Technische Universiteit Delft
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Newton polygonen en parabolische inhomogene symbolen van partiële differentiaalvergelijkingen

(Engelse titel: Newton polygons and parabolic inhomogeneous symbols of partial differential equations)

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"Newton polygonen en parabolische inhomogene symbolen van partiële differentiaalvergelijkingen"

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Preface

In this thesis, I will be researching sections 2.1 and 2.2 of the book *General Parabolic Mixed Order systems in L_p and Applications*, by R. Denk and M. Kaip [1]. These sections consider symbols of partial differential equations, Newton polygons, weight and order functions, parameter-ellipticity and parabolicity. The research I will be doing has two main goals:

1. make the content of the book more accessible for bachelor students. Using examples and sparing as little detail as possible, it should be easier for students to understand what is being covered, and how they are proven.

2. construct a complete proof of the equivalence of being N-parameter-elliptic and having non-vanishing principal parts for a symbol P with a regular representation. As per our first point, this proof needs to be on the level of knowledge that is assumed of a bachelor student, so that they may understand it themselves, and be complete enough to be easily verified.

This thesis is written as part of the course AM3000 Bachelorproject, which is the graduation project for the bachelor studies Applied Mathematics on the TU Delft. The goal of this thesis is to show I can comprehend, apply and rephrase new material, and to show I am capable of finding the missing pieces of the proofs and filling these in myself. Since there was interest in a comprehensive and detailed description of the result that allows us to use principal parts to determine N-parameter-ellipticity in the research group Analysis of the TU Delft, this subject was chosen for this bachelor thesis.

Summary

When transforming PDE problems using Fourier and Laplace transforms, we can find functions that represent the problem, and which can be used to determine properties of the problem. We define such functions as symbols $P(\lambda, z)$. In general, we define the class of symbols $S(L_t \times L_x)$ are all functions which are represented by a polynomial of the form $R_P(\lambda, z) := \sum_{\ell \in I_P} \tau_\ell(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z)$, where $\tau_\ell(\lambda, z)$ are ρ-homogeneous functions of (λ, z) on the cones $L_t \times L_x$, and $\phi_\ell(\lambda)$ and $\psi_\ell(z)$ homogeneous functions of λ on the cone L_t and z on the cone L_x respectively. These functions have a certain γ-order $d_\gamma(P)$ that shows the order of the function relative to a relative weight γ, and a certain γ-principal part $\pi_\gamma P(\lambda, z)$, which is the part of P that causes this γ-order.

For such a symbol, we define its Newton polygon $N(P)$ as a certain convex hull of points on $[0, \infty)^2$, which serves as a geometric description of the order of P. We define the weight function W_P to be a positive polynomial with the orders found on the vertices of the Newton polygon $N(P)$. We define a notion of order functions, and show the order function of P is $d_\gamma(P)$.

We define a notion of parameter-ellipticity and parabolicity for symbols in $S(L_t \times L_x)$ based on the Newton polygon $N(P)$, namely N-parameter-ellipticity and N-parabolicity. Using various results from the work of R. Denk and M. Kaip [1] and results I introduced myself, we then prove the equivalence between N-parameter-ellipticity and having non-vanishing γ-principal parts.

Contents

Introduction

The subject of the book by R. Denk and M. Kaip [1] is partial differential equations: expressions of time and space derivatives. We can take an analytical approach of PDE's by viewing these equations as expressions of differential operators of the form $P(\partial_t, \nabla_x)$, and then applying Laplace transforms to transform time derivatives into multiplication, and Fourier transforms to transform space derivatives into multiplication. Take for instance the heat equation on the half space:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + f(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n_+, \\ u(t=0) = 0, & x \in \mathbb{R}^n_+, \\ u(x_n = 0) = g(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^{n-1}. \end{cases}$$

Here we see the differential operator $P(\partial_t, \nabla_x) = \frac{\partial}{\partial t} - \Delta$, which allows us to write the equation as $P(\partial_t, \nabla_x)u(x,t) = f(x,t)$. We will first apply Fourier transforms $\mathscr{F}_{n-1}[u(x,t)](\xi) = u'(\xi, x_n, t)$ on $x_1, \ldots, x_{n-1}$, then apply a Laplace transform $\mathscr{L}[u'(\xi, x_n, t)](\lambda) = \hat{u}(\lambda, \xi, x_n)$ on t. This will give us the problem

$$\begin{cases} \lambda \hat{u}(\lambda, \xi, x_n) = -\hat{u}(\lambda, \xi, x_n) \sum_{j=1}^{n-1}(\xi_j^2) + \frac{\partial^2 \hat{u}}{\partial x_n^2}(\lambda, \xi, x_n) + \hat{f}(\lambda, \xi, x_n), & x_n \in \mathbb{R}_+, \\ \hat{u}(x_n = 0) = g(\lambda, \xi). \end{cases}$$

We fix the transform variables λ and ξ. We will solve this ODE of the variable x_n by order reduction: we introduce $\hat{v} = \frac{\partial \hat{u}}{\partial x_n}$, so we can write our ODE as

$$\frac{\partial}{\partial x_n} \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} (\lambda, \xi, x_n) = \begin{bmatrix} 0 & 1 \\ \lambda + \sum_{j=1}^{n-1}(\xi_j^2) & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} (\lambda, \xi, x_n) + \begin{bmatrix} 0 \\ \hat{f}(\lambda, \xi, x_n) \end{bmatrix}.$$

A problem of this form is typically solved by solutions of the form $\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = Ce^{\lambda_i x_n} v_i$, where λ_i, v_i are some eigenvalue and eigenvector of the matrix $A := \begin{bmatrix} 0 & 1 \\ \lambda + \sum_{j=1}^{n-1}(\xi_j^2) & 0 \end{bmatrix}$ and C is some unknown function/constant. The eigenvalues of A are $\sqrt{\lambda + \sum_{j=1}^{n-1}(\xi_j^2)}$ and $-\sqrt{\lambda + \sum_{j=1}^{n-1}(\xi_j^2)}$, but eigenvalues with positive real values make no sense since this would contradict the stability of the heat problem. Therefore we have a solution of the form

$$\phi(\lambda, \xi, x_n) = Ce^{-\sqrt{\lambda + \sum_{j=1}^{n-1}(\xi_j^2)}x_n} \begin{bmatrix} 1 \\ -\sqrt{\lambda + \sum_{j=1}^{n-1}(\xi_j^2)} \end{bmatrix}.$$

This solution can then be adapted to the boundary conditions and the non-homogeneous terms, before being transformed back into a solution for the heat equation.

A term that shows up a lot in the calculation of this transformed solution is the term $\lambda + \sum_{j=1}^{n-1}(\xi_j^2) = \lambda + |\xi|^2$. If we were to use $z = i\xi$, we find the term

$$P(\lambda, z) = \lambda + |z|_-^2,$$

where we define $|z|_- = \sqrt{\sum_{j=1}^{n-1} -z_j^2}$. This is called the symbol of the heat equation.

Functions of λ and z that are related to a certain PDE problem through Fourier and Laplace transforms are called symbols, and these symbols are important for determining important properties of the problems they represent. For instance, for simple (quasi-)homogeneous symbols we already have fairly easy conditions of parabolicity and parameter-ellipticity, that help us solve the differential equations by letting us find bounds from above and below. However, there are more difficult symbols to investigate however: take for instance the symbol related to the Stefan problem, defined as

$$P_S(\lambda, z) = \lambda + |z|_-^2 \sqrt{\lambda + |z|_-^2}.$$

In this thesis, we will investigate these kinds of symbols $P(\lambda, z)$ on certain aspects based on sections 2.1 and 2.2 of the work of R. Denk and M. Kaip [1]:

- In section 1.1, we will give an exact definition of homogeneity, quasi-homogeneity, and of the sort of symbols that we will be considering. We will see that if we can represent our symbol $P(\lambda, z)$ as a sum of products of (quasi-)homogeneous functions with a certain weight ρ, we can determine the order of the function by comparing the weight ρ of the function to a relative weight γ. We wil call this order the γ-order. From this we can also determine which part of the symbol determines this order, which will be called the γ-principal part.

- In section 1.2, we will define three tools that will help us determine properties of these symbols. Firstly there is the Newton polygon N, which is a graphical description of the γ-order of the symbol $P(\lambda, z)$. Newton polygons allow us to easily see whether a symbol is regular, and what orders are larger or smaller than the order of the symbol. We also define weight functions $W(\lambda, z)$, which are positive polynomials with the same order as $P(\lambda, z)$, and which can easily be used to estimate P from above or below. Finally we define order functions $\mu(\gamma)$, which can be used to describe Newton polygons as a piece-wise linear and continuous function of γ.

- Based on the Newton polygon and weight function of the symbol, we will assign a new form of parameter-ellipticity and parabolicity in section 1.3 by defining it as the existence of a bound from above and below in the form of $CW(\lambda, z)$ for some constant C. However, as we will proof in this thesis, this is equivalent to the symbol having non-vanishing γ-principal parts. Proving this equivalence is the main result of this thesis.

A Newton polygon is a much more broad term than just the definition of the Polygon used here. Any convex hull of points and their completions in the $[0, \infty)^2$ plane representing the exponents of the monomials of a polynomial expression can be labeled as a Newton polygon (or a Newton polyhedron for higher dimensions.)[3] For instance, as depicted in the article of C. Christensen [2], Newton used a form of Newton polygon to determine the smallest order terms for his approximation of an explicit solution $y(x)$ solving the implicit affected equation

$$y^3 + axy + a^2y - x^3 - 2a^3 = 0.$$

The Newton polygon used here is just convex hull of the points of the polynomial, without including the origin or the projections of the points on the axes like in the work of R. Denk and M. Kaip. This is done so that it is possible to determine which points are closest to the origin, i.e. which exponents are the smallest, instead of which exponents are the largest like we will do here. Based on the slope of the closest points, Newton can then find the exponent of the next term in the approximation.

Even though it's used in a completely different setting, and the Polygon itself is defined in a different way, the idea of the Newton polygon remains the same in our setting: it is a change of perspective on a polynomial, with the goal to determine certain properties of the polynomial.

Chapter 1

Definitions and properties

1.1 Symbols

In general, a symbol is a function $P(\lambda, z)$, where we take λ on some cone L_t, z on some cone L_x, and that has an outcome $P(\lambda, z) \in \mathbb{C}$. Often we will see that $L_t = S_\theta$ the sector, which is the cone on the complex plane $\mathbb{C}$ with arguments between $-\theta$ and θ, and $L_x = \Sigma_\delta^n$ the bisector, which can be described as the values on the complex vector space $\mathbb{C}^n$ with arguments that are at least δ close to $\frac{1}{2}\pi$ or to $-\frac{1}{2}\pi$.

Definition 1.1.1 (Cone, (bi)sector). (i) A *cone* in $\mathbb{C}^n$ is a subset $L \subseteq \mathbb{C}^n$ such that $\eta z \in L$ for all $\eta > 0$ and $z \in L$.

(ii) The *sector* $S_\theta \subseteq \mathbb{C}$ for $\theta \in (0, \pi)$ is defined as

$$S_\theta := \{z \in \mathbb{C} : -\theta < \arg(z) < \theta\}.$$

(iii) The *bisector* $\Sigma_\delta^n \subseteq \mathbb{C}^n$ for $\delta \in (0, \frac{1}{2}\pi)$ and $n \in \mathbb{N}$ is defined as

$$\begin{aligned}\Sigma_\delta^n :=& \{(z_1, \dots, z_n) \in \mathbb{C}^n \setminus \{0\} : \pi - \delta < \arg(z_i) < \pi + \delta, i \in \{1, \dots, n\}\} \\ &\cup \{(z_1, \dots, z_n) \in \mathbb{C}^n \setminus \{0\} : -\pi - \delta < \arg(z_i) < -\pi + \delta, i \in \{1, \dots, n\}\}\end{aligned}$$

For $(\lambda, z) \in L_t \times L_x$, we will treat inhomogeneous symbols $P(\lambda, z)$ that are holomorphic and polynomially bounded on the interior of the cones L_t and L_x. A definition of homogeneity can be found below.

Definition 1.1.2. (i) For L_t, L_x closed cones, define $\mathring{L}_t$ and $\mathring{L}_x$ as the *interior* of the set L_t and L_x. We then define the class of symbols $H(\mathring{L}_t \times \mathring{L}_x)$ as all the symbols $P(\lambda, z)$ with $(\lambda, z) \in L_t \times L_x$ that are holomorphic in $\mathring{L}_t \times \mathring{L}_x$, and $H_P(\mathring{L}_t \times \mathring{L}_x) \subseteq H(\mathring{L}_t \times \mathring{L}_x)$ as all the symbols $P(\lambda, z)$ that are holomorphic and bounded by some polynomial function in $\mathring{L}_t \times \mathring{L}_x$.

(ii) If $L \subseteq \mathbb{C}^n$ is a closed cone, then a function $\psi : L \setminus \{0\} \to \mathbb{C}$ is called *homogeneous of degree* $N \in \mathbb{R}$ if

$$\psi(\eta z) = \eta^N \psi(z), \quad \eta > 0, \ z \in L \setminus \{0\}.$$

We write $S^{(N)}(L)$ for the set of all continuous functions $\psi(z)$ which are homogeneous of degree N and for which we have $\psi(z) \neq 0$ for all $z \in L \setminus \{0\}$.

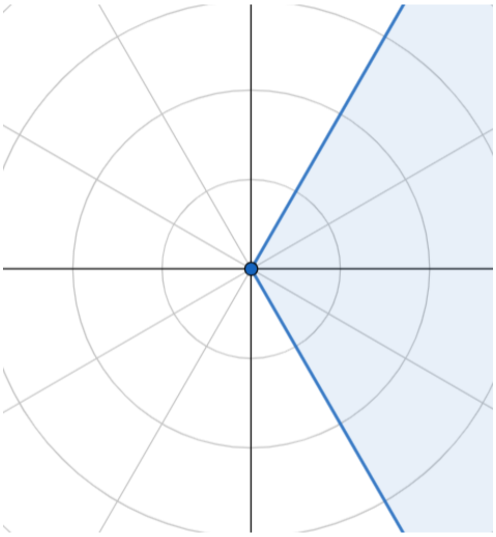

Figure 1.1: A sector S_θ with $\theta = \frac{2}{3}\pi$

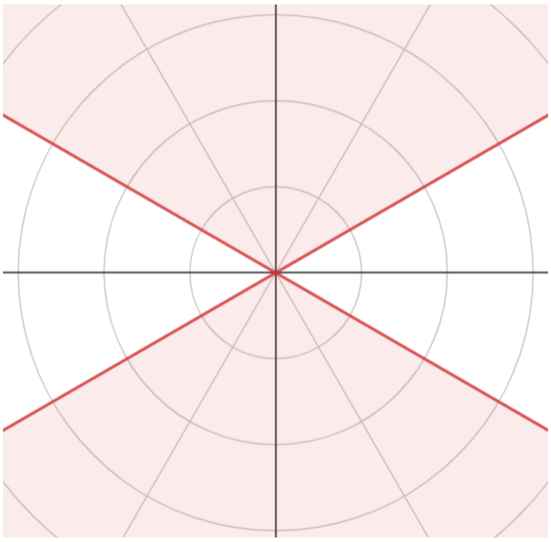

Figure 1.2: A bisector Σ_δ with $\delta = \frac{2}{3}\pi$

(iii) Let $\rho > 0$ and $N \in \mathbb{R}$, and let $L_t \subseteq \mathbb{C}$ and $L_x \subseteq \mathbb{C}^n$ closed cones. Then a function $\tau : (L_t \times L_x) \to \mathbb{C}$ is called *ρ-homogeneous of degree N* if

$$\tau(\eta^\rho \lambda, \eta z) = \eta^N \tau(\lambda, z), \quad \eta > 0,\ (\lambda, z) \in (L_t \times L_x) \setminus \{(0,0)\}.$$

Functions which are ρ-homogeneous are also called *quasi-homogeneous*, and ρ is called the *weight* of λ relative to z. We define $S^{(\rho,N)}(L_t \times L_x)$ as the set of all continuous functions $\tau(\lambda, z)$ which are ρ-homogeneous of degree N and which satisfy $\tau(\lambda, z) \neq 0$ for all $(\lambda, z) \in (L_t \times L_x) \setminus \{(0,0)\}$.

With this definition in mind, we will soon define the inhomogeneous symbols $P(\lambda, z)$ as sums of terms, which are products of the following types functions:

1. $\phi(\lambda)$ homogeneous of a certain degree M, functions of $\lambda \in L_t$.

2. $\psi(z)$ homogeneous of degree L, functions of $z \in L_x$.

3. $\tau(\lambda, z)$ ρ-homogeneous of degree N, functions of λ and z.

Before we can define our symbols, we must first investigate these (ρ-)homogeneous functions.

Lemma 1.1.3. *(i) For $\phi \in S^{(M)}(L_t)$ and $\psi \in S^{(L)}(L_x)$, we can find a constant $C_1, C_2 > 0$ s.t.*

$$C_1|\lambda|^M \leq |\phi(\lambda)| \leq C_2|\lambda|^M, \quad \lambda \in L_t,$$
$$C_1|z|^L \leq |\psi(z)| \leq C_2|z|^L, \quad z \in L_x.$$

(ii) For $\tau \in S^{(\rho,N)}(L_t \times L_x)$, we can find constants $C_1, C_2 > 0$ s.t.

$$C_1(|\lambda|^{\frac{N}{\rho}} + |z|^N) \leq |\tau(\lambda, z)| \leq C_2(|\lambda|^{\frac{N}{\rho}} + |z|^N), \quad (\lambda, z) \in L_t \times L_x. \tag{1.1}$$

Proof. (i) Note that we can rewrite our equations using the definitions of $\phi \in S^{(M)}(L_t)$ and $\psi \in S^{(L)}(L_x)$ as

$$C_1 \leq \frac{|\phi(\lambda)|}{|\lambda|^M} = \left|\phi\left(\frac{\lambda}{|\lambda|}\right)\right| \leq C_2, \quad \lambda \in L_t \setminus \{0\}, \tag{1.2}$$

$$C_1 \leq \frac{|\psi(z)|}{|z|^L} = \left|\psi\left(\frac{z}{|z|}\right)\right| \leq C_2, \quad z \in L_x \setminus \{0\}. \tag{1.3}$$

Taking the sets $K(L) := \{\xi \in L : |\xi| = 1\}$, we see that $\frac{\lambda}{|\lambda|} \in K(L_t)$ and $\frac{z}{|z|} \in K(L_x)$. However, the sets $K(L_t)$ and $K(L_x)$ are bounded and closed, and are therefore compact, and since any continuous function attains it's maximum on a compact set, we know by the definitions of $S^{(M)}(L_t)$ and $S^{(L)}(L_x)$ that $\phi(\lambda)$ and $\psi(z)$ attain a maximum C_2' and C_2'' respectively, and a minimum C_1' and C_1'' respectively, on the sets $K(L_t)$ and $K(L_x)$. Taking $C_2 := \max\{C_2', C_2''\}$ and $C_1 := \min\{C_1', C_1''\}$, we see that equations (1.2) and (1.3) hold, and since $\phi(\lambda) \neq 0$ and $\psi(z) \neq 0$ on their domains by the definitions of $S^{(M)}(L_t)$ and $S^{(L)}(L_x)$, we know that $C_1, C_2 > 0$.

(ii) Since $\tau(\lambda, z) \in S^{(\rho,N)}$, we can rewrite for $(\lambda, z) \in L_t \times L_x \setminus \{(0,0)\}$:

$$\frac{|\tau(\lambda,z)|}{|\lambda|^{\frac{N}{\rho}} + |z|^N} = \left|\tau\left(\frac{\lambda}{(|\lambda|^{\frac{N}{\rho}} + |z|^N)^{\frac{\rho}{N}}}, \frac{z}{(|\lambda|^{\frac{N}{\rho}} + |z|^N)^{\frac{1}{N}}}\right)\right|.$$

For all of these $(\lambda, z) \in L_t \times L_x \setminus \{(0,0)\}$, we see that

$$\left|\frac{\lambda}{(|\lambda|^{\frac{N}{\rho}} + |z|^N)^{\frac{\rho}{N}}}\right|^{\frac{N}{\rho}} + \left|\frac{z}{(|\lambda|^{\frac{N}{\rho}} + |z|^N)^{\frac{1}{N}}}\right|^N = \frac{|\lambda|^{\frac{N}{\rho}} + |z|^N}{|\lambda|^{\frac{N}{\rho}} + |z|^N} = 1,$$

which means that if we instead define $K := \{|\lambda|^{\frac{N}{\rho}} + |z|^N = 1\}$, which is also a compact set, we have

$$\left(\frac{\lambda}{(|\lambda|^{\frac{N}{\rho}} + |z|^N)^{\frac{\rho}{N}}}, \frac{z}{(|\lambda|^{\frac{N}{\rho}} + |z|^N)^{\frac{1}{N}}}\right) \in K,$$

so that

$$\begin{aligned} C_1(|\lambda|^{\frac{N}{\rho}} + |z|^N) \leq |\tau(\lambda, z)| &\leq C_2(|\lambda|^{\frac{N}{\rho}} + |z|^N), \quad (\lambda, z) \in L_t \times L_x, \\ C_1 \leq \frac{|\tau(\lambda,z)|}{|\lambda|^{\frac{N}{\rho}} + |z|^N} &\leq C_2, \quad (\lambda, z) \in L_t \times L_x \setminus \{(0,0)\}, \\ C_1 \leq |\tau(\tilde{\lambda}, \tilde{z})| &\leq C_2, \quad (\tilde{\lambda}, \tilde{z}) \in K \end{aligned}$$

means we must prove the last equation to prove our statement for $(\tilde{\lambda}, \tilde{z}) \in K$. Since $\tau(\lambda, z)$ is a continuous function from the definition of $S^{(\rho,N)}(L_t \times L_x)$, and since K is compact, we know $|\tau(\lambda, z)|$ attains it's maximum and minimum on K. Taking $C_2 := \max_{(\tilde{\lambda},\tilde{z})\in K} |\tau(\tilde{\lambda}, \tilde{z})|$ and $C_1 := \min_{(\tilde{\lambda},\tilde{z})\in K} |\tau(\tilde{\lambda}, \tilde{z})|$, then since $\tau(\lambda, z) \neq 0$ for $(\lambda, z) \in L_t \times L_x \setminus \{(0,0)\} \supseteq K$ by definition of $S^{(\rho,N)}(L_t \times L_x)$, we know $C_1, C_2 > 0$, and thus we have proven our statement. $\square$

This result will be useful later for proving N-parameter-ellipticity.

These $\tau(\lambda, z)$ functions have a certain weight ρ with respect to λ, and we see that they have orders N for z and $\frac{N}{\rho}$ for λ from equation (1.1). Suppose that instead of $\tau(\eta^\rho\lambda, \eta z)$, we write $\tau(\eta^\gamma\lambda, \eta z)$, which essentially compares the weight ρ to a relative weight γ. This allows us to define the γ-order of the function, or the order of the function for this relative weight γ, and the γ-principal part, the part of the function that causes this γ-order. We look at $\gamma \in [0, \infty]$, where $\gamma = 0$ means we look only at the order of z, and $\gamma = \infty$ means we look at the order of λ.

Definition 1.1.4 (γ-order and γ-principal part of ρ-homogeneous symbol)**.** Let τ be a symbol in $S^{(\rho,N)}(L_t \times L_x)$. Then we define for all $\gamma \geq 0$ the *γ-order* by

$$d_\gamma(\tau) := \max\left\{N, \frac{N}{\rho}\gamma\right\} = \begin{cases} N, & \gamma < \rho, \\ \frac{N}{\rho}\gamma, & \gamma \geq \rho, \end{cases}$$

and the γ-*principal part* by

$$\begin{aligned}\pi_\gamma\tau : (L_t \times L_x) &\to \mathbb{C}\\ (\lambda, z) &\mapsto \lim_{\eta\to\infty} \eta^{-d_\gamma(\tau)}\tau(\eta^\gamma\lambda, \eta z)\end{aligned} \tag{1.4}$$

In the same way we define the ∞-*order*

$$d_\infty(\tau) := \frac{N}{\rho}$$

and the ∞-*principal part*

$$\begin{aligned}\pi_\infty\tau : (L_t \times L_x) &\to \mathbb{C}\\ (\lambda, z) &\mapsto \lim_{\eta\to\infty} \eta^{-d_\infty(\tau)}\tau(\eta\lambda, z)\end{aligned} \tag{1.5}$$

These expressions will be used to define the γ-order and γ-principal part of more general symbols. For the γ-order, the part increasing with γ will be seen as the order of λ here, and the part that is constant as the order of z. For large relative weight γ, the order of λ dominates, and for small γ, the order of z dominates. From that we can determine a better expression for γ-principal and ∞-principal part:

Lemma 1.1.5. *For $\tau \in S^{(\rho,N)}(L_t \times L_x)$ and $\gamma \in [0, \infty]$ we have*

$$\pi_\gamma\tau(\lambda, z) = \begin{cases} \tau(0, z), & \gamma < \rho, \\ \tau(\lambda, z), & \gamma = \rho, \\ \tau(\lambda, 0), & \gamma > \rho, \end{cases} \quad (\lambda, z) \in (L_t \times L_x).$$

Proof. For $(\lambda, z) = (0, 0)$ the proof follows immediately. We check $(\lambda, z) \in L_t \times L_x \setminus \{(0, 0)\}$ for $\gamma > \rho$, $\gamma < \rho$ and $\gamma = \rho$.

(I) $\gamma > \rho$. First, let's consider $\gamma < \infty$. Take any $\eta > 0$ and define $\alpha := (\eta^\rho|z|^\rho + \eta^\gamma|\lambda|)^{\frac{1}{\rho}}$. We can then write

$$\begin{aligned}\tau(\eta^\gamma\lambda, \eta z) &= \frac{\alpha^N}{\alpha^N}\tau(\eta^\gamma\lambda, \eta z)\\ &= \alpha^N\tau\left(\frac{\eta^\gamma}{\alpha^\rho}\lambda, \frac{\eta}{\alpha}z\right).\end{aligned}$$

However, since $\gamma > \rho$, we get

$$\begin{aligned}\lim_{\eta\to\infty}\frac{\eta}{\alpha}z &= \lim_{\eta\to\infty}\frac{z}{(|z|^\rho + \eta^{\gamma-\rho}|\lambda|)^{\frac{1}{\rho}}} = 0,\\ \lim_{\eta\to\infty}\frac{\eta^\gamma}{\alpha^\rho}\lambda &= \lim_{\eta\to\infty}\frac{\lambda}{\eta^{\rho-\gamma}|z|^\rho + |\lambda|} = \begin{cases}\frac{\lambda}{|\lambda|}, & \lambda \neq 0,\\ 0, & \lambda = 0.\end{cases}\end{aligned}$$

Since $d_\gamma(\tau) = \frac{N}{\rho}\gamma$, we will get

$$\begin{aligned}\pi_\gamma\tau(\lambda, z) &= \lim_{\eta\to\infty}\eta^{-\frac{N}{\rho}\gamma}\tau(\eta^\gamma\lambda, \eta z)\\ &= \lim_{\eta\to\infty}\eta^{-\frac{N}{\rho}\gamma}\alpha^N\tau\left(\frac{\eta^\gamma}{\alpha^\rho}\lambda, \frac{\eta}{\alpha}z\right)\\ &= \lim_{\eta\to\infty}(\eta^{\rho-\gamma}|z|^\rho + |\lambda|)^{\frac{N}{\rho}}\tau\left(\frac{\eta^\gamma}{\alpha^\rho}\lambda, \frac{\eta}{\alpha}z\right)\\ &= \begin{cases}|\lambda|^{\frac{N}{\rho}}\tau\left(\frac{\lambda}{|\lambda|}, 0\right) = \tau(\lambda, 0), & \lambda \neq 0,\\ 0 = \tau(0, 0), & \lambda = 0.\end{cases}\end{aligned}$$

For $\gamma = \infty$, we instead get $d_\infty(\tau) = \frac{N}{\rho}$ so

$$\begin{aligned}\pi_\infty \tau(\lambda, z) &= \lim_{\eta\to\infty} \eta^{-\frac{N}{\rho}} \tau(\eta\lambda, z)\\ &= \lim_{\eta\to\infty} \tau(\eta^{1-1}\lambda, \eta^{-\frac{1}{\rho}} z)\\ &= \tau(\lambda, 0),\end{aligned}$$

so the statement also holds for $\gamma = \infty$.

(II) $\gamma < \rho$. With the same η and α as before, we can find

$$\begin{aligned}\lim_{\eta\to\infty} \frac{\eta}{\alpha} z &= \lim_{\eta\to\infty} \frac{z}{(|z|^\rho + \eta^{\gamma-\rho}|\lambda|)^{\frac{1}{\rho}}} = \begin{cases} \frac{z}{|z|}, & z \neq 0,\\ 0, & z = 0,\end{cases}\\ \lim_{\eta\to\infty} \frac{\eta^\gamma}{\alpha^\rho} \lambda &= \lim_{\eta\to\infty} \frac{\lambda}{\eta^{\rho-\gamma}|z|^\rho + |\lambda|} = 0,\end{aligned}$$

and since $d_\gamma(\tau) = N$, we get

$$\begin{aligned}\pi_\gamma \tau(\lambda, z) &= \lim_{\eta\to\infty} \eta^{-N} \alpha^N \tau\left(\frac{\eta^\gamma}{\alpha^\rho}\lambda, \frac{\eta}{\alpha} z\right)\\ &= \lim_{\eta\to\infty} (|z|^\rho + \eta^{\gamma-\rho}|\lambda|)^{\frac{N}{\rho}} \tau\left(\frac{\eta^\gamma}{\alpha^\rho}\lambda, \frac{\eta}{\alpha} z\right)\\ &= \begin{cases} |z|\tau\left(0, \frac{z}{|z|}\right) = \tau(0, z), & z \neq 0,\\ 0 = \tau(0, 0), & z = 0.\end{cases}\end{aligned}$$

(III) $\gamma = \rho$. Here $\frac{N}{\rho}\gamma = N$, so we find for $d_\gamma(\tau) = N$ that by definition of ρ-homogeneity we get

$$\pi_\rho \tau(\lambda, z) = \lim_{\eta\to\infty} \eta^{-N} \tau(\eta^\rho \lambda, \eta z) = \lim_{\eta\to\infty} \frac{\eta^N}{\eta^N} \tau(\lambda, z) = \tau(\lambda, z).$$

□

We can then use these functions to create a representation $R_P(\lambda, z)$ of our symbol $P(\lambda, z)$. A symbol may have multiple representations R_P and R'_P, but $R_P = R'_P$ must hold for these symbols.

Definition 1.1.6 (Symbol class $\tilde{S}(L_t \times L_x)$)**.** For $\rho > 0$ we define $\tilde{S}(L_t \times L_x) \subseteq H_P(\mathring{L}_t \times \mathring{L}_x)$ as the set of all functions $P : L_t \times L_x \to \mathbb{C}$ for which a representation $R_P(\lambda, z)$ exists s.t. $P(\lambda, z) = R_P(\lambda, z)$ for all $(\lambda, z) \in L_t \times L_x$, and the representation $R_P(\lambda, z)$ is of the form

$$R_P(\lambda, z) := \sum_{\ell \in I_P} \tau_\ell(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z), \quad (\lambda, z) \in L_t \times L_x, \tag{1.6}$$

where I_P is an arbitrary finite index set and

$$\begin{aligned}&\tau_\ell \in S^{(\rho, N_\ell)}(L_t \times L_x) \cap H(\mathring{L}_t \times \mathring{L}_x), \quad N_\ell \geq 0,\\ &\phi_\ell \in S^{(M_\ell,)}(L_t) \cap H(\mathring{L}_t), \quad M_\ell \geq 0,\\ &\psi_\ell \in S^{(L_\ell)}(L_x) \cap H(\mathring{L}_x), \quad L_\ell \geq 0\end{aligned}$$

for all $\ell \in I_P$. In other words: each τ_ℓ, ϕ_ℓ and ψ_ℓ is (ρ-)homogeneous with degree N_ℓ, M_ℓ and L_ℓ respectively. Note that τ_ℓ is ρ-homogenenous of degree N_ℓ with a fixed weight $\rho > 0$ for all $\ell \in I_P$.

Using the orders and principal parts of the functions, we can also use the relative weights γ on this ρ.

Definition 1.1.7 (γ-order and γ-principal part of symbols in $\tilde{S}(L_t \times L_x)$)**.** For all $\gamma \geq 0$ we define the γ-*order* of the symbol as seen in equation (1.6) by

$$d_\gamma(P) := \max_{\ell \in I_P}(d_\gamma(\tau_\ell) + M_\ell\gamma + L_\ell) = \begin{cases} \max_{\ell \in I_P} M_\ell\gamma + N_\ell + L_\ell, & \gamma < \rho, \\ \max_{\ell \in I_P}(M_\ell + \frac{N_\ell}{\rho})\gamma + L_\ell, & \gamma \geq \rho, \end{cases} \tag{1.7}$$

and for $(\lambda, z) \in L_t \times L_x$, we define $I_\gamma := \{\ell \in I_P : d_\gamma(\tau_\ell) + \gamma M_\ell + L_\ell = d_\gamma(P)\}$, the part of the index I_P for which the maximum of equation (1.7) is attained, so that the γ-*principal part* is defined

$$\pi_\gamma P(\lambda, z) := \lim_{\eta\to\infty} \eta^{-d_\gamma(P)} P(\eta^\gamma\lambda, \eta z) = \sum_{\ell \in I_\gamma} [\pi_\gamma\tau_\ell](\lambda, z)\phi_\ell(\lambda)\psi_\ell(z) \tag{1.8}$$

In the same way we define the ∞-*order* by

$$d_\infty(P) := \max_{\ell \in I_p} \left(\frac{N_\ell}{\rho} + M_\ell\right),$$

and for $(\lambda, z) \in (L_t \times L_x)$ the ∞-*principal part* by

$$\pi_\gamma P(\lambda, z) := \lim_{\eta\to\infty} \eta^{-d_\infty(P)} P(\eta\lambda, z) = \sum_{\ell \in I_\infty} \tau_\ell(\lambda, 0)\phi_\ell(\lambda)\psi_\ell(z) \tag{1.9}$$

with $I_\infty := \{\ell \in I : M_\ell + \frac{N_\ell}{\rho} = d_\infty(P)\}$.

We will later use this γ-order when defining Newton polygons and order functions in sections 1.2.1 and 1.2.3.

We can show that γ-principal parts are always quasi-homogeneous.

Lemma 1.1.8. *(i) Let $\tau \in S^{(\rho,N)}(L_t \times L_x)$. Then*

$$\begin{aligned} &\pi_0\tau(\lambda, z) = \tau(0, z) \in S^{(d_0(P))}(L_x), \\ &\pi_\gamma\tau(\lambda, z) \in S^{(\gamma, d_\gamma(\tau))}(L_t \times L_x), \qquad \gamma \in (0, \infty), \\ &\pi_\infty\tau(\lambda, z) = \tau(\lambda, 0) \in S^{(d_\infty(\tau))}(L_t). \end{aligned}$$

(ii) Let $P \in \tilde{S}(L_t \times L_x)$ with representation $R_P(\lambda, z) = \sum_{\ell \in I} \tau_\ell(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z)$ as in definition 1.1.6. Then the function $\pi_\gamma P(\lambda, z)$ is a quasi-homogeneous function of degree $d_\gamma(P)$ and weight γ for $\gamma \in (0, \infty)$, $\pi_0 P(\lambda, z)$ is a homogeneous function of degree $d_0(P)$ with respect to z and $\pi_\infty P(\lambda, z)$ is a homogeneous function of degree $d_\infty(P)$ with respect to λ.

Proof. (i) Using lemma 1.1.5, we see that for $\gamma \in [0, \infty)$, $(\lambda, z) \in (L_t \times L_x)$ and $\eta > 0$,

$$\pi_\gamma\tau(\eta^\gamma\lambda, \eta z) = \begin{cases} \tau(0, \eta z), & \gamma < \rho, \\ \tau(\eta^\rho\lambda, \eta z), & \gamma = \rho, \\ \tau(\eta^\gamma\lambda, 0), & \gamma > \rho, \end{cases} = \begin{cases} \eta^N\tau(0, z), & \gamma < \rho, \\ \eta^N\tau(\lambda, z), & \gamma = \rho, \\ \eta^{\frac{N}{\rho}\gamma}\tau(\lambda, 0), & \gamma > \rho, \end{cases} = \eta^{d_\gamma(\tau)}\pi_\gamma P(\lambda, z).$$

To show the bottom case is true, one can use $\eta = (\eta^{\frac{\gamma}{\rho}})^{\frac{\rho}{\gamma}} = \alpha^{\frac{\rho}{\gamma}}$, so that $\tau(\eta^\gamma\lambda, 0) = \tau(\alpha^\rho\lambda, 0) = \alpha^N\tau(\lambda, 0) = \eta^{\frac{\gamma}{\rho}N}\tau(\lambda, 0) = \eta^{\frac{N}{\rho}\gamma}$ for $\alpha := \eta^{\frac{\gamma}{\rho}}$. This allows us to conclude $\pi_\gamma\tau(\lambda, z)$ is quasi-homogeneous of degree $d_\gamma(\tau)$ and weight γ, so by $\tau \in S^{(\rho,N)}(L_t \times L_x)$ we get $\pi_\gamma\tau(\lambda, z) \in S^{(\gamma, d_\gamma(\tau))}(L_t \times L_x)$ for $\gamma \in (0, \infty)$, and $\pi_0\tau(\lambda, z) = \tau(0, z) \in S^{(d_0(\tau))}(L_x)$.

Secondly, again using lemma 1.1.5, we see

$$\pi_\infty \tau(\eta\lambda, z) = \tau(\eta\lambda, 0) = \eta^{\frac{N}{\rho}} \tau(\lambda, 0) = \eta^{d_\infty(\tau)} \tau(\lambda, 0).$$

So we have $\pi_\infty \tau(\lambda, z) = \tau(\lambda, 0) \in S^{(d_\infty(\tau))}(L_t)$.

(ii) If we denote $\pi_\gamma P(\lambda, z) = \sum_{\ell \in I_\gamma} [\pi_\gamma \tau_\ell](\lambda, z)\phi_\ell(\lambda)\psi_\ell(z)$ from definition 1.1.7 and use statement (i), then we can conclude for $\gamma \in [0, \infty)$, $(\lambda, z) \in (L_t \times L_x)$ and $\eta > 0$

$$\begin{aligned}
\pi_\gamma P(\eta^\rho \lambda, \eta z) &= \sum_{\ell \in I_\gamma} [\pi_\gamma \tau_\ell](\eta^\gamma \lambda, \eta z)\phi_\ell(\eta^\gamma \lambda)\psi_\ell(\eta z) \\
&= \sum_{\ell \in I_\gamma} \eta^{d_\gamma(\tau_\ell) + M_\ell \gamma + L_\ell} [\pi_\gamma \tau_\ell](\lambda, z)\phi_\ell(\lambda)\psi_\ell(z) \\
&= \sum_{\ell \in I_\gamma} \eta^{d_\gamma(P)} [\pi_\gamma \tau_\ell](\lambda, z)\phi_\ell(\lambda)\psi_\ell(z) \\
&= \eta^{d_\gamma(P)} \sum_{\ell \in I_\gamma} [\pi_\gamma \tau_\ell](\lambda, z)\phi_\ell(\lambda)\psi_\ell(z) \\
&= \eta^{d_\gamma(P)} \pi_\gamma P(\lambda, z).
\end{aligned}$$

Here, we have used the definition of $I_\gamma := \{\ell \in I_P : d_\gamma(\tau_\ell) + M_\ell \gamma + L_\ell = d_\gamma(P)\}$. This immediately implies the statement for $\gamma < \infty$. Taking $\gamma = \infty$, we use the definition 1.1.7 and statement (i) to find

$$\begin{aligned}
\pi_\infty P(\eta\lambda, z) &= \sum_{\ell \in I_\infty} \tau(\eta\lambda, 0)\phi(\eta\lambda)\psi(z) \\
&= \sum_{\ell \in I_\infty} \eta^{\frac{N_\ell}{\rho} + M_\ell} \tau(\lambda, 0)\phi(\lambda)\psi(z) \\
&= \sum_{\ell \in I_\infty} \eta^{d_\infty(P)} \tau(\lambda, 0)\phi(\lambda)\psi(z) \\
&= \eta^{d_\infty(P)} \sum_{\ell \in I_\infty} \tau(\lambda, 0)\phi(\lambda)\psi(z) \\
&= \eta^{d_\infty(P)} \pi_\infty P(\lambda, z)
\end{aligned}$$

Here, we have used the definition of $I_\infty := \{\ell \in I_P : \frac{N_\ell}{\rho} + M_\ell = d_\infty(P)\}$.

□

We now have an arbitrary representation $R_P(\lambda, z)$ of our symbol $P(\lambda, z)$. Symbols usually have multiple representations, however not all these representations are well-behaved: different representations (using other indexes I_P, or other functions) of the same symbol $P(\lambda, z)$ can lead to different γ-orders and γ-principal parts (see for instance the symbol K in the examples.) To fix, this we introduce the notion of regular representations.

Definition 1.1.9 (Regular representation of a symbol)**.** The representation $R_P(\lambda, z)$ of the symbol P in equation (1.6) is said to be *regular* if we have

$$\pi_\gamma P \not\equiv 0$$

for all $\gamma \in [0, \infty]$ (for every $\gamma \in [0, \infty]$, there is a coordinate (λ, z) such that $\pi_\gamma P(\lambda, z) \neq 0$.)

We define the subclass of symbols $S(L_t \times L_x) \subseteq \tilde{S}(L_t \times L_x)$ as the set of all symbols $P \in \tilde{S}(L_t \times L_x)$ for which a regular representation $R_P(\lambda, z)$ exists. We assume that the given representation of $P \in S(L_t \times L_x)$ is always regular.

The following lemma shows that regular representations are well-defined in terms of γ-orders and γ-principal parts.

Lemma 1.1.10. *For two regular representations*

$$\begin{aligned} P(\lambda, z) = R_P(\lambda, z) &:= \sum_{\ell \in I} \tau_\ell(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z) \\ = R'_P(\lambda, z) &:= \sum_{\ell \in I} \tau'_\ell(\lambda, z)\phi'_\ell(\lambda)\psi'_\ell(z), \quad (\lambda, z) \in L_t \times L_x \end{aligned}$$

of the symbol P we have $d_\gamma(P) = d'_\gamma(P)$ and $\pi_\gamma P = \pi'_\gamma P$ for all $\gamma \in [0, \infty]$. Here $d'_\gamma(P)$ and $\pi'_\gamma P$ denote the γ-order and the γ-principal part with respect to the second representation.

Proof. This can be shown by contradiction: for any $\gamma \in [0, \infty)$, assume $d_\gamma(P) \neq d'_\gamma(P)$, so either $d_\gamma(P) < d'_\gamma(P)$ or $d_\gamma(P) > d'_\gamma(P)$. In the first case, we see that for all $(\lambda, z) \in L_t \times L_x$

$$\pi'_\gamma P(\lambda, z) = \lim_{\eta \to \infty} \eta^{d_\gamma(P) - d'_\gamma(P)} \cdot \eta^{d_\gamma} P(\eta^\gamma \lambda, \eta z) = 0 \cdot \pi_\gamma P(\lambda, z) = 0,$$

which is a contradiction to $R'_P(\lambda, z)$ being a regular representation. In the same way, we know that $d_\gamma(P) > d'_\gamma(P)$ is a contradiction to $R_P(\lambda, z)$ being a regular representation. Therfore $d_\gamma(P) = d'_\gamma(P)$, which immediately means $\pi_\gamma P = \pi'_\gamma P$.

For $\gamma = \infty$, the proof is similiar: suppose either $d_\infty(P) < d'_\infty(P)$ or $d_\infty(P) > d'_\infty(P)$, then in the first case we see that for any $(\lambda, z) \in L_t \times L_x$

$$\pi'_\infty P(\lambda, z) = \lim_{\eta \to \infty} \eta^{d_\infty(P) - d'_\infty(P)} \cdot \eta^{d_\infty} P(\eta \lambda, z) = 0 \cdot \pi_\infty P(\lambda, z) = 0$$

which contradicts $R'_P(\lambda, z)$ being regular, and in the same way $d_\infty(P) > d'_\infty(P)$ contradicts $R_P(\lambda, z)$ being regular, implying that $d_\infty(P) = d'_\infty(P)$ so $\pi_\infty P = \pi'_\infty P$. $\square$

Examples

- The heat equation [1] operator $P(\partial_t, \nabla_x) = \frac{\partial}{\partial t} - \Delta$ on $\mathbb{R} \times \mathbb{R}^n$ has the symbol $P(\lambda, z) = \lambda + |z|_-^2$ on $\mathbb{C} \times \mathbb{C}^n$, where we define $|z|_- := \sqrt{\sum_{i=1}^n -z_i^2}$. This symbol is ρ-homogeneous of degree $N = 2$ with weight $\rho = 2$, since

  $$P(\eta^2 \lambda, \eta z) = \eta^2(\lambda + |z|_-^2) = \eta^2 P(\lambda, z).$$

  Therefore, it's γ-order is

  $$d_\gamma(P) = \begin{cases} 2, & 0 \leq \gamma < 2, \\ \gamma, & 2 \leq \gamma < \infty, \\ 1, & \gamma = \infty, \end{cases}$$

  and it's γ-principal part, using lemma 1.1.5:

  $$\pi_\gamma P(\lambda, z) = \begin{cases} |z|_-^2, & \gamma < 2, \\ \lambda + |z|_-^2, & \gamma = 2, \\ \lambda, & \gamma > 2. \end{cases}$$

  Since none of the principal parts are equivalent to 0, this representation is regular.

- The symbol $Q(\lambda, z) = \lambda^3 + |z|^5 - |z|^2\sqrt{\lambda^4 - |z|^4}$ is inhomogeneous, but the function itself is a valid representation R_Q:

1. $\phi_1(\lambda) := \lambda^3$ is homogeneous of degree $M_1 = 3$,
2. $\psi_2(z) := |z|^5$ is homogeneous of degree $L_2 = 5$,
3. $\psi_3(z) := -|z|^2$ is homogeneous of degree $L_3 = 2$,
4. $\tau_3(\lambda, z) := \sqrt{\lambda^4 - |z|^4}$ is ρ-homogeneous of degree $N_3 = 2$ with weight $\rho = 1$,
5. All other $N_\ell, M_\ell, L_\ell = 0$, since these functions $\tau_\ell, \phi_\ell, \psi_\ell = 1 =$ constant.

This means the three terms $\phi_1(\lambda)$, $\psi_2(z)$ and $\psi_3(z)\tau_3(\lambda, z)$ have the γ-orders 3γ, 5 and $\begin{cases} 2+2 & \gamma < 1 \\ 2+2\gamma & \gamma \geq 1 \end{cases}$ respectively. So we can establish the γ-order of Q:

$$d_\gamma(Q) = \begin{cases} 5, & 0 \leq \gamma < \frac{3}{2}, \\ 2+2\gamma, & \frac{3}{2} \leq \gamma < 2, \\ 3\gamma, & 2 \leq \gamma < \infty, \\ 3, & \gamma = \infty. \end{cases}$$

And from that, the γ-principal part by working out equation (1.8):

$$\pi_\gamma Q(\lambda, z) = \begin{cases} |z|^5, & \gamma < \frac{3}{2}, \\ |z|^5 - |z|^2\sqrt{\lambda^4}, & \gamma = \frac{3}{2}, \\ -|z^2|\sqrt{\lambda^4}, & \frac{3}{2} < \gamma < 2, \\ \lambda^3 - |z|^2\sqrt{\lambda^4}, & \gamma = 2, \\ \lambda^3, & \gamma > 2. \end{cases}$$

Again, this is a regular representation.

- $\beta(\lambda) = \sqrt{\lambda^2 - \lambda}$ is inhomogeneous on $\mathbb{C}$:

$$\beta(\eta\lambda) = \sqrt{\eta^2\lambda^2 - \eta\lambda} \neq \eta^N \sqrt{\lambda^2 - \lambda}, \quad \forall N > 0, \lambda \in \mathbb{C} \setminus \{0\}, \eta > 0.$$

We also can not represent this β as a finite sum of homogeneous functions $\phi(\lambda)$: we might use a Taylor series around $\lambda = 2$ to find

$$\beta(\lambda) = \sqrt{2} + \frac{5}{2\sqrt{2}}(\lambda - 2) - \frac{1}{2\sqrt{2}}(\lambda - 2)^2 + \dots,$$

but since this is an infinite sum, it is not a representation $r(\beta)$.

- The symbol $K(\lambda, z)$ on $\mathbb{C} \times \mathbb{C}$ is defined to be equal to the representations $R_K(\lambda, z) := \lambda$ and $R'_K(\lambda, z) := z + (\lambda - z)$. We can easily check that $r(K)$ is a regular representation with γ-order $d_\gamma(K) = \begin{cases} \gamma, & 0 \leq \gamma < \infty, \\ 1, & \gamma = \infty, \end{cases}$ and γ-principal part $\pi_\gamma K(\lambda, z) = \lambda$. The other representation $R'_K(\lambda, z) = z + (\lambda - z)$ however has

$$d'_\gamma(K) := \begin{cases} 1, & 0 \leq \gamma < 1, \\ \gamma, & 1 \leq \gamma < \infty, \\ 1 & \gamma = \infty, \end{cases}$$

$$\pi'_\gamma K(\lambda, z) := \begin{cases} z - z = 0, & \gamma < 1 \\ \lambda, & \gamma \geq 1 \end{cases}$$

From this we see $R'_K(\lambda, z)$ is not a regular representation.

References

R. Denk & M. Kaip, section 2.1a [1]:

1. Definition 2.1, p. 71
2. Definition 1.1, p. 12
3. Remark 2.4, p. 72
4. Definition 2.5, p. 72
5. Lemma 2.7 and proof, p. 73-74
6. Definition 2.8, p. 74
7. Definition 2.10, p. 75
8. Definition 2.11, p. 76
9. Definition 2.14, p. 77
10. Lemma 2.13 and proof, p. 76-77

Lemma 1.1.8 is a result of my work. The proof of lemma 1.1.3 is based on the idea of proof given in Remark 2.4 [1]

1.2 Tools

In this section the different tools will be defined and explained.

1.2.1 Newton polygons

We want to map the orders of $P(\lambda, z)$ into the two-dimensional plane $[0,\infty)^2$, in order to simplify our problem. We will map the order of z in the x-direction, and the order of λ in the y-direction. For $P(\lambda, z) \in S(L_t \times L_x)$, we add points on the $[0,\infty)^2$-plane for every $\ell \in I_P$:

- a point $\mathfrak{u}_\ell$ counting the order of τ_ℓ for z, so $\mathfrak{u}_\ell := (N_\ell + L_\ell, M_\ell)$,
- a point $\mathfrak{v}_\ell$ counting the order of τ_ℓ for λ, so $\mathfrak{v}_\ell := (L_\ell, \frac{N_\ell}{\rho} + M_\ell)$.

We can draw a parallel to the definition of the γ-order in equation (1.7): the part that does not increase with γ is mapped in the x-coordinate, and the part that increases with γ is mapped in the y-coordinate. We gather all these points in a finite set $\nu(P)$. We also extend the set with a few other points:

- the origin $(0,0)$,
- the projection of the points in $\nu(P)$ on the x-axis: $(x, 0)$ for $(x, y) \in \nu(P)$,
- the projection of the points in $\nu(P)$ on the y-axis: $(0, y)$ for $(x, y) \in \nu(P)$.

The Newton polygon $N = N(P)$ is then created by taking the convex hull over $\nu(P)$ and it's extension, so it is the set of all points somewhere between the points chosen, or contained within the area of these points.

Definition 1.2.1 (Newton polygon). (i) Let $\nu := \bigcup_{i=0}^{M} \{(a_i, b_i)\} \subseteq [0,\infty)^2$ be any finite set. Then the *Newton polygon* $N(\nu)$ is defined as

$$N(\nu) := \text{Convex hull}(\nu \cup \{(0,0)\} \cup \bigcup_{i=0}^{M} \{(a_i, 0), (0, b_i)\}). \tag{1.10}$$

Any convex set $N \subseteq [0,\infty)^2$ is called a *Newton polygon* if there exists a finite ν such that $N = N(\nu)$.

(ii) Let $P \in S(L_t \times L_x)$ with a regular representation R_P as seen in definition 1.1.6. For all $\ell \in I_P$, define the points

$$\mathfrak{u}_\ell := (N_\ell + L_\ell, M_\ell),$$
$$\mathfrak{v}_\ell := (L_\ell, \frac{N_\ell}{\rho} + M_\ell).$$

If we set

$$\nu(P) := \bigcup_{\ell \in I_P} \{\mathfrak{u}_\ell, \mathfrak{v}_\ell\}, \tag{1.11}$$

then the *Newton polygon* $N(P)$ is defined as $N(P) := N(\nu(P))$.

Any Newton polygon N will have a finite amount of vertices $v_j = (r_j, s_j)$, which are the "vital" points of the convex hull. What this means is that for a set of vertices N_v of a Newton polygon N is the smallest possible set of $(r, s) \in N$ such that the convex hull of N_v equals the set N. Since the set is finite, we can count the vertices:

1. start at $v_0 = (0, 0)$,
2. follow the edges of $N(P)$ counter-clockwise, and name each vertex $v_1 = (r_1, 0)$ (the furthest point on the x-axis), $v_2 = (r_2, s_2)$, so forth,
3. index the last vertex (the highest vertex on the y-axis) $v_{J+1} = (0, s_{J+1})$, so we have a total amount of vertices equal to $J + 2$.

Definition 1.2.2. For some $J \in \mathbb{N}$, the *vertices* $v_j = (r_j, s_j)$ of the Newton polygon N are counted starting from the origin $v_0 = (0, 0)$ and then following the boundary of the convex hull in the counter-clockwise direction, up until the last vertex $v_{J+1} := (0, s_{J+1})$. We can then take the *set of vertices* as

$$N_v := \bigcup_{j=0}^{J+1} \{v_j\},$$

which is the smallest subset of N that has

$$\text{Convex hull}(N_v) = N.$$

Define the *edge* $[v_j v_{j+1}]$ as all the points that are a linear combination of the vertices v_j and v_{j+1}, so

$$[v_j v_{j+1}] := \{\lambda v_j + (1 - \lambda) v_{j+1} : 0 \le \lambda \le 1\} \subseteq N.$$

In section 1.2.3 we will construct order functions, which in essence serve as functions that map the Newton polygon by comparing the points through the means of inner products with vectors of the form $(1, \gamma)$: as we will see, these order functions are functions of the relative weight γ, and we will see that this γ will be used as a vector $(1, \gamma)$ so we can compare points (r, s) in the Newton polygon by looking at the inner product $\langle (1, \gamma), (r, s) \rangle$. In this spirit, we define a partition of $\gamma \in [0, \infty]$ the following way:

Definition 1.2.3. (i) Let N be a Newton polygon with vertices $(r_j, s_j) \in N_v$. We define a partition of $[0, \infty]$ in the form of $0 = \gamma_0 \leq \gamma_1 < \cdots < \gamma_J \leq \gamma_{J+1} = \infty$ by taking:

$$\begin{aligned}
\gamma_0 &:= 0, \\
\gamma_j &:= \frac{r_j - r_{j+1}}{s_{j+1} - s_j}, \quad j \in \{1, \ldots, J-1\}, \\
\gamma_J &:= \begin{cases} \frac{r_J - r_{J+1}}{s_{J+1} - s_J}, & s_J \neq s_{J+1}, \\ \infty, & s_J = s_{J+1}, \end{cases} \\
\gamma_{J+1} &:= \infty.
\end{aligned}$$

Essentially, γ_j is defined in such a way that $-\frac{1}{\gamma_j}$ is the slope of the edge $[v_j v_{j+1}]$, so the vector $(1, \gamma_j)$ is orthogonal to $[v_j v_{j+1}]$.

(ii) Using this partition, we define the outward facing normal vectors q_j to be

$$\begin{aligned}
q_0 &:= (0, -1), \\
q_j &:= \frac{1}{\|(1, \gamma_j)\|_2}(1, \gamma_j), \quad j \in \{1, \ldots, J-1\}, \\
q_J &:= \begin{cases} \frac{1}{\|(1, \gamma_J)\|_2}(1, \gamma_J), & \gamma_J \neq \infty, \\ (0, 1), & \gamma_J = \infty, \end{cases} \\
q_{J+1} &:= (-1, 0).
\end{aligned}$$

For any $j \in \{0, \ldots, J+1\}$, the vector q_j is orthogonal to the edge $[v_j v_{j+1}]$. For all $q_j = (q_{j,1}, q_{j,2})$, we will also define the normal vectors $q_j^{\perp} := (-q_{j,2}, q_{j,1})$ in section 2.2. These vectors follow the edges of the Newton polygon N.

As noted earlier, this partition will be used to define order functions in section 1.2.3. It can also be noted how the set N_v relates to the partition of $[0, \infty]$: due to the set being the smallest possible set that forms the convex hull, it is not possible that $(r_j, s_j) \in [v_k v_{k+1}]$ for some $j, k \in \{0, \ldots, J+1\}$. Therefore after γ_1, we need to keep find a different larger value γ_2, since the slope of $(1, \gamma_j)$ must change as j gets larger, if we want to keep following the angle of the polygon. The vectors q_j and $q_j^{\perp}$ both play an important role in defining the partition from section 2.2. A figure showing a general Newton polygon N, with vertices v_j and outward vectors q_j, can be found in figure 1.3.

With these definitions in mind, we define regularity:

Definition 1.2.4 (Regular Newton polygon)**.** (i) A Newton polygon N (and associated symbol $P \in S(L_t \times L_x)$) is *regular in time* if $r_1 \neq r_2$. This means the edge $[v_1 v_2]$ is not parallel to the y-axis, or that $\gamma_1 \neq 0$, or that $q_1 \neq (1, 0)$

(ii) A Newton polygon N (and associated symbol $P \in S(L_t \times L_x)$) is *regular in space* if $s_J \neq s_{J+1}$. This means that the edge $[v_J v_{J+1}]$ is not parallel to the x-axis, or that $\gamma_J \neq \infty$, or that $q_J \neq (0, 1)$

(iii) A Newton polygon N (and associated symbol $P \in S(L_t \times L_x)$) is *regular* if it is regular in both time and space.

Remark. Please note that P having a regular representation R_P and P being regular in the sense that $N(P)$ is regular are not the same thing. Currently we only consider $P \in S(L_t \times L_x)$, so if P is regular, it definitely has a regular representation R_P.

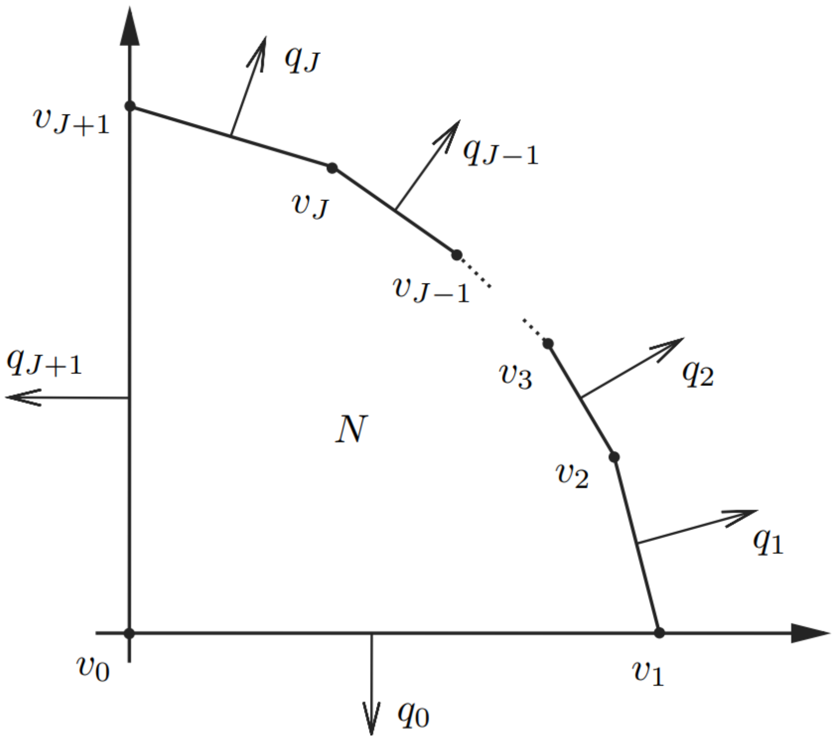

Figure 1.3: A Newton polygon N with vertices v_j and outward vectors q_j. This Newton polygon is regular. [1, Denk, Kaip, 2013, p. 78]

Regularity in time and space have important consequences for definitions and proofs as seen later, and we want to be able to work easily with both regular and irregular Newton polygons and symbols. Therefore, we will define the following indices to allow ourselves to work with these polygons without having to keep bringing up the relations to regularity.

Definition 1.2.5. Let N be a Newton polygon. Define $r_t(N)$ as the *time-regularity index*, or as the first index $j \in \{1, \dots, J+1\}$ that has $\gamma_j > 0$, so

$$r_t(N) := \begin{cases} 1 & \text{if } N \text{ is regular in time,} \\ 2 & \text{if } N \text{ is not regular in time.} \end{cases}$$

Similarly, define $r_x(N)$ as the *space-regularity index*, or as the first index $j \in \{1, \dots, J+1\}$ that has $\gamma_j = \infty$.

$$r_x(N) := \begin{cases} J+1 & \text{if } N \text{ is regular in space,} \\ J & \text{if } N \text{ is not regular in space.} \end{cases}$$

Examples

- The symbol $\omega(\lambda, z) = \lambda|z|$ is a regular representation. The term $\lambda|z|$ yields the point $(1,1) \in \nu(\omega)$. We make the Newton polygon $N(\omega)$ by adding the other needed points: the origin $(0,0)$ and the projections $(1,0)$ and $(0,1)$, and lastly taking the convex hull. This forms a box, as seen in figure 1.4. This Newton polygon is irregular, as the line $[v_1 v_2]$ is perpendicular to the y-axis and the line $[v_2 v_3]$ to the y-axis, and this means that $r_t(N(\omega)) = r_x(N(\omega)) = 2$.

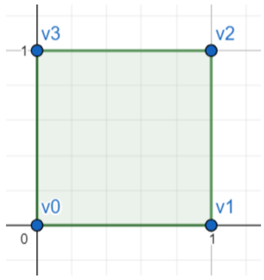

Figure 1.4: The Newton polygon of ω

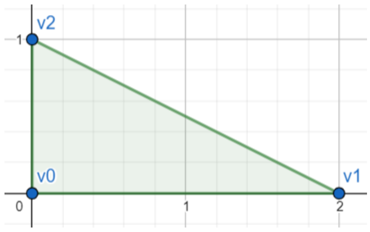

Figure 1.5: The Newton polygon of P

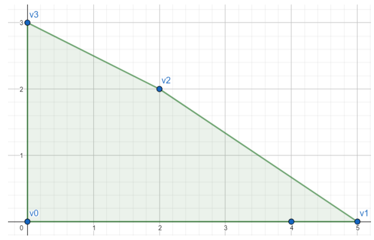

Figure 1.6: The Newton polygon of Q

- The symbol from the heat equation $P(\lambda, z) = \lambda + |z|_-^2$ has the points $(0, 1)$ and $(2, 0)$ from the terms λ and $|z|_-^2$ respectively. Adding $(0, 0)$ we get the convex hull in figure 1.5. This Newton polygon is regular.

- The symbol $Q(\lambda, z) = \lambda^3 + |z|^5 - |z|^2\sqrt{\lambda^4 - |z|^4}$ from the previous section has the points $(5, 0)$ and $(3, 0)$ from λ^3 and $|z|^5$, and the points $(2, 2)$ and $(4, 0)$ from the quasi-homogeneous term $|z|^2\sqrt{\lambda^4 - z^4}$. We add the point $(0, 0)$, and we take the convex hull to form figure 1.6. This is also a regular Newton polygon. Due to taking the convex hull, the point $(4, 0)$ is not a vertex, meaning it will not be included in the set N_v of Q.

References

R. Denk & M. Kaip, section 2.1b [1]:

1. Definition 2.15, p. 77-78
2. Definition 2.18, p. 79
3. Defininition 2.46, p. 98

1.2.2 Weight functions

A very useful property of Newton polygons is that it is a convex set. This means that for any coordinate within the polygon, we can estimate it using only vertices that define the polygon, namely v_0 to v_{J+1}. The general result is shown in the following proposition. Here, ξ represents $|z|$ and η represents $|\lambda|$ (but it works for general $\xi, \eta \geq 0$.)

Proposition 1.2.6. *Let $G \subseteq [0, \infty)^2$ be any convex set with a finite set of vertices G_v. Then for any $(r, s) \in G$ and any $\xi, \eta \geq 0$, we have*

$$\eta^s \xi^r \leq \sum_{(r', s') \in G_v} \eta^{s'} \xi^{r'}.$$

Proof. Since G_v is the set of vertices, we can essentially define G as

$$G := \text{Convex hull}(G_v).$$

Since G_v is finite, we can number the vertices $(r_1, s_1), \ldots, (r_m, s_m)$, where $m := \#G_v$, the amount of vertices in the set G_v. Using convexity, we can find certain $\lambda_1, \ldots, \lambda_m \geq 0$ that have $\sum_{i=1}^m \lambda_i = 1$ s.t.

$$(r, s) = \sum_{i=1}^m \lambda_i (r_i, s_i).$$

For $\xi, \eta \geq 0$, we see that

$$\eta^s \xi^r = \eta^{\sum_{i=1}^m \lambda_i s_i} \xi^{\sum_{i=1}^m \lambda_i r_i} = \prod_{i=1}^m (\eta^{s_i} \xi^{r_i})^{\lambda_i}$$

Here we can apply logarithms to be able to apply logarithmic rules and the concaveness of the logarithm function:

$$\log \left(\prod_{i=1}^m (\eta^{s_i} \xi^{r_i})^{\lambda_i} \right) = \sum_{i=1}^m \lambda_i \log(\eta^{s_i} \xi^{r_i}) \leq \log \left(\sum_{i=1}^m \lambda_i \eta^{s_i} \xi^{r_i} \right).$$

Then since the logarithm is increasing, and $0 \leq \lambda_i \leq 1$ for all $i \in \{1, \dots, m\}$ we know

$$\begin{aligned} \eta^s \xi^r &= \prod_{i=1}^m (\eta^{s_i} \xi^{r_i})^{\lambda_i} \\ &\leq \sum_{i=1}^m \lambda_i \eta^{s_i} \xi^{r_i} \\ &\leq \sum_{i=1}^m \eta^{s_i} \xi^{r_i} = \sum_{(r',s') \in G_v} \eta^{s'} \xi^{r'}. \end{aligned}$$

□

We will use a similar line of thinking when defining the partition of section 2.2. In this section, we use this result to define the weight function, which can be used as an estimate from above for any point in the polygon.

Definition 1.2.7 (Weight Functions)**.** For a Newton polygon N, the corresponding *weight function* is defined by

$$W_N(\lambda, z) := \sum_{(r,s) \in N_v} |\lambda|^s |z|^r = 1 + \sum_{j=1}^{J+1} |\lambda|^{s_j} |z|^{r_j}, \quad (\lambda, z) \in \mathbb{C} \times \mathbb{C}^n.$$

For a finite set $\nu \subseteq [0, \infty)^2$, define $W_\nu := W_{N(\nu)}$. For a symbol $P \in S(L_t \times L_x)$, define $W_P := W_{N(P)}$.

Notice all weight functions are positive due to being sums of positive terms.

Example

- The weight function W_Q of $Q(\lambda, z)$ from previous examples is found by summing over v_0 to v_4 in figure 1.6:

$$\begin{aligned} W_Q(\lambda, z) &= |\lambda|^0 |z|^0 + |\lambda|^0 |z|^5 + |\lambda|^2 |z|^2 + |\lambda|^3 |z|^0 \\ &= 1 + |z|^5 + |\lambda|^2 |z|^2 + |\lambda|^3 \end{aligned}$$

References

R. Denk & M. Kaip, section 2.1b [1]:

1. Remark 2.17, p. 79
2. Definition 2.15, p. 77-78

C. Gindikin & L. R. Volevich, section 1.2.1 [3]:

1. Lemma 1.2.1 and proof, p. 14-15

G. H. Hardy, J. E. Littlewood & G. Pólya, section 2.5 [4]:

1. Lemma 2.5.2, p. 18

The proof of proposition 1.2.6 is based on the proof offered in Remark 2.17 [1], Lemma 1.2.1 [3] and of a result by Hardy, Littlewood and Pólya [4].

1.2.3 Order functions

As stated earlier, we can use order functions to define the order of a symbol given a relative weight γ. However, order functions serve a more general purpose. In this section, we will focus on convex order functions and strictly positive order functions, and show their relation to Newton polygons and to symbols. Firstly, we give a motivation for order functions. We use the following proposition.

Proposition 1.2.8. *Let N be a Newton polygon with vertices $(r_j, s_j) \in N_v$ and let $(r, s) \in [0, \infty)^2$. Then $(r, s) \in N$ if and only if*

$$s\gamma + r \leq \max_{j \in \{1,\dots,J+1\}} s_j\gamma + r_j, \quad 0 < \gamma < \infty.$$

Proof. For any point $(r, s) \in [0, \infty)^2$, we know

$$s\gamma + r = \langle (1, \gamma), (r, s) \rangle$$

for $\gamma > 0$. If we define

$$L_\gamma(c) := \{(r', s') \in [0, \infty)^2 : \langle (1, \gamma), (r', s') \rangle = c\}, \tag{1.12}$$

then we see that by the definition of inner products we have that $L_\gamma(c)$ is a line in the direction of the vector $(-\gamma_j, 1) = (1, \gamma_j)^\perp$.

We want to proof that $\max_{(p,q)\in N} \langle (1, \gamma), (p, q) \rangle = \max_{(p,q)\in N_v} \langle (1, \gamma), (p, q) \rangle$, or in other words, the maximum over the entire polygon is the same as the maximum over the vertices. Define the area $\Gamma(N)$ as the boundary of the Newton polygon N, so $\Gamma(N) := \bigcup_{j=1}^{J} [v_j v_{j+1}]$. We will use an argument of contradiction: suppose there exists an $(r, s) \in N \setminus N_v$ for which the maximum is attained. We can work in two cases:

(I) $(r, s) \in \Gamma(N) \setminus N_v$ s.t. $\langle (1, \gamma), (r, s) \rangle = \max_{(p,q)\in N} \langle (1, \gamma), (p, q) \rangle$. This means that $(r, s) \in [v_j v_{j+1}] \setminus \{v_j, v_j + 1\}$ for some $j \in \{1, \dots, J\}$. Depending on γ we work in three cases:

(1) $\gamma = \gamma_j$. In this case, as we know for definition 1.2.3(i), $L_\gamma(\langle (1, \gamma), (r, s) \rangle)$ will be parallel to the edge $[v_j v_{j+1}]$. This is true, since

$$v_{j+1} - v_j = (r_{j+1} - r_j, s_{j+1} - s_j) = (s_{j+1} - s_j) \left(\frac{r_{j+1} - r_j}{s_{j+1} - s_j}, 1 \right) = (s_{j+1} - s_j)(-\gamma_j, 1).$$

However, since $(r, s) \in [v_j, v_{j+1}]$, We get $[v_j v_{j+1}] \subseteq L_\gamma(\langle (1, \gamma), (r, s) \rangle)$, which in term means

$$\langle (1, \gamma), (r_j, s_j) \rangle = \langle (1, \gamma), (r_{j+1}, s_{j+1}) \rangle = \langle (1, \gamma), (r, s) \rangle,$$

so the maximum is also attained on $v_j = (r_j, s_j)$ and $v_{j+1} = (r_{j+1}, s_{j+1})$.

(2) $\gamma < \gamma_j$. As stated earlier, γ_j is such that $(1, \gamma_j)$ is perpendicular to the edge $[v_j v_{j+1}]$, or $(1, \gamma_j)^{\perp} = (-\gamma_j, 1)$ is parallel to $[v_j v_{j+1}]$. We use this to define

$$\begin{aligned} (r, s) &= v_j + c_1(-\gamma_j, 1), \\ v_{j+1} &= v_j + (s_{j+1} - s_j)(-\gamma_j, 1), \end{aligned}$$

where $c_1 \in (0, s_{j+1} - s_j)$. However, this allows us to find

$$\begin{aligned} \langle (1, \gamma), v_j \rangle &= \langle (1, \gamma), (r, s) - c_1(-\gamma_j, 1) \rangle \\ &= \langle (1, \gamma), (r, s) \rangle - \langle (1, \gamma), c_1(-\gamma_j, 1) \rangle \\ &= \max_{(p,q) \in N} \langle (1, \gamma), (p, q) \rangle - c_1(-\gamma_j + \gamma) \\ &= \max_{(p,q) \in N} \langle (1, \gamma), (p, q) \rangle + c_1(\gamma_j - \gamma) > \max_{(p,q) \in N} \langle (1, \gamma), (p, q) \rangle. \end{aligned}$$

Therefore, we have found a contradiction to (r, s) being the point on which the maximum over N is attained.

(3) $\gamma > \gamma_j$. Using the same proof as (2), we now state that v_{j+1} is a point for which the inner product is larger than the maximum.

$$\begin{aligned} \langle (1, \gamma), v_{j+1} \rangle &= \langle (1, \gamma), v_j + (s_{j+1} - s_j)(-\gamma_j, 1) \rangle \\ &= \langle (1, \gamma), (r, s) + (s_{j+1} - s_j - c_1)(-\gamma_j, 1) \rangle \\ &= \langle (1, \gamma), (r, s) \rangle + \langle (1, \gamma), (s_{j+1} - s_j - c_1)(-\gamma_j, 1) \rangle \\ &= \max_{(p,q) \in N} \langle (1, \gamma), (p, q) \rangle + ((s_{j+1} - s_j) - c_1)(\gamma - \gamma_j) \\ &> \max_{(p,q) \in N} \langle (1, \gamma), (p, q) \rangle. \end{aligned}$$

Therefore we again have a contradiction to (r, s) being the point on which the maximum over N is attained.

(II) $(r, s) \in N \setminus \Gamma(N)$ s.t. $\langle (1, \gamma), (r, s) \rangle = \max_{(p,q) \in N} \langle (1, \gamma), (p, q) \rangle$. In this case, the point lies in the interior of N. However, we can take a $c > 0$ such that the point $(r, s) + c(1, \gamma)$ lies on the boundary $\Gamma(N)$, and for this point, we have

$$\begin{aligned} \langle (1, \gamma), (r, s) + c(1, \gamma) \rangle &= \langle (1, \gamma), (r, s) \rangle + \langle (1, \gamma), (1, \gamma) \rangle \\ &= \max_{(p,q) \in N} \langle (1, \gamma), (p, q) \rangle + \|(1, \gamma)\|^2 > \max_{(p,q) \in N} \langle (1, \gamma), (p, q) \rangle. \end{aligned}$$

so this also contradicts (r, s) being the point on which the point is attained.

In conclusion, the maximum is always attained on one of the vertices, therefore we can write

$$\max_{(p,q) \in N} \langle (1, \gamma), (p, q) \rangle = \max_{(p,q) \in N_v} \langle (1, \gamma), (p, q) \rangle = \max_{j \in \{1, \ldots, J+1\}} \langle (1, \gamma), (r_j, s_j) \rangle.$$

Note that $v_0 = (0, 0)$ is not included in the maximum, simply because the inner product would be equal to 0. A demonstration of how this problem is shaped can be seen in figure 1.7. If $(r, s) \in N$, we get that for any $\gamma > 0$ we have

$$\begin{aligned} s\gamma + r = \langle (1, \gamma), (r, s) \rangle &\le \max_{(p,q) \in N} \langle (1, \gamma), (p, q) \rangle \\ &= \max_{j \in \{1, \ldots, J+1\}} \langle (1, \gamma), (r_j, s_j) \rangle \\ &= \max_{j \in \{1, \ldots, J+1\}} s_j \gamma + r_j. \end{aligned}$$

Conversely if $(r, s) \notin N$, then we know that there exists a constant $c \in (0,1)$ s.t. the point $c(r,s) \in \Gamma(N)$, meaning that for some $j \in \{1, \dots, J\}$, we have $c(r,s) \in [v_j v_{j+1}]$. Taking $\gamma = \gamma_j$ (except for the case where $\gamma_j \in \{0, \infty\}$), we can see that since $(1, \gamma_j)$ is perpendicular to $[v_j v_{j+1}]$, we can take

$$\begin{aligned} c(s\gamma_j + r) &= \langle (1, \gamma_j), c(r,s) \rangle \\ &= \langle (1, \gamma_j), (r_j, s_j) \rangle \\ &= \max_{k \in \{1, \dots, J+1\}} \langle (1, \gamma_j), (r_k, s_k) \rangle, \end{aligned}$$

meaning $s\gamma_j + r > c(s\gamma_j + 1) = \max_{k \in \{1,\dots,J+1\}} s_k \gamma_j + r_k$. However, we also have to cover the cases $\gamma = \infty$ and $\gamma = 0$.

(I) $\gamma_J = \infty$. This means N is not regular in space, i.e. $s_J = s_{J+1}$, and if $c(r,s) \in [v_J v_{J+1}]$ for some constant $c \in (0,1)$, then we must have $cs = s_J = s_{J+1}$, meaning $s > s_{J+1} = s_J$. If we take $\gamma > \max\{\gamma_{J-1}, \frac{|r_J - r|}{s - s_J}\}$, we can use $s > s_J$ to get

$$\begin{aligned} \gamma &> \frac{|r_J - r|}{s - s_J} \geq \frac{r_J - r}{s - s_J}, \\ (s - s_J)\gamma &> r_J - r, \\ s\gamma + r &> s_J \gamma + r_J = \max_{j \in \{1,\dots,J+1\}} s_j \gamma + r_j. \end{aligned}$$

so we know the inequality

(II) $\gamma_1 = 0$, so N is not regular in time, i.e. $r_1 = r_2$. If $c(r,s) \in [v_1 v_2]$ for some constant $c \in (0,1)$, then we must have $cr = r_1 = r_2$, so $r > r_1 = r_2$. Take $0 < \gamma < \min\{\gamma_2, \frac{r - r_2}{|s_2 - s|}\}$, then similar to (I) we get

$$\begin{aligned} \gamma &< \frac{r - r_2}{|s_2 - s|}, \\ (s_2 - s)\gamma &\leq |s_2 - s|\gamma < r - r_2, \\ \max_{j \in \{1,\dots,J+1\}} s_j \gamma + r_j &= s_2 \gamma + r_2 < s\gamma + r. \end{aligned}$$

So in all cases, the inequality $s\gamma + r \leq \max_{j \in \{1,\dots,J+1\}} s_j \gamma + r_j$ does not hold for all $\gamma > 0$.

$\square$

We can use this proposition by looking at functions of the form $\mu(\gamma) = \max_{(r_j, s_j) \in N_v} s_j \gamma + r_j$ to estimate points of the Newton polygon, using the furthest points of the polygon in terms of inner products with $(1, \gamma)$. This is very similar to determining the order of a symbol $P(\lambda, z)$ with respect to a relative weight γ by using $d_\gamma(P) = \max_{\ell \in I_P}(d_\gamma(\tau_\ell) + M_\ell \gamma + L_\ell)$, and we will in fact see that this is exactly the same for order functions related to symbols in lemma 1.2.11. First, we will give a formal definition of an order function, then we will see how this relates to previous work in definition 1.2.10.

Definition 1.2.9 ((Strictly positive) order functions). (i) A continuous and piece-wise linear function $\mu : [0, \infty) \to \mathbb{R}$ is called an *order function* if μ is either convex or concave. For a partition of $[0, \infty]$ of the form $0 = \gamma_0 \leq \gamma_1 < \dots < \gamma_M \leq \gamma_{M+1} = \infty$, an order function can be expressed as

$$\mu(\gamma) := \sum_{i=1}^{M+1} (m_i(\mu)\gamma + b_i(\mu)) \mathbf{1}_{[\gamma_{i-1}, \gamma_i)}(\gamma), \quad 0 \leq \gamma < \infty, \tag{1.13}$$

where $m_i(\mu), b_i(\mu) \in \mathbb{R}$.

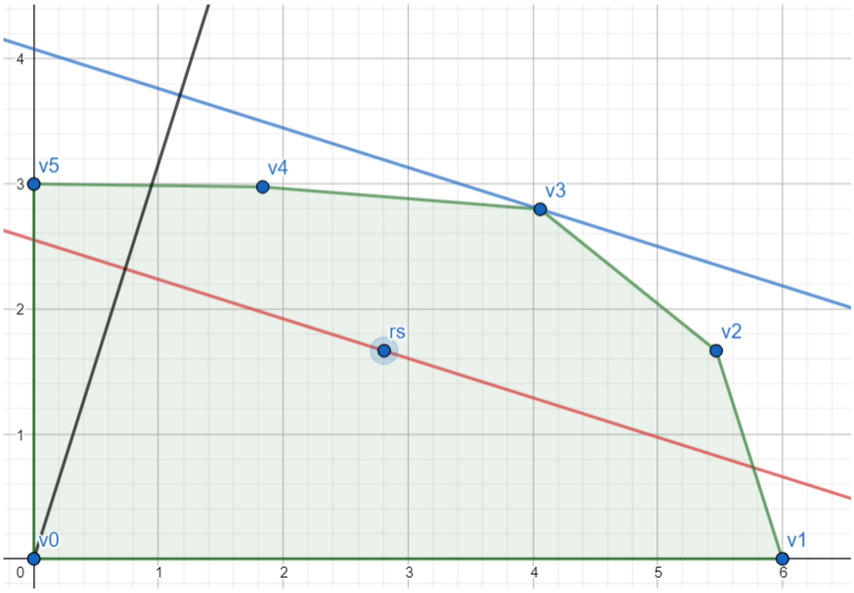

Figure 1.7: An example Newton polygon with $(1,\gamma)$ in black. For any point $(r,s) \in N$, the red perpendicular line $L_\gamma(\langle(1,\gamma),(r,s)\rangle)$ is closer to the origin than the cyan line $L_\gamma(\max_{j\in\{1,\dots,5\}}\langle(1,\gamma),(r_j,s_j)\rangle)$ through the furthest point v_3.

(ii) For a *convex* order function, we have that

$$m_{i-1}(\mu) \leq m_i(\mu), \quad b_{i-1}(\mu) \geq b_i(\mu), \quad i \in \{1,\dots,M+1\}.$$

This allows us to express $\mu(\gamma)$ as

$$\mu(\gamma) = \max_{i\in\{1,\dots,M+1\}} m_i(\mu)\gamma + b_i(\mu), \quad 0 \leq \gamma < \infty.$$

(iii) An order function μ is *strictly positive* if μ is convex, and increasing, and non-negative:

$$m_i(\mu) \geq 0, \quad b_i(\mu) \geq 0, \quad i \in \{1,\dots,M+1\}$$

An order function $\mu(\gamma)$ is a continuous and piece-wise linear function of γ that expresses the order of the symbol $P(\lambda,z)$, or of the Newton polygon N, for each relative weight γ. The following definition shows how this relates to the previous definition.

Definition 1.2.10. (i) Let N be a Newton polygon with vertices

$$N_v = \{(r_j,s_j) : j \in \{0,\dots,J+1\}\} \subseteq [0,\infty)^2,$$

with $J \in \mathbb{N}$, indexed as stated in definition 1.2.2. We define the *associated* order function μ of N as

$$\mu_N(\gamma) := \max_{j\in\{1,\dots,J+1\}} s_j\gamma + r_j, \quad \gamma \geq 0,$$

or for $0 = \gamma_0 \leq \gamma_1 < \cdots < \gamma_J \leq \gamma_{J+1} = \infty$ as defined in definition 1.2.3(i), set

$$\mu_N(\gamma) := \sum_{j=1}^{J+1} (s_j\gamma + r_j)\mathbf{1}_{[\gamma_{j-1},\gamma_j)}(\gamma), \quad \gamma \geq 0 \tag{1.14}$$

(ii) Conversely, let $\mu : [0, \infty) \to \mathbb{R}$ be a strictly positive order function with a partition of $[0, \infty]$ of the form $0 = \gamma_0 \leq \gamma_1 < \cdots < \gamma_J \leq \gamma_{J+1} = \infty$, where μ is defined as in equation (1.13). Using a finite set $\nu(\mu)$ defined as

$$\nu(\mu) := \bigcup_{i=1}^{M+1} \{(b_i(\mu), m_i(\mu))\} \subseteq [0, \infty)^2,$$

we define the *associated* Newton polygon N of μ as $N(\mu) := N(\nu(\mu))$.

(iii) For symbol $P \in S(L_t \times L_x)$, we define the *associated* order function μ of P as

$$\mu_P(\gamma) := \mu_{N(P)}.$$

(iv) For $\mu : [0, \infty) \to \mathbb{R}$ a strictly positive order function, we define the *associated* weight function W_μ as the weight function of the associated Newton polygon N, or in other words:

$$W_\mu := W_{N(\mu)}$$

Remark. It is possible to investigate μ that are not strictly positive, and define weight functions for these order functions. However, since we only use strictly positive order functions and convex order functions for our research of symbols $P(\lambda, z)$, it will not be done here.

We will now show that the definition of the γ-order $d_\gamma(P)$ and of the associated order function μ_P are well defined.

Lemma 1.2.11 (Characterization of symbol associated order functions)**.** *Let $P \in S(L_t \times L_x)$ with representation R_P as in equation (1.6) and Newton polygon $N(P)$ as defined in definition 1.2.1. Then we know that $d_\gamma(P)$ is a strictly positive order function of $\gamma \geq 0$. Moreover, we can find*

$$d_\gamma(P) = \mu_{N(P)}(\gamma) = \sum_{j=1}^{J+1} (s_j \gamma + r_j) \mathbf{1}_{[\gamma_{j-1}, \gamma_j)}(\gamma),$$

where γ_j is as in definition 1.2.3(i) for all $j \in \{0, \ldots, J+1\}$. For $\gamma = \infty$, we have

$$d_\infty(P) = s_{J+1}.$$

Proof. By definition 1.1.7, $d_\gamma(P) := \max_{\ell \in I_P}(d_\gamma(\tau_\ell) + M_\ell \gamma + L_\ell)$. For any $\gamma \geq 0$, it is easy to see that this function of γ is continuous (since it is a maximum of continuous functions of γ), piece-wise linear (whether $d_\gamma(\tau_\ell) = N_\ell$ or $d_\gamma(\tau_\ell) = \frac{N_\ell}{\rho}\gamma$, the terms within the maximum are linear functions), and convex (a maximum of linear functions, which are also convex functions, is again convex). This means $d_\gamma(P)$ is a convex order function, which is moreover strictly positive since $N_\ell, M_\ell, L_\ell \geq 0$. We can therefore show $d_\gamma(P) = \mu_{N(P)}$ by showing that the associated Newton polygon $N(d_\gamma(P))$ of the strictly positive order function $d_{\gamma(P)}$ is equal to $N(P)$, and using the 1-1 relation of order functions and Newton polygons.

First, by $d_\gamma(P)$ being a strictly positive order function, we can write it in the common order function notation: we use some finite $M \in \mathbb{N}$ to rewrite $I_P = \{1, \ldots, M+1\}$. However for every $\ell \in I_P$, we have $d_\gamma(P) = M_\ell \gamma + N_\ell + L_\ell$ or $d_\gamma(P) = (\frac{N_\ell}{\rho} + M_\ell)\gamma + L_\ell$, so we can rewrite our order function as

$$d_\gamma(P) = \max_{\ell \in \{1, \ldots, M+1, M+2, \ldots, 2M+1\}} m_\ell \gamma + b_\ell,$$

where

$$m_\ell = \begin{cases} M_\ell, & \ell \in \{1, \dots, M+1\}, \\ \frac{N_\ell}{\rho} + M_\ell, & \ell \in \{M+2, \dots, 2M+1\}, \end{cases}$$
$$b_\ell = \begin{cases} N_\ell + L_\ell, & \ell \in \{1, \dots, M+1\}, \\ L_\ell, & \ell \in \{M+2, \dots, 2M+1\}. \end{cases}$$

Then we can find $\nu(d_\gamma(P)) = \bigcup_{i=1}^{2M+1}\{(b_i, m_i)\}$ a finite set. However, we can easily see that

$$\nu(d_\gamma(P)) = \nu(P) := \bigcup_{\ell \in I_P} \{(N_\ell + L_\ell, M_\ell), (L_\ell, \frac{N_\ell}{\rho} + M_\ell)\}.$$

This implies $N(d_\gamma(P)) = N(\nu(d_\gamma(P))) = N(\nu(P)) = N(P)$, therefore $d_\gamma(P) = \mu_{N(P)}$.

For $d_\infty(P) = \max_{\ell \in I_P}(\frac{N}{\rho} + M_\ell)$, we see that the points $\mathfrak{v}_\ell := (L_\ell, \frac{N_\ell}{\rho} + M_\ell)$ that have $\ell \in I_\infty := \{\ell \in I_P : \frac{N}{\rho} + M_\ell = d_\infty(P)\}$ are actually points of the form $(L_\ell, d_\infty(P))$, so by the maximum in the definition of $d_\infty(P)$ we see that these points are the points with the highest y-coordinate of all points in $\nu(P)$. By the convexity of $N(\nu(P))$, there can be no point $(r, s) \in N(P)$ that has $s > d_\infty(P)$, so the point $(0, s_{J+1}) \in N_v$ has $s_{J+1} \leq d_\infty(P)$. However, $(0, s_{J+1})$ is a point with the highest y-coordinate of all points in $N(P)$: suppose for a contradiction that there was a point $(r, s) \in N(P)$ with $s > s_{J+1}$. By the definition of $N(P)$ this point would still need a projection onto the y-axis in the form of $(0, s)$, and by the convexity of $N(P)$ this point $(0, s)$ would actually have been the vertex $(0, s_{J+1})$, so $s_{J+1} = s$. This means $s_{J+1} \geq d_{\infty(P)}$, so we really have $s_{J+1} = d_\infty(P)$. □

We can use this last lemma as a better definition of $d_\gamma(P)$, since it allows us to relate points on the Newton polygon $N(P)$ to points $\mathfrak{u}_\ell, \mathfrak{v}_\ell \in \nu(P) \subseteq N(P)$ for $\ell \in I_\gamma$ from the definition 1.2.1, based on the partition of $[0, \infty]$ as defined in 1.2.3(i). The relation we will describe is as follows:

- For $\gamma = \gamma_j = \frac{r_j - r_{j+1}}{s_{j+1} - s_j}$, we see that $(1, \gamma)$ is perpendicular to the slope of the edge $[v_j v_{j+1}]$, meaning all points are equally far with respect to inner products with $(1, \gamma)$. All points $\mathfrak{u}_\ell, \mathfrak{v}_\ell \in \nu(P)$ for $\ell \in I_{\gamma_j}$ with this γ can be found in $[v_j v_{j+1}]$, as will be proven in lemma 1.2.12

- For $\gamma \in (\gamma_{j-1}, \gamma_j)$, we have that $(1, \gamma)$ is such that the vertex v_j is the furthest point in terms of inner products, so points $\mathfrak{u}_\ell, \mathfrak{v}_\ell \in \nu(P)$ for $\ell \in I_\gamma$ with this γ form the vertex v_j. This behaviour will be shown in lemma 1.2.13(ii) and (iii).

Lemma 1.2.12. *Let $P \in S(L_t \times L_x)$ with a Newton polygon $N(P)$.*

For any $j \in \{1, \dots, J\}$, we have

1. *$\mathfrak{u}_\ell := (N_\ell + L_\ell, M_\ell) \in [v_j v_{j+1}]$ for all $\ell \in I_{\gamma_j}$ whenever $\gamma_j \leq \rho$,*

2. *$\mathfrak{v}_\ell := (L_\ell, \frac{N_\ell}{\rho} + M_\ell) \in [v_j v_{j+1}]$ for all $\ell \in I_{\gamma_j}$ whenever $\gamma_j \geq \rho$.*

Proof.

First, fix $j \in \{1, \dots, r_x(N(P)) - 1\}$, meaning $\gamma_j \neq \infty$. We consider the line

$$L := L_{\gamma_j}(d_{\gamma_j}(P)) = \{(r, s) \in [0, \infty)^2 : s\gamma_j + r = d_{\gamma_j}(P)\},$$

where $L_\gamma(c)$ is defined as in equation (1.12). By definition 1.2.3 and lemma 1.2.11, we can prove that

$$\begin{aligned}
d_{\gamma_j}(P) &= r_{j+1} + s_{j+1}\gamma_j = r_{j+1} + s_{j+1}\frac{r_j - r_{j+1}}{s_{j+1} - s_j} \\
&= \frac{r_{j+1}(s_{j+1} - s_j) + s_{j+1}(r_j - r_{j+1})}{s_{j+1} - s_j} \\
&= \frac{s_{j+1}r_j - r_{j+1}s_j}{s_{j+1} - s_j} \\
&= \frac{s_{j+1}r_j - s_jr_j + r_js_j - r_{j+1}s_j}{s_{j+1} - s_j} \\
&= \frac{r_j(s_{j+1} - s_j) + s_j(r_j - r_{j+1})}{s_{j+1} - s_j} \\
&= r_j + s_j\frac{r_j - r_{j+1}}{s_{j+1} - s_j} = r_j + s_j\gamma_j.
\end{aligned}$$

If $\gamma_j \leq \rho$, then for any $\ell \in I_{\gamma_j}$ we can write

$$\begin{aligned}
&d_{\gamma_j}(\tau_\ell) + M_\ell\gamma_j + L_\ell = d_{\gamma_j}(P) \\
&= N_\ell + M_\ell\gamma_j + L_\ell = s_j\gamma_j + r_j = s_{j+1}\gamma_j + r_{j+1}
\end{aligned}$$

Therefore L is a line containing both $(r_j, s_j) = v_j$ and $(r_{j+1}, s_{j+1}) = v_{j+1}$ so $L \supseteq [v_jv_{j+1}]$, but we also have the point $\mathfrak{u}_\ell = (N_\ell + L_\ell, M_\ell) \in L$. We can now proof $\mathfrak{u}_\ell \in [v_jv_{j+1}]$ by contradiction: suppose $\mathfrak{u}_\ell \in L \setminus [v_jv_{j+1}]$. Then either $[v_jv_{j+1}] \subseteq [\mathfrak{u}_\ell v_{j+1}]$, or $[v_jv_{j+1}] \subseteq [v_j\mathfrak{u}_\ell]$, depending on which end of L the point $\mathfrak{u}_\ell$ lies. However, since $\mathfrak{u}_\ell \in N(P)$ which is a convex set, we know that either $v_j \in N(P)$ or $v_{j+1} \in N(P)$ is contained in between two other points of $N(P)$, which contradicts to v_j or v_{j+1} being a vertex of $N(P)$. Therefore, we must conclude $\mathfrak{u}_\ell \in [v_jv_{j+1}]$.

If $\gamma_j \geq \rho$, we can see the following for $\ell \in I_{\gamma_j}$:

$$\begin{aligned}
&d_{\gamma_j}(\tau_\ell) + M_\ell\gamma_j + L_\ell = d_{\gamma_j}(P) \\
&= \left(\frac{N_\ell}{\rho} + M_\ell\right)\gamma_j + L_\ell = s_j\gamma_j + r_j = s_{j+1}\gamma_j + r_{j+1}.
\end{aligned}$$

With the same definition of L, we have that v_j, v_{j+1} and $\mathfrak{v}_\ell$ are contained in the line $L \supseteq [v_jv_{j+1}]$, so by the same contradiction argument we find $\mathfrak{v}_\ell \in [v_jv_{j+1}]$.

Finally, we investigate the case $\gamma_J = \infty$ for space-irregular symbols. Since $\rho < \infty = \gamma_J$, we only consider the points $\mathfrak{v}_\ell$. Using lemma 1.2.11 combined with that $s_J = s_{J+1}$ for space-irregular symbols, we get that for any $\ell \in I_{\gamma_J} = I_\infty$ we have

$$\begin{aligned}
&\frac{N_\ell}{\rho} + M_\ell = d_\infty(P) \\
&= \frac{N_\ell}{\rho} + M_\ell = s_J = s_{J+1}.
\end{aligned}$$

Consider the set $L_\infty = \{(x, y) \in [0, \infty)^2 : y = d_\infty(P)\}$, so (r_J, s_J), $(0, s_{J+1})$ and $(L_\ell, \frac{N_\ell}{\rho} + M_\ell)$ are in this set L_∞. It is clear that L_∞ is a line, so it must contain $[v_Jv_{J+1}]$, therefore we can use the same contradiction argument as before to conclude $\mathfrak{v}_\ell \in [v_Jv_{J+1}]$, which really means $\frac{N_\ell}{\rho} + M_\ell = s_J$ and $L_\ell \in [0, r_J]$. $\square$

If $\gamma \in (\gamma_{j-1}, \gamma_j)$, we also need to think about the position of ρ in the partition of $\gamma \in [0, \infty]$. The first point of the next lemma will help us with this.

Lemma 1.2.13. *Let $P \in S(L_t \times L_x)$ with Newton polygon $N(P)$ and $r_t(N(P))$ and $r_x(N(P))$ as in definition 1.2.5.*

(i) If there exists a $k \in \{r_t(N(P)), \dots, r_x(N(P))\}$ with $\rho \in (\gamma_{k-1}, \gamma_k)$, then for any $\ell \in I_\rho$, we have $N_\ell = 0$, i.e. $\tau_\ell(\lambda, z) = constant$. This also implies $M_\ell = s_k$ and $L_\ell = r_k$.

(ii) Let $\gamma_- \in (\gamma_{j-1}, \gamma_j)$ for $j \in \{r_t(N(P)), \dots, r_x(N(P))\}$. Then we can express I_{γ_-} as a subset of I_{γ_j}:

$$I_{\gamma_-} = \{\ell \in I_{\gamma_j} : \mathbf{1}_{(0,\rho]}(\gamma_j)N_\ell + L_\ell = r_j\} \tag{1.15}$$

$$= \{\ell \in I_{\gamma_j} : \mathbf{1}_{(\rho,\infty]}(\gamma_j)\frac{N_\ell}{\rho} + M_\ell = s_j\}. \tag{1.16}$$

(iii) Let $\gamma_+ \in (\gamma_j, \gamma_{j+1})$ for $j \in \{1, \dots, r_x(N(P)) - 1\}$. Then we can express I_{γ_+} as a subset of I_{γ_j}:

$$I_{\gamma_+} = \{\ell \in I_{\gamma_j} : \mathbf{1}_{[0,\rho)}(\gamma_j)N_\ell + L_\ell = r_{j+1}\} \tag{1.17}$$

$$= \{\ell \in I_{\gamma_j} : \mathbf{1}_{[\rho,\infty)}(\gamma_j)\frac{N_\ell}{\rho} + M_\ell = s_{j+1}\}. \tag{1.18}$$

Proof. (i) Suppose $N_\ell \neq 0$, but $\rho \in (\gamma_{k-1}, \gamma_k)$. Then the order function $d_\gamma(P)$ has a difference in slope for $\gamma < \rho$ and $\gamma > \rho$ due to $d_\gamma(\tau_\ell) = N_\ell$ or $d_\gamma(\tau_\ell) = \frac{N_\ell}{\rho}\gamma$. This difference can only be explained by $\rho = \gamma_j$ for some $j \in \{0, \dots, J+1\}$. This however directly contradicts $\rho \in (\gamma_{k-1}, \gamma_k)$. Therefore, $N_\ell = 0$, so $d_\gamma(\tau_\ell) = 0$ for these $\ell \in I_\rho$, which means that

$$\begin{aligned} I_\rho &= \{\ell \in I_P : M_\ell \rho + L_\ell = d_\rho(P)\} \\ &= \{\ell \in I_P : M_\ell \rho + L_\ell = s_k \rho + r_k\}, \end{aligned}$$

so $M_\ell = s_k$ and $L_\ell = r_k$ for $\ell \in I_\rho$.

(ii) The definition $I_{\gamma_-} := \{\ell \in I_P : d_{\gamma_-}(\tau_\ell) + M_\ell\gamma_- + L_\ell = d_\gamma(P)\}$ can be defined using $d_\gamma(P)$ as in lemma 1.2.11. We split in the case $\gamma_j \in (0, \rho]$ and $\gamma_j \in (\rho, \infty)$.

If $\gamma_j \in (0, \rho]$, then by linearity and continuity in $[\gamma_{j-1}, \gamma_j]$ we get

$$\begin{aligned} I_{\gamma_-} &:= \{\ell \in I_P : d_{\gamma_-}(\tau_\ell) + M_\ell\gamma_- + L_\ell = d_{\gamma_-}(P)\} \\ &= \{\ell \in I_P : M_\ell\gamma_- + N_\ell + L_\ell = s_j\gamma_- + r_j\} \\ &= \{\ell \in I_P : M_\ell\gamma_j + N_\ell + L_\ell = s_j\gamma_j + r_j\}, \end{aligned}$$

and if $\gamma_j \in (\rho, \infty)$, then we can take $\gamma' \in (\max\{\gamma_{j-1}, \rho\}, \gamma_j)$, since if $\gamma_{j-1} < \rho$, we would have $N_\ell = 0$ by (i). Thus we know that by linearity and continuity in $[\gamma_{j-1}, \gamma_j]$ we get

$$\begin{aligned} I_{\gamma_-} &= \left\{\ell \in I_P : \frac{N_\ell}{\rho} + M_\ell\gamma_- + L_\ell = d_{\gamma_-}(P)\right\} \\ &= \left\{\ell \in I_P : \left(\frac{N_\ell}{\rho} + M_\ell\right)\gamma' + L_\ell = s_j\gamma' + r_j\right\} \\ &= \left\{\ell \in I_P : \left(\frac{N_\ell}{\rho} + M_\ell\right)\gamma_j + L_\ell = s_j\gamma_j + r_j\right\}. \end{aligned}$$

A special examination is needed for $j = r_x(N(P))$, since $\gamma_j = \infty$. We know for certain that $\rho < \gamma_j$, so we can take $\gamma' \in (\max\{\gamma_{j-1}, \rho\}, \infty)$:

$$\begin{aligned} I_{\gamma_-} &= \left\{ \ell \in I_P : \frac{N_\ell}{\rho} + M_\ell \gamma_- + L_\ell = d_{\gamma_-}(P) \right\} \\ &= \left\{ \ell \in I_P : \left(\frac{N_\ell}{\rho} + M_\ell \right) \gamma' + L_\ell = s_j \gamma' + r_j \right\} \\ &= \left\{ \ell \in I_P : \frac{N_\ell}{\rho} + M_\ell = s_j,\ L_\ell = r_j \right\} \\ &= \left\{ \ell \in I_P : \frac{N_\ell}{\rho} + M_\ell = d_\infty(P),\ L_\ell = r_j \right\}. \end{aligned}$$

So in all cases, the statement follows: we see that I_{γ_-} is a subset of I_{γ_j} where instead of anywhere on the edge $[v_j v_{j+1}]$, the points $\mathfrak{u}_\ell$ and $\mathfrak{v}_\ell$ for $\ell \in I_{\gamma_-}$ must be on the vertex $v_j = (r_j, s_j)$, meaning we need both $\mathbf{1}_{(0,\rho]}(\gamma_j) N_\ell + L_\ell = r_j$ and $\mathbf{1}_{(\rho,\infty]}(\gamma_j) \frac{N_\ell}{\rho} + M_\ell = s_j$ to hold.

(iii) We follow the same steps as in (ii).

In the case $\gamma_j \in [0, \rho)$, we can take $\gamma' \in (\gamma_j, \min\{\gamma_{j+1}, \rho\})$, since again $\rho < \gamma_{j+1}$ would mean that $N_\ell = 0$ by (i). We can use lemma 1.2.11 and linearity again.

$$\begin{aligned} I_{\gamma_+} &= \{ \ell \in I_P : N_\ell + M_\ell \gamma_+ + L_\ell = d_{\gamma_+}(P) \} \\ &= \left\{ \ell \in I_P : M_\ell \gamma' + N_\ell + L_\ell = s_{j+1} \gamma' + r_{j+1} \right\} \\ &= \{ \ell \in I_P : M_\ell \gamma_j + N_\ell + L_\ell = s_{j+1} \gamma_j + r_{j+1} \}. \end{aligned}$$

In the case $\gamma_j \in [\rho, \infty)$, we can see

$$\begin{aligned} I_{\gamma_+} &= \left\{ \ell \in I_P : \frac{N_\ell}{\rho} \gamma_+ + M_\ell \gamma_+ + L_\ell = d_{\gamma_+}(P) \right\} \\ &= \left\{ \ell \in I_P : \left(\frac{N_\ell}{\rho} + M_\ell \right) \gamma_+ + L_\ell = s_{j+1} \gamma_+ + r_{j+1} \right\} \\ &= \left\{ \ell \in I_P : \left(\frac{N_\ell}{\rho} + M_\ell \right) \gamma_j + L_\ell = s_{j+1} \gamma_j + r_{j+1} \right\}. \end{aligned}$$

Therefore, we see that I_{γ_+} is a subset of I_{γ_j} where both $\mathbf{1}_{[0,\rho)}(\gamma_j) N_\ell + L_\ell = r_{j+1}$ and $\mathbf{1}_{[\rho,\infty)}(\gamma_j) \frac{N_\ell}{\rho} + M_\ell = s_{j+1}$ hold. Again, this means that the points $\mathfrak{u}_\ell$ and $\mathfrak{v}_\ell$ with $\ell \in I_{\gamma_+}$ are found on the vertex $v_{j+1} = (r_{j+1}, s_{j+1})$ instead of anywhere on $[v_j v_{j+1}]$.

□

Both lemma 1.2.12 and 1.2.13 will be used in chapter 2. We define a notion of upper and lower order functions. These will be used for defining a new notion of parameter-ellipticity and parabolicity in section 1.3.

Definition 1.2.14 (Upper and lower order functions). (i) μ is a *lower order function* of $P(\lambda, z) \in H_P(\mathring{L}_t \times \mathring{L}_x)$ if for some constant $C_1 > 0$ and bound $\lambda_0 \geq 0$, we have for all $(\lambda, z) \in \mathring{L}_t \times \mathring{L}_x$ with $|\lambda| \geq \lambda_0$ that

$$|P(\lambda, z)| \geq C_1 W_\mu(\lambda, z).$$

(ii) μ is an *upper order function* of $P(\lambda, z) \in H_P(\mathring{L}_t \times \mathring{L}_x)$ if for some constant $C_2 > 0$ and bound $\lambda_0 \geq 0$, we have for all $(\lambda, z) \in \mathring{L}_t \times \mathring{L}_x$ with $|\lambda| \geq \lambda_0$ that

$$|P(\lambda, z)| \leq C_2 W_\mu(\lambda, z).$$

(iii) μ can also be both: for $C_1, C_2 > 0$ constants and $\lambda_0 \geq 0$ a bound, we have for all $(\lambda, z) \in \mathring{L}_t \times \mathring{L}_x$ with $|\lambda| \geq \lambda_0$ that

$$C_1 W_\mu(\lambda, z) \leq |P(\lambda, z)| \leq C_2 W_\mu(\lambda, z).$$

These bounds from above and below are important, these are exactly the sort of bounds we would want from parabolicity. In fact, the existence of these bounds is exactly how we will define N-parabolicity. The bound from above is needed, but we will prove that all symbols in $S(L_t \times L_x)$ will have a trivial bound from above using order functions.

Lemma 1.2.15. *Let $P \in S(L_t \times L_x)$, and let μ be a strictly positive order function that has*

$$d_\gamma(P) \leq \mu(\gamma), \quad \gamma \geq 0.$$

Then μ is an upper order function of P. Moreover, we can find a constant $C_2 > 0$ so that we have

$$|P(\lambda, z)| \leq C_2 W_P(\lambda, z), \quad (\lambda, z) \in L_t \times L_x. \tag{1.19}$$

Proof. Begin by looking at the Newton polygon $N(\mu)$. Since $d_\gamma(P) \leq \mu$, we have that $(r, s) \in N(P)$ implies $s\gamma + r \leq d_\gamma \leq \mu$ by proposition 1.2.8, and again using this proposition we get that $(r, s) \in N(\mu)$. Therefore $N(P) \subseteq N(\mu)$.

With this in mind, we investigate $|P(\lambda, z)|$. By lemma 1.1.3, it is possible to find constants $C_2', C_2'' > 0$ such that for $\ell \in I_P$ and $(\lambda, z) \in L_t \times L_x$ we have

$$\begin{aligned} |\tau_\ell(\lambda, z)| &\leq C_2'(|\lambda|^{\frac{N_\ell}{\rho}} + |z|^{N_\ell}), \\ |\phi_\ell(\lambda)| &\leq C_2'|\lambda|^{M_\ell}, \\ |\psi_\ell(z)| &\leq C_2''|z|^{L_\ell}. \end{aligned}$$

So taking $C' = \max\{C_2', C_2''\}$, all three hold at the same time. We can then write

$$\begin{aligned} |P(\lambda, z)| &\leq \sum_{\ell \in I_P} |\tau_\ell(\lambda, z)|\,|\phi_\ell(\lambda)|\,|\psi_\ell(z)| \\ &\leq C' \sum_{\ell \in I_P} \left(|\lambda|^{\frac{N_\ell}{\rho}} + |z|^{N_\ell}\right) |\lambda|^{M_\ell}|z|^{L_\ell} \\ &= C' \sum_{\ell \in I_P} \left(|\lambda|^{\frac{N_\ell}{\rho}+M_\ell}|z|^{L_\ell} + |\lambda|^{M_\ell}|z|^{N_\ell+L_\ell}\right) \end{aligned}$$

However, since the points $\mathfrak{u}_\ell = (N_\ell + L_\ell, M_\ell)$ and $\mathfrak{v}_\ell = (L_\ell, \frac{N_\ell}{\rho} + M_\ell)$ are points in the finite set $\nu(P)$, we can use $\nu(P) \subseteq N(P) \subseteq N(\mu)$ to see $\mathfrak{u}_\ell, \mathfrak{v}_\ell \in N(\mu)$. This allows us to use proposition 1.2.6 on the convex set $N(\mu)$ to get

$$\begin{aligned} |P(\lambda, z)| &\leq C' \sum_{\ell \in I_P} \left(|\lambda|^{\frac{N_\ell}{\rho}+M_\ell}|z|^{L_\ell} + |\lambda|^{M_\ell}|z|^{N_\ell+L_\ell}\right) \\ &\leq C' \sum_{\ell \in I_P} \left(2 \cdot W_\mu(\lambda, z)\right). \end{aligned}$$

Defining $\#I_P$ as the size of I_P, we will define $C_2 := 2C' \cdot \#I$ so that

$$|P(\lambda, z)| \leq 2C' \cdot \#I_P \cdot W_\mu(\lambda, z) = C_2 W_\mu.$$

If we use $\mu = d_\gamma(P)$, we find by lemma 1.2.11 that

$$|P(\lambda, z)| \leq C_2 W_{d_\gamma(P)} = C_2 W_{\mu_{N(P)}} = C_2 W_{N(P)} = C_2 W_P.$$

$\square$

This equation (1.19) is used as the upper bound when proving a symbol is N-parameter-elliptic, as seen in section 1.3.

Examples

- The order function belonging to the symbol $Q(\lambda, z)$ belonging to previous examples is:

$$\mu_Q(\gamma) = d_\gamma(Q) = \begin{cases} 5 & 0 \le \gamma < \frac{3}{2} \\ 2 + 2\gamma & \frac{3}{2} \le \gamma < 2 \\ 3\gamma & 2 \le \gamma \end{cases}$$

  It is easy to see this order function is strictly positive. We can also check lemma's 1.2.12 and lemma 1.2.13 in figure 1.6:

  1. If $0 < \gamma < \frac{3}{2} = \gamma_1$, $d_\gamma(P) = L_2 = 5$ meaning $I_\gamma = \{2\}$, and the point $\mathfrak{u}_2 = \mathfrak{v}_2 = (0, 5)$ is the vertex v_1.
  2. If $\gamma = \gamma_1$, then since $\frac{3}{2} > \rho = 1$, we have $d_\gamma(P) = L_2 = \frac{N_3}{\rho}\gamma + L_3$ meaning $I_\gamma = \{2, 3\}$, and the points $(0, 5)$ and $\mathfrak{v}_3 = (2, 2)$ are found on the edge $[v_1 v_2]$.
  3. If $\frac{3}{2} < \gamma < 2 = \gamma_2$, we have $I_\gamma = \{3\}$ so the point $(2, 2)$ forms the vertex v_3.
  4. If $\gamma = \gamma_2$, then $d_\gamma(P) = \frac{N_3}{\rho}\gamma + L_3 = M_1\gamma$ meaning $I_\gamma = \{1, 3\}$, and the points $(2, 2)$ and $\mathfrak{u}_1 = \mathfrak{v}_1 = (0, 3)$ are found on the edge $[v_2 v_3]$.
  5. Lastly $\gamma > 2$ means $I_\gamma = \{1\}$ and we find $(0, 3)$ as the vertex v_3.

- The Newton polygon N_1 in figure 1.8 with the vertices $v_0 = (0, 0)$, $v_1 = (4, 0)$, $v_2 = (3, 2)$, $v_3 = (2, 3)$, $v_0 = (\frac{1}{2}, 4)$ and $v_5 = (0, 4)$ has an associated order function, using definition 1.2.10(i), equal to $\mu_1(\gamma) = \max_{i \in \{0,\ldots,3\}} m_i(\mu_1)\gamma + b_i(\mu_1)$ with $m_i(\mu_1) \in \{0, 2, 3, 4\}$ and $b_i(\mu_1) \in \{4, 3, 2, \frac{1}{2}\}$. This order function can also be defined by partitioning the relative weights $\gamma \in [0, \infty]$ in the partition as defined in 1.2.3(i): $\gamma_0 = 0$, $\gamma_1 = \frac{1}{2}$, $\gamma_2 = 1$, $\gamma_3 = \frac{\frac{3}{2}}{1} = \frac{3}{2}$, $\gamma_4 = \infty$ and $\gamma_5 = \infty$. Then the associated order function is.

$$\mu_1(\gamma) = \begin{cases} 4 & 0 \le \gamma < \frac{1}{2} \\ 3 + 2\gamma & \frac{1}{2} \le \gamma < 1 \\ 2 + 3\gamma & 1 \le \gamma < \frac{3}{2} \\ \frac{1}{2} + 4\gamma & \frac{3}{2} \le \gamma \end{cases}.$$

  Since $\gamma_4 = \infty$, we know this order function is not regular in space. Another way to see that is to see that the last entry still has a term $\frac{1}{2}$.

- We can also construct a Newton polygon from an order function. Consider the strictly positive order function defined by $\mu_2(\gamma) = \max_{j \in \{0,\ldots,3\}} m_j(\mu_2)\gamma + b_j(\mu_2)$ with $m_j(\mu_2) \in \{1, 3, 4, 5\}$ and $b_j(\mu_2) \in \{4, 3, 2, 0\}$, we find the points $(4, 1)$, $(3, 3)$, $(2, 4)$ and $(5, 0)$, and adding the origin $(0, 0)$ and the projection $(4, 0)$, and lastly taking the convex hull, we get the Newton polygon N_2 in figure 1.9.

References

R. Denk & M. Kaip, section 2.1b [1]:

1. Remark 2.20, p. 80-81
2. Definition and Remark 2.21, p. 81

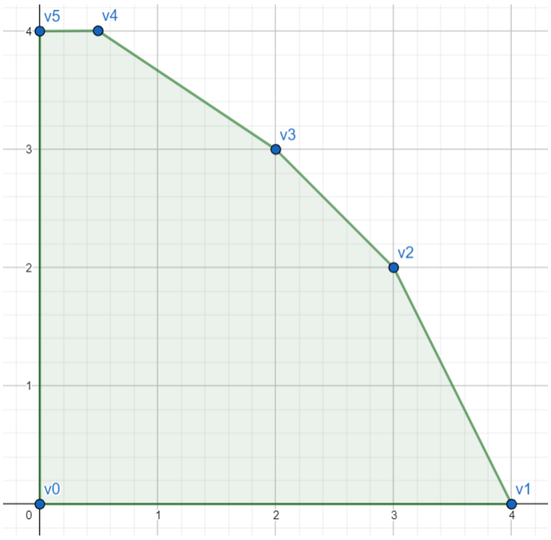

Figure 1.8: A Newton polygon N_1. This polygon is not regular in space.

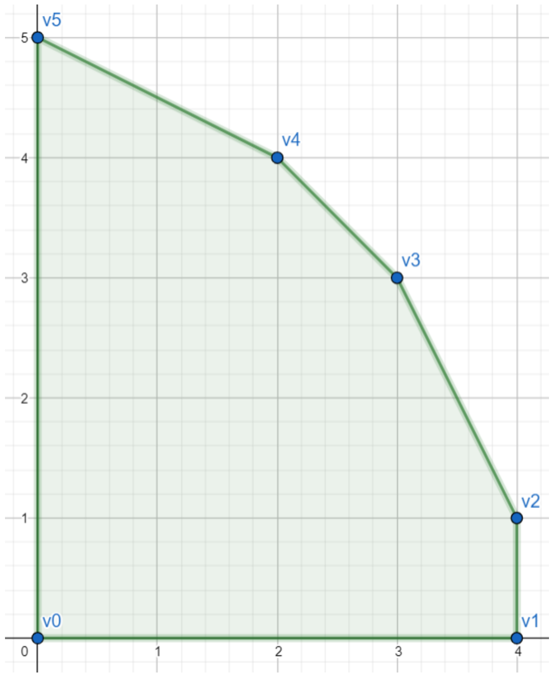

Figure 1.9: The Newton polygon N_2 belonging to μ_2. This polygon is not regular in time.

3. Definition and Remark 2.22, p. 82
4. Definition 2.24, p. 83
5. Remark 2.23, p. 82
6. Definition 2.27, p. 84
7. Lemma 2.30 and proof, p. 85

The proof of proposition 1.2.8 is based on Remark 2.20 [1]. The statements of lemmas 1.2.11 and 1.2.13 are based on Remark 2.23 [1], and the proofs are a result of my work. The statement of lemma 1.2.12 is based on a remark in the proof of Lemma 2.53 and Theorem 2.56 [1], and the proof is a result of my work.

1.3 N-parameter-ellipticity and N-parabolicity

For the quasi-homogeneous symbols $P(\lambda, z)$ we would define parameter-ellipticity as $P(\lambda, z) \neq 0$ for any $(\lambda, z) \in (L_t \times L_x) \setminus \{0, 0\}$, since this would allows us to make estimates from below on $|P(\lambda, z)|$. However, the symbols we are currently investigating are not necessarily (quasi-)homogeneous, and therefore this definition is not sufficient. However, we can use the tools we have defined, the Newton polygon, the weight function and order function, to help us make a different notion of parameter-ellipticity and parabolicity that also allows us to make these estimates from below, while still being well-defined for inhomogeneous symbols $P(\lambda, z)$.

Definition 1.3.1 (N-parameter-elliptic/N-parabolic). (i) Let $P \in S(L_t \times L_x)$ with $\mu_P(\gamma) = d_\gamma(P)$ as the order function. Let $W_P = W_{\mu_P}$ the weight function associated to P. P is then *N-parameter-elliptic* in $\mathring{L}_t \times \mathring{L}_x$ if μ_P is not only an upper order function (see equation (1.19)), but also a lower order function: there exist $C_1, C_2 > 0$ and a bound $\lambda_0 \geq 0$ such that for all $(\lambda, z) \in \mathring{L}_t \times \mathring{L}_x$ with $|\lambda| \geq \lambda_0$, we have that

$$C_1 W_P(\lambda, z) \leq |P(\lambda, z)| \leq C_2 W_P(\lambda, z)$$

Define $S_N(L_t \times L_x) \subseteq S(L_t \times L_x)$ as the subclass of $S(L_t \times L_x)$ containing all N-parameter-elliptic symbols P.

(ii) The symbol P is instead called N-parameter-elliptic of *angle* θ if $L_t \times L_x = \overline{S}_\theta \times \overline{\Sigma}_\delta^n$ for some $\delta > 0$ and $n \in \mathbb{N}$ in the previous definition. This means that P is N-parameter-elliptic in $S_\theta \times \Sigma_\delta^n$.

(iii) The symbol P is instead called *N-parabolic* if it is N-parameter-elliptic of angle θ where $\theta \in (\frac{1}{2}\pi, \pi)$. This means that $\lambda \in S_\theta$ can take on values with negative real parts.

For $P \in H_P(\mathring{L}_t \times L_x)$, the notion $d_\gamma(P) = \mu_P$ is not well-defined, so we instead define N-parameter-ellipticity as the existence of a strictly positive order function μ that is both an upper and lower order function of P. Points (ii) and (iii) are defined in the exact same manner.

A very important equivalent definition of N-parameter-ellipticity is that the γ-principal parts of $P(\lambda, z)$ are non-vanishing on $(L_t \setminus \{0\}) \times (L_x \setminus \{0\})$. This is the main result of this thesis.

Corollary 1.3.2 (Characterization of N-parameter-elliptic symbols)**.** *The symbol class $S_N(L_t \times L_x)$ is equivalent to the class of all symbols $P \in S(L_t \times L_x)$ that have non-vanishing γ-principal parts, i.e. that satisfy*

$$\pi_\gamma P(\lambda, z) \neq 0, \qquad \pi_\infty P(\lambda, 0) \neq 0$$

for all $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$, and $\gamma \in (0, \infty]$.

Proof. Proposition 1.3.3 proves that N-parameter-elliptic symbols have non-vanishing principal parts, and theorem 2.5.1 proves the other way around. For the proofs of these results, see chapter 2, and the next proposition. □

This equivalent definition is very important in proving whether or not a symbol is N-parameter-elliptic/N-parabolic, since it is much easier to check whether the principal parts are not equal to zero on any of the coordinates $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ than it is to find a bound $C_1 W_P(\lambda, z)$ from below. We can first check whether an N-parameter-elliptic function really has non-vanishing principal parts.

Proposition 1.3.3. *Let $R_P(\lambda, z) = \sum_{\ell \in I} \tau_\ell(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z)$, $(\lambda, z) \in L_t \times L_x$, be a regular representation of the symbol $P \in S_N(L_t \times L_x)$. Then we have*

$$\pi_\gamma P(\lambda, z) \neq 0$$

for all $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$, and $\gamma \in (0, \infty)$. For the ∞-principal part we get

$$\pi_\infty P(\lambda, z) \neq 0$$

for all $(\lambda, z) \in (L_t \setminus \{0\}) \times L_x$.

Proof. Since $P(\lambda, z)$ is N-parameter-elliptic, we use the definition 1.3.1: for some constants $C_1, C_2 > 0$ and a bound $\lambda_0 \geq 0$, we have for $(\lambda, z) \in L_t \times L_x$ with $|\lambda| \geq \lambda_0$ that

$$C_1 W_P(\lambda, z) \leq |P(\lambda, z)| \leq C_2 W_P(\lambda, z).$$

We will use this to create a lower bound of $|\pi_\gamma P(\lambda, z)|$ and $|\pi_\infty P(\lambda, z)|$:

$$\begin{aligned} |\pi_\gamma P(\lambda, z)| = \lim_{\eta \to \infty} \eta^{-d_\gamma(P)} |P(\eta^\gamma \lambda, \eta z)| &\geq C_1 \lim_{\eta \to \infty} \eta^{-d_\gamma(P)} W_P(\eta^\gamma \lambda, \eta z) \\ &= C_1 \pi_\gamma W_P(\lambda, z), \quad |\lambda| \geq \lambda_0, \\ |\pi_\infty P(\lambda, z)| = \lim_{\eta \to \infty} \eta^{-d_\infty(P)} |P(\eta \lambda, z)| &\geq C_1 \lim_{\eta \to \infty} \eta^{-d_\infty(P)} W_P(\eta \lambda, z) \\ &= C_1 \pi_\infty W_P(\lambda, z), \quad |\lambda| \geq \lambda_0. \end{aligned}$$

However, these bounds also hold for $0 < |\lambda| < \lambda_0$ because of the (quasi-)homogeneity of the principal parts that we have found in lemma 1.1.8, using an argument of scaling: take $|\lambda| \geq \lambda_0$. Taking $\eta > 0$ very small, we can multiply above equations by $\eta^{d_\gamma(P)}$ and $\eta^{d_\infty(P)}$ respectively.

$$\eta^{d_\gamma}|\pi_\gamma P(\lambda, z)| \geq C_1 \eta^{d_\gamma} \pi_\gamma W_P(\lambda, z),$$
$$\eta^{d_\infty}|\pi_\infty P(\lambda, z)| \geq C_1 \eta^{d_\infty} \pi_\infty W_P(\lambda, z).$$

Since $W_P(\lambda, z)$ is a symbol with $d_\gamma(W_P) = d_\gamma(W_{\mu_P}) = d_\gamma(W_{d_\gamma(P)}) = d_\gamma(P)$ for $\gamma \in (0, \infty]$, we can use the (quasi-)homogeneity from lemma 1.1.8 and apply it to both sides of the inequalities:

$$|\pi_\gamma P(\eta^\gamma \lambda, \eta z)| \geq C_1 \pi_\gamma W_P(\eta^\gamma \lambda, \eta z),$$
$$|\pi_\infty P(\eta\lambda, z)| \geq C_1 \pi_\infty W_P(\eta\lambda, z).$$

This means these inequalities also hold for $0 < \eta^\gamma|\lambda| < \lambda_0$ and $0 < \eta|\lambda| < \lambda_0$, so we must have

$$|\pi_\gamma P(\lambda, z)| \geq C_1 W_P(\lambda, z), \quad (\lambda, z) \in (L_t \setminus \{0\}) \times L_x,$$
$$|\pi_\infty P(\lambda, z)| \geq C_1 W_P(\lambda, z), \quad (\lambda, z) \in (L_t \setminus \{0\}) \times L_x.$$

We will now show, using the Newton polygon $N(P)$, that these principal parts of weight functions can be non-zero for correctly chosen λ and z. Firstly for $\gamma \in (0, \infty)$, we can divide the problem in two cases:

(I) $\gamma = \gamma_j$ for $j \in \{r_t(N(P)), \ldots, r_x(N(P)) - 1\}$ (meaning $0 < \gamma_j < \infty$). Since $d_{\gamma_j}(P) = s_j\gamma_j + r_j = s_{j+1}\gamma_j + r_{j+1}$, as seen in proposition 1.2.11, we know that the terms that define the γ_j-order of $W_P(\lambda, z) = 1 + \sum_{j=1}^{J+1} |\lambda|^{s_j}|z|^{r_j}$ are the terms $|\lambda|^{s_j}|z|^{r_j}$ and $|\lambda|^{s_{j+1}}|z|^{r_{j+1}}$. Therefore
$$\pi_{\gamma_j} W_P(\lambda, z) = |\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}}.$$

(II) $\gamma \in (\gamma_{j-1}, \gamma_j)$ for $j \in \{r_t(N(P)), \ldots, r_x(N(P))\}$. Here, we see $d_\gamma(P) = s_j\gamma + r_j$ from proposition 1.2.11, so only the term $|\lambda|^{s_j}|z|^{r_j}$ defines the γ-order of $W_P(\lambda, z)$. Therefore
$$\pi_\gamma W_P(\lambda, z) = |\lambda|^{s_j}|z|^{r_j}.$$

In either case, if we have $\lambda \neq 0$ and $z \neq 0$, we get

$$|\pi_\gamma P(\lambda, z)| \geq C_1 \pi_\gamma W_P(\lambda, z) > 0$$

Which shows $\pi_\gamma P(\lambda, z) \neq 0$ for all $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$.

For $\gamma = \infty$, we must also split in two cases:

(I) $N(P)$ is regular in space, then $\gamma = \gamma_{J+1} = \infty$ has $d_\gamma(P) = d_\infty(P) = s_{J+1}$ so the ∞-principal part is $\pi_\infty W_P(\lambda, z) = |\lambda|^{s_{J+1}}$.

(II) $N(P)$ is not regular in space, then $\gamma = \gamma_J = \gamma_{J+1} = \infty$, meaning $d_\gamma(P) = d_\infty(P) = s_{J+1} = s_J$. Since the ∞-order is defined by the terms $|\lambda|^{s_J}|z|^{r_J}$ and $|\lambda|^{s_{J+1}}$, we find the ∞-principal part $\pi_\infty W_P(\lambda, z) = |\lambda|^{s_J}|z|^{r_J} + |\lambda|^{s_{J+1}} = |\lambda|^{s_{J+1}}(1 + |z|^{r_J})$.

In both cases, if $\lambda \neq 0$, we get

$$|\pi_\infty P(\lambda, z)| \geq C_1 \pi_\infty W_P(\lambda, z) > 0$$

So $\pi_\infty P(\lambda, z) \neq 0$ for $(\lambda, z) \in (L_t \setminus \{0\}) \times L_x$. $\square$

Remark. The above statement is also true for $\gamma = 0$: we can use the same bound $|\pi_0 P(\lambda, z)| \geq C_1 \pi_0 W_P(\lambda, z)$, use scaling with lemma 1.1.8, then split in two cases:

(I) If $N(P)$ is regular in time, then $\gamma = \gamma_0 = 0$ has $d_0(P) = s_1 \cdot 0 + r_1 = r_1$ which implies $\pi_0 W_P(\lambda, z) = |z|^{r_1}$.

(II) If $N(P)$ is not regular in time, then $\gamma = \gamma_0 = \gamma_1 = 0$ has $d_0(P) = r_1 = r_2$. Sice the 0-order is defined by the terms $|z|^{r_1}$ and $|\lambda|^{s_2}|z|^{r_2}$, we get $\pi_0 W_P(\lambda, z) = |z|^{r_1} + |\lambda|^{s_2}|z|^{r_2} = |z|^{r_1}(1 + |\lambda|^{s_2})$.

In both cases $z \neq 0$ will mean that $|\pi_0 P(\lambda, z)| > 0$.

However the case $\gamma = 0$ will not be needed, as in theorem 2.5.1 we will prove that the condition $\pi_\gamma P(\lambda, z) \neq 0$ for just $\gamma \in (0, \infty]$ is enough to prove that there exists a $\lambda_0 \geq 0$ for which we can prove that P is N-parameter-elliptic. The reason $\gamma = 0$ is not needed comes from the bound $|\lambda| \geq \lambda_0$ and the partition from lemma 2.2.4(iv), where we see that we can partition the $[0, \infty)^2$ plane without using an index j for which $\gamma_j = 0$. Since $\gamma = 0$ is not needed, we choose to prove an equivalence that does not require us to check $\pi_0 P(\lambda, z)$, as seen in corollary 1.3.2 where we only have to check $\gamma \in (0, \infty]$.

Examples

- The symbol $P(\lambda, z) = \lambda + |z|_-^2$ belonging to the Heat equation is a classic example of a parabolic symbol. We can also show it is N-parameter-elliptic/N-parabolic in $S_\theta \times \Sigma_\delta^n$ for $\theta \in (0, \pi)$ and $\delta \in (0, \frac{1}{2}\pi)$ as long as $\pi > \theta + 2\delta$.

  Using definition 1.3.1, we need to prove $\mu_P(\gamma) = d_\gamma(P)$ is an upper and lower order function. Since P is ρ-homogeneous, we can use lemma 1.1.3 here to find $C_1, C_2 > 0$ s.t.

$$C_1(|\lambda| + |z|^2) \leq |P(\lambda, z)| \leq C_2(|\lambda| + |z|^2).$$

  Since the weight function is $W_P(\lambda, z) = 1 + |\lambda| + |z|^2$, we can add the term C_2 to bound on the right to see $|P(\lambda, z)| \leq C_2(1 + |\lambda| + |z|^2) = C_2 W_P$. To get a bound from below, take $\lambda_0 = 1$ so that for $|\lambda| \geq 1$ and $C_1 > 0$ as above, we can use $1 \leq |\lambda| \leq |\lambda| + |z|^2$ to get:

$$\begin{aligned} \frac{C_1}{2}(1 + |\lambda| + |z|^2) \leq \frac{C_1}{2}(2|\lambda| + 2|z|^2) &= C_1(|\lambda| + |z|^2) \\ &\leq |P(\lambda, z)|. \end{aligned}$$

  Then we clearly see that $P(\lambda, z)$ is N-parameter-elliptic.

  Using corollary 1.3.2 we can instead check whether $\pi_\gamma P(\lambda, z) \neq 0$ and $\pi_\infty P(\lambda, 0) \neq 0$ for $\lambda \in \overline{S_\theta} \setminus \{0\}$, $z \in \overline{\Sigma_\delta^n} \setminus \{0\}$ and $\gamma \in (0, \infty]$:

$$\pi_\gamma P(\lambda, z) = \begin{cases} |z|_-^2, & \gamma < 2, \\ \lambda + |z|_-^2, & \gamma = 2, \\ \lambda, & \gamma > 2. \end{cases}$$

  For $\gamma \neq 2$, it is easy to see that $\pi_\gamma P(\lambda, z) \neq 0$ and also $\pi_\infty P(\lambda, 0) \neq 0$. For $\gamma = 2$, we could get an equality to zero if $\lambda = -|z|_-^2 = \sum_{i=1}^n z_i^2$. We can check that this does not happen on $(\overline{S_\theta} \setminus \{0\}) \times (\overline{\Sigma_\delta^n} \setminus \{0\})$ with chosen θ and δ: λ takes values in $\mathbb{C}$ with arguments in $(-\theta, \theta)$, and the z_i^2 take values in $\mathbb{C}$ with arguments in $(-\pi, -\pi + 2\delta) \cup (\pi - 2\delta, \pi]$. If $\pi > \theta + 2\delta$, these arguments never overlap, except for $0 \in \mathbb{C}$, as also seen in figure 1.10. Therefore $\lambda \neq \sum_{i=1}^n z_i^2$, meaning $\pi_2 P(\lambda, z) \neq 0$. Therefore, this also proves that $P(\lambda, z)$ is N-parameter-elliptic in $S_\theta \times \Sigma_\delta^n$ for $\theta \in (0, \pi)$ and $\delta \in (0, \frac{1}{2}\pi)$ as long as $\pi > \theta + 2\delta$. We can denote this as $P \in S_N(S_\theta \times \Sigma_\delta^n)$.

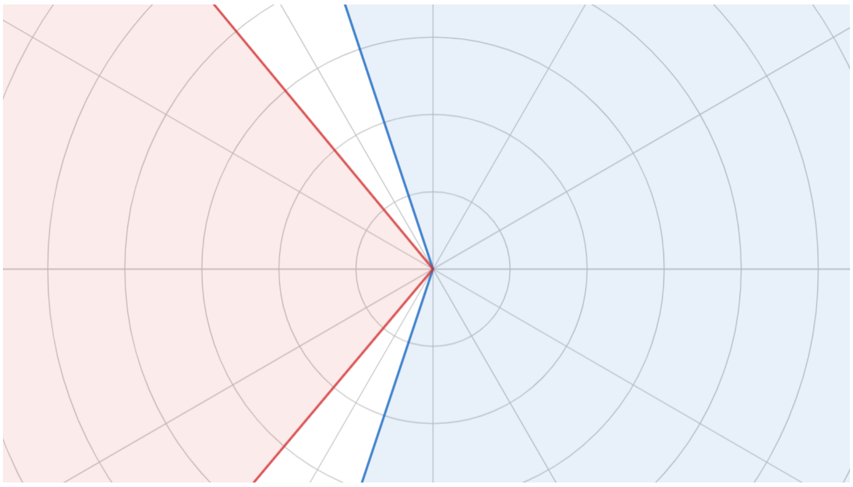

Figure 1.10: The sector S_θ in blue and the possible values of the z_i^2 in red for some θ and δ with $\pi < \theta + 2\delta$. In this case $\theta > \frac{1}{2}\pi$, so resulting $P(\lambda, z)$ is N-parabolic.

- The symbol $Q(\lambda, z) = \lambda^3 + |z|^5 - |z|^2\sqrt{\lambda^4 - |z|^4}$ is not N-parameter-elliptic in $S_\theta \times \Sigma_\delta^n$ for any θ or δ. We will show this using corollary 1.3.2: recall the γ-principal part from the previous examples:

$$\pi_\gamma Q(\lambda, z) = \begin{cases} |z|^5, & \gamma < \frac{3}{2}, \\ |z|^5 - |z|^2\sqrt{\lambda^4}, & \gamma = \frac{3}{2}, \\ -|z|^2\sqrt{\lambda^4}, & \frac{3}{2} < \gamma < 2, \\ \lambda^3 - |z|^2\sqrt{\lambda^4}, & \gamma = 2, \\ \lambda^3, & \gamma > 2. \end{cases}$$

  Take $\lambda = 1 \in \overline{S_\theta} \setminus \{0\}$ and $z_1 = i$, $z_2 = \cdots = z_n = 0$ so $z \in \overline{\Sigma_\delta^n} \setminus \{0\}$ and lastly $\gamma = 2$. Then we get:

$$\pi_2 Q(\lambda, z) = \lambda^3 - |z|^2\sqrt{\lambda^4} = 1^3 - 1^2 \cdot 1^2 = 1 - 1 = 0.$$

  Since not all γ-principal parts of Q are non-vanishing, corollary 1.3.2 implies Q is not N-parameter-elliptic.

- In the investigation of the Stefan problem [1] we deal with a symbol defined as

$$P_S(\lambda, z) = \lambda + |z|_-^2\sqrt{\lambda + |z|_-^2}.$$

  This symbol is very similiar to the heat equation, however unlike the heat equation it is not quasi-homogeneous. We will now prove it is N-parameter-elliptic in $S_\theta \times \Sigma_\delta^n$ with $\theta \in (0, \pi), \delta \in (0, \frac{1}{2}\pi)$ s.t. $\pi > \theta + 2\delta$, like with the heat equation. The symbol consists of the term $\phi_1(\lambda) = \lambda$ and the term $\psi_2(z)\tau_2(\lambda, z) = |z|_-^2\sqrt{\lambda + |z|_-^2}$. $\tau_2(\lambda, z)$ is quasi-homogeneous of degree $N_2 = 1$ and weight $\rho = 2$, since $\sqrt{\eta^2\lambda + |\eta z|_-^2} = \sqrt{\eta^2(\lambda + |z|_-^2)} = \eta\sqrt{\lambda + |z|_-^2}$. With $\phi_1(\lambda)$ homogeneous of degree $M_1 = 1$ and $\psi_2(z)$ of degree $L_2 = 2$, we can see that the Newton polygon $N(P_S)$ is the convex hull of the points $\mathfrak{u}_1 = \mathfrak{v}_1 = (0, 1)$, $\mathfrak{u}_2 = (3, 0)$, $\mathfrak{v}_2 = (2, \frac{1}{2})$ and the origin $(0, 0)$. The Newton polygon can be found in figure 1.11. We then define $\gamma_0 = 0$, $\gamma_1 = \frac{1}{\frac{1}{2}} = 2 = \rho$, $\gamma_2 = \frac{2}{\frac{1}{2}} = 4$ and $\gamma_3 = \infty$, so that we can use

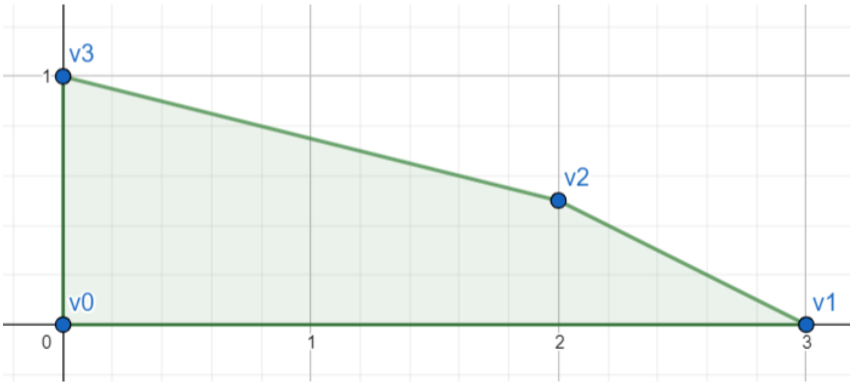

Figure 1.11: The Newton polygon of the Stefan problem

the polygon $N(P_S)$ to find

$$d_\gamma(P_S) = \begin{cases} 3, & 0 \le \gamma < 2, \\ \frac{1}{2}\gamma + 2, & 2 \le \gamma < 4, \\ \gamma, & 4 \le \gamma < \infty, \\ 1, & \gamma = \infty, \end{cases}$$

and with that we can see the γ-principal part

$$\pi_\gamma P_S(\lambda, z) = \begin{cases} |z|_-^2 \sqrt{|z|_-^2}, & \gamma < 2, \\ |z|_-^2 \sqrt{\lambda + |z|_-^2}, & \gamma = 2, \\ |z|_-^2 \sqrt{\lambda}, & 2 < \gamma < 4, \\ \lambda + |z|_-^2 \sqrt{\lambda}, & \gamma = 4, \\ \lambda, & \gamma > 4. \end{cases}$$

By corollary 1.3.2, we check whether $\pi_\gamma P_S(\lambda, z) \neq 0$ for $(\lambda, z) \in (\overline{S_\theta} \setminus \{0\}) \times (\overline{\Sigma_\delta^n} \setminus \{0\})$ for all $\gamma \in (0, \infty]$.

1. For $\gamma \notin \{2, 4\}$, $\pi_\gamma P_S(\lambda, z) \neq 0$ is obvious by $\lambda \neq 0$ and $z \neq 0$.
2. If $\gamma = 2$, we see that $\lambda + |z|_-^2 \neq 0$ since we have taken $\theta \in (0, \pi), \delta \in (0, \frac{1}{2}\pi)$ s.t. $\pi > \theta + 2\delta$. This combined with $\lambda \neq 0$ means $\pi_2 P_S(\lambda, z) \neq 0$.
3. If $\gamma = 4$, we can divide by $\sqrt{\lambda}$, and see that we need to check whether $\sqrt{\lambda} + |z|_-^2$ can be zero, or whether it is possible that $\sqrt{\lambda} = \sum_{i=1}^n z_i^2$. Since $\arg(\sqrt{\lambda}) = \frac{1}{2}\arg(\lambda)$, we know $\arg(\sqrt{\lambda}) \in (-\frac{\theta}{2}, \frac{\theta}{2})$, so since there is no overlap for $\arg(\lambda) \in (-\theta, \theta)$ and $\arg(z_i^2) \in (-\pi, -\pi + 2\delta) \cup (\pi - 2\delta, \pi]$, there definitely is no overlap between $\arg(\sqrt{\lambda})$ and $\arg(z_i^2)$, meaning $\sqrt{\lambda} \neq \sum_{i=1}^n z_i^2$ and therefore $\pi_4 P_S(\lambda, z) \neq 0$.

We also check whether $\pi_\infty P_S(\lambda, 0) \neq 0$ for $\lambda \in \overline{\Sigma_\delta^n} \setminus \{0\}$. Since $\pi_\infty P_S(\lambda, z) = \lambda$ is independent of z, this is also true. Therefore, we know that $P_S \in S_N(S_\theta \times \Sigma_\delta^n)$.

References

R. Denk & M. Kaip, section 2.2a, 2.2c [1]:

1. Definition 2.39, p.92
2. Remark 2.40, p. 93

3. Definition 2.41, p. 93
4. Corollary 2.57, p. 114
5. Proposition 2.47 and proof, p. 98-99

Chapter 2

N-parameter-ellipticity main result

2.1 Introduction

In this chapter we will be completing the proof of corollary 1.3.2 by proving the other direction: we will consider all symbols with non-vanishing γ-principal parts, meaning $\pi_\gamma P(\lambda, z) \neq 0$, for $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ and $\gamma \in (0, \infty]$, and also $\pi_\infty(\lambda, 0) \neq 0$ for $\lambda \in L_t \setminus \{0\}$. In this chapter we proof these symbols are N-parameter-elliptic in $\mathring{L}_t \times \mathring{L}_x$, which means there exists a $C_1 > 0$ and a $\lambda_0 \geq$ s.t. $|P(\lambda, z)| \geq C_1 W_P(\lambda, z)$ for any $(\lambda, z) \in \mathring{L}_t \times \mathring{L}_x$ with $|\lambda| \geq \lambda_0$. We do this the following way:

1. Define a partition of $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ in section 2.2
2. Check the statement for parts of the partition, using bounds from sections 2.3 and 2.4.
3. Complete the proof by showing it on the entirety of the partition.

The proof itself is found in section 2.5.

Firstly, the next proposition allows us to bypass the problem of having to check the ∞-principal parts of space regular symbols.

Proposition 2.1.1. *Let $P \in S(L_t \times L_x)$ with representation R_P as in definition 1.1.6 be a symbol with Newton polygon $N(P)$ as in definition 1.2.1 regular in space. If we take $\gamma \geq \max\{\gamma_J, \rho\}$, then we have*

$$\pi_\infty P(\lambda, z) = \pi_\gamma P(\lambda, z), \quad (\lambda, z) \in L_t \times L_x.$$

Therefore, if we know the γ-principal parts are non-vanishing for $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ and $\gamma \in (0, \infty)$, so

$$\pi_\gamma P(\lambda, z) \neq 0,$$

then we know the ∞-principal part is non-vanishing for $(\lambda, z) \in (L_t \setminus \{0\}) \times L_x$, or

$$\pi_\infty P(\lambda, z) \neq 0.$$

Proof. If $\gamma \geq \gamma_J$ and $\gamma \geq \rho$ so that $d_\gamma(\tau_\ell) = \frac{N_\ell}{\rho}\gamma$, we get $d_\gamma(P) = s_{J+1}\gamma + r_{J+1} = s_{J+1}\gamma = d_\infty(P)\gamma$ from lemma 1.2.11, since P is regular in space. That means that for any $\ell \in I_\infty$, $L_\ell = r_{J+1} = 0$, which means that $\psi_\ell(z)$ must be a constant function, so $\psi_\ell(z) = \psi_\ell(0)$. Therefore, we can use lemma 1.1.5 to conclude

$$\begin{aligned}\pi_\infty P(\lambda,z) &= \sum_{\ell\in I_\infty}[\pi_\infty\tau_\ell](\lambda,z)\phi_\ell(\lambda)\psi_\ell(z)\\ &= \sum_{\ell\in I_\gamma}\tau(\lambda,0)\phi_\ell(\lambda)\psi_\ell(z)\\ &= \sum_{\ell\in I_\gamma}\tau(\lambda,0)\phi_\ell(\lambda)\psi_\ell(0)\\ &= \pi_\gamma P(\lambda,0)\end{aligned}$$

So we see $\pi_\infty P(\lambda,0) = \pi_\infty P(\lambda,z) = \pi_\gamma P(\lambda,z)$ for $\gamma \geq \max\{\gamma_J,\rho\}$. Therefore, if for any $\gamma\in(0,\infty)$ we have

$$\pi_\gamma P(\lambda,z)\neq 0,\quad (\lambda,z)\in(L_t\setminus\{0\})\times(L_x\setminus\{0\}),$$

then for any $\gamma\geq\max\{\gamma_J,\rho\}$ we get

$$\pi_\infty P(\lambda,0)=\pi_\infty P(\lambda,z)=\pi_\gamma P(\lambda,z)\neq 0,\quad (\lambda,z)\in(L_t\setminus\{0\})\times L_x.$$

□

During the following proof, we will not investigate $\gamma=\infty$ for space-regular symbols, but we will occasionally have to make exceptions for space-irregular cases.

Reference

R. Denk & M. Kaip, section 2.2c [1]:

1. Proposition 2.48 and proof, p. 99

2.2 The partition

In this section we are going to construct a partition of (λ,z) on basis of their moduli $(|z|,|\lambda|)$, on which the γ-principal part $\pi_\gamma P(\lambda,z)$ will be fully dominated by either a vertex or an edge of the Newton polygon $N(P)$. We will begin by defining this partition, and then proving some properties that hold on this partition.

Firstly, let's make a partition of $\mathbb{R}^2$ based on a Newton polygon N with vertices $N_v = \{(r_j,s_j), j\in\{0,\dots,J+1\}\}$, and unit vectors $q_j=(q_{j,1},q_{j,2})$, in the direction of $(1,\gamma_j)$, and $q_j^\perp = (-q_{j,2},q_{j,1})$ following the edge $[v_jv_{j+1}]$ of the Newton polygon (as defined in definition 1.2.3(ii).) Based on parameters $\varepsilon_0\in(0,1)$ and $\varepsilon_1\in(0,1)$, we will define half-strips of width $2\log\left(\frac{1}{\varepsilon_0}\right)$, starting from a certain point $\log\left(\frac{1}{\varepsilon_1}\right)$, in the direction of q_j based from the origin. Only half-strips however does not cover $\mathbb{R}^2$ well enough, so we also define cone-shaped area's between the strip of q_j and the strip of q_{j-1}.

Definition 2.2.1. (i) Let N be a Newton polygon with vectors q_j as defined in definition 1.2.3(ii). For $\varepsilon_0,\varepsilon_1\in(0,1)$, we define the *half-strip* $S_j(\varepsilon_0,\varepsilon_1)$, for $j\in\{1,\dots,J\}$, as

$$S_j(\varepsilon_0,\varepsilon_1) := \left\{p\in\mathbb{R}^2 : \log(\varepsilon_0)\leq\langle q_j^\perp,p\rangle\leq\log\left(\varepsilon_0^{-1}\right) \text{ and } \langle q_j,p\rangle\geq\log\left(\varepsilon_1^{-1}\right)\right\}.$$

(ii) We also define the *cone-shaped area's* $C_j(\varepsilon_0)$ as the area between $S_{j-1}(\varepsilon_0,\varepsilon_1)$ and $S_j(\varepsilon_0,\varepsilon_1)$, for $j\in\{1,\dots,J+1\}$:

$$C_j(\varepsilon_0) := \left\{p\in\mathbb{R}^2 : \langle q_{j-1}^\perp,p\rangle\geq\log\left(\varepsilon_0^{-1}\right) \text{ and } \langle q_j^\perp,p\rangle\leq\log(\varepsilon_0)\right\}.$$

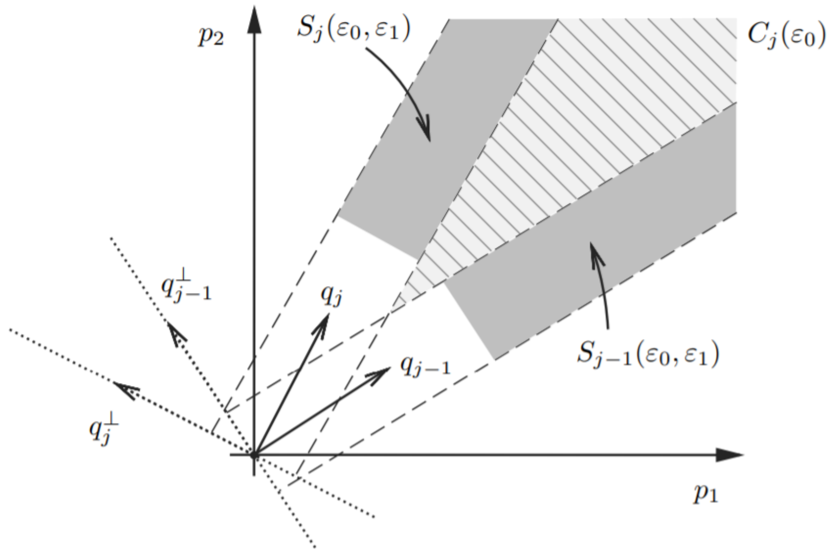

Figure 2.1: The half-strips S_j and cone-shaped area's C_j based on the vectors q_j. [1, Denk, Kaip, 2013, p. 94]

A good representation of this partition of $\mathbb{R}^2$ can be found in figure 2.1.

Remark. The partition does not cover the entirety of $\mathbb{R}^2$, however as we will see in lemma 2.2.4(iv) this is also not needed.

We will now create a partition of $(|z|, |\lambda|) \in (0, \infty)^2$, using the previous partition of $\mathbb{R}^2$. Let ξ represent $|z|$, and let η represent $|\lambda|$.

Definition 2.2.2 (Partition of $(0, \infty)^2$)**.** Let N be a Newton polygon with vectors q_j as defined in definition 1.2.3(ii). For $\varepsilon_0 \in (0,1), \varepsilon_1 \in (0,1)$, we use $S_j(\varepsilon_0, \varepsilon_1)$ and $C_j(\varepsilon_0)$ from definition 2.2.1 to define:

$$G_k(\varepsilon_0, \varepsilon_1) := \{(\xi, \eta) \in (0,\infty)^2 : (\log(\xi), \log(\eta)) \in S_k(\varepsilon_0, \varepsilon_1)\},$$
$$\widetilde{G}_j(\varepsilon_0) := \{(\xi, \eta) \in (0,\infty)^2 : (\log(\xi), \log(\eta)) \in C_j(\varepsilon_0)\},$$

where $k \in \{1, \dots, J\}$ and $j \in \{1, \dots, J+1\}$.

There is a definition easier to use available, as seen in the next lemma.

Lemma 2.2.3. *(i) Let N be a Newton polygon that is regular in space with vectors q_j as defined in definition 1.2.3(ii). For $\varepsilon_0, \varepsilon_1 \in (0,1)$, we can define $G_j(\varepsilon_0, \varepsilon_1)$ as*

$$G_j(\varepsilon_0, \varepsilon_1) = \left\{ (\xi, \eta) \in (0,\infty)^2 : \varepsilon_0^{\frac{1}{q_{j,1}}} \xi^{\gamma_j} \le \eta \le \varepsilon_0^{-\frac{1}{q_{j,1}}} \xi^{\gamma_j} \text{ and } \xi \eta^{\gamma_j} \ge \varepsilon_1^{-\frac{1}{q_{j,1}}} \right\},$$

for $j \in \{1, \dots, J\}$, and $\widetilde{G}_j(\varepsilon_0)$ as

$$\widetilde{G}_1(\varepsilon_0) = \left\{ (\xi, \eta) \in (0,\infty)^2 : \eta \le \varepsilon_0^{\frac{1}{q_{1,1}}} \xi^{\gamma_1} \text{ and } \xi \ge \varepsilon_0^{-1} \right\},$$
$$\widetilde{G}_j(\varepsilon_0) = \left\{ (\xi, \eta) \in (0,\infty)^2 : \varepsilon_0^{-\frac{1}{q_{j-1,1}}} \xi^{\gamma_{j-1}} \le \eta \le \varepsilon_0^{\frac{1}{q_{j,1}}} \xi^{\gamma_j} \right\},$$
$$\widetilde{G}_{J+1}(\varepsilon_0) = \left\{ (\xi, \eta) \in (0,\infty)^2 : \varepsilon_0^{-\frac{1}{q_{J,1}}} \xi^{\gamma_J} \le \eta \text{ and } \eta \ge \varepsilon_0^{-1} \right\},$$

for $j \in \{2, \dots, J\}$.

(ii) If N is not regular in space, we must define the last few indices differently. We instead get

$$G_J(\varepsilon_0,\varepsilon_1)=\left\{(\xi,\eta)\in(0,\infty)^2:\varepsilon_0\le\xi\le\varepsilon_0^{-1}\ \textit{and}\ \eta\ge\varepsilon_1^{-1}\right\}$$

and

$$\widetilde{G}_J(\varepsilon_0)=\left\{(\xi,\eta)\in(0,\infty)^2:\varepsilon_0^{-\frac{1}{q_{J-1,1}}}\xi^{\gamma_{J-1}}\le\eta\ \textit{and}\ \xi\ge\varepsilon_0^{-1}\right\},$$
$$\widetilde{G}_{J+1}(\varepsilon_0)=\left\{(\xi,\eta)\in(0,\infty)^2:\xi\le\varepsilon_0\ \textit{and}\ \eta\ge\varepsilon_0^{-1}\right\}.$$

Proof. (i) We investigate $(\xi,\eta)\in G_j(\varepsilon_0,\varepsilon_1)$ for $j\in\{1,\dots,J\}$, which is equivalent to $(\log(\xi),\log(\eta))\in S_j(\varepsilon_0,\varepsilon_1)$. This is equivalent to (ξ,η) satisfying the conditions

$$\begin{aligned}\log(\varepsilon_0)\le -q_{j,2}\log(\xi)+q_{j,1}\log(\eta)&\le\log\left(\varepsilon_0^{-1}\right),\\ q_{j,1}\log(\xi)+q_{j,2}\log(\eta)&\ge\log\left(\varepsilon_1^{-1}\right)\end{aligned}\tag{2.1}$$

as seen in definition 2.2.1(i). If N is regular in space, then for $j\in\{1,\dots,J\}$, we can use the definition 1.2.3(ii) to write $q_j=(q_{j,1},q_{j,2})=q_{j,1}(1,\gamma_j)$, or in other words, $q_{j,2}=q_{j,1}\gamma_j$. This means our conditions become

$$\begin{aligned}\log(\varepsilon_0)\le -q_{j,1}\gamma_j\log(\xi)+q_{j,1}\log(\eta)&\le\log\left(\varepsilon_0^{-1}\right),\\ q_{j,1}\log(\xi)+q_{j,1}\gamma_j\log(\eta)&\ge\log\left(\varepsilon_1^{-1}\right).\end{aligned}$$

Divide the equations by $q_{j,1}$, and apply logarithmic rules to get

$$\begin{aligned}\log\left(\varepsilon_0^{\frac{1}{q_{j,1}}}\right)\le\log\left(\xi^{-\gamma_j}\right)+\log(\eta)&\le\log\left(\varepsilon_0^{-\frac{1}{q_{j,1}}}\right),\\ \log(\xi)+\log(\eta^{\gamma_j})&\ge\log\left(\varepsilon_1^{-\frac{1}{q_{j,1}}}\right).\end{aligned}$$

Now the property is found by taking the exponent of the equation(this is allowed since the $\exp(x)$ function is increasing).

$$\begin{aligned}\varepsilon_0^{\frac{1}{q_{j,1}}}\le\xi^{-\gamma_j}\eta&\le\varepsilon_0^{-\frac{1}{q_{j,1}}},\\ \xi\eta^{\gamma_j}&\ge\varepsilon_1^{-\frac{1}{q_{j,1}}},\end{aligned}$$

which is rewritten as

$$\varepsilon_0^{\frac{1}{q_{j,1}}}\xi^{\gamma_j}\le\eta\le\varepsilon_0^{-\frac{1}{q_{j,1}}}\xi^{\gamma_j}\text{ and }\xi\eta^{\gamma_j}\ge\varepsilon_1^{-\frac{1}{q_{j,1}}}.$$

Thus we have found this condition is equivalent with $(\xi,\eta)\in G_j(\varepsilon_0,\varepsilon_1)$.

We also check $(\xi,\eta)\in\widetilde{G}_j(\varepsilon_0)$, which is equivalent to the conditions

$$\begin{aligned}-q_{j-1,2}\log(\xi)+q_{j-1,1}\log(\eta)&\ge\log\left(\varepsilon_0^{-1}\right),\\ -q_{j,2}\log(\xi)+q_{j,1}\log(\eta)&\le\log(\varepsilon_0)\end{aligned}\tag{2.2}$$

as seen in definition 2.2.1(ii). We go through the different cases.

(I) For $j\in\{2,\dots,J\}$, the proof for $(\xi,\eta)\in\widetilde{G}_j(\varepsilon_0)$ follows the same steps as the proof for $(\xi,\eta)\in G_j(\varepsilon_0,\varepsilon_1)$: use $q_{j,2}=q_{j,1}\gamma_j$, divide by $q_{j,1}$, use logarithmic rules and take the exponent. After rewriting, one can easily find the equivalence

$$(\xi,\eta)\in\widetilde{G}_j(\varepsilon_0)\Leftrightarrow\varepsilon_0^{-\frac{1}{q_{j-1,1}}}\xi^{\gamma_{j-1}}\le\eta\le\varepsilon_0^{\frac{1}{q_{j,1}}}\xi^{\gamma_j}.$$

(II) For $j = 1$, we have that $q_{j-1} = q_0 = (0, -1)$. Using this on the found conditions for $(\xi, \eta) \in \widetilde{G}_1(\varepsilon_0)$, we get

$$\begin{aligned} \log(\xi) &\geq \log\left(\varepsilon_0^{-1}\right), \\ -q_{j,2}\log(\xi) + q_{j,1}\log(\eta) &\leq \log\left(\varepsilon_0\right). \end{aligned}$$

The bottom equation is solved as before, and the top equation is immediately equivalent to $\xi \geq \varepsilon_0^{-1}$. Then we get the equivalence

$$(\xi, \eta) \in \widetilde{G}_1(\varepsilon_0) \Leftrightarrow \eta \leq \varepsilon_0^{\frac{1}{q_{1,1}}} \xi^{\gamma_1} \text{ and } \xi \geq \varepsilon_0^{-1}.$$

(III) For $j = J + 1$, we have $q_j = q_{J+1} = (-1, 0)$, so the conditions are

$$\begin{aligned} -q_{J,2}\log(\xi) + q_{J,1}\log(\eta) &\geq \log\left(\varepsilon_0^{-1}\right), \\ -\log(\eta) &\leq \log\left(\varepsilon_0\right). \end{aligned}$$

Top equation is solved as before, but bottom equation is solved by multiplying with -1 (a negative number!) to get $\log(\eta) \geq \log\left(\varepsilon_0^{-1}\right)$ meaning $\eta \geq \varepsilon_0^{-1}$. This means we have the equivalence

$$(\xi, \eta) \in \widetilde{G}_{J+1}(\varepsilon_0) \Leftrightarrow \varepsilon_0^{-\frac{1}{q_{J,1}}} \xi^{\gamma_J} \leq \eta \text{ and } \eta \geq \varepsilon_0^{-1}.$$

(ii) If N is not regular in space, then we have the problem that $q_J = (0, 1) \neq q_{J,1}(1, \gamma_J)$. We have to account for this in our conditions where q_J appears.

(I) $(\xi, \eta) \in G_J(\varepsilon_0, \varepsilon_1)$ is equivalent to the conditions

$$\begin{aligned} \log\left(\varepsilon_0\right) \leq -q_{J,2}\log(\xi) + q_{J,1}\log(\eta) &\leq \log\left(\varepsilon_0^{-1}\right), \\ q_{J,1}\log(\xi) + q_{J,2}\log(\eta) &\geq \log\left(\varepsilon_1^{-1}\right), \end{aligned}$$

which become

$$\begin{aligned} \log\left(\varepsilon_0\right) \leq -\log(\xi) &\leq \log\left(\varepsilon_0^{-1}\right), \\ \log(\eta) &\geq \log\left(\varepsilon_1^{-1}\right). \end{aligned}$$

Multiplying the top equation by -1, and taking the exponent of both equations, we find the equivalence

$$(\xi, \eta) \in G_J(\varepsilon_0, \varepsilon_1) \Leftrightarrow \varepsilon_0 \leq \xi \leq \varepsilon_0^{-1} \text{ and } \eta \geq \varepsilon_1^{-1}.$$

(II) $(\xi, \eta) \in \widetilde{G}_J(\varepsilon_0)$ is equivalent to

$$\begin{aligned} -q_{J-1,2}\log(\xi) + q_{J-1,1}\log(\eta) &\geq \log\left(\varepsilon_0^{-1}\right), \\ -q_{J,2}\log(\xi) + q_{J,1}\log(\eta) &\leq \log\left(\varepsilon_0\right), \end{aligned}$$

which becomes

$$\begin{aligned} -q_{J-1,2}\log(\xi) + q_{J-1,1}\log(\eta) &\geq \log\left(\varepsilon_0^{-1}\right), \\ -\log(\xi) &\leq \log\left(\varepsilon_0\right). \end{aligned}$$

Solving the top equation as before, and finding $\xi \geq \varepsilon_0^{-1}$, we find the equivalence

$$(\xi, \eta) \in \widetilde{G}_J(\varepsilon_0) \Leftrightarrow \varepsilon_0^{-\frac{1}{q_{J-1,1}}} \xi^{\gamma_{J-1}} \leq \eta \text{ and } \xi \geq \varepsilon_0^{-1}.$$

(III) $(\xi,\eta) \in \widetilde{G}_{J+1}(\varepsilon_0)$ is equivalent to

$$\begin{aligned} -q_{J,2}\log(\xi) + q_{J,1}\log(\eta) &\geq \log\left(\varepsilon_0^{-1}\right), \\ -\log(\eta) &\leq \log\left(\varepsilon_0\right), \end{aligned}$$

which becomes

$$\begin{aligned} -\log(\xi) &\geq \log\left(\varepsilon_0^{-1}\right), \\ -\log(\eta) &\leq \log\left(\varepsilon_0\right). \end{aligned}$$

So we find the equivalence

$$(\xi,\eta) \in \widetilde{G}_{J+1}(\varepsilon_0) \Leftrightarrow \xi \leq \varepsilon_0 \text{ and } \eta \geq \varepsilon_0^{-1}.$$

□

Finally, we can find some useful properties of our partition $G(\varepsilon_0,\varepsilon_1)$ and $\widetilde{G}(\varepsilon_0)$ related to a Newton polygon $N(\nu)$ made from a finite set $\nu \subseteq [0,\infty)^2$ (see definition 1.2.1.)

Lemma 2.2.4 (Properties of the partition)**.** *Let $\nu \subseteq [0,\infty)^2$ be a finite set, and $N(\nu)$ its Newton polygon with vertices $(r_j,s_j) \in N_v$ for $j \in \{0,\ldots,1\}$, and the partition $G(\varepsilon_0,\varepsilon_1)$ and $\widetilde{G}(\varepsilon_0)$. For the following statements, let $\varepsilon > 0$ be arbitrary.*

(i) There exists a bound $\hat{\varepsilon}_0 > 0$ s.t. for arbitrary $j \in \{1,\ldots,J+1\}$ and $0 < \varepsilon_0 \leq \hat{\varepsilon}_0$, we have that for the point $v_j = (r_j,s_j) \in N_v$ and any point $(r,s) \in \nu \setminus \{v_j\}$ the following must hold for $(\xi,\eta) \in \widetilde{G}_j(\varepsilon_0)$:

$$\eta^s\xi^r \leq \varepsilon \cdot \eta^{s_j}\xi^{r_j}$$

(ii) Denote L as the line through the vertices v_j and v_{j+1} for some fixed $j \in \{1,\ldots,J\}$. For every $\alpha,\beta \in L$, there exists a constant $\theta \in \mathbb{R}$ such that for any $\varepsilon_0 > 0, \varepsilon_1 > 0$, and $(\xi,\eta) \in G_j(\varepsilon_0,\varepsilon_1)$, we get

$$\eta^{\alpha_2}\xi^{\alpha_1} \leq \varepsilon_0^{-|\theta|}\eta^{\beta_2}\xi^{\beta_1}.$$

(iii) For all $\varepsilon_0 > 0$, there is a constant $\varepsilon_1 > 0$ s.t. for arbitrary $j \in \{1,\ldots,J\}$, we can take any $(\xi,\eta) \in G_j(\varepsilon_0,\varepsilon_1)$, any $(r,s) \in \nu \cap [v_jv_{j+1}]$ and any $(r',s') \in \nu \setminus [v_jv_{j+1}]$, and get the property

$$\eta^{s'}\xi^{r'} \leq \varepsilon \cdot \eta^s\xi^r.$$

(iv) For all $\varepsilon_0,\varepsilon_1 > 0$, there is a bound $\mu_0 > 0$ such that

$$\{(\xi,\eta) \in (0,\infty)^2 : \eta \geq \mu_0\} \subseteq \bigcup_{j=1}^{J}\left\{G_j(\varepsilon_0,\varepsilon_1) \cup \widetilde{G}_j(\varepsilon_0)\right\} \cup \widetilde{G}_{J+1}(\varepsilon_0).$$

If N is not regular in time, $G_1(\varepsilon_0,\varepsilon_1)$ and $\widetilde{G}_1(\varepsilon_0)$ are redundant for covering ths set. Therefore, we can then have

$$\{(\xi,\eta) \in (0,\infty)^2 : \eta \geq \mu_0\} \subseteq \bigcup_{j=2}^{J}\left\{G_j(\varepsilon_0,\varepsilon_1) \cup \widetilde{G}_j(\varepsilon_0)\right\} \cup \widetilde{G}_{J+1}(\varepsilon_0).$$

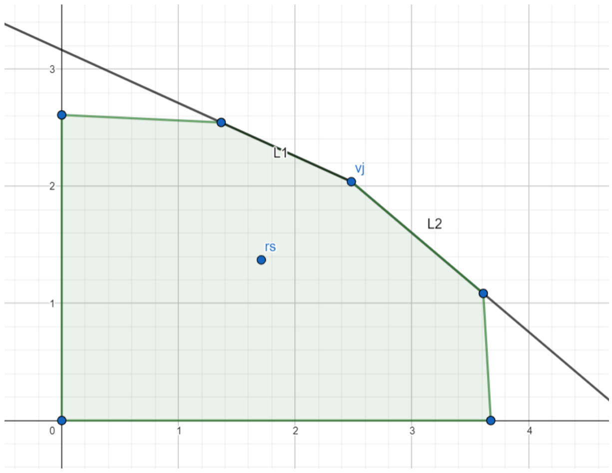

Figure 2.2: An arbitrary Newton polygon with L_1 and L_2 from the proof of lemma 2.2.4(i). Taking $\delta_1, \ldots, \delta_4$ on these lines allows us to encompass the point (r, s) in the Newton Polygon.

Proof. (i) Take arbitrary $j \in \{1, \ldots, J+1\}$, and set $v_j = (r_j, s_j) \in N_v$. Choose any $(r, s) \in \nu \setminus \{v_j\}$. We define L_1, L_2 as two half-lines following the edges $[v_j v_{j+1}]$ and $[v_{j-1} v_j]$ of $N(\nu)$ respectively:

$$L_1 := \{\alpha \in \mathbb{R}^2 : \alpha = v_j + t q_j^{\perp}, t > 0\}$$
$$L_2 := \{\alpha \in \mathbb{R}^2 : \alpha = v_j - t q_{j-1}^{\perp}, t > 0\}$$

Since we follow the edge of a finite convex object, if we take the convex hull of $L_1 \cup L_2$, this hull will contain the entire Newton polygon $N(\nu)$, so also any point (r, s). More precisely, we can always pick four points that have (r, s) in their hull: take $\delta_1 = v_j + t_1 q_j^{\perp}$, $\delta_2 = v_j + t_2 q_j^{\perp} \in L_1$ and $\delta_3 = v_j - t_3 q_{j-1}^{\perp}$, $\delta_4 = v_j - t_4 q_{j-1}^{\perp} \in L_2$, so that $(r, s) \in \text{Convex hull}\left(\bigcup_{i=1}^4 \delta_i\right)$. This convex hull means that for some $\lambda_1, \ldots, \lambda_4$ that have $\sum_{i=1}^4 \lambda_i = 1$, we can set $(r, s) = \sum_{i=1}^4 \lambda_i \delta_i$. See also figure 2.2.

Set $\chi := \min_{i \in \{1,2,3,4\}} t_i$, and choose a bound $\hat{\varepsilon}_{0,j} > 0$ small enough s.t. $(\hat{\varepsilon}_{0,j})^{\chi} \leq \varepsilon$. let $0 < \varepsilon_0 \leq \hat{\varepsilon}_{0,j}$ be arbitrary, so that we can take $(\xi, \eta) \in \widetilde{G}_j(\varepsilon_0)$. We first use convexity, then by the definition of δ_i and $q_j^{\perp}$ we get

$$\begin{aligned}
\eta^s \xi^r &= \eta^{\sum_{i=1}^4 \delta_{i,2} \lambda_i} \xi^{\sum_{k=1}^4 \delta_{k,1} \lambda_k} = \prod_{i=1}^4 \left(\eta^{\delta_{i,2}} \xi^{\delta_{i,1}}\right)^{\lambda_i} \\
&= \eta^{s_j} \xi^{r_j} \left(\prod_{i=1}^2 \left(\eta^{q_{j,1} t_i} \xi^{-t_i q_{j,2}}\right)^{\lambda_i}\right) \cdot \left(\prod_{i=3}^4 \left(\eta^{-t_i q_{j-1,1}} \xi^{t_i q_{j-1,2}}\right)^{\lambda_i}\right).
\end{aligned}$$

This is rewritten using logarithms, so that we can use the conditions of $(\xi, \eta) \in \widetilde{G}_j(\varepsilon_0)$ as found in equation (2.2).

$$\begin{aligned}
\eta^s\xi^r &= \eta^{s_j}\xi^{r_j}\left(\prod_{i=1}^{2}\left(\eta^{q_{j,1}}\xi^{-q_{j,2}}\right)^{\lambda_i t_i}\right)\cdot\left(\prod_{i=3}^{4}\left(\eta^{-q_{j-1,1}}\xi^{q_{j-1,2}}\right)^{\lambda_i t_i}\right)\\
&= \eta^{s_j}\xi^{r_j}\left(\prod_{i=1}^{2}\left(\exp\left[q_{j,1}\log(\eta)-q_{j,2}\log(\xi)\right]\right)^{t_i\lambda_i}\right)\\
&\qquad\cdot\left(\prod_{i=3}^{4}\left(\exp\left[-q_{j-1,1}\log(\eta)+q_{j-1,2}\log(\xi)\right]\right)^{t_i\lambda_i}\right),
\end{aligned}$$

and applying equation (2.2) to get

$$\begin{aligned}
\eta^s\xi^r &\le \eta^{s_j}\xi^{r_j}\prod_{i=1}^{4}(\exp[\log(\varepsilon_0)])^{t_i\lambda_i} = \eta^{s_j}\xi^{r_j}\varepsilon_0^{\sum_{i=1}^4 t_i\lambda_i}\\
&\le \eta^{s_j}\xi^{r_j}\varepsilon_0^{\chi\sum_{i=1}^4\lambda_i} = \varepsilon_0^{\chi}\cdot\eta^{s_j}\xi^{r_j}\\
&\le (\hat{\varepsilon}_{0,j})^{\chi}\cdot\eta^{s_j}\xi^{r_j} \le \varepsilon\cdot\eta^{s_j}\xi^{r_j}.
\end{aligned}$$

We wanted to proof there is a bound $\hat{\varepsilon}_0 > 0$ for which the property holds for all $j \in \{1,\ldots,J+1\}$, but we can easily set

$$\hat{\varepsilon}_0 := \min_{j\in\{1,\ldots,J+1\}}\hat{\varepsilon}_{0,j}.$$

(ii) Let L be a line through v_j and v_{j+1} for some fixed $j \in \{1,\ldots,J\}$, and take $\alpha = (\alpha_1,\alpha_2)$ and $\beta = (\beta_1,\beta_2)$ on this line L. Remember that $q_j^{\perp}$ is the vector following the edge $[v_jv_{j+1}]$, so it also follows the line L: this allows us to find a $\theta \in \mathbb{R}$ that makes $\alpha-\beta = \theta q_j^{\perp}$. For any $(\xi,\eta) \in (0,\infty)^2$, we can use logarithms to denote

$$\begin{aligned}
\eta^{\alpha_2}\xi^{\alpha_1}\eta^{-\beta_2}\xi^{-\beta_1} &= \exp\left[(\alpha_2-\beta_2)\log(\eta)+(\alpha_1-\beta_1)\log(\xi))\right]\\
&= \exp\left[\theta(q_{j,1}\log(\eta)-q_{j,2}\log(\xi))\right]
\end{aligned}$$

If we instead take $(\xi,\eta) \in G_1(\varepsilon_0,\varepsilon_1)$ for any $\varepsilon_0,\varepsilon_1 > 0$, we can use equation (2.1) to find the inequality

$$\theta(q_{j,1}\log(\eta)-q_{j,2}\log(\xi)) \le |\theta|\log\left(\varepsilon_0^{-1}\right) = -|\theta|\log(\varepsilon_0).$$

This allows us to get

$$\begin{aligned}
\eta^{\alpha_2}\xi^{\alpha_1}\eta^{-\beta_2}\xi^{-\beta_1} &= \exp\left[\theta(q_{j,1}\log(\eta)-q_{j,2}\log(\xi))\right]\\
&\le \exp\left[-|\theta|\log(\varepsilon_0)\right] = \varepsilon_0^{-|\theta|},
\end{aligned}$$

or in other words,

$$\eta^{\alpha_2}\xi^{\alpha_1} \le \varepsilon_0^{-|\theta|}\eta^{\beta_2}\xi^{\beta_1}.$$

(iii) Take $\varepsilon_0 > 0$ and $j \in \{1,\ldots,J\}$ arbitrary, and take any $(r,s) \in \nu\cap[v_jv_{j+1}]$ and $(r',s') \in \nu\setminus[v_jv_{j+1}]$. We can represent (r',s') using it's projection on the line L through $[v_jv_{j+1}]$: there is a point $\delta \in L \subseteq \mathbb{R}^2$ and a constant $t > 0$ s.t.

$$(r',s') = \delta - tq_j.$$

For points (r,s) and δ on the line L, we can use part (ii) to find a θ for any $\varepsilon_1 > 0$. We choose a $\varepsilon_{1,j} > 0$ small enough s.t. $\varepsilon_0^{-|\theta|}\varepsilon_{1,j}^t \leq \varepsilon$. Take $(\xi,\eta) \in G_j(\varepsilon_0, \varepsilon_{1,j})$. We can first rewrite this, then use equation (2.1).

$$\begin{aligned}
\eta^{s'}\xi^{r'} &= \exp\left[s'\log(\eta) + r'\log(\xi)\right] = \exp\left[(\delta_2 - tq_{j,2})\log(\eta) + (\delta_1 - tq_{j,1})\log(\xi)\right] \\
&= \eta^{\delta_2}\xi^{\delta_1}\exp\left[-t(q_{j,2}\log(\eta) + q_{j,1}\log(\xi))\right] \\
&\leq \eta^{\delta_2}\xi^{\delta_1}\exp\left[-t\log\left(\varepsilon_{1,j}^{-1}\right)\right] = \eta^{\delta_2}\xi^{\delta_1}\varepsilon_{1,j}^t.
\end{aligned}$$

Next, since $\delta \in L$ and $(r,s) \in L$, we can use part (ii), and our choice of $\varepsilon_{1,j}$:

$$\begin{aligned}
\eta^{s'}\xi^{r'} &\leq \varepsilon_{1,j}^t \cdot \eta^{\delta_2}\xi^{\delta_1} \\
&\leq \varepsilon_0^{-|\theta|}\varepsilon_{1,j}^t \cdot \eta^s\xi^r \\
&\leq \varepsilon \cdot \eta^s\xi^r
\end{aligned}$$

Since, similar to (i), we want a choice ε_1 that works for all $j \in \{1,\ldots,J\}$, we take

$$\varepsilon_1 = \min_{j\in\{1,\ldots,J\}} \varepsilon_{1,j}.$$

(iv) Let $\varepsilon_0, \varepsilon_1 > 0$ be arbitrary. Define

$$\begin{aligned}
A(\mu_0) &:= \{(\xi,\eta) \in (0,\infty)^2 : \eta \geq \mu_0)\}, \\
B(y_0) &:= \{(x,y) \in \mathbb{R}^2 : y \geq y_0\}, \\
\mathcal{G}(\varepsilon_0,\varepsilon_1) &:= \bigcup_{j=1}^{J}\left\{G_j(\varepsilon_0,\varepsilon_1) \cup \widetilde{G}_j(\varepsilon_0)\right\} \cup \widetilde{G}_{J+1}(\varepsilon_0) \subseteq (0,\infty)^2, \\
\mathcal{S}(\varepsilon_0,\varepsilon_1) &:= \bigcup_{j=1}^{J}\left\{S_j(\varepsilon_0,\varepsilon_1) \cup C_j(\varepsilon_0)\right\} \cup C_{J+1}(\varepsilon_0) \subseteq \mathbb{R}^2.
\end{aligned}$$

We show the set $A(\mu_0)$ can be covered by a set $\mathcal{G}(\varepsilon_0,\varepsilon_1)$ by showing the set $B(y_0)$ can be covered by $\mathcal{S}(\varepsilon_0,\varepsilon_1)$. Then, since $A(\mu_0) = \{(\xi,\eta) \in (0,\infty) : (\log(\xi),\log(\eta)) \in B(\log(\mu_0))\}$ and $\mathcal{G}(\varepsilon_0,\varepsilon_1) = \{(\xi,\eta) \in (0,\infty) : (\log(\xi),\log(\eta)) \in \mathcal{S}(\varepsilon_0,\varepsilon_1)\}$, we will have proven our statement (iv) for time-regular Newton polygons $N(\nu)$.

The advantage of working with $S_j(\varepsilon_0,\varepsilon_1)$ and $C_j(\varepsilon_1)$ is that we can work with angles of vectors $\mathbf{x} = (x,y) \in \mathbb{R}^2$. We will work with an arbitrary sequence of points $(x_n,y_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^2$ that has $\lim_{n\to\infty} y_n = \infty$, so that for any chosen y_0, we can find a bound $n_0 \in \mathbb{N}$ s.t. $(x_n,y_n) \in B(y_0)$ for all $n \geq n_0$.

We investigate the limit of the argument, or angle, of the vector $\mathbf{x}_n = (x_n,y_n)$. Since $(x_n,y_n) \in \mathbb{R}^2$, we know the vector $\mathbf{x}'_n = \frac{\mathbf{x}_n}{|\mathbf{x}_n|}$ has the same argument as $\mathbf{x}_n$, but is also bounded to the compact circle $K := \{(x,y) \in \mathbb{R}^2 : |(x,y)| = 1\}$, which by Bolzano-Weierstrass on $\mathbb{R}^2$ implies it has a convergent subsequence $\mathbf{x}'_{n_k} \to \mathbf{x}$ for some limit $\mathbf{x}$ (or rather, $\mathbf{x}_n$ has a subsequence $\mathbf{x}_{n_k}$ for which this property holds.) However, this also means that as $k \to \infty$, we get

$$\arg(\mathbf{x}_{n_k}) = \arg(\mathbf{x}'_{n_k}) \to \arg(\mathbf{x}) \in (-\pi,\pi].$$

In conclusion, $\lim_{k\to\infty} \arg(\mathbf{x}_{n_k}) \in (-\pi,\pi]$ exists. From now, we will denote $\mathbf{x}_k := \mathbf{x}_{n_k}$.

We can split $\lim_{k\to\infty} \arg(\mathbf{x}_k) \in (-\pi,\pi]$ into two cases: either $\lim_{k\to\infty} \arg(\mathbf{x}_k) = \arg(q_j)$ for $j \in \{1,\ldots,J\}$, or $\lim_{k\to\infty} \arg(\mathbf{x}_k) \in (\arg(q_{j-1}),\arg(q_j))$ for $j \in \{1,\ldots,J+1\}$.

(I) $\lim_{k\to\infty}\arg(\mathbf{x}_k)=\arg(q_j)$ for $j\in\{1,\dots,J\}$. We know $|\mathbf{x}_k|\to\infty$ from $y_k\to\infty$, which imply $\langle q_j^\perp,\mathbf{x}_k\rangle\to 0$ and $\langle q_j,\mathbf{x}_k\rangle\to\infty$. Therefore we can find a bound $s_j\in\mathbb{N}$ s.t.

$$\begin{aligned}\log(\varepsilon_0)\leq&\langle q_j^\perp,\mathbf{x}_k\rangle\leq\log\bigl(\varepsilon_0^{-1}\bigr),\\ &\langle q_j,\mathbf{x}_k\rangle\geq\log\bigl(\varepsilon_1^{-1}\bigr)\end{aligned}$$

for all $k\geq s_j\in\mathbb{N}$. For these $k\in\mathbb{N}$, we have $\mathbf{x}_k=(x_k,y_k)\in S_j(\varepsilon_0,\varepsilon_1)$.

(II) $\lim_{k\to\infty}\arg(\mathbf{x}_k)\in(\arg(q_{j-1}),\arg(q_k))$, or $\arg(q_{j-1})\leq\lim_{k\to\infty}\arg(\mathbf{x}_k)\leq\arg(q_j)$ for $j\in\{1,\dots,J+1\}$. Here, $|\mathbf{x}_k|\to\infty$ means we can take a bound $c_j\in\mathbb{N}$ s.t.

$$\begin{aligned}&\langle q_{j-1}^\perp,\mathbf{x}_k\rangle\geq\log\bigl(\varepsilon_0^{-1}\bigr),\\ &\langle q_j^\perp,\mathbf{x}_k\rangle\leq\log(\varepsilon_0)\end{aligned}$$

for all $k\geq c_j\in\mathbb{N}$. For these $k\in\mathbb{N}$, we have $\mathbf{x}_k=(x_k,y_k)\in C_j(\varepsilon_0)$.

Either way, we can take $k_0:=\max\left(\bigcup_{j=1}^J\{s_j,c_j\}\cup\{c_{J+1}\}\right)$ so that for $k\geq k_0$, $\mathbf{x}_k=(x_k,y_k)\in\mathcal{S}(\varepsilon_0,\varepsilon_1)$. Since this holds for any sequence $(x_n,y_n)_{n\in\mathbb{N}}$ with $y_n\to\infty$, and since we can describe $B(y_0)$ as the set of all possible values $(x_k,y_k)\in\mathbb{R}^2$ with $k\geq k_0$ giving $y_k\geq y_0$ for sufficiently large y_0, generated by these sequences $(x_n,y_n)_{n\in\mathbb{N}}$, we can imply

$$(x,y)\in B(y_0)\Rightarrow(x,y)=(x_k,y_k)\in\mathcal{S}(\varepsilon_0,\varepsilon_1).$$

Therefore $B(y_0)\subseteq\mathcal{S}(\varepsilon_0,\varepsilon_1)$ for some $y_0\geq 0$, and as earlier mentioned, $A(\mu_0)\subseteq\mathcal{G}(\varepsilon_0,\varepsilon_1)$ for some μ_0.

Suppose we have that $N(\nu)$ is not regular in time, or in other words $q_1=(1,0)$ and $q_1^\perp=(0,1)$. Then, if we take y_0 large enough, or more precisely, $y_0>\log\bigl(\varepsilon_0^{-1}\bigr)$, we get that for any $(x,y)\in B(y_0)$ that

$$\langle q_1^\perp,(x,y)\rangle=y\geq y_0>\log\bigl(\varepsilon_0^{-1}\bigr)>\log(\varepsilon_0),$$

which means that, as seen in lemma 2.2.3, we have $(x,y)\notin S_1(\varepsilon_0,\varepsilon_1)$ and $(x,y)\notin C_1(\varepsilon_0)$. This means we can instead find a cover

$$B(y_0)\subseteq\bigcup_{j=2}^J\{S_j(\varepsilon_0,\varepsilon_1)\cup C_j(\varepsilon_0)\}\cup C_{J+1}(\varepsilon_0)$$

for some $y_0\in\mathbb{R}$, and likewise we can always cover $A(\mu_0)$ by

$$A(\mu_0)\subseteq\bigcup_{j=2}^J\left\{G_j(\varepsilon_0,\varepsilon_1)\cup\widetilde{G}_j(\varepsilon_0)\right\}\cup\widetilde{G}_{J+1}(\varepsilon_0)$$

for some $\mu_0>0$.

□

Example

The symbol $P_S(\lambda,z)=\lambda+|z|_-^2\sqrt{\lambda+|z|_-^2}$ from the Stefan problem has the Newton polygon N as in figure 1.11. From this Newton polygon, and definition 1.2.3, we can find the weights γ_j

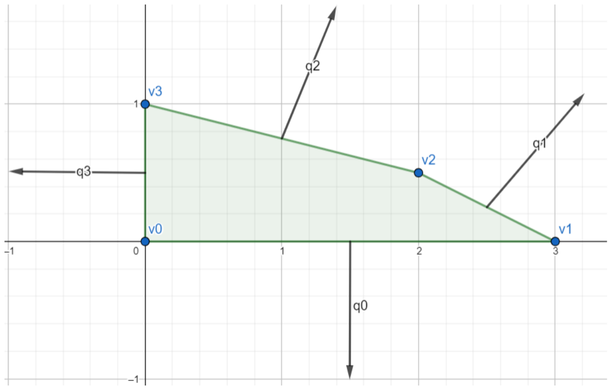

Figure 2.3: The Newton Polygon $N(P_S)$, vertices v_j and outward vectors q_j of the symbol of the Stefan problem.

and the outward vectors q_j are

$$\begin{aligned}
\gamma_0 &= 0, & q_0 &= (0,-1)\\
\gamma_1 &= 2, & q_1 &= \frac{1}{\sqrt{5}}(1,2)\\
\gamma_2 &= 4, & q_2 &= \frac{1}{\sqrt{17}}(1,4)\\
\gamma_3 &= \infty, & q_3 &= (-1,0).
\end{aligned}$$

In figure 2.3 a graphical description of the polygons with the q_j vectors can be seen.

Using lemma 2.2.3, we can then define the partition $G(\varepsilon_0,\varepsilon_1)$ and $\widetilde{G}(\varepsilon_0)$ as

$$\begin{aligned}
G_1(\varepsilon_0,\varepsilon_1) &= \left\{(\xi,\eta)\in(0,\infty)^2 : \varepsilon_0^{\sqrt{5}}\xi^2 \le \eta \le \varepsilon_0^{-\sqrt{5}}\xi^2 \text{ and } \xi\eta^2 \ge \varepsilon_1^{-\sqrt{5}}\right\},\\
G_2(\varepsilon_0,\varepsilon_1) &= \left\{(\xi,\eta)\in(0,\infty)^2 : \varepsilon_0^{\sqrt{17}}\xi^4 \le \eta \le \varepsilon_0^{-\sqrt{17}}\xi^4 \text{ and } \xi\eta^4 \ge \varepsilon_1^{-\sqrt{17}}\right\},
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{G}_1(\varepsilon_0) &= \left\{(\xi,\eta)\in(0,\infty)^2 : \eta \le \varepsilon_0^{\sqrt{5}}\xi^2 \text{ and } \xi \ge \varepsilon_0^{-1}\right\},\\
\widetilde{G}_2(\varepsilon_0) &= \left\{(\xi,\eta)\in(0,\infty)^2 : \varepsilon_0^{-\sqrt{5}}\xi^2 \le \eta \le \varepsilon_0^{\sqrt{17}}\xi^4\right\},\\
\widetilde{G}_3(\varepsilon_0) &= \left\{(\xi,\eta)\in(0,\infty)^2 : \varepsilon_0^{-\sqrt{17}}\xi^4 \le \eta \text{ and } \eta \ge \varepsilon_0^{-1}\right\}.
\end{aligned}$$

This is also shown for $\varepsilon_0 = \varepsilon_1 = \frac{1}{2}$ in figure 2.4.

We can show this partition does indeed cover $\{(\xi,\eta)\in(0,\infty)^2 : \eta \ge \mu_0\}$ for some $\mu_0 > 0$. As seen in figure 2.5, we can take any $\mu_0 > 3$ such that we will have no holes in the partition for $\eta > 3$.

The finite set $\nu \subseteq [0,\infty)^2$ belonging to $P(\lambda, z)$ is $\nu(P) = \{(0,1),(2,\frac{1}{2}),(3,0)\}$. Take $\varepsilon = 1$, then by lemma 2.2.4(i), there are $\varepsilon_0 > 0$ available such that we can compare $v_1 = (3,0) \in N$ to $(0,1),(2,\frac{1}{2}) \in \nu \subseteq \{(3,0)\}$ on the values in the partition $(\tilde{\xi},\tilde{\eta}) \in \widetilde{G}_1(\varepsilon_0)$:

$$\tilde{\eta} \le \tilde{\xi}^3, \qquad \tilde{\eta}^{\frac{1}{2}}\tilde{\xi}^2 \le \tilde{\xi}^3.$$

We can also use lemma 2.2.4(iii) for the edge $[v_1v_2]$, with the points $(3,0)$ and $(2,\frac{1}{2})$, using the same ε_0 as previously, and some chosen $\varepsilon_1 > 0$. We use the edge $[v_1v_2]$ and the values in the

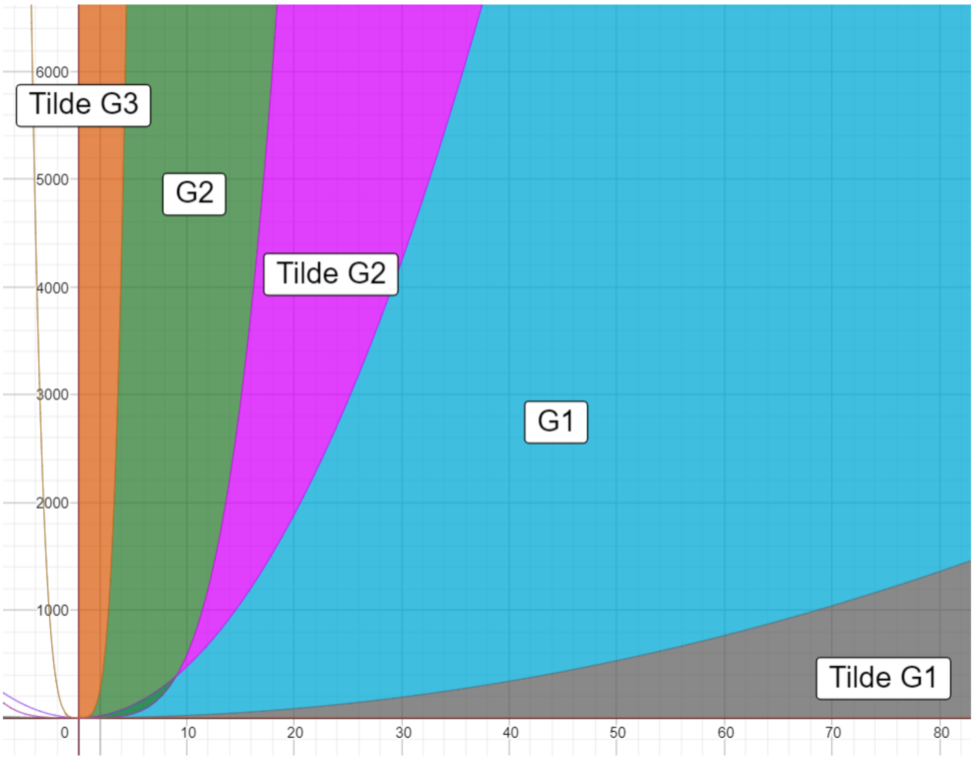

Figure 2.4: A graphical description of (ξ, η) in G_1, G_2 and $\widetilde{G}_1$, $\widetilde{G}_2$ and $\widetilde{G}_1$ of the Stefan problem, for $\varepsilon_0 = \varepsilon_1 = \frac{1}{2}$. As is visible, there is some overlap of strips G_1 and G_2 near the origin, but it is possible to choose ε_1 in such way that this does not happen.

Figure 2.5: The same partition of the Stefan problem, zoomed in around the origin. Any $\mu_0 > 2.5$ will satisfy 2.2.4(iv) for these $\varepsilon_0, \varepsilon_1$.

partition $(\xi, \eta) \in G_1(\varepsilon_0, \varepsilon_1)$, so that for the point $(0,1) \in \nu \setminus [v_1 v_2]$, we get the inequalities

$$\eta \leq \xi^3, \qquad \eta \leq \eta^{\frac{1}{2}} \xi^2.$$

References

R. Denk & M. Kaip, section 2.2b [1]:

1. Definition 2.43, p. 94
2. Remark 2.44, p. 95
3. Lemma 2.45 and proof, p. 95-98

The proof of lemma 2.2.3 is based on the idea of proof given in Remark 2.44 [1].

2.3 γ-principal parts on the partition

In this section we define several helpful lemmas that allow us to proof the other direction, based on the partition of the previous section 2.2.

We first prove certain bounds for ρ-homogeneous functions $\tau(\lambda, z)$ and their γ-principal parts, in the following three cases:

1. $\gamma = \gamma_j$,
2. $\gamma \in (\gamma_{j-1}, \gamma_j)$ and $\rho \notin (\gamma_{j-1}, \gamma_j)$,
3. $\gamma = \rho$.

The case $\gamma \in (\gamma_{j-1}, \gamma_j)$ and $\rho \in (\gamma_{j-1}, \gamma_j)$ is not treated, as we can bypass this by using lemma 1.2.13 in a clever way.

Lemma 2.3.1. *Let N be an arbitrary Newton polygon and $\tau(\lambda, z) \in S^{(\rho,M)}(L_t \times L_x)$ for some $M \geq 0$. For $\vartheta, \varepsilon_0, \varepsilon_1 > 0$ arbitrary, there exists a bound $\mu_1 > 0$ such that for any $j \in \{r_t(N), \ldots J\}$ and $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ that have $(|z|, |\lambda|) \in G_j(\varepsilon_0, \varepsilon_1)$ and $|\lambda| \geq \mu_1$, we have the property*

$$|\tau(\lambda, z) - \pi_{\gamma_j} \tau(\lambda, z)| \leq \vartheta |\pi_{\gamma_j} \tau(\lambda, z)|.$$

Proof. We define the difference function $\Gamma_j(\lambda, z)$ as

$$\Gamma_j(\lambda, z) := \frac{\tau(\lambda, z) - \pi_{\gamma_j} \tau(\lambda, z)}{\pi_{\gamma_j} \tau(\lambda, z)}, \quad (\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}),$$

for a fixed $j \in \{r_t(N), \ldots, J\}$ and γ_j of the Newton polygon N. We distinguish three cases related to ρ:

(I) $\gamma_j < \rho$. We use lemma 1.1.3 to get a bound $|\tau(\lambda, z)| \geq C_2(|\lambda|^{\frac{M}{\rho}} + |z|^M)$. Now using lemma 1.1.5, we see $|\pi_{\gamma_j} \tau(\lambda, z)| = |\tau(0, z)| \geq C_2 |z|^M$. We apply this estimate on $|\Gamma_j(\lambda, z)|$, and use the fact that $\tau \in S^{(\rho,M)}(L_t \times L_x)$:

$$|\Gamma_j(\lambda, z)| \leq \frac{|\tau(\lambda, z) - \tau(0, z)|}{C_1 |z|^M} = \frac{1}{C_1} \left| \tau\left(\frac{\lambda}{|z|^\rho}, \frac{z}{|z|} \right) - \tau\left(0, \frac{z}{|z|}\right) \right|.$$

We can find a bound of $\frac{|\lambda|}{|z|^\rho}$ using the definition of $(|z|,|\lambda|) \in G_j(\varepsilon_0,\varepsilon_1)$ as seen in lemma 2.2.3.

$$\begin{aligned}
|\lambda| &\leq \varepsilon_0^{-\frac{1}{q_{j,1}}} |z|^{\gamma_j}, \\
|\lambda|^{\frac{\rho}{\gamma_j}} &\leq \varepsilon_0^{-\frac{\rho}{\gamma_j q_{j,1}}} |z|^{\rho}, \\
|\lambda| &\leq \varepsilon_0^{-\frac{\rho}{\gamma_j q_{j,1}}} |z|^{\rho} |\lambda|^{1-\frac{\rho}{\gamma_j}}, \\
\frac{|\lambda|}{|z|^\rho} &\leq \varepsilon_0^{-\frac{\rho}{\gamma_j q_{j,1}}} |\lambda|^{1-\frac{\rho}{\gamma_j}} = \varepsilon_0^{-\frac{\rho}{\gamma_j q_{j,1}}} |\lambda|^{\frac{\gamma_j-\rho}{\gamma_j}}.
\end{aligned}$$

Next, we move towards setting a bound $|\Gamma_j(\lambda,z)| \leq \frac{\vartheta}{C_1}$. Consider the set

$$K_\lambda = \{(\lambda',z') \in L_t \times L_x : |\lambda'| \leq 1, |z'| = 1\}. \tag{2.3}$$

This set is compact, so $\tau(\lambda,z)$, being a continuous function, is uniformly continuous on K_λ. Therefore, we can find a $\delta > 0$ for each $\vartheta > 0$ such that $|\lambda_1'-\lambda_2'|+|z_1'-z_2'| \leq \delta$ implies $|\tau(\lambda_1',z_1') - \tau(\lambda_2',z_2')| \leq C_1\vartheta$ for $(\lambda_1',z_1'),(\lambda_2',z_2') \in K_\lambda$. Since $\frac{|\lambda|}{|z|^\rho} \leq \varepsilon_0^{-\frac{\rho}{\gamma_j q_{j,1}}} |\lambda|^{1-\frac{\rho}{\gamma_j}}$, we can choose a bound $\mu_j > 0$ large enough s.t. for $|\lambda| > \mu_j$, we get

$$\frac{|\lambda|}{|z|^\rho} \leq \varepsilon_0^{-\frac{\rho}{\gamma_j q_{j,1}}} |\lambda|^{\frac{\gamma_j-\rho}{\gamma_j}} \leq \min\{\delta,1\}$$

using $\gamma_j < \rho$. This means $(\frac{\lambda}{|z|^\rho},\frac{z}{|z|}) \in K_\lambda$ and $(0,\frac{z}{|z|}) \in K_\lambda$, and $|\frac{\lambda}{|z|^\rho}-0|+|0-0| = \frac{|\lambda|}{|z|^\rho} \leq \delta$. Thus for $(\lambda,z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ with $(|z|,|\lambda|) \in G_j(\varepsilon_0,\varepsilon_1)$ and $|\lambda_0| \geq \mu_j$, we can use the uniform continuity of $\tau(\lambda,z)$ on the set K_λ to get

$$|\Gamma_j(\lambda,z)| \leq \frac{1}{C_1}\left|\tau\left(\frac{\lambda}{|z|^\rho},\frac{z}{|z|}\right) - \tau\left(0,\frac{z}{|z|}\right)\right| \leq \frac{C_1\vartheta}{C_1} = \vartheta.$$

(II) $\gamma_j > \rho$. In this case, $\pi_{\gamma_j}\tau(\lambda,z) = \tau(\lambda,0)$. Our argumentation follows largely the same steps. We get the bound

$$|\Gamma_j(\lambda,z)| \leq \frac{1}{C_1}\left|\tau\left(\frac{\lambda}{|\lambda|},\frac{z}{|\lambda|^{\frac{1}{\rho}}}\right) - \tau\left(\frac{\lambda}{|\lambda|},0\right)\right|.$$

Next, we have to take regularity in space in consideration, as is seen in lemma 2.2.3 for the definition of $(|z|,|\lambda|) \in G_j(\varepsilon_0,\varepsilon_1)$ to get slightly different bounds. For N regular in space, we get

$$\begin{aligned}
\varepsilon_0^{\frac{1}{q_{j,1}}} |z|^{\gamma_j} &\leq |\lambda|, \\
|z| &\leq \varepsilon_0^{-\frac{1}{\gamma_j q_{j,1}}} |\lambda|^{\frac{1}{\gamma_j}}, \\
\frac{|z|}{|\lambda|^{\frac{1}{\rho}}} &\leq \varepsilon_0^{-\frac{1}{\gamma_j q_{j,1}}} |\lambda|^{\frac{1}{\gamma_j}-\frac{1}{\rho}} = \varepsilon_0^{-\frac{1}{\gamma_j q_{j,1}}} |\lambda|^{\frac{\rho-\gamma_j}{\gamma_j\rho}},
\end{aligned}$$

or if $j = J$ for a Newton polygon that is not regular in space,

$$\begin{aligned}
|z| &\leq \varepsilon_0^{-1}, \\
\frac{|z|}{|\lambda|^{\frac{1}{\rho}}} &\leq \frac{\varepsilon_0^{-1}}{|\lambda|^{\frac{1}{\rho}}}.
\end{aligned}$$

Either way, if μ_j is chosen large enough for a $\delta > 0$ belonging to a $\vartheta > 0$, we can take $|\lambda| \geq \mu_j$ and use $\gamma_j > \rho$ to get

$$\frac{|z|}{|\lambda|^{\frac{1}{\rho}}} \leq \min\{\delta, 1\},$$

so that the compact set

$$K_z = \{(\lambda', z') \in L_t \times L_x : |\lambda'| = 1, |z'| \leq 1\} \tag{2.4}$$

allows us to conclude that

$$|\Gamma_j(\lambda, z)| \leq \vartheta$$

in a similar way as above.

(III) $\gamma_j = \rho$. In this case $\pi_{\gamma_j}\tau(\lambda, z) = \tau(\lambda, z)$, so $|\Gamma_j(\lambda, z)| = 0 \leq \vartheta$.

To complete, we need to find a μ_1 that is independent of chosen j, so we take

$$\mu_1 = \max_{j \in \{r_t(N), \ldots, J\}} \mu_j.$$

□

Now for $\gamma \in (\gamma_{j-1}, \gamma_j)$, we can instead use $\widetilde{G}_j(\varepsilon_0)$ to make a similar estimate.

Lemma 2.3.2. *Let N be a Newton polygon and $\tau(\lambda, z) \in S^{(\rho,M)}(L_t \times L_x)$ for some $M \geq 0$.*

(i) For fixed $j \in \{r_t(N) \ldots, r_t(N)\}$ that has $\rho \notin (\gamma_{j-1}, \gamma_j)$ and an arbitrary $\vartheta > 0$, there exists a bound $\varepsilon_j > 0$ such that for any $0 < \varepsilon_0 \leq \varepsilon_j$, $\gamma \in (\gamma_{j-1}, \gamma_j)$ and any $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ that have $(|z|, |\lambda|) \in \widetilde{G}_j(\varepsilon_0)$ and $|\lambda| \geq 1$, we have the property

$$|\tau(\lambda, z) - \pi_\gamma\tau(\lambda, z)| \leq \vartheta |\pi_\gamma\tau(\lambda, z)|.$$

(ii) If N is not regular in space, we can instead find an $\varepsilon_{J+1} > 0$ such that for any $0 < \varepsilon_0 \leq \varepsilon_{J+1}$ and $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ that have $(|z|, |\lambda|) \in \widetilde{G}_{J+1}(\varepsilon_0)$ and $|\lambda| \geq 1$, the property for $j = J + 1$:

$$|\tau(\lambda, z) - \tau(\lambda, 0)| \leq \vartheta |\tau(\lambda, 0)|$$

Proof. (i) Fix $j \in \{r_t(N), \ldots, r_x(N)\}$, that has $\rho \notin (\gamma_{j-1}, \gamma_j)$, and take $\gamma \in (\gamma_{j-1}, \gamma_j)$. Define

$$\Gamma(\lambda, z) := \frac{\tau(\lambda, z) - \pi_\gamma\tau(\lambda, z)}{\pi_\gamma\tau(\lambda, z)}, \quad (\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}).$$

Like in the proof of lemma 2.3.1, we use lemma 1.1.3 to get $|\tau(\lambda, z)| \geq C_1(|\lambda|^{\frac{M}{\rho}} + |z|^M)$. We use a compact set $K := K_\lambda \cup K_z$, defined as in equations (2.3) and (2.4).

$$K = \{(\lambda', z') \in L_t \times L_x : |\lambda'| \leq 1, |z'| = 1 \text{ or } |\lambda'| = 1, |z'| \leq 1\}.$$

$\tau(\lambda, z)$ is uniformly continuous on this set, so for any $\vartheta > 0$, there exists a $\delta > 0$ so that $|\lambda_1' - \lambda_2'| + |z_1' - z_2'| \leq \delta$ implies $|\tau(\lambda_1', z_1') - \tau(\lambda_2', z_2')| \leq C_1\vartheta$ for $(\lambda_1', z_1'), (\lambda_2', z_2') \in K$. For this proof, we only consider take $0 < \delta < 1$, so that we remain in the set K. Since we can take δ as small as possible, this is not a problem for the proof.

Now we investigate the cases related to ρ.

(I) $\gamma_j \leq \rho$. Since $\rho \in (0,\infty)$, we must have $\gamma_j \neq \infty$, which allows us to use the definition for G_j with $j < J+1$. It also means $\gamma < \rho$, so we use lemma 1.1.3 like we did in the proof of lemma 2.3.1 for $\gamma_j < \rho$ to get

$$|\Gamma(\lambda,z)| \leq \frac{1}{C_1}\left|\tau\left(\frac{\lambda}{|z|^\rho},\frac{z}{|z|}\right) - \tau\left(0,\frac{z}{|z|}\right)\right|.$$

Take $\varepsilon_j := \delta^{\frac{\gamma_j q_{j,1}}{\rho}}$, and let $0 < \varepsilon_0 \leq \varepsilon_j$. We deduce a bound for $\frac{|\lambda|}{|z|^\rho}$ with $(|z|,|\lambda|) \in \widetilde{G}_j(\varepsilon_0)$, using its definition from lemma 2.2.3.

$$\begin{aligned} |\lambda| &\leq \varepsilon_0^{\frac{1}{q_{j,1}}}|z|^{\gamma_j} \\ \frac{|\lambda|^{\frac{\rho}{\gamma_j}}}{|z|^\rho} &\leq \varepsilon_0^{\frac{\rho}{\gamma_j q_{j,1}}} \\ \frac{|\lambda|}{|z|^\rho} &\leq \varepsilon_0^{\frac{\rho}{\gamma_j q_{j,1}}}|\lambda|^{1-\frac{\rho}{\gamma_j}}, \end{aligned}$$

which means that if $|\lambda| \leq 1$, we have

$$\frac{|\lambda|}{|z|^\rho} \leq \varepsilon_0^{\frac{\rho}{\gamma_j q_{j,1}}}|\lambda|^{1-\frac{\rho}{\gamma_j}} \leq \varepsilon_0^{\frac{\rho}{\gamma_j q_{j,1}}} \leq \varepsilon_j^{\frac{\rho}{\gamma_j q_{j,1}}} = \delta < 1.$$

Therefore, $\left\|\left(\frac{\lambda}{|z|^\rho},\frac{z}{|z|}\right) - \left(0,\frac{z}{|z|}\right)\right\| \leq \delta$, and both points are in K.

(II) $\gamma_j > \rho$. Since we picked $j \leq k_x(N)$, we know that $0 < \rho < \gamma_{j-1} < \gamma_j \leq \infty$, so $\gamma_{j-1} \notin \{0,\infty\}$. We have $\gamma > \rho$, so we get

$$|\Gamma(\lambda,z)| \leq \frac{1}{C_1}\left|\tau\left(\frac{\lambda}{|\lambda|},\frac{z}{|\lambda|^{\frac{1}{\rho}}}\right) - \tau\left(\frac{\lambda}{|\lambda|},0\right)\right|.$$

Take $\varepsilon_j := \delta^{\gamma_{j-1}q_{j-1,1}}$ and any $0 < \varepsilon_0 \leq \varepsilon_j$, so we can bind $\frac{|z|}{|\lambda|^{\frac{1}{\rho}}}$ with $(|z|,|\lambda|) \in \widetilde{G}_j(\varepsilon_0)$ by using lemma 2.2.3 to find

$$\begin{aligned} \varepsilon_0^{-\frac{1}{q_{j-1,1}}}|z|^{\gamma_{j-1}} &\leq |\lambda| \\ |z| &\leq \varepsilon_0^{\frac{1}{\gamma_{j-1}q_{j-1,1}}}|\lambda|^{\frac{1}{\gamma_{j-1}}} \\ \frac{|z|}{|\lambda|^{\frac{1}{\rho}}} &\leq \varepsilon_0^{\frac{1}{\gamma_{j-1}q_{j-1,1}}}|\lambda|^{\frac{1}{\gamma_{j-1}}-\frac{1}{\rho}}, \end{aligned}$$

so that if $|\lambda| \leq 1$, we get

$$\frac{|z|}{|\lambda|^{\frac{1}{\rho}}} \leq \varepsilon_0^{\frac{1}{\gamma_{j-1}q_{j-1,1}}}|\lambda|^{\frac{1}{\gamma_{j-1}}-\frac{1}{\rho}} \leq \varepsilon_0^{\frac{1}{\gamma_{j-1}q_{j-1,1}}} \leq \varepsilon_j^{\frac{1}{\gamma_{j-1}q_{j-1,1}}} = \delta < 1.$$

Therefore, $\left\|\left(\frac{\lambda}{|\lambda|},\frac{z}{|\lambda|^{\frac{1}{\rho}}}\right) - \left(\frac{\lambda}{|\lambda|},0\right)\right\| \leq \delta$, and both points are in K.

In both cases, we can use the uniform continuity of $\tau(\lambda, z)$ on the set K so that for any $\vartheta > 0$, we can find $0 < \delta < 1$ s.t. for $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ with $(|z|, |\lambda|) \in \widetilde{G}_j(\varepsilon_0)$ and $|\lambda| \geq 1$, we get

$$|\Gamma(\lambda, z)| \leq \frac{C_1 \vartheta}{C_1} = \vartheta.$$

(ii) If N is not regular in space, we instead take $\varepsilon_{J+1} = \delta^{-1-\frac{1}{\rho}}$. Since $\gamma_{J+1} = \gamma_J = \infty > \rho$, we only investigate $\gamma = \infty > \rho$. Therefore we use the bound from (II) from the previous point:

$$|\Gamma(\lambda, z)| \leq \frac{1}{C_1} \left| \tau\left(\frac{\lambda}{|\lambda|}, \frac{z}{|\lambda|^{\frac{1}{\rho}}}\right) - \tau\left(\frac{\lambda}{|\lambda|}, 0\right) \right|.$$

For any $0 < \varepsilon_0 \leq \varepsilon_{J+1}$, use lemma 2.2.3(ii) to find that for any $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ with $(|z|, |\lambda|) \in \widetilde{G}_{J+1}(\varepsilon_0)$, we get

$$|\lambda| \geq \varepsilon_0^{-1}, \quad |z| \leq \varepsilon_0,$$
$$\frac{|z|}{|\lambda|^{\frac{1}{\rho}}} \leq \varepsilon_0^{1+\frac{1}{\rho}} \leq \varepsilon_{J+1}^{1+\frac{1}{\rho}} = \delta.$$

Therefore using the uniform continuity of $\tau(\lambda, z)$ on the compact set K, we conclude $|\Gamma(\lambda, z)| \leq \frac{C_1 \vartheta}{C_1} = \vartheta$

□

Remark. Note that the case $\gamma \in (\gamma_{j-1}, \gamma_j)$ with $\rho \in (\gamma_{j-1}, \gamma_j)$ is not covered in this section. However, this is since $\pi_\rho \tau(\lambda, z) = \tau(\lambda, z)$, which means we will not need an estimate for τ. We will need one for $\pi_\rho P$, as we see in the next section.

References

R. Denk & M. Kaip, section 2.2c [1]:

1. Lemma 2.51 and proof, p. 100-101
2. Lemma 2.52 and proof, p. 101-103

2.4 Estimates of symbols $P \in S(L_t \times L_x)$

Next we will consider $P \in S(L_t \times L_x)$. We will use the same cases of γ as in the previous section, and work these cases through for a bound on which to bind the γ-principal parts of $P(\lambda, z)$.

Lemma 2.4.1. *Let $P \in S(L_t \times L_x)$ be a symbol with*

$$\pi_\gamma P(\lambda, z) \neq 0, \quad \pi_\infty P(\lambda, 0) \neq 0, \quad (\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}),\ \gamma \in (0, \infty].$$

For every $j \in \{r_t(N(P)), \ldots, J\}$, there exists a constant $C_j > 0$ s.t.

$$|\pi_{\gamma_j} P(\lambda, z)| \geq C_j (|\lambda|^{s_j} |z|^{r_j} + |\lambda|^{s_{j+1}} |s|^{r_{j+1}}), \quad (\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}).$$

Proof. First, fix $j \in \{r_t(N(P)), \ldots, r_x(N(P)) - 1\}$, so we can treat the case $j = J$ and $\gamma_J = \infty$ at a later time. We will go through the cases $\gamma_j > \rho, \gamma_j < \rho$ and $\gamma_j = \rho$, and try to contradict the non-vanishing principal parts of $P(\lambda, z)$, making use of some scaled principal part function $\widetilde{P}(\lambda, z)$.

(I) $\gamma_j > \rho$. Then we can use the definition of the γ-principal part from definition 1.1.7:

$$\pi_{\gamma_j} P(\lambda, z) = \sum_{\ell \in I_\gamma} \tau_\ell(\lambda, 0)\phi_\ell(\lambda)\psi_\ell(z).$$

Since $\gamma_j > \rho$, we have that all points $\mathfrak{v}_\ell := (L_\ell, \frac{N_\ell}{\rho} + M_\ell) \in [v_j v_{j+1}]$ for any $\ell \in I_{\gamma_j}$ by lemma 1.2.12. This means that we have

$$L_\ell \geq r_{j+1}, \quad \frac{N_\ell}{\rho} + M_\ell \geq s_j, \quad \ell \in I_{\gamma_j}.$$

For $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$, define

$$\widetilde{P}(\lambda, z) := \sum_{\ell \in I_{\gamma_j}} \left[|\lambda|^{\frac{N_\ell}{\rho} + M_\ell - s_j} |z|^{L_\ell - r_{j+1}} \cdot \tau_\ell\left(\frac{\lambda}{|\lambda|}, 0\right) \phi_\ell\left(\frac{\lambda}{|\lambda|}\right) \psi_\ell\left(\frac{z}{|z|}\right) \right],$$

which means that by (quasi-)homogeneity, we have $|\lambda|^{s_j}|z|^{r_{j+1}}\widetilde{P}(\lambda, z) = \pi_{\gamma_j} P(\lambda, z) \neq 0$. This allows us to transform our problem into trying to proof a bound of the form

$$\begin{aligned}
|\pi_{\gamma_j} P(\lambda, z)| &\geq C_j(|\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}}), \\
|\widetilde{P}(\lambda, z)| &\geq C_j \frac{|\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}}}{|\lambda|^{s_j}|z|^{r_{j+1}}}, \\
|\widetilde{P}(\lambda, z)| &\geq C_j(|\lambda|^{s_{j+1} - s_j} + |z|^{r_j - r_{j+1}}), \quad (\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})
\end{aligned}$$

for some constant $C_j > 0$. We can however write this bound in another form, using the compact set

$$\Omega_j := \{(\alpha, \zeta) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}) : |\alpha|^{s_{j+1} - s_j} + |\zeta|^{r_j - r_{j+1}} = 1\},$$

and writing

$$\begin{aligned}
&\sum_{\ell \in I_{\gamma_j}} \left[|\lambda|^{\frac{N_\ell}{\rho} + M_\ell - s_j} |z|^{L_\ell - r_{j+1}} \cdot \tau_\ell\left(\frac{\lambda}{|\lambda|}, 0\right) \phi_\ell\left(\frac{\lambda}{|\lambda|}\right) \psi_\ell\left(\frac{z}{|z|}\right) \right] \geq C_j(|\lambda|^{s_{j+1} - s_j} + |z|^{r_j - r_{j+1}}), \\
&\sum_{\ell \in I_{\gamma_j}} \left[\frac{|\lambda|^{\frac{N_\ell}{\rho} + M_\ell - s_j} |z|^{L_\ell - r_{j+1}}}{|\lambda|^{s_{j+1} - s_j} + |z|^{r_j - r_{j+1}}} \cdot \tau_\ell\left(\frac{\lambda}{|\lambda|}, 0\right) \phi_\ell\left(\frac{\lambda}{|\lambda|}\right) \psi_\ell\left(\frac{z}{|z|}\right) \right] \geq C_j, \\
&\sum_{\ell \in I_{\gamma_j}} \left[|\alpha|^{\frac{N_\ell}{\rho} + M_\ell - s_j} |\zeta|^{L_\ell - r_{j+1}} \cdot \tau_\ell\left(\frac{\alpha}{|\alpha|}, 0\right) \phi_\ell\left(\frac{\alpha}{|\alpha|}\right) \psi_\ell\left(\frac{\zeta}{|\zeta|}\right) \right] = \widetilde{P}(\alpha, \zeta) \geq C_j.
\end{aligned}$$

We will now proof $\widetilde{P}(\alpha, \zeta) \geq C_j$ for $(\alpha, \zeta) \in \Omega_j$ is true by contradiction. Assume there exists a sequence $(\alpha_n, \zeta_n)_{n \in \mathbb{N}} \subseteq \Omega_j$ that has $\widetilde{P}(\alpha_n, \zeta_n) \to 0$. Similiar to what we did in the proof of lemma 2.2.4(iv), we can find a subsequence $(\alpha_{n_k}, \zeta_{n_k})$ that has $\frac{\alpha_{n_k}}{|\alpha_{n_k}|}$ and $\frac{\zeta_{n_k}}{|\zeta_{n_k}|}$ converging. Take this subsequence as the sequence (α_k, ζ_k), so we can define

$$\begin{aligned}
(\alpha_0, \zeta_0) &:= \lim_{k \to \infty} (\alpha_k, \zeta_k) \in \Omega_j, \\
\alpha' &:= \lim_{k \to \infty} \frac{\alpha_k}{|\alpha_k|} \neq 0, \\
\zeta' &:= \lim_{k \to \infty} \frac{\zeta_k}{|\zeta_k|} \neq 0.
\end{aligned}$$

Note that since Ω_j is compact, this limit (α_0, ζ_0) always exists for this subsequence (a_k, ζ_k) with converging quotients $(\frac{\alpha_k}{|\alpha_k|}, \frac{\zeta_k}{|\zeta_k|}) \to (\alpha', \zeta')$.

We must consider the three possible cases, in which we find a contradiction to having non-vanishing γ-principal parts:

(1) $\alpha_0 \neq 0, \zeta_0 \neq 0$. We use the continuity of $\widetilde{P}$ on Ω_j to get

$$\begin{aligned}\lim_{k\to\infty} \widetilde{P}(\alpha_k, \zeta_k) &= \widetilde{P}(\alpha_0, \zeta_0)\\ &= 0.\end{aligned}$$

This also means $\pi_{\gamma_j} P(\alpha_0, \zeta_0) = 0$, but since $(\alpha_0, \zeta_0) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$, this is a contradiction to $\pi_\gamma P \neq 0$.

(2) $\alpha_0 = 0$. Then for (α_0, ζ_0) to be in Ω_j, we need $|\zeta_0| = 1$, which means $\zeta_0 = \zeta'$. Then we can write

$$\begin{aligned}\lim_{k\to\infty} \widetilde{P}(\alpha_k, \zeta_k) &= \lim_{k\to\infty} \sum_{\ell \in I_{\gamma_j}} \left[|\alpha_k|^{\frac{N_\ell}{\rho} + M_\ell - s_j} |\zeta_k|^{L_\ell - r_{j+1}} \cdot \tau_\ell\left(\frac{\alpha_k}{|\alpha_k|}, 0\right) \phi_\ell\left(\frac{\alpha_k}{|\alpha_k|}\right) \psi_\ell\left(\frac{\zeta_k}{|\zeta_k|}\right)\right]\\ &= \sum_{\substack{\ell \in I_{\gamma_j},\\ \frac{N_\ell}{\rho} + M_\ell - s_j = 0}} |\zeta_0|^{L_\ell - r_{j+1}} \cdot \tau_\ell\left(\alpha', 0\right) \phi_\ell\left(\alpha'\right) \psi_\ell\left(\zeta'\right)\\ &= \sum_{\substack{\ell \in I_{\gamma_j},\\ \frac{N_\ell}{\rho} + M_\ell = s_j}} \tau_\ell\left(\alpha', 0\right) \phi_\ell\left(\alpha'\right) \psi_\ell\left(\zeta_0\right)\\ &= 0.\end{aligned}$$

We know $\gamma_j > \rho$, so taking any $\gamma \in (\max\{\gamma_{j-1}, \rho\}, \gamma_j)$, we can find the γ-principal part using lemma 1.2.13(ii) equation (1.16):

$$\begin{aligned}\pi_\gamma P(\alpha', \zeta_0) &= \sum_{\ell \in I_\gamma} \tau_\ell\left(\alpha', 0\right) \phi_\ell\left(\alpha'\right) \psi_\ell\left(\zeta_0\right)\\ &= \sum_{\substack{\ell \in I_{\gamma_j},\\ \frac{N_\ell}{\rho} + M_\ell = s_j}} \tau_\ell\left(\alpha', 0\right) \phi_\ell\left(\alpha'\right) \psi_\ell\left(\zeta_0\right)\\ &= 0,\end{aligned}$$

which is a contradiction to $\pi_\gamma P \neq 0$.

(3) $\zeta_0 = 0$. Here we instead have $|\alpha_0| = 1$, and $\alpha' = \alpha_0$, and we get

$$\begin{aligned}\lim_{k\to\infty} \widetilde{P}(\alpha_k, \zeta_k) &= \sum_{\substack{\ell \in I_{\gamma_j},\\ L_\ell - r_{j+1} = 0}} |\alpha_0|^{\frac{N_\ell}{\rho} + M_\ell - s_j} \cdot \tau_\ell\left(\alpha', 0\right) \phi_\ell\left(\alpha'\right) \psi_\ell\left(\zeta'\right)\\ &= \sum_{\substack{\ell \in I_{\gamma_j},\\ L_\ell = r_{j+1}}} \tau_\ell\left(\alpha_0, 0\right) \phi_\ell\left(\alpha_0\right) \psi_\ell\left(\zeta'\right)\\ &= 0,\end{aligned}$$

and since for $\gamma \in (\gamma_j, \gamma_{j+1})$, we can use lemma 1.2.13(iii) equation (1.17) and $\gamma_j > \rho$ to find the principal part

$$\begin{aligned}\pi_\gamma P(\alpha_0, \zeta') &= \sum_{\ell \in I_\gamma} \tau_\ell\left(\alpha_0, 0\right) \phi_\ell\left(\alpha_0\right) \psi_\ell\left(\zeta'\right) \\ &= \sum_{\substack{\ell \in I_{\gamma_j},\\ L_\ell = r_{j+1}}} \tau_\ell\left(\alpha_0, 0\right) \phi_\ell\left(\alpha_0\right) \psi_\ell\left(\zeta'\right) \\ &= 0,\end{aligned}$$

which is again a contradiction to $\pi_\gamma P \neq 0$.

In conclusion, the bound $\widetilde{P}(\alpha, \zeta) \geq C_j$ holds for some $C_j > 0$, and all $(\alpha, \zeta) \in \Omega_j$, and therefore $|\pi_{\gamma_j} P(\lambda, z)| \geq C_j(|\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}})$ holds for $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ and $\gamma_j > \rho$.

(II) $\gamma_j < \rho$. Then by lemma 1.2.12, we have $\mathfrak{u}_\ell := (N_\ell + L_\ell, M_\ell) \in [v_j v_{j+1}]$ for all $\ell \in I_{\gamma_j}$, so $N_\ell + L_\ell \geq r_{j+1}$ and $M_\ell \geq s_j$. Here,

$$\pi_{\gamma_j} P(\lambda, z) = \sum_{\ell \in I_{\gamma_j}} \tau_\ell(0, z) \phi_\ell(\lambda) \psi_\ell(z),$$

so define

$$\widetilde{P}(\lambda, z) := \sum_{\ell \in I_{\gamma_j}} \left[|\lambda|^{M_\ell - s_j} |z|^{N_\ell + L_\ell - r_{j+1}} \cdot \tau_\ell\left(0, \frac{z}{|z|}\right) \phi_\ell\left(\frac{\lambda}{|\lambda|}\right) \psi_\ell\left(\frac{z}{|z|}\right) \right]$$

for $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$, so that again $|\lambda|^{s_j}|z|^{r_{j+1}} \widetilde{P}(\lambda, z) = \pi_{\gamma_j} P(\lambda, z) \neq 0$, and therefore we prove $\widetilde{P}(\alpha, \zeta) \geq C_j$ is true for some $C_j > 0$ and all $(\alpha, \zeta) \in \Omega_j$, with Ω_j as in (I), using the same form of contradiction: assume there exists a sequence $(\alpha_n, \zeta_n)_{n \in \mathbb{N}} \subseteq \Omega_j$ with $\widetilde{P}(\alpha_n, \zeta_n) \to 0$. Define (α_k, ζ_k), (α_0, ζ_0) and (α', ζ') as in (I).

(1) $\alpha_0 \neq 0, \zeta_0 \neq 0$. Again using continuity, we have $\lim_{k\to\infty} \widetilde{P}(\alpha_k, \zeta_k) = \widetilde{P}(\alpha_0, \zeta_0) = 0$, so $\pi_{\gamma_j} P(\alpha_0, \zeta_0) = 0$, which is a contradiction to $\pi_\gamma P \neq 0$.

(2) $\alpha_0 = 0$. Then again we have $|\zeta_0| = 1$ and $\zeta_0 = \zeta'$. We write

$$\begin{aligned}\lim_{k\to\infty} \widetilde{P}(\alpha_k, \zeta_k) &= \lim_{k\to\infty} \sum_{\ell \in I_{\gamma_j}} \left[|\alpha_k|^{M_\ell - s_j} |\zeta_k|^{N_\ell + L_\ell - r_{j+1}} \cdot \tau_\ell\left(0, \frac{\zeta_k}{|\zeta_k|}\right) \phi_\ell\left(\frac{\alpha_k}{|\alpha_k|}\right) \psi_\ell\left(\frac{\zeta_k}{|\zeta_k|}\right) \right] \\ &= \sum_{\substack{\ell \in I_{\gamma_j},\\ M_\ell = s_j}} \tau_\ell\left(0, \zeta_0\right) \phi_\ell\left(\alpha'\right) \psi_\ell\left(\zeta_0\right) \\ &= 0.\end{aligned}$$

For $\gamma_j > \rho$, take $\gamma \in (\gamma_{j-1}, \gamma_j)$, we use lemma 1.2.13(ii) equation (1.16):

$$\begin{aligned}\pi_\gamma P(\alpha', \zeta_0) &= \sum_{\ell \in I_\gamma} \tau_\ell\left(0, \zeta_0\right) \phi_\ell\left(\alpha'\right) \psi_\ell\left(\zeta_0\right) \\ &= \sum_{\substack{\ell \in I_{\gamma_j},\\ M_\ell = s_j}} \tau_\ell\left(0, \zeta_0\right) \phi_\ell\left(\alpha'\right) \psi_\ell\left(\zeta_0\right) \\ &= 0,\end{aligned}$$

which is a contradiction to $\pi_\gamma P \neq 0$.

(3) $\zeta_0 = 0$. We have $|\alpha_0| = 1$, and $\alpha' = \alpha_0$, and we get

$$\begin{aligned}\lim_{k\to\infty} \widetilde{P}(\alpha_k, \zeta_k) &= \sum_{\substack{\ell \in I_{\gamma_j},\\ N_\ell + L_\ell = r_{j+1}}} \tau_\ell\left(0, \zeta'\right) \phi_\ell\left(\alpha_0\right) \psi_\ell\left(\zeta'\right)\\ &= 0,\end{aligned}$$

and taking $\gamma \in (\gamma_j, \min\{\gamma_{j+1}, \rho\})$ and using lemma 1.2.13(iii) equation (1.17) with $\gamma_j < \rho$ to find the principal part

$$\begin{aligned}\pi_\gamma P(\alpha_0, \zeta') &= \sum_{\ell \in I_\gamma} \tau_\ell\left(0, \zeta'\right) \phi_\ell\left(\alpha_0\right) \psi_\ell\left(\zeta'\right)\\ &= \sum_{\substack{\ell \in I_{\gamma_j},\\ N_\ell + L_\ell = r_{j+1}}} \tau_\ell\left(0, \zeta'\right) \phi_\ell\left(\alpha_0\right) \psi_\ell\left(\zeta'\right)\\ &= 0,\end{aligned}$$

which is again a contradiction to $\pi_\gamma P \neq 0$.

So, in this case we have $\widetilde{P}(\alpha, \zeta) \geq C_j$, so $|\pi_{\gamma_j} P(\lambda, z)| \geq C_j(|\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}})$ holds for $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ and $\gamma_j < \rho$.

(III) $\gamma_j = \rho$. We are now investigating

$$\pi_\rho P(\lambda, z) = \sum_{\ell \in I_\rho} \tau(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z).$$

In this case, for all $\ell \in I_\rho$, we have $(N_\ell + L_\ell, M_\ell) \in [v_j v_{j+1}]$ and $(L_\ell, \frac{N_\ell}{\rho} + M_\ell) \in [v_j v_{j+1}]$, which implies that $L_\ell \geq r_{j+1}$ and $M_\ell \geq s_j$. For $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$, define

$$\widetilde{P}(\lambda, z) := \sum_{\ell \in I_\rho} \left[|\lambda|^{M_\ell - s_j}|z|^{L_\ell - r_{j+1}} \tau_\ell\left(\lambda, z\right) \phi_\ell\left(\frac{\lambda}{|\lambda|}\right) \psi_\ell\left(\frac{z}{|z|}\right)\right].$$

Again, $|\lambda|^{s_j}|z|^{r_{j+1}}\widetilde{P}(\lambda, z) = \pi_\rho P(\lambda, z) \neq 0$, and we proof a bound $\widetilde{P}(\alpha, \zeta) \geq C_j$ for some constant $C_j > 0$ and $(\alpha, \zeta) \in \Omega_j$ by contradiction: assume there exists a sequence $(\alpha_n, \zeta_n)_{n\in\mathbb{N}} \subseteq \Omega_j$ with $\widetilde{P}(\alpha_n, \zeta_n) \to 0$. Define (α_k, ζ_k), (α_0, ζ_0) and (α', ζ') as in (I).

(1) $\alpha_0 \neq 0, \zeta_0 \neq 0$. Again using continuity, we have $\lim_{k\to\infty} \widetilde{P}(\alpha_k, \zeta_k) = \widetilde{P}(\alpha_0, \zeta_0) = 0$, so $\pi_\rho P(\alpha_0, \zeta_0) = 0$, which is a contradiction to $\pi_\gamma P \neq 0$.

(2) $\alpha_0 = 0$. Then again we have $|\zeta_0| = 1$ and $\zeta_0 = \zeta'$. We write

$$\begin{aligned}\lim_{k\to\infty} \widetilde{P}(\alpha_k, \zeta_k) &= \lim_{k\to\infty} \sum_{\ell \in I_{\gamma_j}} \left[|\alpha_k|^{M_\ell - s_j}|\zeta_k|^{L_\ell - r_{j+1}} \cdot \tau_\ell\left(\alpha_k, \zeta_k\right) \phi_\ell\left(\frac{\alpha_k}{|\alpha_k|}\right) \psi_\ell\left(\frac{\zeta_k}{|\zeta_k|}\right)\right]\\ &= \sum_{\substack{\ell \in I_{\gamma_j},\\ M_\ell = s_j}} \tau_\ell\left(\alpha_0, \zeta_0\right) \phi_\ell\left(\alpha'\right) \psi_\ell\left(\zeta_0\right)\\ &= \sum_{\substack{\ell \in I_{\gamma_j},\\ M_\ell = s_j}} \tau_\ell\left(0, \zeta_0\right) \phi_\ell\left(\alpha'\right) \psi_\ell\left(\zeta_0\right)\\ &= 0.\end{aligned}$$

Take any $\gamma \in (\gamma_{j-1}, \gamma_j)$ (meaning $\gamma < \rho$) and use lemma 1.2.13(ii) equation (1.16):

$$\begin{aligned}\pi_\gamma P(\alpha', \zeta_0) &= \sum_{\ell \in I_\gamma} \tau_\ell(0, \zeta_0)\, \phi_\ell(\alpha')\, \psi_\ell(\zeta_0) \\ &= \sum_{\substack{\ell \in I_{\gamma_j},\\ M_\ell = s_j}} \tau_\ell(0, \zeta_0)\, \phi_\ell(\alpha')\, \psi_\ell(\zeta_0) \\ &= 0,\end{aligned}$$

which is a contradiction to $\pi_\gamma P \neq 0$.

(3) $\zeta_0 = 0$. We have $|\alpha_0| = 1$, and $\alpha' = \alpha_0$, and we get

$$\begin{aligned}\lim_{k\to\infty} \widetilde{P}(\alpha_k, \zeta_k) &= \sum_{\substack{\ell \in I_{\gamma_j},\\ L_\ell = r_{j+1}}} \tau_\ell(\alpha_0, 0)\, \phi_\ell(\alpha_0)\, \psi_\ell(\zeta') \\ &= 0,\end{aligned}$$

and taking $\gamma \in (\gamma_j, \gamma_{j+1})$ (meaning $\gamma > \rho$) and using lemma 1.2.13(iii) equation (1.17) to find the principal part

$$\begin{aligned}\pi_\gamma P(\alpha_0, \zeta') &= \sum_{\ell \in I_\gamma} \tau_\ell(\alpha_0, 0)\, \phi_\ell(\alpha_0)\, \psi_\ell(\zeta') \\ &= \sum_{\substack{\ell \in I_{\gamma_j},\\ L_\ell = r_{j+1}}} \tau_\ell(\alpha_0, 0)\, \phi_\ell(\alpha_0)\, \psi_\ell(\zeta') \\ &= 0,\end{aligned}$$

which is again a contradiction to $\pi_\gamma P \neq 0$.

Therefore in all three cases, we get $|\pi_{\gamma_j} P(\lambda, z)| \geq C_j(|\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}})$ for $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ and any $j \in \{r_t(N(P)), \dots, r_x(N(P)) - 1\}$.

We must also investigate the case $j = J$ and N is irregular in space, so $\gamma_J = \infty$. For any $\ell \in I_{\gamma_J} = I_\infty$, we have $\frac{N_\ell}{\rho} + M_\ell = d_\infty(P) = s_{J+1} = s_J$, and we know $L_\ell \in [0, r_J]$ by lemma 1.2.12. Here, we have to proof

$$|\pi_\infty P(\lambda, z)| \geq C_J |\lambda|^{s_J}(1 + |z|^{r_J}), \quad (\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}).$$

For these (λ, z), define

$$\widetilde{P}(\lambda, z) := \sum_{\ell \in I_\infty} \left[\frac{|z|^{L_\ell}}{|z|^{r_J} + 1} \cdot \tau_\ell\left(\frac{\lambda}{|\lambda|}, 0\right) \phi_\ell\left(\frac{\lambda}{|\lambda|}\right) \psi_\ell\left(\frac{z}{|z|}\right) \right],$$

so that

$$\begin{aligned}|\lambda|^{s_J}(|z|^{r_J} + 1)\widetilde{P}(\lambda, z) &= \sum_{\ell \in I_\infty} \left[|\lambda|^{\frac{N_\ell}{\rho} + M_\ell} |z|^{L_\ell} \cdot \tau_\ell\left(\frac{\lambda}{|\lambda|}, 0\right) \phi_\ell\left(\frac{\lambda}{|\lambda|}\right) \psi_\ell\left(\frac{z}{|z|}\right) \right] \\ &= \sum_{\ell \in I_\infty} \tau_\ell(\lambda, 0)\, \phi_\ell(\lambda)\, \psi_\ell(z) \\ &= \pi_\infty P(\lambda, z),\end{aligned}$$

which transforms our problem into proving $|\widetilde{P}(\lambda, z)| \geq C_J$. We once again use contradiction: assume there is a $(\lambda_n, z_n)_{n\in\mathbb{N}} \subseteq (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ that has $P(\lambda_n, z_n) \to 0$. On the compact set

$$\Omega_\infty := \{(\alpha, \zeta) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}) : |\alpha| = |\zeta| = 1\}, \tag{2.5}$$

we can see that the sequence $\left(\frac{\lambda_n}{|\lambda_n|}, \frac{z_n}{|z_n|}\right)_{n\in\mathbb{N}} \subseteq \Omega_\infty$ has a convergent subsequence $\left(\frac{\lambda_{n_k}}{|\lambda_{n_k}|}, \frac{z_{n_k}}{|z_{n_k}|}\right)_{k\in\mathbb{N}}$ with limits $(\lambda', z') \in \overline{\Omega_\infty}$. Again, we only consider the sequence $(\lambda_k, z_k) := (\lambda_{n_k}, z_{n_k})$. We now consider only two cases.

(1) $\lim_{k\to\infty} |z_k| = \infty$. Then

$$\begin{aligned}
\lim_{k\to\infty} \widetilde{P}(\lambda_k, z_k) &= \lim_{k\to\infty} \sum_{\ell\in I_\infty} \left[\frac{|z_k|^{L_\ell}}{|z_k|^{r_J}+1} \cdot \tau_\ell\left(\frac{\lambda_k}{|\lambda_k|}, 0\right) \phi_\ell\left(\frac{\lambda_k}{|\lambda_k|}\right) \psi_\ell\left(\frac{z_k}{|z_k|}\right)\right] \\
&= \sum_{\substack{\ell\in I_\infty \\ L_\ell = r_J}} \tau_\ell\left(\lambda', 0\right) \phi_\ell\left(\lambda'\right) \psi_\ell\left(z'\right) \\
&= 0.
\end{aligned}$$

Taking $\gamma \in (\max\{\gamma_{J-1}, \rho\}, \infty)$ and using lemma 1.2.13(ii) equation (1.15) to get

$$\begin{aligned}
\pi_\gamma P(\lambda', z') &= \sum_{\ell\in I_\gamma} \tau_\ell(\lambda', 0)\phi_\ell(\lambda')\psi_\ell(z') \\
&= \sum_{\substack{\ell\in I_\infty, \\ L_\ell = r_J}} \tau_\ell(\lambda', 0)\phi_\ell(\lambda')\psi_\ell(z') \\
&= 0,
\end{aligned}$$

which is a contradiction to $\pi_\gamma P(\lambda, z) \neq 0$, since $(\lambda', z') \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$.

(2) $\lim_{k\to\infty} |z_k| \neq \infty$. This means $(|z_k|)_{k\in\mathbb{N}}$ is bounded, and we can use Bolzano-Weierstrass to find a subsequence $z_p = z_{n_{k_p}}$ for which $|z_p| \to \tilde{z}$ for some limit $\tilde{z} \geq 0$.

Suppose $\tilde{z} = 0$, then

$$\begin{aligned}
\lim_{p\to\infty} \widetilde{P}(\lambda_p, z_p) &= \lim_{p\to\infty} \sum_{\ell\in I_\infty} \left[\frac{|z_p|^{L_\ell}}{|z_p|^{r_J}+1} \cdot \tau_\ell\left(\frac{\lambda_p}{|\lambda_p|}, 0\right) \phi_\ell\left(\frac{\lambda_p}{|\lambda_p|}\right) \psi_\ell\left(\frac{z_p}{|z_p|}\right)\right] \\
&= \sum_{\substack{\ell\in I_\infty \\ L_\ell = 0}} \tau_\ell\left(\lambda', 0\right) \phi_\ell\left(\lambda'\right) \psi_\ell\left(z'\right) \\
&= 0.
\end{aligned}$$

Since $L_\ell = 0$, $\phi_\ell(z') = \phi_\ell(0) =$ constant, and $\phi_k(0) = 0$ for any k that has $L_k > 0$, we can conclude this is equal to

$$\begin{aligned}
\pi_\infty P(\lambda', 0) &= \sum_{\ell\in I_\infty} \tau_\ell(\lambda', 0)\phi_\ell(\lambda')\psi_\ell(0) \\
&= \sum_{\substack{\ell\in I_\infty \\ L_\ell = 0}} \tau_\ell\left(\lambda', 0\right) \phi_\ell\left(\lambda'\right) \psi_\ell\left(z'\right) \\
&= 0,
\end{aligned}$$

which is a contradiction to $\pi_\infty P(\lambda, z) \neq 0$, since $(\lambda', 0) \in (L_t \setminus \{0\}) \times L_x$.

Instead suppose $\tilde{z} > 0$, then $z_p = \frac{z_p}{|z_p|} \cdot |z_p| \to z' \cdot \tilde{z}$. So we can see

$$\begin{aligned}\lim_{p\to\infty} \widetilde{P}(\lambda_p, z_p) &= \lim_{p\to\infty} \sum_{\ell\in I_\infty} \left[\frac{1}{|z_p|^{r_J}+1} \cdot \tau_\ell\left(\frac{\lambda_p}{|\lambda_p|}, 0\right)\phi_\ell\left(\frac{\lambda_p}{|\lambda_p|}\right)\psi_\ell\left(z_p\right)\right] \\ &= \frac{1}{\tilde{z}^{r_J}+1} \sum_{\ell\in I_\infty} \tau_\ell\left(\lambda', 0\right)\phi_\ell\left(\lambda'\right)\psi_\ell\left(z'\cdot\tilde{z}\right) \\ &= 0 \\ &= \frac{1}{\tilde{z}^{r_J}+1}\pi_\infty P(\lambda', z'\cdot\tilde{z}),\end{aligned}$$

which again is a contradiction to $\pi_\infty P(\lambda, z) \neq 0$, since $(\lambda', z'\cdot\tilde{z}) \in (L_t \setminus \{0\}) \times L_x$.

In conclusion, for all $j \in \{r_t(N(P)), \dots, J\}$, we have $\widetilde{P} \geq C_j$ for some $C_j > 0$, and thus $|\pi_{\gamma_j} P(\lambda, z)| \geq C_J(|\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}})$ for $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$. $\square$

Now for $\gamma \in (\gamma_{j-1}, \gamma_j)$, we will make a similar estimate. We will proof that this estimate holds whether or not ρ appears in this interval, as we see in point (ii).

Lemma 2.4.2. *Let $P \in S(L_t \times L_x)$ be a symbol with*

$$\pi_\gamma P(\lambda, z) \neq 0, \quad (\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}),\ \gamma \in (0, \infty].$$

(i) For every $j \in \{r_t(N(P)), \dots, r_x(N(P))\}$ with $\rho \notin (\gamma_{j-1}, \gamma_j)$, there exists a constant $C_j > 0$ s.t.

$$|\pi_\gamma P(\lambda, z)| \geq C_j |\lambda|^{s_j}|z|^{r_j}, \quad (\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}),\ \gamma \in (\gamma_{j-1}, \gamma_j). \tag{2.6}$$

(ii) For the $j \in \{r_t(N(P)), \dots, r_x(N(P))\}$ that has $\rho \in (\gamma_{j-1}, \gamma_j)$, there exists a constant $C_j > 0$ s.t.

$$|\pi_\rho P(\lambda, z)| \geq C_j |\lambda|^{s_j}|z|^{r_j}, \quad (\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}),$$

and equation (2.6) also holds for other $\gamma \in (\gamma_{j-1}, \gamma_j)$.

Proof. (i) Take any $j \in \{r_t(N(P)), \dots, r_x(N(P))\}$ with $\rho \notin (\gamma_{j-1}, \gamma_j)$, and take any $\gamma \in (\gamma_{j-1}, \gamma_j)$. We split in the cases $\gamma_j > \rho$ and $\gamma_j \leq \rho$.

(I) $\gamma_j > \rho$. Then $0 < \rho \leq \gamma_{j-1} < \gamma < \gamma_j$. Using lemma 1.2.13(iii) equation (1.17) we know that for any $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ we can write

$$\begin{aligned}\pi_\gamma P(\lambda, z) &= \sum_{\substack{\ell\in I_{\gamma_{j-1}},\\ L_\ell = r_j}} \tau_\ell(\lambda, 0)\phi_\ell(\lambda)\psi_\ell(z) \\ &= \sum_{\substack{\ell\in I_{\gamma_{j-1}},\\ L_\ell = r_j}} \left[|\lambda|^{\frac{N_\ell}{\rho}+M_\ell}|z|^{L_\ell} \cdot \tau_\ell\left(\frac{\lambda}{|\lambda|}, 0\right)\phi_\ell\left(\frac{\lambda}{|\lambda|}\right)\psi_\ell\left(\frac{z}{|z|}\right)\right] \\ &= |\lambda|^{s_j}|z|^{r_j} \sum_{\substack{\ell\in I_{\gamma_{j-1}},\\ L_\ell = r_j}} \tau_\ell\left(\frac{\lambda}{|\lambda|}, 0\right)\phi_\ell\left(\frac{\lambda}{|\lambda|}\right)\psi_\ell\left(\frac{z}{|z|}\right) \\ &= |\lambda|^{s_j}|z|^{r_j}\pi_\gamma P\left(\frac{\lambda}{|\lambda|}, \frac{z}{|z|}\right) \neq 0.\end{aligned}$$

We use another compactness argument like in lemma 2.4.1 using the set Ω_∞ as in equation (2.5): $\Omega_\infty := \{(\alpha, \zeta) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}) : |\alpha| = |\zeta| = 1\}$. We can then rewrite our problem to finding a constant $C_j > 0$ s.t.

$$\begin{aligned} &\pi_\gamma P(\lambda, z) \geq C_j |\lambda|^{s_j} |z|^{r_j}, \quad && (\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}), \\ &\pi_\gamma P(\alpha, \zeta) \geq C_j > 0, && (\alpha, \zeta) \in \Omega_\infty. \end{aligned}$$

Since $\lambda = 0$ and $z = 0$ both imply $(\lambda, z) \notin \overline{\Omega_\infty} \subseteq (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$, we know the bottom statement is true from $\pi_\gamma P(\lambda, z) \neq 0$ on $(L_t \setminus \{0\}) \times (L_x \setminus \{0\})$, and taking some $0 < C_j \leq \min_{(\alpha,\zeta)\in\Omega_\infty} |\pi_\gamma P(\alpha, \zeta)| > 0$, which can be done since $\pi_\gamma P$ is a continuous function.

(II) $\gamma_j \leq \rho$. Then $\gamma_{j-1} < \gamma < \gamma_j \leq \rho < \infty$. Using lemma 1.2.13(ii) equation (1.16) we get

$$\begin{aligned} \pi_\gamma P(\lambda, z) &= \sum_{\substack{\ell \in I_{\gamma_j}, \\ M_\ell = s_j}} \tau_\ell(0, z) \phi_\ell(\lambda) \psi_\ell(z) \\ &= \sum_{\substack{\ell \in I_{\gamma_j}, \\ M_\ell = s_j}} \left[|\lambda|^{M_\ell} |z|^{N_\ell + L_\ell} \cdot \tau_\ell \left(0, \frac{z}{|z|}\right) \phi_\ell \left(\frac{\lambda}{|\lambda|}\right) \psi_\ell \left(\frac{z}{|z|}\right) \right] \\ &= |\lambda|^{s_j} |z|^{r_j} \pi_\gamma P \left(\frac{\lambda}{|\lambda|}, \frac{z}{|z|} \right) \neq 0. \end{aligned}$$

Using the exact same argument as in (I), we prove $\pi_\gamma P(\lambda, z) \geq C_j |\lambda|^{s_j} |z|^{r_j}$ for $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$.

(ii) For $j \in \{r_t(N(P)), \ldots, r_x(N(P))\}$ with $\rho \in (\gamma_{j-1}, \gamma_j)$, lemma 1.2.13(i) tells us that $N_\ell = 0$, and $\tau_\ell(\lambda, z) = \text{constant} = a_\ell$ for $\ell \in I_\rho$. For $a_\ell \neq 0$, we have

$$\begin{aligned} \pi_\rho P(\lambda, z) &= \sum_{\ell \in I_\rho} a_\ell \phi_\ell(\lambda) \psi_\ell(z) \\ &= \sum_{\ell \in I_\rho} \left[|\lambda|^{M_\ell} |z|^{L_\ell} \cdot a_\ell \phi_\ell \left(\frac{\lambda}{|\lambda|}\right) \psi_\ell \left(\frac{z}{|z|}\right) \right] \\ &= |\lambda|^{s_j} |z|^{r_j} \pi_\rho P \left(\frac{\lambda}{|\lambda|}, \frac{z}{|z|} \right). \end{aligned}$$

Using the exact same argument from Ω_∞ as in (i), we conclude $|\pi_\rho P(\lambda, z)| \geq C_j |\lambda|^{s_j} |z|^{r_j}$ for $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$. Since for any other $\gamma \in (\gamma_{j-1}, \gamma_j)$ we have $I_\gamma = I_\rho$ from lemma 1.2.13, we also have $\pi_\gamma P = \pi_\rho P$, meaning the same bound also holds for $|\pi_\gamma P(\lambda, z)|$. □

References

R. Denk & M. Kaip, section 2.2c [1]:

1. Lemma 2.53 and proof, p. 103-107
2. Lemma 2.54 and proof, p. 108
3. Lemma 2.55 and proof, p. 109

2.5 N-parameter-ellipticity $\Leftarrow$ non-vanishing principal parts

Finally, we can prove the other direction:

Theorem 2.5.1. *Let $P \in S(L_t \times L_x)$ be a symbol satisfying*

$$\pi_\gamma P(\lambda, z) \neq 0, \quad \pi_\infty P(\lambda, 0) \neq 0, \qquad (\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}),\ \gamma \in (0, \infty].$$

Then P is N-parameter-elliptic in $\mathring{L}_t \times \mathring{L}_x$.

Proof. This proof is divided into several parts for the sake of readability.

Decomposition of P

Take any $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ and $\gamma \in (0, \infty]$. We write

$$\begin{aligned}
|P(\lambda, z)| &= \left| \sum_{\ell \in I_\gamma} \tau_\ell(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z) + \sum_{\ell \in I_P \setminus I_\gamma} \tau_\ell(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z) \right| \\
&= \left| \pi_\gamma P(\lambda, z) - \pi_\gamma P(\lambda, z) - \sum_{\ell \in I_\gamma} \tau_\ell(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z) - \sum_{\ell \in I_P \setminus I_\gamma} \tau_\ell(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z) \right| \\
&\geq |\pi_\gamma P(\lambda, z)| - \left| \pi_\gamma P(\lambda, z) - \sum_{\ell \in I_\gamma} \tau_\ell(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z) \right| - \left| \sum_{\ell \in I_P \setminus I_\gamma} \tau_\ell(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z) \right|. \\
&\geq |\pi_\gamma P(\lambda, z)| - T_\gamma(\lambda, z) - V_\gamma(\lambda, z),
\end{aligned}$$

where we define

$$T_\gamma(\lambda, z) := \sum_{\ell \in I_\gamma} |\pi_\gamma \tau_\ell(\lambda, z) - \tau_\ell(\lambda, z)|\, |\phi_\ell(\lambda)|\, |\psi_\ell(z)|\,, \tag{2.7}$$

$$V_\gamma(\lambda, z) := \sum_{\ell \in I_P \setminus I_\gamma} |\tau_\ell(\lambda, z)|\, |\phi_\ell(\lambda)|\, |\psi_\ell(z)|\,. \tag{2.8}$$

Choose a bound $\hat{C}_2 > 0$ from lemma 1.1.3(i) and (ii), so that the following bounds hold for all $\ell \in I_P$ and $(\lambda, z) \in L_t \times L_x$:

$$|\tau_\ell(\lambda, z)| \leq \hat{C}_2 \left(|\lambda|^{\frac{N_\ell}{\rho}} + |z|^{N_\ell} \right), \tag{2.9}$$

$$|\phi_\ell(\lambda)| \leq \hat{C}_2 |\lambda|^{M_\ell}, \tag{2.10}$$

$$|\psi_\ell(z)| \leq \hat{C}_2 |z|^{L_\ell}. \tag{2.11}$$

Next choose a bound $\hat{C}_1 > 0$ small enough to satisfy lemmas 2.4.1 and 2.4.2 for $(L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ and all required j:

$$|\pi_{\gamma_j} P(\lambda, z)| \geq \hat{C}_1 \left(|\lambda|^{s_j} |z|^{r_j} + |\lambda|^{s_{j+1}} |z|^{r_{j+1}} \right), \ j \in \{r_t(N(P)), \dots, J\}, \tag{2.12}$$

$$|\pi_\gamma P(\lambda, z)| \geq \hat{C}_1 |\lambda|^{s_j} |z|^{r_j}, \qquad j \in \{r_t(N(P)), \dots, r_x(N(P))\},\ \rho \notin (\gamma_{j-1}, \gamma_j), \tag{2.13}$$

$$|\pi_\rho P(\lambda, z)| \geq \hat{C}_1 |\lambda|^{s_j} |z|^{r_j}, \qquad j \in \{r_t(N(P)), \dots, r_x(N(P))\},\ \rho \in (\gamma_{j-1}, \gamma_j). \tag{2.14}$$

We will firstly create a partition $G(\varepsilon_0, \varepsilon_1)$ and $\widetilde{G}(\varepsilon_0)$.

The partition

We create the partition of $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$, as seen in section 2.2. Lemma 2.2.4 works for any $\varepsilon > 0$, so we will work with $\varepsilon = \frac{\hat{C}_1}{4\hat{C}_2^3 \cdot \# I_P}$, where $\# I_P$ is the size of the index set I_P, and $\vartheta := \frac{\varepsilon}{4}$. Using 2.2.4(i), we can choose a bound $\hat{\varepsilon}_0 > 0$. We will now choose a $0 < \varepsilon_0 \leq \hat{\varepsilon}_0$ small enough s.t. lemma 2.3.2 gives us that for any $\ell \in I_P$, for any $j \in \{r_t(N(P)), \dots, r_x(N(P))\}$ that has $\rho \notin (\gamma_{j-1}, \gamma_j)$, for any $\gamma \in (\gamma_{j-1}, \gamma_j)$, and for any $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ that has $(|z|, |\lambda|) \in \widetilde{G}_j(\varepsilon_0)$ and $|\lambda| \geq 1$, we have

$$|\tau_\ell(\lambda, z) - \pi_\gamma \tau_\ell(\lambda, z)| \leq \vartheta |\pi_\gamma \tau_\ell(\lambda, z)|. \tag{2.15}$$

ε_0 is chosen by making sure $\varepsilon_0 \leq \min_{j \in \{r_t(N(P)), \dots, r_x(N(P))\}} \varepsilon_j$, with ε_j defined as in lemma 2.3.2. In the case that $N(P)$ is not regular in space, we also need to make sure that $\varepsilon_0 \leq \varepsilon_{J+1}$ from lemma 2.3.2(ii), such that for any $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ that has $(|z|, |\lambda|) \in \widetilde{G}_{J+1}(\varepsilon_0)$ and $|\lambda| \geq 1$, we have

$$|\tau_\ell(\lambda, z) - \tau_\ell(\lambda, 0)| \leq \vartheta |\tau_\ell(\lambda, 0)|. \tag{2.16}$$

Next, choose a $\varepsilon_1 > 0$ small enough so that lemma 2.2.4(iii) holds. By lemma 2.3.1, we can define a $\mu_1 > 0$ large enough, so that for any $\ell \in I_P$, for any $j \in \{r_t(N(P)), \dots, J\}$, and for any $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ that has $(|z|, |\lambda|) \in G_j(\varepsilon_0, \varepsilon_1)$ and $|\lambda| \geq \mu_1$, we have

$$|\tau_\ell(\lambda, z) - \pi_{\gamma_j} \tau_\ell(\lambda, z)| \leq \vartheta |\pi_{\gamma_j} \tau_\ell(\lambda, z)|. \tag{2.17}$$

Now, we use the partition $G(\varepsilon_0, \varepsilon_1)$ and $\widetilde{G}(\varepsilon_0)$, which is still a partition of $(0, \infty)^2$, to define a partition of $(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$:

$$\begin{aligned} U_k &:= \{(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}) : (|z|, |\lambda|) \in G_k(\varepsilon_0, \varepsilon_1)\}, \\ \widetilde{U}_j &:= \{(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}) : (|z|, |\lambda|) \in \widetilde{G}_j(\varepsilon_0)\}, \end{aligned}$$

where $k \in \{1, \dots, J\}$ and $j \in \{1, \dots, J+1\}$. From lemma 2.2.4(iv), we can pick a $\mu_0 > 0$ large enough such that we have

$$\{(\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}) : |\lambda| \geq \mu_0\} \subseteq \bigcup_{j = r_t(N(P))}^{J} \{U_j \cup \widetilde{U}_j\} \cup U_{J+1} \tag{2.18}$$

Finally we pick the bound $\lambda_0 := \max\{1, \mu_0, \mu_1\}$, so that $|\lambda| \geq \lambda_0$ satisfies equations (2.15), (2.16), (2.17) and (2.18). We will now proof

$$|P(\lambda, z)| \geq C \cdot W_P(\lambda, z), \quad (\lambda, z) \in \mathring{L}_t \times \mathring{L}_x, \ |\lambda| \geq \lambda_0$$

is true for some $C > 0$ in the following way:

1. proof above estimate is true for $(\lambda, z) \in U_j$
2. proof above estimate is true for $(\lambda, z) \in \widetilde{U}_j$
3. proof above estimate is true for $(\lambda, z) \in \widetilde{U}_{J+1}$, even if $N(P)$ is not regular in space.

We can then define C and conclude that P is N-parameter-elliptic on $\mathring{L}_t \times \mathring{L}_x$.

Estimate on U_j

Suppose $(\lambda, z) \in U_j$, $|\lambda| \geq \lambda_0$ for some $j \in \{r_t(N(P)), \dots, J\}$. We then pick the decomposition

$$|P(\lambda, z)| \geq |\pi_{\gamma_j} P(\lambda, z)| - T_{\gamma_j}(\lambda, z) - V_{\gamma_j}(\lambda, z).$$

We take T_{γ_j} and V_{γ_j} as in equations (2.7) and (2.8), and we try to find an estimate from above.

(I) T_{γ_j}. Because of equation (2.17), we can write

$$\begin{aligned} T_{\gamma_j}(\lambda, z) &:= \sum_{\ell \in I_\gamma} \left|\pi_{\gamma_j} \tau_\ell(\lambda, z) - \tau_\ell(\lambda, z)\right| |\phi_\ell(\lambda)| \, |\psi_\ell(z)| \\ &\leq \vartheta \sum_{\ell \in I_\gamma} \left|\pi_{\gamma_j} \tau_\ell(\lambda, z)\right| |\phi_\ell(\lambda)| \, |\psi_\ell(z)| . \end{aligned}$$

Using the bounds from equations (2.9), (2.10) and (2.11), and lemma 1.1.5 we can write for $\ell \in I_{\gamma_j}$:

$$\begin{aligned} \left|\pi_{\gamma_j} \tau_\ell(\lambda, z)\right| |\phi_\ell(\lambda)| \, |\psi_\ell(z)| &= \begin{cases} |\tau_\ell(0, z)| \, |\phi_\ell(\lambda)| \, |\psi_\ell(z)| \, , & \gamma_j < \rho, \\ |\tau_\ell(\lambda, z)| \, |\phi_\ell(\lambda)| \, |\psi_\ell(z)| \, , & \gamma_j = \rho, \\ |\tau_\ell(\lambda, 0)| \, |\phi_\ell(\lambda)| \, |\psi_\ell(z)| \, , & \gamma_j > \rho \end{cases} \\ &\leq \hat{C}_2^3 \begin{cases} |\lambda|^{M_\ell} |z|^{N_\ell + L_\ell}, & \gamma_j < \rho, \\ \left(|\lambda|^{\frac{N_\ell}{\rho}} + |z|^{N_\ell}\right) |\lambda|^{M_\ell} |z|^{L_\ell}, & \gamma_j = \rho, \\ |\lambda|^{\frac{N_\ell}{\rho} + M_\ell} |z|^{L_\ell}, & \gamma_j > \rho \end{cases} \\ &= \hat{C}_2^3 \begin{cases} |\lambda|^{M_\ell} |z|^{N_\ell + L_\ell}, & \gamma_j < \rho, \\ |\lambda|^{\frac{N_\ell}{\rho} + M_\ell} |z|^{L_\ell} + |\lambda|^{M_\ell} |z|^{N_\ell + L_\ell}, & \gamma_j = \rho, \\ |\lambda|^{\frac{N_\ell}{\rho} + M_\ell} |z|^{L_\ell}, & \gamma_j > \rho \end{cases} \end{aligned}$$

We know that in these cases, either or both $\mathfrak{u}_\ell := (N_\ell + L_\ell, M_\ell)$ and $\mathfrak{v}_\ell := (L_\ell, \frac{N_\ell}{\rho} + M_\ell)$ are in the convex set $[v_j v_{j+1}]$ by lemma 1.2.12. This allows us to apply the estimate of proposition 1.2.6:

$$\begin{aligned} \left|\pi_{\gamma_j} \tau_\ell(\lambda, z)\right| |\phi_\ell(\lambda)| \, |\psi_\ell(z)| &\leq 2\hat{C}_2^3 \left(|\lambda|^{s_j} |z|^{r_j} + |\lambda|^{s_{j+1}} |z|^{r_{j+1}}\right) \\ &\leq 4\hat{C}_2^3 \left(|\lambda|^{s_j} |z|^{r_j} + |\lambda|^{s_{j+1}} |z|^{r_{j+1}}\right) \end{aligned}$$

Then we get

$$\begin{aligned} T_{\gamma_j}(\lambda, z) &\leq \vartheta \sum_{\ell \in I_{\gamma_j}} \left|\pi_{\gamma_j} \tau_\ell(\lambda, z)\right| |\phi_\ell(\lambda)| \, |\psi_\ell(z)| \\ &\leq \frac{\varepsilon}{4} \sum_{\ell \in I_{\gamma_j}} 4\hat{C}_2^3 \left(|\lambda|^{s_j} |z|^{r_j} + |\lambda|^{s_{j+1}} |z|^{r_{j+1}}\right) \\ &= \frac{\hat{C}_1}{4\hat{C}_2^3 \cdot \# I_P} \cdot \# I_{\gamma_j} \cdot \hat{C}_2^3 \left(|\lambda|^{s_j} |z|^{r_j} + |\lambda|^{s_{j+1}} |z|^{r_{j+1}}\right) \\ &\leq \frac{\hat{C}_1}{4 \cdot \# I_P} \cdot \# I_P \cdot \left(|\lambda|^{s_j} |z|^{r_j} + |\lambda|^{s_{j+1}} |z|^{r_{j+1}}\right) \\ &= \frac{\hat{C}_1}{4} \left(|\lambda|^{s_j} |z|^{r_j} + |\lambda|^{s_{j+1}} |z|^{r_{j+1}}\right) . \end{aligned} \tag{2.19}$$

(II) V_{γ_j}. Firstly, let's apply (2.9), (2.10) and (2.11):

$$\begin{aligned} V_{\gamma_j} &:= \sum_{\ell \in I_P \setminus I_\gamma} |\tau_\ell(\lambda, z)|\,|\phi_\ell(\lambda)|\,|\psi_\ell(z)| \\ &\leq \hat{C}_2^3 \sum_{\ell \in I_P \setminus I_\gamma} |\lambda|^{\frac{N_\ell}{\rho}+M_\ell}|z|^{L_\ell} + |\lambda|^{M_\ell}|z|^{N_\ell+L_\ell}. \end{aligned}$$

For any $\ell \in I_P \setminus I_{\gamma_j}$, we have $\mathfrak{u}_\ell, \mathfrak{v}_\ell \in \nu(P) \setminus [v_j v_{j+1}]$ (since if $\mathfrak{u}_\ell, \mathfrak{v}_\ell \in [v_j v_{j+1}]$, we would have $\ell \in I_{\gamma_j}$.) Therefore, we can use lemma 2.2.4(iii) to get for $(\lambda, z) \in U_j$ that

$$\begin{aligned} V_{\gamma_j} &\leq \hat{C}_2^3 \sum_{\ell \in I_P \setminus I_\gamma} |\lambda|^{\frac{N_\ell}{\rho}+M_\ell}|z|^{L_\ell} + |\lambda|^{M_\ell}|z|^{N_\ell+L_\ell} \\ &\leq \varepsilon \hat{C}_2^3 \sum_{\ell \in I_P \setminus I_\gamma} |\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}} \\ &\leq \frac{\hat{C}_1}{4 \cdot \#I_P} \cdot \#I_P(|\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}}) \\ &= \frac{\hat{C}_1}{4}(|\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}}). \end{aligned} \tag{2.20}$$

We now combine equations (2.14), (2.19) and (2.20) to get

$$\begin{aligned} |P(\lambda, z)| &\geq |\pi_{\gamma_j} P(\lambda, z)| - T_{\gamma_j}(\lambda, z) - V_{\gamma_j}(\lambda, z) \\ &\geq \left(\hat{C}_1 - \frac{\hat{C}_1}{4} - \frac{\hat{C}_1}{4}\right)(|\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}}) \\ &= \frac{\hat{C}_1}{2}(|\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}}). \end{aligned}$$

We look at $W_P(\lambda, z)$. Since $(\lambda, z) \in U_j$, we can again use lemma 2.2.4(iii) to obtain

$$\begin{aligned} W_P(\lambda, z) &:= \sum_{(r,s) \in N_v} |\lambda|^s |z|^r \\ &= |\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}} + \sum_{(r,s) \in N_v \setminus [v_j v_{j+1}]} |\lambda|^s |z|^r \\ &\leq |\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}} + \varepsilon \cdot (\#N_v \setminus \{v_j, v_{j+1}\}) \cdot (|\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}}) \\ &= (1 + \varepsilon J) \cdot (|\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}}). \end{aligned}$$

Here we used $\#N_v = J + 2$. Taking $C_j := \frac{\hat{C}_1}{2(1+\varepsilon J)}$, we get that for all $(\lambda, z) \in U_j$ with $|\lambda| \geq \lambda_0$:

$$\begin{aligned} |P(\lambda, z)| &\geq \frac{\hat{C}_1}{2}(|\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}}) \\ &= C_j(1 + \varepsilon J)|\lambda|^{s_j}|z|^{r_j} + |\lambda|^{s_{j+1}}|z|^{r_{j+1}} \\ &\geq C_j W_P(\lambda, z). \end{aligned} \tag{2.21}$$

Estimate on $\widetilde{U}_j$

First, take $j \in \{r_t(N(P)), \ldots, r_x(N(P))\}$ that has $\rho \notin (\gamma_{j-1}, \gamma_j)$. Take an arbitrary $\gamma \in (\gamma_{j-1}, \gamma_j)$ and let $(\lambda, z) \in \widetilde{U}_j$ with $|\lambda| \geq \lambda_0$. We pick the decomposition

$$|P(\lambda, z)| \geq |\pi_\gamma P(\lambda, z)| - T_\gamma(\lambda, z) - V_\gamma(\lambda, z).$$

We investigate $T_\gamma(\lambda, z)$ and $V_\gamma(\lambda, z)$.

(I) $T_\gamma(\lambda, z)$. For any $\ell \in I_\gamma$ we apply (2.9), (2.10) and (2.11), and also use lemma 1.2.13(ii) on $\mathfrak{u}_\ell$ and $\mathfrak{v}_\ell$ to get

$$\begin{aligned}|\pi_\gamma\tau_\ell(\lambda, z)|\,|\phi_\ell(\lambda)|\,|\psi_\ell(z)| &\leq \hat{C}_2^3\begin{cases}|\lambda|^{M_\ell}|z|^{N_\ell+L_\ell}, & \gamma_j \leq \rho,\\ |\lambda|^{\frac{N_\ell}{\rho}+M_\ell}|z|^{L_\ell}, & \gamma_j > \rho\end{cases}\\ &= \hat{C}_2^3 \cdot |\lambda|^{s_j}|z|^{r_j}\\ &\leq 2\hat{C}_2^3 \cdot |\lambda|^{s_j}|z|^{r_j}.\end{aligned}$$

This combined with equation (2.15) is used to conclude

$$\begin{aligned}T_\gamma(\lambda, z) &\leq \vartheta \sum_{\ell\in I_\gamma} |\pi_\gamma\tau_\ell(\lambda, z)|\,|\phi_\ell(\lambda)|\,|\psi_\ell(z)|\\ &\leq \frac{\varepsilon}{4} \cdot 2\hat{C}_2^3 \cdot \#I_\gamma \cdot |\lambda|^{s_j}|z|^{r_j}\\ &\leq \frac{\hat{C}_1}{8}|\lambda|^{s_j}|z|^{r_j}. \end{aligned}\tag{2.22}$$

(II) $V_\gamma(\lambda, z)$. By lemma 1.2.13(ii), we know that for $\ell \in I_\gamma$ is equivalent to $\mathfrak{u}_\ell = v_j$ in the case $\gamma_j \leq \rho$ and $\mathfrak{v}_\ell = v_j$ in the case $\gamma_j > \rho$. Thus, $\ell \in I_P \setminus I_\gamma$ means in these cases $\mathfrak{u}_\ell \neq v_j$ and $\mathfrak{v}_\ell \neq v_j$, or in other words $\mathfrak{u}_\ell, \mathfrak{v}_\ell \in \nu(P) \setminus \{v_j\}$. Therefore, we can use lemma 2.2.4(i) since $(\lambda, z) \in \widetilde{U}_j$, after using the bounds as before:

$$\begin{aligned}V_\gamma(\lambda, z) &\leq \hat{C}_2^3 \sum_{\ell\in I_p\setminus I_\gamma} |\lambda|^{M_\ell}|z|^{N_\ell+L_\ell} + |\lambda|^{\frac{N_\ell}{\rho}+M_\ell}|z|^{L_\ell}\\ &\leq 2\hat{C}_2^3 \cdot \varepsilon \cdot (\#I_P \setminus I_\gamma) \cdot |\lambda|^{s_j}|z|^{r_j}\\ &\leq \hat{C}_2^3\frac{\hat{C}_1}{2\hat{C}_2^3} \cdot |\lambda|^{s_j}|z|^{r_j} = \frac{\hat{C}_1}{2}|\lambda|^{s_j}|z|^{r_j}. \end{aligned}\tag{2.23}$$

Combining equations (2.13), (2.22) and (2.23), we get

$$|P(\lambda, z)| \geq \left(\hat{C}_1 - \frac{1}{8}\hat{C}_1 - \frac{1}{2}\hat{C}_1\right)|\lambda|^{s_j}|z|^{r_j} = \frac{3}{8}\hat{C}_1 \cdot |\lambda|^{s_j}|z|^{r_j}.$$

We also look at the weight function with the help of lemma 2.2.4(i). For $(\lambda, z) \in \widetilde{U}_j$ we get

$$\begin{aligned}W_P(\lambda, z) &= |\lambda|^{s_j}|z|^{r_j} + \sum_{(r,s)\in N_v\setminus\{v_j\}} |\lambda|^s|z|^r\\ &\leq |\lambda|^{s_j}|z|^{r_j} + \varepsilon(\#N_v \setminus \{v_j\}) \cdot |\lambda|^{s_j}|z|^{r_j} = (1 + \varepsilon(J + 1)) \cdot |\lambda|^{s_j}|z|^{r_j},\end{aligned}$$

meaning that for $\widetilde{C}_j := \frac{3}{8(1+\varepsilon(J+1))}\hat{C}_1$ we get for all $(\lambda, z) \in \widetilde{U}_j$ with $|\lambda| \geq \lambda_0$ that

$$|P(\lambda, z)| \geq \frac{3}{8}\hat{C}_1 \cdot |\lambda|^{s_j}|z|^{r_j} \geq \widetilde{C}_j W_P(\lambda, z). \tag{2.24}$$

Now to investigate the case where there exists a $k \in \{r_t(N(P)), \ldots, r_x(N(P))\}$ s.t. $\rho \in (\gamma_{k-1}, \gamma_k)$. Taking $(\lambda, z) \in \widetilde{U}_k$ with $|\lambda| \geq \lambda_0$, we know by lemma 1.2.13(ii) that $I_\gamma = I_\rho$ for any other $\gamma \in (\gamma_{k-1}, \gamma)$, we only have to consider $\gamma := \rho$. We use the same decomposition as earlier, to again look into $T_\rho(\lambda, z)$ and $V_\rho(\lambda, z)$.

(I) $T_\rho(\lambda, z)$. Since $\pi_\rho \tau_\ell(\lambda, z) = \tau_\ell(\lambda, z)$ for any $\ell \in I_\rho$, we immediately see that $T_\gamma(\lambda, z) = 0$.

(II) $V_\rho(\lambda, z)$. In the exact way as done before, we find the same bound as in equation (2.23), namely $V_\rho \le \frac{\hat{C}_1}{2}|\lambda|^{s_j}|z|^{r_j}$.

Using the same weight function bound $W_P(\lambda, z) \le (1 + \varepsilon(J+1)) \cdot |\lambda|^{s_j}|z|^{r_j}$, we can define $\widetilde{C}_k := \frac{1}{2(1+\varepsilon(J+1))} \cdot \hat{C}_1$, to find for all $(\lambda, z) \in \widetilde{U}_k$ with $|\lambda| \ge \lambda_0$

$$|P(\lambda, z)| \ge \frac{1}{2}\hat{C}_1 \cdot |\lambda|^{s_j}|z|^{r_j} \ge \widetilde{C}_k W_P(\lambda, z). \tag{2.25}$$

Estimate on $\widetilde{U}_{J+1}$ in the space-irregular case

If $N(P)$ is not regular in space, we need slightly different argumentation in the case $j = J+1$. Take $(\lambda, z) \in \widetilde{U}_{J+1}$ with $|\lambda| \ge \lambda_0$. This time, we decompose P by

$$\begin{aligned}
|P(\lambda, z)| &= \left| \sum_{\substack{\ell \in I_\infty, \\ L_\ell = 0}} \tau_\ell(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z) + \sum_{\substack{\ell \in I_P \setminus I_\infty \\ \text{OR } L_\ell > 0}} \tau_\ell(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z) \right| \\
&= \left| \pi_\infty P(\lambda, z) - \pi_\infty P(\lambda, z) - \sum_{\substack{\ell \in I_\infty, \\ L_\ell = 0}} \tau_\ell(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z) - \sum_{\substack{\ell \in I_P \setminus I_\infty \\ \text{OR } L_\ell > 0}} \tau_\ell(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z) \right| \\
&\ge |\pi_\infty P(\lambda, 0)| - \hat{T}_\infty(\lambda, z) - \hat{V}_\infty(\lambda, z),
\end{aligned}$$

where we instead take

$$\begin{aligned}
\hat{T}_\infty(\lambda, z) &:= \sum_{\substack{\ell \in I_\infty, \\ L_\ell = 0}} |\tau_\ell(\lambda, 0) - \tau_\ell(\lambda, z)|\, |\phi_\ell(\lambda)|\, |\psi_\ell(z)|\,, \\
\hat{V}_\infty(\lambda, z) &:= \sum_{\substack{\ell \in I_P \setminus I_\infty \\ \text{OR } L_\ell > 0}} |\tau_\ell(\lambda, z)|\, |\phi_\ell(\lambda)|\, |\psi_\ell(z)|\,.
\end{aligned}$$

By equation (2.12), we get

$$\begin{aligned}
\pi_{\gamma_J} P(\lambda, z) = \pi_\infty P(\lambda, z) &\ge \hat{C}_1(|\lambda|^{s_J}|z|^{r_J} + |\lambda|^{s_{J+1}}) \\
\pi_\infty P(\lambda, 0) &\ge \hat{C}_1|\lambda|^{s_{J+1}}
\end{aligned}$$

We investigate $\hat{T}_\infty(\lambda, z)$ and $\hat{V}_\infty(\lambda, z)$.

(I) $\hat{T}_\infty(\lambda, z)$. Using the bounds (2.9), (2.10) and (2.11), we have that for $\ell \in I_\infty$ with $L_\ell = 0$ that

$$|\tau_\ell(\lambda, 0)|\, |\phi_\ell(\lambda)|\, |\psi_\ell(z)| \le \hat{C}_2^3 \cdot |\lambda|^{\frac{N_\ell}{\rho} + M_\ell} = \hat{C}_2^3 \cdot |\lambda|^{s_{J+1}} \le 2\hat{C}_2^3 \cdot |\lambda|^{s_{J+1}}.$$

Then we can use equation (2.16) to get

$$\begin{aligned}
\hat{T}_\infty(\lambda, z) &\le \vartheta \sum_{\substack{\ell \in I_\infty, \\ L_\ell = 0}} |\tau_\ell(\lambda, 0)|\, |\phi_\ell(\lambda)|\, |\psi_\ell(z)| \\
&\le 2\hat{C}_2^3 \frac{\varepsilon}{4} \cdot (\#\{\ell \in I_\infty : L_\ell = 0\}) \cdot |\lambda|^{s_{J+1}} \\
&\le \frac{\hat{C}_1}{8}|\lambda|^{s_{J+1}}.
\end{aligned} \tag{2.26}$$

(II) $\hat{V}_\infty(\lambda, z)$. Since $\frac{N_\ell}{\rho} + M_\ell \neq s_{J+1} = d_\infty(P)$ implies $\mathfrak{v}_\ell \notin I_\infty$ and $L_\ell > 0 = r_{J+1}$, we know that $\mathfrak{v}_\ell \in \nu(P) \setminus \{v_{J+1}\}$ for all $\ell \in I_P \setminus I_\infty$ and all $\ell \in I_P$ with $L_\ell > 0$. Additionally, we know that for all of these ℓ, we know that either:

1. $N_\ell = 0$, which means $\mathfrak{u}_\ell = \mathfrak{v}_\ell \in \nu(P) \setminus \{v_{J+1}\}$.
2. $N_\ell \neq 0$, which means $M_\ell \neq \frac{N_\ell}{\rho} + M_\ell$, so that even if $\frac{N_\ell}{\rho} + M_\ell = s_{J+1} = d_\infty$, we still have $\mathfrak{u}_\ell \in \nu(P) \setminus \{v_{J+1}\}$.

Since $\mathfrak{u}_\ell, \mathfrak{v}_\ell \in \nu(P) \setminus \{v_{J+1}\}$, we can apply lemma 2.2.4(i) after applying the bounds to get

$$\begin{aligned}
\hat{V}_\infty(\lambda, z) &\leq \hat{C}_2^3 \sum_{\substack{\ell \in I_P \setminus I_\infty \\ \text{OR } L_\ell > 0}} |\lambda|^{\frac{N_\ell}{\rho} + M_\ell} |z|^{L_\ell} + |\lambda|^{M_\ell} |z|^{N_\ell + L_\ell} \\
&\leq 2\hat{C}_2^3 \cdot \varepsilon \cdot (\#\{\ell \in I_P : \ell \notin I_\infty \text{ OR } L_\ell > 0\}) \cdot |\lambda|^{s_{J+1}} \\
&\leq 2\hat{C}_2^3 \cdot \frac{\hat{C}_1}{4\hat{C}_2^3 \cdot \#I_P} \cdot \#I_P \cdot |\lambda|^{s_{J+1}} = \frac{\hat{C}_1}{2} |\lambda|^{s_{J+1}}.
\end{aligned}$$

Using the same weight function bound $W_P(\lambda, z) \leq (1 + \varepsilon(J+1))|\lambda|^{s_{J+1}}$ as we found in the previous section, we can again define $\widetilde{C}_{J+1} := \frac{3}{8(1+\varepsilon(J+1))}$ to find for all $(\lambda, z) \in \widetilde{U}_{J+1}$ with $|\lambda| \geq \lambda_0$ that

$$|P(\lambda, z)| \geq \frac{3}{8} \hat{C}_1 \cdot |\lambda|^{s_j} |z|^{r_j} \geq \widetilde{C}_{J+1} W_P(\lambda, z). \tag{2.27}$$

Conclusion

Take $(\lambda, z) \in \mathring{L}_t \times \mathring{L}_x \subseteq (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ with $|\lambda| \geq \lambda_0$. Then by equation (2.18), we know that (λ, z) must be in U_j for some $j \in \{r_t(N(P)), \ldots, J\}$ or in $\widetilde{U}_j$ for some $j \in \{r_t(N(P)), \ldots, J+1\}$. Whichever is the case, if we take

$$C := \min \left(\bigcup_{j=1}^{J} \{C_j, \widetilde{C}_j\} \cup \widetilde{C}_{J+1} \right),$$

Then we find

$$|P(\lambda, z)| \geq C W_P(\lambda, z).$$

□

References

R. Denk & M. Kaip, section 2.2c [1]:

1. Theorem 2.56 and proof, p. 109-114

Conclusion

We reflect on what we have seen so far. We have seen that we can symbols $P \in S(L_t \times L_x)$ as

$$P(\lambda, z) = \sum_{\ell \in I_P} \tau_\ell(\lambda, z)\phi_\ell(\lambda)\psi_\ell(z),$$

which have a γ-order $d_\gamma(P)$ that functions as the order function of the symbol and have a γ-principal part $\pi_\gamma P(\lambda, z)$, which are the terms that determine the γ-order. For this symbol, we can define a Newton polygon $N(P)$ with vertices $v_j \in N_v$ for $j \in \{0, \dots, J+1\}$, and a weight function $W_P(\lambda, z) = \sum_{i=1}^{J+1} |\lambda|^{s_j}|z|^{r_j}$. We investigated order functions, and used this knowledge to determine an upper bound of the form

$$|P(\lambda, z)| \leq C_2 W_P(\lambda, z).$$

We defined a notion of N-parameter-ellipticity as the existence of a lower bound

$$|P(\lambda, z)| \geq C_1 W_P(\lambda, z), \quad (\lambda, z) \in \mathring{L}_t \times \mathring{L}_x, \; |\lambda| \geq \lambda_0.$$

and we proved that this is equivalent to the symbol $P(\lambda, z)$ having non-vanishing γ-principal parts on $(L_t \setminus \{0\}) \times (L_x \setminus \{0\})$, i.e.

$$\pi_\gamma P(\lambda, z) \neq 0, \quad (\lambda, z) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\}), \; \gamma \in (0, \infty].$$

We have proven this equivalence by showing that N-parameter-ellipticity must mean that we can estimate the γ-principal parts of P from below with non-negative γ-principal parts of W_P, and conversely by using a partition of $(\lambda, z) \in \mathring{L}_t \times \mathring{L}_x$ with $|\lambda| \geq \lambda_0$ based on their moduli $(|z|, |\lambda|) \in (0, \infty)^2$, which allowed us to decompose $|P(\lambda, z)|$ into parts that could be estimated from below using certain lemmas and this partition.

We can also compare my work to that of R. Denk & M. Kaip [1]. We see that I have managed to make the proofs more complete by filling in the details that R. Denk & M. Kaip may have left out. We can also see that due to being more complete and more elaborate in the details, we have made our work easier to understand and therefore more accessible to bachelor students, so that they can read into the subject before conducting their own research on partial differential equations.

Discussion and future research

In my literature analysis of these chapters, there is quite a bit I have left out. Two main things are there to mention:

1. concave and strictly negative order functions and corresponding weight functions.
   When analysing mixed order systems (see section 2.3 [1],) we also need a notion of strictly

negative order functions. In section 2.1, we already see how these order functions are defined in Definition and Remark 2.21[1], but I have left out these definitions since they were not needed to proof the result, and they would make the content less accessible to bachelor students. Future research should also look into the subject of creating weight functions corresponding to order functions that are not necessarily strictly positive, for the same reason as above. One can refer to Definition 2.24 [1].

2. operations on Newton polygons and order functions.
   There are some very useful results on parameter shifts, additions and inequalities of order functions described in section 2.1 [1] that I could not get into for the exact same reasons: they were not needed for the result, and they would add more content that is harder to understand. These results however are useful when researching the quotient of symbols, as can be seen Proposition 2.42 [1].

Bibliography

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List of symbols

Symbols, functions, domains and function classes

$d_\gamma(P)$	γ-order of the symbol $P \in \tilde{S}(L_t \times L_x)$, page 18.
$d_\gamma(\tau)$	γ-order of the function $\tau \in S^{(\rho,N)}(L_t \times L_x)$, page 15.
$d_\infty(P)$	∞-order of the symbol $P \in \tilde{S}(L_t \times L_x)$, page 18.
$d_\infty(\tau)$	∞-order of the function $\tau \in S^{(\rho,N)}(L_t \times L_x)$, page 16.
$H(\mathring{L}_t \times \mathring{L}_x)$	Class of symbols that are holomorphic in $\mathring{L}_t \times \mathring{L}_x$, 13.
$H_P(\mathring{L}_t \times \mathring{L}_x)$	Class of symbols that are holomorphic and polynomially bound in $\mathring{L}_t \times \mathring{L}_x$, page 13.
I_P	Finite index set belonging to $P \in \tilde{S}(L_t \times L_x)$, page 17.
I_γ	Part of I_P determining the γ-principal part, page 18.
I_∞	Part of I_P determining the ∞-principal part, page 18.
L_t, L_x	Closed cones, page 13.
$\mathring{L}_t$, $\mathring{L}_x$	Interiors of L_t and L_x, page 13.
$\pi_\gamma P$	γ-principal part of the symbol $P \in \tilde{S}(L_t \times L_x)$, page 18.
$\pi_\gamma \tau$	γ-principal part of the function $\tau \in S^{(\rho,N)}(L_t \times L_x)$, page 16.
$\pi_\infty P$	∞-principal part of the symbol $P \in \tilde{S}(L_t \times L_x)$, page 18.
$\pi_\infty \tau$	∞-principal part of the function $\tau \in S^{(\rho,N)}(L_t \times L_x)$, page 16.
R_P	Representation of a symbol $P \in \tilde{S}(L_t \times L_x)$, page 17.
$S(L_t \times L_x)$	Class of symbols P for which a regular representation R_P exists, page 19.
$\tilde{S}(L_t \times L_x)$	Class of symbols P for which a representation R_P exists, page 17.
$S_N(L_t \times L_x)$	Class of N-parameter-elliptic symbols P for which a regular representation R_P exists, page 40.
$S^{(N)}(L)$	Class of continuous non-vanishing homogeneous functions with degree N on the cone L, page 13.
$S^{(\rho,N)}(L_t \times L_x)$	Class of continuous non-vanishing ρ-homogeneous functions with degree N, page 14.
S_θ	Sector in $\mathbb{C}$ with angle $\theta \in (0,\pi)$, page 13.
Σ_δ^n	Bi-sector in $\mathbb{C}^n$ with angle $\delta \in (0, \frac{1}{2}\pi)$, page 13.
τ_ℓ , ϕ_ℓ, ψ_ℓ	Functions that build up a representation R_P with degrees N_ℓ, M_ℓ and L_ℓ respectively for each $\ell \in I_P$, page 17.

Newton polygons, weight and order functions

$b_i(\mu)$	Value associated to the order function μ, page 30.
γ_j	j-th part of the partition of $[0,\infty]$ based on a Newton polygon N, page 24.
$m_i(\mu)$	Value associated to the order function μ, page 30.
μ	An order function, page 30.
μ_N	Order function associated to Newton polygon N, page 31.
μ_P	Order function associated to Newton polygon $N(P)$, page 32.
N	A Newton polygon, page 22.
$N(\nu)$	Newton polygon of the finite set $\nu \subseteq [0,\infty)^2$, page 22.
$N(\mu)$	Newton polygon associated with order function μ, page 32.
$N(P)$	Newton polygon of the symbol $P \in S(L_t \times L_x)$, page 22.
N_v	Set of vertices of the Newton polygon N, page 23.
ν	A finite set of points in $[0,\infty)^2$, page 22.
$\nu(\mu)$	Finite set of points associated to μ, page 32.
$\nu(P)$	Finite set of points $\mathfrak{u}_\ell$ and $\mathfrak{v}_\ell$ for $\ell \in I_P$, page 22.
q_j	j-th outward normal vector from the edge $[v_j v_{j+1}]$ of the Newton polygon N, page 24.
$q_j^\perp$	Orthogonal normal vector of q_j following the edge $[v_j v_{j+1}]$ of the Newton polygon N, page 24.
r_j	First coordinate of v_j, page 23.
s_j	Second coordinate of v_j, page 23.
$\mathfrak{u}_\ell$	Point in $[0,\infty)^2$ that counts the order of τ_ℓ for z, page 22.
v_j	j-th vertex in N_v, page 23.
$[v_j v_{j+1}]$	j-th edge of the Newton polygon, page 23.
$\mathfrak{v}_\ell$	Point in $[0,\infty)^2$ that counts the order of τ_ℓ for λ, page 22.
W_N	Weight function corresponding to the Newton polygon N, page 27.
W_μ	Weight function corresponding to the Newton polygon $N(\mu)$, page 32.
W_P	Weight function corresponding to the Newton polygon $N(P)$, page 27.

Other

$C_j(\varepsilon_0)$	Cone-shaped area in between $S_{j-1}(\varepsilon_0,\varepsilon_1)$ and $S_j(\varepsilon_0,\varepsilon_1)$, page 48.
$G_j(\varepsilon_0,\varepsilon_1), \widetilde{G}_j(\varepsilon_0)$	Parts of the partition of $(0,\infty)^2$ based on N, page 49.
γ	A relative weight in $[0,\infty]$, page 15.
J	A number in $\mathbb{N}$ indicating how many vertices the Newton polygon N has, page 23.
N_ℓ, M_ℓ, L_ℓ	Degrees of τ_ℓ, ϕ_ℓ and ψ_ℓ respectively for $\ell \in I_P$, page 17.
$r_t(N)$	Time-regularity index of the Newton polygon N, page 25.
$r_x(N)$	Space-regularity index of the Newton polygon N, page 25.
ρ	Weight of a ρ-homogeneous function, page 14.
$S_j(\varepsilon_0,\varepsilon_1)$	Half-strip in direction of q_j, page 48.