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The influence of time dependent vertical mixing on suspended sediment in the Ems estuary

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"The influence of time dependent vertical mixing on suspended sediment in the Ems estuary"

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Abstract

Over time, the width-averaged depth of estuaries changes due to a complex interaction of hydrodynamics and suspended sediment transport. In many estuaries one specific location with a suspended sediment concentration (SSC) higher than in the sea or in the upstream river, is found, which is called sediment trapping. The location of the maximum SSC is called the estuary turbidity maximum (ETM). Understanding the dynamics is important to maintain a healthy ecosystem while making anthropogenic changes. To investigate such changes, a two-dimensional model is developed, considering the Ems estuary as a case study. The model equations consist of the width-averaged shallow water equations and a SSC equation. Assuming a morphodynamic equilibrium, these equations are solved mostly analytically by making a regular expansion of each physical variable in a relatively small parameter. Using this method, we are able to gain insight into the fundamental physical processes resulting in sediment trapping in an estuary by studying the influence of various forcings separately. One of the hydrodynamic forces is vertical mixing. This force has been assumed to be constant over time in previous studies [1]. In this thesis vertical mixing as a function that varies on the tidal timescale has been added to the model and is analysed. As a result of the salinity gradient in the estuary the mixing is stronger during flood and weaker during ebb. Using the model it is found that time variations in vertical mixing result in tidally averaged non-zero, and therefore contributing, transports. They cause a narrowing of the location where sediment is trapped. If vertical mixing is exactly maximal when the flood is maximal and minimal when ebb is maximal, the ETM shifts downstream. But if the vertical mixing is lagging the tidal stream, which is more plausible, the ETM will stay or shift upstream.

Contents

Chapter 1

Introduction

In many estuaries one specific location is found with a higher concentration of suspended sediment (SS) than in the sea or upstream in the river. The exact location of the maximum concentration, called the estuarine turbidity maximum (ETM), is a result of a complex interaction of water motions and sediment dynamics in the estuary; caused by anthropogenic and natural changes, the location of the ETM can change with possible consequences for the quality of live in the estuary: at the ETM less light penetrates in the water column, resulting in less phytoplankton growth and therefore less oxygen in the water. An example of an ETM is shown in figure 1.1 for the Humber estuary, UK. Here, especially in figure (C), but also in figure (A), an ETM is measured at approximately 30 km from the tidal limit, since the SSC is higher at this point than upstream the river or downstream at sea.

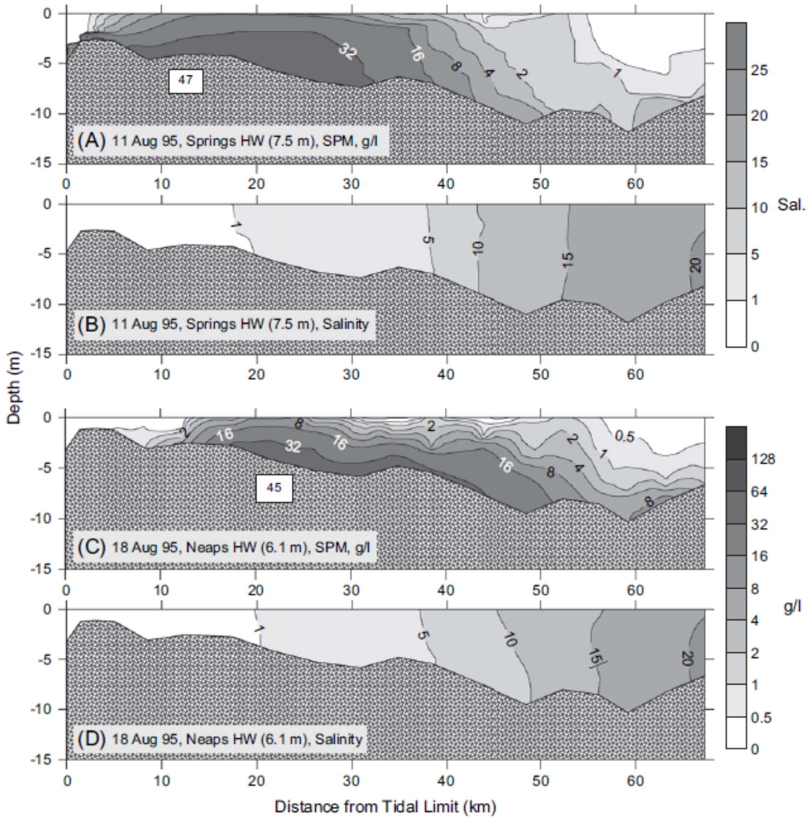


Figure 1.1: A cross section of the Humber estuary, UK. At the right side the entrance to the sea is located, and at the left side is the most upstream location where the river is affected by tidal fluctuations, called the tidal limit. Measured suspended particulate matter (SPM) concentration is plotted in figure (A) and (C) for tow different moments in the year. [6]

Because of its high environmental and economical impact, it is important to understand the physical mechanisms of water motions and sediment dynamics in estuaries. Employing numerical models it is possible to simulate most of the processes (baroclinic circulation, tidal straining, tidal pumping, flocculation, settling and scour lag, non-linear reactions, etc). However, these models require high computational capacity to calculate the evolution of the physical equations in time. Another problem with these models is that it is hard to understand which processes are most significant. In more idealized models, methods like scale analysis and expansions of the physical parameters are used to obtain solutions that can largely be found analytic. This results in a much faster model and the different processes can be analysed separately due to the linearity, what makes it possible to compare their relative importance.

An idealized model as described above has been developed in 2010 [1], with the Ems estuary, Germany, as a reference case. One of the processes related to the trapping of suspended sediment is vertical mixing, i.e. the eddy diffusion. In this work [1], it was assumed that the quantity of the mixing term was constant over time. However, this is not realistic since the mixing is influenced by the water motions and salinity gradient. In this thesis we will add the dependency of time of the vertical mixing to the model from 2010 [1] and analyse the influences. The Ems estuary will be used as a reference case again.

In chapter 2 we will derive the set of equations describing the water motions and suspended sediment balance including the time dependent mixing term, then we will rewrite them in chapter 3 with a scaling method to obtain differential equations which can be solved mostly analytically. In chapter 4 the results of the model will be presented, followed by a discussion and conclusion in chapter 5.

Chapter 2

Model formulation

2.1 Geometry

The estuary is considered to have length L, with a width that is allowed to vary in the along-estuary direction. The seaward side of the estuary is located at $x = 0$. At $x = L$, the most upstream side of the estuary, a weir is found. The width is assumed to be constant over time, and vary exponentially as

$$B(x) = B_0 e^{-x/L_b}, \tag{2.1}$$

with B_0 the width at the seaward side and L_b the exponential convergence length. As sketched in figure 2.1 the vertical direction is denoted by z, where $z = 0$ is the undisturbed surface. $z = -H(x)$ describes the location of the bed at location x, and H_0 is the depth at the seaward side. The surface elevation is denoted by $z = \zeta(x, t)$, and deviates from zero because of tides, wind, etc.. The central axis of the estuary is denoted by $y = 0$, and the shores of the estuary are located at $y = \frac{1}{2}B$ and $y = -\frac{1}{2}B$. Furthermore $\mathbf{u} = (u, v, w)$ describes the velocity in the x, y and z direction respectively. All the velocities are functions of x, y, z and the time t.

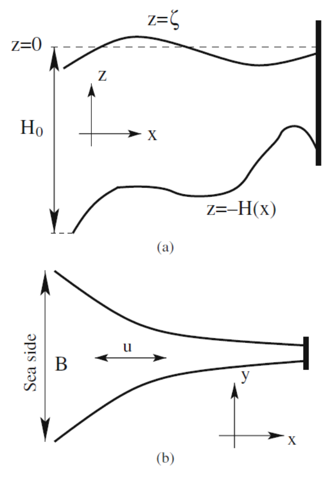


Figure 2.1: Scetch of the model geometry[1].

2.2 Time dependent mixing

Consider a tank filled with water, see figure 2.2. In the middle of the tank is a barrier, the left side of the tank is filled with salt water the right side with fresh water. When the barrier is removed, first the lighter fresh water will flow over the heavier salt water. Next, due to turbulent motions the fresher upper layer will mix with the salty layer. In the estuary the fresh water from the river meets the salt sea water, resulting in similar movements.

The eddy diffusion coefficient K_v describes the efficiency of vertical mixing of the sediment, due to turbulent motions. In the model described in [1] it has been assumed that the vertical eddy diffusion

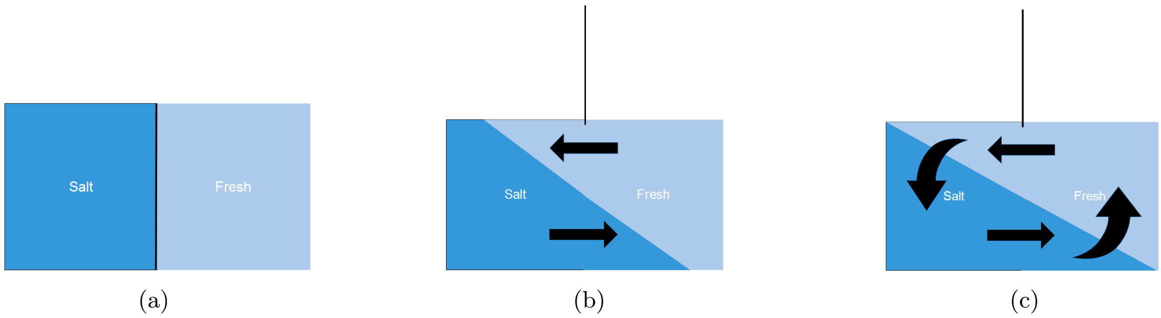


Figure 2.2: A tank filled with salt and sweet water. Removing the barrier results in turbulent motions.

coefficient K_v equals the time independent eddy viscosity coefficient A_v. However it is possible to argue that both A_v and K_v should be time dependent.

To illustrate the plausibility of time dependent eddy viscosity and diffusivity, consider an estuary which is well-mixed in the vertical, with high salinities at the seaward side and low salinities at the landward side, as illustrated in figure 2.3. Since the water contains more salt at the seaside of the estuary, it is heavier than the fresh river water. Also the amount of sediment concentration in the water is dependent of z, as a result of gravity. The closer to the bottom the higher the concentration of sediment. The SS profile is indicated by the yellow line in figure 2.3.

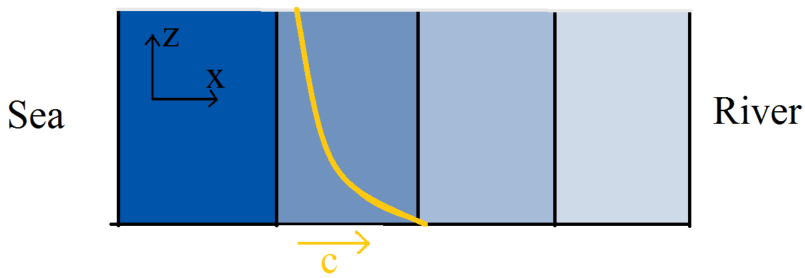


Figure 2.3: Sketch of the estuary with no current. Vertical is the z-direction and horizontal the x-direction. Left is the seaside of the estuary and right the weir. The yellow line gives an indication of the amount of sediment in the river plotted against the z-axis. The darker blue columns indicate higher salinity of the water. Closer to the sea the water contains more salt, indicated with the darker colour.

During flood water is transported into the estuary. Because of the friction with the bottom the velocity is lower near the bottom than higher up in the water column. As a result the heavier salty water moves over the lighter fresh water, see figure 2.4. This is an unstable situation, resulting in an enhanced vertical mixing. The sediment - which was first generally at the bottom - will now mix higher in the water column. Since both the velocities and sediment concentrations are enhanced due to stronger vertical mixing, there is enhanced transport of suspended sediments in the landward direction.

During ebb the water flows back to the sea, in this case the lighter water will move over the heavy water, sea figure 2.5. This is a stable situation, resulting in suppression of mixing and therefore the sediment will stay closer to the bottom. This means that the transport of SS in the direction of the sea during ebb is smaller than the transport in the landward direction during flood. Hence, a tidally averaged transport of SS in the upstream direction is expected.

To describe this tidal mixing asymmetry, the eddy diffusion coefficient will be prescribed as

$$K_v = K_v^0(x) + \hat{K}_v^1(x,t), \tag{2.2}$$

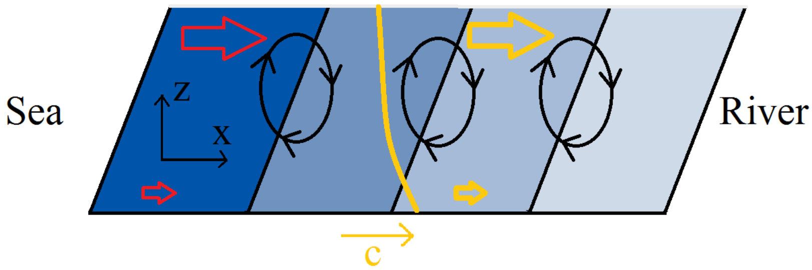


Figure 2.4: Sketch of the estuary during flood. The red arrows represent the current, the round arrows indicate the mixing and the yellow arrows represent suspended sediment transport.

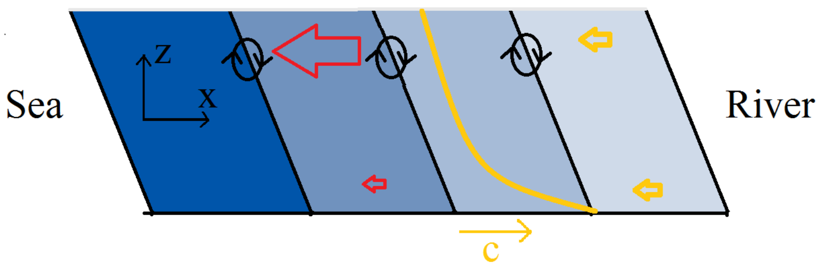


Figure 2.5: Sketch of the estuary during ebb. The size and direction of some arrows has changed compared to the situation during flood, which means velocities, transport and/or mixing have increased or decreased.

where K_v^0 is the tidally average mixing coefficient and the variation in mixing is denoted by $\hat{K}_v^1$, that is a tidally periodic function. Since the mixing during flood is related to the salinity gradient we assume $\hat{K}_v^1$ to be proportional to the gradient of salinity. It seems plausible that K_v has the same period as the period T of the main tidal component with $T \approx 12.5$h. Probably the mixing is not at a maximum at maximal flood but a bit later, because the mixing is not instantaneous. Therefore the temporal part of the eddy diffusivity has a phase difference $\Delta\phi$ with the horizontal velocity, hence

$$\hat{K}_v^1 = K_v^1 \cos(\sigma t - \phi_u - \Delta\phi), \tag{2.3}$$

with $K_v^1 \sim \frac{\mathrm{d}s}{\mathrm{d}x}$ where $\frac{\mathrm{d}s}{\mathrm{d}x}$ is the salinity gradient, $\sigma = 2\pi/T = 1.4 \cdot 10^{-4}\mathrm{s}^{-1}$, and ϕ_u the phase of horizontal velocity of the water motion. For simplicity define $\phi_{K_v^1} := \phi_u + \Delta\phi$. Assuming $\Delta\phi = 0°$ physically means that the mixing is maximal during maximal flood, and minimal during maximal ebb. If $\Delta\phi = 20°$ is chosen, the maximal mixing happens $\frac{20}{360} \cdot T \approx 40$ minutes after the flood is maximal. In this case the mixing needs some time to get started. The correct value for $\Delta\phi$ is unknown and can be determined from measurements.

2.3 Equations governing the water motion

The width-averaged water motion in the estuary is described by the continuity and momentum equation, see appendix A for a more detailed derivation,

$$u_x + w_z - \frac{u}{L_b} = 0, \tag{2.4}$$

$$u_t + uu_x + wu_z + g\zeta_x - \frac{g\rho_x}{\rho_0}(z - \zeta) - (A_v u_z)_z = 0, \tag{2.5}$$

where $g \sim 10$ m^2/s is the gravitational acceleration, $\rho(x, z, t)$ the water density, $\rho_0 \sim 1{,}020$ kg m^3 the reference density and A_v is the vertical eddy viscosity coefficient that parameterizes the strength of (unresolved) small scale turbulence. Following previous study [4], we assume that

$$A_v(x) = A_{v0}\frac{H(x)}{H_0}, \tag{2.6}$$

with A_{v0} the eddy viscosity at the seaside. This means that the viscosity is linearly proportional to the depth of the estuary.

It is assumed that density only depends on salinity, thus neglecting e.g. temperature or suspended sediment concentration (SSC). This results in the following expression for the density

$$\rho(s) = \rho_0(1 + \beta < s(x) >), \tag{2.7}$$

with $\beta \sim 7.6 \cdot 10^{-4}\text{psu}^{-1}$ which converts salt to density. The tidal average is denoted by $< . >$. Hence the density in equation (2.7) depends on $\langle s(x) \rangle$, the observed along-channel, time and depth-averaged salinity profile which describes the gradual decrease of the salinity from the sea to the river.

At the free surface $z = \zeta$ no water will leave the system:

$$\frac{\mathrm{d}}{\mathrm{d}t}(z - \zeta) = 0 \text{ at } z - \zeta = 0, \tag{2.8}$$

with $\frac{\mathrm{d}}{\mathrm{d}t}$ the total derivative. Using the chain rule, the kinematic boundary condition

$$w = \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} \qquad \text{at } z = \zeta. \tag{2.9}$$

is derived from equation (2.8)

Furthermore, a no stress condition is imposed:

$$A_v u_z = 0 \qquad \text{at } z = \zeta. \tag{2.10}$$

At the bottom $z = -H(x)$ the water does not penetrate the sand, which means we have to prescribe an impermeability condition (2.11). Furthermore, the partial slip condition (2.12) is imposed

$$w = -u\frac{\partial H}{\partial x}, \tag{2.11}$$

$$\tau_b \equiv \rho_0 A_v u_z = \rho_0 s u. \tag{2.12}$$

Here, τ_b is the bottom shear stress. The latter condition gives a (linear) relation between the bed shear stress and the velocity at the top of the bottom boundary layer [7], with s the stress or slip parameter, which is assumed to be $s = s_0\frac{H(x)}{H_0}$. If $s \to 0$, this condition reduces to a free slip boundary condition, for $s \to \infty$ a no-slip boundary condition is found.

The water motion is forced both at the sea and riverine side. At the seaward side, a tidal elevation is prescribed that consists of two tidal frequencies, the semi-diurnal tide (M_2) and its first overtide (M_4)

$$\zeta(t) = A_{M_2}\cos(\sigma t) + A_{M_4} cos(2\sigma t - \phi) \qquad \text{at } x = 0, \tag{2.13}$$

where ϕ is the relative phase difference in the phases of the M_2 and the M_4 tidal constituents and A_{M_2} and A_{M_4} the amplitudes of respectively the semi-diurnal and its first overtide.

At the upstream side we prescribe a constant inflow of water Q from the river,

$$B(x)\int_{-H}^{\zeta} u dz = Q \qquad \text{at } x = L. \tag{2.14}$$

2.4 Suspended sediment concentration (SSC) equation

The equation describing the width averaged SSC reads:

$$c_t + uc_x + wc_z = w_s c_z + (K_h c_x)_x + (K_v c_z)_z - \frac{1}{L_b}K_h c_x, \tag{2.15}$$

where $c(t, x, z)$ denotes the SSC, $w_s \sim 0.2 - 5$ mm s^{-1} the settling velocity, K_h the horizontal turbulent eddy diffusion coefficient and K_v the vertical one. A derivation of this expression can be found in appendix

B. The tidally average vertical turbulent eddy diffusion K_v^0, see equation (2.2), is assumed to be equal to A_v.

At the surface it is required that no sediment particles can leave the water,i.e. the normal component of the diffusive and the settling flux must vanish:

$$\left\langle \begin{pmatrix} -\zeta_x \\ 1 \end{pmatrix}, -cw_s \begin{pmatrix} 0 \\ 1 \end{pmatrix} - K_h \begin{pmatrix} c_x \\ 0 \end{pmatrix} - K_v \frac{\partial c}{\partial z} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = 0 \qquad \text{at } z = \zeta,$$
$$\Rightarrow w_s c + K_v c_z - K_h c_x \zeta_x = 0 \qquad \text{at } z = \zeta. \tag{2.16}$$

Sediment will be suspended form the bed due to erosion. This is described by

$$E_s \equiv -K_v c_z n^z - K_h c_x n^x = w_s c_* \text{ at } z = -H(x), \tag{2.17}$$

where $\mathbf{n} = (n^x, n^z) = (H_x, -1)$ is the downward pointing vector on the bottom. The concentration c_* is the reference concentration, defined as

$$c_*(t, x) = \rho_s \frac{|\tau_b(t, x)|}{\rho_0 g' d_s} a(x). \tag{2.18}$$

Here, ρ_s is the density of sediment, $\frac{|\tau_b(t,x)|}{\rho_0 g' d_s}$ is the dimensionless bed shear stress in which τ_b is the bottom stress as defined in equation (2.12), $g' = g(\rho_s - \rho_0)/\rho_0$ is the reduced gravity and d_s is the the average grain size of the sediment. The erosion coefficient $a(x)$ models the along-channel distribution of easily erodible sediment, available in mud reaches [1]. The greater this variable, the more sediment can be eroded and contribute to concentration, and sediment transport.

Following studies by Friedrichs et al. [3] and Huijts et al. [5], the model considers the system to be in morphodynamic equilibrium, which means that there is no evolution of the bed over a tidal period. This is a valid assumption when the the duration of redistribution of easily erodible sediment is much smaller than the typical timescale at which external forces change significantly. Hence, there exists a balance between erosion and deposition of the sediment at the bed. The erosion flux is defined in equation (2.17) and the deposition flux has been defined as

$$\vec{F}_s = -cw_s \vec{e}_z, \tag{2.19}$$

(also see Appendix B, equation (10b)). The morphodynamic equilibrium imposes

$$\langle E \rangle - \langle D \rangle = 0. \tag{2.20}$$

Integrating the sediment mass balance equation (2.15) over depth, using boundary conditions (2.9), (2.11) and (2.16) the following equilibrium is obtained after tidally averaging:

$$\left\langle \int_{-H}^{\zeta} (uc - K_h c_x) dz \right\rangle = 0, \tag{2.21}$$

where we assumed that there is no residual sediment flux at the weir.

The morphodynamic equilibrium condition still dependents on the unknown erosion coefficient $a(x)$. Later will be derived that the sediment concentration depends linearly on the coefficient. Hence we can rewrite the equilibrium condition as a first order differential equation for $a(x)$:

$$\left\langle \int_{-H}^{\zeta} \left(au\frac{c}{a} - aK_h \left(\frac{c}{a}\right)_x - K_h a_x \frac{c}{a} \right) dz \right\rangle = 0, \tag{2.22}$$

$$\Rightarrow Fa_x + Ta = 0, \tag{2.23}$$

where

$$F = \left\langle \int_{-H}^{\zeta} -K_h \frac{c}{a} dz \right\rangle, \tag{2.24}$$

$$T = \left\langle \int_{-H}^{\zeta} \left(u\frac{c}{a} - K_h \left(\frac{c}{a}\right)_x \right) dz \right\rangle. \tag{2.25}$$

Prescribing the average amount of sediment available for resuspension a_*, the integration constant can be determined by requiring

$$\frac{\int_0^L B(x)a(x)dx}{\int_0^L B(x)dx} = a_*. \tag{2.26}$$

Chapter 3

Scaling

In this section we will find semi-analytic solutions for the water motion and SSC equations using a scaling technique, which allows for a systematic way to assess the relative importance of the different terms in the equations. As a first step the variables are made dimensionless, i.e. by dividing them by their typical unit value. For example for the vertical variable $z = H_0\tilde{z}$, where the tilde indicates that the variable is dimensionless. An overview for al the variables can be seen in table 3.1 based on the article by Chernetsky et al.[1].

Scaling			
Physical quantity	Typical scale	Symbol	Variable
Time	M_2 tidal frequency	σ	$t = \sigma^{-1}\tilde{t}$
Sea surface elevation	M_2 tidal amplitude	A_{M_2}	$\zeta = A_{M_2}\tilde{\zeta}$
Local water depth	Water depth at entrance	H_0	$H = H_0\tilde{H}$
Vertical coordinate	Water depth at entrance	H_0	$z = H_0\tilde{z}$
Horizontal coordinate	Minimum of the estuary length or convergence length	l	$x = l\tilde{x}$
Vertical velocity	Obtained from the width-averaged continuity equation (2.4) by requiring an approximate balance between the first and second term	$W = \frac{H_0}{l}U$	$w = W\tilde{w}$
Horizontal velocity	Follows from the integration of the continuity equation (2.4) over depth and requiring an approximate balance between the resulting contributions	$U = \frac{\sigma A_{M_2} l}{H_0}$	$u = U\tilde{u}$
Sediment concentration	Typical magnitude of the quantity under consideration	$C = \frac{\rho_s A_v U a_*}{H_0 g' d_s}$	$c = C\tilde{c}$
Salinity gradient		S_x	$\langle s \rangle_x = S_x\widetilde{\langle s \rangle_x}$
Erosion coefficient	The average amount of sediment avalable for resuspension	a_*	$a = a_* a$

Table 3.1: Scaling of variables

Using the relations between the dimensional and dimensionless variables, equations (2.4) and (2.5) can be rewritten as

$$\begin{cases} \frac{U}{l}\tilde{u}_{\tilde{x}} + \frac{U}{l}\tilde{w}_{\tilde{z}} - \frac{U\tilde{u}}{L_b} = 0, \\ \frac{U}{L}\tilde{u}_{\tilde{t}} + \frac{U^2}{l}(\tilde{u}\tilde{u}_{\tilde{x}} + \tilde{w}\tilde{u}_{\tilde{z}}) + \frac{gA_{M_2}}{l}\zeta_x - g\beta S_x\widetilde{\langle s \rangle_x}(H_0\tilde{z} - A_{M_2}\tilde{\zeta}) - \left(\frac{A_v H}{H_0^2}\tilde{u}_{\tilde{z}}\right)_{\tilde{z}} = 0, \end{cases} \tag{3.1}$$

$$\Rightarrow \begin{cases} \tilde{u}_{\tilde{x}} + \tilde{w}_{\tilde{z}} - \frac{l\tilde{u}}{L_b} = 0, \\ \tilde{u}_{\tilde{t}} + \frac{U}{\sigma l}(\tilde{u}\tilde{u}_{\tilde{x}} + \tilde{w}\tilde{u}_{\tilde{z}}) + \lambda^{-2}\zeta_x - \frac{U_d}{U}\tilde{\langle}s\rangle_x(\tilde{z} - \frac{A_{M_2}}{H_0}\tilde{\zeta}) - \left(\frac{A_v}{\sigma H_0^2}\tilde{u}_{\tilde{z}}\right)_{\tilde{z}} = 0, \end{cases} \tag{3.2}$$

where $\lambda := l/L_w$, in which L_w is the frictionless tidal wavelength and $U_d := \frac{gH_0\beta S_x}{\sigma}$ is the typical velocity scale for the density driven residual circulation.

Similarly all the boundary conditions have to be written in terms of the dimensionless variables. The prescribed forcing of the surface due to tide, equation (2.13), becomes

$$\tilde{\zeta} = \cos(\tilde{t}) + \frac{A_{M_4}}{A_{M_2}}\cos(2\tilde{t} - \phi) \text{ at } \tilde{x} = 0, \tag{3.3}$$

and the boundary condition at the riverine side, equation (2.14), is

$$\int_{-\tilde{H}}^{\varepsilon\tilde{\zeta}} \tilde{u}d\tilde{z} = \frac{Q}{UH_0B} \text{ at } \tilde{x} = 1, \tag{3.4}$$

where $\varepsilon = \frac{A_{M_2}}{H_0}$.

At the free surface $\tilde{z} = \varepsilon\tilde{\zeta}$, the boundary conditions described in equation (2.9) are

$$\tilde{w} = \tilde{\zeta}_{\tilde{t}} + \frac{A_{M_2}}{H_0}\tilde{u}\tilde{\zeta}_{\tilde{x}}, \tag{3.5}$$

$$A_v\tilde{u}_{\tilde{z}} = 0. \tag{3.6}$$

At the bottom $\tilde{z} = -\tilde{H}$ equations (2.11) and (2.12) become

$$\tilde{w} = -\tilde{u}\tilde{H}_{\tilde{x}}, \tag{3.7}$$

$$\tilde{u}_{\tilde{z}} = \frac{sH_0}{A_v}\tilde{u}. \tag{3.8}$$

The equations for the SSC, (2.15), (2.16) and (2.17) become

$$\tilde{c}_{\tilde{t}} + \frac{U}{\sigma l}(\tilde{u}\tilde{c}_{\tilde{x}} + \tilde{w}\tilde{c}_{\tilde{z}}) - \frac{w_s}{\sigma H_0}\tilde{c}_{\tilde{z}} - \frac{K_h}{\sigma l^2}\tilde{c}_{\tilde{x}\tilde{x}} - \frac{K_v}{\sigma H_0^2}\tilde{c}_{\tilde{z}\tilde{z}} - \frac{K_h}{\sigma l L_b}\tilde{c}_{\tilde{x}} = 0, \tag{3.9}$$

$$\frac{w_s}{\sigma H_0}\tilde{c} + \frac{K_v}{\sigma H_0^2}\tilde{c}_{\tilde{z}} - \frac{K_h A_{M_2}}{\sigma l^2 H_0}\tilde{c}_{\tilde{x}}\tilde{\zeta}_{\tilde{x}} = 0 \qquad \text{at } \tilde{z} = \varepsilon\tilde{\zeta}, \tag{3.10}$$

$$-\frac{K_v}{\sigma H_0^2}\tilde{c}_{\tilde{z}} - \frac{K_h}{\sigma l^2}\tilde{c}_{\tilde{x}}\tilde{H}_{\tilde{x}} = \frac{w_s}{\sigma H_0}|\tilde{u}_{\tilde{z}}|\tilde{a} \qquad \text{at } \tilde{z} = -\tilde{H}. \tag{3.11}$$

The morphodanymic equilibrium condition and the integral condition for preserving the available amount of sediment for resuspension become

$$\left\langle \int_{-\tilde{H}}^{\varepsilon\tilde{\zeta}} (\tilde{u}\tilde{c} - \frac{K_h}{lU}\tilde{c}_{\tilde{x}})d\tilde{z} \right\rangle = 0, \tag{3.12}$$

and

$$\frac{\int_0^1 \tilde{a}e^{-\frac{l}{L_b}\tilde{x}}d\tilde{x}}{\int_0^1 e^{-\frac{l}{L_b}\tilde{x}}d\tilde{x}} = 1. \tag{3.13}$$

Using the dimensionless equations it is possible to assess the relative importance of each term. In most estuaries the amplitude of the semi-diurnal constituent A_{M_2} is much smaller than the depth of the estuary H_0, which means that $\varepsilon := A_{M_2}/H_0 \ll 1$. Comparing the other coefficients in the above equations to this parameter, an estimate of their order of magnitude in terms of ε can be made, see table 3.2. Notice that the dimensionless slip parameter $\frac{sH_0}{A_v}$ is allowed to vary from zero to a large value. Furthermore we assume that $\frac{Q}{UH_0B}$ is at most of order ε.

The time dependent part of the eddy diffusion coefficient K_v is assumed one order higher than the average K_v^0. In section 2.2 we defined K_v as

$$K_v(x,t) = K_v^0(x) + K_v^1(x)\cos(\sigma t - \phi_{K_v^1}) \tag{3.14}$$

Non-dimensional parameter	Value	Order
$\varepsilon := A_{M_2}/H_0$	0.14	$\mathcal{O}(\varepsilon)$
$U/\sigma l$	0.14	$\mathcal{O}(\varepsilon)$
l/L_b	1	$\mathcal{O}(1)$
U_d/U	0.27	$\mathcal{O}(\varepsilon)$
$A_v/\sigma H_0 = K_v^0/\sigma H_0^2$	1.57	$\mathcal{O}(1)$
$w_s/\sigma H_0$	.14-3.57	$\mathcal{O}(1)$
$K_h/\sigma l^2$	$7.9 \cdot 10^{-4}$	$\mathcal{O}(\varepsilon^3)$
$K_h/\sigma l L b$	$7.9 \cdot 10^{-4}$	$\mathcal{O}(\varepsilon^3)$
A_{M_4}/A_{M_2}	0.17	$\mathcal{O}(\varepsilon)$
K_h/lU	0.006	$\mathcal{O}(\varepsilon^2)$
$A_{M_2}K_h/\sigma l^2 H_0$	$1.1 \cdot 10^{-4}$	$\mathcal{O}(\varepsilon^4)$

Table 3.2: The orders of the magnitude of the parameters and their exact values [1]

.

with K_v^0 the tidally averaged mixing, K_v^1 the amplitude of the time-varying diffusivity, and $\phi_{K_v^1}$ the phase. Recall that the average mixing K_v^0 is assumed to be equal to A_v. To make the time dependent part of one order higher $K_v^1/\sigma H_0^2$ must be of $\mathcal{O}(\varepsilon)$. The parameters K_v^1 and are prescribed and will be varied to assess their influence on the sediment trapping.

Next, the dimensionless physical variables are rewritten as power series of ε, i.e.

$$\tilde{u} = \tilde{u}^0 + \varepsilon\tilde{u}^1 + \varepsilon^2\tilde{u}^2 + \ldots, \tag{3.15}$$

$$\tilde{w} = \tilde{w}^0 + \varepsilon\tilde{w}^1 + \varepsilon^2\tilde{w}^2 + \ldots, \tag{3.16}$$

$$\tilde{\zeta} = \tilde{\zeta}^0 + \varepsilon\tilde{\zeta}^1 + \varepsilon^2\tilde{\zeta}^2 + \ldots, \tag{3.17}$$

$$\tilde{c} = \tilde{c}^0 + \varepsilon\tilde{c}^1 + \varepsilon^2\tilde{c}^2 + \ldots. \tag{3.18}$$

The superscripts describe the order of the variable. Substituting these in the equations and collecting terms of the same order results in systems of equations at each order in ε.

The system of differential equations will be solved at each different order of ε separately. First the differential equations will be solved at the leading order ($\mathcal{O}(1)$), since higher order terms are negligible at this order. Knowing the solutions of the leading order, solutions at the first order can be derived. In general, knowing solutions of the equations at order n, the solutions at order $n+1$ can be derived.

3.1 Leading order system of equations

In this section only the leading order system of equations for the water motion is considered. First we transform the dimensionless equations on order $\mathcal{O}(1)$ back to dimension equation. The continuity and momentum equation at leading order are

$$u_x^{02} + w_z^{02} - \frac{u^{02}}{L_b} = 0, \tag{3.19a}$$

$$u_t^{02} + g\zeta_x^{02} - (A_v u_z^{02})_z = 0, \tag{3.19b}$$

where the first superscript describes the order of epsilon and the second the frequency the solution contributes to. Because the water motion at leading order is only forced by an M_2 tidal constituent at the seaward side, all the variables have a 2 as second superscript (3.21).

The boundary conditions at leading order are:

$$w^{02} = \zeta_t^{02} \quad \text{at } z = 0, \tag{3.20a}$$
$$A_v u_z^{02} = 0 \quad \text{at } z = 0, \tag{3.20b}$$
$$w^{02} = -u^{02} H_x \quad \text{at } z = -H, \tag{3.20c}$$
$$A_v u_z^{02} = s u^{02} \quad \text{at } z = -H, \tag{3.20d}$$
$$\zeta^{02} = A_{M_2} \cos(\sigma t) \quad \text{at } x = 0, \tag{3.20e}$$
$$\int_{-H}^{0} u^{02} dz = 0 \quad \text{at } x = L. \tag{3.20f}$$

These differential equations allow solutions of the form

$$(u^{02}, w^{02}, \zeta^{02}) = \Re\left\{ \left(\hat{u}^{02}(x,z), \hat{w}^{02}(x,z), \hat{\zeta}^{02}(x) \right) e^{i\sigma t} \right\}, \tag{3.21}$$

where $\Re\{.\}$ denotes the real part of the expression between the braces. Substituting these expressions in equations (3.19) results in:

$$\hat{u}_x^{02} + \hat{w}_z^{02} - \frac{\hat{u}^{02}}{L_b} = 0, \tag{3.22a}$$
$$i\sigma \hat{u}^{02} + g\hat{\zeta}_x^{02} - (A_v \hat{u}_z^{02})_z = 0. \tag{3.22b}$$

with boundary conditions:

$$\hat{w}^{02} = i\sigma\hat{\zeta}^{02} \quad \text{at } z = 0, \tag{3.23a}$$
$$A_v \hat{u}_z^{02} = 0 \quad \text{at } z = 0, \tag{3.23b}$$
$$\hat{w}^{02} = -\hat{u}^{02} H_x \quad \text{at } z = -H, \tag{3.23c}$$
$$A_v \hat{u}_z^{02} = s\hat{u}^{02} \quad \text{at } z = -H, \tag{3.23d}$$
$$\hat{\zeta}^{02} = A_{M_2} \quad \text{at } x = 0, \tag{3.23e}$$
$$\int_{-H}^{0} \Re\{\hat{u}^{02} e^{i\sigma t}\} dz = 0 \quad \text{at } x = L. \tag{3.23f}$$

Notice that this set of differential equations can be solved as an ordinary differential equation with respect to z, for a given $\zeta(x)$. First solve equation (3.22b) subject to the boundary conditions (3.23b) and (3.23d). Using inspection a particular solution $\hat{u}_{part}^{02} = -\frac{g\zeta_x}{i\sigma}$ can be found. The homogeneous solution is of the form

$$\hat{u}_{hom}^{02} = c_1 e^{\sqrt{i\sigma/A_v}z} + c_2 e^{-\sqrt{i\sigma/A_v}z}, \tag{3.24}$$

with c_1 and c_2 constants with respect to z. Define $\beta(x) = \sqrt{i\sigma/A_v}$. Substituting equation (3.24) in the boundary conditions gives the solutions for c_1 and c_2, resulting in

$$\hat{u}^{02} = -\frac{g\zeta_x}{i\sigma}(1 - \cosh(\beta z)\alpha(x)), \tag{3.25}$$

with $\alpha(x) = s/(s\cosh(\beta H) + A_v \beta \sinh(\beta H))$. Now a solution for $\hat{w}^{02}$ can be found using equation (3.22a) with boundary condition (3.23a). This is a first-order ordinary differential equation, which can be solved using an integration factor, resulting in

$$\begin{aligned} \hat{w}^{02} = -\frac{g}{i\sigma} \Bigg(& \left(\frac{\hat{\zeta}_x^{02}}{L_b} - \hat{\zeta}_{xx}^{02} \right) \left(z - \frac{\alpha}{\beta}\sinh(\beta z) \right) + \\ & \frac{\sinh(\beta z)\hat{\zeta}_x^{02}}{\beta} \left(\alpha_x - \frac{\alpha\beta_x}{\beta} \right) + \frac{\hat{\zeta}_x^{02}\alpha\beta_x z}{\beta}\cosh(\beta z) \Bigg) + i\sigma\hat{\zeta}^{02}. \end{aligned} \tag{3.26}$$

To satisfy the boundary condition (3.23c), substitute $\hat{u}^{02}$ and $\hat{w}^{02}$ in it. An ordinary differential equation for $\hat{\zeta}^{02}$ is found:

$$\begin{aligned} \hat{\zeta}^{02}_{xx}\left(-H+\frac{\alpha}{\beta}\sinh(\beta H)\right) - \\ \hat{\zeta}^{02}_{x}\left(\frac{1}{L_b}\cdot\left(-H+\frac{\alpha}{\beta}\sinh(\beta H)\right)-\frac{\sinh(\beta H)}{\beta}\left(\alpha_x-\frac{\alpha\beta_x}{\beta}\right)\right. \\ \left.-\frac{\alpha\beta_x H}{\beta}\cosh(\beta H)+H_x(1-\cosh(\beta H)\alpha)\right)-\frac{\sigma^2}{g}\hat{\zeta}^{02}=0. \end{aligned} \tag{3.27}$$

This ODE, subject to the boundary conditions (3.23e) and (3.23f), can be solved numerically. Notice that boundary condition (3.23f) at the riverine side can be written as $\hat{\zeta}^{02}_x = 0$ since $\hat{u}^{02}$ is lineair proportional to $\hat{\zeta}^{02}_x$.

The SSC equation at leading order reads

$$c^0_t - w_s c^0_z = (K^0_v c^0_z)_z, \tag{3.28}$$

with boundary conditions

$$w_s c^0 + K^0_v c^0_z = 0 \qquad \text{at } z = 0, \tag{3.29}$$

$$-K^0_v c^0_z = w_s \rho_s \frac{s|u^{02}(t,x)|}{g' d_s} a(x) \qquad \text{at } z = -H(x). \tag{3.30}$$

As will be explained in section 3.3, only the residual and the M_4 contribution of the leading order SSC solutions need to be evaluated. Therefore the solution is of the form

$$c^0 = c^{00} + c^{04} \tag{3.31}$$

$$= c^{00}(x,z) + \Re\left\{\hat{c}^{04}(x,z)e^{2i\sigma t}\right\}. \tag{3.32}$$

Substituting this expression the following system of equations is obtained to be solved for the residual component

$$\begin{cases} w_s c^{00}_z = (K^0_v c^{00}_z)_z, & \\ w_s c^{00} + K^0_v c^{00}_z = 0 & \text{at } z = 0, \\ -K^0_v c^{00}_z = w_s \rho_s \frac{s a_0}{g' d_s} a(x) & \text{at } z = -H(x), \end{cases} \tag{3.33}$$

where a_0 is a coefficient in the Fourier series of the absolute value of the M_2 velocity at the bottom

$$|u^{02}(t,-H)| = a_0 + a_1\cos(\sigma t) + b_1\sin(\sigma t) + a_2\cos(2\sigma t) + b_2\sin(2\sigma t) + \dots. \tag{3.34}$$

The solution of equations (3.33) is

$$c^{00} = \frac{a\rho_s s a_0}{g' d_s} e^{\frac{-w_s(H+z)}{K^0_v}}. \tag{3.35}$$

To find the solution of the M_4-contribution of the SSC at leading order the following set of equations has to be solved:

$$\begin{cases} 2i\sigma\hat{c}^{04} - w_s\hat{c}^{04}_z = (K^0_v\hat{c}^{04}_z)_z, & \\ w_s\hat{c}^{04} + K^0_v\hat{c}^{04}_z = 0 & \text{at } z = 0, \\ -K^0_v\hat{c}^{04}_z = w_s\rho_s\frac{s(a_2-ib_2)}{g' d_s}a(x) & \text{at } z = -H(x), \end{cases} \tag{3.36}$$

with a_2 and b_2 defined in equation (3.34). The solution is

$$\hat{c}^{04} = a\left(A_1 e^{\frac{\lambda_1 - w_s}{2K^0_v}z} + A_2 e^{\frac{-\lambda_1 - w_s}{2K^0_v}z}\right), \tag{3.37}$$

with $\lambda_1 = \sqrt{w_s^2 + 8i\sigma K^0_v}$ and A_1 and A_2 are constants with respect to z which can be determined using the boundary condition, resulting in

$$A_2 = \frac{-2w_s\rho_s s(a_2 - ib_2)(w_s+\lambda_1)}{g' d_s\left((\lambda_1 - w_s)^2 e^{\frac{(w_s-\lambda_1)H}{2K^0_v}} - (\lambda_1 + w_s)^2 e^{\frac{(w_s+\lambda_1)H}{2K^0_v}}\right)}, \tag{3.38}$$

$$A_1 = \frac{-A_2(w_s - \lambda_1)}{w_s + \lambda_1}. \tag{3.39}$$

3.2 First order system of equations

3.2.1 First order water motion equations

The water motion equations at first order read

$$u_x^1 + w_z^1 - \frac{u^1}{L_b} = 0, \tag{3.40}$$

$$u_t^1 + u^{02}u_x^{02} + w^{02}u_z^{02} + g\zeta_x^1 - g\beta\langle s\rangle_x z - (A_v u_z^1)_z = 0. \tag{3.41}$$

For the boundary conditions at the surface $\tilde{z} = \varepsilon\tilde{\zeta}$ we write $\tilde{u}$ and $\tilde{w}$ as Taylor expansions around 0. The following result for $\tilde{w}$ is obtained:

$$\tilde{w}(x, \varepsilon\tilde{\zeta}) = \tilde{w}(x, 0) + \varepsilon\tilde{\zeta}\tilde{w}_z(x, 0) + \mathcal{O}(\varepsilon^2), \tag{3.42}$$

$$= w^0(x, 0) + \varepsilon w^1(x, 0) + \varepsilon\zeta^{02}w_z^{02}(x, 0) + \mathcal{O}(\varepsilon^2), \tag{3.43}$$

and the same applies for $\tilde{u}$.

Using these expressions the boundary conditions (2.9)-(2.14) become

$$w_1 + \zeta^{02}w_z^{02} = \zeta_t^1 + u^{02}\zeta_x^{02} \qquad \text{at } z = 0, \tag{3.44a}$$

$$A_v u_z^1 + A_v\zeta^{02}u_{zz}^{02} = 0 \qquad \text{at } z = 0, \tag{3.44b}$$

$$w^1 = -u^1 H_x \qquad \text{at } z = -H, \tag{3.44c}$$

$$A_v u_z^1 = s u^1 \qquad \text{at } z = -H, \tag{3.44d}$$

$$\zeta^1 = A_{M_4} cos(2\sigma t - \phi) \qquad \text{at } x = 0, \tag{3.44e}$$

$$\int_{-H}^{0} u^1 dz = \frac{Q}{B} \qquad \text{at } x = L. \tag{3.44f}$$

Using the leading order solution, it is found that first order solutions consist of a residual (i.e., a time independent) contribution and a M_4 contribution. For example the first order horizontal velocity can be written as $u^1 = u^{10} + u^{14} = u^{10}(x, z) + \Re\{\hat{u}^{14}(x, z)e^{i2\sigma t}\}$ and the same applies for w^1 and ζ^1. The first order system of equations is solved for u^{10} and u^{14} separately in section 3.2.1 and in section 3.2.1 respectively.

Residual contributions of the water motion

Using averaging over a tidal cycle, the following equations are obtained for the residual contributions $(u^{10}, w^{10}, \zeta^{10})$:

$$u_x^{10} + w_z^{10} - \frac{u^{10}}{L_b} = 0, \tag{3.45}$$

$$\langle u^{02}u_x^{02} + w^{02}u_z^{02}\rangle + g\zeta_x^{10} - g\beta\langle s\rangle_x z - (A_v u_z^{10})_z = 0, \tag{3.46}$$

where the brackets $\langle . \rangle$ denote tidal averaging. The boundary conditions become

$$w_{10} = -\langle\zeta^{02}w_z^{02} - u^{02}\zeta_x^{02}\rangle \qquad \text{at } z = 0, \tag{3.47a}$$

$$A_v u_z^{10} + \langle A_v\zeta^{02}u_{zz}^{02}\rangle = 0 \qquad \text{at } z = 0, \tag{3.47b}$$

$$w^{10} = -u^{10}H_x \qquad \text{at } z = -H, \tag{3.47c}$$

$$A_v u_z^{10} = s u^{10} \qquad \text{at } z = -H, \tag{3.47d}$$

$$\zeta^{10} = 0 \qquad \text{at } x = 0, \tag{3.47e}$$

$$\int_{-H}^{0} u^{10} dz = \frac{Q}{B} \qquad \text{at } x = L. \tag{3.47f}$$

These equations can be solved analogously to the leading order system.

M_4 contributions of the water motion

For the M_4 contributions we obtain the following equations.

$$u_x^{14} + w_z^{14} - \frac{u^{14}}{L_b} = 0, \tag{3.48}$$

$$u_t^{14} + [u^{02}u_x^{02} + w^{02}u_z^{02}] + g\zeta_x^{14} - (A_v u_z^{14})_z = 0. \tag{3.49}$$

where the braces [.] denote the M_4-contribution.

$$\begin{aligned}
w_{14} &= \zeta_t^{14} + [u^{02}\zeta_x^{02} - \zeta^{02}w_z^{02}] && \text{at } z = 0, && (3.50a)\\
A_v u_z^{14} + A_v[\zeta^{02}u_{zz}^{02}] &= 0 && \text{at } z = 0, && (3.50b)\\
w^{14} &= -u^{14}H_x && \text{at } z = -H, && (3.50c)\\
A_v u_z^{14} &= su^{14} && \text{at } z = -H, && (3.50d)\\
\zeta^{14} &= A_{M_4}\cos(2\sigma t - \phi) && \text{at } x = 0, && (3.50e)\\
\int_{-H}^{0} u^{14}dz &= 0 && \text{at } x = L. && (3.50f)
\end{aligned}$$

To solve this system a solution of the form

$$(u^{14}, w^{14}, \zeta^{14}) = \Re\left\{(\hat{u}^{14}, \hat{w}^{14}, \hat{\zeta}^{14})e^{i2\sigma t}\right\} \tag{3.51}$$

is allowed. In this system of differential equations there are four inhomogeneities, in equations (3.49), (3.50a), (3.50b), (3.50e). The problem has to be split into four systems of equations in which each of the systems includes one forcing term (inhomogeneity). The sum of these four solutions will give $(\hat{u}^{14}, \hat{w}^{14}, \hat{\zeta}^{14})$.

3.2.2 First order SSC equations

The system for the SSC on the first order equals

$$\begin{cases}
c_t^1 - w_s c_z^1 = K_v^0 c_{zz}^1 + \underbrace{\frac{\partial}{\partial z}\left(K_v^1\cos(\sigma t - \phi_{K_v^1})c_z^0\right)}_{I}, & \\
w_s c^1 + K_v^0 c_z^1 + \underbrace{\left(K_v^1\cos(\sigma t - \phi_{K_v^1})c_z^0\right)}_{II} = 0 & \text{at } z = 0,\\
-K_v^0 c_z^1 \underbrace{- \left(K_v^1\cos(\sigma t - \phi_{K_v^1})c_z^0\right)}_{III} = \underbrace{w_s\rho_s s u^1 \frac{u^{02}}{|u^{02}|g' d_s}a(x)}_{IV} & \text{at } z = -H.
\end{cases} \tag{3.52}$$

The solution of this problem is a sum of all the contributions on different frequencies:

$$c^1 = c^{10} + c^{12} + \cdots.$$

In section 3.3 it will be explained that only the M_2 contribution (c^{12}) needs to be solved. Since the problem contains four different forcings terms (I, II, III and IV), the solution of c^{12} can be found by deriving the solutions to the systems with only one of these forcing terms. The sum of these four solutions will then be the total solution Hence,

$$c^{12} = c^{I(12)} + c^{II(12)} + c^{III(12)} + c^{IV(12)} = \Re\left\{(\hat{c}^I + \hat{c}^{II} + \hat{c}^{III} + \hat{c}^{IV})(x, z)e^{i\sigma t}\right\}, \tag{3.53}$$

where e.g. $c^{I(12)}$ is the solution of the system with only forcing term I. In the next paragraphs we will derive these different

Solution of the first order SSC with constant mixing

The solution $c^{IV(12)}$ describes the SSC solution at first order when there is no variation in mixing, i.e. $K_v^1 = 0$. The set of equations needed to solve this component of the solution is:

$$\begin{cases}
i\sigma\hat{c}^{IV} - w_s\hat{c}_z^{IV} = (K_v\hat{c}_z^{IV})_z, & \\
w_s\hat{c}^{IV} + K_v\hat{c}_z^{IV} = 0 & \text{at } z = 0,\\
-K_v\hat{c}_z^{IV} = w_s\rho_s s\frac{p_1 - d_1}{g' d_s}a(x) & \text{at } z = -H.
\end{cases} \tag{3.54}$$

where p_1 and d_1 are the coefficients of the following Fourier series

$$\hat{u}^1(x,-H)\frac{\hat{u}^{02}(x,-H)}{|\hat{u}^{02}(x,-H)|} = p_0 + p_1\cos(\sigma t) + d_1\sin(\sigma t) + p_2\cos(2\sigma t) + \cdots .$$

The solution of this ODE is

$$\hat{c}^{IV} = \frac{w_s\rho_s sa(p_1 - id_1)}{g'd_s}\left(B_1 e^{r_1 z} + B_2 e^{r_2 z}\right), \tag{3.55}$$

where $r_1 = \frac{\lambda - w_s}{2K_v^0}$, $r_2 = -\frac{\lambda + w_s}{2K_v^0}$ and $\lambda = \sqrt{w_s^2 + 4K_v i\sigma}$, and B_1 and B_2 constants defined as

$$B_2 = \frac{-2(w_s+\lambda)}{(\lambda - w_s)^2 e^{-r_1 H} - (w_s+\lambda)^2 e^{-r_2 H}}, \tag{3.56}$$

$$B_1 = -\frac{w_s - \lambda}{w_s + \lambda}B_2. \tag{3.57}$$

Solution of the internal forcing due to the time dependent mixing term

To find the solution of $c^{I(12)}$ the following system needs to be solved

$$\begin{cases} i\sigma\hat{c}^I - w_s\hat{c}^I_z = K_v^0\hat{c}^I_{zz} + \left[\frac{\partial}{\partial z}\left(\hat{K}_v^1\cos(\sigma t - \phi_{K_v^1})c_z^0\right)\right], & \\ w_s\hat{c}^I + K_v^0\hat{c}^I_z = 0 & \text{at } z = 0, \\ -K_v^0\hat{c}^I_z = 0 & \text{at } z = -H. \end{cases} \tag{3.58}$$

where [.] denotes the M_2 contribution. Rewrite:

$$\frac{\partial}{\partial z}\left(\hat{K}_v^1\cos(\sigma t - \phi_{K_v^1})c_z^0\right) = \hat{K}_v^1\cos(\sigma t - \phi_{K_v^1})(c_{zz}^{00} + c_{zz}^{04}). \tag{3.59}$$

Since the linearity in the leading order SSC solution (c^0) it follows that the solution of $\hat{c}^I$ can be separated into two parts

$$\hat{c}^I = \hat{c}_0^{I(12)} + \hat{c}_4^{I(12)}, \tag{3.60}$$

where $\hat{c}_0^{I(12)}$ is the solution depending on the residual part of the sediment concentration at leading order and $\hat{c}_4^{I(12)}$ depending on the M_4 contribution of the sediment concentration at leading order.

The calculation of the M_2 contribution of the expression in equation (3.59) has been elaborated in Appendix C, resulting in

$$[K_v^1\cos(\sigma t - \phi_{K_v^1})c_{zz}^{00}] = K_v^1 c_{zz}^{00} e^{-i\phi_{K_v^1}}, \tag{3.61}$$

$$[K_v^1\cos(\sigma t - \phi_{K_v^1})c_{zz}^{04}] = K_v^1\frac{1}{2}\overline{c_{zz}^{04}}e^{-i\phi_{K_v^1}}. \tag{3.62}$$

The resulting solutions are

$$\hat{c}_0^I = \frac{K_v^1 e^{-i\phi_{K_v^1}}}{i\sigma}\left(C_1 e^{r_1 z} + C_2 e^{r_2 z} + c_{zz}^{00}\right), \tag{3.63}$$

$$\tag{3.64}$$

where

$$C_1 = \frac{\lambda - w_s}{\lambda + w_s}C_2 - \frac{2\left(\mathbf{w_s}c_{zz}^{00}(x,0) + \mathbf{K_v^0}c_{zzz}^{00}(x,0)\right)}{(\lambda + w_s)}, \tag{3.65}$$

$$= \frac{\lambda - w_s}{\lambda + w_s}C_2, \tag{3.66}$$

$$C_2 = \frac{2(\lambda - w_s)\left(\mathbf{w_s}c_{zz}^{00}(x,0) + \mathbf{K_v^0}c_{zzz}^{00}(x,0)\right)e^{-r_1 H} - 2(\lambda + w_s)K_v^0 c_{zzz}^{00}(x,-H)}{(\lambda - w_s)^2 e^{-r_1 H} - (\lambda + w_s)^2 e^{-r_2 H}}, \tag{3.67}$$

$$= \frac{-2(\lambda + w_s)K_v^0 c_{zzz}^{00}(x,-H)}{(\lambda - w_s)^2 e^{-r_1 H} - (\lambda + w_s)^2 e^{-r_2 H}}, \tag{3.68}$$

and

$$\hat{c}_4^I = D_1 e^{r_1 z} + D_2 e^{r_2 z} + P_1 e^{\overline{R_1} z} + P_2 e^{\overline{R_2} z}, \tag{3.69}$$

where P_1 and P_2 are defined

$$P_i = \frac{\frac{1}{2} K_v^1 e^{-i\phi_{K_v^1}} a\overline{A_i}}{i\sigma - w_s \overline{R_i} - K_v^0 \overline{R_i}^2}, \text{ with } i \in \{1, 2\}, \tag{3.70}$$

with A_1 and A_2 defined in equations (3.38) and (3.39), $R_1 = \frac{\lambda_1 - w_s}{K_v^0}$, $R_2 = -\frac{\lambda_1 + w_s}{K_v^0}$ and $\lambda_1 = \sqrt{w_s^2 + 8iK_v^0}$. The integration constants D_1 and D_2 are determined by substituting the solution into the boundary conditions, resulting in

$$D_1 = \frac{\lambda - w_s}{\lambda + w_s} D_2 - \frac{2w_s \left(P_1 + P_2\right) + 2K_v^0 \left(P_1 \overline{R_1} + P_2 \overline{R_2}\right)}{\lambda + w_s}, \tag{3.71}$$

$$D_2 = \frac{(\lambda - w_s)\left(2w_s\left(P_1 + P_2\right) + 2K_v^0\left(P_1\overline{R_1} + P_2\overline{R_2}\right)\right)e^{-r_1 H} - 2K_v^0(\lambda + w_s)\left(P_1\overline{R_1}e^{-\overline{R_1}H} + P_2\overline{R_2}e^{-\overline{R_2}H}\right)}{((\lambda - w_s)^2 e^{-r_1 H} - (\lambda + w_s)^2 e^{-r_2 H})}. \tag{3.72}$$

Solution of the surface forcing due to the time dependent mixing term

For the solution of $\hat{c}^{II(12)}$ the following set of equations needs to be solved:

$$\begin{cases} i\sigma\hat{c}^{II} - w_s v_z = K_v^0 \hat{c}_{zz}^{II}, & \\ w_s \hat{c}^{II} + K_v^0 \hat{c}_z^{II} + \left[\hat{K}_v^1 \cos(\sigma t - \phi_{K_v^1}) c_z^0\right] = 0 & \text{at } z = 0, \\ -K_v^0 \hat{c}_z^{II} = 0 & \text{at } z = -H. \end{cases} \tag{3.73}$$

Using the previous calculations to obtain the M_2 contributions, the following results

$$\left[\hat{K}_v^1 \cos(\sigma t - \phi_{K_v^1}) c_z^0\right] = \left[\hat{K}_v^1 \cos(\sigma t - \phi_{K_v^1})(c_z^{00} + c_z^{04})\right] \tag{3.74}$$

$$= K_v^1 c_z^{00} e^{-i\phi_{K_v^1}} + K_v^1 \frac{1}{2} \overline{c_z^{04}} e^{-i\phi_{K_v^1}} \tag{3.75}$$

The solution of this problem is

$$\hat{c}^{II(12)} = \hat{c}_0^{II} + \hat{c}_4^{II}, \tag{3.76}$$

where

$$\hat{c}_0^{II} = K_v^1 c_z^{00}(x, 0) e^{-i\phi_{K_v^1}} \left(E_1 e^{r_1 z} + E_2 e^{r_2 z}\right), \tag{3.77}$$

$$\hat{c}_4^{II} = \frac{1}{2} K_v^1 \overline{c_z^{04}}(x, 0) e^{-i\phi_{K_v^1}} \left(E_1 e^{r_1 z} + E_2 e^{r_2 z}\right), \tag{3.78}$$

with

$$E_1 = \frac{\lambda - w_s}{\lambda + w_s} E_2 - \frac{2}{\lambda + w_s}, \tag{3.79}$$

$$E_2 = \frac{2(\lambda - w_s) e^{-r_1 H}}{(\lambda - w_s)^2 e^{-r_1 H} - (\lambda + w_s)^2 e^{-r_2 H}}. \tag{3.80}$$

Solution of the bottom forcing due to the time dependent mixing term

The differential equation to solve $\hat{c}^{III}$ is

$$\begin{cases} i\sigma\hat{c}^{III} - w_s \hat{c}_z^{III} = K_v^0 \hat{c}_{zz}^{III}, & \\ w_s \hat{c}^{III} + K_v^0 \hat{c}_z^{III} = 0 & \text{at } z = 0, \\ -K_v^0 \hat{c}_z^{III} - \left[\hat{K}_v^1 \cos(\sigma t - \phi_{K_v^1}) c_z^0\right] = 0 & \text{at } z = -H. \end{cases} \tag{3.81}$$

Notice that this system of equations is analogous to the equations (3.54). Hence the obtained solutions are analogous:

$$\hat{c}_0^{III} = K_v^1 c_z^{00}(x,-H) e^{-i\phi_{K_v^1}} \left(B_1 e^{r_1 z} + B_2 e^{r_2 z}\right), \tag{3.82}$$

$$\hat{c}_4^{III} = \frac{1}{2} K_v^1 \overline{c_z^{04}}(x,-H) e^{-i\phi_{K_v^1}} \left(B_1 e^{r_1 z} + B_2 e^{r_2 z}\right). \tag{3.83}$$

When having a closer look to the solution for equation (3.54), we see the slip parameter s is assumed to be constant at the bottom. But since we take the mixing term time dependent at the bottom, this slip parameter should be taken time dependent as well. We will assume that the time dependent part of IV is equal to III and will therefore balance out in equation (3.52). This implies solution $\hat{c}^{III}$ is not needed.

3.3 Transport

Our main interest is to obtain the suspended sediment transports (uc). For the morphodynamic equilibrium, equation (2.21), only residual transports ($\langle uc \rangle$) contribute. When multiplying a M_i contribution of the velocity field by an M_j contribution of the sediment concentration the frequencies of the obtained transport are M_{i+j} and $M_{|i-j|}$. This means only multiplications between sediment concentrations and velocities of the same frequency will result a residual transport (M_0).

For the horizontal velocity in leading order only a M_2 contribution exists and in first order there are only a residual and a M_4 contribution.For the SSC the contributions for the leading order are M_0, M_4, M_8 and all other contributions which are a multiplication of four. For the first order SSC the contributions are M_0, M_2, M_6 etc.. General they are M_{4n+2} with $n \in \mathbb{N}_0$. An overview can be viewed in table 3.3.

	$\mathcal{O}(1)$	$\mathcal{O}(\varepsilon)$
u	M_2	M_0, M_4
c	M_0, M_4, M_8,...	M_0,M_2,M_6,...

Table 3.3: Frequencies of the horizontal velocity u and sediment concentration c on different orders of significance.

We see for the velocity field on order $\mathcal{O}(1)$ component only a multiplication with a sediment concentration at first order will result in a residual transport, being of $\mathcal{O}(\varepsilon)$. The velocity field on first order has a M_0 and an M_4 contribution which has a transport contribution of order ε when combined with the M_0 and M_4 contribution of the sediment concentration of $\mathcal{O}(1)$. Since we are only interested in the most important residual transports, the only frequencies of the sediment concentration that need to be evaluated are M_0 and M_4 at leading order and M_2 at first order. As a result from the morphodynamic equilibrium, equation (3.12) and the fact that the transports with a resulting residual transports are at least of order ε. The morphodynamic equilibrium is $\mathcal{O}\varepsilon^2$ reads

$$\int_{-H}^{0} (u^{10}c^{00} + \langle u^{02}c^{12} \rangle + \langle u^{14}c^{04} \rangle - K_h \langle c_x^{00} \rangle) dz + \langle \zeta^0 [u^{02}c^0]_{z=0} \rangle = 0. \tag{3.84}$$

Note that the last contribution is a result of a Taylor expansion around $z = 0$ at the upper boundary. The solutions of c are known, except for the unknown erosion coefficient $a(x)$, i.e. $c^{00} = a(x)c^{00a}$, $c^{04} = a(x)c^{04a}$, $c^{12} = a(x)c^{12a}$. Substituting these expressions in equation (3.84), leads to the differential equation

$$F\frac{da}{dx} + Ta = 0, \tag{3.85}$$

where

$$F = \left\langle \int_{-H}^{0} -K_h c^{00a} dz \right\rangle, \tag{3.86}$$

$$T = \int_{-H}^{0} \left(u^{10}c^{00a} + \langle u^{02}c^{12a} \rangle + \langle u^{14}c^{04a} \rangle - K_h \langle c_x^{00a} \rangle\right) dz + \langle \zeta^0 [u^{02}c^{0a}]_{z=0} \rangle. \tag{3.87}$$

This linear first-order differential equation can be solved using the method of separation of variables. The solution reads

$$a(x) = a_0 \exp\left(\int -T/F dx \right) \equiv a_0 I(x), \tag{3.88}$$

where the integration constant is denoted by a_0, which can be determined from the integral condition (equation (2.26)), resulting in

$$a_0 = \frac{a_* \int_0^L B(x)dx}{\int_0^L B(x)I(x)dx}. \tag{3.89}$$

Chapter 4

Results

4.1 Reference case

In this section the model obtained in section 3 is analysed to gain insight in the effect of adding the time dependent vertical mixing term. As a case study we use the Ems estuary. The exact values of the parameters can be found in table 4.1. The solutions of $c^{I(12)}$ and $c^{II(12)}$, as defined in equation 3.53 had to be added to the model.

Parameter	*Symbol*	*Value*
Geometry		
Length of estuary	L	64 km
Width at the seaside	B_0	670 m
Exponential convergence length	L_b	30 km
Depth at the seaside	H_0	10.85 m
Hydrodynamics		
M_2 tidal amplitude at entrance	A_{M_2}	1.39
M_4 tidal amplitude at entrance	A_{M_4}	0.22
Relative phase at entrance between M_2 and $M4$ tidal	ϕ	-173°
Inflow of water from the river	Q	65 m^3/s
Turbulence		
Eddy viscosity at the seaside	A_v^0	0.019 m^2/s
Stress parameter	s_0	0.05 m/s
Salinity		
Along-estuary residual salinity gradient	$\langle s \rangle_x$	$0.5 \cdot 10^{-3}$ psu/m
Sediment		
Average amount of sediment available for resuspension	a_*	$1.0 \cdot 10^{-5}$
Settling velocity	w_s	$0.5{\cdot}10^{-3}$ m/s
Horizontal diffusivity	K_h	100 m^2/s
Average vertical diffusivity	K_v^0	0.019 m^2/s

Table 4.1: Values of the parameters regarding the Ems estuary.

Two different cases will be compared, one with a constant mixing term ($K_v := K_v^0$), describing the model by Chernetsky et. al. [1], and one with a time dependent mixing term ($K_v := K_v^0 + K_v^1 \cos(\sigma t - \phi_{u^{02}} - \Delta\phi)$). Because the phase of the vertical mixing is assumed to be a slightly lagging the phase of the horizontal velocity, the relative phase $\Delta\phi$ is chosen to be 20°. The salinity gradient is shown in figure 4.1. Since we assumed the amplitude K_v^1 to be of one order lower than K_v^0. This means $K_v^1 = C \frac{\mathrm{d}s}{\mathrm{d}x} \approx \varepsilon K_v^0 = 0.0028$, where C is the proportionality constant, resulting in an estimate $C = 2$ m^4/s/psu.

The absolute values of the residual contribution of the SSC at leading order (c^{00}) for both cases are shown in figure 4.2. The ETM has not significantly shifted as a result of the time dependent mixing term.

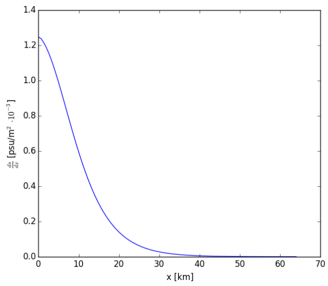
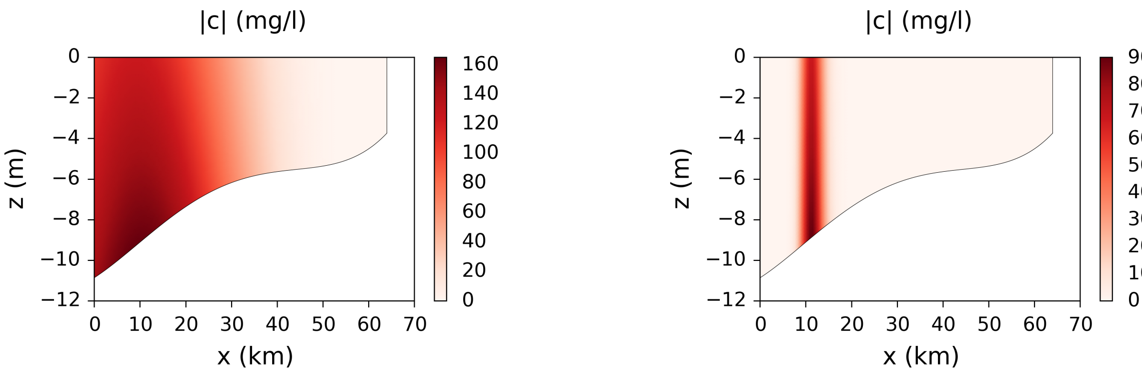


Figure 4.1: Salinity gradient

The change we see is that the column of SS has narrowed. In section 2.2 we expected the ETM to move upstream caused by an increase of upstream transport. Why are the results different?

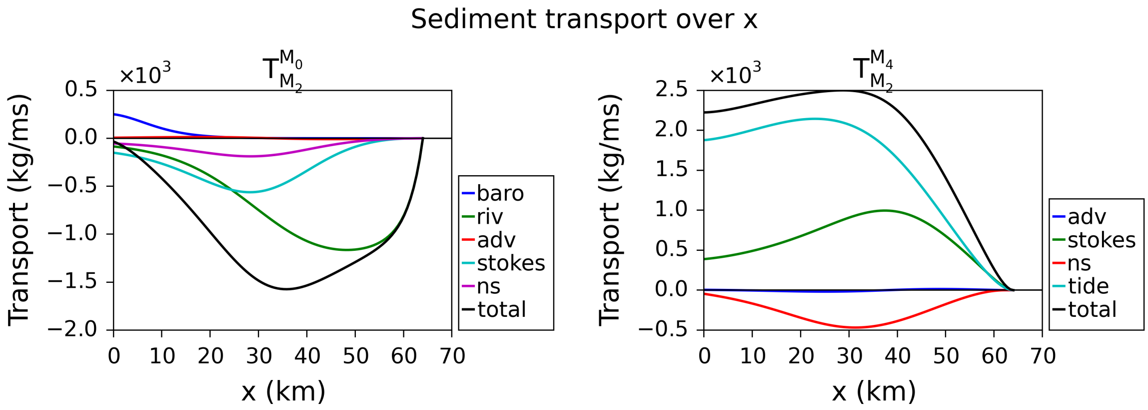


(a) (b)

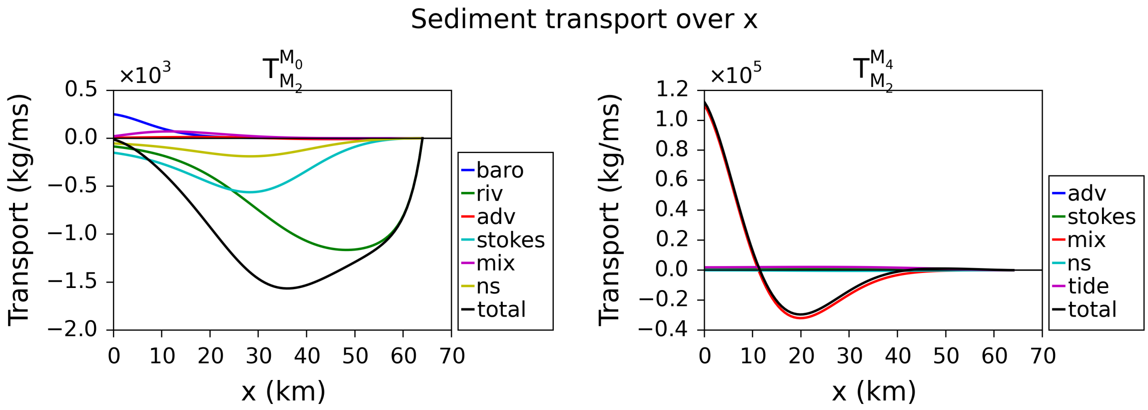
Figure 4.2: The width-averaged cross-section of the Ems. The soil profile is represented with the white plane. The absolute value of c^{00} without (a) and with (b) mixing is shown in the graphs. Remark that the scale for the SSC is different in both figures.

Perhaps an insight in the present transports can clarify the narrowing. The time dependent mixing term only influences the first order sediment solution. This means the transport term T defined in differential equation (3.85) is only effected on the $M2$ contribution $(u^{02}c^{12})$. Because we have split the solutions of first order SSC in two parts, one depending on the $M4$ and one on the M_2 contribution of the SSC on leading order, what was done in equation (3.60), the results can be considered separately as well. They are shown in figure 4.3. All the transports in figure 4.3a and 4.3b remain the same except for the mixing term which is added in the latter. It can be seen in the figure that the mixing term due to the M_0 SSC on leading order is almost equal to zero. The transport due to the M_4 contribution is more present. At the seaward side of the estuary the transport is positive and upstream it is negative. Where the transport changes sign (around $x = 11$ km) the ETM was found. Because al the transports caused by vertical mixing point towards the ETM the SS column in figure 4.2 is squeezed. Increasing the amplitude K_v^1 would increase the effect of time dependent vertical mixing and will therefore enhance te narrowing of the column. We see the amplitude of the transport is greater at the seaward side of the estuary. This is a result of the salinity gradient profile, to which the time dependent vertical mixing is linear proportional. The gradient is higher close to the seaward side of the estuary than upstream.

The real part of the solutions $\hat{c}_0^{I(12)}$ (Eq. (3.63)), $\hat{c}_4^{I(12)}$ (Eq. (3.69)), $\hat{c}_0^{II(12)}$ (Eq. (3.77)) and $\hat{c}_4^{II(12)}$ (Eq. (3.78)) are plotted in figure 4.4. Again we see that the solutions have the most influence at the entrance. We see that the solution of the equations with the internal forcing due to c^{04}, that is $\hat{c}_4^I$, has the biggest influence on the system.

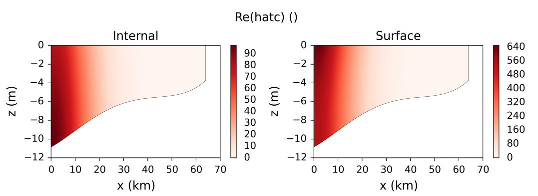


(a)

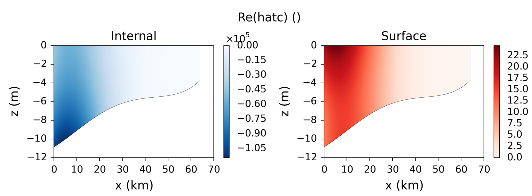


(b)

Figure 4.3: Height-averaged transports of M_2 frequencies ($\langle u^{02}c^{12}\rangle$) without (a) and with (b) mixing. The contribution caused by vertical mixing is indicated with 'mix', which is pink and red in respectively the left and right graph in figure (b). The total transports are marked by a black line. We are not interested in all the other occurring transports. T_{M_2} indicates that the transports are due to M_2 contribution of both the velocity and sediment concentration. The figures with $T_{M_2}^{M_0}$ and $T_{M_2}^{M_4}$ represent the transports due to the M_0 and M_4 contribution of the leading order sediment concentration in the processes respectively. A positive transport represents an upstream transport, a negative represents a downstream transport. Remark that the scales differ, the transport $T_{M_2}^{M_4}$ is much more extreme than the other transports.



(a)



(b)

Figure 4.4: The real part of the solutions $\hat{c}_0^{I(12)}$ (Eq. (3.63)), $\hat{c}_4^{I(12)}$ (Eq. (3.69)), $\hat{c}_0^{II(12)}$ (Eq. (3.77)) and $\hat{c}_4^{II(12)}$ (Eq. (3.78)). The solutions dependent on the M_2 (M_4) contribution of SSC on leading order are show in figure a (b). The solutions to system I (II) are represented in the plots with 'Internal' ('Surface').

4.2 Sensitivity of the parameter $\phi_{K_v^1}$

In this section the consequences of varying the phase of the vertical mixing will be analyzed. We will vary the value for $\Delta\phi$ from $0°$ to $360°$. The proportionality constant has been lowered to $C = 10^{-2}$ m^4/s/psu, such that the results are easier to observe. Varying $\Delta\phi$ has consequences on the location of the ETM as can be seen in figure 4.5.

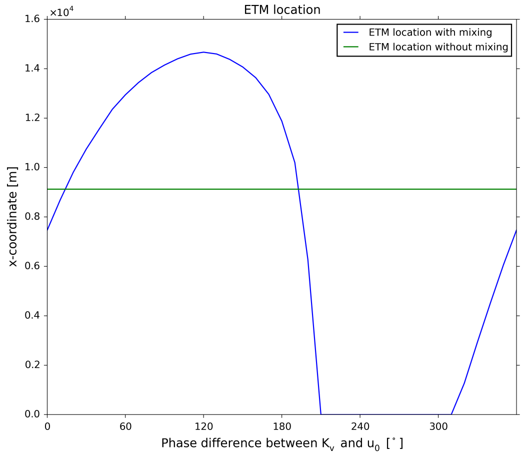


Figure 4.5: Relation between the variation of $\Delta\phi$ and the location of the ETM. On the x-axis lies $\Delta\phi$ and on the y-axis lies the x-coordinate of the ETM. The ETM for the model without time dependent vertical mixing is plotted with the green line. The blue line represents the location of the ETM when varying $\Delta\phi$. Here, $C = 10^{-2}$ m^4/s/psu.

The ETM shifts down stream when there is no phase difference between the vertical mixing and the horizontal tidal velocity, compared to the model without time dependent vertical mixing. This is shown in the figure, because the blue line is lower than the green line for $\Delta\phi = 0$. When the phase difference $\Delta\phi$ is approximately 120 degrees, the ETM is maximally shifted upstream. For a phase difference of approximately 270 degrees the ETM is shifted into the sea and there is no sediment trapping in the estuary. We expected the ETM to move upstream when the phase of the horizontal velocity (u^{02}) and the mixing term were equal. However, in the figure it is observed that the ETM shifts slightly downstream for a phase difference of $0°$. For the most realistic phase difference ($\Delta\phi = 20°$) the ETM will not shift at all compared to the model without vertical mixing that varies on the tidal timescale. An upstream shift occurs when the phase of the mixing term is lagging the horizontal velocity with more than $20°$. In the solutions of the SSC on first order with a forcing term of vertical mixing in the internal partial differential equation, equations (3.63) and (3.69), we see a deviation by i, since $\frac{1}{i} = e^{-\frac{\pi}{2}i}$ this can be interpreted as a phase shift of $90°$. Resulting in a phase shift of the SSC, which means the SSC is lagging behind the horizontal velocity. This is why the resulting upstream transport happens when the mixing is lagging as well.

In figure 4.6 the horizontal velocity, the SSC and the corresponding transports over one tide are shown at the entrance of the estuary. If $\Delta\phi$ equals $90°$, the ETM is more upstream inside the estuary, so that explains why the concentration of SS is lower for this value of $\Delta\phi$ than for the other ones plotted. If $\Delta\phi = 270°$ the ETM has shifted out of the estuary which results in the higher amount of SSC at the seaside of the estuary. Because the SSC are results of many complex effects, it is hard to understand why they behave exactly as plotted in the second subplot of figure 4.6, which is left for further research. The transports shown in figure (4.6) are a simple multiplication of the horizontal velocity in the first subplot and the sediment concentrations in the second subplot. The average transports in the third subplot or table 4.2 agree with the results in figure 4.5, since the transport for $\Delta\phi = 0°$ is slightly negative the transport for $\Delta\phi = 90$ is positive and the transport for $\Delta\phi = 270$ is the highest amount of negative transport, which relates with the absence of trapping belonging to this phase difference. For $\Delta\phi = 180°$ the transport caused by vertical mixing is negative at the entrance of the estuary, which can be seen in figure 4.7. However, upstream the estuary vertical mixing results in an upstream transport, which still induces an upstream shift of the ETM as was seen in figure 4.5.

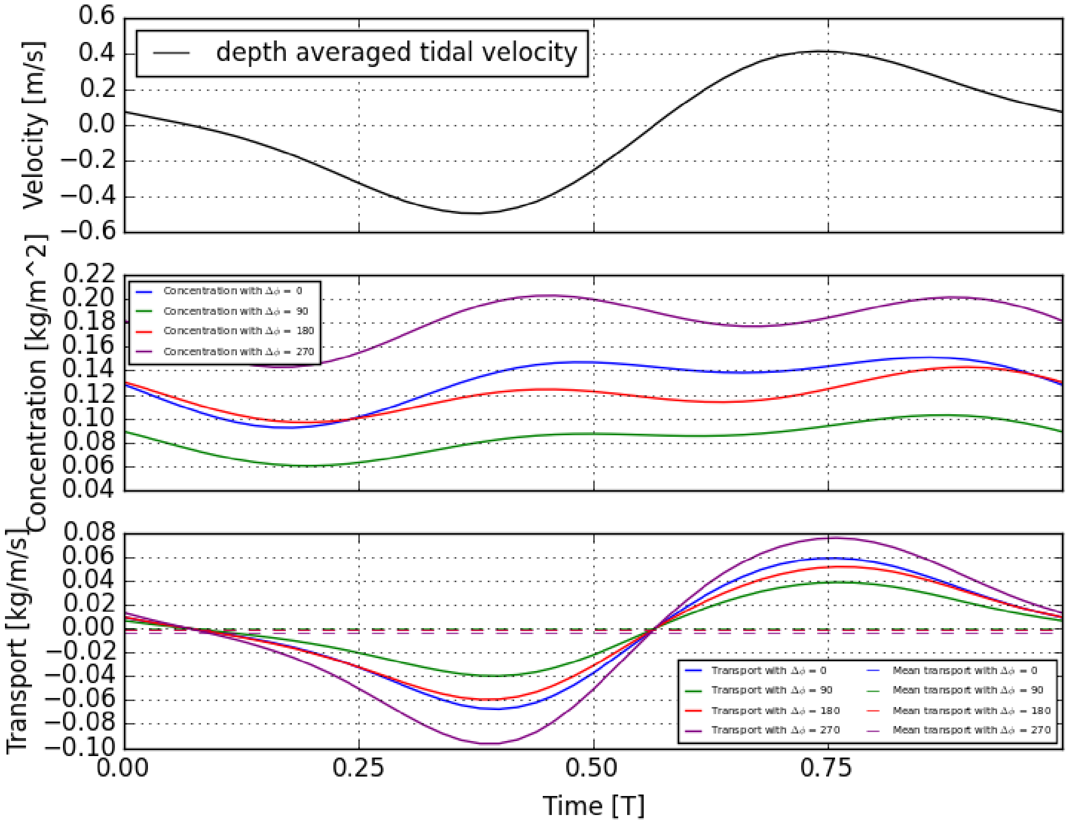


Figure 4.6: From the top down: the depth averaged tidal velocity, the depth averaged SSC for different values for the phase difference $\Delta\phi$ and the corresponding transports over one tide T at the seaside of the estuary ($x = 0$ km). In the figure $C = 0.01$ m^4/s/psu. The average transports can also be found in table 4.2.

Phase difference $\Delta\phi$ [°]	Average transport [10^{-4} kg/m/s]
0	-7.0451
90	4.3141
180	-9.536
270	-33.5282

Table 4.2: Average transports of SS at the estuary seaside for different values of $\Delta\phi$.

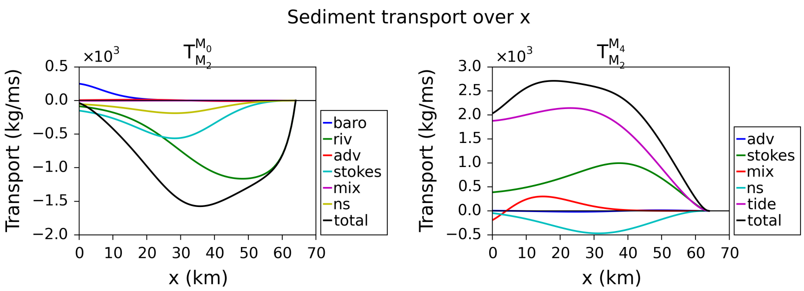


Figure 4.7: The height-averaged transports of M_2 frequencies with $\Delta\phi = 180°$ and $C = 10^{-2}$.

Chapter 5

Discussion & conclusion

A model for the hydrodynamics and sediment transport in the estuary of a river consist of a complex set of partial differential equations. These are solved by an idealized model where the equations are firstly scaled, before solving them. In this thesis the model as described by Chernetsky et al. [1] has been improved. In the model the vertical mixing had been assumed to be constant over time.

However, as a result from the salinity gradient in the estuary, an increase of vertical mixing is expected during flood and a decrease during ebb, which would result in more SS in higher layers of the water column, and therefore more transport during flood, resulting in a tidally averaged upstream transport. In this thesis the vertical mixing has been redefined as a function that varies on the tidal timescale. Solving the idealized model results in extra terms to be added to the original SSC solution.

In many estuaries, like the Ems estuary, trapping of suspended sediment is found. The original model was already able to declare an appearance of an ETM. However, when adding the extra solutions, some changes occur to the ETM. The results show that to the vertical mixing induces a squeezing of the column of higher concentrations SS around the ETM. We assume the vertical mixing is lagging the horizontal velocity slightly, which means that the mixing needs time to start up. The maximal mixing happens later than the maximal flood and the minimal mixing happens later than the maximal ebb. This means the phase of the vertical mixing is different from the phase of the horizontal velocities. Their phase difference $\Delta\phi$ is assumed to be 20° on the tidal scale, which equals approximately 40 minutes. The ETM will not shift to another location with this phase difference. If we vary the phase difference $\Delta\phi$ the ETM will shift. If $\Delta\phi$ is more than 20° but less then 200° it will shift upstream. For the other phase differences the ETM shifts downstream and for phases around 270°, it will even leave the estuary, which means there is no trapping of sediment in the estuary.

The results indicate that vertical mixing results in more stabilization of the ETM and even an upstream shift if the mixing is lagging enough behind the horizontal tidal velocity. Determination of the exact value for $\Delta\phi$ is left for further research. To calculate effects of anthropogenic changes to remove the ETM, the time dependent vertical mixing should be taken into account, since its effects are clearly present. The addition of time dependent vertical mixing gives the model more accuracy.

In the model some assumptions have been made which could be improved in further research; in section 3.2.2 the solution resulting from to the bottom forcing has been neglected. For a better model the slip parameter s should be considered time dependent as well. In that case the solution to the bottom forcing could also be taken into account, which would result in a more accurate model. Also external factors like wind, rain and temperatures are neglected. Adding these factors would make the model more accurate.

We can conclude that vertical mixing has non-negligible influences on the tidally average transport of suspended sediment. It is recommended to take this process into account when using the model and in further research.

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Appendices

A Derivation of the water motion equations

The width-averaged continuity equation (2.4) can be derived from the three dimensional mass conservation equation, which reads [2, p. 704]

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0, \tag{1}$$

with $\rho(x, y, z, t)$ the density of the water at time t and place (x, y, z). This equation describes changes in density due to convergences of density. In most geophysical systems, the fluid density varies only slightly from place to place and over time:

$$\rho(x, y, z, t) = \rho_0 + \rho'(x, y, z, t), \tag{2}$$

with ρ_0 the average density and ρ' the density variations with $|\rho'| << \rho_0$.

Now, equation (1) becomes:

$$\rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \rho' \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \left(\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} + w\frac{\partial \rho'}{\partial z} \right) = 0. \tag{3}$$

From geophysical literature [2, p.77] it follows that the third term is always of equal order or much smaller than the second, because the total difference of density is much less or equal than the relative variations in the velocity field. Using the assumption $|\rho'| << \rho_0$, it follows that the second term is much smaller than te first. So neglecting smaller terms, the mass conservation equation becomes a volume conservation equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{4}$$

Because both shores are assumed impermeable, i.e. no water fluxes through the shores, the following boundary conditions have to be imposed,

$$\begin{cases} v = -\frac{1}{2}u\frac{\partial B}{\partial x} & \text{at } y = \frac{1}{2}B, \\ v = \frac{1}{2}u\frac{\partial B}{\partial x} & \text{at } y = -\frac{1}{2}B. \end{cases} \tag{5}$$

The velocities can be decomposed as

$$u = \bar{u}(x, z) + u'(x, z, y), \tag{6}$$

$$w = \bar{w}(x, z) + w'(x, z, y), \tag{7}$$

where $\bar{u}$ and $\bar{w}$ are the width averaged velocities and u' and w' the velocity variations in the lateral direction. This means that $\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} u'dy = 0$ and $\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} w'dy = 0$. When integrating equation (4) in the lateral-direction, using the Leibniz integral rule to interchange integration and differentiation and the above boundary conditions it follows that

$$\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dy = 0,$$

$$\Rightarrow B\frac{\partial \bar{u}}{\partial x}(-u\frac{1}{2}B_x|_{y=\frac{1}{2}B} - u\frac{1}{2}B_x|_{y=-\frac{1}{2}B} + [v]_{-\frac{1}{2}B}^{\frac{1}{2}B}) + B\frac{\partial \bar{w}}{\partial z} = 0,$$

$$\Rightarrow \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} + \frac{\bar{u}}{B}B_x = 0,$$

$$\Rightarrow \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} - \frac{\bar{u}}{L_b} = 0. \tag{8}$$

where in the last derivation equation (2.1) has been used. In the following, the bars will be left out for simplicity, resulting in equation (2.4).

A similar derivation can be made to obtain the width averaged momentum equation (2.5) from the three dimensional shallow water equations [2]. Using the Navier-Stokes equations, neglecting the horizontal viscosity, assuming a hydrostatic balance and using the Boussinesq approximation, the momentum equation is found after width averaging.

B Derivation of the SSC equation

The SSC follows from the conservation law stating that the mass of the sediment is conserved. This equation is described by[1]

$$\frac{\partial c}{\partial t} + \vec{\nabla} \cdot \vec{F} = 0, \tag{9}$$

with $c(x, y, z, t)$ the sediment concentration at (x, y, z) and time t, and $\vec{F}$ is the local sediment flux. This flux is the sum of an advective flux $\vec{F}_a$, a settling flux $\vec{F}_s$ and a diffusive flux $\vec{F}_d$. The advective flux is the flux due to the advection of the sediment by the water flow. The settling flux accounts for the settling of sediment due to gravity, with the settling velocity given by w_s. The settling velocity depends on the SSC, on grain characteristics and settling conditions, e.g. size and shape of the settling volume, mixing etc.. The diffusive flux is proportional to the suspend sediment concentration gradient. The fluxes are given by

$$\vec{F}_a = c\vec{u} + cw\vec{e}_z, \tag{10a}$$

$$\vec{F}_s = -cw_s\vec{e}_z, \tag{10b}$$

$$\vec{F}_d = -K_h\vec{\nabla}c - K_v\frac{\partial c}{\partial z}\vec{e}_z, \tag{10c}$$

where $\vec{u} = (u, v)$ describes the horizontal velocities and $\vec{e}_z$ is the unit vector in the z-direction. The turbulent vertical eddy diffusion coefficient K_v is allowed to be time dependent, see section 2.2 for a depth discussion. The horizontal eddy diffusion coefficient is denoted by K_h.

Since there is no sediment flux through the sides of the estuary, it follows that, using (10),

$$\vec{n} \cdot (c\vec{u} + K_h\vec{\nabla}c) = 0 \text{ at } y = \pm\frac{1}{2}B. \tag{11}$$

with $\vec{n}$ the normal outward pointing vector on the boundaries. For the boundary $y = \frac{1}{2}B$, $\vec{n} = (-\frac{1}{2}B_x, 1)$ and for $y = -\frac{1}{2}B$, $\vec{n} = -(\frac{1}{2}B_x, 1)$.

Substituting (10) into equation (9), one finds

$$c_t + (uc)_x + (vc)_y + (c(w - w_s))_z = (K_hc_x)_x + (K_hc_y)_y + (K_vc_z)_z. \tag{12}$$

To get a width-averaged sediment model the above equation is integrated over the width:

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} c dy + \int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} (uc)_x dy + [vc]_{y=-\frac{1}{2}B}^{y=\frac{1}{2}B} + \frac{\partial}{\partial z}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} (c(w-w_s))dy \\ &\qquad - \int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} (K_h c_x)_x dy - [K_h c_y]_{y=-\frac{1}{2}B}^{y=\frac{1}{2}B} - \frac{\partial}{\partial z}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} K_v c_z dy = 0, \\ \Rightarrow &\frac{\partial}{\partial t}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} c dy + \frac{\partial}{\partial x}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} (uc) dy + [vc - K_h c_y - uc\frac{1}{2}B_x + K_h c_x \frac{1}{2}B_x]_{y=-\frac{1}{2}B}^{y=\frac{1}{2}B} \\ &+\frac{\partial}{\partial z}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} (c(w-w_s))dy - \frac{\partial}{\partial x}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} (K_h c_x) dy - \frac{\partial}{\partial z}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} K_v c_z dy = 0. \qquad (13)\end{aligned}$$

In the second step the Leibniz rule has been used. Using the boundary conditions equation (13) can be reduced to

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} c dy + \frac{\partial}{\partial x}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} (uc) dy + \frac{\partial}{\partial z}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} (c(w-w_s))dy \\ &\qquad -\frac{\partial}{\partial x}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} (K_h c_x) dy - \frac{\partial}{\partial z}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} K_v c_z dy = 0. \qquad (14)\end{aligned}$$

Similar to the decomposition of u and w in eqn. (6), the concentration c is decomposed as $c = \bar{c} + c'$, with $\bar{c}$ is the width averaged sediment concentration,

$$\bar{c} = \frac{1}{B(x)}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} c dy. \qquad (15)$$

and c' is the deviation of c from the mean $\bar{c}$ in the y direction. Now, equation (14) can be rewritten into

$$B\hat{c}_t + (B\bar{u}\bar{c})_x + B(\bar{c}(\bar{w}-w_s))_z - (BK_h\hat{c}_x)_x - B(K_v\hat{c}_z)_z = 0. \qquad (16)$$

where we used the calculations:

$$\begin{aligned}\frac{\partial}{\partial x}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} uc dy &= \frac{\partial}{\partial x}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} \bar{u}\bar{c} dy + \frac{\partial}{\partial x}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} \bar{u}c' dy + \frac{\partial}{\partial x}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} \bar{c}u' dy + \frac{\partial}{\partial x}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} u'c' dy \quad (17)\\ &= (B\bar{u}\bar{c})_x + \frac{\partial}{\partial x}\left(\bar{u}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} c' dy\right) + \frac{\partial}{\partial x}\left(\bar{c}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} u' dy\right) \quad (18)\\ &= (B\bar{u}\bar{c})_x. \quad (19)\end{aligned}$$

Here it is used that $\bar{u}$ and $\bar{c}$ are independent from y and it is assumed that the width-correlations, e.g. $\frac{\partial}{\partial x}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} u'c' dy$, behave as an effective diffusive flux in the x-direction, which cancels out. And

$$\begin{aligned}\frac{\partial}{\partial x}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} K_v\bar{c}_x dy &= \frac{\partial}{\partial x}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} K_h\bar{c}_x dy + \frac{\partial}{\partial x}\int_{-\frac{1}{2}B(x)}^{\frac{1}{2}B(x)} K_h c'_x dy \quad (20)\\ &= (BK_h\bar{c}_x)_x \quad (21)\end{aligned}$$

where the second term is neglected.

Using $B(x) = B_0 e^{-x/L_b}$, the width-averaged sediment concentration equation (16) can be rewritten as

$$c_t + uc_x + wc_z = w_s c_z + (K_h c_x)_x + (K_v c_z)_z - \frac{1}{L_b}K_h c_x, \qquad (22)$$

suppressing the bars for simplicity, resulting in equation (2.15). In this derivation the continuity equation (8) has been used.

C Calculation of the M_2-contribution

The M_2 contribution of equation (3.59) due to c^{00} is

$$[K_v^1 \cos(\sigma t - \phi_{K_v^1}) c_{zz}^{00}] = \left[K_v^1 c_{zz}^{00} \left(\cos(\sigma t)\cos(\phi_{K_v^1}) + \sin(\sigma t)\sin(\phi_{K_v^1})\right)\right] \quad (23)$$

$$= K_v^1 c_{zz}^{00} (\cos(\phi_{K_v^1}) - i \sin(\phi_{K_v^1})) \quad (24)$$

$$= K_v^1 c_{zz}^{00} e^{-i\phi_{K_v^1}} \quad (25)$$

For the M_2 contribution of equation (3.59) due to c^{04} a different calculation needs to be done. Write

$$\begin{aligned}
\Re\{\hat{c}^{04} e^{2i\sigma t}\} \cdot \cos(\sigma t - \phi) =& \Re\{\hat{c}^{04}(\cos(2\sigma t) + i \sin(2\sigma t))\} \cdot \cos(\sigma t - \phi) \\
=& \left(\Re\{\hat{c}^{04}\}\cos(2\sigma t) - \Im\{\hat{c}^{04}\}\sin(2\sigma t)\right) \cdot \cos(\sigma t - \phi) \\
=& \left(r \cdot \frac{e^{i2\sigma t} + e^{-i2\sigma t}}{2} - m \cdot \frac{e^{i2\sigma t} - e^{-i2\sigma t}}{2i}\right) \frac{e^{i(\sigma t - \phi)} + e^{-i(\sigma t-\phi)}}{2} \\
=& \frac{1}{4}\left(e^{2i\sigma t}(r - \frac{m}{i}) + e^{-2i\sigma t}(r + \frac{m}{i})\right)\left(e^{i(\sigma t-\phi)} + e^{-i(\sigma t-\phi)}\right) \\
=& \frac{1}{4}\left(e^{i(3\sigma t-\phi)}(r - \frac{m}{i}) + e^{-i(3\sigma t-\phi)}(r + \frac{m}{i}) + \right. \\
& \left. e^{i(\sigma t-\phi)}(r - \frac{m}{i}) + e^{-i(\sigma t-\phi)}(r + \frac{m}{i})\right),
\end{aligned}$$

where

$$r := \Re\{\hat{c}^{04}\},$$
$$m := \Im\{\hat{c}^{04}\}.$$

Only the contribution concerning the semi-diurnal frequencies are to be calculated, hence

$$\begin{aligned}
\frac{1}{4}\left(e^{i(\sigma t-\phi)}(r - \frac{m}{i}) + e^{-i(\sigma t-\phi)}(r + \frac{m}{i})\right) &= \frac{r}{2}\cos(\sigma t - \phi) - \frac{m}{2}\sin(\sigma t - \phi) \\
&= \Re\{\frac{1}{2}(r - mi)e^{i(\sigma t - \phi)}\} \\
&= \Re\{\frac{1}{2}\overline{\hat{c}^{04}} e^{-i\phi} e^{i\sigma t}\}.
\end{aligned}$$

Which means

$$[K_v^1 \cos(\sigma t - \phi_{K_v^1}) c_{zz}^{04}] = K_v^1 \frac{1}{2} \overline{c_{zz}^{04}} e^{-i\phi_{K_v^1}}. \quad (26)$$