



Technische Universiteit Delft  
Faculteit Elektrotechniek, Wiskunde en Informatica  
Delft Institute of Applied Mathematics

**Duality in interacting particle systems out  
of equilibrium.**

Verslag ten behoeve van het  
Delft Institute of Applied Mathematics  
als onderdeel ter verkrijging

van de graad van

**BACHELOR OF SCIENCE**  
in  
**TECHNISCHE WISKUNDE**

door  
**Sjoerd Friso Wiarda**

**Delft, Nederland**  
**Juli 2019**

Copyright © 2019 door Sjoerd Wiarda. Alle rechten  
voorbehouden.





## **BSc verslag TECHNISCHE WISKUNDE**

**“Duality in interacting particle systems out of equilibrium.”**

Sjoerd Friso Wiarda

**Technische Universiteit Delft**

### **Begeleider**

Dr. G. Carinci

### **Overige commissieleden**

Dr. Ir. W. G. M. Groenevelt

E. M. van Elderen

Juli, 2019

Delft

## Abstract

In this thesis we study a class of interacting particle systems sharing a duality property. This class includes the Symmetric Inclusion Process ( $\text{SIP}(2k)$ ), the Symmetric Exclusion Process ( $\text{SEP}(2j)$ ) and the Independent Random Walkers (IRW). When these systems are in equilibrium (namely they are isolated from the exterior) they admit stationary measures that are also reversible product space-homogeneous measures. However when the systems are in contact with reservoirs that keep them out of equilibrium, the reversibility is lost and the stationary microscopic distributions are unknown with the exception for the  $\text{SEP}(1)$  (the Symmetric Exclusion Process) and the IRW. In order to examine the non-equilibrium stationary distributions for the whole class of processes, we make use of duality between these processes and particle systems with absorbing boundaries. Our first main results are in section 6, where we give explicit formulas for the absorption probabilities for the dual systems with 1 and 2 particles. Then we use this result to compute an explicit formula for the variance and covariance of the sites occupation numbers for the many-particles systems with density reservoirs in the non-equilibrium stationary distribution in section 7. Then we identify three possible scaling regimes for the density field: a deterministic regime where the variance vanishes and particles are expected to converge to independent Brownian motions; a sticky regime where the variance is finite and particles are expected to converge to sticky Brownian motions; and finally an absorbing regime where the variance is infinite and particles are expected to converge to coalescing Brownian motions. In the last part of the thesis (section 8) we start the analysis of the dynamics of the particle systems.

# Contents

<b>1</b>	<b>Introduction</b>	<b>5</b>
<b>2</b>	<b>Basic definitions and notations for probability theory and Markov Processes</b>	<b>8</b>
2.1	Random variables . . . . .	8
2.2	Markov Processes . . . . .	9
<b>3</b>	<b>Interacting particle systems</b>	<b>13</b>
3.1	Standard Generator . . . . .	13
3.2	The Reference Process . . . . .	13
<b>4</b>	<b>Stationary measures of the interacting particle systems at equilibrium</b>	<b>19</b>
<b>5</b>	<b>Duality</b>	<b>22</b>
<b>6</b>	<b>Distance and sum processes</b>	<b>25</b>
6.1	Discrete transition probabilities . . . . .	28
6.2	The reflection principle and $\mathbf{q}_1 - \mathbf{q}_2$ . . . . .	29
6.3	Probability of $\mathbf{q}_0, \mathbf{q}_1$ and $\mathbf{q}_2$ . . . . .	31
<b>7</b>	<b>Stationary profile of particle systems out of equilibrium</b>	<b>35</b>
7.1	Expectation . . . . .	35
7.2	Covariance . . . . .	37
7.3	Variance . . . . .	39
7.4	Density field . . . . .	41
<b>8</b>	<b>The dynamics of the particle system</b>	<b>46</b>
8.1	Brownian motion and random walkers . . . . .	46
8.2	The expectation of the sites . . . . .	47
<b>9</b>	<b>Conclusion</b>	<b>49</b>
<b>A</b>	<b>Appendix</b>	<b>51</b>
A.1	Derivation of the exact formula for $\mathbf{q}_0$ . . . . .	51
A.2	Code . . . . .	57

# 1 Introduction

The world around us is made out of interacting particles. These particles differ in their size and behaviour, and their behaviour ultimately determine the observations that we make in the real world. However, describing these behaviours has been a tough task within mathematics due to the complexity that arises even from simple systems.

There have been attempts to describe the physical phenomena of the world through the behaviour of fundamental particles, with specifically Leucippus and Democritus talking about *atomos* in the 5th to 4th century BC. Ever since then, many philosophers and scientist have proposed their own models about the behaviour of these particles, including Newton, Leibniz and Einstein. The latter of which proposed a mathematical description of these particles which ultimately served as enough evidence to suggest that atoms exist.

In this thesis, we deal with interacting particle systems. These are Markov processes modeling the motion of particles in a lattice. Particles start interacting when they are close to each other, according to some stochastic rule. These models have been introduced originally in statistical mechanics where the primary interest was the derivation of the macroscopic laws of thermodynamics starting from the underlying macroscopic structure of the matter. The main interest is the understanding of the collective behaviour of a huge number of interacting particles, and in the derivation of the crucial macroscopic properties that do not depend on the microscopic details. Nowadays the interest for such systems goes beyond statistical mechanics, as their relevance in various research fields such as biology, ecology and social sciences, has come to light. Particles indeed might model atoms, molecules, polymers, dust grains, but also individuals in a population or economic agents. In this thesis I focus in particular on out-of-equilibrium systems. A system is out of equilibrium when it is put in contact with external reservoirs, which generate a current flow of particles throughout the system. The current pushes the particles from the reservoir at higher particle density towards the reservoir at lower density. Differently from the equilibrium systems, in the case of the non-equilibrium systems there are still many open questions. One of these is the understanding of the properties of the non-equilibrium stationary distributions, for which, with the exception of some particular models, there are no exact formulas known. This is one of the main interests in this thesis. We will study the non-equilibrium stationary distribution for a particular class of particles systems, that are particle systems with duality property. This class includes three models: the symmetric inclusion process (SIP) (introduced in [7]), the symmetric exclusion process (SEP) (introduced in [12]) and independent random walkers (IRW) see figure 1. Duality is a property relating a system with many particles with a system composed by a few particles.

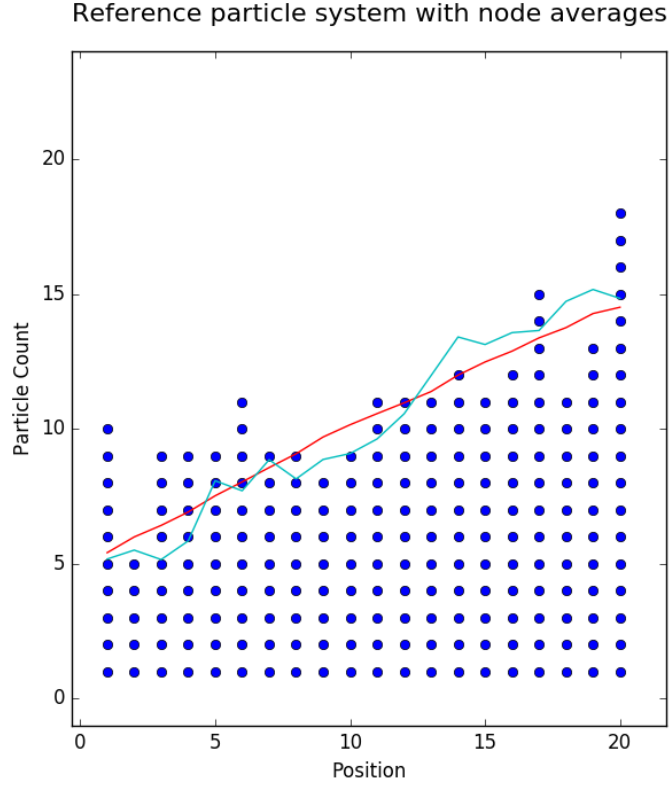


Figure 1: Visualization of the Independent Random Walkers through simulation. Every blue dot represents a particle that is located on one of the 20 sites within the interval. The red line approximates the average amount of particles at each site for the last 8000 time-steps. The blue line approximates the average amount of particles at each site for the last 100 time-steps.

We will make use of duality properties of these particle systems, which we use to describe the behaviour of complex particle systems, by first examining simpler systems. Specifically we will be looking at a simple system with one or two particles with absorbing boundaries. Using absorption probabilities from this simpler particle system, we will use duality to talk about the lower moments of stationary solutions of the more complex systems. We will also be examining limiting behaviours of the particle systems with respect to the space that they occupy as well as their limiting behaviour with respect to their attracting or repelling interaction. This leads us to the following question.

**What is the behaviour of the stationary distribution of the IRW, SIP and SEP out of equilibrium?**

To answer this question we will first go through the basic definitions and notations for probability theory and Markov Processes that we will use throughout this thesis in section 2. We will then introduce the particle generators for the particle systems out of equilibrium that we wish to analyze in section 3. After that we will talk about the stationary measures that have been found for these particle systems in equilibrium in section 4. We will then introduce simpler particle systems that we can analyze in order to study these complex systems by linking them with a property called duality in section 5. We will then analyze these simpler particle systems by in particular looking at a two particle system, and we will analyze the absorption probabilities for this system in section 6. We will then talk about the stationary distribution of the out of equilibrium particle systems by making use of the previous results in section 7.1, 7.2 and 7.3. We will examine the stationary distribution of the density field in section 7.4. We will briefly talk about the dynamics of the particle systems in section 8. We will then restate all of the findings in the conclusion in section 9. Finally we have a detailed proof for one of the absorption probabilities and the code for our simulation in the appendix.



## 2 Basic definitions and notations for probability theory and Markov Processes

In order to describe particle systems and their behaviour we will first need to describe the fundamental mathematics behind these models. In these sections we will first give some general definitions and theorems of probability theory and we will then talk about Markov Processes, which are fundamental to the way particle systems are defined.

### 2.1 Random variables

**Definition 2.1** (Random variable). A random variable  $X : \Omega \rightarrow D$  is a measurable function defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a measurable space  $(D, \mathcal{D})$ .

Next we wish to define the expectation of a random variable.

**Definition 2.2** (Expectation of a random variable). The expectation of a random variable  $X$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is defined by:

$$\mathbb{E}[X] := \int_{\Omega} X d\mathbb{P}$$

Where we take the Lebesgue integral of the variable  $X$  over the domain  $\Omega$  with measure  $\mathbb{P}$ .

The expectation can be viewed as average value you get from many iterations of a random variable  $X$ . If  $X$  were to be a fair coin, which is a coin which has an equal probability of heads or tails and is not influenced by previous throws. And  $X = 1$  would mean heads and  $X = 0$  would mean tails, then  $\mathbb{E}[X] = 0.5$ . In order to say something about how much a random variable deviates from its expectation we make use of the variance of a random variable.

**Definition 2.3** (Variance of a random variable). The variance of a random variable  $X$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is defined by:

$$\text{Var}[X] := \mathbb{E}[(X - \mathbb{E}[X])^2] = \int_{\Omega} (X - \mathbb{E}[X])^2 d\mathbb{P}$$

Next we make use of independence of two random variables.

**Definition 2.4** (Independence of two random variables). We say that two variables  $X, Y : \mathcal{F} \rightarrow D$  on probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a measurable space  $(D, \mathcal{D})$  are independent when  $\forall D_1, D_2 \in \mathcal{D}$  we have:

$$\mathbb{P}(X \in D_1, Y \in D_2) = \mathbb{P}(X \in D_1)P(Y \in D_2)$$

We will also make use of the covariance of two random variables. The covariance says something about how correlated two variables are. However, if two variables have 0 covariance, then this does not mean that they are independent.

**Definition 2.5** (Covariance of two random variables). The covariance of the random variables  $X, Y$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is defined by:

$$\text{Cov}[X, Y] := \mathbb{E} [(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \int_{\Omega} (X - \mathbb{E}[X])(Y - \mathbb{E}[Y]) d\mathbb{P}$$

In particular we have that  $\text{Cov}(X, X) = \text{Var}(X)$ .

## 2.2 Markov Processes

Particle systems are processes with random behaviour over time, in order to properly define them, we first define stochastic processes.

**Definition 2.6** (Stochastic process). A stochastic process is a collection of random variables  $\{X_t : t \geq 0\}$  defined on a common probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  taking values on the same measurable space  $(D, \mathcal{D})$ , which is named the state space. If time is discrete a stochastic process is a sequence  $\{X_n : n \in \mathbb{N}_0\}$ .

In this thesis we will use the following notation for Stochastic Processes with state space  $D$ :

$$\mathbb{P}_x(X_t = y) := \mathbb{P}(X_t = y | X_0 = x) \quad \forall x, y \in D \text{ and } t \geq 0$$

Let  $\mu$  denote a probability measure on  $\mathcal{D}$ , then we denote by  $\mathbb{E}_{\mu}[X_t]$  the expectation of a stochastic process  $X_t$  initialized from a value  $X_0$  distributed according to  $\mu$ .

In a similar manner we use the following notation for the conditional variance in which  $X_0$  is distributed according to  $\mu$ :

$$\text{Var}_{\mu}[X_t] := \mathbb{E}_{\mu} [(X_t - \mathbb{E}_{\mu}[X_t])^2]$$

We then also adapt the notation for the conditional covariance in which  $(X_0, Y_0)$  are distributed according to  $\mu$ :

$$\text{Cov}_{\mu}[X_t] := \mathbb{E}_{\mu} [(X - \mathbb{E}_{\mu}[X])(Y - \mathbb{E}_{\mu}[Y])]$$

Throughout this thesis we consider stochastic processes satisfying the so-called *Markov Property*. This property means that only the last known state of the system has any influence on the next unknown state. The formal definition is as follows:

**Definition 2.7** (Markov Property). Let  $\{X_n : n \in \mathbb{N}_0\}$  be a discrete-time stochastic process on a discrete state space  $D$ . We say that such a sequence satisfies the Markov Property if for all  $x_0, x_1, \dots, x_n \in D$ :

$$\begin{aligned} \mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2} \dots X_1 = x_1, X_0 = x_0) \\ = \mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1}) \end{aligned}$$

We can extend this definition to include processes which work on continuous time. We call a continuous stochastic process  $\{X_t, t \geq 0\}$  a continuous Markov Process if for all  $t_n > t_{n-1} > \dots > t_1 > t_0 \geq 0$  it holds that:

$$\mathbb{P}(X_{t_n} = x_n \mid X_{t_{n-1}} = x_{n-1}, \dots, X_{t_0} = x_0) = \mathbb{P}(X_{t_n} = x_n \mid X_{t_{n-1}} = x_{n-1})$$

A stochastic process with this property is called a *Markov Process*. We will also define *time-homogeneity*, which is the property for a stochastic process that the next unknown state is only influenced by how much time has passed since the previous known state. And that this unknown state is not dependent on the total amount of time that has passed since the start of the process.

**Definition 2.8** (Time-homogeneity). A stochastic process with state space  $D$  has the time-homogeneity property if  $\forall x, y \in D$ , and  $t > s$  with  $t, s \in \mathbb{N}_0$  (for discrete Markov Processes) or  $t, s \in \mathbb{R}_0$  (for continuous Markov Processes).

$$\mathbb{P}(X_t = x \mid X_s = y) = \mathbb{P}(X_{t-s} = x \mid X_0 = y)$$

For the rest of this thesis we will assume that all the Markov Processes have the time-homogeneity property.

We can then denote all of the possible transition probabilities in the following manner  $\forall x, y \in D$  and  $t \geq 0$ :

$$p_t(x, y) := \mathbb{P}(X_t = y \mid X_0 = x)$$

This defines the (possibly infinitely sized) transition matrices  $P_t$ . It turns out that for Markov Processes these matrices satisfy the *semigroup property*:

**Theorem 1.** *The transition matrices  $P_t$  of Markov Processes with state space  $D$  satisfy the semigroup property  $\forall t, u \geq 0$ :*

$$P_{t+u} = P_t P_u$$

Or expressed differently:

$$p_{t+u}(x, y) = \sum_{z \in D} p_t(z, y) p_u(x, z)$$

*Proof.* This property relies on the Markov property and time-homogeneity:

$$\begin{aligned}
p_{t+s}(x, y) &= \mathbb{P}(X_{t+s} = y | X_0 = x) \\
&= \frac{\mathbb{P}(X_{t+s} = y, X_0 = x)}{\mathbb{P}(X_0 = x)} \\
&= \sum_{z \in D} \frac{\mathbb{P}(X_{t+s} = y, X_s = z, X_0 = x)}{\mathbb{P}(X_0 = x)} = \\
&= \sum_{z \in D} \frac{\mathbb{P}(X_{t+s} = y | X_s = z, X_0 = x) \mathbb{P}(X_s = z, X_0 = x)}{\mathbb{P}(X_0 = x)} \\
&= \sum_{z \in D} \mathbb{P}(X_{t+s} = y | X_s = z, X_0 = x) \mathbb{P}(X_s = z | X_0 = x) \\
&\stackrel{\text{Markov property}}{=} \sum_{z \in D} \mathbb{P}(X_{t+s} = y | X_s = z) \mathbb{P}(X_s = z | X_0 = x) \\
&\stackrel{\text{Time-homogeneity}}{=} \sum_{z \in D} \mathbb{P}(X_t = y | X_0 = z) \mathbb{P}(X_s = z | X_0 = x) \\
&= \sum_{z \in D} p_t(z, y) p_s(x, z)
\end{aligned}$$

□

This property implies that there is a matrix  $L$  such that:

$$P_t = e^{Lt}$$

Where we define the exponential of a matrix by means of its Taylor series expansion around  $t = 0$ .

$$e^{Lt} := \sum_{n=0}^{\infty} \frac{(Lt)^n}{n!}$$

This matrix  $L$  is called the infinitesimal generator of a continuous-time Markov Process. And we can find its coefficients by calculating the time derivative of  $P_t$ .

$$\frac{dP_t}{dt} = Le^{tL} = LP_t$$

And evaluating it at time  $t = 0$ :

$$\left. \frac{dP_t}{dt} \right|_{t=0} = L$$

The non-diagonal elements of  $L$  represent the jump rates of the continuous-time Markov Process:

$$c(x, y) := L_{xy} \quad \text{for } x \neq y$$

The diagonal elements of this matrix are equal to the negative sum of the other elements on the same row; indeed:

$$p_t(x, x) + \sum_{\substack{y \in D \\ y \neq x}} p_t(x, y) = 1$$

Taking the derivatives with respect to time and evaluating them at  $t = 0$  gives us

$$L_{xx} + \sum_{\substack{y \in D \\ y \neq x}} c(x, y) = 0$$

thus

$$L_{xx} = - \sum_{\substack{y \in D \\ y \neq x}} c(x, y)$$

These rates define the entire process, as these define  $L$ , which in turn defines  $P_t$ , which holds all of the transition probabilities due to the Markov Property and time-homogeneity. One way of describing the matrix  $L$ , is by looking at how it acts on functions  $f : D \rightarrow \mathbb{R}$  in the following manner:

$$[Lf](x) = \sum_{\substack{y \in D \\ y \neq x}} c(x, y)(f(y) - f(x))$$

We will also define  $f_t(x) = [P_t f](x) = \mathbb{E}[f(X_t) | X_0 = x]$  this gives us:

$$\frac{df_t(x)}{dt} = Lf_t$$

$f_t(x)$  is called the time-evolution of  $f(x)$ .

### 3 Interacting particle systems

We can now define interacting particle systems in terms of continuous-time Markov processes.

**Definition 3.1** (Interacting particle system). An interacting particle system  $\{\eta(t), t \geq 0\}$  is a Markov Process modeling the interaction of particles moving in a lattice  $V \subseteq \mathbb{Z}^d$  and jumping among the sites of the lattice according to some stochastic rules. In this thesis we consider the one dimensional case  $d = 1$  where each  $\eta_t$  is defined on a measurable space  $(S^V, \mathcal{P}(S^V))$ . For which  $S \subseteq \mathbb{N}_0$  is a subset of the natural numbers called *the space of occupation numbers* and  $\mathcal{P}(S^V)$  is the power set of  $S^V$ .

#### 3.1 Standard Generator

We examine the interacting particle systems which are defined on the set  $V = \mathbb{Z}$  and for which  $S \subseteq \mathbb{N}_0$ . Then  $\forall \eta \in S^V$  we denote by  $\eta_i \in S$  the amount of particles at site  $i \in \mathbb{Z}$ .

We would like that our particle system only has interactions between neighbouring sites, we want particles to only be able to jump to the sites left and right of them. To describe these jumps we use a notation that describes the configuration in which a particle has jumped one spot to the left or right with respect to the former state  $\eta \in S^V$ . This way we can for all  $\eta \in S^V$  denote  $\eta^{i,j}$  to be the configuration obtained by moving a particle from site  $i$  to  $j$ .

$$\eta_k^{i,j} = \begin{cases} \eta_i - 1 & k = i \\ \eta_j + 1 & k = j \\ \eta_k & \text{otherwise} \end{cases}$$

We denote by  $c_+(\eta_i, \eta_{i+1})$  the rate of particles jumping from  $i$  to  $i+1$  and with  $c_-(\eta_{i+1}, \eta_i)$  the rate to describe a particle jumping from site  $i+1$  to  $i$ . This gives us the following generator:

$$[\mathcal{L}f](\eta) = \sum_{i \in \mathbb{Z}} \{c_+(\eta_i, \eta_{i+1})[f(\eta^{i,i+1}) - f(\eta^i)] + c_-(\eta_{i+1}, \eta_i)[f(\eta^{i+1,i}) - f(\eta^i)]\}$$

#### 3.2 The Reference Process

In this thesis we will study an interacting particle which has linear rates living on a finite lattice  $V_N = \{1, \dots, N\}$ . It is of the following form:

$$[\mathcal{L}_{int}^\theta f](\eta) = \frac{\alpha}{2} \sum_{i=1}^{N-1} \{\eta_i(1 + \theta\eta_{i+1})[f(\eta^{i,i+1}) - f(\eta^i)] + \eta_{i+1}(1 + \theta\eta_i)[f(\eta^{i+1,i}) - f(\eta^i)]\}$$

The reason that we choose our particle system to have these rates is because we can then make use of the duality property, which we will describe in section 5.

These generators have two free parameters:  $\alpha$  and  $\theta$ .  $\alpha$  determines how fast the process evolves. If  $\alpha$  is larger, then the process will evolve faster, and if it is smaller, the process will evolve slower.  $\alpha$  therefore serves as nothing more than a time-scaling parameter. The sign of  $\theta$  modules the nature of the interaction between particles. It relates to the ‘attractiveness’ that the particles have between each other. We divide these differing behaviours into 3 categories. *The Symmetric Inclusion Process (SIP)*, *the Symmetric Exclusion Process (SEP)* and *the Independent Random Walkers (IRW)*.

So far we have been examining particles systems in which the total amount of particles remains constant over time. We will now be examining particle systems in which the particles have reservoirs which interact with a finite lattice of length  $N$ . Our lattice  $V_N$  has  $N$  sites on which particles move  $V_N = \{1, 2, \dots, N-1, N\}$  and on the site 1 and  $N$ , particles will interact with ‘reservoirs’. On these sites, the particles can jump out of the finite lattice or jump in based on birth rates and death rates. In general the generator of our particle systems will be written in the following manner:

$$\mathcal{L}_R = \mathcal{L}_0 + \mathcal{L}_{int}^\theta + \mathcal{L}_{N+1}$$

Where

$$\begin{aligned} [\mathcal{L}_0 f](\eta) &= b(\eta_1)(f(\eta^{0,1}) - f(\eta)) + d(\eta_1)(f(\eta^{1,0}) - f(\eta)) \\ [\mathcal{L}_{N+1} f](\eta) &= \bar{b}(\eta_N)(f(\eta^{N+1,N}) - f(\eta)) + \bar{d}(\eta_N)(f(\eta^{N,N+1}) - f(\eta)) \end{aligned}$$

With  $b, d, \bar{b}, \bar{d}$  being functions from  $S \subseteq \mathbb{N}_0$  to  $\mathbb{R}$ . If we specifically choose  $b = \bar{b} = 0$  we will have absorbing boundaries. Particles will be able to jump out of the lattice, but not into it.

We can now discuss the 3 different particle systems in more detail.

**The Symmetric Inclusion Process SIP( $2k$ ).** If  $\theta > 0$  then when particles are adjacent on the lattice, the jump rate onto each other will increase. This will result in the particles performing attractive behaviour. If  $\theta$  gets larger, the likelihood of the particles separating once they are adjacent becomes lower. The space of occupation numbers is  $S = \mathbb{N}_0$ . A visualization of the rates can be seen in Figure 2. The inclusion process (without reservoirs) was defined first in [7], and examined further in [8]. This system is often described with a parameter  $k$ , with  $\theta = \frac{1}{2k}$  and  $\alpha = 4k$ .

The inclusion generator has the following properties for the reservoirs  $b(x) = \alpha(2k + x)$ ,  $d(x) = \gamma x$ ,  $\bar{b}(x) = \delta(2k + x)$ ,  $\bar{d}(x) = \beta x$ . All of this gives us the following generator:

$$\begin{aligned}
[\mathcal{L}_R^{SIP(2k)} f](\eta) &= \alpha(2k + \eta_1)(f(\eta^{0,1}) - f(\eta)) + \gamma\eta_1(f(\eta^{1,0}) - f(\eta)) \\
&+ \sum_{i=1}^{N-1} \{ \eta_i(2k + \eta_{i+1})[f(\eta^{i,i+1}) - f(\eta^i)] + \eta_{i+1}(2k + \eta_i)[f(\eta^{i+1,i}) - f(\eta^i)] \} \\
&+ \delta(2k + \eta_N)(f(\eta^{N+1,N}) - f(\eta)) + \beta\eta_N(f(\eta^{N,N+1}) - f(\eta))
\end{aligned}$$

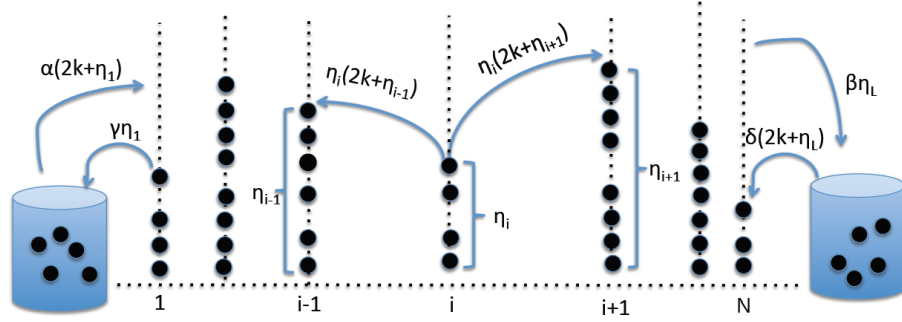


Figure 2: Figure from [6] showing the transitions and corresponding rates between the sites and the reservoirs of the Symmetric Inclusion Process (SIP(2k)).

This process, is generally defined with parameter  $2k$ , instead we will use the parameter  $\frac{1}{|\theta|}$ , with  $\theta > 0$ . We are interested in the cases when  $\gamma - \alpha = \beta - \delta = \frac{1}{|\theta|}$ . When we make use of these constraints we get the following equality:

$$\begin{aligned}
[\theta \mathcal{L}_R^{SIP(\frac{1}{\theta})} f](\eta) &= \alpha(1 + \theta\eta_1)(f(\eta^{0,1}) - f(\eta)) + (1 + \alpha\theta)\eta_1(f(\eta^{1,0}) - f(\eta)) \\
&+ \sum_{i=1}^{N-1} \{ \eta_i(1 + \theta\eta_{i+1})[f(\eta^{i,i+1}) - f(\eta^i)] + \eta_{i+1}(1 + \theta\eta_i)[f(\eta^{i+1,i}) - f(\eta^i)] \} \\
&+ \delta(1 + \theta\eta_N)(f(\eta^{N+1,N}) - f(\eta)) + (1 + \delta\theta)\eta_N(f(\eta^{N,N+1}) - f(\eta))
\end{aligned} \tag{1}$$

**The Symmetric Exclusion Process SEP(2j).** If  $\theta < 0$  and  $\frac{1}{|\theta|} \in \mathbb{N}$  then the particles will repel each other. If for example  $\theta = -1$  then each site can at most hold 1 particle, because as soon as a second particle is adjacent to this one, the rate becomes 0. A schematic overview of the processes can be seen in Figure 3. In general, with  $\frac{1}{|\theta|}$  being a natural number, each  $\eta_i$  takes values in  $S = \{1, 2, \dots, \frac{1}{|\theta|}\}$ . The case  $\theta = -1$  has been studied by H. Spohn [14], after which the general model was introduced by G. Schütz and S. Sandow [12]. This process is generally defined with parameter  $2j$ .



The reservoirs are defined by the following rates  $b(x) = \alpha(2k-x)$ ,  $d(x) = \gamma x$ ,  $\bar{b}(x) = \delta(2k-x)$ ,  $\bar{d}(x) = \beta x$ . Which gives the following generator:

$$\begin{aligned} [\mathcal{L}_R^{SEP(2j)} f](\eta) &= \alpha(2j - \eta_1)(f(\eta^{0,1}) - f(\eta)) + \gamma\eta_1(f(\eta^{1,0}) - f(\eta)) \\ &+ \sum_{i=1}^{N-1} \{ \eta_i(2j - \eta_{i+1})[f(\eta^{i,i+1}) - f(\eta^i)] + \eta_{i+1}(2j - \eta_i)[f(\eta^{i+1,i}) - f(\eta^i)] \} \\ &+ \delta(2j - \eta_N)(f(\eta^{N+1,N}) - f(\eta)) + \beta\eta_N(f(\eta^{N,N+1}) - f(\eta)) \end{aligned}$$

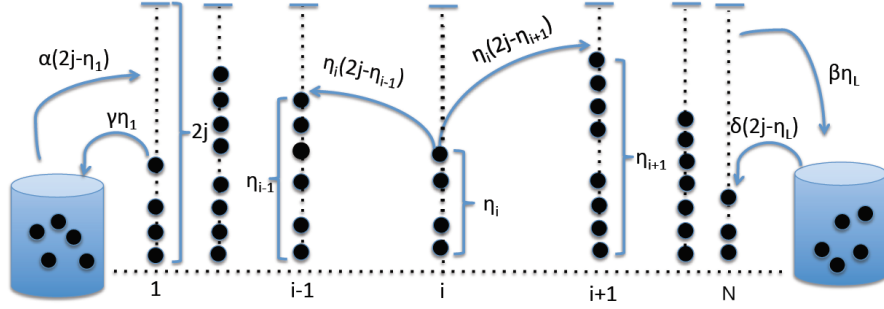


Figure 3: Figure from [6] showing the transitions and corresponding rates between the sites and the reservoirs of the Symmetric Exclusion Process (SEP(2j)).

We will yet again replace this parameter with  $\frac{1}{|\theta|}$ , with  $\theta < 0$  and we are interested in the cases when we take  $\gamma + \alpha = \beta + \delta = \frac{1}{|\theta|}$ . We will rewrite the particle system into the following form:

$$\begin{aligned} [|\theta| \mathcal{L}_R^{SEP(\frac{1}{|\theta|})} f](\eta) &= \alpha(1 + \theta\eta_1)(f(\eta^{0,1}) - f(\eta)) + (1 + \alpha\theta)\eta_1(f(\eta^{1,0}) - f(\eta)) \\ &+ \sum_{i=1}^{N-1} \{ \eta_i(1 + \theta\eta_{i+1})[f(\eta^{i,i+1}) - f(\eta^i)] + \eta_{i+1}(1 + \theta\eta_i)[f(\eta^{i+1,i}) - f(\eta^i)] \} \\ &+ \delta(1 + \theta\eta_N)(f(\eta^{N+1,N}) - f(\eta)) + (1 + \delta\theta)\eta_N(f(\eta^{N,N+1}) - f(\eta)) \end{aligned} \quad (2)$$

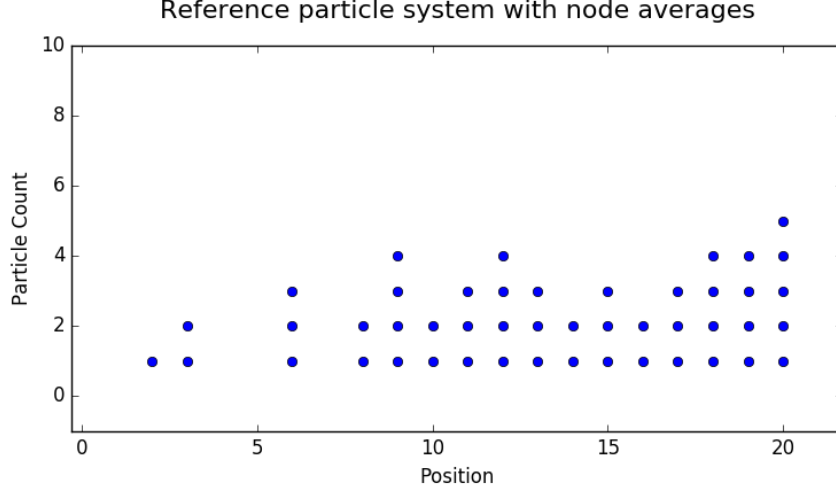


Figure 4: Visualization of Symmetric Exclusion Process with  $N = 20$ ,  $\theta = \frac{1}{5}$ ,  $\alpha = 0$  and  $\delta = 5$  through simulation. Every blue dot represents a particle that is located on a site in  $V_N$  with the x-axis representing the position of this site. Note that there are no sites with more than 5 particles.

**Independent Random Walkers IRW.** If  $\theta = 0$ , then the particles in the system become independent random walkers. The state space where  $\eta_i$  takes values is  $S = \mathbb{N}_0$ . This model was first researched by F. Spitzer [13], after which the system was investigated with boundary conditions by E. Levine et al. [3]. The process that we are interested in has the following boundary conditions  $b(x) = \alpha$ ,  $d(x) = \gamma x$ ,  $\bar{b}(x) = \delta$ ,  $\bar{d}(x) = \beta x$ .

$$\begin{aligned} [\mathcal{L}_R^{IRW} f](\eta) &= \alpha(f(\eta^{0,1}) - f(\eta)) + \gamma\eta_1(f(\eta^{1,0}) - f(\eta)) \\ &+ \sum_{i=1}^{N-1} \{\eta_i[f(\eta^{i,i+1}) - f(\eta^i)] + \eta_{i+1}[f(\eta^{i+1,i}) - f(\eta^i)]\} \\ &+ \delta(f(\eta^{N+1,N}) - f(\eta)) + \beta\eta_N(f(\eta^{N,N+1}) - f(\eta)) \end{aligned}$$

For the independent random walker process we are interested in the cases when  $\gamma = \beta = 1$  in this case the generator take the following form:

$$\begin{aligned} [\mathcal{L}_R^{IRW} f](\eta) &= \alpha(f(\eta^{0,1}) - f(\eta)) + \eta_1(f(\eta^{1,0}) - f(\eta)) \\ &+ \sum_{i=1}^{N-1} \{\eta_i[f(\eta^{i,i+1}) - f(\eta^i)] + \eta_{i+1}[f(\eta^{i+1,i}) - f(\eta^i)]\} \\ &+ \delta(f(\eta^{N+1,N}) - f(\eta)) + \eta_N(f(\eta^{N,N+1}) - f(\eta)) \end{aligned} \quad (3)$$

**Reference generator.** What we now notice by examining equation 1, 2 and 3 is that all of these generators reduce to the same form depending on a parameter  $\theta$ . We can describe the above three cases using the reference generator in the following manner:

$$[\mathcal{L}_R^{REF(\theta)}](\eta) = \left\{ \begin{array}{ll} [\mathcal{L}_R^{IRW}](\eta) & \theta = 0, \eta \in \mathbb{N}^{\{1,2,\dots,N\}}, \gamma = \beta = 1 \\ [\theta \mathcal{L}_R^{SIP(\frac{1}{\theta})}](\eta) & \theta > 0, \eta \in \mathbb{N}^{\{1,2,\dots,N\}}, \gamma - \alpha = \delta - \beta = \frac{1}{\theta} \\ [|\theta| \mathcal{L}_R^{SEP(\frac{1}{|\theta|})}](\eta) & \theta < 0, \frac{1}{|\theta|} \in \mathbb{N}, \eta \in \{1, 2, \dots, \frac{1}{|\theta|}\}^{\{1,2,\dots,N\}}, \gamma + \alpha = \delta + \beta = \frac{1}{|\theta|} \end{array} \right\} \quad (4)$$

The reservoirs on the sides of the interval control the average or expected amount of particles at these sides. We will call these average amounts, the reservoir densities of the particle systems. With  $\rho_a$  being the expected amount of particles at the left reservoir and  $\rho_b$  being the expected amount of particles in the right reservoir.

System	$\rho_a$	$\rho_b$
SIP	$2k \frac{\alpha}{\gamma - \alpha}$	$2k \frac{\delta}{\beta - \delta}$
SEP	$2j \frac{\alpha}{\gamma + \alpha}$	$2j \frac{\delta}{\beta + \delta}$
IRW	$\frac{\alpha}{\gamma}$	$\frac{\delta}{\beta}$

Table 1: Definition of  $\rho_a$  and  $\rho_b$ .

Note that the SIP process requires  $\gamma > \alpha$  and  $\beta > \delta$ . This condition turns out to be necessary in order for the system to reach a stationary state. Under the restrictions imposed by our reference generator (equation 4) we have that  $\rho_a = \alpha$  and  $\rho_b = \delta$ . We would now like to examine the limiting distributions of the particle systems with respect to these reservoir densities.

## 4 Stationary measures of the interacting particle systems at equilibrium

As  $t \rightarrow \infty$ , the distribution of particles will converge to a stationary distribution. Stationary measures are known for the particular case where the system is kept at equilibrium. This means that the boundary densities imposed by the reservoirs are the same:  $\rho_a = \rho_b$ . These measures are also called reversible measures, as they have been found by imposing the detailed balance condition for which it is required that for the generators  $\mathcal{L}$ , with configurations  $\eta, \eta' \in \Omega$  and probability measure  $\mathbb{P}$ :

$$\mathcal{L}(\eta, \eta')\mathbb{P}(\eta) = \mathcal{L}(\eta', \eta)\mathbb{P}(\eta')$$

For the generators defined in section 3, we have the following stationary measures:

### Symmetric Inclusion Process SIP(2k)

The process with generator  $\mathcal{L}_R^{SIP(2k)}$  with parameters  $\alpha, \gamma$  and  $\beta, \delta$  such that  $\alpha\beta - \gamma\delta = 0$ , has a reversible stationary measure given by products of generalized *Negative Binomial* measures with parameters  $2k > 0$ ,  $p = \frac{\alpha}{\gamma} = \frac{\delta}{\beta}$ ,  $\alpha < \gamma$  and  $\delta < \beta$ .

$$\mathbb{P}(\eta) = \prod_{i=1}^N \frac{p^{\eta_i}}{\eta_i!} \frac{\Gamma(2k + \eta_i)}{\Gamma(2k)} (1-p)^{2k}.$$

### Symmetric Exclusion Process SEP(2j)

The process with generator  $\mathcal{L}_R^{SEP(2j)}$  with parameters  $\alpha, \gamma$  and  $\beta, \delta$  such that  $\alpha\beta - \gamma\delta = 0$ , has a reversible stationary measure given by products of *Binomial* measures with parameters  $2j \in \mathbb{N}$  and  $p = \frac{\alpha}{\gamma + \alpha} = \frac{\delta}{\beta + \delta}$

$$\mathbb{P}(\eta) = \prod_{i=1}^N \frac{\left(\frac{p}{1-p}\right)^{\eta_i}}{\eta_i!} \frac{\Gamma(2j + 1)}{\Gamma(2j + 1 - \eta_i)} (1-p)^{2j}.$$

### Independent Random Walkers IRW

The process with generator  $\mathcal{L}_R^{IRW}$  with  $\frac{\alpha}{\gamma} = \frac{\delta}{\beta}$  has a reversible stationary measure given by products of *Poisson* distributions with parameter  $\lambda := \frac{\alpha}{\gamma} = \frac{\delta}{\beta}$  i.e.

$$\mathbb{P}(\eta) = \prod_{i=1}^N \frac{\lambda^{\eta_i}}{\eta_i!} e^{-\lambda}.$$

All of these stationary measures are for particle systems which are in equilibrium, which means that  $\rho_a = \rho_b$ . However, when we try to examine systems

out of equilibrium in which  $\rho_a \neq \rho_b$ . We lose the product structure of the stationary measure, and everything becomes much more complicated. We can see an example of the stationary distribution in equilibrium in Figure 5.

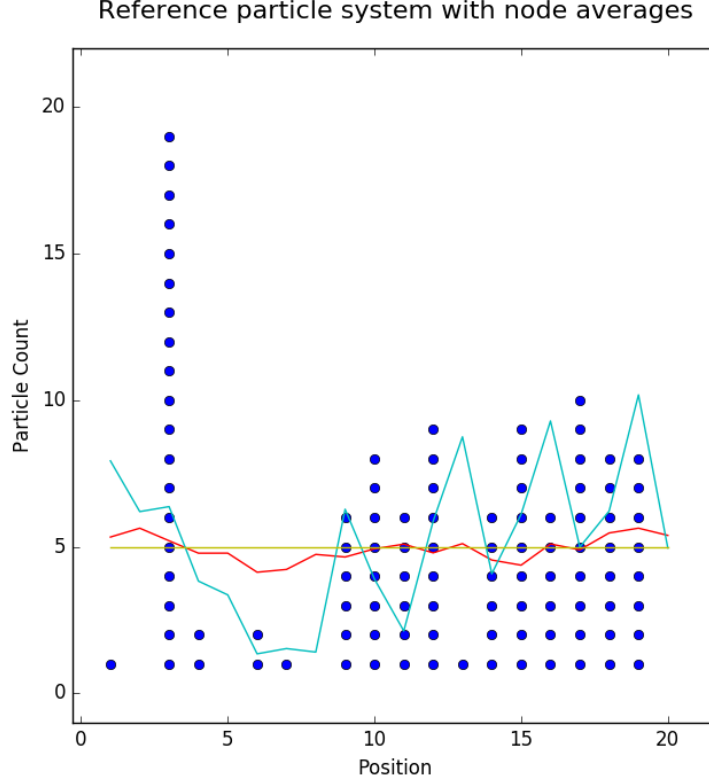


Figure 5: Visualization of the stationary distribution of the Symmetric Inclusion Process in equilibrium with  $N = 20$ ,  $\theta = 0.5$ ,  $\alpha = 5$  and  $\delta = 5$  through simulation. Every blue dot represents a particle that is located on a site in  $V_N$  with the x-axis representing the position of this site. The yellow line is the analytically calculated expectation for the stationary profile. The red line is an approximation for the expected amount of particles by averaging over the last 8000 timesteps. The blue line is an approximation for the expected amount of particles by averaging over the last 100 timesteps. We can see the expected amount of particles is the same for all sites, as the distribution of particles is the same at each site.

At the moment there are only exact results for the out of equilibrium particle

systems in two cases:  $\theta = 0$  and  $\theta = -1$ . For the  $\theta = 0$  case, the product structure still holds, this case has been studied by L. Bertini et al. [10]. The  $\theta = -1$  case (where there can be at most 1 particle at each site) has been studied by B. Derrida et al. [1], where a matrix product solution was found. In order to say something about the distribution of these stationary measures for particle systems out of equilibrium we will make use of a duality.

## 5 Duality

It can be very difficult to say something about the moments of complex particle systems with many particles. Even more in the non-equilibrium case where we do not have a reversible measure. Instead, we would like to reduce complex systems to simpler ones and link properties between the two. One way of setting up such a link is through duality of Markov Processes.

**Definition 5.1** (Duality between Markov Processes). We say that two Markov Processes  $\eta_t$  and  $\xi_t$  are dual with a duality function  $D(\eta, \xi) : \Omega \times \Omega' \rightarrow \mathbb{R}$  if for all of the starting configurations  $\eta \in \Omega$  and  $\xi \in \Omega'$  and for all  $t \geq 0$  we have the following property:

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t))$$

There are several duality functions that have been found. The following duality relationships are from C. Giardinà et al. [2].

**Symmetric Inclusion Process (SIP(2k)).** We have a duality function that links an inclusion particle system with reservoirs as described in formula (1) with generators of the form:

$$\begin{aligned} [\mathcal{L}_{Dual}^{SIP(2k)} f](\xi) &= (\gamma - \alpha)\xi_1(f(\xi^{1,0}) - f(\xi)) \\ &+ \sum_{i=1}^{N-1} \{\xi_i(2k + \xi_{i+1})[f(\xi^{i,i+1}) - f(\xi^i)] + \xi_{i+1}(2k + \xi_i)[f(\xi^{i+1,i}) - f(\xi^i)]\} \\ &+ (\beta - \delta)\xi_N(f(\xi^{N,N+1}) - f(\xi)) \end{aligned}$$

with duality function:

$$D^{SIP(2k)}(\eta, \xi) = \left(\frac{\alpha}{\gamma - \alpha}\right)^{\xi_0} \prod_{i=1}^N \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma(2k)}{\Gamma(2k + \xi_i)} \left(\frac{\delta}{\beta - \delta}\right)^{\xi_{N+1}}$$

**Symmetric Exclusion Process (SEP(2k)).** We also have a duality function for the exclusion process as described in formula (2) and generators of the form:

$$\begin{aligned} [\mathcal{L}_{Dual}^{SEP(2j)} f](\xi) &= (\gamma + \alpha)\xi_1(f(\xi^{1,0}) - f(\xi)) \\ &+ \sum_{i=1}^{N-1} \{\xi_i(2j - \xi_{i+1})[f(\xi^{i,i+1}) - f(\xi^i)] + \xi_{i+1}(2j - \xi_i)[f(\xi^{i+1,i}) - f(\xi^i)]\} \\ &+ (\beta + \delta)\xi_N(f(\xi^{N,N+1}) - f(\xi)) \end{aligned}$$

with duality function:

$$D^{SEP(2j)}(\eta, \xi) = \left(\frac{\alpha}{\gamma + \alpha}\right)^{\xi_0} \prod_{i=1}^N \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma(2j + 1 - \xi_i)}{\Gamma(2j + 1)} \left(\frac{\delta}{\beta + \delta}\right)^{\xi_{N+1}}$$

**Independent Random Walkers (IRW).** Finally we also have a duality function for the independent random walkers as described in formula (3) and generators of the form:

$$\begin{aligned} [\mathcal{L}_{Dual}^{IRW} f](\xi) &= \gamma \xi_1 (f(\xi^{1,0}) - f(\xi)) \\ &+ \sum_{i=1}^{N-1} \{ \xi_i [f(\xi^{i,i+1}) - f(\xi^i)] + \xi_{i+1} [f(\xi^{i+1,i}) - f(\xi^i)] \} \\ &+ \beta \xi_N (f(\xi^{N,N+1}) - f(\xi)) \end{aligned}$$

with duality function:

$$D^{IRW}(\eta, \xi) = \left(\frac{\alpha}{\gamma}\right)^{\xi_0} \prod_{i=1}^N \frac{\eta_i!}{(\eta_i - \xi_i)!} \left(\frac{\delta}{\beta}\right)^{\xi_{N+1}}$$

We notice that when we use the constraints for  $\gamma, \alpha, \beta, \delta$  from equation 1, 2 and 3, that all of these dual processes have the same form as the reference generator in equation 4.

$$[\mathcal{L}_{Dual}^{REF(\theta)}](\eta) = \left\{ \begin{array}{ll} [\mathcal{L}_{Dual}^{IRW}](\eta) & \theta = 0, \eta \in \mathbb{N}^{\{1,2,\dots,N\}}, \gamma = \beta = 1 \\ [\theta \mathcal{L}_{Dual}^{SIP(\frac{1}{\theta})}](\eta) & \theta > 0, \eta \in \mathbb{N}^{\{1,2,\dots,N\}}, \gamma - \alpha = \delta - \beta = \frac{1}{\theta} \\ [|\theta| \mathcal{L}_{Dual}^{SEP(\frac{1}{|\theta|})}](\eta) & \theta < 0, \frac{1}{|\theta|} \in \mathbb{N}, \eta \in \{1, 2, \dots, \frac{1}{|\theta|}\}^{\{1,2,\dots,N\}}, \gamma + \alpha = \delta + \beta = \frac{1}{|\theta|} \end{array} \right\}$$

And in a similar way we can characterize all of the duality functions:

$$D^\theta(\eta, \xi) = \left\{ \begin{array}{ll} D^{IRW}(\eta, \xi) & \theta = 0, \eta, \xi \in \mathbb{N}^{\{1,2,\dots,N\}}, \gamma = \beta = 1 \\ D^{SIP(\frac{1}{\theta})}(\eta, \xi) & \theta > 0, \eta, \xi \in \mathbb{N}^{\{1,2,\dots,N\}}, \gamma - \alpha = \delta - \beta = \frac{1}{\theta} \\ D^{SEP(\frac{1}{|\theta|})}(\eta, \xi) & \theta < 0, \frac{1}{|\theta|} \in \mathbb{N}, \eta \in \{1, 2, \dots, \frac{1}{|\theta|}\}^{\{1,2,\dots,N\}}, \gamma + \alpha = \delta + \beta = \frac{1}{|\theta|} \end{array} \right\}$$

We can now make use of the duality function to extract information about expected number of particles in the sites of the particle systems. For example if we take  $x \in V_N$  and  $\xi = \delta_x$ .  $\xi$  is now a configuration with a single particle at site  $x$ . We can then get the first moment of site  $x$  in  $\eta$ :

$$\eta_x = \frac{1}{|\theta + 1_{\theta=0}|} D^\theta(\eta, \delta_x)$$

Note that  $\xi$  now has 1 particle and that  $\eta$  can have an arbitrary amount of particles.

In order to examine higher moments we take  $x, y \in V_n$ ,  $x \neq y$  and  $\xi = \delta_x + \delta_y$ .  $\xi$  is now a configuration with two particles at two different sites  $x$  and  $y$ . We can get the second moments for differing sites in  $\eta$ :



$$\eta_x \eta_y = \frac{1}{(\theta + 1_{\theta=0})^2} D^\theta(\eta, \delta_x + \delta_y)$$

Finally we can take  $x \in V_N$  and  $\xi = 2\delta_x$ .  $\xi$  is now a configuration with two particles at site  $x$ . We can then get the second moment of a single site  $x$  in  $\eta$ :

$$D^\theta(\eta, 2\delta_x) = \frac{(\theta + 1_{\theta=0})^2 \eta_x (\eta_x - 1)}{\theta + 1}$$

and thus

$$\eta_x^2 = \eta_x + \frac{\theta + 1}{(\theta + 1_{\theta=0})^2} D^\theta(\eta, 2\delta_x)$$

In the next section we will examine the dual process in order to then use that information for examining the moments for more complex systems like  $\eta$  with an arbitrary amount of particles.

## 6 Distance and sum processes

We now examine the following generator, which is the dual process to the processes described in section 3, as described in section 5.

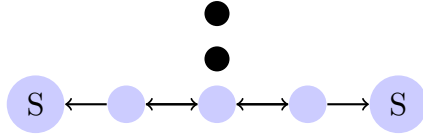
$$[\mathcal{L}_{Dual}^{REF(\theta)} f](\xi) = \xi_1(f(\xi^{1,0}) - f(\xi)) + [\mathcal{L}_{int}^\theta f] + \xi_N(f(\xi^{N,N+1}) - f(\xi))$$

This process is a special case of the reference process with  $\alpha = 2$ ,  $N$  internal points and absorbing boundaries with the death rate being equal to  $d(x) = \bar{d}(x) = x$ . We want to examine this system with two particles in such a way that we can get information about the second moments of the original process  $\eta$ . Instead of examining the position coordinates processes of the two particles  $(x(t), y(t))$  we will instead examine the distance and sum processes of the particles  $(w(t), u(t))$ .

$$\begin{aligned} w(t) &= |x(t) - y(t)| \\ u(t) &= x(t) + y(t) - (N + 1) \end{aligned}$$

With  $x(t)$  being the position of the leftmost particle and  $y(t)$  being the position of the rightmost particle. The jump rates in terms of the distance process and sum process breaks down into three cases. The numbers above the arrows ( $\overset{\cdot}{\rightarrow}$ ) indicate the rates for these transitions

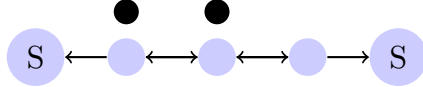
Case 1, two particles at the same node  $w = 0$ :



In this case each particle can jump to the right or the left with an equal rate 1. So

$$\begin{aligned} (w, u) &\overset{2}{\rightarrow} (w + 1, u + 1) && \text{a particle jumps to the right} \\ (w, u) &\overset{2}{\rightarrow} (w + 1, u - 1) && \text{a particle jumps to the left} \end{aligned}$$

Case 2, two particles next to each other  $w = 1$ :

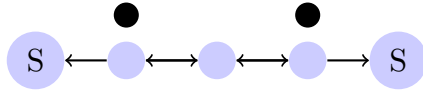


In this case the rate of jumping onto the other particle is slightly different than

them jumping away.

$$\begin{array}{ll}
(w, u) \xrightarrow{1} (w+1, u-1) & \text{left particle jumps to the left} \\
(w, u) \xrightarrow{1} (w+1, u+1) & \text{right particle jumps to the right} \\
(w, u) \xrightarrow{1+\theta} (w-1, u+1) & \text{left particle jumps to the right} \\
(w, u) \xrightarrow{1+\theta} (w-1, u-1) & \text{right particle jumps to the left}
\end{array}$$

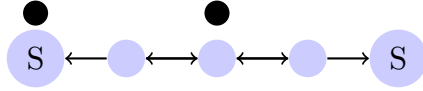
Case 3, two particles far away from each other  $w \geq 2$ :



In this case the particles move independent from each other with equal rate.

$$\begin{array}{ll}
(w, u) \xrightarrow{1} (w+1, u-1) & \text{left particle jumps to the left} \\
(w, u) \xrightarrow{1} (w+1, u+1) & \text{right particle jumps to the right} \\
(w, u) \xrightarrow{1} (w-1, u+1) & \text{left particle jumps to the right} \\
(w, u) \xrightarrow{1} (w-1, u-1) & \text{right particle jumps to the left}
\end{array}$$

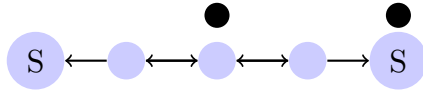
Case 4, a particle is stuck in the left boundary:



In this case the left particle can not move and the right one moves independently.

$$\begin{array}{ll}
(w, u) \xrightarrow{1} (w+1, u+1) & \text{right particle jumps to the right} \\
(w, u) \xrightarrow{1} (w-1, u-1) & \text{right particle jumps to the left}
\end{array}$$

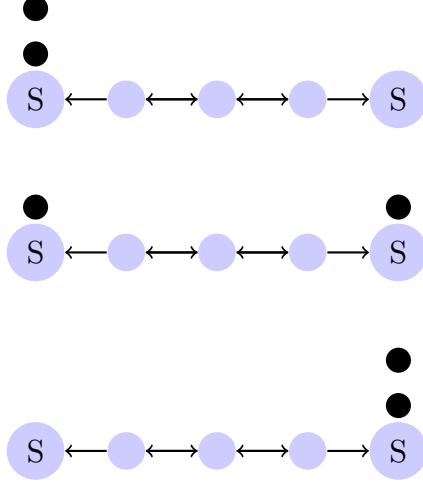
Case 5, a particle is stuck in the right boundary:



In this case the right particle can not move and the left one moves independently.

$$\begin{array}{ll}
(w, u) \xrightarrow{1} (w+1, u-1) & \text{left particle jumps to the left} \\
(w, u) \xrightarrow{1} (w-1, u+1) & \text{left particle jumps to the right}
\end{array}$$

Case 6, the particles are both stuck in the boundaries:



In this case none of the particles can move and all of the rates are zero

The coordinates are defined on a space  $\Lambda_N$ , this forms a triangle, as can be seen in Figure 6.  $\Lambda_N$  is defined in the following manner:

$$\Lambda_N := \begin{cases} \{(w, u) \in \mathbb{N}_0 \times \mathbb{Z} : w = u \bmod 2, 0 \leq w \leq N + 1 - |u|\} & \text{if } N \bmod 2 = 0 \\ \{(w, u) \in \mathbb{N}_0 \times \mathbb{Z} : w = u + 1 \bmod 2, 0 \leq w \leq N + 1 - |u|\} & \text{if } N \bmod 2 = 1 \end{cases}$$

We will denote all the configuration on this axis  $\{u = 0\}$  by  $\Lambda_0 = \{(w, u) \in \Lambda_N : u = 0\}$ . All the configurations on the right of this axis by  $\Lambda_+ = \{(w, u) \in \Lambda_N : u > 0\}$  and all of the configurations on the left of this axis by  $\Lambda_- = \{(w, u) \in \Lambda_N : u < 0\}$ .

We can now write down the generator for the sum and distance process  $(w(t), u(t))$ .

$$[\mathcal{L}_{dual}^{(sum, dist)} f](w, u) = \left\{ \begin{array}{ll} f(w+1, u-1) - 2f(w, u) + f(w-1, u+1), & w = (N+1) - |u| \\ & \text{and } (w \neq 0 \text{ or } w \neq N+1) \\ 0, & (w, u) \in S \\ 2(f(w+1, u-1) - 2f(w, u) + f(w+1, u+1)), & w = 0 \text{ and } w \neq (N+1) - |u| \\ (1+\theta)(f(w-1, u-1) + f(w-1, u+1)) + \\ f(w+1, u-1) + f(w+1, u+1) - (4+2\theta)f(w, u), & w = 1 \text{ and } w \neq (N+1) - |u| \\ f(w-1, u-1) + f(w+1, u-1) + \\ f(w-1, u+1) + f(w+1, u+1) - 4f(w, u) & w = 2 \text{ and } w \neq (N+1) - |u| \end{array} \right\}$$

Where we have  $S = S_0 \cup S_1 \cup S_2$ .

$S_0 \in \Lambda_N$  represents the state in which one particle is in the left reservoir and the other one is in the right one, and corresponds with  $w = N + 1$  and  $u = 0$ .

$S_1 \in \Lambda_N$  represents the state in which both particles are stuck in the right reservoir and corresponds with  $w = 0$  and  $u = N + 1$ .

$S_2 \in \Lambda_N$  represents the one for which both particles are in the left reservoir and  $w = 0$  and  $u = -(N + 1)$ .

Putting all of these cases together, we can examine the full transition graph:

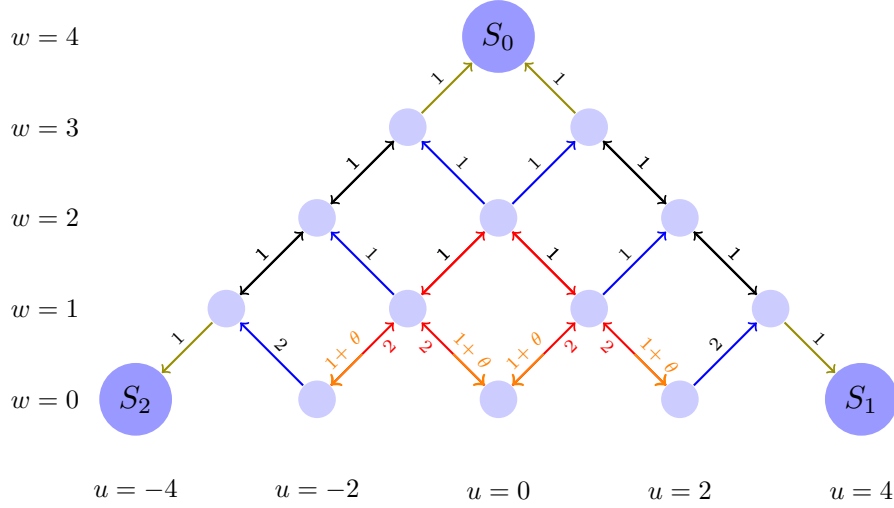


Figure 6: Transition graph with rates for the distance and sum process of the two particle dual process with  $N=3$

We can see that the process gets stuck on the line  $w = u + (N + 1)$  and  $w = -u + (N + 1)$ . And we can also see the 3 absorbing states.  $S_0, S_1$  and  $S_2$ .

Our aim is to compute the probabilities for each of the sinks to eventually be reached. Because there is no time dependence on these probabilities we will examine the corresponding discrete-time process (or Skeleton Process).

## 6.1 Discrete transition probabilities

We now convert these continuous probability processes into discrete ones using the following notation:

$$P((w, u), (w', u')) = P(w_{n+1} = w', u_{n+1} = u' | w_n = w, u_n = u)$$

It can now be seen that

for  $(w = u + (N + 1) \text{ and } (w \neq 0 \text{ or } w \neq N + 1))$

$$P((w, u), (w + 1, u + 1)) = P((w, u), (w - 1, u - 1)) = \frac{1}{2}$$

for  $(w = -u + (N + 1) \text{ and } (w \neq 0 \text{ or } w \neq N + 1))$

$$P((w, u), (w - 1, u + 1)) = P((w, u), (w + 1, u - 1)) = \frac{1}{2}$$

for  $(w = u + (N + 1) \text{ or } w = -u + (N + 1)) \text{ and } (w = 0 \text{ or } w = N + 1)$

$$P((w, u), (w, u)) = 1$$

for  $w = 0 \text{ and } w \neq (N + 1) - |u|$

$$P((0, u), (1, u + 1)) = P((0, u), (1, u - 1)) = \frac{1}{2}$$

for  $w = 1 \text{ and } w \neq (N + 1) - |u|$

$$P((1, u), (2, u + 1)) = P((1, u), (2, u - 1)) = \frac{1}{4 + 2\theta}$$

$$P((1, u), (0, u + 1)) = P((1, u), (0, u - 1)) = \frac{1 + \theta}{4 + 2\theta}$$

for  $w \geq 2 \text{ and } w \neq (N + 1) - |u|$

$$P((w, u), (w + 1, u + 1)) = P((w, u), (w + 1, u - 1)) =$$

$$P((w, u), (w - 1, u + 1)) = P((w, u), (w - 1, u - 1)) = \frac{1}{4}$$

We wish to determine the probability for the two particles to be absorbed in each of the absorbing states. We denote the probability of going into  $S_0$  by  $q_0(w, u)$  (with  $(w, u) \in \Lambda_N$ ) and for  $S_1, S_2$  by  $q_1(w, u), q_2(w, u)$  respectively. We have three unknowns, so we need to get three constraints in order to solve these unknowns. The first constraint is obvious as eventually one of the three states must be reached so  $q_0(w, u) + q_1(w, u) + q_2(w, u) = 1$ .

## 6.2 The reflection principle and $q_1 - q_2$

The second constraint comes from the symmetric properties in the transition graph. This gives us a relation between the probability of  $q_1(w, u)$  and  $q_2(w, u)$ . We were able to verify this relation with the recurrence relations which were found by G. Carinic et al [5].

**Theorem 2.** *For the skeleton process defined in section 6.1 the following holds for any  $(w, u) \in \Lambda_N$ :*

$$q_1(w, u) - q_2(w, u) = \frac{u}{N + 1}$$

*Proof.* With abuse of notation we will use  $\xi$  to denote the respective state in the coordinates  $(w, u)$ .

We first note a symmetry in the transition graph. If we start out with a state

$\xi = (w, u) \in \Lambda_+$  and compare it with the reflected state  $R\xi = (w, -u) \in \Lambda_-$  along the axis  $\Lambda_0$ , the odds of reaching  $S_1$  and  $S_2$  are reversed and the odds of reaching  $S_0$  remains the same.

We denote by  $\tau$  the first hitting time of the axis  $\Lambda_0$ . We then note that the odds of getting absorbed in state  $S_1$  can be written as:

$$q_1(\xi) = P_\xi(\xi(\infty) = S_1) = P_\xi(\xi(\infty) = S_1, \tau = \infty) + P_\xi(\xi(\infty) = S_1, \tau < \infty) \quad (5)$$

For the reflected state  $R\xi$  to reach  $S_1$ , it must pass through  $\Lambda_0$ , so  $\tau < \infty$  for all of

$$q_2(\xi) = P_\xi(\xi(\infty) = S_2) = P_{R\xi}(\xi(\infty) = S_1, \tau < \infty) \quad (6)$$

**Reflection principle.** We note that the probability for  $\tau$  to be finite is the same for  $\xi$  and  $R\xi$  because of the symmetry of the transition graph along  $\Lambda_0$ . The probability distribution along the axis  $\Lambda_0$  is the same because of this symmetry. And thus if we know that we have reached this axis  $\Lambda_0$ , we now know that the probability distribution of hitting the sinks is the same for  $\xi$  and  $R\xi$ , we now have the following:

$$P_{R\xi}(\xi(\infty) = S_1, \tau < \infty) = P_\xi(\xi(\infty) = S_1, \tau < \infty) \quad (7)$$

This gives us the final relationship by combining equation 5, 6 and 7:

$$q_1(\xi) - q_2(\xi) = P_\xi(\xi(\infty) = S_1, \tau = \infty)$$

The right hand side turns out to be relatively easy to solve. We examine the same Markov Process, but this time with an absorbing border  $\Lambda_0$ . If any of the absorbing states on the line  $\Lambda_0$  is hit, then we have for the corresponding normal Markov Process that either  $\tau < \infty$  or  $\xi(\infty) = S_1$ . Solving the probability for the particle to end up in the right reservoir is the same as calculating the right hand side of the above expression.

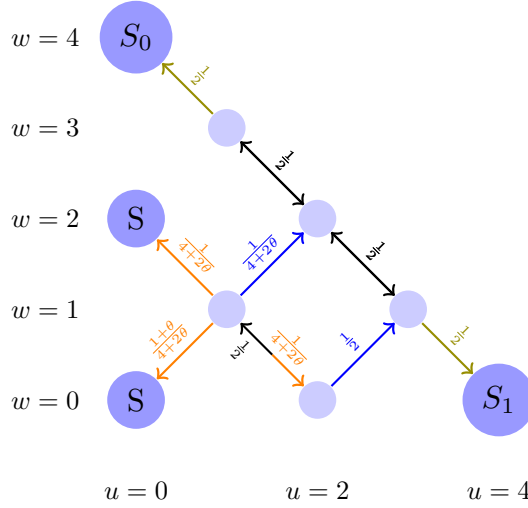


Figure 7: Transition graph for the discrete-time distance and sum process of the two particle inclusion process with  $N=3$ , with  $\Lambda_0$  absorbing. Note that at each node there is an equal chance of  $w$  increasing or decreasing.

The question now reduces to whether  $\xi(\infty) \in \Lambda_0$  or  $\xi(\infty) = S_1 = (w, N+1)$ . We can now see that at every state, there is a probability of a  $\frac{1}{2}$  for  $u$  to increase by 1 and a probability of  $\frac{1}{2}$  for  $u$  to decrease by 1, it does not depend on  $w$ . We only need to know when  $u = 0$  or  $u = N+1$ , so this reduces the problem to solving a 1 dimensional random walk on  $N+2$  nodes (including the absorbing boundaries). The coordinates perform an autonomous discrete-time symmetric random walk on  $\{0, \dots, N+1\}$  with 0 and  $N+1$  being absorbing states. We know for such a simple system that the probability of reaching the right boundary is equal to distance from the left boundary divided by the amount of nodes or  $\frac{u}{N+1}$ . With  $u$ , defined by the starting configuration  $\xi = (w, u)$ . The symmetry of the problem shows the same argument holds under the assumption that  $\xi \in \Lambda_-$ , so:

$$P_\xi(\xi(\infty) = S_1, \tau < \infty) = \frac{u}{N+1}$$

This gives us the following expression:

$$q_1(w, u) - q_2(w, u) = \frac{u}{N+1} \quad (8)$$

□

### 6.3 Probability of $q_0, q_1$ and $q_2$

Finally we would like to derive an explicit expression for  $q_0$ , such that we have 3 restrictions, namely  $q_0 + q_1 + q_2 = 1$  and equation (8) with which we can fully determine the other two probabilities  $q_1$  and  $q_2$ .

We have the following result:



**Theorem 3.** *A particle system based on the reference generator with absorbing boundaries at  $\{0, N + 1\}$  as defined in section 5 with 2 particles in  $V_N$  in the starting configuration has the following property:*

$$q_0(w, u) = \frac{\gamma_N P^{\theta=\infty}(w, u) + P^{IRW}(w, u)}{\gamma_N + 1}$$

with

$$\begin{aligned}\gamma_N &= \frac{\theta}{N + 1} \\ P^{\theta=\infty}(w, u) &= \frac{w + 1_{w=0}}{N + 1} \\ P^{IRW}(w, u) &= \frac{w^2 - u^2 + (N + 1)^2}{2(N + 1)^2}\end{aligned}$$

The same probability can also be expressed in positional coordinates.

$$\begin{aligned}P^{\theta=\infty}(x, y) &= \frac{|x - y| + 1_{x=y}}{N + 1} \\ P^{IRW}(x, y) &= \left(\frac{x}{N + 1}\right)\left(1 - \frac{y}{N + 1}\right) + \left(\frac{y}{N + 1}\right)\left(1 - \frac{x}{N + 1}\right)\end{aligned}$$

*Proof.* The proof of this theorem can be found in section A.1. □

When  $\theta = 0$  the expression for  $q_0(w, u)$  reduces to two independent random walkers. This has the following probability:

$$\begin{aligned}P^{IRW}(w, u) &= \frac{w^2 - u^2 + (N + 1)^2}{2(N + 1)^2} \\ P^{IRW}(x, y) &= \left(\frac{x}{N + 1}\right)\left(1 - \frac{y}{N + 1}\right) + \left(\frac{y}{N + 1}\right)\left(1 - \frac{x}{N + 1}\right)\end{aligned}$$

We see that as  $\theta \rightarrow \infty$  that  $q_0 \rightarrow P^{\theta=\infty}(w, u) = \frac{w + 1_{w=0}}{N + 1}$  for internal points. It can be easily verified that this should be true.

When  $\theta \rightarrow \infty$  if we reach a state on  $w = 0$  or  $w = 1$  we will automatically end up in  $(w = 1, u = N)$  or  $(w = 1, u = 1)$ . Which has a probability of  $\frac{1}{N + 1}$  of reaching  $S_0$  and otherwise if we don't reach  $w = 0$  or  $w = 1$  we have a random walk on  $w = 1, 2, \dots, N + 1$  where  $N + 1$  and 1 are absorbing boundaries. As such the final probability of reaching  $S_0$  becomes:

$$\begin{aligned}
P^{\theta=\infty}(w, u) &= \frac{w-1}{N} + \frac{N-w+1}{N} \frac{1}{N+1} \\
&= \frac{(w-1)(N+1)}{N(N+1)} + \frac{N(N-w+1)}{N(N+1)} \\
&= \frac{w}{N+1}
\end{aligned}$$

and as such

$$P^{\theta=\infty}(w, u) = \frac{w + 1_{w=0}}{N+1}$$

This allows us to rewrite  $q_0$  as a linear combination of the two solutions for  $\theta = \infty$  and  $\theta = 0$  with  $\gamma_N = \frac{\theta}{N+1}$

$$q_0(w, u) = \frac{\gamma_N P^{\theta=\infty}(w, u) + P^{IRW}(w, u)}{\gamma_N + 1}$$

Now if one of the particles starts in one of the reservoirs, or expressed differently, if it holds for our  $(w, u) \in \Lambda_N$  that  $w = N+1 - |u|$ , the system becomes an independent random walk again so we finally get the following formula for all  $(w, u) \in \Lambda_N$ :

$$\begin{aligned}
q_0(w, u) &= \frac{\gamma_N P^{\theta=\infty}(w, u) 1_{w \neq N+1-|u|} + P^{IRW}(w, u)}{\gamma_N 1_{w \neq N+1-|u|} + 1} \\
&= \frac{\frac{(w+1_{w=0})}{N+1} \gamma_N 1_{w \neq N+1-|u|} + \frac{w^2 - u^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N 1_{w \neq N+1-|u|} + 1}
\end{aligned}$$

Equivalently we can keep the definition original definition of  $q_0(w, u)$  and rewrite  $P^{\theta=\infty}(w, u)$  to take into account the absorbing states at  $w = 0$ .

$$P^{\theta=\infty}(w, u) = \frac{w + 1_{w=0 \text{ and } u \neq N+1 \text{ and } u \neq -(N+1)}}{N+1}$$

Now that we have defined  $q_0(w, u)$  for all  $(w, u) \in \Lambda_N$ , we can do the same for  $q_1(w, u)$  and  $q_2(w, u)$ .

**Corollary 3.1.** *A particle system based on the reference generator with absorbing boundaries at  $\{0, N+1\}$  as defined in section 5 with 2 particles in the starting configuration has the following property:*

$$\begin{aligned}
q_1(w, u) &= \frac{1}{2} + \frac{u}{2(N+1)} - \frac{q_0(w, u)}{2} \\
q_2(w, u) &= \frac{1}{2} - \frac{u}{2(N+1)} - \frac{q_0(w, u)}{2}
\end{aligned}$$

And thus in the positional coordinates they can be expressed as:

$$q_1(x, y) = \frac{x + y}{2(N + 1)} - \frac{q_0(x, y)}{2}$$

$$q_2(x, y) = 1 - \frac{x + y}{2(N + 1)} - \frac{q_0(x, y)}{2}$$

*Proof.* Adding  $q_1 - q_2 = \frac{u}{N+1}$  and  $q_0 + q_1 + q_2 = 1$  we get:

$$q_0 + 2q_1 = 1 + \frac{u}{N + 1}$$

rearranging these terms we get:

$$q_1 = \frac{1}{2} + \frac{u}{2(N + 1)} - \frac{q_0}{2}$$

Subtracting  $q_1 - q_2 = \frac{u}{N+1}$  from  $q_0 + q_1 + q_2 = 1$  we get:

$$q_0 + 2q_1 = 1 - \frac{u}{N + 1}$$

and thus after reordering these terms we get:

$$q_2 = \frac{1}{2} - \frac{u}{2(N + 1)} - \frac{q_0}{2}$$

□

This gives us all of the probabilities for the sink states  $q_0, q_1$  and  $q_2$ .

## 7 Stationary profile of particle systems out of equilibrium

In order to describe the stationary measures of the reference particle systems out of equilibrium from equation 4 we will first examine the expectation and covariance/variance of the individual sites  $\eta_i$ , with  $i \in V_N$ .

In order to calculate  $\mathbb{E}_\mu[\eta_x]$  and  $\text{Cov}_\mu[\eta_x, \eta_y]$  for all  $x, y \in \{1, 2, 3, \dots, N\}$ , where  $\mu$  is a stationary distribution, we will make use of the duality property from section 5.

We are interested in the stationary distributions  $\mu$  of the processes. Because the particle process is irreducible, our limiting distribution is the same as the stationary distribution. We can therefore say that for any initial configuration  $\eta$  the following holds:

$$\mathbb{E}_\mu[f(\eta)] = \lim_{t \rightarrow \infty} \mathbb{E}_\eta[f(\eta(t))]$$

We first observe that for a process  $\xi(t)$ , with  $|\xi|$  particles and absorbing boundaries at  $\{0, N+1\}$  after an infinite amount of time, all of the particles will have to be absorbed by one of the two boundaries. This causes the duality function to reduce to a simpler expression:

$$\begin{aligned} \mathbb{E}_\mu[D^\theta(\eta, \xi)] &= \lim_{t \rightarrow \infty} \mathbb{E}_\xi[D^\theta(\eta, \xi(t))] \\ &= \sum_{i=0}^{|\xi|} (\alpha|\theta + 1_{\theta=0}|)^i (\delta|\theta + 1_{\theta=0}|)^{|\xi|-i} P(\xi_0(\infty) = i | \xi(0) = \xi) \\ &= (\delta|\theta + 1_{\theta=0}|)^{|\xi|} \sum_{i=0}^{|\xi|} \left(\frac{\alpha}{\delta}\right)^i P(\xi_0(\infty) = i | \xi(0) = \xi) \end{aligned}$$

The term  $P(\xi_0(\infty) = i | \xi(0) = \xi)$  corresponds to the probability that  $i$  particles will end up in the left sink after an infinite amount of time with initial configuration  $\xi$ . We will repeatedly make use of this in the following sections.

### 7.1 Expectation

In order to calculate the expectation of the individual site  $\eta_i$  under the stationary measure  $\mu$  we first observe that if we take  $\xi = \delta_x$  (a single particle system), with  $x \in \{1, 2, 3, \dots, N\}$  then we have the following:

$$\eta_x = \frac{1}{|\theta + 1_{\theta=0}|} D^\theta(\eta, \delta_x)$$

So in order to calculate  $\mathbb{E}_\mu[\eta_x]$  we get:

$$\begin{aligned}
\mathbb{E}_\mu[\eta_x] &= \frac{1}{|\theta + 1_{\theta=0}|} \mathbb{E}_\mu[D^\theta(\eta, \delta_x)] \\
&= \frac{1}{|\theta + 1_{\theta=0}|} \lim_{t \rightarrow \infty} \mathbb{E}_\eta[D^\theta(\eta(t), \delta_x)] \\
&\stackrel{\text{Duality}}{=} \frac{1}{|\theta + 1_{\theta=0}|} \lim_{t \rightarrow \infty} \mathbb{E}_{\delta_x}[D^\theta(\eta, \delta_{X(t)})] \\
&= \frac{1}{|\theta + 1_{\theta=0}|} \delta |\theta + 1_{\theta=0}| (P(\xi_0(\infty) = 0 | \xi(0) = \delta_x) + \frac{\alpha}{\delta} P(\xi_0(\infty) = 1 | \xi(0) = \delta_x)) \\
&= \delta \frac{x}{N+1} + \alpha (1 - \frac{x}{N+1}) \\
&= \alpha + (\delta - \alpha) \frac{x}{N+1}
\end{aligned}$$

Where we have that  $\delta_{X(t)}$  is a symmetric random walk on  $V_N$  with absorbing boundaries at  $\{0, N+1\}$ . The expectation of the stationary profile is thus a linear equation, which is equal to  $\alpha$  at the left boundary and  $\delta$  at the right boundary.

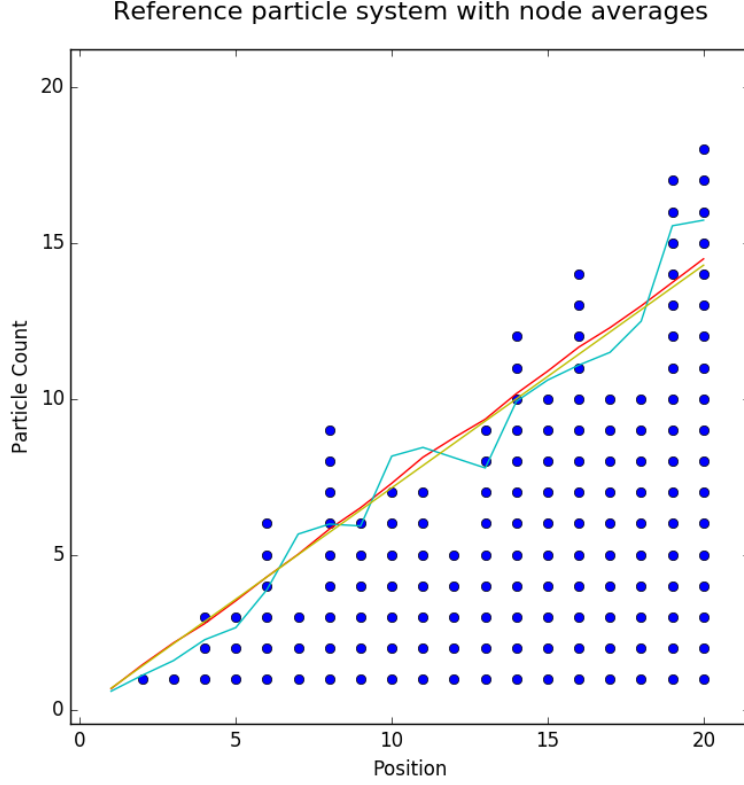


Figure 8: Visualization of the stationary distribution of Independent Random Walkers with  $N = 20$ ,  $\alpha = 0$  and  $\delta = 15$  through simulation. Every blue dot represents a particle that is located on a site in  $V_N$  with the x-axis representing the position of this site. The yellow line is the analytically calculated expectation for the stationary profile. The red line is an approximation for the expected amount of particles by averaging over the last 8000 timesteps. The blue line is an approximation for the expected amount of particles by averaging over the last 100 timesteps. It can be seen that the long term approximation very accurately covers the stationary profile.

## 7.2 Covariance

In order to calculate the  $\text{Cov}[\eta_x, \eta_y]$ , because  $\text{Cov}[\eta_x, \eta_y] = \mathbb{E}[\eta_x \eta_y] - \mathbb{E}[\eta_x] \mathbb{E}[\eta_y]$  we need to calculate the second moments of the sites as well. We start with the second moments of the internal points where we take  $\xi = \delta_x + \delta_y$ , with  $x \neq y$ :

$$\eta_x \eta_y = \frac{1}{(\theta + 1_{\theta=0})^2} D^\theta(\eta, \delta_x + \delta_y)$$

Then we perform the calculations in a similar manner:

$$\begin{aligned}
\mathbb{E}_\mu[\eta_x \eta_y] &= \frac{1}{(\theta + 1_{\theta=0})^2} \mathbb{E}_\mu[D^\theta(\eta, \delta_x + \delta_y)] \\
&= \frac{1}{(\theta + 1_{\theta=0})^2} \lim_{t \rightarrow \infty} \mathbb{E}_\eta[D^\theta(\eta(t), \delta_x + \delta_y)] \\
&\stackrel{\text{Duality}}{=} \frac{1}{(\theta + 1_{\theta=0})^2} \lim_{t \rightarrow \infty} \mathbb{E}_{\delta_x + \delta_y}[D^\theta(\eta, \delta_{X(t)} + \delta_{Y(t)})] \\
&= \frac{1}{(\theta + 1_{\theta=0})^2} ((\theta + 1_{\theta=0})\delta)^2 [P(\xi_0(\infty) = 0 | \xi(0) = \delta_x + \delta_y) \\
&\quad + \frac{\alpha}{\delta} P(\xi_0(\infty) = 1 | \xi(0) = \delta_x + \delta_y) \\
&\quad + \frac{\alpha^2}{\delta^2} P(\xi_0(\infty) = 2 | \xi(0) = \delta_x + \delta_y)] \\
&= \delta^2 q_1 + \alpha \delta q_0 + \alpha^2 q_2 \\
&\stackrel{\text{Corollary 3.1}}{=} \delta^2 \left( \frac{x+y}{2(N+1)} - \frac{q_0}{2} \right) + \alpha \delta q_0 + \alpha^2 \left( 1 - \frac{x+y}{2(N+1)} - \frac{q_0}{2} \right) \\
&= \alpha^2 + \frac{(\delta^2 - \alpha^2)}{2} \frac{x+y}{N+1} + \left( \alpha \delta - \frac{\alpha^2 + \delta^2}{2} \right) q_0 \\
&\stackrel{\text{Theorem 3}}{=} \alpha^2 + \frac{(\delta^2 - \alpha^2)}{2} \frac{x+y}{N+1} - \frac{(\delta - \alpha)^2}{2} \frac{\gamma_N \frac{|x-y|}{N+1} + \frac{x+y}{N+1} + \frac{-2xy}{(N+1)^2}}{\gamma_N + 1}
\end{aligned}$$

Where we have that  $\delta_{X(t)} + \delta_{Y(t)}$  is a symmetric random walk on  $V_N$  which interact when they are close to each other, with absorbing boundaries at  $\{0, N+1\}$  and starting configuration  $\delta_x + \delta_y$ .

And because  $\mathbb{E}_\mu[\eta_x] \mathbb{E}_\mu[\eta_y] = (\alpha + (\delta - \alpha) \frac{x}{N+1})(\alpha + (\delta - \alpha) \frac{y}{N+1}) = \alpha^2 + \alpha(\delta - \alpha) \frac{x+y}{N+1} + \frac{(\delta - \alpha)^2}{2} \frac{2xy}{(N+1)^2}$ , we can now calculate the covariance of the internal

points for  $x \neq y$ .

$$\begin{aligned}
& \text{Cov}_\mu[\eta_x, \eta_y] \\
&= \mathbb{E}_\mu[\eta_x \eta_y] - \mathbb{E}_\mu[\eta_x] \mathbb{E}_\mu[\eta_y] \\
&= \alpha^2 + \frac{(\delta^2 - \alpha^2)}{2} \frac{x+y}{N+1} - \frac{(\delta - \alpha)^2}{2} \frac{\gamma_N \frac{|x-y|}{N+1} + \frac{x+y}{N+1} + \frac{-2xy}{(N+1)^2}}{\gamma_N + 1} \\
&\quad - \alpha^2 - \alpha(\delta - \alpha) \frac{x+y}{N+1} - \frac{(\delta - \alpha)^2}{2} \frac{2xy}{(N+1)^2} \\
&= \left( \frac{(\delta^2 - \alpha^2)}{2} + \alpha^2 - \alpha\delta \right) \frac{x+y}{N+1} - \frac{(\delta - \alpha)^2}{2} \left( \frac{2xy}{(N+1)^2} + \frac{\gamma_N \frac{|x-y|}{N+1} + \frac{x+y}{N+1} - \frac{2xy}{(N+1)^2}}{\gamma_N + 1} \right) \\
&= \frac{(\delta - \alpha)^2}{2} \left( \frac{x+y}{N+1} - \frac{2xy}{(N+1)^2} \right) - \frac{(\delta - \alpha)^2}{2} \frac{\gamma_N \frac{|x-y|}{N+1} + \frac{x+y}{N+1} - \frac{2xy}{(N+1)^2}}{\gamma_N + 1} \\
&= \frac{(\delta - \alpha)^2}{2} \left( \frac{x+y}{N+1} - \frac{2xy}{(N+1)^2} - \frac{\gamma_N \frac{|x-y|}{N+1} + \frac{x+y}{N+1} - \frac{2xy}{(N+1)^2}}{\gamma_N + 1} \right) \\
&= \frac{\gamma_N}{\gamma_N + 1} \frac{(\delta - \alpha)^2}{2} \left( \frac{x+y}{N+1} - \frac{2xy}{(N+1)^2} - \frac{|x-y|}{N+1} \right) \\
&= \frac{\gamma_N}{\gamma_N + 1} \frac{(\delta - \alpha)^2}{2} (P^{IRW}(x, y) - P^{\theta=\infty}(x, y))
\end{aligned}$$

Note that if  $\theta = 0$  then the covariances are zero as well. Also note that these terms are bounded for all  $\theta$ .

### 7.3 Variance

Lastly we need to calculate the covariance for the internal points  $x = y$  which is the same as the variance of a single point  $x$ . We begin with calculating the second moment of a single point, for this we take  $\xi = 2\delta_x$ . We then have

$$\eta_x^2 = \eta_x + \frac{\theta + 1}{(\theta + 1_{\theta=0})^2} D^\theta(\eta, 2\delta_x)$$



$$\begin{aligned}
& \mathbb{E}_\mu[\eta_x^2] \\
&= \mathbb{E}_\mu[\eta_x] + \frac{\theta + 1}{(\theta + 1_{\theta=0})^2} \mathbb{E}_\mu[D^\theta(\eta, 2\delta_x)] \\
&= \mathbb{E}_\mu[\eta_x] + \frac{\theta + 1}{(\theta + 1_{\theta=0})^2} \lim_{t \rightarrow \infty} \mathbb{E}_\eta[D^\theta(\eta(t), 2\delta_x)] \\
&= \mathbb{E}_\mu[\eta_x] + \frac{\theta + 1}{(\theta + 1_{\theta=0})^2} \lim_{t \rightarrow \infty} \mathbb{E}_{2\delta_x}[D^\theta(\eta, \delta_{X(t)} + \delta_{Y(t)})] \\
&= \mathbb{E}_\mu[\eta_x] + \frac{\theta + 1}{(\theta + 1_{\theta=0})^2} ((\theta + 1_{\theta=0})\delta)^2 [P(\xi_0(\infty) = 0 | \xi(0) = 2\delta_x) \\
&\quad + \frac{\alpha}{\delta} P(\xi_0(\infty) = 1 | \xi(0) = 2\delta_x) \\
&\quad + \frac{\alpha^2}{\delta^2} P(\xi_0(\infty) = 2 | \xi(0) = 2\delta_x)] \\
&= \mathbb{E}_\mu[\eta_x] + (1 + \theta)(\delta^2 q_1 + \alpha \delta q_0 + \alpha^2 q_2) \\
&\stackrel{\text{Corollary 3.1}}{=} \mathbb{E}_\mu[\eta_x] + (1 + \theta)(\delta^2(\frac{2x}{2(N+1)} - \frac{q_0}{2}) + \alpha \delta q_0 + \alpha^2(1 - \frac{2x}{2(N+1)} - \frac{q_0}{2})) \\
&= \mathbb{E}_\mu[\eta_x] + (1 + \theta)(\alpha^2 + \frac{(\delta^2 - \alpha^2)}{2} \frac{2x}{N+1} + (\alpha \delta - \frac{\alpha^2 + \delta^2}{2}) q_0) \\
&\stackrel{\text{Theorem 3}}{=} \mathbb{E}_\mu[\eta_x] + (1 + \theta)(\alpha^2 + \frac{(\delta^2 - \alpha^2)}{2} \frac{2x}{N+1} - \frac{(\delta - \alpha)^2}{2} \frac{\gamma_N \frac{1}{N+1} + \frac{2x}{N+1} - \frac{2x^2}{(N+1)^2}}{\gamma_N + 1})
\end{aligned}$$

Where we have that  $\delta_{X(t)} + \delta_{Y(t)}$  is a symmetric random walk on  $V_N$  which interact when they are close to each other, with absorbing boundaries at  $\{0, N+1\}$  and starting configuration  $2\delta_x$ . And because  $\mathbb{E}_\mu[\eta_x]^2 = (\alpha + (\delta - \alpha)\frac{x}{N+1})^2$ , we can now calculate the variance of a single point  $x$ .

$$\begin{aligned}
\text{Var}_\mu[\eta_x] &= \mathbb{E}_\mu[\eta_x^2] - \mathbb{E}_\mu[\eta_x]^2 = \\
&= \mathbb{E}_\mu[\eta_x] - (\alpha + (\delta - \alpha)\frac{x}{N+1})^2 + \\
&\quad (1 + \theta)(\alpha^2 + \frac{(\delta^2 - \alpha^2)}{2} \frac{2x}{N+1} - \frac{(\delta - \alpha)^2}{2} \frac{\gamma_N \frac{1}{N+1} + \frac{2x}{N+1} - \frac{2x^2}{(N+1)^2}}{\gamma_N + 1}) = \\
&\quad \frac{\gamma_N}{\gamma_N + 1} \frac{(\delta - \alpha)^2}{2} (P^{IRW}(x, y) - P^{\theta=\infty}(x, y)) \\
&\quad + \mathbb{E}_\mu[\eta_x] + \theta(\alpha^2 + \frac{(\delta^2 - \alpha^2)}{2} \frac{2x}{N+1} - \frac{(\delta - \alpha)^2}{2} \frac{\gamma_N \frac{1}{N+1} + \frac{2x}{N+1} - \frac{2x^2}{(N+1)^2}}{\gamma_N + 1})
\end{aligned}$$

We notice in the final two lines of the above equations that the first term is the same term as the one we discovered for the covariance with  $x \neq y$ . Then

we have a bias term  $\mathbb{E}_\mu[\eta_x]$  and finally we have a term that is dependent on  $\theta$ . We now have that for  $\theta = 0$  that  $\text{Var}_\mu(\eta_x) = \mathbb{E}_\mu[\eta_x]$  and for  $\theta \gg 0$  we have that  $\text{Var}_\mu(\eta_x) \gg 0$  as the variance tends to infinity due to the term that is dependent on  $\theta$ .

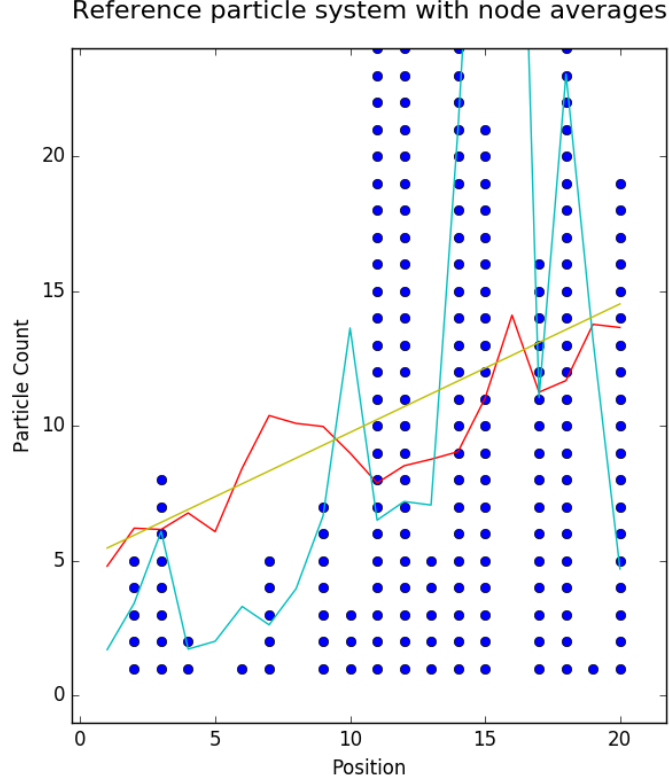


Figure 9: Visualization of the stationary distribution of the Symmetric Inclusion Process with  $N = 20$ ,  $\theta = 2$ ,  $\alpha = 5$  and  $\delta = 15$  through simulation. Every blue dot represents a particle that is located on a site in  $V_N$  with the x-axis representing the position of this site. The yellow line is the analytically calculated expectation for the stationary profile. The red line is an approximation for the expected amount of particles by averaging over the last 8000 timesteps. The blue line is an approximation for the expected amount of particles by averaging over the last 100 timesteps. It can be seen that the long term approximation insufficiently approximates the stationary profile due to the high variance.

## 7.4 Density field

We would like to examine the stationary behavior of the particle systems on a real numbered interval instead of just an integer lattice. By taking the limit in

space ( $N \rightarrow \infty$ ) we would like to get the probability density function  $p(x)$  of the particles. In order to examine the probability density function  $p(x)$ , we will look at the way Schwartz functions  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  (with  $\Phi \in S(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \|f\|_{\alpha,\beta} < \infty \forall \alpha, \beta \in \mathbb{N}\}$ ,  $\|f\|_{\alpha,\beta} = \sup_x |x^\alpha \frac{d^\beta}{dx^\beta} f(x)|$ ) behave. In order to extract information about the density function. We will define the density field as:

$$\mathcal{X}_N(\Phi, \eta(t)) = \frac{1}{N} \sum_{x=1}^N \Phi\left(\frac{x}{N}\right) \eta_x(t)$$

And we will refer to  $\mathcal{X}_N(\Phi, \eta(t))$  under the limiting distribution  $\mu$  as  $\mathcal{X}_N(\Phi, \eta)$ .

In the following theorem we provide a result concerning the expectation and variance of the density field. We prove that there are three different behaviour depending on the limit of  $\gamma_N$  as  $N \rightarrow \infty$ .

**Theorem 4** (Variance of the stationary density field for different scaling regimes). *For particle systems with generators of the form as in equation 4 we have, for the expectation of the density field:*

$$\lim_{N \rightarrow \infty} \mathbb{E}_\mu[\mathcal{X}_N(\Phi, \eta)] = \int_0^1 \Phi(x) [\alpha + (\delta - \alpha)x] dx$$

Whereas for the variance we have three different scaling regimes:

**The deterministic regime:**  $\lim_{N \rightarrow \infty} \gamma_N = 0$ . In this regime we have that

$$\lim_{N \rightarrow \infty} \text{Var}_\mu[\mathcal{X}_N(\Phi, \eta)] = 0 \quad (9)$$

and then

$$\mathcal{X}_N(\Phi, \eta) \rightarrow \int_0^1 \Phi(x) [\alpha + (\delta - \alpha)x] dx \quad \text{in probability as } N \rightarrow \infty \quad (10)$$

**The sticky regime:**  $\gamma := \lim_{N \rightarrow \infty} \gamma_N \in (0, \infty)$ . In this regime we have that

$$\begin{aligned} \lim_{N \rightarrow \infty} \text{Var}_\mu[\mathcal{X}_N(\Phi, \eta)] &= \frac{\gamma}{\gamma + 1} \int_0^1 \int_0^1 \Phi(x) \Phi(y) \frac{(\delta - \alpha)^2}{2} (x + y - 2xy - |x - y|) dx dy + \\ &\quad \gamma \int_0^1 \Phi(x)^2 [(\alpha^2 + (\delta^2 - \alpha^2)x - (\delta - \alpha)^2 \frac{x - x^2}{\gamma + 1})] dx \end{aligned}$$

**The absorbing regime:**  $\gamma = \lim_{N \rightarrow \infty} \gamma_N = \infty$ . In this regime we have that

$$\lim_{N \rightarrow \infty} \text{Var}_\mu[\mathcal{X}_N(\Phi, \eta)] = +\infty$$

.

*Proof.* We have

$$\begin{aligned}
\mathbb{E}_\mu[\mathcal{X}_N(\Phi, \eta)] &= \frac{1}{N} \sum_{x=1}^N \Phi\left(\frac{x}{N}\right) \mathbb{E}_\mu[\eta_x] \\
\text{Var}_\mu[\mathcal{X}_N(\Phi, \eta)] &= \mathbb{E}_\mu[(\mathcal{X}_N(\Phi, \eta) - \mathbb{E}_\mu[\mathcal{X}_N(\Phi, \eta)])^2] \\
&= \mathbb{E}_\mu[\mathcal{X}_N(\Phi, \eta)^2] - \mathbb{E}_\mu[\mathcal{X}_N(\Phi, \eta)]^2 \\
&= \frac{1}{N^2} \sum_{y=1}^N \sum_{x=1}^N \Phi\left(\frac{y}{N}\right) \Phi\left(\frac{x}{N}\right) \mathbb{E}_\mu[\eta_x \eta_y] - \frac{1}{N^2} \sum_{y=1}^N \sum_{x=1}^N \Phi\left(\frac{y}{N}\right) \Phi\left(\frac{x}{N}\right) \mathbb{E}_\mu[\eta_x] \mathbb{E}_\mu[\eta_y] \\
&= \frac{1}{N^2} \sum_{y=1}^N \sum_{x=1}^N \Phi\left(\frac{y}{N}\right) \Phi\left(\frac{x}{N}\right) \text{Cov}_\mu[\eta_x, \eta_y]
\end{aligned}$$

We will first examine the expectation value using the expectations from section 7.1.

$$\begin{aligned}
\mathbb{E}_\mu[\mathcal{X}(\Phi, \eta)] &= \lim_{N \rightarrow \infty} \mathbb{E}_\mu[\mathcal{X}_N(\Phi, \eta)] \\
&= \lim_{N \rightarrow \infty} \mathbb{E}_\mu\left[\frac{1}{N} \sum_{x=1}^N \Phi\left(\frac{x}{N}\right) \eta_x\right] \\
&= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{x=1}^N \Phi\left(\frac{x}{N}\right) \mathbb{E}_\mu[\eta_x] \\
&= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{x=1}^N \Phi\left(\frac{x}{N}\right) [\alpha + (\delta - \alpha) \frac{x}{N+1}] \\
&= \int_0^1 \Phi(x) [\alpha + (\delta - \alpha)x] dx
\end{aligned}$$

We then find the density function  $p(x) = \alpha + (\delta - \alpha)x$ .

Next we will look at the variance of  $\mathcal{X}(\Phi, \eta)$  by splitting up the covariances for the diagonal and the rest of the sum and using the results from section 7.2 and section 7.3.

$$\begin{aligned}
\text{Var}_\mu[\mathcal{X}(\Phi, \eta)] &= \lim_{N \rightarrow \infty} \text{Var}_\mu[\mathcal{X}_N(\Phi, \eta)] \\
&= \lim_{N \rightarrow \infty} \text{Var}_\mu\left[\frac{1}{N} \sum_{x=1}^N \Phi\left(\frac{x}{N}\right) \eta_x\right] = \\
&= \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{y=1}^N \sum_{x=1}^N \Phi\left(\frac{y}{N}\right) \Phi\left(\frac{x}{N}\right) \text{Cov}_\mu[\eta_x, \eta_y] = \\
&= \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{y=1}^N \sum_{\substack{x=1 \\ x \neq y}}^N \Phi\left(\frac{y}{N}\right) \Phi\left(\frac{x}{N}\right) \text{Cov}_\mu[\eta_x, \eta_y] + \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{x=1}^N \Phi\left(\frac{x}{N}\right)^2 \text{Var}_\mu[\eta_x]
\end{aligned}$$

We will have to calculate two terms. We start with  $\lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{x=1}^N \Phi(\frac{x}{N})^2 \text{Var}_\mu[\eta_x]$ .

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{x=1}^N \Phi(\frac{x}{N})^2 \text{Var}_\mu[\eta_x] \\
&= \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{x=1}^N \Phi(\frac{x}{N})^2 \left( \mathbb{E}_\mu[\eta_x] + \frac{\gamma_N}{\gamma_N + 1} \frac{(\delta - \alpha)^2}{2} (P^{IRW}(x, y) - P^{\theta=\infty}(x, y)) \right) \\
&+ \lim_{N \rightarrow \infty} \frac{\theta}{N^2} \sum_{x=1}^N \Phi(\frac{x}{N})^2 \left( \alpha^2 + \frac{(\delta^2 - \alpha^2)}{2} \frac{2x}{N+1} - \frac{(\delta - \alpha)^2}{2} \frac{\gamma_N \frac{1}{N+1} + \frac{2x}{N+1} - \frac{2x^2}{(N+1)^2}}{\gamma_N + 1} \right) \\
&= \lim_{N \rightarrow \infty} \frac{\gamma_N(N+1)}{N^2} \sum_{x=1}^N \Phi(\frac{x}{N})^2 \left( \alpha^2 + \frac{(\delta^2 - \alpha^2)}{2} \frac{2x}{N+1} - \frac{(\delta - \alpha)^2}{2} \frac{\gamma_N \frac{1}{N+1} + \frac{2x}{N+1} - \frac{2x^2}{(N+1)^2}}{\gamma_N + 1} \right) \\
&= \gamma \int_0^1 \Phi(x)^2 \left[ (\alpha^2 + (\delta^2 - \alpha^2)x - (\delta - \alpha)^2 \frac{x - x^2}{\gamma + 1}) \right] dx
\end{aligned}$$

Next we examine  $\lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{y=1}^N \sum_{\substack{x=1 \\ x \neq y}}^N \Phi(\frac{y}{N}) \Phi(\frac{x}{N}) \text{Cov}_\mu[\eta_x, \eta_y]$ .

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{y=1}^N \sum_{\substack{x=1 \\ x \neq y}}^N \Phi(\frac{y}{N}) \Phi(\frac{x}{N}) \text{Cov}_\mu[\eta_x, \eta_y] \\
&= \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{y=1}^N \sum_{\substack{x=1 \\ x \neq y}}^N \Phi(\frac{y}{N}) \Phi(\frac{x}{N}) (P^{IRW}(x, y) - P^{\theta=\infty}(x, y)) \\
&= \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{y=1}^N \sum_{\substack{x=1 \\ x \neq y}}^N \Phi(\frac{y}{N}) \Phi(\frac{x}{N}) \frac{\gamma_N}{\gamma_N + 1} \frac{(\delta - \alpha)^2}{2} \left( \frac{x+y}{N+1} - \frac{2xy}{(N+1)^2} - \frac{|x-y|}{N+1} \right) \\
&= \frac{\gamma}{\gamma + 1} \int_0^1 \int_0^1 \Phi(x) \Phi(y) \frac{(\delta - \alpha)^2}{2} (x + y - 2xy - |x - y|) dx dy
\end{aligned}$$

Now we have a full expression for the variance of  $\mathcal{X}(\Phi, \eta)$ . Then the variance is related to the limit  $\gamma = \lim_{N \rightarrow \infty} \frac{\theta}{N+1}$ , where we make  $\theta$  dependent on  $N$ . Depending on the value of this limit we get different behaviours for the stationary density field  $\mathcal{X}(\Phi, \eta)$ . If  $\gamma = \infty$  it is clear that

$$\lim_{N \rightarrow \infty} \text{Var}_\mu[\mathcal{X}_N(\Phi, \eta)] = +\infty.$$

If  $\gamma \in [0, \infty)$  then

$$\begin{aligned}
\lim_{N \rightarrow \infty} \text{Var}_\mu[\mathcal{X}_N(\Phi, \eta)] &= \frac{\gamma}{\gamma + 1} \int_0^1 \int_0^1 \Phi(x) \Phi(y) \frac{(\delta - \alpha)^2}{2} (x + y - 2xy - |x - y|) dx dy + \\
&\quad \gamma \int_0^1 \Phi(x)^2 \left[ (\alpha^2 + (\delta^2 - \alpha^2)x - (\delta - \alpha)^2 \frac{x - x^2}{\gamma + 1}) \right] dx
\end{aligned}$$

We notice that when the particle system is in equilibrium ( $\delta = \alpha$ ), then the variance reduces to  $\text{Var}_\mu[\mathcal{X}(\Phi, \eta)] = \gamma \int_0^1 \Phi(x)^2 \alpha^2 dx$ .

In particular, if  $\gamma = 0$  then, using Chebychev's we can say more about how the density field varies from its expected value:

$$P(|\mathcal{X}_N(\Phi, \eta) - \mathbb{E}_\mu[\mathcal{X}_N(\Phi, \eta)]| > \epsilon) \leq \frac{1}{\epsilon^2} \text{Var}_\mu[\mathcal{X}_N(\Phi, \eta)]$$

where

$$\lim_{N \rightarrow \infty} \text{Var}_\mu[\mathcal{X}_N(\Phi, \eta)] = 0 \tag{11}$$

then

$$\lim_{N \rightarrow \infty} P(|\mathcal{X}_N(\Phi, \eta) - \mathbb{E}_\mu[\mathcal{X}_N(\Phi, \eta)]| > \epsilon) = 0 \tag{12}$$

from which we have the convergence in probability in the deterministic regime.  $\square$

We know from [11] that in the sticky regime, the particles will form piles which distance processes behave like one-sided sticky Brownian motion. We expect a similar thing to be happening for the sticky regime of the out of equilibrium particle systems. Indeed this is consistent with our result from which emerges a positive finite variance of the density field in the sticky regime.

## 8 The dynamics of the particle system

After having examined the stationary distribution in the previous section, we would like to examine the dynamics of the particle system and see if the results are consistent with what we have found. Before we can do that, we will first define introduce some background information about Brownian motion.

### 8.1 Brownian motion and random walkers

A one-dimensional random walker is a Markov Process on a (possibly infinite) discrete lattice. It describes a point that starts somewhere on the lattice and randomly jumps to one of its neighbouring points. We would like to examine the limiting behaviour of the random walker  $\{X_t, t \geq 0\}$  on a finite lattice  $V_N = \{0, \dots, N+1\}$  with absorbing boundaries and symmetric jump rates as we take  $N \rightarrow \infty$ . In order to do this, we try to examine the generators. The generator for such a simple random walker is the following:

$$[L_R^N f](x) = \left\{ \frac{1}{2}f(x+1) - f(x) + \frac{1}{2}f(x-1) \right\} 1_{x \in V_N}$$

We rescale the random process such that the values stay in a bounded interval for all  $N \geq 1$ :  $X_t \in \{0, \frac{1}{N}, \dots, 1, \frac{N+1}{N}\}$ . Furthermore, we will be making use of the critical time-scaling of the random walker, this means that we multiply the time variable by  $N^2$ . So we will be examining the process  $B^N(t) = \frac{1}{N}X(tN^2)$ . Which has the following generator:

$$[L^N f](x) = N^2 \left\{ \frac{1}{2}f\left(x + \frac{1}{N}\right) - f(x) + \frac{1}{2}f\left(x - \frac{1}{N}\right) \right\} 1_{x \in \{\frac{1}{N}, \dots, 1\}}$$

We will also define the following limiting time-evolution on f:

$$f_t^N(x_N) := \mathbb{E}\left[f\left(\frac{1}{N}X(N^2t)\right) \mid \frac{1}{N}X(0) = x_N\right]$$

$$f_t(x) := \lim_{N \rightarrow \infty} f_t^N(x_N)$$

If we then take  $N \rightarrow \infty$  we get

$$[L_B f](x) := \lim_{N \rightarrow \infty} [L^N f](x) = \frac{1}{2}\Delta f(x) \quad \text{for } x \in [0, 1]$$

And because  $[L^N f](0) = [L^N f](\frac{N+1}{N}) = 0$  for all  $N$ , we also have  $[L_B f](0) = [L_B f](1) = 0$  and thus  $\Delta f_t(0) = \Delta f_t(1) = 0$ . Because the generators converge, this also means that the semigroup converges [4]. And this  $L_B$  is the infinitesimal generator of the Brownian motion in  $[0, 1]$  with absorbing boundary conditions. A detailed proof of this can be found in [9]. Indeed if we denote by  $B(t)$  such Brownian motion, then its semigroup  $f_t(x) := \mathbb{E}_x[B(t)]$  satisfies the following two-boundary heat equation problem:

$$\frac{df_t(x)}{dt} = [L_B f_t](x) = \frac{1}{2}\Delta f_t(x) \quad \text{for } x \in [0, 1] \quad (13)$$

$$\Delta f_t(0) = \Delta f_t(1) = 0 \quad (14)$$

$$f_0(x) = f(x) \quad (15)$$

## 8.2 The expectation of the sites

We now wish to examine the time-dependent density field when the system is initialized from a probability measures  $\nu_N$  on the configuration space  $\Omega$ . We will again make use of the duality property. We first note that for  $x \in \{0, 1, 2, \dots, N, N+1\}$  we have:

$$\frac{1}{|\theta + 1_{\theta=0}|} D^\theta(\eta, \delta_x) = \begin{cases} \alpha & x = 0 \\ \eta_x & x \in \{1, 2, \dots, N\} \\ \delta & x = N+1 \end{cases}$$

Because  $\eta_0$  and  $\eta_{N+1}$  are undefined, we can define them with  $\eta_0 := \delta$  and  $\eta_{N+1} := \gamma$ . By doing this, we get the simplified expression.

$$\frac{1}{|\theta + 1_{\theta=0}|} D^\theta(\eta, \delta_x) = \eta_x$$

We can now analyze all of the processes at the same time. We will make use of the duality property. Let  $X(t)$  be the position of a random walker in  $\{0, \dots, N+1\}$  with absorbing boundaries at  $\{0, N+1\}$ . We denote by  $\mathbf{E}_x$  with respect to  $X(t)$  with  $X(0) = x$ . We define  $\tau$  as the absorption time of  $X(t)$  into the absorbing boundaries. We now have:

$$\begin{aligned} \mathbb{E}_\eta[\eta_x(t)] &= \frac{1}{|\theta + 1_{\theta=0}|} \mathbb{E}_\eta[D^\theta(\eta(t), \delta_x)] \\ &= \frac{1}{|\theta + 1_{\theta=0}|} \mathbf{E}_x[D^\theta(\eta, \delta_{X(t)})] \\ &= \mathbf{E}_x[\eta_{X(t)}] \\ &= \mathbf{E}_x[\eta_{X(t)}, 1_{t < \tau}] + \mathbf{E}_x[\eta_{X(t)}, 1_{t \geq \tau, X(\tau)=0}] + \mathbf{E}_x[\eta_{X(t)}, 1_{t \geq \tau, X(\tau)=N+1}] \\ &= \mathbf{E}_x[\eta_{X(t)}, 1_{t < \tau}] + \alpha P_x(X(\tau) = 0, t \geq \tau) + \delta P_x(X(\tau) = N+1, t \geq \tau) \end{aligned}$$

We assume the expectation of the number of particles at each site with respect to  $\nu_N$  to be a slowly varying function:

$$\rho\left(\frac{x}{N}\right) := \mathbb{E}_{\nu_N}[\eta_x]$$

Lastly we will perform critical time-scaling for the random walker  $X(t)$  starting at  $x$  as  $N \rightarrow \infty$ . Let  $B(t)$  be the Brownian motion defined in section 8.1. This means that we will scale time by a factor  $N^2$ , and space by a factor  $N^{-1}$ . As  $N \rightarrow \infty$  this causes the limit object to become a Brownian motion process  $B(t)$  with absorbing boundaries at 0 and 1. We then take  $\mathbb{E}_\eta[\eta_x(t)]$  and then average out the initial configuration  $\eta$  with respect to the initial probability measure  $\nu_N$ .

$$\begin{aligned} \mathbb{E}_{\nu_N}[\eta_x(t)] &= \mathbf{E}_x\left[\rho\left(\frac{X(t)}{N}\right), 1_{t < \tau}\right] \\ &\quad + \alpha P_x(X(\tau) = 0, t \geq \tau) + \delta P_x(X(\tau) = N+1, t \geq \tau) \end{aligned}$$



We now examine the density field of the particle system starting from a probability measure  $\nu_N$  as described above.

$$\begin{aligned}\mathbb{E}_{\nu_N}[\mathcal{X}(\phi, \eta(N^2t))] &= \frac{1}{N} \sum_{i=1}^N \Phi\left(\frac{x}{N}\right) \mathbb{E}_{\nu_N}[\eta_x(N^2t)] \\ &= \frac{1}{N} \sum_{x=1}^N \Phi\left(\frac{x}{N}\right) \mathbf{E}_x\left[\rho\left(\frac{X(N^2t)}{N}\right)\right]\end{aligned}$$

In order to examine what happens with the limit as  $N \rightarrow \infty$ , we will have to look at the limit of  $\mathbf{E}_x[\rho(\frac{X(N^2t)}{N})]$ .

Then we define  $\tau'$  to be the absorption time of the Brownian motion  $B(t)$  into its boundaries.

$$\begin{aligned}\rho(x, t) := \lim_{N \rightarrow \infty} \mathbf{E}_x\left[\rho\left(\frac{X(N^2t)}{N}\right)\right] &= \mathbf{E}_x[\rho(B(t)), 1_{t < \tau'}] \\ &\quad + \alpha P_x(B(\tau') = 0, t \geq \tau') + \delta P_x(B(\tau') = 1, t \geq \tau')\end{aligned}$$

$\rho(x, t)$  satisfies the following two-boundary heat equation problem:

$$\begin{aligned}\frac{\partial \rho(x, t)}{\partial t} &= \frac{1}{2} \Delta \rho(x, t) \\ \rho(0, t) &= \alpha \\ \rho(1, t) &= \delta \\ \rho(x, 0) &= \rho(x)\end{aligned}$$

A detailed proof of this can be found in [9].

The unique stationary solution to these equations is  $\alpha + (\delta - \alpha)x$ . This is consistent with the scaling limit of the stationary density field that we found in section 7.1.

Now we take the limit  $N \rightarrow \infty$  and we get the following:

$$\begin{aligned}&\lim_{N \rightarrow \infty} \mathbb{E}_{\nu_N}[\mathcal{X}(\phi, \eta(N^2t))] \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{x=1}^N \Phi\left(\frac{x}{N}\right) \mathbf{E}_x\left[\rho\left(\frac{X(N^2t)}{N}\right)\right] \\ &= \int_0^1 \Phi(x) \rho(x, t) dx\end{aligned}$$

This gives a lot of information as to how the field density function evolves over time as the the heat equation has been studied thoroughly.

## 9 Conclusion

Within this thesis we examined the behaviour of the stationary distribution for the out of equilibrium Symmetric Inclusion Process (SIP), Symmetric Exclusion Process (SEP) and Independent Random Walkers (IRW).

We achieved this by first examining the generators of the reference processes with reservoirs and linking these systems with the dual processes with absorbing boundaries.

By examining the behaviour of a two particle system we came up with the following theorems regarding the absorption probabilities for the particles.

**Theorem 3.** *A particle system based on the reference generator with absorbing boundaries at  $\{0, N + 1\}$  as defined in section 5 with 2 particles in  $V_N$  in the starting configuration has the following property:*

$$q_0(w, u) = \frac{\gamma_N P^{\theta=\infty}(w, u) + P^{IRW}(w, u)}{\gamma_N + 1}$$

with

$$\begin{aligned}\gamma_N &= \frac{\theta}{N + 1} \\ P^{\theta=\infty}(w, u) &= \frac{w + 1_{w=0}}{N + 1} \\ P^{IRW}(w, u) &= \frac{w^2 - u^2 + (N + 1)^2}{2(N + 1)^2}\end{aligned}$$

The same probability can also be expressed in positional coordinates.

$$\begin{aligned}P^{\theta=\infty}(x, y) &= \frac{|x - y| + 1_{x=y}}{N + 1} \\ P^{IRW}(x, y) &= \left(\frac{x}{N + 1}\right)\left(1 - \frac{y}{N + 1}\right) + \left(\frac{y}{N + 1}\right)\left(1 - \frac{x}{N + 1}\right)\end{aligned}$$

**Corollary 3.1.** *A particle system based on the reference generator with absorbing boundaries at  $\{0, N + 1\}$  as defined in section 5 with 2 particles in the starting configuration has the following property:*

$$\begin{aligned}q_1(w, u) &= \frac{1}{2} + \frac{u}{2(N + 1)} - \frac{q_0(w, u)}{2} \\ q_2(w, u) &= \frac{1}{2} - \frac{u}{2(N + 1)} - \frac{q_0(w, u)}{2}\end{aligned}$$

And thus in the positional coordinates they can be expressed as:

$$q_1(x, y) = \frac{x + y}{2(N + 1)} - \frac{q_0(x, y)}{2}$$

$$q_2(x, y) = 1 - \frac{x + y}{2(N + 1)} - \frac{q_0(x, y)}{2}$$

We were able use these theorems together with duality to describe the first and second moments of the stationary distribution of complex reference processes out of equilibrium.

We described the behaviour of the out of equilibrium stationary density fields and their variance. We noted three different scaling regimes depending on  $\gamma_N = \frac{\theta}{N+1}$ , relating the strength of the interaction  $\theta$  to the amount of nodes on the lattice  $V_N$ .

We noticed that for the regimes where  $\lim_{N \rightarrow \infty} \gamma_N = 0$  the variance tends to 0. This means that the fluctuations around the macroscopic profile (that is the linear function connecting the reservoirs densities) are small and tend to 0 as  $N \rightarrow \infty$ .

We noticed for the sticky regimes  $\lim_{N \rightarrow \infty} \gamma_N \in (0, \infty)$  that the stationary density field would get a finite non-zero variance. We know from [11] that the stationary regimes for particle systems on an infinite lattice become ‘piles’ which distance processes perform one sided sticky Brownian motion with non-zero local time. By examining the simulations, we expect the same thing to happen with the particle systems with reservoirs. This picture would be consistent with the emergence of a positive finite macroscopic variance.

Finally we have the absorbing regime  $\lim_{N \rightarrow \infty} \gamma_N = \infty$ . For this regime we have infinite variance for the stationary density field. We know from [11] that in the limiting distribution on the infinite lattice there will be similar ‘piles’ as with the sticky regime, however the distance processes will perform absorbing Brownian motion, meaning that once their distance become 0, their distance will remain 0 forever.

Finally we also examined the dynamics of the particle systems and we discovered how the expectation of the density field would converge to the heat equation with Dirichlet boundary conditions, which is consistent with our findings about the linear profile of the stationary distribution.

There is still further research that needs to be done towards the distribution of the stationary density field for out of equilibrium particle systems. Specifically, by finding relationships between the absorption probabilities for the dual process as was done in [5], will allow us to get the all of the higher moments in the particle system as well. Ultimately allowing us to get the full distribution of the stationary density field.

## A Appendix

### A.1 Derivation of the exact formula for $q_0$

*Proof.* In order to prove this formula for  $q_0$  we make use of the Markov property. We denote with abuse of notation  $S_0$  as the event that  $(w_n, u_n)$  is at some point equal to the sink  $S_0$  (which means  $\xi = (w, u) = (N + 1, 0)$ ). We denote with  $\Omega$  the state space of the particle system.

$$\begin{aligned}
q_0(\xi) &= P_\xi(S_0) \\
&= P(S_0 | \xi_0 = \xi) \\
&= \sum_{\xi' \in \Omega} P(S_0 | \xi_1 = \xi', \xi_0 = \xi) P(\xi_1 = \xi' | \xi_0 = \xi) \\
&= \sum_{\xi' \in \Omega} P(S_0 | \xi_1 = \xi') P(\xi_1 = \xi' | \xi_0 = \xi) \\
&= \sum_{\xi' \in \Omega} P_{\xi'}(S_0) P(\xi, \xi') \\
&= \sum_{\xi' \in \Omega} q_0(\xi') P(\xi, \xi')
\end{aligned}$$

Thus

$$q_0(\xi) = \sum_{\xi' \in \Omega} q_0(\xi') P(\xi, \xi')$$

This formula must hold for all  $N$  and for all states  $\xi = (w, u) \in \Lambda_N$ , as long as the two particles are on internal nodes, which means that  $w \neq N + 1 - |u|$ . We know all of the values of  $P(\xi, \xi')$  from section 6.1.

We start with the case  $w = 0$ . In this case we have two states we can transition to with non-zero probability. One or both of these states can be on  $w = N + 1 - |u|$ . We first check the case for when both of them are not on  $w = N + 1 - |u|$ .

$$\begin{aligned}
\sum_{\xi' \in \Omega} q_0(\xi') P((0, u), \xi') &= \frac{1}{2} q_0(1, u + 1) + \frac{1}{2} q_0(1, u - 1) \\
&= \frac{\gamma_N \frac{1}{N+1} + \frac{1 - (u+1)^2 + (N+1)^2}{2(N+1)^2}}{2(\gamma_N + 1)} + \frac{\gamma_N \frac{1}{N+1} + \frac{1 - (u-1)^2 + (N+1)^2}{2(N+1)^2}}{2(\gamma_N + 1)} \\
&= \frac{\gamma_N \frac{1}{N+1} + \frac{1 - \frac{(u+1)^2 + (u-1)^2}{2} + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{1}{N+1} + \frac{u^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= q_0(0, u)
\end{aligned}$$

We then check the case with  $w = 0$  when only the term  $q_0(1, u + 1)$  is on  $w = N + 1 - |u|$ . Which means  $u = N - 1$

$$\begin{aligned}
\sum_{\xi' \in \Omega} q_0(\xi') P((0, N - 1), \xi') &= \frac{1}{2} q_0(1, N) + \frac{1}{2} q_0(1, N - 2) \\
&= \frac{1}{2} \frac{1}{N + 1} + \frac{1}{2} \frac{\gamma_N \frac{1}{N+1} + \frac{1 - (N-2)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{1}{2} \frac{\gamma_N \frac{1}{N+1} + \frac{2N+2}{2(N+1)^2}}{\gamma_N + 1} + \frac{1}{2} \frac{\gamma_N \frac{1}{N+1} + \frac{1 - (N-2)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{1}{N+1} + \frac{2N+3 - (N-2)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{1}{N+1} + \frac{4N}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{1}{N+1} + \frac{-(N-1) + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} = q_0(0, N - 1)
\end{aligned}$$

We then check the case  $w = 0$  when both  $q_0(1, u + 1)$  and  $q_0(1, u - 1)$  are on  $w = N + 1 - |u|$ . This means  $N = 1$  and  $u = 0$ .

$$\begin{aligned}
\sum_{\xi' \in \Omega} q_0(\xi') P((0, 0), \xi') &= \frac{1}{2} q_0(1, 1) + \frac{1}{2} q_0(1, -1) \\
&= \frac{1}{2} \frac{1}{N + 1} + \frac{1}{2} \frac{1}{N + 1} \\
&= \frac{1}{N + 1} = q_0(0, 0)
\end{aligned}$$

We then move on to the case  $w = 1$  with all the adjacent states being internal

points as well.

$$\begin{aligned}
& \sum_{\xi' \in \Omega} q_0(\xi') P((1, u), \xi') \\
&= \frac{1+\theta}{4+2\theta} q_0(0, u+1) + \frac{1+\theta}{4+2\theta} q_0(0, u-1) + \frac{1}{4+2\theta} q_0(2, u+1) + \frac{1}{4+2\theta} q_0(2, u-1) \\
&= \frac{1+\theta}{4+2\theta} \frac{\gamma_N \frac{1}{N+1} + \frac{-(u+1)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} + \frac{1+\theta}{4+2\theta} \frac{\gamma_N \frac{1}{N+1} + \frac{-(u-1)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&+ \frac{1}{4+2\theta} \frac{\gamma_N \frac{2}{N+1} + \frac{4-(u+1)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} + \frac{1}{4+2\theta} \frac{\gamma_N \frac{2}{N+1} + \frac{4-(u-1)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{2+2\theta}{4+2\theta} \frac{\gamma_N \frac{1}{N+1} + \frac{-1-u^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} + \frac{2}{4+2\theta} \frac{\gamma_N \frac{2}{N+1} + \frac{3-u^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{1}{N+1}}{\gamma_N + 1} + \frac{2}{4+2\theta} \frac{\gamma_N \frac{1}{N+1}}{\gamma_N + 1} + \frac{2+2\theta}{4+2\theta} \frac{\frac{-1-u^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} + \frac{2}{4+2\theta} \frac{\frac{3-u^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{1}{N+1}}{\gamma_N + 1} + \frac{2}{4+2\theta} \frac{\gamma_N \frac{1}{N+1}}{\gamma_N + 1} + \frac{2\theta}{4+2\theta} \frac{\frac{-1-u^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} + \frac{4}{4+2\theta} \frac{\frac{1-u^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{1}{N+1} + \frac{1-u^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} + \frac{2}{4+2\theta} \frac{\gamma_N \frac{1}{N+1}}{\gamma_N + 1} + \frac{2\theta}{4+2\theta} \frac{\frac{-2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{1}{N+1} + \frac{1-u^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= q_0(1, u)
\end{aligned}$$

We then have the case with  $w = 1$  when only the term  $q_0(2, u+1)$  is on

$w = N + 1 - |u|$ . Which means  $u = N - 2$

$$\begin{aligned}
& \sum_{\xi' \in \Omega} q_0(\xi') P((1, u), \xi') \\
&= \frac{1+\theta}{4+2\theta} q_0(0, N-1) + \frac{1+\theta}{4+2\theta} q_0(0, N-3) + \frac{1}{4+2\theta} q_0(2, N-1) + \frac{1}{4+2\theta} q_0(2, N-3) \\
&= \frac{1+\theta}{4+2\theta} \frac{\gamma_N \frac{1}{N+1} + \frac{-(N-1)^2+(N+1)^2}{2(N+1)^2}}{\gamma_N + 1} + \frac{1+\theta}{4+2\theta} \frac{\gamma_N \frac{1}{N+1} + \frac{-(N-3)^2+(N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&+ \frac{1}{4+2\theta} \frac{2}{N+1} + \frac{1}{4+2\theta} \frac{\gamma_N \frac{2}{N+1} + \frac{4-(N-3)^2+(N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{1}{N+1}}{\gamma_N + 1} + \frac{1}{4+2\theta} \frac{2}{N+1} + \frac{1+\theta}{4+2\theta} \frac{\frac{-(N-1)^2+(N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&+ \frac{1+\theta}{4+2\theta} \frac{\frac{-(N-3)^2+(N+1)^2}{2(N+1)^2}}{\gamma_N + 1} + \frac{1}{4+2\theta} \frac{\frac{4-(N-3)^2+(N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{1}{N+1}}{\gamma_N + 1} + \frac{1}{4+2\theta} \frac{\frac{4\theta}{2(N+1)^2} + \frac{4(N+1)}{2(N+1)^2}}{\gamma_N + 1} \\
&+ \frac{1}{4+2\theta} \frac{\frac{4-(N-1)^2-2(N-3)^2+3(N+1)^2}{2(N+1)^2}}{\gamma_N + 1} + \frac{\theta}{4+2\theta} \frac{\frac{-(N-1)^2-(N-3)^2+2(N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{1}{N+1}}{\gamma_N + 1} + \frac{4}{4+2\theta} \frac{\frac{-N^2+4N-3+(N+1)^2}{2(N+1)^2}}{\gamma_N + 1} + \frac{2\theta}{4+2\theta} \frac{\frac{-N^2+4N-3+(N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{1}{N+1} + \frac{1-(N-2)^2+(N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= q_0(1, N-2)
\end{aligned}$$

We then check the case  $w = 1$  when both  $q_0(2, u+1)$  and  $q_0(2, u-1)$  are on

$w = N + 1 - |u|$ . This means  $N = 2$  and  $u = 0$ .

$$\begin{aligned}
& \sum_{\xi' \in \Omega} q_0(\xi') P((1, 0), \xi') \\
&= \frac{1+\theta}{4+2\theta} q_0(0, 1) + \frac{1+\theta}{4+2\theta} q_0(0, -1) + \frac{1}{4+2\theta} q_0(2, 1) + \frac{1}{4+2\theta} q_0(2, -1) \\
&= \frac{1+\theta}{4+2\theta} \frac{\gamma_N \frac{1}{3} + \frac{4}{9}}{\gamma_N + 1} + \frac{1+\theta}{4+2\theta} \frac{\gamma_N \frac{1}{3} + \frac{4}{9}}{\gamma_N + 1} + \frac{1}{4+2\theta} \frac{2}{3} + \frac{1}{4+2\theta} \frac{2}{3} \\
&= \frac{2+2\theta}{4+2\theta} \frac{\gamma_N \frac{1}{3} + \frac{4}{9}}{\gamma_N + 1} + \frac{4}{4+2\theta} \frac{1}{3} \\
&= \frac{2\gamma_N^2 + \frac{14\gamma_N}{3} + \frac{20}{9}}{(4+6\gamma_N)(\gamma_N + 1)} \\
&= \frac{\gamma_N \frac{1}{3} + \frac{5}{9}}{\gamma_N + 1} \\
&= q_0(1, 0)
\end{aligned}$$

We then check the case  $w \geq 2$  with all the adjacent points being internal nodes.

$$\begin{aligned}
& \sum_{\xi' \in \Omega} q_0(\xi') P((w, u), \xi') \\
&= \frac{1}{4} q_0(w-1, u+1) + \frac{1}{4} q_0(w-1, u-1) + \frac{1}{4} q_0(w+1, u+1) + \frac{1}{4} q_0(w+1, u-1) \\
&= \frac{1}{4} \frac{\gamma_N \frac{w}{N+1} + \frac{(w-1)^2 - (u+1)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} + \frac{1}{4} \frac{\gamma_N \frac{w}{N+1} + \frac{(w-1)^2 - (u-1)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&+ \frac{1}{4} \frac{\gamma_N \frac{w}{N+1} + \frac{(w+1)^2 - (u+1)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} + \frac{1}{4} \frac{\gamma_N \frac{w}{N+1} + \frac{(w+1)^2 - (u-1)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{w}{N+1}}{\gamma_N + 1} + \frac{1}{4} \frac{\frac{2(w-1)^2 + 2(w+1)^2 - 2(u+1)^2 - 2(u-1)^2 + 4(N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{w}{N+1}}{\gamma_N + 1} + \frac{1}{4} \frac{\frac{4w^2 - 4u^2 + 4(N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{w}{N+1} + \frac{w^2 - u^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= q_0(w, u)
\end{aligned}$$

We then have the case with  $w \geq 2$  when only the term  $q_0(w+1, u+1)$  is on



$w = N + 1 - |u|$ . Which means  $u = N - w - 1$

$$\begin{aligned}
& \sum_{\xi' \in \Omega} q_0(\xi') P((w, N - w - 1), \xi') \\
&= \frac{1}{4} q_0(w - 1, N - w) + \frac{1}{4} q_0(w - 1, N - w - 2) \\
&+ \frac{1}{4} q_0(w + 1, N - w) + \frac{1}{4} q_0(w + 1, N - w - 2) \\
&= \frac{1}{4} \frac{\gamma_N \frac{w}{N+1} + \frac{(w-1)^2 - (N-w)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} + \frac{1}{4} \frac{\gamma_N \frac{w}{N+1} + \frac{(w-1)^2 - (N-w-2)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&+ \frac{1}{4} \frac{w}{N+1} + \frac{1}{4} \frac{\gamma_N \frac{w}{N+1} + \frac{(w+1)^2 - (N-w-2)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{1}{4} \frac{3\gamma_N \frac{w}{N+1} + \frac{2(w-1)^2 + (w+1)^2 - 2(N-w-2)^2 - (N-w)^2 + 3(N+1)^2}{2(N+1)^2}}{\gamma_N + 1} + \frac{1}{4} \frac{\gamma_N \frac{w}{N+1} + \frac{2w(N+1)}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{w}{N+1}}{\gamma_N + 1} + \frac{1}{4} \frac{\frac{2(w-1)^2 + (w+1)^2 - 2(N-w-2)^2 - (N-w)^2 + 2w(N+1) + 3(N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{w}{N+1}}{\gamma_N + 1} + \frac{\frac{-N^2 + 2Nw + 2N - 2w - 1 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{w}{N+1} + \frac{w^2 - (N-w-1)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= q_0(w, N - w - 1)
\end{aligned}$$

We finally check the case  $w = 1$  when both  $q_0(w+1, u+1)$  and  $q_0(w+1, u-1)$

are on  $w = N + 1 - |u|$ . This means  $u = 0$  and  $w = N - 1$ .

$$\begin{aligned}
& \sum_{\xi' \in \Omega} q_0(\xi') P((N-1, 0), \xi') \\
&= \frac{1}{4} q_0(N-2, 1) + \frac{1}{4} q_0(N-2, -1) + \frac{1}{4} q_0(N, 1) + \frac{1}{4} q_0(N, -1) \\
&= \frac{1}{4} \frac{\gamma_N \frac{N-2}{N+1} + \frac{(N-2)^2 - 1^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&+ \frac{1}{4} \frac{\gamma_N \frac{N-2}{N+1} + \frac{(N-2)^2 - (-1)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} - \frac{1}{4} \frac{N}{N+1} + \frac{1}{4} \frac{N}{N+1} \\
&= \frac{1}{2} \frac{\gamma_N \frac{N-2}{N+1} + \frac{(N-2)^2 - 1 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} + \frac{1}{2} \frac{\gamma_N \frac{N}{N+1} + \frac{N}{N+1}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{N-1}{N+1}}{\gamma_N + 1} + \frac{1}{2} \frac{\frac{(N-2)^2 - 1 + (N+1)^2}{2(N+1)^2} + \frac{2N(N+1)}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{N-1}{N+1} + \frac{2N^2 + 2}{2(N+1)^2}}{\gamma_N + 1} \\
&= \frac{\gamma_N \frac{N-1}{N+1} + \frac{(N-1)^2 + (N+1)^2}{2(N+1)^2}}{\gamma_N + 1} \\
&= q_0(N-1, 0)
\end{aligned}$$

□

## A.2 Code

The following code is used for the particle simulation. The code is written in python 2.7.12, using Numpy 1.11.2, Matplotlib 1.5.3 and Scipy 0.18.1.

```

import numpy as np
from scipy.spatial.distance import pdist, squareform

import matplotlib.pyplot as plt
import scipy.integrate as integrate
import matplotlib.animation as animation
import random as rnd

class ParticleLine:
    def __init__(self, init_state = [1,2,3,4],
                 alphas = 15,
                 theta = 55, alpha=0, gamma=0, delta=0, beta= 0):

```

```

#self.length is equal to the length N of the lattice V_N.
self.length = len(init_state)
#self.rates is an array of dimension [N,2] and represents the jump
#rate of each particle with [i,0] being the jump rate of particle i
#to the left and [i,1] being the jump rate of particle i to the right.
self.rates = np.zeros([self.length,2])
#self.chances is an array of dimension [N+2,2] and it represents
#a sample of exponential distributions of each of the particles.
#The first and last element [0,j] and [N+1,j] represent the odds
#of a particle jumping from a reservoir onto the lattice.
self.chances = np.zeros([self.length + 2,2])

#self.init_state is an array of length N which indicates the amount
#of particles at each site at the start of the simulation.
self.init_state = np.asarray(init_state, dtype=int)
#self.init_state is an array of length N which indicates the amount
#of particles at each site during the simulation.
self.state = self.init_state.copy()
#self.time_elapsed keeps track of the time t that has
#elapsed since the start of the simulation.
self.time_elapsed = 0

#self.alpharate is the rate that indicates the speed of the simulation.
#The alpharate is multiplied against the entire generator,
#so no additional scaling of parameters is required.
self.alpharate = alpharate
#self.theta is the theta parameter for the generator and
#indicates the attractiveness between the particles.
self.theta = theta

#self.alpha is the left reservoir birth rate
self.alpha = alpha
#self.gamma is the left reservoir death rate
self.gamma = gamma

#self.delta is the left reservoir birth rate
self.delta = delta
#self.beta is the right reservoir death rate
self.beta = beta

#self.timeleft indicates how much time there is left in the frame
self.timeleft = 0
#self.updated indicates if the changes that have been made
#during the previous frame have been applied
self.updated = True

```

```

#self.minx and self.y are used for determining which
#particle should jump and to what location
self.minx = -99
self.miny = -99

#self.boundratea indicates the calculated rate for the left
#boundary, and self.boundrateb respectively for the right boundary
self.boundratea = 0
self.boundrateb = 0

#calcRates is used for calculating the rates of the all
#of the particles for the general processes
def calcRates(self):

    #We will first go through the rates of all of the internal nodes
    for i in range(len(self.state)):
        # i == 0 indicates the case of the leftmost internal node
        #and takes into account the death rate of the left reservoir
        if i == 0:
            self.rates[i] = [ self.theta*self.gamma*self.state[i], \
                               self.state[i]*(1+self.theta*self.state[i+1])]

            continue
        # i == self.length-1 indicates the case of the rightmost internal
        #node and takes into account the death rate of the left reservoir
        if i == self.length-1:
            self.rates[i] = [ self.state[i]*(1+self.theta*self.state[i-1]), \
                               self.theta*self.beta*self.state[i]]

            continue
        # This line is for calculating the rates on the internal nodes
        self.rates[i] = [ self.state[i]*(1+self.theta*self.state[i-1]), \
                           self.state[i]*(1+self.theta*self.state[i+1])]

    # In this line we speed up all the rates depending
    # on the paramater self.alpharate
    self.rates = self.rates*self.alpharate/2

    # We assign the rates for the left and right boundary
    self.boundratea = self.alpha* \
        (1+self.theta*self.state[0])*self.alpharate/2
    self.boundrateb = self.delta* \
        (1+self.theta* self.state[len(self.state)-1])*self.alpharate/2

#calcRates is used for calculating the rates of the all
#of the particles for the reference processes
def calcRefRates(self):

```

```

#We will first go through the rates of all of the internal nodes
for i in range(len(self.state)):
    # i == 0 indicates the case of the leftmost internal node
    #and takes into account the death rate of the left reservoir
    if i == 0:
        self.rates[i] = [ (1+self.theta*self.alpha)*self.state[i], \
                           self.state[i]*(1+self.theta*self.state[i+1])]
        continue
    # i == self.length-1 indicates the case of the rightmost internal
    #node and takes into account the death rate of the left reservoir
    if i == self.length-1:
        self.rates[i] = [ self.state[i]*(1+self.theta*self.state[i-1]),\
                           (1+self.theta*self.delta)*self.state[i]]
        continue

    self.rates[i] = [ self.state[i]*(1+self.theta*self.state[i-1]), \
                      self.state[i]*(1+self.theta*self.state[i+1])]

# In this line we speed up all the rates depending
# on the paramater self.alpharate
self.rates = self.rates*self.alpharate/2

# We assign the rates for the left and right boundary
self.boundratea = self.alpha* \
    (1+self.theta*self.state[0])*self.alpharate/2
self.boundrateb = self.delta* \
    (1+self.theta*self.state[len(self.state)-1])*self.alpharate/2

# In ratesToExp we use the rates to sample from the exponential
#distribution to see which which particle should jump first
def ratesToExp(self):

    # This line samples the exponential distribution for the left boundary
    self.chances[0] = [ float("inf"), float("inf") \
                        if self.boundratea==0 else rnd.expovariate(self.boundratea)]

    # This line samples the exponential distribution for the internal nodes
    for i in range(len(self.state)):
        self.chances[i+1] = [ float("inf") if self.rates[i][0]==0 \
                               else rnd.expovariate(self.rates[i][0]),
                               float("inf") if self.rates[i][1]==0 \
                               else rnd.expovariate(self.rates[i][1])]

    # This line samples the exponential distribution for the right boundary

```

```

self.chances[len(self.state)+1] = [ float("inf") if self.boundrateb==0\
                                     else rnd.expovariate(self.boundrateb), float("inf") ]

# This function performs one timestep of the simulation ,
#the timestep length is given by dt
def step(self , dt):

    self.time_elapsed += dt
    self.timeleft += dt

    # Timeleft indicates how much time we still have for this timestep ,
    #when we have no time left , we will go to the next frame
    while self.timeleft > 0:

        # This function checks if we applied the last changes from the
        #previous frame, in case we had no time anymore
        if not self.updated:

            # We first check if we have to remove a particle , if the
            #jump event came from one of the internal nodes
            if (self.minx > 0) and (self.minx < len(self.chances)-1 ):
                self.state[self.minx-1 ] -= 1
            # We then check if we need to add a patricl , if the jump
            #event is into one of the internal nodes
            if ( (self.minx-1 + self.miny*2-1) > -1) and \
                ( (self.minx-1 + self.miny*2-1) < len(self.state) ):
                self.state[self.minx-1 + self.miny*2-1] += 1
            self.updated = True

        # We first update the rates to our current configuration
        self.calcRefRates()
        # We can uncomment self.calcRates if we want to make use of
        # the general particle simulator
        #self.calcRates()
        # We then calculate the waiting time for each event
        self.ratesToExp()

    minvalue = float("inf") ;

    # We then look up what event has the lowest waiting
    # time and save its index in self.minx and self.miny
    # which wil be used in the next update

```

```

        minvalue = np.amin(self.chances)
        result = np.where(self.chances == minvalue)
        indices = zip(result[0], result[1])
        self.minx = indices[0][0]
        self.miny = indices[0][1]

    self.updated = False
    self.timeleft -= minvalue

#-----
# Set up initial state
np.random.seed(0)

box = []

#RenderExpectation is for rendering the Analytical expectation
#RenderExpectationLongApprox is for rendering the
#long term approximation for the expectation
#RenderExpectationApprox is for rendering the
#short term approximation for the expectation
RenderExpectation = True
RenderExpectationLongApprox = True
RenderExpectationApprox = True

# We then have several processes which we can run
if 0:
    #Expectation inclusion process in equilibrium
    init_state = [4]*20

    Theta = 0.5
    Alpha = 5
    Delta = 5

    box = ParticleLine(init_state, alpharate = 1, \
                        theta = Theta,
                        alpha = Alpha, gamma = Alpha + float(1)/Theta, \
                        delta = Delta, beta = Delta+ + float(1)/Theta)

elif 0:

```

```

#Expectation inclusion process out of equilibrium
init_state = [4]*20

Theta = 2
Alpha = 5
Delta = 15

box = ParticleLine(init_state , alpharate = 11, \
                    theta = Theta, \
                    alpha = Alpha, gamma = float(1)/Theta, \
                    delta = Delta, beta = Delta+ + float(1)/Theta)

elif 0:
    #Expectation exclusion process out of equilibrium, theta = -1
    init_state = [0]*20
    init_state[10] = 1
    Theta = -1
    Alpha = 0
    Delta = 1

    box = ParticleLine(init_state , alpharate = 100, \
                        theta = Theta, \
                        alpha = Alpha, gamma = -float(1)/Theta - Alpha, \
                        delta = Delta, beta = -float(1)/Theta - Delta )

elif 0:
    #Expectation exclusion process out of equilibrium, theta = -0.2
    RenderExpectation = False
    RenderExpectationLongApprox = False
    RenderExpectationApprox = False
    init_state = [3]*20
    Theta = -0.2
    Alpha = 0
    Delta = 5

    box = ParticleLine(init_state , alpharate = 10, \
                        theta = Theta, \
                        alpha = Alpha, gamma = -float(1)/Theta - Alpha, \
                        delta = Delta, beta = -float(1)/Theta - Delta )

elif 1:
    #Expectation independent walker out of equilibrium
    RenderExpectation = False
    init_state = [0]*20
    Theta = 0
    Alpha = 5
    Delta = 15

```



```

box = ParticleLine(init_state , alphasrate = 20, \
                    theta = Theta,
                    alpha = Alpha, gamma = 1, \
                    delta = Delta, beta = 1 )

else:
    #Expectation inclusion process theta >> 0
    init_state = [4]*20

    Theta = 200
    Alpha = 5
    Delta = 15

    box = ParticleLine(init_state , alphasrate = 1, \
                        theta = Theta,
                        alpha = Alpha + float(1)/Theta, \
                        delta = Delta, beta = Delta+  + float(1)/Theta)

dt = 1. / 30 # 30fps

#-----
# Setting up the plot and animation
fig = plt.figure(figsize=(8, 7))

fig.subplots_adjust(left=0.08, right=.92, bottom=0.08, top=.92)
fig.suptitle('Reference_particle_system_with_node_averages', fontsize=16)

ax = fig.add_subplot(111, aspect='equal', autoscale_on=False, \
xlim=(-.3, len(init_state)+1.7), ylim=(-1, 6 + 1.6*max(box.delta ,box.alpha)))

ax.set_xlabel('Position')
ax.set_ylabel('Particle_Count')

# Particles holds the locations of the particles

```

```

particles , = ax.plot([], [], 'bo', ms=6)
# Avg holds the average of the sites for the long term
avg, = ax.plot([], [], 'r', ms=16)
# Avg holds the average of the sites for the short term
avg2, = ax.plot([], [], 'c', ms=16)

# This plots the analytical expectation
if RenderExpectation:
    linex = np.linspace(1, len(box.state), 100)
    liney = box.alpha + (box.delta - box.alpha) * linex / (len(box.state) + 1)
    plt.plot(linex, liney, '-y', label='y=2x+1')

timestep = 1
#stephistory indicates the amount of timesteps for the long-term approximation
stephistory = 8000
avgdata = np.zeros((stephistory, len(init_state)))
#stephistory indicates the amount of timesteps for the short-term approximation
stephistory2 = 100
avgdata2 = np.zeros((stephistory2, len(init_state)))
def animate(i):
    """perform animation step"""

    global box, line, dt, ax, fig, timestep, avgdata
    timestep += 1

    if timestep % 1000 == 0:
        print "Current_frame:", timestep

    box.step(dt)

    #ms = int(fig.dpi * 2 * fig.get_figwidth()
    #        / np.diff(ax.get_xbound())[0])

    x = []
    y = []

    #This is for generating all the particle points for the plot
    for i in range(0, len(box.state)):
        for j in range(0, box.state[i]):
            x = x + [i+1]
            y = y + [j+1]

```

```

#particles.set_data(range(0,len(box.state)), box.state)
particles.set_data( x, y)

#This is for rendering the long term approximation of the expectation
if RenderExpectationApprox:
    for i in range(0,stephistory-1):
        avgdata[i] = avgdata[i+1]
    avgdata[stephistory-1] = np.copy(box.state)
    temp = np.sum(avgdata,0).astype(float)/min(stephistory ,timestep)
    avg.set_data(range(1,len(box.state)+1), temp )

#This is for rendering the short term approximation of the expectation
if RenderExpectationLongApprox:
    for i in range(0,stephistory2-1):
        avgdata2[i] = avgdata2[i+1]
    avgdata2[stephistory2-1] = np.copy(box.state)
    temp2 = np.sum(avgdata2,0).astype(float)/min(stephistory2 ,timestep)
    avg2.set_data(range(1,len(box.state)+1), temp2 )

return particles

ani = animation.FuncAnimation(fig , animate ,
                             interval=dt*1000, blit=False)

plt.show()

```

## References

- [1] V. Hakim B. Derrida, M.R. Evans and V. Pasquier. Exact solution of a 1d assymetric exclusion model using a matrix formulation. *Journal of Statistical Physics*, A26:1493–1517, 1993.
- [2] F. Redig C. Giardinà, J. Kurchan and K. Vafayi. Duality and hidden symmetries in interacting particle systems. *Journal of Statistical Physics*, 135:25–55, 2009.
- [3] D. Mukamel E. Levine and G. M. Schütz. Zero-range process with open boundaries. *Journal of Statistical Physics*, 120(5):759–778, 2005.
- [4] S. N. Ethier and T. G. Kurtz. *Markov processes: Characterization and convergence*. 1986.
- [5] C. Giardinà G. Carinci and F. Redig. Consistent particle systems and duality. in preparation.
- [6] C. Giberti G. Carinci, C. Giardinà and F. Redig. Duality for stochastic models of transport. *Journal of Statistical Physics*, 152, 2012.
- [7] C. Giardinà, J. Kurchan, and F. Redig. Duality and exact correlations for a model of heat conduction. *J. Math. Phys.*, 48, 2007.
- [8] C. Giardinà, F. Redig, and K. Vafayi. Correlation inequalities for interacting particle systems with duality. *Journal of Statistical Physics*, 141(2):242–263, 2010.
- [9] F. B. Knight. *Essentials of Brownian motion and diffusion*. American Mathematical Society, 2009.
- [10] D. Gabrielli L. Bertini, A. De Sole, G. Jona-Lasinio, and C. Landim. Stochastic interacting particle systems out of equilibrium. *Journal of Statistical Mechanics: Theory and Experiment*, 2007(07):P07014–P07014, 2007.
- [11] F. Redig S. Grosskinsky and K. Vafayi. Condensation in the inclusion process and related models. *Journal of Statistical Physics*, 142(5):952–974, 2011.
- [12] G. Schütz and S. Sandow. Non-abelian symmetries of stochastic processes: Derivation of correlation functions for random-vertex models and disordered-interacting-particle systems. *Phys. Rev. E*, 49:2726–2741, 1994.
- [13] F. Spitzer. Interaction of markov processes. *Advances in Mathematics*, 5(2):246 – 290, 1970.
- [14] H. Spohn. Long range correlations for stochastic lattice gases in a non-equilibrium steady state. *Journal of Physics A: Mathematical and General*, 16(18):4275–4291, 1983.