DELFT UNIVERSITY OF TECHNOLOGY

REPORT 08-06

LIMIT CYCLE BIFURCATIONS IN A QUADRATIC SYSTEM
WITH TWO PARALLEL STRAIGHT LINE-ISOCLINES

V.A. GAIKO

[Limit Cycle Bifurcations in a Quadratic System with Two Parallel Straight Line-Isoclines]

[V.A. Gaiko]

Limit Cycle Bifurcations in a Quadratic System with Two Parallel Straight Line-Isoclines [⋆]

Valery A. Gaiko [1]

Belarusian State University of Informatics and Radioelectronics,
Department of Mathematics, L. Beda Str. 6–4, Minsk 220040, Belarus

Abstract

In this paper, a quadratic system with two parallel straight line-isoclines is considered. This system corresponds to the system of class II in the classification of Ye Yanqian [13]. Using the field rotation parameters of the constructed canonical system and geometric properties of the spirals filling the interior and exterior domains of its limit cycles, we prove that the maximum number of limit cycles in a quadratic system with two parallel straight line-isoclines and two finite singular points is equal to two. Besides, we obtain the same result in a different way: applying the Wintner–Perko termination principle for multiple limit cycles and using the methods of global bifurcation theory developed in [7].

Keywords: planar quadratic dynamical system; isocline; field rotation parameter; bifurcation; limit cycle; Wintner–Perko termination principle

1 Introduction

We consider the system of differential equations

$$
\begin{aligned}
\dot{x} &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \\
\dot{y} &= b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2
\end{aligned}
\tag{1.1}
$$

with the real coefficients a_{ij}, b_{ij} in the real variables x, y, where at least one quadratic term has a coefficient unequal to zero. Such a system will be referred to as a quadratic system. The main problem of the qualitative theory of system (1.1) is *Hilbert's Sixteenth Problem* on the maximum number and relative position of its limit cycles, i.e., closed isolated trajectories of (1.1) [1], [7], [10]–[13]. The solution of this problem could give us all possible phase pertraits of system (1.1) and, thus, could complete its qualitative analysis.

Earlier [4]–[9], we studied a quadratic system with two intersecting straight line-isoclines. In this paper, we will study a quadratic system with two parallel straight line-isoclines. Such a system corresponds to the system of class II in the classification of Ye Yanqian [13]:

$$
\begin{aligned}
\dot{x} &= \lambda\, x - y + l\, x^2 + m\, xy + n\, y^2, \\
\dot{y} &= x + x^2.
\end{aligned}
\tag{1.2}
$$

Applying a new geometric approach to the study of limit cycle bifurcations developed in [9], we will prove that a quadratic system with two parallel sraight line-isoclines and two finite singular points has at most two limit cycles (a quadratic system with one sraight line-isocline was studied in detail in [1]). The same result will be obtained in a different way: applying the Wintner–Perko termination principle for multiple limit cycles [11] and using the methods of global bifurcation theory developed in [7].

In particular, in Section 2, we construct a canonical system with field rotation parameters corresponding to the system of class II in the classification of Ye Yanqian [13]. In Section 3, using the canonical system and geometric properties of the spirals filling the interior and exterior domains of limit cycles, we obtain the main result of this paper on the maximum number of limit cycles of a quadratic system with two parallel straight line-isoclines. In Section 4, we obtain the same result applying the Wintner–Perko termination principle for multiple limit cycles.

2 Canonical systems

First, we have to construct canonical quadratic systems with field rotation parameters for studying limit cycle bifurcations. The following theorem is valid.

Theorem 2.1. *A quadratic system with limit cycles can be reduced to the*

canonical form

$$\dot{x} = -y\,(1 + x + \alpha\,y),$$
$$\dot{y} = x + (\lambda + \beta + \gamma)y + a\,x^2 + (\alpha + \beta + \gamma)xy + c\,\gamma\,y^2 \qquad (2.1)$$

or

$$\dot{x} = -y\,(1 + \nu\,y) \equiv P, \quad \nu = 0;\,1,$$
$$\dot{y} = x + (\lambda + \beta + \gamma)y + a\,x^2 + (\beta + \gamma)xy + c\,\gamma\,y^2 \equiv Q. \qquad (2.2)$$

Proof. As was shown in [7], by means of Erugin's two-isocline method [3], an arbitrary quadratic system with limit cycles can be reduced to the form

$$\dot{x} = -y + mxy + ny^2,$$
$$\dot{y} = x + \lambda y + ax^2 + bxy + cy^2, \qquad (2.3)$$

where $m = -1$ or $m = 0$.

Input the field rotation parameters into this system so that (2.1) corresponds to the case of $m = -1$ and (2.2) corresponds to the case of $m = 0$.

Compare (2.1) with (2.3) when $m = -1$. Firstly, we have changed several parameters: n by $-\alpha$; b by β; c by $c\,\gamma$. Secondly, we have input additional terms into the expression for $\dot{y}$: $(\beta + \gamma)\,y$ and $(\alpha + \gamma)\,xy$. Similar transformations have been made in system (2.3) when $m = 0$; but in this case, we have denoted n by ν assigning two principal values to this parameter: 0 and 1. It is obvious that all these transformations do not restrict generality of systems (2.1) and (2.2) in comparison with system (2.3), which proves the theorem. $\square$

System (2.1) is a basic system for studying limit cycle bifurcations. It contains four field rotation parameters: λ, α, β, γ. This system has been considered in [9]. Now we will consider system (2.2). The following lemma is valid for (2.2).

Lemma 2.1. *Each of the parameters λ, β, and γ rotates the vector field of system (2.2) in the domains of existence of its limit cycles, under the fixed other parameters of this system, namely: when the parameter λ, β, or γ increases (decreases), the field is rotated in positive (negative) direction, i. e., counterclockwise (clockwise), in the domains, respectively:*

$$1 + \nu\,y < 0 \ (> 0);$$
$$(1 + x)(1 + \nu\,y) < 0 \ (> 0);$$
$$(1 + x + c\,y)(1 + \nu\,y) < 0 \ (> 0).$$

3

Proof. Using the definition of a field rotation parameter [2], [7], we can calculate the following determinants:

$$\Delta_\lambda = PQ'_\lambda - QP'_\lambda = -y^2(1 + \nu\,y);$$
$$\Delta_\beta = PQ'_\beta - QP'_\beta = -y^2(1 + x)(1 + \nu\,y);$$
$$\Delta_\gamma = PQ'_\gamma - QP'_\gamma = -y^2(1 + x + c\,y)(1 + \nu\,y).$$

Since, by definition, the vector field is rotated in positive direction (counterclockwise) when the determinant is positive and in negative direction (clockwise) when the determinant is negative [2], [7] and since the obtained domains correspond to the domains of existence of limit cycles of (2.2), the lemma is proved. □

3   Limit cycle bifurcations

We will study limit cycle bifurcations of canonical system (2.2) with two parallel straight line-isoclines and three field rotation parameters. We will consider the case when system (2.2) has only two finite singularities: a saddle and an anti-saddle (all other cases can be considered in an absolutely similar way). Let us prove the following theorem.

Theorem 3.1. *System (2.2) with two parallel straight line-isoclines and two finite singular points can have at least two limit cycles surrounding the origin.*

Proof. To prove the theorem, fix, for example, $a = 1$ and take $c > 1$ in system (2.2) for $\nu = 1$. Then vanish all field rotation parameters of (2.2), $\beta = \gamma = \lambda = 0$:

$$\dot{x} = -y\,(1 + y),$$
$$\dot{y} = x + x^2. \tag{3.1}$$

We have got a system with the zero divergence and four finite singular points: two centers and two saddles (a Hamiltonian case).

Input, for example, a positive parameter γ into system (3.1):

$$\dot{x} = -y\,(1 + y),$$
$$\dot{y} = x + \gamma\,y + x^2 + \gamma\,xy + c\,\gamma\,y^2. \tag{3.2}$$

On increasing the parameter γ, the vector field of (3.2) is rotated in negative direction (clockwise) and the center at the origin turns into an unstable focus.

4

Suppose that γ satisfies the condition

$$-1 + 2\left(c - \sqrt{c(c-1)}\right) < \gamma < -1 + 2\left(c + \sqrt{c(c-1)}\right). \qquad (3.3)$$

In this case we will have only two finite sigularities: a saddle $(-1, 0)$ and an anti-saddle $(0, 0)$.

Fix γ and input a negative parameter β into system (3.2):

$$
\begin{aligned}
\dot{x} &= -y\,(1 + y), \\
\dot{y} &= x + (\beta + \gamma)\,y + x^2 + (\beta + \gamma)\,xy + c\,\gamma\,y^2.
\end{aligned}
\qquad (3.4)
$$

On decreasing the parameter β, the vector field of (3.4) is rotated in positive direction, and, for some value β_S of the parameter β, a separatrix loop is formed around the origin generating a stable limit cycle for $\beta < \beta_S$ (an unstable limit cycle cannot appear from the origin because of the negative first focus quantity at the origin for $\gamma > 0$ when $\beta + \gamma = 0$ [1]).

Fix β satisfying the condition $0 < \beta + \gamma \ll 1$ and input a positive parameter λ into system (3.4):

$$
\begin{aligned}
\dot{x} &= -y\,(1 + y), \\
\dot{y} &= x + (\lambda + \beta + \gamma)\,y + x^2 + (\beta + \gamma)\,xy + c\,\gamma\,y^2.
\end{aligned}
\qquad (3.5)
$$

To have still two finite singularities, we also suppose that the parameters β, γ, and λ satisfy the condition

$$-1 - \sqrt{4c\gamma - 1} < \beta + \gamma + \lambda < -1 + \sqrt{4c\gamma - 1}. \qquad (3.6)$$

On increasing the parameter λ, the vector field of (3.5) is rotated in negative direction, and, for some value $\lambda = \lambda_S$, a separatrix loop is formed around the origin again generating an unstable limit cycle for $\lambda > \lambda_S$ (the stable limit cycle cannot disappear through the loop because of the positive divergence at the saddle $(-1, 0)$ for $\lambda > 0$ [1]).

Thus, we have obtained at least two limit cycles surrounding the focus $(0, 0)$, which proves the theorem. $\square$

Let us prove now a much stronger theorem (it is the main result of our paper).

Theorem 3.2. *System (2.2) with two parallel straight line-isoclines and two finite singular points has at most two limit cycles surrounding the origin.*

Proof. Consider again system (2.2) for $\nu = 1$, $a = 1$, and $c > 1$ supposing

5

that condition (3.6) is also valid. All other particular cases of (2.2) can be considered in a similar way.

Vanishing all field rotation parameters of system (3.1), $\beta = \gamma = \lambda = 0$, we have got again Hamiltonian system (3.1) with four finite singular points: two centers and two saddles. Let us input successively the field rotation parameters into (3.1).

Begin, for example, with the parameter γ supposing that $\gamma > 0$. Then we get system (3.2). On increasing the parameter γ, the vector field of (3.2) is rotated in negative direction (clockwise) and the center at the origin turns into an unstable focus. We also suppose that γ satisfies condition (3.3) and that we have only two finite sigularities in this case: a focus $(0,0)$ and a saddle $(-1,0)$.

Fix γ and input a negative parameter β getting system (3.4). On decreasing the parameter β, the vector field of (3.4) is rotated in positive direction, and, for some value β_S of the parameter β, a separatrix loop is formed around the origin generating a stable limit cycle for $\beta < \beta_S$. As we noted above, a limit cycle cannot appear from the origin because of the negative first focus quantity at the origin for $\gamma > 0$ when $\beta + \gamma = 0$ [1].

Denote the limit cycle by Γ_1, the domain inside the cycle by D_1, the domain outside the cycle by D_2 and consider logical possibilities of the appearance of other (semi-stable) limit cycles from a "trajectory concentration" surrounding the focus $(0,0)$. It is clear that on decreasing β, a semi-stable limit cycle cannot appear in the domain D_2, since the outside spirals winding onto the cycle will untwist and the distance between their coils will increase because of the vector field rotation in positive direction.

By contradiction, we can also prove that a semi-stable limit cycle cannot appear in the domain D_1. Suppose it appears in this domain for some values of the parameters $\gamma^* > 0$ and $\beta^* < 0$. Return to initial system (3.1) and change the order of inputting the field rotation parameters. Input first the parameter $\beta < 0$:

$$\begin{aligned} \dot{x} &= -y\,(1+y), \\ \dot{y} &= x + \beta\,y + x^2 + \beta\,xy. \end{aligned} \tag{3.7}$$

Fix it under $\beta = \beta^*$. The vector field of (3.7) is rotated in negative direction and $(0,0)$ becomes a stable focus. Inputting the parameter $\gamma > 0$ into (3.7), we have got again system (3.4), the vector field of which is rotated in positive direction. Under this rotation, for $\gamma = -\beta$, the focus $(0,0)$ changes the character of its stability, and a stable limit cycle Γ_1 appears from the origin. This cycle will expand, the focus spirals will untwist, and the distance between their coils will increase on increasing the parameter γ to the value $\gamma = \gamma^*$. It

follows that there are no values of $\gamma = \gamma^*$ and $\beta = \beta^*$, for which a semi-stable limit cycle could appear in the domain D_1.

This contradiction proves the uniqueness of a limit cycle surrounding the focus $(0,0)$ in system (3.4) for any values of the parameters γ and λ of different signs. Obviously, if these parameters have the same sign, system (3.4) has no limit cycles surrounding $(0,0)$ at all.

Let system (3.4) have the unique limit cycle Γ_1. Fix the parameters $\gamma > 0$, $\beta < 0$ and input the third parameter, $\lambda > 0$, getting system (3.5). On increasing the parameter λ, the vector field of (3.5) is rotated in negative direction, and, for some value $\lambda = \lambda_S$, a separatrix loop is formed around the origin again generating an unstable limit cycle for $\lambda > \lambda_S$. Note again that a limit cycle cannot disappear through the loop because of the positive divergence at the saddle $(-1, 0)$ for $\lambda > 0$ [1]. Denote this cycle by Γ_2. On further increasing λ, the limit cycle Γ_2 will join with Γ_1 forming a semi-stable limit cycle, Γ_{12}, which will disappear in a "trajectory concentration" surrounding the origin $(0,0)$. Can another semi-stable limit cycle appear around the origin in addition to Γ_{12}? It is clear that such a limit cycle cannot appear either in the domain D_1 bounded by the origin and Γ_1 or in the domain D_3 bounded on the inside by Γ_2 because of the increasing distance between the spiral coils filling these domains on increasing λ.

To prove impossibility of the appearance of a semi-stable limit cycle in the domain D_2 bounded by the cycles Γ_1 and Γ_2 (before their joining), suppose the contrary, i. e., for some set of values of the parameters $\gamma^* > 0$, $\beta^* < 0$, and $\lambda^* > 0$, such a semi-stable cycle exists. Return to system (3.1) again and input first the parameters $\gamma > 0$ and $\lambda > 0$:

$$
\begin{aligned}
\dot{x} &= -y\,(1+y), \\
\dot{y} &= x + (\lambda + \gamma)\,y + x^2 + \gamma\,xy + c\,\gamma\,y^2.
\end{aligned}
\tag{3.8}
$$

Both parameters act in a similar way: they rotate the vector field of (3.7) in negative direction turning the origin $(0,0)$ into an unstable focus.

Fix these parameters under $\gamma = \gamma^*$, $\lambda = \lambda^*$ and input the parameter $\beta < 0$ into (3.8) getting again system (3.5). Since, in our assumption, this system has two limit cycles for $\beta < \beta^*$, there exists some value of the parameter, β_{12} ($\beta^* < \beta_{12} < 0$), for which a semi-stable limit cycle, Γ_{12}, appears in system (3.5) and then splits into a stable cycle, Γ_1, and an unstable cycle, Γ_2, on further decreasing β. The formed domain D_2 bounded by the limit cycles Γ_1, Γ_2 and filled by the spirals will enlarge, since, by the properties of a field rotation parameter, the interior stable limit cycle Γ_1 will contract and the exterior unstable limit cycle Γ_2 will expand on decreasing β. The distance between the spirals of the domain D_2 will naturally increase, which will prohibit the

7

appearance of a semi-stable limit cycle in this domain for $\beta < \beta_{12}$. Thus, there are no such values of the parameters, $\gamma^* > 0$, $\lambda^* > 0$, $\beta^* < 0$, for which system (3.5) would have an additional semi-stable limit cycle.

Obviously, there are no other values of the parameters λ, β, γ, for which system (3.5) would have more than two limit cycles surrounding the origin $(0,0)$. It follows that system (3.5) and, hence, system (2.2) can have at most two limit cycles. The theorem is proved. $\qquad\square$

4   The Wintner–Perko termination principle

In [7], for the global analysis of limit cycle bifurcations, we used the Wintner–Perko termination principle which was stated for relatively prime, planar, analytic systems and which connected the main bifurcations of limit cycles [11]. Let us formulate this principle for the polynomial system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}), \qquad\qquad (4.1_{\boldsymbol{\mu}})$$

where $\boldsymbol{x} \in \mathbf{R}^2$; $\boldsymbol{\mu} \in \mathbf{R}^n$; $\boldsymbol{f} \in \mathbf{R}^2$ ($\boldsymbol{f}$ is a polynomial vector function).

Theorem 4.1 (Wintner–Perko termination principle). *Any one-parameter family of multiplicity-m limit cycles of relatively prime polynomial system $(4.1_{\boldsymbol{\mu}})$ can be extended in a unique way to a maximal one-parameter family of multiplicity-m limit cycles of $(4.1_{\boldsymbol{\mu}})$ which is either open or cyclic.*

If it is open, then it terminates either as the parameter or the limit cycles become unbounded; or, the family terminates either at a singular point of $(4.1_{\boldsymbol{\mu}})$, which is typically a fine focus of multiplicity m, or on a (compound,) separatrix cycle of $(4.1_{\boldsymbol{\mu}})$, which is also typically of multiplicity m.

The proof of the Wintner–Perko termination principle for general polynomial system $(4.1_{\boldsymbol{\mu}})$ with a vector parameter $\boldsymbol{\mu} \in \mathbf{R}^n$ parallels the proof of the planar termination principle for the system

$$\dot{x} = P(x, y, \lambda), \quad \dot{y} = Q(x, y, \lambda) \qquad\qquad (4.1_{\lambda})$$

with a single parameter $\lambda \in \mathbf{R}$ (see [7], [11]), since there is no loss of generality in assuming that system $(4.1_{\boldsymbol{\mu}})$ is parameterized by a single parameter λ; i. e., we can assume that there exists an analytic mapping $\boldsymbol{\mu}(\lambda)$ of $\mathbf{R}$ into $\mathbf{R}^n$ such that $(4.1_{\boldsymbol{\mu}})$ can be written as $(4.1_{\boldsymbol{\mu}(\lambda)})$ or even (4.1_{λ}) and then we can repeat everything, which had been done for system (4.1_{λ}) in [11]. In particular, if λ is a field rotation parameter of (4.1_{λ}), the following Perko's theorem on monotonic families of limit cycles is valid.

Theorem 4.2. *If L_0 is a nonsingular multiple limit cycle of (4.1_0), then L_0 belongs to a one-parameter family of limit cycles of (4.1_λ); furthermore:*

1) if the multiplicity of L_0 is odd, then the family either expands or contracts monotonically as λ increases through λ_0;

2) if the multiplicity of L_0 is even, then L_0 bifurcates into a stable and an unstable limit cycle as λ varies from λ_0 in one sense and L_0 disappears as λ varies from λ_0 in the opposite sense; i.e., there is a fold bifurcation at λ_0.

In [4]–[9], using Theorems 4.1 and 4.2, we have proved the following theorem.

Theorem 4.3. *There exists no quadratic system having a swallow-tail bifurcation surface of multiplicity-four limit cycles in its parameter space. In other words, a quadratic system cannot have either a multiplicity-four limit cycle or four limit cycles around a singular point (focus), and the maximum multiplicity or the maximum number of limit cycles surrounding a focus is equal to three.*

Applying the same approach, let us give an alternative proof of Theorem 3.2.

Proof (an alternative proof of Theorem 3.2). The proof of this theorem is carried out by contradiction. Consider canonical system (2.2) with three field rotation parameters, λ, β, γ, and suppose that (2.2) has three limit cycles around the origin. Then we get into some domain of the field rotation parameters being restricted by definite conditions on two other parameters, a and c, corresponding to one of that cases of finite singularities which were considered in [7]. We can fix both of these parameters putting, for example, $a = 1$ and $c > 1$ ($\nu = 1$) and supposing that system (2.2) has only two finite singularities: a saddle and an anti-saddle. Thus, there is a domain bounded by two fold bifurcation surfaces forming a cusp bifurcation surface of multiplicity-three limit cycles in the space of the field rotation parameters λ, β, and γ.

The corresponding maximal one-parameter family of multiplicity-three limit cycles cannot be cyclic, otherwise there will be at least one point corresponding to the limit cycle of multiplicity four (or even higher) in the parameter space. Extending the bifurcation curve of multiplicity-four limit cycles through this point and parameterizing the corresponding maximal one-parameter family of multiplicity-four limit cycles by a field-rotation parameter, according to Theorem 4.2, we will obtain two monotonic curves of multiplicity-three and one, respectively, which, by the Wintner–Perko termination principle (Theorem 4.1), terminate either at the origin or on a separatrix loop surrounding the origin. Since we know at least the cyclicity of the singular point which is equal to two (see [1]), we have got a contradiction with the termination

9

principle stating that the multiplicity of limit cycles cannot be higher than the multiplicity (cyclicity) of the singular point in which they terminate.

If the maximal one-parameter family of multiplicity-three limit cycles is not cyclic, using the same principle (Theorem 4.2), this again contradicts with the cyclicity of the origin [1] not admitting the multiplicity of limit cycles to be higher than two. This contradiction completes the proof. $\qquad\square$

References

[1] N. N. Bautin, E. A. Leontovich, *Methods and Ways of the Qualitative Analysis of Dynamical Systems in a Plane*, Nauka, Moscow, 1990 (in Russian).

[2] G. F. D. Duff, Limit-cycles and rotated vector fields, *Ann. Math.* **67** (1953), 15–31.

[3] N. P. Erugin, Some questions of motion stability and qualitative theory of differential equations on the whole, *Prikl. Mat. Mekh.* **14** (1950), 459–512 (Russian).

[4] V. A. Gaiko, Qualitative theory of two-dimensional polynomial dynamical systems: problems, approaches, conjectures, *Nonlinear Anal.* **30** (1997), 1385–1394.

[5] V. A. Gaiko, Hilbert's sixteenth problem and global bifurcations of limit cycles, *Nonlinear Anal.* **47** (2001), 4455–4466.

[6] V. A. Gaiko, Global bifurcation families of multiple limit cycles in polynomial dynamical systems, *Nonlinear Phenom. Complex Syst.* **6** (2003), 734–745.

[7] V. A. Gaiko, *Global Bifurcation Theory and Hilbert's Sixteenth Problem*, Kluwer, Boston, 2003.

[8] V. A. Gaiko, Wintner–Perko termination principle, parameters rotating a field, and limit-cycle problem, *J. Math. Sci.* **126** (2005), 1259–1266.

[9] V. A. Gaiko, Limit cycles of quadratic systems, *Nonliner Anal.* (2007), doi:10.1016/j.na.2007.07.052.

[10] Yu. Ilyashenko, Centennial history of Hilbert's 16th problem, *Bul. Amer. Math. Soc.* **39** (2002), 301–354.

[11] L. Perko, *Differential Equations and Dynamical Systems,* Springer, New York, 2002.

[12] J. W. Reyn, *Phase Portraits of Planar Quadratic Systems*, Springer, New York, 2007.

[13] Ye Yanqian et al., *Theory of Limit Cycles*, Transl. Math. Monogr., Vol. 66, American Mathematical Society, Providence, RI, 1986.