}\left(\mathbb{E}\|u_0\|^p_{L^2(\mathscr{D})} + \mathbb{E}\left(\int_0^T \|f_t\|^2_{H^{-1}(\mathscr{D})}\mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|g_t\|^2_{\ell^2(\mathbb{N}; H_0^1(\mathscr{D}))}\mathrm{d}t\right)^{\frac{p}{2}}\right), \end{aligned} \tag{4.7}$$

*where $C$ depends on $\theta$ and $p$.*

**Remark 4.1.** Setting $f = g = 0$ and assuming that all $b^k$ are not space dependent, we can actually use remark 3.3 to include the endpoint $p_0 = \infty$. That is, we can prove that for any $p_0 \in [2, \infty]$ and $u_0 \in L^{p_0}(\Omega; L^2(\mathscr{D}))$, there exists a unique solution $u$ and the following estimate holds for $p \in [2, p_0]$:

$$\|u\|_{L^p(\Omega; C([0,T]; H))} \leq C\|u_0\|_{L^p(\Omega; H)}, \tag{4.8}$$

where $C$ only depends on $\theta$.

*Proof.* It suffices to show that assumptions H1 to H5 hold for equation (4.1) without $f$ and $g$. By applying Theorem 3.3, the result will immediately follow.

For H1, hemicontinuity is immediately clear from the definition of $A$. For H2, we let $u, v \in H_0^1(\mathscr{D})$ and use the uniform ellipticity type condition to derive:

$$\begin{aligned}
&2\langle A(u)-A(v), u-v\rangle + \sum_{k=1}^{\infty}\left\|\sum_{i=1}^{d} b^{ik}\partial_i(u-v)\right\|_{L^2(\mathscr{D})}^2 \\
&\overset{(4.3)}{=} -\sum\int_{\mathscr{D}} 2a^{ij}(x)(\partial_i u-\partial_i v)(\partial_j u-\partial_j v)\mathrm{d}x + \sum_{k=1}^{\infty}\int_{\mathscr{D}}\left(\sum_{i=1}^{d} b^{ik}\partial_i(u-v)\right)^2 \mathrm{d}x \\
&= -\sum\int_{\mathscr{D}} 2a^{ij}(x)(\partial_i u-\partial_i v)(\partial_j u-\partial_j v)\mathrm{d}x + \sum_{k=1}^{\infty}\int_{\mathscr{D}}\sum_{i,j=1}^{d} b^{ik}b^{jk}\partial_i(u-v)\partial_j(u-v)\mathrm{d}x \qquad (4.9)\\
&\overset{(4.5)}{=} \sum_{i,j=1}^{d}\int_{\mathscr{D}}(-2a^{ij}+\sigma^{ij})\partial_i(u-v)\partial_j(u-v)\mathrm{d}x \\
&\overset{(4.6)}{\le} -\theta\sum\int_{\mathscr{D}}|\partial_i(u-v)|^2 dx \\
&= -\theta\|u-v\|_{H_0^1(\mathscr{D})}^2
\end{aligned}$$

We now move to H3. For the first two terms of H3, we can take (4.9) with $v=0$, since all terms are linear. Therefore, we are only left to derive an expression for $\frac{\|B_t(v)^* v\|_{\ell^2}^2}{\|v\|_{L^2(\mathscr{D})}^2}$, where $v \in H_0^1(\mathscr{D})$. Now, we use integration by parts and smoothness of the coefficients $b^{ik}$ to obtain:

$$\begin{aligned}
(B_t(v)^* v)_k &= \sum_{i=1}^{d}\int_{\mathscr{D}}(b^{ik}\partial_i v)v\mathrm{d}x \\
&= \frac{1}{2}\sum_{i=1}^{d}\int_{\mathscr{D}}\partial_i(b^{ik}v^2)-(\partial_i b^{ik})v^2\mathrm{d}x \\
&= -\frac{1}{2}\sum_{i=1}^{d}\int_{\mathscr{D}}\partial_i(b^{ik})v^2\mathrm{d}x \qquad (4.10)\\
&\le \frac{1}{2}\sum_{i=1}^{d}\|\partial_i b^{ik}\|_{L^\infty(\mathscr{D})}\int_{\mathscr{D}}v^2\mathrm{d}x \\
&= \frac{1}{2}\sum_{i=1}^{d}\|\partial_i b^{ik}\|_{L^\infty(\mathscr{D})}\|v\|_{L^2(\mathscr{D})}^2
\end{aligned}$$

Therefore, the coercivity condition is satisfied in the following way:

$$\begin{aligned}
&2\langle A(v), v\rangle + \sum_{k=1}^{\infty}\|\sum_{i=1}^{d} b^{ik}\partial_i v\|_{L^2(\mathscr{D})}^2 + (p_0-2)\frac{\|(B_t(v)^* v)\|_{\ell^2(\mathbb{N};L^2(\mathscr{D}))}}{\|v\|_{L^2(\mathscr{D})}^2} \\
&\le -\theta\|v\|_{H_0^1(\mathscr{D})}^2 + C(p_0-2)\|v\|_{L^2(\mathscr{D})}^2
\end{aligned} \qquad (4.11)$$

For H4, let $u, v \in H_0^1(\mathscr{D})$. Then,

$$|\langle A(u), v\rangle| \le \sum_{i,j=1}^{d}\|a^{ij}\|_{L^\infty}\|u\|_{H_0^1(\mathscr{D})}\|v\|_{H_0^1(\mathscr{D})} \qquad (4.12)$$

We conclude that $\|A(u)\|^2_{H^{-1}(\mathscr{D})} \leq \left(\sum \|a^{ij}\|_{L^\infty}\right)^2 \|u\|^2_{H^1_0(\mathscr{D})}$, so H4 is satisfied with $\alpha = 2$. Similarly, H5 holds:

$$\|B_t(v)\|^2_{\ell^2(\mathbb{N};L^2(\mathscr{D}))} \leq \left\| \sum_{i,j=1}^{d} \sigma^{ij} \right\|_\infty \|v\|^2_{H^1_0(\mathscr{D})} \tag{4.13}$$

□

## 4.2. The stochastic heat equation with Neumann boundary conditions

The second equation we consider is the same stochastic heat equation equation as before, but now with Neumann boundary conditions on a $C^1$ domain $\mathscr{D} \subseteq \mathbb{R}^d$. For completeness, this equation is:

$$\mathrm{d}u_t = \left( \sum_{i,j=1}^{d} \partial_i (a^{ij} \partial_j u_t) + f_t \right) \mathrm{d}t + \left( \sum_{k=1}^{\infty} \sum_{i=1}^{d} b^{ik} \partial_i u_t + g^k_t \right) \mathrm{d}W^k_t. \tag{4.14}$$

Most assumptions and computations will be similar as before. However, the coercivity condition will turn out to be $p$-dependent.

**Assumptions 4.2.** *Consider equation* (4.14) *on a $d$-dimensional $C^1$ domain $\mathscr{D}$ with Neumann boundary conditions and initial condition $u_0 \in L^{p_0}(\Omega; H)$ where $p_0 \in [2,\infty)$. We reformualte this equation into an equation of the form*

$$\mathrm{d}u_t = A_t(u_t)\,\mathrm{d}t + \sum_{k=1}^{\infty} B^k_t(u_t)\,\mathrm{d}W^k_t. \tag{4.15}$$

*The Gelfand triple used for this example is*

$$H^1(\mathscr{D}) \subseteq L^2(\mathscr{D}) \subseteq H^{-1}(\Omega).$$

*This allows us to define the determinstic part of the equation as an operator $A_t : H^1(\mathscr{D}) \to H^{-1}_0(\mathscr{D})$ with:*

$$\langle A_t(u), v \rangle = -\sum \int_{\mathscr{D}} a^{ij}(t,x) \partial_i u \partial_j v \,\mathrm{d}x + \langle f_t, v \rangle \qquad \text{for } u, v \in H^1(\mathscr{D}). \tag{4.16}$$

*The coefficients $a^{ij}$ depend on $x, \omega$ and $t$, and are bounded almost surely. Another assumption is that $f \in L^{p_0}(\Omega; L^2([0,T]; H^{-1}_0(\mathscr{D})))$. The diffusion part of the equation has operators $B^k : H^1(\mathscr{D}) \to L^2(\mathscr{D})$ as components defined as:*

$$B^k_t(v) = \sum_{i=1}^{d} b^{ik} \partial_i v + g^k_t \qquad \text{for } v \in H^1(\mathscr{D}) \tag{4.17}$$

*It is assumed the coefficients $b^{ik}$ depend on $x$, $\omega$ and $t$, almost surely bounded and are smooth for all $i, k \in \mathbb{N}$. The collection $W^k$ is assumed to consist of real-valued, independent Wiener processes. Furthermore, we assume $g \in L^{p_0}(\Omega; L^2([0,T]; \ell^2(\mathbb{N}; H^1(\mathscr{D}))))$. We also assume a $p$-dependent uniform ellipticity condition holds for the coefficients $a^{ij}$ and $b^{ik}$ in the following way:*

$$\sum_{i,j=1}^{d} \left( 2a^{ij} - (p_0 - 1)\sigma^{ij} \right) \xi_i \xi_j \geq \theta |\xi|^2 \qquad \forall \xi \in \mathbb{R}^d$$

*where* $\sigma^{ij} = \sum_k b^{ik} b^{jk}$, $p_0 \in [2,\infty)$ *and* $\theta > 0$.

**Proposition 4.2.** *Suppose the assumptions in Assumptions 4.2 hold,* $u_0 \in L^{p_0}(\Omega; L^2(\mathscr{D}))$, *where* $p_0$ *is the same as in Assumptions 4.2. Then, a unique solution* $u$ *of equation* (4.14) *exists and the following estimate holds for all* $p \in [2, p_0]$*:*

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]} \|u_t\|^p_{L^2(\mathscr{D})} + \mathbb{E}\int_0^T \|u_t\|^{p-2}_{L^2(\mathscr{D})} \|u_t\|^2_{H^1(\mathscr{D})} \mathrm{d}t + \mathbb{E}\left(\int_0^T \|u_t\|^2_{H^1(\mathscr{D})} \mathrm{d}t\right)^{\frac{p}{2}} \\
&\le Ce^{CT}\left(\mathbb{E}\|u_0\|^p_{L^2(\mathscr{D})} + \mathbb{E}\left(\int_0^T \|f_t\|^2_{H^{-1}(\mathscr{D})} \mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|g_t\|^2_{\ell^2(\mathbb{N};H^1(\mathscr{D}))} \mathrm{d}t\right)^{\frac{p}{2}}\right)
\end{aligned} \tag{4.18}$$

*Proof.* We show that assumptions H1 to H5 hold, where we set $f, g = 0$. Application of theorem 3.3 gives the result. We see that H1, H4 and H5 are similar to the proof of proposition 4.1. To prove H2, we require an extra step in inequality (4.9). Note that the same sequence of inequalities hold, since we only use the uniform ellipticity condition. This condition also follows from the new uniform ellipticity condition in assumptions 4.2. Let $u, v \in H^1(\mathscr{D})$. Using inequality (4.9), this results in:

$$\begin{aligned}
&2\langle A(u) - A(v), u - v\rangle + \sum_{k=1}^{\infty} \left\| \sum_{i=1}^{d} b^{ik}\partial_i(u-v)\right\|^2_{L^2(\mathscr{D})} \\
&\le -\theta \sum_{i=1}^{d} \int_{\mathscr{D}} |\partial_i(u-v)|^2 \mathrm{d}x \\
&\le -\theta\|u-v\|^2_{H^1(\mathscr{D})} + \theta\|u-v\|^2_{L^2(\mathscr{D})}
\end{aligned} \tag{4.19}$$

We are only left to prove H3. We again take $v = 0$ in H2 to get part of H3. It only remains to inspect the term $\|(B_t(v))^* v\| / \|v\|^2$ where $v \in H^1(\mathscr{D})$. It is sometimes tempting to do the following trivial estimate

$$\frac{\|B_t(v)^* v\|^2_{\ell^2}}{\|v\|^2_{L^2(\mathscr{D})}} \le \|B_t(v)\|^2_{L_2(L^2(\mathscr{D}),\ell^2)}.$$

However, this makes the coercivity condition immediately $p_0$ dependent in this case, which might not be optimal. Doing this estimate for now results in the following preliminary estimate for H3:

$$\begin{aligned}
&2\langle A(v), v\rangle + (p_0-1)\sum_{k=1}^{\infty}\left\|\sum_{i=1}^{d} b^{ik}\partial_i v\right\|^2_{L^2(\mathscr{D})} \\
&\overset{\text{def.}}{=} -2\sum_{i,j=1}^{d}\int_{\mathscr{D}} a^{ij}\partial_i v \partial_j v \, \mathrm{d}x + (p_0-1)\sum_{k=1}^{\infty}\sum_{i,j=1}^{d}\int_{\mathscr{D}} b^{ik}b^{jk}\partial_i v\partial_j v \mathrm{d}x \\
&\le \int_{\mathscr{D}}\sum_{i,j=1}^{d}\left(-2a^{ij} + (p_0-1)\sigma^{ij}\right)\partial_i v\partial_j v \mathrm{d}x \\
&\le -\theta\sum\int_{\mathscr{D}} |\partial_i v|^2 \mathrm{d}x \\
&= -\theta\|v\|^2_{H^1_0(\mathscr{D})}
\end{aligned} \tag{4.20}$$

Therefore, we have shown that H3 holds. Applying Theorem 3.3 results in the statement we wanted to prove. □

## 4.3. Stochastic Burgers' equation

One of the most important equations in fluid dynamics is Burgers' equation. Introduced by J.M. Burgers in [8] as a simplification of the Navier-Stokes equations (which will be extended next section), it was studied to understand how dissaptive and non-linear inertial forces interact in a fluid. The equation fails to capture the intriguing phenomenon of turbulence, however. Therefore, adding stochastic forcing to the equation might be an interesting generalization. This was first studied by the authors of [5] and later by [11]. In this section, we will discuss existence, uniqueness and energy estimates for such an equation in our framework on the domain $\mathscr{D} = (0,1)$ with Dirichlet boundary conditions. In particular, we take the following stochastic form of Burgers' equation:

$$\mathrm{d}u_t = (\Delta u_t + u_t Du_t)\,\mathrm{d}t + \gamma Du_t\,\mathrm{d}W_t, \tag{4.21}$$

where $D$ denotes the spatial derivative and $\gamma \in (-1,1)$. It will turn out that we are able to obtain $L^\infty(\Omega; C([0,T]; L^2(0,1)))$ estimates for this particular example. This is in correspondence with what has been shown in [5], albeit obtained in a different way.

**Assumptions 4.3.** *Consider equation* (4.21) *with* $\gamma \in (-1,1)$ *on an interval* $(0,1)$ *with Dirichlet boundary conditions and initial condition* $u_0 \in L^{p_0}(\Omega; L^2(0,1))$ *for some* $p_0 \in [2,\infty]$. *We reformulate this equation into an equation of the form*

$$\mathrm{d}u_t = A(u_t)\,\mathrm{d}t + B(u_t)\mathrm{d}W_t. \tag{4.22}$$

*The Gelfand triple that will use for this is* $\big(H_0^1(0,1), L^2(0,1), H^{-1}(0,1)\big)$. *We interpret the deterministic term as an operator* $A\colon H_0^1(0,1) \to H^{-1}(0,1)$ *with*

$$\langle A(u), v\rangle = -\int_{(0,1)} Du\, Dv\,\mathrm{d}x + \int_{(0,1)} u\, Du\, v\,\mathrm{d}x \qquad \text{for } u, v \in H_0^1(0,1). \tag{4.23}$$

*The diffusion part of the equation is defined as an operator* $B : H_0^1(0,1) \to L^2(0,1)$ *with*

$$B(v) = \gamma Dv \qquad \text{for } v \in H_0^1(0,1). \tag{4.24}$$

**Proposition 4.3.** *Let* $p_0 \in [2,\infty)$ *and* $u_0 \in L^{p_0}(\Omega; L^2(0,1))$. *Then for* $p \in [2, p_0]$, *a unique solution of equation* (4.21) *in* $L^p(\Omega; C([0,T]; L^2(0,1)))$ *exists. Furthermore, we get the following energy estimate:*

$$\|u\|_{L^p(\Omega;C([0,T];L^2(0,1)))} + \|u\|_{L^p(\Omega;L^2([0,T];H_0^1(0,1)))} \leq C\sqrt{p}\|u_0\|_{L^p(\Omega;L^2(0,1)} \tag{4.25}$$

*where* $C$ *only depends* $\gamma$. *Moreover, if* $u_0 \in L^\infty(\Omega; L^2(0,1))$, *then for any* $p \in [2,\infty]$:

$$\|u\|_{L^p(\Omega;C([0,T];L^2(0,1)))} \leq C'\|u_0\|_{L^p(\Omega;L^2(0,1))} \tag{4.26}$$

*where* $C'$ *only depends on* $\gamma$.

*Proof.* As in previous instances, it suffices to check assumptions H1 to H5, in order to apply Theorem 3.2. Specifically, we want to apply remark 3.3, which means we also need to show that $K_B = K_c = 0$ in H3 and H5. We skip H1 to move to H2 immediately. For $u, v \in H_0^1(0,1)$, we have:

$$\begin{aligned}\langle A(u)-A(v),u-v\rangle &= -\int_{(0,1)} D(u-v)D(u-v)\mathrm{d}x + \int_{(0,1)} (uDu-vDv)(u-v)\mathrm{d}x \\ &= -\|u-v\|^2_{H^1_0(0,1)} - \int_{(0,1)} (uDu-vDv)(u-v)\mathrm{d}x.\end{aligned} \tag{4.27}$$

We analyze the second term on the RHS. Using integration by parts, we obtain:

$$\begin{aligned}\int_{(0,1)} (uDu-vDv)(u-v)\mathrm{d}x &= \int_{(0,1)} \frac{1}{2}D(u^2-v^2)(u-v)\mathrm{d}x \\ &= \int_{(0,1)} \frac{1}{2}(u^2-v^2)(D(u-v))\mathrm{d}x \\ &= \int_{(0,1)} \frac{1}{2}(u-v)^2 D(u-v)\mathrm{d}x + \int_{(0,1)} v(u-v)D(u-v)\mathrm{d}x.\end{aligned} \tag{4.28}$$

The first term on the RHS is zero by using integration by parts:

$$\begin{aligned}\int_{(0,1)} \frac{1}{2}(u-v)^2 D(u-v)\mathrm{d}x &= -\int_{(0,1)} \frac{1}{2}D((u-v)^2)(u-v)\mathrm{d}x \\ &= \int_{(0,1)} (u-v)^2 D(u-v)\mathrm{d}x.\end{aligned} \tag{4.29}$$

By substracting the LHS from the RHS, we obtain

$$\frac{1}{2}\int_{(0,1)} (u-v)^2 D(u-v)\mathrm{d}x = 0, \tag{4.30}$$

which implies that the first term in expression (4.28) is zero. Returning to equation (4.27), we get:

$$\begin{aligned}\langle A(u)-A(v),u-v\rangle &= -\|u-v\|^2_{H^1_0(0,1)} - \int_{(0,1)} v(u-v)D(u-v)\mathrm{d}x \\ &\le -\|u-v\|^2_{H^1_0(0,1)} + \|v\|_{L^4(0,1)}\|u-v\|_{L^4}\|u-v\|_{H^1_0(0,1)}\end{aligned} \tag{4.31}$$

We use the Sobolev-Gagliardo-Nirenberg and Poincaré inequality to obtain the estimate:

$$\|u\|_{L^4(0,1)} \le C\|u\|^{\frac{3}{4}}_{L^2(0,1)}\|u\|^{\frac{1}{4}}_{H^1_0(0,1)} \le C'\|u\|^{\frac{1}{2}}_{L^2(0,1)}\|u\|^{\frac{1}{2}}_{H^1_0(0,1)}, \tag{4.32}$$

where $C, C' \ge 0$. This can be used to obtain H2 in the following way:

$$\begin{aligned}\langle A(u)-A(v),u-v\rangle &\le -\|u-v\|^2_{H^1_0(0,1)} + \|v\|_{L^4(0,1)}\|u-v\|_{L^4}\|u-v\|_{H^1_0(0,1)} \\ &\le -\|u-v\|^2_{H^1_0(0,1)} + C\|v\|_{L^4(0,1)}\|u-v\|^{\frac{1}{2}}_{L^2(0,1)}\|u-v\|^{\frac{3}{2}}_{H^1_0(0,1)} \\ &\overset{\text{Young}}{\le} (\varepsilon-1)\|u-v\|^2_{H^1_0(0,1)} + C_\varepsilon\|v\|^4_{L^4(0,1)}\|u-v\|^2_{L^2(0,1)} \\ &\le (\varepsilon-1)\|u-v\|^2_{H^1_0(0,1)} + C_\varepsilon\|v\|^2_{L^2(0,1)}\|v\|^2_{H^1_0(0,1)}\|u-v\|^2_{L^2(0,1)},\end{aligned} \tag{4.33}$$

where we used Young's inequality in the third line, choosing some $\varepsilon \in (0,1)$. Now we combine with (4.24) to get:

$$
\begin{aligned}
&2\langle A(u)-A(v),u-v\rangle+\|B(u)-B(v)\|^2_{L^2(0,1)}\\
&\le(\gamma^2+2\varepsilon-2)\|u-v\|^2_{H^1_0(0,1)}+C_\varepsilon(1+\|v\|^2_{L^2(0,1)}\|v\|^2_{H^1_0(0,1)})\|u-v\|^2_{L^2(0,1)}\\
&\le(\gamma^2+2\varepsilon-2)\|u-v\|^2_{H^1_0(0,1)}+C_\varepsilon\left(1+\|v\|^2_{L_2(0,1)}\right)\left(1+\|v\|^2_{H^1_0(0,1)}\right)\|u-v\|^2_{L^2(0,1)}
\end{aligned}
\tag{4.34}
$$

where $u,v\in H^1_0(0,1)$. We see that H2 holds with $\alpha=2$, $\beta=2$, since $\varepsilon\in(0,1)$.

For H3, we first inspect the quantity $\frac{\|B(v)^*v\|^2_U}{\|v\|^2_H}$ with $v\in H^1_0(0,1)$ and $v\neq 0$. Now, note that the following holds by using integration by parts

$$
\int_{(0,1)}(\gamma Dv)v\mathrm{d}x=-\int_{(0,1)}(\gamma Dv)v\mathrm{d}x=0 \tag{4.35}
$$

In order to derive H3, take $v\in H^1_0(0,1)$, $v\neq0$. This leads to:

$$
\begin{aligned}
&2\langle A(v),v\rangle+\|B(v)\|^2_{L^2(0,1)}+(p_0-2)\frac{\|B(v)^*v\|^2_U}{\|v\|^2_H}\\
&=-2\|v\|^2_{H^1_0(0,1)}+\int_{(0,1)}v^2Dv\mathrm{d}x+\gamma^2\|v\|^2_{H^1_0(0,1)}
\end{aligned}
\tag{4.36}
$$

The middle term on the RHS can be treated by using integration by parts:

$$
\int_{(0,1)}v^2Dv\,\mathrm{d}x=-\int_{(0,1)}2v^2Dv\,\mathrm{d}x. \tag{4.37}
$$

By adding the RHS to the LHS, we see that this quantity vanishes. Therefore,

$$
\begin{aligned}
&2\langle A(v),v\rangle+\|B(v)\|^2_{L^2(0,1)}+(p_0-2)\frac{\|B(v)^*v\|^2_U}{\|v\|^2_H}\\
&=(-2+\gamma^2)\|v\|^2_{H^1_0(0,1)}
\end{aligned}
\tag{4.38}
$$

We see that H3 holds with $\alpha=2$. We also require $\gamma^2<2$. Let $u,v\in H^1_0(0,1)$. For H4, we inspect the following quantity:

$$
|\langle A(u),v\rangle|\le\int_{(0,1)}|Du||Dv|\mathrm{d}x+\left|\int_{(0,1)}u(Du)v\mathrm{d}x\right|. \tag{4.39}
$$

By Cauchy-Schwarz, the first term on the RHS is estimated as:

$$
\int_{(0,1)}|Du||Dv|\mathrm{d}x\le\|u\|_{H^1_0(0,1)}\|v\|_{H^1_0(0,1)}. \tag{4.40}
$$

The second term is estimated by using integration by parts, Hölder's inequality and the Sobolev-Gagliardo-Nirenberg inequality:

$$
\begin{aligned}
\left|\int_{(0,1)}uDuv\mathrm{d}x\right|&=\left|\int_{(0,1)}\frac{1}{2}u^2Dv\mathrm{d}x\right|\\
&\le\frac{1}{2}\|u\|^2_{L^4(0,1)}\|v\|_{H^1_0(0,1)}\\
&\le C\|u\|_{L^2(0,1)}\|u\|_{H^1_0(0,1)}\|v\|_{H^1_0(0,1)}
\end{aligned}
\tag{4.41}
$$

This results in the following estimate for H4:

$$|\langle A(u), v| \leq \left(\|u\|_{H_0^1(0,1)} + C\|u\|_{L^2(0,1)}\|u\|_{H_0^1(0,1)}\right)\|v\|_{H_0^1(0,1)} \tag{4.42}$$

We use that $\alpha$ was set $\alpha = 2$ in H2, to obtain:

$$\begin{aligned}\|A(u)\|^2_{H^{-1}(0,1)} &\leq C'\left(\|u\|_{H_0^1(0,1)} + \|u\|_{L^2(0,1)}\|u\|_{H_0^1(0,1)}\right)^2 \\ &\leq C''\left(1+\|u\|^2_{H_0^1(0,1)}\right)\left(1+\|u\|^2_{L^2(0,1)}\right)\end{aligned} \tag{4.43}$$

Last but not least, for $v \in H_0^1(0,1)$ it is clear that $\|B(v)\|^2_{L^2(0,1)} = \gamma^2\|v\|^2_{H_0^1(0,1)}$. Therefore, H5 holds with $\alpha = 2$. Since we have shown all assumptions for our theory, we can apply Theorem 3.2 and remark 3.3. □

## 4.4. Stochastic Navier-Stokes equations in 2D

Having treated the stochastic Burgers' equation, we now turn to the stochastic Navier-Stokes equations. The deterministic version is a system of PDEs given by:

$$\begin{cases}\frac{\partial \mathbf{u}_t}{\partial t} + \mathbf{u_t}\cdot\nabla\mathbf{u_t} - \nu\Delta\mathbf{u}_t = -\nabla p \\ \nabla\cdot\mathbf{u} = 0,\end{cases} \tag{4.44}$$

with initial condition

$$\mathbf{u}(x,0) = u_0(x). \tag{4.45}$$

and viscosity $\nu > 0$. We use boldface typography $\mathbf{u}$ to denote that the above equation is really a system of equations. This notation will *not* be used in the sequel. One solves this system of equations by finding a velocity field $\mathbf{u}$ and a pressure function $p$ that satisfy the above equation. In $\mathbb{R}^3$, this leads to one of the famous open Millenium problems posed by the Clay Mathematics Institute. Given any $C^\infty$, divergence free vector field $u_0(x)$, one has to show that there exist smooth functions $p$ and $\mathbf{u}$ such that the above equation is satisfied (see [15] for the official formulation). In $\mathbb{R}^2$, this equation has already been solved in the deterministic setting ($\mathbb{R}^1$ is not interesting because of the divergence-free condition), but these methods do not seem to generalize to $\mathbb{R}^3$. We will treat the stochastic version in 2D as introduced in [5].

It turns out that the stochastic Navier-Stokes equations with multiplicative noise arise naturally from physical considerations as shown in [5]. This can be done by considering that the supposedly solution of the Navier-Stokes equations can be decomposed as the sum of an average field and a fluctuating field. Therefore, we are led to the following equation:

$$\mathrm{d}u_t = (\nu\Delta u_t - (u,\nabla)u)\mathrm{d}t + \sum_{k=1}^{\infty}[(b^k,\nabla)u]\mathrm{d}W^k(t) - (\nabla p)\mathrm{d}t \tag{4.46}$$

where the components $b_k$ are vectors of divergence free vector fields.

**Assumptions 4.4.** *Consider equation* (4.46) *on a domain $\mathscr{D} \subseteq \mathbb{R}^2$ with $C^1$ boundary, $\nu > 0$ and vectors $b^k$ consisting of divergence free vector fields. We denote its components by $b^{ik}$,* *where $i \in \{1,2\}$. Since $b^{ik}$ produces a vector in $\mathbb{R}^2$, we denote the components of that vector by $b^{ik}_\gamma$, where $\gamma \in \{1,2\}$. We take the following Gelfand triple, following [4]. Define*

$$V = \{v \in W^{1,2}_0(\mathscr{D};\mathbb{R}^2) : \nabla \cdot v = 0 \quad a.e. \ on\, \mathscr{D}\}, \quad \|v\|_V := \left(\int_{\mathscr{D}} |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}}, \tag{4.47}$$

*and set $H$ to be the closure of $V$ with respect to the norm*

$$\|v\|_H := \left(\int_{\mathscr{D}} |v|^2 \mathrm{d}x\right)^{\frac{1}{2}}. \tag{4.48}$$

*In order to formally treat equation* (4.46)*, we introduce the Helmholtz-Hodge projection $\mathbb{P}_{HL}$ which is defined as the orthogonal projection*

$$\mathbb{P}_{HL} : L^2(\mathscr{D};\mathbb{R}^2) \to H. \tag{4.49}$$

*This is well-defined, since $H$ is a closed subspace of the Hilbert space $L^2(\mathscr{D};\mathbb{R}^2)$. Applying the Helmholtz-Leray projection to both sides of* (4.46)*, removes the pressure term and leads us to consider the following equation:*

$$\mathrm{d}u_t = (\nu \mathbb{P}_{HL}(\Delta u_t) - \mathbb{P}_{HL}[(u,\nabla)u])\mathrm{d}t + \sum_{k=1}^{\infty} \mathbb{P}_{HL}[(b^k,\nabla)u]\mathrm{d}W^k(t) \tag{4.50}$$

*We reformulate this into an equation of the form*

$$\mathrm{d}u_t = (Au_t + F(u_t))\mathrm{d}t + \sum_{k=1}^{\infty} B^k(u_t)\mathrm{d}W^k(t) \tag{4.51}$$

*where we define $A$ to be an operator $A : V \to V^*$ given by*

$$Au = \nu \mathbb{P}_{HL}(\Delta u), \quad u \in V.$$

*We define $F$ to be a nonlinear operator $F : V \to V^*$ given by*

$$F(u) = -\mathbb{P}_{HL}[(u,\nabla)u], \quad u \in V.$$

*Finally, we define the components $B^k$ to be operators $B^k : V \to H$ given by*

$$B^k(u) = \mathbb{P}_{HL}[(b^k,\nabla)u], \quad u \in V.$$

*We assume the following coercivity condition:*

$$\left(2\nu - \sum_{k=1}^{\infty}\sum_{i,j=1}^{2} b^{ik}_\gamma b^{jk}_\gamma\right)\xi_i\xi_j(\eta^\gamma)^2 \geq \kappa|\xi|^2|\eta|^2, \qquad \textit{for all}\ \xi,\eta \in \mathbb{R}^2 \tag{4.52}$$

**Proposition 4.4.** *Given assumptions 4.4, let $p_0 \in [2,\infty)$. Then, for any $u_0 \in L^{p_0}(\Omega;H)$, there exists a unique solution $u$ to equation* (4.50)*. Furthermore, there exists a constant $C$ only depending on $\kappa$ such that for all $p \in [2,p_0]$:*

$$\|u\|_{L^p(\Omega;C([0,T];H))} + \|u\|_{L^p(\Omega;L^2([0,T];V))} \leq C\sqrt{p}\left[\|u_0\|_{L^p(\Omega;H)}\right]. \tag{4.53}$$

*In particular, for any $u_0 \in L^\infty(\Omega;H)$, there exists a unique solution $u$ to equation* (4.50) *with energy estimate*

$$\|u\|_{L^\infty(\Omega;C([0,T];H))} \leq C\left[\|u_0\|_{L^\infty(\Omega;H)}\right]. \tag{4.54}$$

**Remark 4.2.** A similar result can be found in [5] using the semigroup approach.

**Remark 4.3.** One can also consider extra body forces in the equation by using extension 3.3.

**Remark 4.4.** The above coercivity condition is slightly artificial and the author was not able to find a better condition in the literature. Intuitively, the condition makes sense, since the noise is not allowed to dominate the smoothing of the equation.

*Proof.* In order to apply our framework (theorem 3.2), we need to show that assumptions H1 to H5 hold. Specifically, we want to use Remark 3.3, which means that $K_B = K_c = 0$ in H1 to H5. We will only show H2, H3 and H5, since the other assumptions have already been shown in [28] and [4]. Let $u, v \in V$. We will first consider the quantity $\langle A(u) - A(v), u - v\rangle$. Then,

$$\begin{aligned}\langle A(u) - A(v), u - v\rangle &= \nu\langle \mathbb{P}_{HL}(\Delta(u - v)), u - v\rangle \\ &= \nu(\Delta(u - v), u - v) \\ &= -\nu\int_{\mathscr{D}} \nabla(u - v)\cdot\nabla(u - v)\mathrm{d}x \\ &= -\nu\|u - v\|_V^2,\end{aligned} \tag{4.55}$$

where the second line follows by definition of $A$ and the fact that orthogonal projections are self-adjoint. The last line follows by definition of $V$. We also need to inspect $\langle F(u) - F(v), u - v\rangle$. From [28] we see that

$$\langle F(u) - F(v), u - v\rangle \le \frac{\nu}{2}\|u - v\|_V^2 + \frac{C}{\nu^3}\|v\|^4_{L^4(\mathscr{D};\mathbb{R}^2)}\|u - v\|_H^2 \tag{4.56}$$

where $C \in (0, \infty)$. We continue by inspecting the second term for H2:

$$\begin{aligned}\sum_{k=1}^{\infty}\|B^k(u) - B^k(v)\|_H^2 &\le \sum_{k=1}^{\infty}\|[(b^k, \nabla)(u - v)]\|_H^2 \\ &= \sum_{k=1}^{\infty}\sum_{i,j=1}^{2}\int_{\mathscr{D}} b_\gamma^{ik} b_\gamma^{jk}\partial_i(u - v)^\gamma\partial_j(u - v)^\gamma\mathrm{d}x,\end{aligned} \tag{4.57}$$

where the first line follows since projections are contractive. The second line follows by writing out the terms. Combining all terms in H2 and applying the coercivity assumption (4.52), we obtain

$$\begin{aligned}&2\langle A(u) + F(u) - (A(v) + F(v)), u - v\rangle + \sum_{k=1}^{\infty}\|B^k(u) - B^k(v)\|_H^2 \\ &\le -\nu\int_{\mathscr{D}}\nabla(u - v)\cdot\nabla(u - v)\mathrm{d}x + \frac{\nu}{2}\|u - v\|_V^2 + \frac{C}{\nu^3}\|v\|^4_{L^4(\mathscr{D};\mathbb{R}^2)}\|u - v\|_H^2 \\ &= -\frac{\nu}{2}\|u - v\|_V^2 + \frac{C}{\nu^3}\|v\|^4_{L^4(\mathscr{D};\mathbb{R}^2)}\|u - v\|_H^2 \\ &\le -\frac{\nu}{2}\|u - v\|_V^2 + \frac{C'}{\nu^3}\|v\|_V^2\|v\|_H^2\|u - v\|_H^2 \\ &\le -\frac{\nu}{2}\|u - v\|_V^2 + \frac{C'}{\nu^3}(1 + \|v\|_V^2)(1 + \|v\|_H^2)\|u - v\|_H^2,\end{aligned} \tag{4.58}$$

where $C' \in (0,\infty)$. The third line follows by Sobolev embedding (see [28], appendix H). The above implies that H2 holds with $\alpha = 2$. To show H3, we note that from (4.55) it follows that

$$\langle A(u), u\rangle = -\nu \|u\|_V^2. \tag{4.59}$$

We also note that $\langle F(u), u\rangle = 0$, which follows by writing out the definition and using integration by parts. Therefore, the only term that remains to be estimated is

$$\frac{\|B^*(u)u\|^2_{\ell^2(\mathbb{N})}}{\|u\|_H^2}.$$

This will turn out to be 0, by using that the components of $b^k$ are divergence free vector fields. Consider $(B^*(u)u)_k$. Then,

$$\begin{aligned}
&(B^*(u)u)_k \\
&= \int_{\mathscr{D}} \left[(b^k,\nabla)u\right]\cdot u \mathrm{d}x \\
&= \underbrace{\int_{\mathscr{D}} (b_1^{1k}\partial_1 u^1)u^1 + (b_2^{1k}\partial_2 u^1)u^1 \mathrm{d}x}_{\boxed{\text{A}}} + \int_{\mathscr{D}} (b_1^{2k}\partial_1 u^2)u^2 + (b_2^{2k}\partial_2 u^2)u^2 \mathrm{d}x.
\end{aligned} \tag{4.60}$$

We will only treat $\boxed{\text{A}}$, since the other term is treated in a similar way. Using integration by parts, we see:

$$\begin{aligned}
\boxed{\text{A}} &= -\int_{\mathscr{D}} \partial_1(b_1^{1k}u^1)u^1 + \partial_2(b_2^{ik}u^1)u^1 \mathrm{d}x \\
&= -\int_{\mathscr{D}} \partial_1(b_1^{1k})(u^1)^2 + b_1^{1k}(\partial_1 u^1)u^1 \mathrm{d}x - \int_{\mathscr{D}} \partial_2(b_2^{ik})(u^1)^2 + b_2^{ik}(\partial_2 u^1)u^1 \mathrm{d}x \\
&= -\int_{\mathscr{D}} \left(\partial_1(b_1^{1k}) + \partial_2(b_2^{2k})\right)(u^1)^2 \mathrm{d}x - \int_{\mathscr{D}} (b_1^{1k}\partial_1 u^1)u^1 + (b_2^{1k}\partial_2 u^1)u^1 \mathrm{d}x \\
&= -\int_{\mathscr{D}} (b_1^{1k}\partial_1 u^1)u^1 + (b_2^{1k}\partial_2 u^1)u^1 \mathrm{d}x.
\end{aligned} \tag{4.61}$$

Since the last line and $\boxed{\text{A}}$ are equal, this implies that $(B^*(u)u)_k = 0$ for all $k \in \mathbb{N}$. We therefore conclude that the coercivity condition H3 is as follows:

$$2\langle A(u) + F(u), u\rangle + \sum_{k=1}^{\infty} \|B^k(u)\|^2_{L^2(\mathscr{D};\mathbb{R}^2)} \leq -\kappa\|u\|_V^2. \tag{4.62}$$

We are only left to show H5, for which we can re-use H3. From H3, it follows that

$$\begin{aligned}
\sum_{k=1}^{\infty} \|B^k(u)\|^2_{L^2(\mathscr{D};\mathbb{R}^2)} &\leq -\kappa\|u\|_V^2 + 2|\langle A(u), u\rangle| \\
&\leq (2\nu - \kappa)\|u\|_V^2
\end{aligned} \tag{4.63}$$

We see that H3 and H5 hold with $K_B = K_c = 0$. Therefore, we can use remark 3.3 to conclude that the theorem holds. □

## 4.5. Stochastic $p$-Laplacian

Instead of using the Laplacian as a driver for a parabolic equation, one can also admit more nonlinear operators. One such variant is the so-called $p$-Laplacian. This is an equation on $\mathbb{R}^d$ of the form

$$\frac{\partial u_t}{\partial t} = \nabla \cdot (|\nabla u_t|^{\alpha-2} \nabla u_t)$$

where $\alpha > 2$. Since we want to reserve $p$ for the coercivity condition H3, we use $\alpha$ instead of $p$ in the $p$-Laplacian. We present the following stochastic version of the $p$-Laplacian:

$$\mathrm{d}u_t = \nabla \cdot (|\nabla u_t|^{\alpha-2} \nabla u_t)\mathrm{d}t + \sum_{k=1}^{\infty} B^k(u_t)\mathrm{d}W_t^k. \tag{4.64}$$

We will specify restriction on the operators $B^k$ in the assumptions section. Subsequently, we will prove existence, uniqueness and another energy estimate. Some of the exposition is based on work David Šiška and Neelima Varshney [38].

**Assumptions 4.5.** *Let $\alpha > 2$ and consider equation* (4.64) *on a $C^1$-domain $\mathscr{D}$ with Dirichlet boundary conditions. The collection $\{W^k\}_{k\geq 1}$ consists of real-valued, independent Wiener processes. The Gelfand triple used for this equation is*

$$(W_0^{1,\alpha}(\mathscr{D}), L^2(\mathscr{D}), W^{-1,\alpha}(\mathscr{D})).$$

*Here, $W^{-1,\alpha}(\mathscr{D})$ is the dual of $W_0^{1,\alpha}(\mathscr{D})$. We define the operator $A: W_0^{1,\alpha}(\mathscr{D}) \to W^{-1,\alpha}(\mathscr{D})$ as:*

$$\langle A(u), v\rangle = -\int_{\mathscr{D}} |\nabla u|^{\alpha-2} \nabla u \cdot \nabla v \mathrm{d}x \qquad \textit{for all } u, v \in W_0^{1,\alpha}(\mathscr{D}).$$

*Furthermore, we assume that the operators $B^k : W_0^{1,\alpha}(\mathscr{D}) \to L^2(\mathscr{D})$ satisfy two conditions: $B(0) = 0$ and the following bound for $u, v \in W_0^{1,\alpha}(\mathscr{D})$:*

$$\|B^k(u) - B^k(v)\|_{L^2(\mathscr{D})}^2 \leq \gamma_k^2 \| |\nabla u|^{\frac{\alpha}{2}} - |\nabla v|^{\frac{\alpha}{2}} \|_{L^2(\mathscr{D})}^2 + C_k^2 \|u - v\|_{L^2(\mathscr{D})}^2,$$

*where we assume $\sum \gamma_k^2 \leq \gamma^2$ with $\gamma^2 < \frac{8(\alpha-1)}{\alpha^2}$ and $\sum C_k^2 < \infty$. Last but not least, we let $p_0 \in \left[2, \frac{2}{\gamma^2} + 1\right)$ and $u_0 \in L^{p_0}(\Omega; L^2(\mathscr{D}))$.*

**Proposition 4.5.** *Given assumptions 4.5, there exists a unique solution $u$ to equation* (4.64)*. Furthermore, for any $p \in [2, p_0]$, there exists a constant $C$ depending on $\gamma$, $\alpha$ and $p$ such that the following estimate holds:*

$$\begin{aligned} &\mathbb{E} \sup_{t\in[0,T]} \|u_t\|_{W_0^{1,\alpha}(\mathscr{D})}^p + \mathbb{E}\int_0^T \|u_t\|_{L^2(\mathscr{D})}^{p-2} \|u_t\|_{W_0^{1,\alpha}(\mathscr{D})}^\alpha \mathrm{d}t + \mathbb{E}\left(\int_0^T \|u_t\|_{W_0^{1,\alpha}(\mathscr{D})}^\alpha \mathrm{d}t\right)^{\frac{p}{2}} \\ &\leq Ce^{CT}\mathbb{E}\|u_0\|_{L^2(\mathscr{D})}^p \end{aligned} \tag{4.65}$$

**Remark 4.5.** One could pick $B^k(u) = \gamma^k |\nabla u|^{\frac{\alpha}{2}}$ as noise operator, for instance.

It is interesting to see how there is an interplay between the constants $\alpha$, $\gamma$ and the integrability $p_0$. For example, a very large $\alpha$ will also allow for large $\gamma$. The problem is that a $\gamma$ that is too large will not allow for integrability in $\Omega$. It is of course very well possible that the condition derived on $p_0$ based on the coercivity condition is not optimal. We also observe that we don't recover the results from section 4.1 if we let $\alpha \to 2$, though the noise is different.

*Proof.* We show that H1 to H5 hold for equation (4.64) and can therefore apply Theorem 3.2. Assumption H1 can be found in [28][p. 82]. For H2, take $u, v \in W_0^{1,\alpha}(\mathscr{D})$ and consider the following inequality which follows from [28][p. 82]:

$$2\langle A(u)-A(v), u-v\rangle \leq -2\int_{\mathscr{D}} \left(|\nabla u|^{\alpha-1}-|\nabla v|^{\alpha-1}|\right)(|\nabla u|-|\nabla v|)\mathrm{d}x \tag{4.66}$$

We now consider the other term for H2:

$$\begin{aligned}\|B(u)-B(v)\|^2 &\leq \sum_{k=1}^{\infty}\|B^k(u)-B^k(v)\|^2_{L^2(\mathscr{D})}\\ &\leq \sum_{k=1}^{\infty}\gamma_k^2\||\nabla u|^{\frac{\alpha}{2}}-|\nabla v|^{\frac{\alpha}{2}}\|^2_{L^2(\mathscr{D})}+\sum_{i=1}^{k}C_k^2\|u-v\|^2_{L^2(\mathscr{D})}\\ &\leq \gamma^2\||\nabla u|^{\frac{\alpha}{2}}-|\nabla v|^{\frac{\alpha}{2}}\|^2_{L^2(\mathscr{D})}+C\|u-v\|^2_{L_2(\mathscr{D})}\end{aligned} \tag{4.67}$$

Combining bounds (4.66) and (4.67), we obtain:

$$\begin{aligned}&2\langle A(u)-A(v), u-v\rangle+\|B(u)-B(v)\|^2\\ &\leq -\int_{\mathscr{D}}\left(|\nabla u|^{\alpha-1}-|\nabla v|^{\alpha-1}|\right)(|\nabla u|-|\nabla v|)+\gamma^2(|\nabla u|^{\frac{\alpha}{2}}-|\nabla v|^{\frac{\alpha}{2}})^2\mathrm{d}x+C\|u-v\|^2_{L^2(\mathscr{D})}\end{aligned} \tag{4.68}$$

To finish H2, it is sufficient to prove that the first term on the RHS of the inequality is negative. To do this, it suffices to show the following real-valued inequality for all $x, y > 0$:

$$2\left(x^{\alpha-1}-y^{\alpha-1}\right)(x-y)-\gamma^2(x^{\frac{\alpha}{2}}-y^{\frac{\alpha}{2}})^2 \geq 0. \tag{4.69}$$

Without loss of generality, we can take $x > y$. By homogeneity, it suffices to prove the following inequality for $x > 1$:

$$2(x^{\alpha-1}-1)(x-1)-\gamma^2(x^{\frac{\alpha}{2}}-1)^2 \geq 0 \tag{4.70}$$

For proof, we refer to the appendix (A.1). Since the inequality holds, H2 has been shown.

We continue by showing H3. Let $v \in W_0^{1,\alpha}(\mathscr{D})$. For the first terms in H3, we have:

$$2\langle A(v), v\rangle = -2\int_{\mathscr{D}}|\nabla v|^{\alpha}\mathrm{d}x = -2\|v\|^{\alpha}_{W_0^{1,\alpha}(\mathscr{D})} \tag{4.71}$$

We only need to inspect the $p$-dependent term. Using Cauchy-Schwarz, we obtain:

$$\frac{\|(B_t(v))^* v\|^2_{\mathbb{R}^d}}{\|v\|^2_{L^2(\mathscr{D})}} \leq \|B_t(v)\|^2_{\ell^2(\mathbb{N};L^2(\mathscr{D}))} \tag{4.72}$$

Therefore, we get the following $p$-dependent condition for H3:

$$\begin{aligned}&2\langle A(v), v\rangle+\|B_t(v)\|^2_{L^2(\mathscr{D})}+(p_0-2)\frac{\|(B_t(v))^* v\|^2_{\mathbb{R}^d}}{\|v\|^2_{L^2(\mathscr{D})}}\\ &\leq \left((p_0-1)\gamma^2-2\right)\|v\|^{\alpha}_{W^{1,\alpha}(\mathscr{D})}+C\|v\|^2_{L^2(\mathscr{D})}\end{aligned} \tag{4.73}$$

The first term on the RHS is negative by assumption. Therefore, H3 holds with $\alpha' = \alpha$ and $f = 0$. We are only left to show H4 and H5. For $v \in W_0^{1,\alpha}(\mathscr{D})$, we use Hölder's inequality to obtain:

$$
\begin{aligned}
|\langle A(u), v\rangle| &\leq \left|\int_{\mathscr{D}} |\nabla u|^{\alpha-2}\nabla u \cdot \nabla v \mathrm{d}x\right| \\
&\leq \int_{\mathscr{D}} |\nabla u|^{\alpha-1}|\nabla v|\mathrm{d}x \\
&\leq \left(\int_{\mathscr{D}} |\nabla u|^{\alpha}\mathrm{d}x\right)^{\frac{\alpha-1}{\alpha}} \left(\int_{\mathscr{D}} |\nabla v|^{\alpha}\mathrm{d}x\right)^{\frac{1}{\alpha}} \\
&\leq \|u\|^{\alpha-1}_{W_0^{1,\alpha}(\mathscr{D})} \|v\|_{W_0^{1,\alpha}(\mathscr{D})}
\end{aligned} \tag{4.74}
$$

Therefore, it follows for all $v \in W^{1,\alpha}(\mathscr{D})$ that

$$
\|A(v)\|^{\frac{\alpha}{\alpha-1}}_{W^{-1,\alpha}(\mathscr{D})} \leq \|v\|^{\alpha}_{W_0^{1,\alpha}(\mathscr{D})} \tag{4.75}
$$

We omit H5, since it is clear from assumption. Application of Theorem 3.2 finishes the proof. □

## 4.6. An application to systems of SPDEs

Our theory can also be used to easily show certain results from other previous papers. In particular, the authors in [13] attempted to construct a $C^{2+\delta}$ theory for systems of SPDEs. In order to construct a $C^{2+\delta}$ theory, they need estimates for the system of SPDEs which can be found using our framework (Theorem 3.1 of [13]). One of the underlying assumptions is a so-called modified stochastic parabolicity condition, which will be similar to our coercivity condition.

**Assumptions 4.6.** *Consider the random field*

$$
\mathbf{u} = (u^1, \dots, u^N)' : \mathbb{R}^d \times [0,\infty) \times \Omega \to \mathbb{R}^N
$$

*described by the following linear system of SPDEs:*

$$
\mathrm{d}u^{\alpha} = \left(a^{ij}_{\alpha\beta}\partial_{ij}u^{\beta} + f_{\alpha}\right)\mathrm{d}t + \left(\sigma^{ik}_{\alpha\beta}\partial_i u^{\beta} + g^k_{\alpha}\right)\mathrm{d}W^k_t \tag{4.76}
$$

*where the collection $\{W^k\}_{k\geq 1}$ are countably independent Wiener processes on some complete filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. We also use Einstein's summation convention (sum whenever one sees a repeated index) with*

$$
i, j = 1, 2, \dots, d; \quad \alpha, \beta = 1, 2, \dots, N; \quad k = 1, 2, \dots
$$

*We now define the deterministic part of the equation as an operator $A : H^{m+1}(\mathbb{R}^d;\mathbb{R}^N) \to H^{m-1}(\mathbb{R}^d;\mathbb{R}^N)$ such that for any $u, v \in H^{m+1}(\mathbb{R}^d;\mathbb{R}^N)$:*

$$
\langle A(u), v\rangle = -\int_{\mathbb{R}^d} a^{ij}_{\alpha\beta}\partial_i u^{\beta}\partial_j u^{\alpha}\mathrm{d}x. \tag{4.77}
$$

*The diffusion part of the equation is defined as an operator* $B : H^{m+1}(\mathbb{R}^d;\mathbb{R}^N) \to \ell^2(\mathbb{N}; H^m(\mathbb{R}^d;\mathbb{R}^N))$ *such that for any* $u \in H^{m+1}(\mathbb{R}^d;\mathbb{R}^N)$*:*

$$B^k_\alpha(u) = \sigma^{ik}_{\alpha\beta}\partial_i u^\beta. \tag{4.78}$$

*We assume that the coefficients* $a^{ij}_{\alpha\beta}$ *and* $\sigma^{ik}_{\alpha\beta}$ *depend only on* $(t,\omega)$ *and satisfy the following condition, called stochastic modified parabolicity:*

**Definition 4.1.** Let $p \in [2,\infty)$. The coefficients $a = (a^{ij}_{\alpha\beta})$ and $\sigma = (\sigma^{ik}_{\alpha\beta})$ are said to satisfy the modified stochastic parabolicity (MSP) condition if there are measurable functions $\lambda^{ik}_{\alpha\beta} : \mathbb{R}^d \times [0,\infty) \times \Omega \to \mathbb{R}$ with $\lambda^{ik}_{\alpha\beta} = \lambda^{ik}_{\beta\alpha}$ such that

$$\mathscr{A}^{ij}_{\alpha\beta} = 2a^{ij}_{\alpha\beta} - \sigma^{ik}_{\gamma\alpha}\sigma^{jk}_{\gamma\beta} - (p-2)(\sigma^{ik}_{\gamma\alpha} - \lambda^{ik}_{\gamma\alpha})(\sigma^{jk}_{\gamma\beta} - \lambda^{jk}_{\gamma\beta}) \tag{4.79}$$

satisfy the Legendre-Hadamard condition: there exists a constant $\kappa > 0$ such that

$$\mathscr{A}^{ij}_{\alpha\beta}\xi_i\xi_j\eta^\alpha\eta^\beta \geq \kappa|\xi|^2|\eta|^2 \quad \forall \xi \in \mathbb{R}^d, \eta \in \mathbb{R}^N \tag{4.80}$$

everywhere on $\mathbb{R}^d \times [0,\infty) \times \Omega$

We are now in a position to restate Theorem 3.1 from [13]

**Proposition 4.6.** *Let* $p \in [2,\infty)$ *and* $m \geq 0$*. Suppose* $f \in L^p(\Omega; L^2([0,T]; H^{m-1}(\mathbb{R}^d;\mathbb{R}^N)))$ *and* $g \in L^p(\Omega;\ell^2(\mathbb{N}; L^2([0,T]; H^m(\mathbb{R}^d;\mathbb{R}^N))))$*. Taking the initial condition* $u_0 = 0$*, equation* (4.76) *has a unique solution* $u \in L^p(\Omega; L^\infty([0,T]; H^m(\mathbb{R}^d;\mathbb{R}^N))) \cap L^p(\Omega; L^2([0,T]; H^{m+1}(\mathbb{R}^d;\mathbb{R}^N)))$*. Moreover, for any multi-index* $\mathfrak{s}$ *with* $|\mathfrak{s}| \leq m$*, there exists a constant* $C$ *depending on* $d$*,* $p$*,* $\kappa$ *and* $K$ *such that:*

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}\|\partial^{\mathfrak{s}}u_t\|^p_{L^2(\mathbb{R}^d;\mathbb{R}^N)} + \mathbb{E}\left(\int_0^T \|\partial^{\mathfrak{s}}\partial_x u_t\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^N)}\mathrm{d}t\right)^{\frac{p}{2}}\\ &\leq Ce^{CT}\left(\mathbb{E}\left(\int_0^T \|\partial^{\mathfrak{s}} f_t\|^2_{H^{-1}(\mathbb{R}^d;\mathbb{R}^N)}\mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|\partial^{\mathfrak{s}} g_t\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^N)}\mathrm{d}t\right)^{\frac{p}{2}}\right)\end{aligned} \tag{4.81}$$

**Remark 4.6.** The formulation of this proposition is different in a few ways from Theorem 3.1 in [13]. First of all, we have raised all quantities to the power $p$, since it adapts better to our framework. Furthermore, we work on the whole space $\mathbb{R}^d$ instead of choosing some region $\mathscr{O}$ and extending to 0 outside.

**Remark 4.7.** The supremum estimate in the proposition actually also holds for $p = \infty$, but since we want to draw a comparison between the results in [13] and this master's thesis, we only mention it in this remark.

*Proof.* Without loss of generality, we can restrict to the case $m = 0$. By differentiating the equation, we can obtain the other cases. We proceed to show that assumptions H1 to H5 hold for equation (4.76). We do this by first showing H1 to H5 for the equation

$$\mathrm{d}u^\alpha = a^{ij}_{\alpha\beta}\partial_{ij}u^\beta\mathrm{d}t + \sigma^{ik}_{\alpha\beta}\partial_i u^\beta \mathrm{d}W^k_t \tag{4.82}$$

Recall from section 3.4 that the results with added $f$ and $g$ follow easily once H1 to H5 have been shown for the above equation. We only check H3, since H1, H2, H4 and H5 are very similar to the stochastic heat equation treated in section 4.1 under $\alpha = 2$, $\beta = 0$. To this end, let $v \in H^1(\mathbb{R}^d;\mathbb{R}^N)$ and consider the following:

$$\begin{aligned} 2\langle A(v), v\rangle &= -2\int_{\mathbb{R}^d} a^{ij}_{\alpha\beta}\partial_i v^\beta \partial_j v^\alpha \mathrm{d}x \\ &= -2a^{ij}_{\alpha\beta}\int_{\mathbb{R}^d} \partial_i v^\beta \partial_j v^\alpha \mathrm{d}x \end{aligned} \tag{4.83}$$

We emphasize again the use of the Einstein summation convention. In this equation, we therefore sum over $\alpha, \beta, i$ and $j$. Next, we use definition (4.104) to consider the term $\|B_t(v)\|^2$:

$$\begin{aligned} \|B_t(v)\|^2_{\ell^2(\mathbb{N};L^2(\mathbb{R}^d;\mathbb{R}^N))} &= \sum_{k=1}^{\infty} \|(B_t(v)))_k\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^N)} \\ &= \sum_{k=1}^{\infty}\sum_{\gamma=1}^{N} \left\|\sigma^{ik}_{\gamma\beta}\partial_i v^\beta\right\|^2_{L^2(\mathbb{R}^d)} \\ &= \sum_{k=1}^{\infty}\sum_{\gamma=1}^{N} \int_{\mathbb{R}^d} \sigma^{ik}_{\gamma\alpha}\sigma^{jk}_{\gamma\beta}\partial_i v^\alpha \partial_j v^\beta \mathrm{d}x \end{aligned} \tag{4.84}$$

Last but not least, we look at the term $\|(B_t(v))^* v\|^2/\|v\|^2$. Let $v \in H^1(\mathbb{R}^d;\mathbb{R}^N)$, $v \neq 0$, then we obtain the following:

$$\begin{aligned} \|(B_t(v))^* v\|^2_{\ell^2(\mathbb{N})} &= \sum_{k=1}^{\infty} |((B_t(v))^* v)_k|^2 \\ &= \sum_{k=1}^{\infty}\left(\int_{\mathbb{R}^d} \sigma^{ik}_{\gamma\beta}\partial_i v^\beta v^\gamma \mathrm{d}x\right)^2 \end{aligned} \tag{4.85}$$

Now, note that the following identity holds:

$$\sigma^{ik}_{\gamma\beta}\partial_i v^\beta v^\gamma = (\sigma^{ik}_{\gamma\beta} - \lambda^{ik}_{\gamma\beta}) v^\gamma \partial_i v^\beta + \frac{1}{2}\lambda^{ik}_{\gamma\beta}\partial_i(v^\gamma v^\beta) \tag{4.86}$$

Integrating both sides of the above expression over $\mathbb{R}^d$, we can use this expression in equation (4.85) to find:

$$\begin{aligned} \|(B_t(v))^* v\|^2_{\ell^2(\mathbb{N})} &= \sum_{k=1}^{\infty}\left(\int_{\mathbb{R}^d} (\sigma^{ik}_{\gamma\beta} - \lambda^{ik}_{\gamma\beta})\partial_i v^\beta v^\gamma \mathrm{d}x\right)^2 \\ &\overset{\text{C.S.}}{\leq} \sum_{k=1}^{\infty}\left(\int_{\mathbb{R}^d}\left(\sum_{\gamma=1}^{N}(v^\gamma)^2\right)^{\frac{1}{2}} \left(((\sigma^{ik}_{\gamma\beta} - \lambda^{ik}_{\gamma\beta})\partial_i v^\beta)^2\right)^{\frac{1}{2}} \mathrm{d}x\right)^2 \\ &\overset{\text{C.S.}}{\leq} \sum_{k=1}^{\infty}\left(\int_{\mathbb{R}^d}\sum_{\gamma=1}^{N}(v^\gamma)^2 \mathrm{d}x\right)\left(\int_{\mathbb{R}^d}((\sigma^{ik}_{\gamma\beta} - \lambda^{ik}_{\gamma\beta})\partial_i v^\beta)^2 \mathrm{d}x\right) \\ &= \sum_{k=1}^{\infty}\left(\int_{\mathbb{R}^d}\sum_{\gamma=1}^{N}(v^\gamma)^2 \mathrm{d}x\right)\left(\int_{\mathbb{R}^d}(\sigma^{ik}_{\gamma\beta} - \lambda^{ik}_{\gamma\beta})(\sigma^{jk}_{\gamma\alpha} - \lambda^{jk}_{\gamma\alpha})\partial_i v^\beta \partial_j v^\alpha \mathrm{d}x\right) \end{aligned} \tag{4.87}$$

We can therefore conclude the following:

$$\frac{\|(B_t(v))^* v\|^2_{\ell^2(\mathbb{N})}}{\|v\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^N)}} \leq \sum_{k=1}^{\infty}\left(\int_{\mathbb{R}^d}(\sigma^{ik}_{\gamma\beta}-\lambda^{ik}_{\gamma\beta})(\sigma^{jk}_{\gamma\alpha}-\lambda^{jk}_{\gamma\alpha})\partial_i v^\beta \partial_j v^\alpha \mathrm{d}x\right) \tag{4.88}$$

We are now finally in the position to derive the coercivity condition H3 from the MSP condition.

$$\begin{aligned}
&2\langle A(v),v\rangle + \|(B_t(v))\|^2_{\ell^2(\mathbb{N};L^2(\mathbb{R}^d;\mathbb{R}^N))} + (p-2)\frac{\|(B_t(v))^* v\|^2_{\ell^2(\mathbb{N})}}{\|v\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^N)}}\\
&\leq \int_{\mathbb{R}^d}\left(-2a^{ij}_{\alpha\beta}+\sigma^{ik}_{\gamma\alpha}\sigma^{jk}_{\gamma\beta}+(p-2)(\sigma^{ik}_{\gamma\beta}-\lambda^{ik}_{\gamma\beta})(\sigma^{jk}_{\gamma\alpha}-\lambda^{jk}_{\gamma\alpha})\right)\partial_i v^\beta\partial_j v^\alpha \mathrm{d}x\\
&\leq -\kappa\int_{\mathbb{R}^d}|\partial_i v^\alpha|^2\mathrm{d}x\\
&\leq -\kappa\|v\|^2_{H^1(\mathbb{R}^d;\mathbb{R}^N)}
\end{aligned} \tag{4.89}$$

Choosing $\theta=\kappa$, we have shown that H3 can be derived from the MSP condition. □

# 4.7. An application to higher order SPDEs

Another result that appears in a paper by Wang and Du [39] can be obtained using our framework (Lemma 3.1 of [39]). This paper treats higher order SPDEs of the following form:

$$\mathrm{d}u_t = \left[(-1)^{m+1}\sum_{|\alpha|,|\beta|=m}A_{\alpha\beta}(t)D^{\alpha+\beta}u_t+f_t\right]\mathrm{d}t+\sum_{k=1}^{\infty}\left[\sum_{|\alpha|=m}B^k_\alpha(t)D^\alpha u_t+g^k_t\right]\mathrm{d}W^k_t. \tag{4.90}$$

The authors introduce an assumption on the coefficients to obtain existence, uniqueness and an energy estimate. We will show, using our framework, that is assumption is a very natural one.

**Assumptions 4.7.** *Consider equation* (4.90) *on* $\mathbb{R}^d$. *The collection* $\{W^k\}$ *are countably independent Wiener processes on some complete filtered probability space* $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},\mathbb{P})$. *Furthermore, we assume that the coefficients* $A_{\alpha\beta}$ *and* $B^k_\alpha$ *only depend on* $t$ *and* $\omega$. *We also place the following restriction on the coefficients* $A_{\alpha\beta}$ *and* $B^k_\alpha$, *which is assumed in [39] as well:*

$$2\sum_{|\alpha|,|\beta|=m}A_{\alpha\beta}\xi_\alpha\xi_\beta-\lambda\sum_{|\alpha|=m}|\xi_\alpha|^2\geq\frac{p+(-1)^m(p-2)}{2}\sum_{k=1}^{\infty}\left|\sum_{|\alpha|=m}B^k_\alpha\xi_\alpha\right|^2 \tag{4.91}$$

*where* $\lambda\geq 0$. *Formally, the deterministic part of the equation is defined as a time-dependent linear operator* $A_t: H^{l+m}(\mathbb{R}^d)\to H^{l-m}(\mathbb{R}^d)$, *where for all* $u,v\in H^{l+m}(\mathbb{R}^d)$*:*

$$\begin{aligned}
\langle A_t(u),v\rangle &= -\sum_{|\alpha|,|\beta|=m}\langle A_{\alpha\beta}(t)D^\alpha u, D^\beta v\rangle\\
&= -\sum_{|\alpha|,|\beta|=m}\int_{\mathbb{R}^d}A_{\alpha\beta}(t)D^\alpha u D^\beta v\mathrm{d}x.
\end{aligned} \tag{4.92}$$

*Similarly, the stochastic part is defined as a time-dependent linear operator* $B^k: H^{l+m}(\mathbb{R}^d)\to H^l(\mathbb{R}^d)$, *where for all* $u\in H^{l+m}(\mathbb{R}^d)$*:*

$$B^k_t(u)=\sum_{|\alpha|=m}B^k_\alpha(t)D^\alpha u. \tag{4.93}$$

**Proposition 4.7.** *Let $p \in [2,\infty)$ and $l, m \geq 0$. Suppose $f \in L^p(\Omega; L^2([0,T]; H^{l-m}(\mathbb{R}^d)))$ and $g \in L^p(\Omega; \ell^2(\mathbb{N}; L^2([0,T]; H^l(\mathbb{R}^d))))$. With $u_0 = 0$, there exists a unique solution $u \in L^p(\Omega; L^\infty([0,T]; H^l(\mathbb{R}^d)))$ and $u \in L^p(\Omega; L^2([0,T]; H^{l+m}(\mathbb{R}^d)))$ for any $|\beta| \leq l$ such that:*

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}\|u_t\|^p_{L^2(\mathbb{R}^d)} + \mathbb{E}\left(\int_0^T \|u_t\|^2_{H^m(\mathbb{R}^d)}\mathrm{d}t\right)^{\frac{p}{2}} \\ &\leq C\left(\mathbb{E}\left(\int_0^T \|f_t\|^2_{H^{l-m}(\mathbb{R}^d)}\mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|g_t\|^2_{\ell^2(\mathbb{N};H^l(\mathbb{R}^d))}\mathrm{d}t\right)^{\frac{p}{2}}\right)\end{aligned} \tag{4.94}$$

*Proof.* Without loss of generality we can set $l = 0$. By differentiating we can still obtain the above theorem. Furthermore, we can also set $f = g = 0$ by using the extension we made in chapter 3.4. Now, consider the following Gelfand triple:

$$(H^m(\mathbb{R}^d), L^2(\mathbb{R}^d), H^{-m}(\mathbb{R}^d)). \tag{4.95}$$

We need to check assumptions H1 to H5 to apply our framework. We only check H3. From now on, consider an arbitrary $v \in H^m(\mathbb{R}^d)$. From (4.92), we see that

$$2\langle A_t(v), v\rangle = -2\sum_{|\alpha|,|\beta|=m}\int_{\mathbb{R}^d} A_{\alpha\beta}(t)(D^\alpha v)(D^\beta v)\mathrm{d}x. \tag{4.96}$$

For $\|B_t(v)\|^2_{\ell^2(\mathbb{N};L^2(\mathbb{R}^d))}$, we obtain:

$$\begin{aligned}\|B_t(v)\|^2_{\ell^2(\mathbb{N};L^2(\mathbb{R}^d))} &= \sum_{k=1}^\infty \left\|\sum_{|\alpha|=m} B^k_\alpha(t)D^\alpha v\right\|^2_{L^2(\mathbb{R}^d)} \\ &= \sum_{k=1}^\infty \int_{\mathbb{R}^d}\sum_{|\alpha|,|\beta|=m} B^k_\alpha(t)B^k_\beta(t)(D^\alpha v)(D^\beta v)\mathrm{d}x.\end{aligned} \tag{4.97}$$

Now, the last term we need to inspect is $\|(B_t(v))^* v\|^2_{\ell^2(\mathbb{N})}$. For any $k \in \mathbb{N}$,

$$\begin{aligned}((B_t(v))^* v)_k &= \int_{\mathbb{R}^d}\sum_{|\alpha|=m} B^k_\alpha(t)(D^\alpha v)v\mathrm{d}x \\ &= (-1)^m\int_{\mathbb{R}^d}\sum_{|\alpha|=m} B^k_\alpha(t)(D^\alpha v)v\mathrm{d}x,\end{aligned} \tag{4.98}$$

where we used integration by parts in the last step. Note that for $m$ odd, we can conclude that $((B_t(v))^* v)_k = 0$ for all $k \in \mathbb{N}$. However, if $m$ is even, the above doesn't tell us anything. Therefore, we just use Cauchy-Schwarz to make an estimate in the case $m$ is even:

$$\frac{\|(B_t(v))^* v\|^2_{\ell^2(\mathbb{N})}}{\|v\|_{L^2(\mathbb{R}^d)}}^2 \leq \|B_t(v)\|^2_{\ell^2(\mathbb{N};L^2(\mathbb{R}^d))}. \tag{4.99}$$

We also note that we have the following:

$$\frac{p + (-1)^m(p-2)}{2} = \begin{cases} p-1, & \text{if } m \text{ even} \\ 1, & \text{if } m \text{ odd}\end{cases} \tag{4.100}$$

Using the assumption made on the coefficients in 4.7, we can combine all terms to get the following sequence of inequalities for the coercivity condition H3:

$$\begin{aligned}
&2\langle A_t(v), v\rangle + \|B_t(v)\|^2_{\ell^2(\mathbb{N};L^2(\mathbb{R}^d))} + (p-2)\frac{\|(B_t(v))^* v\|^2_{\ell^2(\mathbb{N})}}{\|v\|_{L^2(\mathbb{R}^d)}} \\
&\leq \sum_{|\alpha|,|\beta|=m} \int_{\mathbb{R}^d} \left(-2A_{\alpha\beta}(t) + \frac{p+(-1)^m(p-2)}{2} \sum_{k=1}^{\infty} B^k_\alpha(t) B^k_\beta(t)\right) D^\alpha v D^\beta v \mathrm{d}x \\
&\leq -\lambda \sum_{|\alpha|=m} \int_{\mathbb{R}^d} |D^\alpha v|^2 \mathrm{d}x \\
&= -\lambda \|v\|^2_{H^m(\mathbb{R}^d)}
\end{aligned} \tag{4.101}$$

Therefore, the coercivity condition H3 holds with $\theta = \lambda$. Invoking theorem 3.2 we obtain existence, uniqueness and the energy estimate stated in the theorem. □

## 4.8. An application to higher order systems of SPDEs

It is natural to ask for an extension of the two previous sections. In this case, one would consider the following higher order system of SPDEs:

$$\mathrm{d}u^\eta = \left[(-1)^{m+1} \sum_{|\alpha|,|\beta|=m} A^{\eta\gamma}_{\alpha\beta}(t) D^{\alpha+\beta} u^\gamma + f^\gamma\right] \mathrm{d}t + \sum_{k=1}^{\infty} \left[\sum_{|\alpha|=m} B^{k,\eta\gamma}_\alpha(t) D^\alpha u^\gamma + g^{k,\gamma}\right] \mathrm{d}w^k_t, \tag{4.102}$$

where $\eta$ and $\gamma$ denote the equation number in the system, ranging from 1 to $N$ for some $N \in \mathbb{N}$. Since this equation covers both examples from sections 4.6 and 4.7, we must at minimum take the least restrictive condition such that we can apply our framework. Since the modified stochastic parabolicity condition in second order systems ($m = 1$ in our setting, see 4.6) is always $p$-dependent, we can't expect a condition that is $p$-dependent only for odd $m$ like in the higher order equation (see 4.7). Using a different version of the MSP condition in 4.6, we can also obtain existence, uniqueness and an energy estimate using our framework. However, it is well-possible that the condition stated is not optimal.

**Assumptions 4.8.** *Consider the random field*

$$\mathbf{u} = (u^1, \dots, u^N)' : \mathbb{R}^d \times [0,\infty) \times \Omega \to \mathbb{R}^N$$

*described by the linear system of SPDEs* (4.102). *The collection* $\{W^k\}$ *are again countably independent Wiener processes on some complete filtered probability space* $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. *We repeat the use of Einstein's summation convention (sum whenever one sees a repeated index) with*

$$\gamma, \eta = 1, 2, \dots, N; \quad k = 1, 2, \dots$$

*We also note that* $\alpha$, $\beta \in \mathbb{N}^d$ *with* $|\alpha| = |\beta| = m$ *and* $D^\alpha u := D^{\alpha_1} D^{\alpha_2} \dots D^{\alpha_d} u$. *We define the deterministic part of the equation as time-dependent linear operator* $A_t : H^{l+m}(\mathbb{R}^d; \mathbb{R}^N) \to H^{l-m}(\mathbb{R}^d; \mathbb{R}^N)$ *such that for any* $u, v \in H^{l+m}(\mathbb{R}^d; \mathbb{R}^N)$*:*

$$\langle A_t(u), v\rangle = -\sum_{|\alpha|,|\beta|=m} \int_{\mathbb{R}^d} A^{\eta\gamma}_{\alpha\beta}(t) D^\alpha u^\gamma D^\beta u^\eta \mathrm{d}x. \tag{4.103}$$

*The diffusion part of the equation is defined as an operator* $B^k : H^{l+m}(\mathbb{R}^d;\mathbb{R}^N) \to H^l(\mathbb{R}^d;\mathbb{R}^N)$ *such that for any* $u \in H^{l+m}(\mathbb{R}^d;\mathbb{R}^N)$*:*

$$B^k_\alpha(u) = \sum_{|\alpha|=m} B^{k,\gamma\eta}_\alpha(t) D^\alpha u^\gamma. \tag{4.104}$$

*We assume that the coefficients* $A^{\eta\gamma}_{\alpha\beta}$ *and* $B^{k,\gamma\eta}_\alpha$ *only depend on* $t$ *and* $\omega$*, and satisfy the following condition for all* $\xi \in \mathbb{R}^d$*,* $\zeta \in \mathbb{R}^N$*:*

$$(2A^{\eta\gamma}_{\alpha\beta} - (p-1)B^{k,\mu\gamma}_\alpha B^{k,\mu\eta}_\beta)\xi^\alpha\xi^\beta\zeta^\gamma\zeta^\eta \geq \kappa|\xi|^2|\zeta|^2 \tag{4.105}$$

**Proposition 4.8.** *Let* $p \in [2,\infty)$ *and* $l, m \geq 0$*. Suppose* $f \in L^p(\Omega; L^2([0,T]; H^{l-m}(\mathbb{R}^d;\mathbb{R}^N)))$ *and* $g \in L^p(\Omega; \ell^2(\mathbb{N}; L^2([0,T]; H^l(\mathbb{R}^d;\mathbb{R}^N))))$*. With* $u_0 = 0$*, there exists a unique solution* $u \in L^p(\Omega; L^\infty([0,T]; H^l(\mathbb{R}^d;\mathbb{R}^N)))$ *and* $u \in L^p(\Omega; L^2([0,T]; H^{l+m}(\mathbb{R}^d;\mathbb{R}^N)))$ *for any* $|\beta| \leq l$ *such that:*

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]} \|u_t\|^p_{H^l(\mathbb{R}^d;\mathbb{R}^N)} + \mathbb{E}\left(\int_0^T \|u_t\|^2_{H^{l+m}(\mathbb{R}^d;\mathbb{R}^N)}\mathrm{d}t\right)^{\frac{p}{2}}\\ &\leq C\left(\mathbb{E}\left(\int_0^T \|f_t\|^2_{H^{l-m}(\mathbb{R}^d;\mathbb{R}^N)}\mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|g_t\|^2_{\ell^2(\mathbb{N};H^l(\mathbb{R}^d;\mathbb{R}^N))}\mathrm{d}t\right)^{\frac{p}{2}}\right)\end{aligned} \tag{4.106}$$

*Proof.* using the extension for additive noise, we can consider $f$ and $g$ to be zero. As always, the coercivity condition H3 is the main trouble. Now,

$$2\langle A(v), v\rangle = -2\int_{\mathbb{R}^d} \sum_{|\alpha|,|\beta|=m} A^{\eta\gamma}_{\alpha\beta}(t) D^\alpha v^\gamma D^\beta v^\eta \mathrm{d}x \tag{4.107}$$

and

$$\begin{aligned}\|B_t(v)\|^2_{\ell^2(\mathbb{N};L^2(\mathbb{R}^d;\mathbb{R}^N))} &= \sum_{k=1}^\infty \left\|\sum_{|\alpha|=m} B^{k,\eta\gamma}_\alpha(t) D^\alpha u^\gamma\right\|^2_{L^2(\mathbb{R}^d;\mathbb{R}^N)}\\ &= \sum_{k=1}^\infty\sum_{\eta=1}^N \left\|\sum_{|\alpha|=m} B^{k,\eta\gamma}_\alpha(t) D^\alpha u^\gamma\right\|^2_{L^2(\mathbb{R}^d)}\\ &= \sum_{k=1}^\infty\sum_{\eta=1}^N \int_{\mathbb{R}^d}\left(\sum_{|\alpha|}^m B^{k,\eta\gamma}_\alpha(t) D^\alpha u^\gamma\right)^2 \mathrm{d}x\\ &= \sum_{k=1}^\infty\sum_{\mu=1}^N \int_{\mathbb{R}^d}\sum_{|\alpha|,|\beta|=m} B^{k,\mu\gamma}_\alpha(t) B^{k,\mu\eta}_\beta D^\alpha u^\gamma D^\beta u^\eta \mathrm{d}x\end{aligned} \tag{4.108}$$

Both previous terms are just as before, now problematic term is:

$$\begin{aligned}\|(B_t(v))^* v\|^2_{\ell^2(\mathbb{N})} &= \sum_{k=1}^\infty |(B_t(v))^* v)_k|^2\\ &= \sum_{k=1}^\infty \left(\int_{\mathbb{R}^d}\sum_{|\alpha|=m} B^{k,\eta\gamma}_\alpha(t) D^\alpha u^\gamma u^\eta \mathrm{d}x\right)^2\end{aligned} \tag{4.109}$$

We estimate this term by using Cauchy-Schwarz and then invoke the coercivity assumption. This leads to:

$$2\langle A(v), v\rangle + \|B_t(v)\|_{\ell^2}^2 + (p-2)\frac{\|(B_t(v))^* v\|_{\ell^2}^2}{\|v\|_{H^m(\mathbb{R}^d;\mathbb{R}^N)}^2} \leq -\kappa\|v\|_{H^m(\mathbb{R}^d;\mathbb{R}^N)}^2 \tag{4.110}$$

□

I do not suspect that there is a way to obtain a nice condition that reduces to the conditions that worked in the previous two sections. For example, if we take $m = 1$ in the higher order section, we obtain a condition that does not even coincide in the case $N = 1$ of the second order systems of SPDEs. With this information, we can choose to just do a Cauchy-Schwarz estimate in (4.109), which also yields a certain condition. I am not sure how to proceed otherwise.

# 5
# Higher order regularity for stochastic heat equations

One of the examples that was discussed in the previous chapter was the stochastic heat equation with Dirichlet boundary conditions on a $C^1$ domain $\mathscr{D}$. For completeness, this is an equation of the form

$$\mathrm{d}u_t = (a_{ij} u_{x_i x_j} + b^i u_{x_i} + cu + f_t)\mathrm{d}t + (\sigma^{ik} u_{x_i} + \nu_k u + g_t^k)\mathrm{d}W_t^k, \tag{5.1}$$

where the coefficients depend on $x$, $tand\omega$ and satisfy a coercivity condition. For this equation, we obtained existence and uniqueness in $L^p(\Omega; L^2([0,T]; H_0^1(\mathscr{D})$ and $L^p(\Omega; C([0,T]; L^2(\mathscr{D})))$ for all $p \geq 2$. In the deterministic setting, we know that the space regularity can be improved. In the stochastic setting, Krylov has shown for $p = 2$ that we can also obtain higher order regularity in space [20], though weighted Sobolev spaces are needed for this. To illustrate why is this is the case, we will repeat a simple example given by Krylov in [20] that shows why unweighted Sobolev spaces do not suffice. Consider the following equation:

$$\mathrm{d}u_t = u_{xx}\mathrm{d}t + \mathrm{d}W_t, \tag{5.2}$$

on $(0,1)$ with Dirichlet boundary conditions and $u(0,x) = 0$ for all $x \in (0,1)$, and where $W_t$ is a real-valued Wiener process. This equation clearly has a solution, but the function $u_{xx}$ can never be continuous for $x \in [0,1]$. For if it were continuous, then the following equation shows that we can express $W_t$ as an integral against a continuous function:

$$\int_0^t u_{xx}(s,0)\mathrm{d}s + W_t = u(t,0) - u(0,0) = 0. \tag{5.3}$$

This implies that $W_t$ is of bounded variation, which is a contradiction. THerefore, $u_{xx}$ can not be continuous. As Krylov shows in [20], weighted Sobolev spaces are sufficient to treat higher order regularity for (5.1). This chapter will show that we can also obtain higher order regularity for all $p \geq 2$. In order to prove this, we will take approximately the same steps as Krylov, though we can improve his argument at some points. For example, we are able to reuse the framework that we have built in this thesis. We will first prove higher regularity for a simplified version of the above equation on $\mathbb{R}_+^d$ where the coefficients do not depend on $x$. Subsequently, we improve the argument to space-dependent coefficients for the same equation on $\mathbb{R}_+^d$. Finally, we go to domains by straightening the boundary. In this thesis, we only present the first step, since the other steps have not been made as of yet.

## 5.1. The heat equation on $\mathbb{R}^d_+$

In the case $\mathbb{R}^d_+$, we set $b^i = 0$, $c = 0$, $\nu_k = 0$, so we consider an equation of the form

$$\mathrm{d}u_t = (a_{ij}u_{x_ix_j} + f_t)\mathrm{d}t + (\sigma^{ik}u_{x_i} + g^k_t)\mathrm{d}W^k_t. \tag{5.4}$$

Let $\alpha$ be a multi-index such that $|\alpha| = m$, where $m \in \mathbb{N}$. We then apply $D^\alpha$ and multiply by the weight $\psi^m$, where $\psi(x) = x_1$. This results in the following equation:

$$\mathrm{d}(\psi^m D^\alpha u_t) = (\psi^m D^\alpha(a_{ij}u_{x_ix_j} + f_t))\mathrm{d}t + (\psi^m D^\alpha(\sigma^{ik}u_{x_i} + g^k_t))\mathrm{d}W^k_t. \tag{5.5}$$

We still need to justify that we can actually do this. We reformulate the above equation so that we can use our old framework. We need the following identities. When $i, j \neq 1$:

$$(\psi^m D^\alpha u)_{x_ix_j} = \psi^m D^\alpha u_{x_ix_j}. \tag{5.6}$$

If $i = 1, j \neq 1$,

$$\begin{aligned}(\psi^m D^\alpha u)_{x_1x_j} &= (m\psi^{m-1}D^\alpha u + \psi^m D^\alpha u_{x_1})_{x_j}\\ &= m\psi^{m-1}D^\alpha u_{x_j} + \psi^m D^\alpha u_{x_1x_j}.\end{aligned} \tag{5.7}$$

If $i = 1, j = 1$,

$$\begin{aligned}(\psi^m D^\alpha u)_{x_1x_1} &= (m\psi^{m-1}D^\alpha u + \psi^m D^\alpha u_{x_1})_{x_1}\\ &= m(m-1)\psi^{m-2}D^\alpha u + 2m\psi^{m-1}D^\alpha u_{x_1} + \psi^m D^\alpha u_{x_1x_1}.\end{aligned} \tag{5.8}$$

We also decompose the first term of the SPDE as follows:

$$\begin{aligned}&\sum_{i=1}^d\sum_{j=1}^d a_{ij}\psi^m D^\alpha u_{x_ix_j}\\ &= \sum_{i=1}^d\left(\sum_{j=2}^d \psi^m a_{ij}D^\alpha u_{x_ix_j} + \psi^m a_{i1}D^\alpha u_{x_ix_1}\right)\\ &= \sum_{i=2}^d\left(\sum_{j=2}^d \psi^m a_{ij}D^\alpha u_{x_ix_j} + \psi^m a_{i1}D^\alpha u_{x_ix_1}\right) + \left(\sum_{j=2}^d \psi^m a_{1j}D^\alpha u_{x_1x_j} + \psi^m a_{11}D^\alpha u_{x_1x_1}\right)\\ &= \sum_{i,j=2}^d a_{ij}\psi^m D^\alpha u_{x_ix_j} + \sum_{i=2}^d \psi^m a_{i1}D^\alpha u_{x_ix_1} + \sum_{j=2}^d \psi^m a_{1j}D^\alpha u_{x_1x_j} + \psi^m a_{11}D^\alpha u_{x_1x_1}.\end{aligned} \tag{5.9}$$

We can combine this decomposition and the above identities to rewrite the first term of the SPDE (5.5). This results in:

$$\begin{aligned}
&\sum_{i,j=1}^{d} a_{ij}\psi^m D^\alpha u_{x_i x_j} \\
&= \sum_{i,j=2}^{d} a_{ij}(\psi^m D^\alpha u)_{x_i x_j} + \sum_{i=2}^{d} a_{i1}((\psi^m D^\alpha u)_{x_i x_1} - m\psi^{m-1} D^\alpha u_{x_i}) \\
&\quad + \sum_{j=2}^{d} a_{1j}((\psi^m D^\alpha u)_{x_1 x_j} - m\psi^{m-1} D^\alpha u_{x_j}) \\
&\quad + (\psi^m D^\alpha u)_{x_1 x_1} - m(m-1)\psi^{m-2} D^\alpha - 2m\psi^{m-1} D^\alpha u_{x_1} \\
&= \sum_{i,j=1}^{d} a_{ij}(\psi^m D^\alpha)_{x_i x_j} - \sum_{i=2}^{d} a_{i1} m\psi^{m-1} D^\alpha u_{x_i} - \sum_{j=2}^{d} a_{1j} m\psi^{m-1} D^\alpha u_{x_j} \\
&\quad - m(m-1)\psi^{m-2} D^\alpha u - 2m\psi^{m-1} D^\alpha u_{x_1}.
\end{aligned} \tag{5.10}$$

This quantity can be reinserted into the SPDE (5.5). We need to take similar steps for the first term in the stochastic part of the equation. First consider the following identities. For $i \neq 1$:

$$(\psi^m D^\alpha u)_{x_i} = \psi^m D^\alpha u_{x_i}. \tag{5.11}$$

For $i = 1$:

$$(\psi^m D^\alpha u)_{x_i} = m\psi^{m-1} D^\alpha u + \psi^m D^\alpha u_{x_i} \tag{5.12}$$

Therefore,

$$\sum_{i=1}^{d} \psi^m D^\alpha (\sigma^{ik} u_{x_i}) = \sum_{i=1}^{d} \sigma^{ik} (\psi^m D^\alpha u)_{x_i} - m\psi^{m-1} D^\alpha u \tag{5.13}$$

Using identities (5.10) and (5.13), we can rewrite SPDE (5.5) as:

$$\begin{aligned}
&\mathrm{d}(\psi^m D^\alpha u) \\
&= \Big( \sum_{i,j=1}^{d} a_{ij}(\psi^m D^\alpha u)_{x_i x_j} - \sum_{i=2}^{d} a_{i1} m\psi^{m-1} D^\alpha u_{x_i} - \sum_{j=2}^{d} a_{1j} m\psi^{m-1} D^\alpha u_{x_j} \\
&\quad - m(m-1)\psi^{m-2} D^\alpha u - 2m\psi^{m-1} D^\alpha u_{x_1} + \psi^m D^\alpha f_t \Big) \mathrm{d}t \\
&\quad + \left( \sum_{i=1}^{d} \sigma^{ik} (\psi^m D^\alpha u)_{x_i} - m\sigma^{1k} \psi^{m-1} D^\alpha u + \psi^m g^k_t \right) \mathrm{d}W^k_t
\end{aligned} \tag{5.14}$$

Setting $v = \psi^m D^\alpha u$, we obtain the following equation:

$$\begin{aligned}
\mathrm{d}v_t = &\Big( \sum_{i,j=1}^{d} a_{ij} v_{x_i x_j} - \sum_{i=2}^{d} a_{i1} m\psi^{m-1} D^\alpha u_{x_i} - \sum_{j=2}^{d} a_{1j} m\psi^{m-1} D^\alpha u_{x_j} \\
&- m(m-1)\psi^{m-2} D^\alpha u - 2m\psi^{m-1} D^\alpha u_{x_1} + \psi^m D^\alpha f_t \Big) \mathrm{d}t \\
&+ \left( \sum_{i=1}^{d} \sigma^{ik} v_{x_i} - m\sigma^{1k} \psi^{m-1} D^\alpha u + \psi^m g^k_t \right) \mathrm{d}W^k_t.
\end{aligned} \tag{5.15}$$

We apply the a priori estimates from our framework inductively. We first do this for $\alpha = 0$ and $m \geq 2$. Therefore, we first have to show the base cases $m = 0$ and $m = 1$. For $m = 0, \alpha = 0$ SPDE (5.15) is as follows, where $v = u$:

$$\mathrm{d}v_t = \left(\sum_{i,j=1}^{d} a_{ij} v_{x_i x_j} + f_t\right)\mathrm{d}t + \left(\sum_{i=1}^{d} \sigma^{ik} v_{x_i} + g_t^k\right)\mathrm{d}W_t^k \tag{5.16}$$

This equation was also treated before, from which we obtained the existence of a unique solution $v \in L^p(\Omega; C([0,T]; L^2(\mathbb{R}^d_+)))$ and $v \in L^p(\Omega; L^2([0,T]; H^1_0(\mathbb{R}^+_d)))$ for all $p \geq 2$ and we can obtain the following a priori estimate:

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \|u_t\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\left(\int_0^T \|u_t\|^2_{H^1_0(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} \\
&\leq \mathbb{E}\|\phi\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\left(\int_0^T \|f_t\|^2_{H^{-1}(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|g_t\|^2_{\ell^2(\mathbb{N};L^2(\mathbb{R}^d_+))}\mathrm{d}t\right)^{\frac{p}{2}}.
\end{aligned} \tag{5.17}$$

We now move to $m = 1, \alpha = 0$, which results in the following version of SPDE (5.15), where $v = \psi u$:

$$\begin{aligned}
\mathrm{d}v_t = &\left(\sum_{i,j=1}^{d} a_{ij} v_{x_i x_j} - \sum_{i=2}^{d} a_{i1} u_{x_i} - \sum_{j=2}^{d} a_{1j} u_{x_j} - 2u_{x_1} + \psi f_t\right)\mathrm{d}t \\
&+ \left(\sum_{i=1}^{d} \sigma^{ik} v_{x_i} - \sigma^{1k} u + \psi g_t^k\right)\mathrm{d}W_t^k
\end{aligned} \tag{5.18}$$

From the case $m = 1$, we know that $u \in H^1_0(\mathbb{R}^d_+)$, so $u_{x_i} \in L^2(\mathbb{R}^d_+)$ for all $i \in \{1,\dots,m\}$. Therefore, the above equation can be put into our framework and we obtain $v \in L^p(\Omega; C([0,T]; L^2(\mathbb{R}^d_+)))$ and $v \in L^p(\Omega; L^2([0,T]; H^1_0(\mathbb{R}^+_d)))$ with a priori estimate

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \|\psi u\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\left(\int_0^T \|\psi u\|^2_{H^1_0(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} \\
&\leq \mathbb{E}\|\psi\phi\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\left(\int_0^T \|u_x\|^2_{L^2(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|u\|^2_{L^2(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} \\
&\quad + \mathbb{E}\left(\int_0^T \|\psi f_t\|^2_{H^{-1}(\mathscr{D})}\mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|\psi g_t\|^2_{\ell(\mathbb{N};L^2(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}}
\end{aligned} \tag{5.19}$$

Using the case $m = 0, \alpha = 0$, we can further estimate the above inequality to obtain:

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \|\psi u\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\left(\int_0^T \|\psi u\|^2_{H^1_0(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} \\
&\leq \mathbb{E}\|\psi\phi\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\|\phi\|^p_{L^2(\mathbb{R}^d_+)} \\
&\quad + \mathbb{E}\left(\int_0^T \|\psi f_t\|^2_{H^{-1}(\mathscr{D})}\mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|\psi g_t\|^2_{\ell(\mathbb{N};L^2(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} \\
&\quad + \mathbb{E}\left(\int_0^T \|f_t\|^2_{H^{-1}(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|g_t\|^2_{\ell^2(\mathbb{N};L^2(\mathbb{R}^d_+))}\mathrm{d}t\right)^{\frac{p}{2}}.
\end{aligned} \tag{5.20}$$

We now continue to the case $m \geq 2, \alpha = 0$. Suppose (5.15) with $m = k$ has a unique solution $v \in L^p(\Omega; C([0,T]; L^2(\mathbb{R}^d_+)))$ and $L^p(\Omega; L^2([0,T]; H^1_0(\mathbb{R}^d_+)))$ with energy estimate

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \|\psi^k u\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\left(\int_0^T \|\psi^k u\|^2_{H^1_0(\mathbb{R}^d_+)} \mathrm{d}t\right)^{\frac{p}{2}} \\
&\leq \sum_{i=0}^{k} \left( \mathbb{E}\|\psi^i \phi\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\left(\int_0^T \|\psi^i f_t\|^2_{H^{-1}(\mathbb{R}^d_+)} \mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|\psi^i g_t\|^2_{\ell^2(\mathbb{N};L^2(\mathbb{R}^d_+))} \mathrm{d}t\right)^{\frac{p}{2}} \right).
\end{aligned} \tag{5.21}$$

We now consider the SPDE (5.15) for $m = k+1$. This results in the following SPDE:

$$\begin{aligned}
\mathrm{d}v_t = \Big( &\sum_{i,j=1}^d a_{ij} v_{x_i x_j} - \sum_{i=2}^d a_{i1}(k+1)\psi^k u_{x_i} - \sum_{j=2}^d a_{1j}(k+1)\psi^k u_{x_j} \\
&- (k+1)k\psi^{k-1} u - 2(k+1)\psi^k u_{x_1} + \psi^{k+1} f_t \Big)\mathrm{d}t \\
&+ \left( \sum_{i=1}^d \sigma^{ik} v_{x_i} - (k+1)\sigma^{1k}\psi^k u + \psi^{k+1} g^k_t \right) \mathrm{d}W^k_t.
\end{aligned} \tag{5.22}$$

By assumption, we know that $\psi^k u, \psi^k u_{x_i} \in L^2(\mathbb{R}^d_+)$. Therefore, we can apply the framework to obtain existence of a unique solution $v \in L^p(\Omega; C([0,T]; L^2(\mathbb{R}^d_+)))$ and $L^p(\Omega; L^2([0,T]; H^1_0(\mathbb{R}^d_+)))$ with energy estimate

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \|\psi^{k+1} u\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\left(\int_0^T \|\psi^{k+1} u\|^2_{H^1_0(\mathbb{R}^d_+)} \mathrm{d}t\right)^{\frac{p}{2}} \\
&\leq \mathbb{E}\|\psi^{k+1} u\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\left(\int_0^T \|\psi^k u_x\|^2_{L^2(\mathbb{R}^d_+)} \mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|\psi^{k-1} u\|^2_{L^2(\mathbb{R}^d_+)} \mathrm{d}t\right)^{\frac{p}{2}} \\
&\quad + \mathbb{E}\left(\int_0^T \|\psi^{k+1} f_t\|^2_{H^{-1}(\mathbb{R}^d_+)} \mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|\psi^{k+1} g_t\|^2_{\ell^2(\mathbb{N};L^2(\mathbb{R}^d_+))} \mathrm{d}t\right)^{\frac{p}{2}}.
\end{aligned} \tag{5.23}$$

We can use the energy estimate from the assumption to arrive at the following estimate:

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \|\psi^k u\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\left(\int_0^T \|\psi^k u\|^2_{H^1_0(\mathbb{R}^d_+)} \mathrm{d}t\right)^{\frac{p}{2}} \\
&\leq \sum_{i=0}^{k+1} \left( \mathbb{E}\|\psi^i \phi\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\left(\int_0^T \|\psi^i f_t\|^2_{H^{-1}(\mathbb{R}^d_+)} \mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|\psi^i g_t\|^2_{\ell^2(\mathbb{N};L^2(\mathbb{R}^d_+))} \mathrm{d}t\right)^{\frac{p}{2}} \right).
\end{aligned} \tag{5.24}$$

For $m = 1, |\alpha| = 1$, we have the following SPDE:

$$\begin{aligned}
\mathrm{d}v_t = \Big( &\sum_{i,j=1}^d a_{ij} v_{x_i x_j} - \sum_{i=2}^d a_{i1} D^\alpha u_{x_i} - \sum_{j=2}^d a_{1j} D^\alpha u_{x_j} - 2D^\alpha u_{x_1} + \psi D^\alpha f_t \Big)\mathrm{d}t \\
&+ \left( \sum_{i=1}^d \sigma^{ik} v_{x_i} - \sigma^{1k} D^\alpha u + \psi D^\alpha g^k_t \right) \mathrm{d}W^k_t.
\end{aligned} \tag{5.25}$$

Since the case $m = 0, |\alpha| = 0$ shows us that $u \in L^p(\Omega; L^2([0,T]; H^1_0(\mathbb{R}^d_+)))$, we see that $D^\alpha u_{x_i} \in L^p(\Omega; L^2([0,T]; H^{-1}(\mathbb{R}^d_+)))$. Therefore, we can use our framework to solve the above equation. This means that $\psi D^\alpha u \in L^p(\Omega; C([0,T]; L^2(\mathbb{R}^d_+)))$ and $\psi D^\alpha u \in L^p(\Omega; L^2([0,T]; H^1_0(\mathbb{R}^d_+)))$

for all $p \geq 2$. Also,

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\|\psi D^\alpha u\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\left(\int_0^T \|\psi D^\alpha u\|^2_{H^1_0(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} \\
&\leq \mathbb{E}\|\psi D^\alpha \phi\|^p_{L^2(\mathbb{R}^d_+)} + \sum_{i=1}^d \mathbb{E}\left(\int_0^T \|D^\alpha u_{x_i}\|^2_{H^{-1}(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|D^\alpha u\|^2_{L^2(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} \\
&\quad + \mathbb{E}\left(\int_0^T \|\psi D^\alpha f_t\|^2_{H^{-1}(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|\psi D^\alpha g^k_t\|^2_{\ell^2(\mathbb{N};L^2(\mathbb{R}^d_+))}\mathrm{d}t\right)^{\frac{p}{2}}
\end{aligned} \tag{5.26}$$

Now, we find for any multi-index $\alpha$ such that $|\alpha| = 1$, we have

$$\begin{aligned}
|(D^\alpha u_{x_i}, v)| &= \left|\int_{\mathbb{R}^d_+} u_{x_i} D^\alpha v \mathrm{d}x\right| \\
&\leq \|u_{x_i}\|_{L^2(\mathbb{R}^d_+)}\|D^\alpha\|_{L^2(\mathbb{R}^d_+)} \\
&\leq \|u_{x_i}\|_{L^2(\mathbb{R}^d_+)}\|D^\alpha\|_{H^1_0(\mathbb{R}^d_+)}
\end{aligned} \tag{5.27}$$

Therefore, the above energy inequality can be simplified to:

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\|\psi D^\alpha u\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\left(\int_0^T \|\psi D^\alpha u\|^2_{H^1_)(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} \\
&\leq \mathbb{E}\|\psi D^\alpha \phi\|^p_{L^2(\mathbb{R}^d_+)} + \sum_{i=1}^d \mathbb{E}\left(\int_0^T \|u_{x_i}\|^2_{L^2(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|D^\alpha u\|^2_{L^2(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} \\
&\quad + \mathbb{E}\left(\int_0^T \|\psi D^\alpha f_t\|^2_{H^{-1}(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|\psi D^\alpha g_t\|^2_{\ell^2(\mathbb{N};L^2(\mathbb{R}^d_+))}\mathrm{d}t\right)^{\frac{p}{2}}.
\end{aligned} \tag{5.28}$$

We can then re-use the energy inequality from the case $m = 0, \alpha = 0$ to obtain:

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\|\psi D^\alpha u\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\left(\int_0^T \|\psi D^\alpha u\|^2_{H^1_)(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} \\
&\leq \mathbb{E}\|\psi D^\alpha \phi\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\|\phi\|^p_{L^2(\mathbb{R}^d_+)} \\
&\quad + \mathbb{E}\left(\int_0^T \|f_t\|^2_{H^{-1}(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|g_t\|^2_{\ell^2(\mathbb{N};L^2(\mathbb{R}^d_+))}\mathrm{d}t\right)^{\frac{p}{2}} \\
&\quad + \mathbb{E}\left(\int_0^T \|\psi D^\alpha f_t\|^2_{H^{-1}(\mathbb{R}^d_+)}\mathrm{d}t\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T \|\psi D^\alpha g_t\|^2_{\ell^2(\mathbb{N};L^2(\mathbb{R}^d_+))}\mathrm{d}t\right)^{\frac{p}{2}}.
\end{aligned} \tag{5.29}$$

Having proven the cases $m = |\alpha| = 0$ and $m = |\alpha| = 1$, we proceed to $m = |\alpha| \geq 2$. Let $m = |\alpha| = k$ with $k \geq 2$ and consider SPDE (5.15). We assume that there exists a unique solution

$v \in L^p(\Omega; C([0,T]; L^2(\mathbb{R}^d_+)))$ and $v \in L^p(\Omega; L^2([0,T]; H^1_0(\mathbb{R}^d_+)))$ with energy estimate

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \|\psi^k D^\alpha u\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\left(\int_0^T \|\psi^k D^\alpha u\|^2_{H^1_0(\mathbb{R}^d_+)} \mathrm{d}t\right)^{\frac{p}{2}} \\
&\leq \sum_{|\beta|\leq k} \mathbb{E}\|\psi^{|\beta|} D^\beta \phi\|^p_{L^2(\mathbb{R}^d_+)} + \mathbb{E}\left(\int_0^T \|\psi^{|\beta|} D^\beta f_t\|^2_{H^{-1}(\mathbb{R}^d_+)} \mathrm{d}t\right)^{\frac{p}{2}} \\
&\quad + \mathbb{E}\left(\int_0^T \|\psi^{|\beta|} D^\beta g_t\|^2_{\ell^2(\mathbb{N}; L^2(\mathbb{R}^d_+))} \mathrm{d}t\right)^{\frac{p}{2}}
\end{aligned} \tag{5.30}$$

We want to prove existence and uniqueness with the same type of energy estimate for $m = |\alpha| = k+1$. Now consider equation (5.15) with $m = |\alpha| = k+1$. This results in the following SPDE:

$$\begin{aligned}
\mathrm{d}v_t = \Big(&\sum_{i,j=1}^d a_{ij} v_{x_i x_j} - \sum_{i=2}^d a_{i1}(k+1)\psi^k D^\alpha u_{x_i} - \sum_{j=2}^d a_{1j}(k+1)\psi^k D^\alpha u_{x_j} \\
&- (k+1)k\psi^{k-1} D^\alpha u - 2(k+1)\psi^k D^\alpha u_{x_1} + \psi^{k+1} D^\alpha f_t\Big)\mathrm{d}t \\
&+ \left(\sum_{i=1}^d \sigma^{ik} v_{x_i} - (k+1)\sigma^{1k}\psi^k D^\alpha u + \psi^{k+1} g^k_t\right)\mathrm{d}W^k_t.
\end{aligned} \tag{5.31}$$

By assumption, we find that $\psi^k D^\gamma u_{x_i} \in L^2(\mathbb{R}^d_+)$ and $\psi^{k-1} D^\gamma u$ where $|\gamma| = k$

# 6

# Conclusion

This master's thesis introduces a new $p$-dependent coercivity condition that allows to prove $L^p(\Omega; L^2([0,T];X))$ estimates for SPDEs fitting the variational framework. First, a priori estimates are obtained on the solution of the SPDE. Subsequently, these a priori estimates are used to prove existence and uniqueness through a Galerkin approximation argument. The framework that has been built with the new coercivity condition is applied to many equations, such as the stochastic heat equation with Dirichlet boundary conditions as well as the same equation with Neumann boundary conditions. Other equations include Burger's equation and the Navier-Stokes equations in 2D. Last but not least, we re-obtain results from the literature concerning systems of SPDEs and higher order SPDEs.

# A

# Basic inequality

This appendix shows proofs of some of the inequalities used throughout this thesis.

**Proposition A.1.** *Given $\alpha > 2$, the following inequality holds for all $x \geq 1$ if and only if $\gamma^2 < \frac{8(\alpha-1)}{\alpha^2}$:*

$$2(x^{\alpha-1}-1)(x-1)-\gamma^2(x^{\frac{\alpha}{2}}-1)^2 \geq 0. \tag{A.1}$$

*Proof.* We first prove the sufficiency. Define the function $f(x) = 2(x^{\alpha-1}-1)(x-1)-\gamma^2(x^{\frac{\alpha}{2}}-1)^2$. Since $f(1) = 0$, it suffices to show that $f'(x) > 0$ for all $x > 1$. Now, calculating $f'(x)$ results in:

$$f'(x) = 2(\alpha x^{\alpha-1}-(\alpha-1)x^{\alpha-2}-1)-\gamma^2\left(\alpha x^{\alpha-1}-\alpha x^{\frac{\alpha}{2}-1}\right).$$

We notice that $f'(1) = 0$. Therefore, it suffices to prove that $f''(x) > 0$ for all $x > 1$. We calculate $f''(x)$:

$$\begin{aligned}
f''(x) &= 2(\alpha(\alpha-1)x^{\alpha-2}-(\alpha-1)(\alpha-2)x^{\alpha-3})-\gamma^2\left(\alpha(\alpha-1)x^{\alpha-2}-\alpha\left(\frac{\alpha-2}{2}\right)x^{\frac{\alpha}{2}-2}\right)\\
&= x^{\frac{\alpha}{2}-2}\left[2\left(\alpha(\alpha-1)x^{\frac{\alpha}{2}}-(\alpha-1)(\alpha-2)x^{\frac{\alpha}{2}-1}\right)-\gamma^2\left(\alpha(\alpha-1)x^{\frac{\alpha}{2}}-\alpha\left(\frac{\alpha-2}{2}\right)\right)\right]\\
&= x^{\frac{\alpha}{2}-2}g(x),
\end{aligned}$$

where we define $g(x) = 2\left(\alpha(\alpha-1)x^{\frac{\alpha}{2}}-(\alpha-1)(\alpha-2)x^{\frac{\alpha}{2}-1}\right)-\gamma^2\left(\alpha(\alpha-1)x^{\frac{\alpha}{2}}-\alpha\left(\frac{\alpha-2}{2}\right)\right)$. It can be shown that $g(x_0) = 0$ for some $x_0 \in (0,1)$. Therefore, it suffices to show that $g'(x) > 0$ for $x > 1$. Now, we calculate $g'(x)$:

$$\begin{aligned}
g'(x) &= 2\left(\alpha(\alpha-1)\frac{\alpha}{2}x^{\frac{\alpha}{2}-1}-(\alpha-1)(\alpha-2)\left(\frac{\alpha}{2}-1\right)x^{\frac{\alpha}{2}-2}\right)-\gamma^2\left(\alpha(\alpha-1)\frac{\alpha}{2}x^{\frac{\alpha}{2}-1}\right)\\
&= x^{\frac{\alpha}{2}-2}\left[2(\alpha(\alpha-1)\frac{\alpha}{2}x-(\alpha-1)(\alpha-2)(\frac{\alpha}{2}-1))-\gamma^2\alpha(\alpha-1)\frac{\alpha}{2}x\right]\\
&= x^{\frac{\alpha}{2}-2}h(x)
\end{aligned}$$

where $h(x) = 2(\alpha(\alpha-1)\frac{\alpha}{2}x - (\alpha-1)(\alpha-2)(\frac{\alpha}{2}-1)) - \gamma^2\alpha(\alpha-1)\frac{\alpha}{2}x$. Similarly for $g$, $h(x_0) = 0$ for some $x_0 \in (0,1)$ as long as $\gamma^2 < 2$. Then, it suffices to show that $h'(x) > 0$ for all $x > 1$. Therefore, we are only left to calculate $h'(x)$. It follows that

$$h'(x) = 2(\alpha(\alpha-1)\frac{\alpha}{2}) - \gamma^2\alpha(\alpha-1)\frac{\alpha}{2} \tag{A.2}$$

Using the assumption $\gamma^2 < \frac{8(\alpha-1)}{\alpha^2}$, it follows that $h'(x) > 0$ for all $x \in \mathbb{R}$. This means we are done.

In order to show the necessity of the condition, suppose $\gamma^2 \geq \frac{8(\alpha-1)}{\alpha^2}$. We do a Taylor expansion of $f(x)$ around $x = 1$. This results in:

$$f(x) = f''(1)(x-1)^2 + \mathcal{O}((x-1)^3). \tag{A.3}$$

If we can show that $f''(1) \leq 0$, then $f(x) \leq 0$ for some $x \geq 1$. Using the above, we find that

$$\begin{aligned} f''(1) &= 2(\alpha(\alpha-1) - (\alpha-1)(\alpha-2)) - \gamma^2(\alpha(\alpha-1) - \alpha(\tfrac{\alpha-2}{2})) \\ &\leq 2(\alpha(\alpha-1) - (\alpha-1)(\alpha-2)) - 4(\alpha-1) \\ &= 0. \end{aligned} \tag{A.4}$$

□

# Bibliography


[1] G.K. Batchelor and Cambridge University Press. *The Theory of Homogeneous Turbulence*. Cambridge Science Classics. Cambridge University Press, 1953. ISBN 9780521041171. URL `https://books.google.nl/books?id=POG-LSUgYckC`.

[2] A. Bensoussan. *Filtrage optimal des systèmes linéaires*. Méthodes mathématiques de l'informatique. Dunod, 1971.

[3] A. Bensoussan and R. Temam. Equations aux derivees partielles stochastiques non lineaires. *Israel Journal of Mathematics*, 11(1):95–129, March 1972. ISSN 1565-8511. doi:10.1007/BF02761449.

[4] Zdzisław Brzeźniak, Wei Liu, and Jiahui Zhu. Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise. *Nonlinear Analysis: Real World Applications*, 17:283–310, June 2014. ISSN 1468-1218. doi:10.1016/j.nonrwa.2013.12.005.

[5] Z. Brzeźniak, M. Capiński, and F. Flandoli. Stochastic partial differential equations and turbulence. *Math. Models Methods Appl. Sci.*, 1(1):41–59, 1991. ISSN 0218-2025. doi:10.1142/S0218202591000046.

[6] Z. Brzeźniak, M. Capiński, and F. Flandoli. Stochastic partial differential equations and turbulence. *Math. Models Methods Appl. Sci.*, 1(1):41–59, 1991. ISSN 0218-2025. doi:10.1142/S0218202591000046.

[7] Zdzislaw Brzezniak and Mark Veraar. Is the stochastic parabolicity condition dependent on p and q? *Electronic Journal of Probability*, 17(none), January 2012. ISSN 1083-6489. doi:10.1214/EJP.v17-2186.

[8] J. M. Burgers. A mathematical model illustrating the theory of turbulence. In *Advances in Applied Mechanics*, pages 171–199. Academic Press, Inc., New York, N. Y., 1948. edited by Richard von Mises and Theodore von Kármán,.

[9] John B. Conway. *A course in functional analysis.*, volume 96. New York etc.: Springer-Verlag, 1990. ISBN 0-387-97245-5.

[10] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*. Number 45 in Encyclopedia of mathematics and its applications. Cambridge Univ. Press, Cambridge, digitally printed version (with corrections) edition, 2008. ISBN 978-0-521-05980-0. OCLC: 612518037.

[11] Giuseppe Da Prato, Arnaud Debussche, and Roger Temam. Stochastic Burgers' equation. *NoDEA Nonlinear Differential Equations Appl.*, 1(4):389–402, 1994. ISSN 1021-9722. doi:10.1007/BF01194987.

[12] Egbert Dettweiler. Stochastic integration relative to Brownian motion on a general Banach space. *Doğa Mat.*, 15(2):58–97, 1991. ISSN 1010-7622.

[13] Kai Du, Jiakun Liu, and Fu Zhang. Stochastic Hölder continuity of random fields governed by a system of stochastic PDEs. *Ann. Inst. Henri Poincaré Probab. Stat.*, 56(2): 1230–1250, 2020. ISSN 0246-0203. doi:10.1214/19-AIHP1000.

[14] Lawrence C. Evans. *Partial differential equations*, volume 19. Providence, RI: American Mathematical Society (AMS), 2010. ISBN 978-0-8218-4974-3.

[15] Charles L. Fefferman. Existence and smoothness of the Navier-Stokes equation. In *The Millennium Prize problems*, pages 57–67. Providence, RI: American Mathematical Society (AMS); Cambridge, MA: Clay Mathematics Institute, 2006. ISBN 0-8218-3679-X.

[16] Alan Hájek. Interpretations of Probability. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Fall 2019 edition, 2019.

[17] Kiyosi Itô. Differential equations determining Markov processes. *Zenkoku Shijo Sugaku Danwakai*, 1077:1352–1400, 1942.

[18] Kiyosi Itô. Stochastic integral. *Proceedings of the Imperial Academy. Tokyo*, 20:519–524, 1944. ISSN 0369-9846.

[19] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus.*, volume 113. New York etc.: Springer-Verlag, 1991. ISBN 0-387-97655-8.

[20] N. V. Krylov. A $W_2^n$-theory of the Dirichlet problem for SPDEs in general smooth domains. *Probab. Theory Related Fields*, 98(3):389–421, 1994. ISSN 0178-8051. doi:10.1007/BF01192260.

[21] N. V. Krylov. An analytic approach to SPDE's. In R. Carmona and B. L. Rozovskiĭ, editors, *Stochastic partial differential equations: six perspectives*, number v. 64 in Mathematical surveys and monographs, pages 183–242. American Mathematical Society, Providence, R.I, 1999. ISBN 978-0-8218-0806-1.

[22] N. V. Krylov. A relatively short proof of Itô's formula for SPDEs and its applications. *Stoch. Partial Differ. Equ., Anal. Comput.*, 1(1):152–174, 2013. ISSN 2194-0401. doi:10.1007/s40072-013-0003-5.

[23] N. V. Krylov and B. L. Rozovskii. Stochastic evolution equations. *Journal of Soviet Mathematics*, 16(4):1233–1277, July 1981. ISSN 1573-8795. doi:10.1007/BF01084893.

[24] Nicolai Krylov. A brief overview of the $L_p$-theory of SPDEs. *Theory Stoch. Process.*, 14 (2):71–78, 2008. ISSN 0321-3900.

[25] Paul Langevin. Sur la théorie du mouvement brownien. *Comptes rendus de l'Académie des Sciences*, (146):530–533, 1908.

[26] Pierre-Simon Laplace. *Théorie Analytique des Probabilités*. 1812.

[27] Don S. Lemons and Anthony Gythiel. Paul langevin's 1908 paper "on the theory of brownian motion" ["sur la théorie du mouvement brownien," c. r. acad. sci. (paris) 146, 530–533 (1908)]. *American Journal of Physics*, 65(11):1079–1081, 1997. doi:10.1119/1.18725.

[28] Wei Liu and Michael Röckner. *Stochastic partial differential equations: an introduction.* Universitext. Springer, Cham, 2015. ISBN 978-3-319-22353-7; 978-3-319-22354-4. doi:10.1007/978-3-319-22354-4.

[29] Jan Van Neerven, Mark Veraar, Tuomas Hytönen, and Lutz Weis. *Analysis in Banach Spaces*, volume I. 2016. ISBN 9783319485195.

[30] B. Øksendal. *Stochastic Differential Equations: An Introduction with Applications.* Universitext (1979). Springer, 2003. ISBN 9783540047582. doi:10.1007/978-3-642-14394-6.

[31] Etienne Pardoux. *Équations aux dérivées partielles stochastiques non linéaires monotones: étude de solutions fortes de type Ito.* PhD thesis, 1975. OCLC: 489811603.

[32] Georgio Parisi, Yong Shi Wu, et al. Perturbation theory without gauge fixing. *Sci. Sin*, 24(4):483–496, 1981.

[33] Jan Prüss and Gieri Simonett. *Moving interfaces and quasilinear parabolic evolution equations*, volume 105. Basel: Birkhäuser/Springer, 2016. ISBN 978-3-319-27697-7; 978-3-319-27698-4. doi:10.1007/978-3-319-27698-4.

[34] Boris Rozovskii. *Stochastic Evolution Systems.* Springer, 1990. ISBN 9789401057035. doi:10.1007/978-94-011-3830-7.

[35] Walter Rudin. *Functional analysis.* New York, NY: McGraw-Hill, 1991. ISBN 0-07-054236-8.

[36] Jan Van Neerven. *Functional analysis (In progress).* 2021.

[37] Jan van Neerven, Mark Veraar, and Lutz Weis. Stochastic integration in Banach spaces—a survey. In *Stochastic analysis: a series of lectures*, volume 68 of *Progr. Probab.*, pages 297–332. Birkhäuser/Springer, Basel, 2015. doi:10.1007/978-3-0348-0909-2_11.

[38] Neelima Varshney and David Šiška. Coercivity condition for higher moment a priori estimates for nonlinear SPDEs and existence of a solution under local monotonicity. *Stochastics*, 92(5):684–715, July 2020. ISSN 1744-2508. doi:10.1080/17442508.2019.1650043.

[39] Yuxing Wang and Kai Du. Schauder-type estimates for higher-order parabolic SPDEs. *Journal of Evolution Equations*, 20(4):1453–1483, December 2020. ISSN 1424-3199, 1424-3202. doi:10.1007/s00028-020-00562-5.