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An integral Nash equilibrium control scheme for a class of multi-agent linear systems

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Abstract: We propose an integral Nash equilibrium seeking control (I-NESC) law which steers the multi-agent system composed of a special class of linear agents to the neighborhood of the Nash equilibrium in noncooperative strongly monotone games. First, we prove that there exist parameters of the integral controller such that the system converges to the Nash equilibrium in the full-information case, in other words, without the parameter estimation scheme used in extremum seeking algorithms. Then we prove that there exist parameters of the I-NESC such that the system converges to the neighborhood of the Nash equilibrium in the limited information case where parameter estimation is used. We provide a simulation example that demonstrates that smaller perturbation frequencies and amplitudes are needed to attain similar convergence speed as the existing state-of-the-art algorithm.

Keywords: Nash equilibrium seeking, Extremum seeking, Multi-agent systems

1. INTRODUCTION

Extremum seeking control (ESC) is a class of data-driven, adaptive control techniques used in optimization problems where the cost is a function of the states of a dynamical system. The method is a *zero-order* method which means it only uses the value of the cost function for optimization and no a priori knowledge of the cost function is needed, except for some basic assumptions. There was no analytical proof of stability of ESC for general nonlinear systems until the paper [Krstić and Wang [2000]]. This sparked renewed interest into further development of this type of control. Most of the research was based on the original paper by Krstić and Wang, e.g. [Tan et al. [2006]], [Ghaffari et al. [2012]], etc. There were also methods based on different ideas, such as [Dürr et al. [2013]], where the authors proposed an extremum seeking scheme based on Lie algebra, which turned out to be equivalent to the Krstić-Wang scheme. Based on a parameter estimation scheme, Guay and Dochain propose an extremum seeking scheme [Guay and Dochain [2017]] which does not use singular perturbation and averaging theory. As a result, a faster convergence rate is obtained. This fact motivates further research on such type of ESC.

Nash equilibrium problems (NEP) are different from (distributed) optimization problems, as they are characterized by a number of selfish agents whose goal is to optimize their individual cost functions, possibly dependent on the decision variables of other agents. In NEPs, the constraints of each agent are independent of other agents, while in generalized Nash equilibrium problems (GNEPs), they are coupled. Recent interest in GNEPs is motivated many engineering problems, such as demand-side management in the smart grids [Saad et al. [2012]], charging/discharging

of electric vehicles [Belgioioso and Grammatico [2017]] and formation control [Lin et al. [2014]].The literature on (G)NEPs mostly ignores the dynamics of individual agents, which may be a problem in multi-agent systems with non-negligible dynamics. The small portion of literature on (G)NEPs with dynamical agents can be divided into two groups: passivity-based first-order algorithms and extremum seeking *zero-order* algorithms.

By using a passivity property, the authors in [Gadjov and Pavel [2018]] design a control law that guarantees convergence to the Nash equilibrium (NE) of a multi-agent system with single-integrator dynamics over a network. In [Romano and Pavel [2019]], the authors extend the result to the multi-integrator case. The network topology is time invariant in both cases. In [De Persis and Grammatico [2019]], the authors extend the results of [Gadjov and Pavel [2018]] by designing a network weight adaptation scheme. In [Bianchi and Grammatico [2019]], a controller is proposed which guarantees convergence to a GNE of a multi-agent system with integrator dynamics over a network. Most prominently, extremum seeking was used for NE seeking in [Frihauf et al. [2011]] where it is proven that the extremum seeking control, under certain conditions on the individual cost functions, converges to a neighborhood of the NE for general nonlinear agents. In [Liu and Krstić [2011]], it is proven that the use of stochastic perturbation signals also induces convergence to a neighborhood of the NE. The authors in [Poveda and Teel [2017]] propose a framework for the synthesis of a hybrid controller which may be used for NEPs with nonlinear agents. All of the mentioned extremum seeking controllers are based on [Krstić and Wang [2000]].

Contribution Motivated by the recent research interest in NEPs, we adapt the ESC proposed in [Guay and Dochain [2017]], [Guay et al. [2018]]. Specifically, our contributions are the following: (i) We extend a known proportional-integral extremum seeking control scheme to

strongly monotone NEPs for a multi-agent linear systems and we prove a practical convergence to a Nash equilibrium; (ii) We numerically observe an improved performance with respect to [Frihauf et al. [2011]], as smaller amplitudes and frequencies of the sinusoidal perturbations signals are needed for a comparable convergence rate.

2. PROBLEM SETUP

We consider a multi-agent system with N agents indexed by $\mathcal{I} = \{1, 2, \ldots, N\}$, each with the following dynamics:
$$\dot{x}_i = -x_i + B_i u_i, \quad y_i = h_i(x_i, \boldsymbol{x}_{-i}), \qquad (1)$$
where $x_i \in \mathbb{R}^{n_i}$ is the state vector, $u_i \in \mathbb{R}^{m_i}$ is the control input, $y_i \in \mathbb{R}$ is the output variable which evaluates the cost function $h_i : \mathbb{R}^{n_i} \times \mathbb{R}^{n_{-i}} \to \mathbb{R}$. Let us also define $n := \sum n_i$, $n_{-i} := \sum_{j \neq i} n_j$ and $m := \sum m_i$.

Standing Assumption 1 (Regularity)
For each $i \in \mathcal{I}$, the function h_i in (1) is continuous, differentiable in x_i and its partial gradient $\nabla_{x_i} h_i$ is Lipschitz continuous in x_i and $\boldsymbol{x}_{-i}$. □

A common assumption amongst the extremum seeking literature (for example [Krstić and Wang [2000]], [Guay and Dochain [2017]], [Poveda and Teel [2017]]) is the existence of the steady-state mapping, which tells us to which state(s) the system converges when a constant input is applied. For our subsystems (1), there exists a mapping
$$\pi(\boldsymbol{u}) := \text{col}\left((\pi_i(u_i))_{i \in \mathcal{I}}\right) = \text{col}\left((B_i u_i)_{i \in \mathcal{I}}\right) \qquad (2)$$
such that for every $i \in \mathcal{I}$, $\pi_i(u_i) = B_i u_i$. Let us also define
$$\pi_{-i}(\boldsymbol{u}_{-i}) := \text{col}\left((\pi_j(u_j))_{j \neq i}\right). \qquad (3)$$

In this paper, we assume that the goal of each agent is to minimize its own steady-state cost function, i.e.,
$$\min_{u_i \in \mathbb{R}^{m_i}} h_i(\pi_i(u_i), \pi_{-i}(\boldsymbol{u}_{-i})), \qquad (4)$$
which depends on the inputs of some other agents as well. From a game-theoretic perspective, we consider the problem to compute a Nash equilibrium (NE).

Definition 1. (Nash equilibrium). A collective input $\boldsymbol{u}^*$ is a NE of the game (4) if for all $i \in \mathcal{I}$
$$h_i(\pi_i(u_i^*), \pi_{-i}(\boldsymbol{u}_{-i}^*)) \leq \inf_{u_i \in \mathbb{R}^{n_i}} h_i(\pi_i(u_i), \pi_{-i}(\boldsymbol{u}_{-i}^*)). □$$

In plain words, a set of inputs is a NE if no agent can improve its steady-state cost function by unilaterally changing its input. Since the steady-state cost functions are differentiable in u_i, it follows from Thm. 16.3 in [Bauschke et al. [2011]] that a collective vector $\boldsymbol{u}^*$ is a NE if and only if
$$\nabla_{u_i} h_i(\pi_i(u_i^*), \pi_{-i}(\boldsymbol{u}_{-i}^*)) = 0. \qquad (5)$$
In view of (5), we can stack all of the partial gradients into a single vector and form the so-called pseudo-gradient mapping of the steady-state cost functions:
$$F(\boldsymbol{u}) := \text{col}\left((\nabla_{u_i} h_i(\pi_i(u_i), \pi_{-i}(\boldsymbol{u}_{-i})))_{i \in \mathcal{I}}\right) \qquad (6)$$
Therefore, by (5) and (6), we note that the problem of finding a Nash equilibrium of the game in (4) is equivalent to finding $\boldsymbol{u}^*$ such that $F(\boldsymbol{u}^*) = 0$, which is the problem of finding a zero of F in (6), $\boldsymbol{u}^* \in \text{zer}(F)$.

A relatively standard assumption in modern game theory literature [Yu et al. [2017]], [Yi and Pavel [2019]] is strong monotonicity of the pseudo-gradient mapping:

Standing Assumption 2 (Strong monotonicity)
The mapping F in (6) is strongly monotone, i.e.,
$$(F(\boldsymbol{u}) - F(\boldsymbol{v}))^\top (\boldsymbol{u} - \boldsymbol{v}) \geq \mu \|\boldsymbol{u} - \boldsymbol{v}\|^2, \qquad (7)$$
for all $(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^{2m}$, for some $\mu > 0$. □

Let us also define the pseudo-gradient of the cost functions
$$F_{\text{x}}(\boldsymbol{x}) := \text{col}\left((\nabla_{x_i} h_i(x_i, \boldsymbol{x}_{-i}))_{i \in \mathcal{I}}\right). \qquad (8)$$
We note that, in general, monotonicity of $F_{\text{x}}(\boldsymbol{x})$ does not imply monotonicity of $F(\boldsymbol{u})$.

3. INTEGRAL NASH EQUILIBRIUM SEEKING CONTROL

3.1 Full-information case

In this subsection, we assume that every agent knows the analytic expression of its partial gradient and has access to the inputs of the other agents. The integral Nash equilibrium control in the next subsection will approximate the gradient and inputs of the other players and use the approximations in the same control law as the full-information case. Our proposed control law is inspired by the extremum seeking control in [Guay and Dochain [2017]], [Guay et al. [2018]]:
$$\forall i \in \mathcal{I}: \quad \dot{u}_i = -\tau_i^{-1} B_i^\top \nabla_{x_i} h_i(x_i, \boldsymbol{x}_{-i}) \qquad (9)$$
or in collective vector form
$$\dot{\boldsymbol{u}} = -\boldsymbol{\tau}^{-1} \boldsymbol{B}^\top F_{\text{x}}(\boldsymbol{x}), \qquad (10)$$
where $\boldsymbol{B} := \text{diag}(B_1, \ldots, B_N)$ and $\boldsymbol{\tau} := \text{diag}(\tau_1, \ldots, \tau_N)$. Unlike [Guay and Dochain [2017]], we do not use the proportional part, as it does not help with the convergence to the Nash equilibrium.

Theorem 1. Let the S.A. hold and let $(\boldsymbol{x}(t), \boldsymbol{u}(t))$, $t \geq 0$, be the closed-loop solution to the dynamics in (1) with control law in (9)–(10). Then, there exists $\tau^* \leq \min_{i \in \mathcal{I}} \tau_i$, such that $(\boldsymbol{x}(t), \boldsymbol{u}(t))$ converges to $(\boldsymbol{x}^*, \boldsymbol{u}^*) = (\pi(\boldsymbol{u}^*), \boldsymbol{u}^*)$, where $\boldsymbol{u}^*$ is a Nash equilibrium of the game in (4). □

3.2 Limited information case

Next, we consider that the agents have access to their own cost output only. We emphasize that they neither know the actions of other agents, nor they know the analytic expressions of their partial gradients. This is the standard setup used in extremum seeking, e.g. [Krstić and Wang [2000]], [Guay and Dochain [2017]], [Poveda and Teel [2017]]. The extremum seeking control proposed by [Guay and Dochain [2017]] assumes that the cost function of the system has a strong relative degree of value one. This means that the first derivative of the cost function has a direct influence on the input to the system. In the case of multi-agent systems, where the cost functions do not depend only on the states of their agent but also of the others, we make an analogous assumption:

Assumption 1 (Degree of the output)
For every $i \in \mathcal{I}$, $\nabla_{x_i} h_i(x_i, \boldsymbol{x}_{-i})^\top B_i \neq 0$ for all $(x_i, \boldsymbol{x}_{-i}) \in \mathbb{R}^n \setminus \{\boldsymbol{x}^*\}$. □

Let us first evaluate the derivative of the cost functions:
$$\dot{y}_i = -\sum_{j=1}^N \nabla_{x_j} h_i(\boldsymbol{x})^\top x_i + \sum_{j \neq i}^N \nabla_{x_j} h_i(\boldsymbol{x})^\top B_j u_j + \nabla_{x_i} h_i(\boldsymbol{x})^\top B_i u_i, \qquad (11)$$

and introduce the following variables:

$$\theta_i^0 := -\sum_{j=1}^N \nabla_{x_j} h_i(\boldsymbol{x})^\top x_i + \sum_{j\neq i}^N \nabla_{x_j} h_i(\boldsymbol{x})^\top B_j u_j,$$
$$\theta_i^1 := \nabla_{x_i} h_i(\boldsymbol{x})^\top B_i. \tag{12}$$

The variable θ_i^0 measures the effect of the autonomous dynamics of agent i on its cost function and the effects of inputs of the other agents. The variable θ_i^1 measures the effect of the input of agent i on the cost output y_i. By substituting θ_i^0 and θ_i^1 in (11), the derivative reads as

$$\dot{y}_i = \theta_i^0 + \theta_i^1 u_i = [1, u_i^\top]\theta_i, \tag{13}$$

Note that θ_i^1 is proportional to the right-hand side in (9). To estimate the local θ_i^0 and θ_i^1, we use a time-varying parameter estimation approach such as the one proposed in [Guay and Dochain [2017]]. Let us provide a basic intuition. Let $\hat{y}_i$ and $\hat{\theta}_i$ be estimations of the output y_i and the variable θ_i respectively and let $e_i = y_i - \hat{y}_i$ be the estimation error. Then, the estimator model of (13) for agent i is given by

$$\dot{\hat{y}}_i = [1, u_i^\top]\hat{\theta}_i + K_i e_i + c_i^\top \dot{\hat{\theta}}_i, \tag{14}$$

where K_i is a free design parameter. Note that the first two terms on the right-hand side resemble high gain observer schemes. As the structure of the problem does not allow the use of high gain observers, it is necessary to introduce some other dynamics into the estimation. This is the primary role of the third term in (14). Therefore, the dynamics of $c_i(t)$ are choosen as

$$\dot{c}_i^\top = -K_i c_i^\top + [1, u_i^\top]. \tag{15}$$

Let us introduce an auxiliary variable η_i, with dynamics $\dot{\eta}_i = -K_i \eta_i - c_i^\top \dot{\theta}$, and its estimate $\hat{\eta}_i$, with dynamics

$$\dot{\hat{\eta}}_i = -K_i \hat{\eta}_i. \tag{16}$$

It is also necessary to define a symmetric, positive definite scaling matrix variable $\Sigma_i \in \mathbb{R}^{m_i+1 \times m_i+1}$ with dynamics

$$\dot{\Sigma}_i = c_i c_i^\top - k_i^\mathrm{T}\Sigma_i + \sigma_i I \qquad \Sigma_i(0) = \alpha_1^i, \tag{17}$$

where $k_i^\top$, σ_i and α_1^i are free design parameters. The third term is added so that the matrix is always invertible.

Equations (14)-(17) form the parameter update law presented in [Adetola and Guay [2008]]:

$$\dot{\hat{\theta}}_i = \Pi_{\Theta_i}\left(\hat{\theta}_i, \Sigma_i^{-1}(c_i(e_i - \hat{\eta}_i) - \sigma_i\hat{\theta}_i)\right), \tag{18}$$

where $\Pi_{\Theta_i}(\hat{\theta}, v)$ denotes the projection of the vector v onto the tangent cone of the set Θ_i at $\hat{\theta}$, as defined by Equation 2.14 in [Nagurney and Zhang [2012]]. This implies that if the starting value $\hat{\theta}_i(0)$ is in Θ_i, so will $\hat{\theta}_i(t)$ for all t.

We are finally ready to propose an integral decentralized Nash equilibrium seeking control law of the form

$$\forall i \in \mathcal{I} : u_i = \hat{u}_i + d_i(t), \quad \dot{\hat{u}}_i = -\tau_i^{-1}\hat{\theta}_i^1, \tag{19}$$

together with Equations (14)–(18). In the collective vector form, Equation (19) read as

$$\boldsymbol{u} = \hat{\boldsymbol{u}} + \boldsymbol{d}(t), \quad \dot{\hat{\boldsymbol{u}}} = -\boldsymbol{\tau}^{-1}\hat{\theta}^1 \tag{20}$$

As in [Guay and Dochain [2017]], for the parameter estimation scheme to converge, a persistency of excitation (PE) assumption for every agent is introduced.

Assumption 2 (Persistence of excitation)
For every $i \in \mathcal{I}$, there exist constants α_i^2 and T_i such that

$$\int_t^{t+T_i} c_i(\tau)c_i(\tau)^\top d\tau \geq \alpha_i^2 I, \qquad \forall t > 0, \tag{21}$$

where $c_i(\tau)$ is the solution to (15). □

We conclude the section with the main theoretical result of the paper, namely, the convergence of the closed-loop dynamics to a Nash equilibrium of the game.

Theorem 2. Let the S. A. and Assumptions 1, 2 hold and let $(\boldsymbol{x}(t), \boldsymbol{u}(t))$ be the closed-loop solution to the dynamics (1) with control law in (14) – (18), (19). Let π be the steady-state mapping in (2) and let D be the largest amplitude of the perturbation signals $\{d_i(t)\}_{i\in\mathcal{I}}$. Then, there exist gains $(K_i, k_i^\mathrm{T}, \sigma_i)_{i\in\mathcal{I}}$ and $\tau^* \leq \min_{i\in\mathcal{I}} \tau_i$ such that $(\boldsymbol{x}(t), \boldsymbol{u}(t))$ converges towards the $\mathcal{O}(D^2)$ neighborhood of some $(\boldsymbol{x}^*, \boldsymbol{u}^*) = (\pi(\boldsymbol{u}^*), \boldsymbol{u}^*)$, where $\boldsymbol{u}^*$ is a Nash equilibrium of the game in (4). □

4. SIMULATION EXAMPLE

Consider a three agent system with dynamics:

$$\dot{x}_i = -x_i + u_i, \text{ for } i \in \{1, 2, 3\}. \tag{22}$$

The cost functions of agents are given by

$$y_1 = 1.5(x_1 - 1)^2 + 1.5x_1x_2 + x_1x_3$$
$$y_2 = -2x_2x_1 + 1.5(x_2 - 2)^2 + x_2x_3$$
$$y_3 = -2.5x_3x_1 - x_3x_2 + 1.5(x_3 - 3)^2. \tag{23}$$

Two types of controllers were simulated to have a comparison; the limited information controller proposed by this paper and the controller from [Frihauf et al. [2011]] with additional low-pass and high-pass filters as in [Krstić and Wang [2000]] in order to improve the performance. The latter can be described by the following equations

$$\dot{\eta}_i = -\omega_h^i\eta_i + \omega_h^i y_i,$$
$$\dot{\xi}_i = -\omega_l^i\xi_i + \omega_l^i(y_i - \eta_i)A_i\sin(\omega_i t),$$
$$\dot{\hat{u}}_i = -k_iA_i\xi_i, \ u_i = \hat{u}_i + A_i\sin(\omega_i t). \tag{24}$$

For our controller, the following parameters were chosen: $\sigma_1 = \sigma_2 = \sigma_3 = 10^{-6}$, $K_1 = K_2 = K_3 = 50$, $k_T^1 = k_T^2 = k_T^3 = 50$, $\alpha_I^1 = \alpha_I^2 = \alpha_I^3 = 0.1$, $\tau_1 = 5$, $\tau_I^2 = 10$, $\tau_I^3 = 15$, $d_1(t) = \frac{1}{2}\sin(40t)$, $d_2(t) = \frac{1}{2}\sin(50t)$ and $d_3(t) = \frac{1}{2}\sin(60t)$. Initial states of $\boldsymbol{x}$, $\hat{\boldsymbol{u}}$, $\boldsymbol{c}$, $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ were set to zero. The parameters K, k^T and τ_I were initially chosen large enough to ensure stability. Then τ_I was decreased to speed up the convergence. Further decreases in τ_I were making the states oscillate; further decreasing of K and k^T did not improve the performance of the algorithm. For the Frihauf et al., the following parameters were chosen: $\omega_h^1 = 180$, $\omega_h^2 = 200$, $\omega_h^3 = 220$, $\omega_l^1 = 45$, $\omega_l^2 = 50$, $\omega_l^3 = 55$, $\omega_1 = 90$, $\omega_2 = 100$, $\omega_3 = 110$, $k_1 = k_2 = k_3 = 0.5$ and $A_1 = A_2 = A_3 = 5$. Firstly, we chose ω_1, ω_2 and ω_3 such that highest convergence rate for a fixed A_i was observed. Higher perturbation frequencies facilitate faster learning of the gradient, but also higher frequencies get damped out. At the chosen frequencies, best trade-off was observed. Next, the amplitude was increased to speed up the convergence. After $A_i = 5$, non-significant increase in performance was observed. The results of the numerical simulations can be seen in Figures 1 and 2. While the convergence speed of both algorithms is similar, the frequency and amplitude of the sinusoidal perturbation signals are much lower with our I-NESC law.
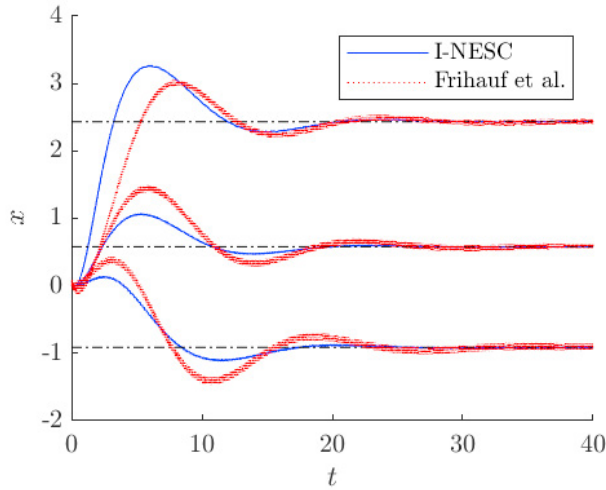
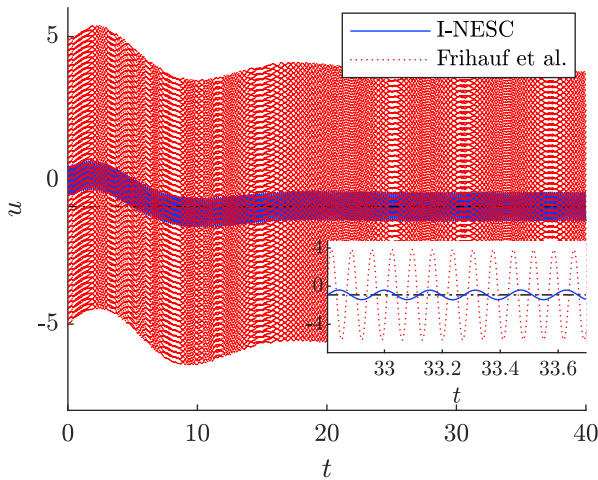Fig. 1. State trajectories of the three agents under I-NESC (solod blue) and [Frihauf et al. [2011]] (doted red)

Fig. 2. Input of the first agent under I-NESC (solid blue) and [Frihauf et al. [2011]] (doted red)

5. CONCLUSION

Nash equilibrium problems can be solved efficiently via extremum seeking if the agents belong to a certain class of linear dynamics with strongly monotone and Lipschitz continuous game mapping.

Appendix A. PROOF OF THEOREM 1

Stability of equilibrium points $\pi_i(u_i)$ for every agent $i \in \mathcal{I}$ can be characterised by the following Lyapunov function:

$$V_i(x_i, u_i) = \frac{\beta}{2}(x_i - B_i u_i)^\top (x_i - B_i u_i), \qquad (A.1)$$

where $\beta > 0$. The derivative of (A.1) is equal to

$$\dot{V}_i(x_i, u_i) = -\beta \|x_i - \pi_i(u_i)\|^2 - \beta(x_i - \pi_i(u_i))^\top B \dot{u}_i. \qquad (A.2)$$

For very slow changes of the input u_i, we expect the subsystems will converge to a small neighborhood of equilibrium points $\pi_i(u_i)$. We consider the controller in (10). Its goal is to estimate the Nash equilibrium input $\boldsymbol{u}^*$

and to preserve the stability of the subsystems. Therefore, we construct the following Lyapunov function candidate:

$$W(\boldsymbol{x}, \boldsymbol{u}) := T(\boldsymbol{u}) + V(\boldsymbol{x}, \boldsymbol{u}) = \frac{1}{2}\tau_{\min}^{-1}\tilde{\boldsymbol{u}}^\top \boldsymbol{\tau}\tilde{\boldsymbol{u}} + \sum_{i=1}^N V_i(x_i, u_i), \qquad (A.3)$$

where $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}^*$ and $\tau_{\min} = \min\{\tau_1, \ldots, \tau_N\}$. Now, we bound the derivative of T. By adding and subtracting $F(\boldsymbol{u})$ to (10), $\dot{\boldsymbol{u}}$ reads as

$$\dot{\boldsymbol{u}} = -\boldsymbol{\tau}^{-1}F(\boldsymbol{u}) - \boldsymbol{\tau}^{-1}(\boldsymbol{B}^\top F_{\mathrm{x}}(\boldsymbol{x}) - F(\boldsymbol{u})). \qquad (A.4)$$

From (A.3) and (A.4), we have

$$\dot{T}(\boldsymbol{x}, \boldsymbol{u}) = -\tau_{\min}^{-1}\tilde{\boldsymbol{u}}^\top F(\boldsymbol{u}) - \tau_{\min}^{-1}\tilde{\boldsymbol{u}}^\top (\boldsymbol{B}F_{\mathrm{x}}(\boldsymbol{x}) - F(\boldsymbol{u})). \qquad (A.5)$$

Considering $F(\boldsymbol{u})$ is strongly monotone and we have $F(\boldsymbol{u}^*) = 0$, since $\boldsymbol{u}^* \in \mathrm{zer}(F)$, (7) reads as

$$F(\boldsymbol{u})^\top (\boldsymbol{u} - \boldsymbol{u}^*) = F(\boldsymbol{u})^\top \tilde{\boldsymbol{u}} \geq \mu \|\boldsymbol{u} - \boldsymbol{u}^*\|^2. \qquad (A.6)$$

To bound the second term in (A.5), we use the identity:

$$\nabla_{u_i} h_i(\pi_i(u_i), \pi_{-i}(\boldsymbol{u}_{-i}))^\top = \nabla_{x_i} h_i(\pi_i(u_i), \pi_{-i}(\boldsymbol{u}_{-i}))^\top B_i. \qquad (A.7)$$

By using the relations (6), (8) and (A.7), it follows that:

$$\boldsymbol{B}^\top F_{\mathrm{x}}(\pi(\boldsymbol{u})) = F(\boldsymbol{u}). \qquad (A.8)$$

By exploiting (A.6) and (A.8), from (A.5) we have

$$\dot{T} \leq -\mu\tau_{\min}^{-1}\|\boldsymbol{u} - \boldsymbol{u}^*\|^2$$
$$- \tau_{\min}^{-1}\tilde{\boldsymbol{u}}^\top(\boldsymbol{B}^\top F_{\mathrm{x}}(\boldsymbol{x}) - \boldsymbol{B}^\top F_{\mathrm{x}}(\pi(\boldsymbol{u}))). \qquad (A.9)$$

Since all of the functions are Lipschitz continuous, the right-hand side in (A.9) can be upper bounded as follows:

$$\dot{T} \leq -\mu\tau_{\min}^{-1}\|\tilde{\boldsymbol{u}}\|^2 + L\tau_{\min}^{-1}\|\tilde{\boldsymbol{u}}\|\|\boldsymbol{x} - \pi(\boldsymbol{u})\|, \qquad (A.10)$$

where $L > 0$ is the Lipschitz constant of the mapping $\boldsymbol{B}^\top \circ F_{\mathrm{x}}$. Now, we turn our attention to the full Lyapunov function candidate W. The derivative is bounded as

$$\dot{W}(\boldsymbol{x}, \boldsymbol{u}) \leq -\beta\|\boldsymbol{x} - \pi(\boldsymbol{u})\|^2 - \mu\tau_{\min}^{-1}\|\tilde{\boldsymbol{u}}\|^2$$
$$+ L\tau_{\min}^{-1}\|\tilde{\boldsymbol{u}}\|\|\boldsymbol{x} - \pi(\boldsymbol{u})\| - \beta(\boldsymbol{x} - \pi(\boldsymbol{u}))^\top \boldsymbol{B}\dot{\boldsymbol{u}}. \qquad (A.11)$$

To complete the proof, we bound the derivative of V caused by the change of inputs:

$$-\beta(\boldsymbol{x} - \pi(\boldsymbol{u}))^\top \boldsymbol{B}\dot{\boldsymbol{u}} \leq \beta\|\boldsymbol{B}\|\|\dot{\boldsymbol{u}}\|\|\boldsymbol{x} - \pi(\boldsymbol{u})\|. \qquad (A.12)$$

By using (A.4), the norm of the derivative is bounded:

$$\|\dot{\boldsymbol{u}}\| \leq \tau_{\min}^{-1}\|F(\boldsymbol{u})\| + \tau_{\min}^{-1}\|(\boldsymbol{B}^\top F_{\mathrm{x}}(\boldsymbol{x}) - F(\boldsymbol{u}))\|. \qquad (A.13)$$

Again, since all of the functions are Lipschitz continuous, the right-hand side of the previous equation can be bounded as follows

$$\|\dot{\boldsymbol{u}}\| \leq L_{\mathrm{F}}\tau_{\min}^{-1}\|\tilde{\boldsymbol{u}}\| + L\tau_{\min}^{-1}\|\boldsymbol{x} - \pi(\boldsymbol{u})\|, \qquad (A.14)$$

where $L_{\mathrm{F}} > 0$ is the Lipschitz constant of F. By using the bounds (A.12) and (A.14), $\dot{W}$ can be bounded as follows:

$$\dot{W}(\boldsymbol{x}, \boldsymbol{u}) \leq -\begin{bmatrix} \|\tilde{\boldsymbol{u}}\| \\ \|\boldsymbol{x} - \pi(\boldsymbol{u})\| \end{bmatrix}^\top M \begin{bmatrix} \|\tilde{\boldsymbol{u}}\| \\ \|\boldsymbol{x} - \pi(\boldsymbol{u})\| \end{bmatrix} \qquad (A.15)$$

where

$$M = \begin{bmatrix} \beta - L\beta\|\boldsymbol{B}\|\tau_{\min}^{-1} & -\frac{1}{2}(L + \beta\|\boldsymbol{B}\|L_{\mathrm{F}})\tau_{\min}^{-1} \\ -\frac{1}{2}(L + \beta\|\boldsymbol{B}\|L_{\mathrm{F}})\tau_{\min}^{-1} & \mu\tau_{\min}^{-1} \end{bmatrix}. \qquad (A.16)$$

Thus, if $\tau_{\min} \geq \frac{(L + \beta\|\boldsymbol{B}\|L_{\mathrm{F}})^2 + 4L\beta\|\boldsymbol{B}\|}{4\beta\mu}$, then the matrix M in (A.16) is positive definite, which in turn implies that $\dot{W}$ is negative definite, which concludes the proof. ∎

Appendix B. PROOF OF THEOREM 2

The proof is similar to the full-information case proof, but unlike the full-information case, our inputs use the estimation of the θ_i variables. Let us consider a Lyapunov function candidate of the form $L = W + V + T$, where

$$W(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\theta}}) = \sum_{i=1}^{N} \left(\tfrac{1}{2} \tilde{\eta}_i^{\top} \tilde{\eta}_i + \tfrac{1}{2} \tilde{\theta}_i^{\top} \Sigma_i \tilde{\theta}_i \right), \tag{B.1}$$

$$V(\boldsymbol{x}, \hat{\boldsymbol{u}}) = \sum_{i=1}^{N} V_i(x_i, \hat{u}_i), \tag{B.2}$$

$$T(\hat{\boldsymbol{u}}) = \frac{1}{2} \tau_{\min}^{-1} (\hat{\boldsymbol{u}} - \boldsymbol{u}^*)^{\top} \boldsymbol{\tau} (\boldsymbol{u} - \boldsymbol{u}^*) = \frac{1}{2} \tau_{\min}^{-1} \tilde{\boldsymbol{u}}^{\top} \boldsymbol{\tau} \tilde{\boldsymbol{u}}. \tag{B.3}$$

Therefore, our candidate consists of a parameter estimation term (W), a local state-input Lyapunov term (V) and the Nash equilibrium estimation error term (T).

Parameter estimation term: We bound the time derivative of the W function similarly to [Guay and Dochain [2017]] and [Guay et al. [2018]] with the only difference that we let each agent choose their own parameters $(\sigma_i, K_i, k_i^{\mathrm{T}})$. The Lyapunov derivative reads as follows:

$$\dot{W}(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\theta}}) \le -k_{\mathrm{a}} \|\tilde{\boldsymbol{\eta}}\|^2 - k_{\mathrm{b}} \|\tilde{\boldsymbol{\theta}}\|^2 + k_{\mathrm{c}} \|\dot{\boldsymbol{\theta}}\|^2 + \frac{\sigma}{2} \|\boldsymbol{\theta}\|^2, \tag{B.4}$$

where $k_{\mathrm{a}} := \min_i \left(K_i - \tfrac{1}{2} - \tfrac{k_1 \zeta_i}{2} \right)$, $k_{\mathrm{b}} := \min_i \left(\tfrac{k_i^{T'} \gamma_{1i}}{2} \right)$, $k_{\mathrm{c}} := \max_i \left(\tfrac{1}{2k_1} + \tfrac{\gamma_{2i}}{2k_2} \right)$ and $\sigma := \max_i \sigma_i$.

Local state-input Lyapunov term: The derivative of the Lyapunov term $V_i(x_i, u_i)$ in (B.2) is

$$\dot{V}_i(x_i, u_i) = -\beta \|x_i - \pi_i(\hat{u}_i)\|^2 + \beta (x_i - \pi_i(\hat{u}_i))^{\top} B_i d_i(t) \\ - \beta (x_i - \pi_i(\hat{u}_i))^{\top} B \dot{\hat{u}}_i. \tag{B.5}$$

The first addend is equal to the complete derivative of the Lyapunov function in the case of constant inputs, the amplitude of the second component is proportional to the amplitude of the perturbations and the amplitude of the third component is equal to the amplitude of the derivative of the input u_i. To bound the third component, we need to bound $\dot{\hat{u}}_i$, hence $\dot{\hat{\boldsymbol{u}}}$, which reads as $\dot{\hat{\boldsymbol{u}}} = -\boldsymbol{\tau}^{-1}(\boldsymbol{B}^{\top} F_{\mathrm{x}}(\boldsymbol{x}) + \tilde{\boldsymbol{\theta}}^1)$. By using the same argument as in (A.14), it follows

$$\dot{\hat{\boldsymbol{u}}} \le L_{\mathrm{F}} \tau_{\min}^{-1} \|\tilde{\boldsymbol{u}}\| + L \tau_{\min}^{-1} \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\| + \tau_{\min}^{-1} \|\tilde{\boldsymbol{\theta}}^1\|. \tag{B.6}$$

By using the previous equation, it is possible to bound the second addend in (B.5):

$$\beta(\boldsymbol{x} - \pi(\boldsymbol{u}))^{\top} \boldsymbol{B} \dot{\hat{\boldsymbol{u}}} \le L_{\mathrm{F}} \beta \tau_{\min}^{-1} \|\boldsymbol{B}\| \|\tilde{\boldsymbol{u}}\| \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\|$$

$$+ L\beta \tau_{\min}^{-1} \|\boldsymbol{B}\| \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\|^2 + \beta \tau_{\min}^{-1} \|\boldsymbol{B}\| \|\tilde{\boldsymbol{\theta}}^1\| \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\|.$$

Therefore, the derivative of V can be bounded as

$$\dot{V}(\boldsymbol{x}, \boldsymbol{u}) \le -\beta \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\|^2 + L_{\mathrm{F}} \beta \tau_{\min}^{-1} \|\boldsymbol{B}\| \|\tilde{\boldsymbol{u}}\| \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\|$$

$$+ L\beta \tau_{\min}^{-1} \|\boldsymbol{B}\| \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\|^2 + \beta \tau_{\min}^{-1} \|\boldsymbol{B}\| \|\tilde{\boldsymbol{\theta}}^1\| \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\|$$

$$- \beta(\boldsymbol{x} - \pi(\hat{\boldsymbol{u}}))^{\top} \boldsymbol{B} d(t). \tag{B.7}$$

The last term can be bounded by using the Cauchy—Bunyakovsky—Schwarz inequality and the inequality $\|x\| \|y\| \le \frac{1}{2k} \|x\|^2 + \frac{k}{2} \|y\|^2$ to conclude the desired bound:

$$\dot{V} \le -\left(\beta - \frac{L\beta \|\boldsymbol{B}\|}{\tau_{\min}} - \frac{\beta \|\boldsymbol{B}\|}{2\tau_{\min} k_3} - \frac{\beta \|\boldsymbol{B}\|}{2k_4} \right) \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\|^2$$

$$+ L_{\mathrm{F}} \beta \tau_{\min}^{-1} \|\boldsymbol{B}\| \|\tilde{\boldsymbol{u}}\| \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\| + \frac{1}{2} \beta k_3 \tau_{\min}^{-1} \|\boldsymbol{B}\| \|\tilde{\boldsymbol{\theta}}^1\|^2$$

$$+ \frac{1}{2} \beta k_4 \|\boldsymbol{B}\| \|\boldsymbol{d}(t)\|^2. \tag{B.8}$$

Nash equilibrium estimation error term: The parameter estimation also has an influence on the Nash equilibrium estimation error. The derivative of (B.3) is equal to

$$\dot{T}(\boldsymbol{x}, \hat{\boldsymbol{u}}) = -\boldsymbol{\tau}^{-1} \tilde{\boldsymbol{u}} (\boldsymbol{B}^{\top} F_{\mathrm{x}}(\boldsymbol{x}) + \tilde{\boldsymbol{\theta}}_i^1). \tag{B.9}$$

By the same method as in (A.10), (B.8), it follows that

$$\dot{T}(\hat{\boldsymbol{u}}) \le -\mu \tau_{\min}^{-1} \|\tilde{\boldsymbol{u}}\|^2 + L \tau_{\min}^{-1} \|\tilde{\boldsymbol{u}}\| \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\|$$

$$+ \frac{1}{2\tau_{\min} k_5} \|\tilde{\boldsymbol{u}}\|^2 + \frac{k_5}{2\tau_{\min}} \|\tilde{\boldsymbol{\theta}}^1\|^2. \tag{B.10}$$

The full Lyapunov candidate: With the bounds (B.4), (B.8) and (B.10), the derivative of the full Lyapunov candidate function is bounded as follows:

$$\dot{L} \le -k_{\mathrm{a}} \|\tilde{\boldsymbol{\eta}}\|^2 - k_{\mathrm{b}} \|\tilde{\boldsymbol{\theta}}^0\|^2 - \left(\mu \tau_{\min}^{-1} - \frac{1}{2\tau_{\min} k_5} \right) \|\tilde{\boldsymbol{u}}\|^2$$

$$- \left(\beta - \frac{L\beta \|\boldsymbol{B}\|}{\tau_{\min}} - \frac{\beta \|\boldsymbol{B}\|}{2\tau_{\min} k_3} - \frac{\beta \|\boldsymbol{B}\|}{2k_4} \right) \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\|^2$$

$$- \left(k_{\mathrm{b}} - \frac{k_5}{2\tau_{\min}} - \frac{\beta \|\boldsymbol{B}\| k_3}{2\tau_{\min}} \right) \|\tilde{\boldsymbol{\theta}}^1\|^2 + k_{\mathrm{c}} \|\dot{\boldsymbol{\theta}}\|^2 + \frac{\sigma}{2} \|\boldsymbol{\theta}\|^2$$

$$+ \frac{L_{\mathrm{F}} \beta \|\boldsymbol{B}\| + L}{\tau_{\min}} \|\tilde{\boldsymbol{u}}\| \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\| + \frac{\beta \|\boldsymbol{B}\| k_4}{2} \|\boldsymbol{d}(t)\|^2. \tag{B.11}$$

We are left with determining bounds on $\|\boldsymbol{\theta}\|$ and $\|\dot{\boldsymbol{\theta}}\|$. Since all of the considered functions (and their composition) in (1), (12) and (20) are Lipschitz continuous, it follows

$$\|\boldsymbol{\theta}\|^2 \le L_1 \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\|^2 + L_2 \|\tilde{\boldsymbol{u}}\|^2 \tag{B.12}$$

$$\|\dot{\boldsymbol{\theta}}\|^2 \le L_3 \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\|^2 + L_4 \|\tilde{\boldsymbol{u}}\|^2, \tag{B.13}$$

for some $L_1, L_2, L_3, L_4 > 0$. Substituting (B.12) and (B.13) into (B.11), we obtain

$$\dot{L} \le -\left(\frac{\mu}{\tau_{\min}} - \frac{1}{2\tau_{\min} k_5} - \frac{L_2 \sigma}{2} - k_{\mathrm{c}} L_4 \right) \|\tilde{\boldsymbol{u}}\|^2 - \left(\beta - \frac{L_1 \sigma}{2} \right.$$

$$\left. - \frac{L\beta \|\boldsymbol{B}\|}{\tau_{\min}} - \frac{\beta \|\boldsymbol{B}\|}{2\tau_{\min} k_3} - \frac{\beta \|\boldsymbol{B}\|}{2k_4} - k_{\mathrm{c}} L_3 \right) \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\|^2$$

$$- \left(k_{\mathrm{b}} - \frac{k_5}{2\tau_{\min}} - \frac{\beta \|\boldsymbol{B}\| k_3}{2\tau_{\min}} \right) \|\tilde{\boldsymbol{\theta}}^1\|^2 - k_{\mathrm{a}} \|\tilde{\boldsymbol{\eta}}\|^2 - k_{\mathrm{b}} \|\tilde{\boldsymbol{\theta}}^0\|^2$$

$$+ \frac{L_{\mathrm{F}} \beta \|\boldsymbol{B}\| + L}{\tau_{\min}} \|\tilde{\boldsymbol{u}}\| \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\| + \frac{\beta \|\boldsymbol{B}\| k_4}{2} \|\boldsymbol{d}(t)\|^2. \tag{B.14}$$

Now we prove that there exist parameters K, k^{T} and $\tau_{\min}$ such that the RHS in (B.14), apart from the term with $\|\boldsymbol{d}(t)\|^2$, is negative definite. The proof goes by the same lines as in [Guay and Dochain [2017]]. Consider the following reformulation of (B.14):

$$\dot{L} \le -k_{\mathrm{a}} \|\tilde{\boldsymbol{\eta}}\|^2 - k_{\mathrm{b}} \|\tilde{\boldsymbol{\theta}}^0\|^2 - \left(-\frac{L_2 \sigma}{2} - k_{\mathrm{c}} L_4 \right) \|\tilde{\boldsymbol{u}}\|^2$$

$$- \left(-\frac{\beta \|\boldsymbol{B}\|}{2k_4} - \frac{L_1 \sigma}{2} - k_{\mathrm{c}} L_3 \right) \|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\|^2$$

$$- \left(k_{\mathrm{b}} - \frac{k_5}{2\tau_{\min}} - \frac{\beta \|\boldsymbol{B}\| k_3}{2\tau_{\min}} \right) \|\tilde{\boldsymbol{\theta}}^1\|^2 + \frac{\beta \|\boldsymbol{B}\| k_4}{2} \|\boldsymbol{d}(t)\|^2$$

$$- \begin{bmatrix} \|\tilde{\boldsymbol{u}}\| \\ \|\boldsymbol{x} - \pi(\boldsymbol{u})\| \end{bmatrix}^{\top} M \begin{bmatrix} \|\tilde{\boldsymbol{u}}\| \\ \|\boldsymbol{x} - \pi(\boldsymbol{u})\| \end{bmatrix}, \tag{B.15}$$

where

$$\boldsymbol{M} = \begin{bmatrix} \beta - \dfrac{L\beta\|\boldsymbol{B}\|}{\tau_{\min}} - \dfrac{\beta\|\boldsymbol{B}\|}{2\tau_{\min}k_3} & -\dfrac{L + \beta\|\boldsymbol{B}\|L_{\mathrm{F}}}{2\tau_{\min}} \\ -\dfrac{L + \beta\|\boldsymbol{B}\|L_{\mathrm{F}}}{2\tau_{\min}} & \dfrac{2\mu - 1/k_5}{2\tau_{\min}} \end{bmatrix}.$$
(B.16)

The parameter k_3 can be chosen arbitrarily, while k_5 has to be chosen such that lower diagonal element in $\boldsymbol{M}$ is positive, i.e. $2\mu - 1/k_5 > 0$. Also, in order for $\boldsymbol{M}$ to be positive definite, the following condition must be satisfied

$$v_1(L, \beta\|\boldsymbol{B}\|, L_{\mathrm{F}}, k_3, k_5, \beta, \mu) =$$
$$\frac{(L + \beta\|\boldsymbol{B}\|L_{\mathrm{F}})^2 + 2L\beta\|\boldsymbol{B}\| + \beta\|\boldsymbol{B}\|/k_3}{2\beta(2\mu - 1/k_5)} \leq \tau_{\min}. \quad \text{(B.17)}$$

From Equation (B.15), it can be also concluded that

$$\tau_{\min} \geq \frac{Lk_3 + k_5}{k_{\mathrm{b}}} = v_2(L, k_3, k_5, k_{\mathrm{b}}). \quad \text{(B.18)}$$

Therefore, $\tau_{\min} \geq \max(v_1, v_2)$. Let $\lambda = \sigma_{\min}(\boldsymbol{M})$. The inequality (B.15) can be reformulated as

$$\dot{L} \leq -k_{\mathrm{a}}\|\tilde{\boldsymbol{\eta}}\|^2 - k_{\mathrm{b}}\|\tilde{\boldsymbol{\theta}}^0\|^2 - \left(\lambda - \frac{L_2\sigma}{2} - k_{\mathrm{c}}L_4\right)\|\tilde{\boldsymbol{u}}\|^2$$
$$- \left(\lambda - \frac{\beta\|\boldsymbol{B}\|}{2k_4} - \frac{L_1\sigma}{2} - k_{\mathrm{c}}L_3\right)\|\boldsymbol{x} - \pi(\hat{\boldsymbol{u}})\|^2$$
$$- \left(k_{\mathrm{b}} - \frac{k_5}{2\tau_{\min}} - \frac{\beta\|\boldsymbol{B}\|k_3}{2\tau_{\min}}\right)\|\tilde{\boldsymbol{\theta}}^1\|^2 + \frac{\beta\|\boldsymbol{B}\|k_4}{2}\|\boldsymbol{d}(t)\|^2.$$

The parameters σ, k_{c} must be chosen small enough, while k_4 must be large enough such that the following equations hold true: $0 < \lambda - \frac{L_2\sigma}{2} - k_{\mathrm{c}}L_4$, $0 < \lambda - \frac{\beta\|\boldsymbol{B}\|}{2k_4} - \frac{L_1\sigma}{2} - k_{\mathrm{c}}L_3$. The parameter σ is a free design parameter; the parameter k_{c} can be made arbitrarily small by increasing the gains K and k^{T} (or to be more precise K_i and k_i^{T}, see [Guay and Dochain [2017]] for more details). The parameter k_4 can be arbitrarily chosen. Therefore, it is possible to choose the controller parameters σ, k^{T} and K such that all of the constants that multiply the squares of the norms in (B.15) (except for $\boldsymbol{d}(t)$) are positive.

Next, we consider the Lyapunov functions of the subsystems in (B.2), the bounds on matrices Σ_i and the quadratic elements of the Lyapunov function candidate L. Let D be the largest amplitude of all the perturbation signals $d_i(t)$. Then it can be concluded that there exists a positive constant α_L such that $\dot{L} \leq -\alpha_L L + \frac{\beta\|\boldsymbol{B}\|k_4 D^2}{2}$. With $z = \left(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\theta}}, \boldsymbol{x}, \tilde{\boldsymbol{u}}\right) \in \mathbb{R}^N \times \mathbb{R}^{m+N} \times \mathbb{R}^n \times \mathbb{R}^m$, let us define the set $\Omega_\gamma = \{z \mid L(z) \leq \gamma\}$. We choose γ such that $z \in \Omega_\gamma \Rightarrow \hat{\boldsymbol{\theta}} \in \Theta_1 \times \Theta_2 \times \cdots \times \Theta_N$. It follows that the trajectories $\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\theta}}, \boldsymbol{x}, \tilde{\boldsymbol{u}}$ enter the set $\Omega_{\gamma_0} = \left\{z \mid L(z) \leq \frac{\beta\|\boldsymbol{B}\|k_4 D^2}{2\alpha_L}\right\}$. Thus, for D chosen such that $\Omega_{\gamma_0} \subset \Omega_\gamma$, the set Ω_{γ_0}, which is contained in a ball containing the point $(0, 0, \boldsymbol{x}^*, 0)$ with radius of order $\mathcal{O}\left(D^2\right)$, is exponentially stable for the closed-loop system. ∎

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