Literature review on beam buckling lengths and methods to calculate the real buckling length

Literature assignment ME54010

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Abstract

In this literature study the buckling of frames is investigated. Two frames are considered, where both consists of two columns and a girder. The first step was looking at the behavior of a single column under an compressive load. This results in so called effective lengths for different sets of boundary conditions on the column. Because in a frame the multiple columns and girders work together, the buckling mode also depends on the frame itself and the neighboring beams. There are two types of frames according to their buckling mode, a sway and a non sway frame. For determining the effective buckling lengths Eurocode 3 is studied for general rules for frames and calculation methods for effective lengths. Also the method from the AISC standard is described shortly. After that, it is investigated what methods the FEM programs use. Beside these methods, another method has been found and will be demonstrated on both situations. In the last chapter the results are summarized and a final conclusion and recommendation are made.

Contents

1

Introduction

1.1. Motivation of assignment

1.1.1. Buckling

Buckling is in structural engineering a sudden change in shape (deformation) of a structural component under load. It can also be seen as loss of stability or as a failure mode of the structural component and so possibly the entire structure. A common example is a column under compression (treated in the next chapter) as can be seen in the left example from the figure below. This figure shows that the column under compression losses its stability and buckles to the side. When the column buckles it losses part of its stiffness and the column must resists bending moment instead of only normal stress

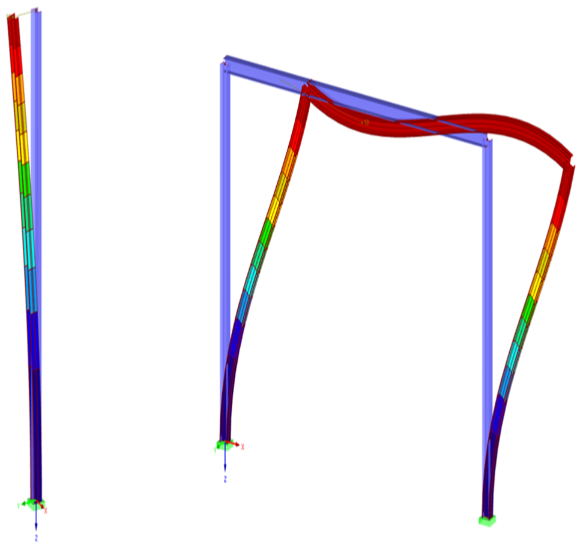


Figure 1.1: Single column buckling (left) and frame buckling (right)

Looking at the example in the figure on the right it can be seen that when the frame buckles, both columns and the girder lose there stability. All members of the frame thus interact with each other and the buckling behavior of each individual column is not only dependent on its own parameters (E, I, A,ect.) and its compressive load but also on the frame itself and the neighboring beams.

The figure above shows only two examples, but there are many kinds of frames which all have multiple buckling modes. This makes it hard to determine a critical load where the structure is likely to buckle.

1.1.2. Sway and non sway frame

A frame consists of several beams joined together at so called joints. In the literature two different types of frame buckling modes are distinguished, namely the sway mode and the non sway mode, where a frame which tends to buckle in a sway mode is called a sway frame and a frame which tends to buckle in a non sway mode is called a non sway frame

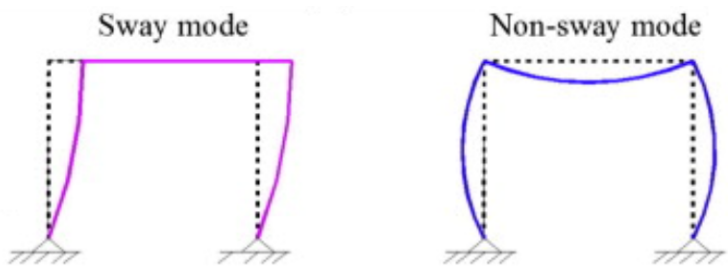


Figure 1.2: Sway mode (left) and non sway mode (right)

The difference between sway and non sway frames can be described as follows [**?**]

- *Sway frame*: The frames in which longitudinal deflection takes place when the horizontal force is applied are known as sway frames. Sway frames provide lateral resistance only though columns and they lake adequate stiffness against horizontal loads. Sway frames are also called unbraced frames.
- *Non sway frame*: In non sway frames longitudinal deflection is restrained by supports when the horizontal load is applied. It is sufficiently braced by lateral bracing elements like structural walls and they have enough stiffness to tackle horizontal forces. Non sway frames are also called braced frames.

Checking for a sway frame A classical method of checking if a frame is a sway frame is given in [**?**]. According to this method, one should compare the deformations of the frame with and without bracing against a load applied in the direction of the sway mode. If a frame without the bracings deforms 5 or less times more than the one with the bracings, this is a non- sway frame. So the frame in the figure below is a sway frame when

$$B > 5 \cdot A \tag{1.1}$$

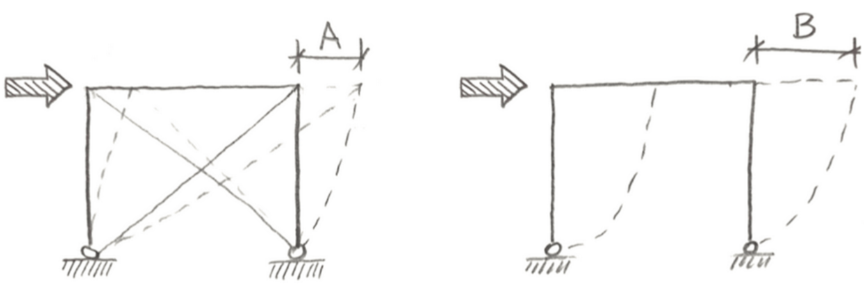


Figure 1.3: Braced frame and unbraced frame for checking of the corresponding frame is a sway frame

This method is not very accurate and it is not guaranteed that a sway frame satisfies this criterion. Also, an additional vertical load in the direction of the buckling mode has been applied in this example above. This means that the direction of the sway mode must be known in order to apply the additional load, which can be difficult for some situations. Sometimes, a value of 10 or an other factor is used instead of 5 [**?**].

There are also other possibilities of determining if the sway or non sway frame calculation has to be used

- Perform both and use the worst case scenario
- Preform a buckling analysis with a finite element analysis program and observe if the first buckling mode is a sway or non sway mode
- Use own insight if the frame has the possibility to sway when it buckles. If not, the non sway frame methods need to be used

1.2. Approach

1.2.1. Research question

Determining the critical load for which a frame buckles can be a complex problem. Considering a single column, this critical load can be determined if the so called effective length is known. For a single column this effective length can be easily be determined by using the constraints on the column. For a frame structure, determining the boundary conditions of the considered column under compression is more difficult because this column has interaction with the neighboring beams. The goal of this literature assignment is to investigate which methods are available to determine the effective length of a column under compression as part of a frame. This effective length can then be used to determine critical load on the column and so the corresponding load in the frame by using a static analysis. The associated research question is defined as

How can the effective length of a column under compression, as part of a frame, be determined?

This research question will be answered throughout this report, where a conclusion is made at the final chapter.

1.2.2. Scope

For the sake of simplicity only two situations will be considered: one situation for each type of buckling mode. The situations can be seen below

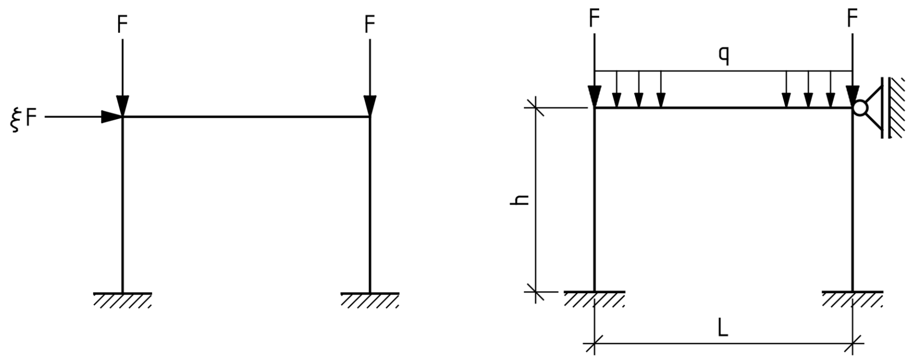


Figure 1.4: Situation 1 or Sway frame (left) and situation 2 or non sway frame (right)

- *Situation 1*: situation 1 has two columns with each a fixed connection with the ground. These column are connected to each other with a girder. The connection of each column to the girder is fixed. At each end point of the girder, a vertical load F is applied. At the left upper corner also a horizontal load with a very small fraction ξ of F has been applied. This horizontal loads has been applied to led the structure buckle to right side (instead of a random one) and to look at the effect on the first order elastic moment lines and the buckling mode. ξF can also be seen as an imperfection and therefore it is feasible to apply this load because in reality noting is perfect and there always is an imperfection.

- *Situation 2*: The column and girder construction of situation 2 is the same as situation 1. However, a slider which allows displacement in the vertical direction, is applied at the right top corner to prevent the sway mode. Beside the vertical load F, also a distributed load q has been applied over the total length.

For both situations the same dimensions and beams are used: the height (length) of the column equals $h = L_{cln} = 1m$, the length of the girder equals $L = L_{bm} = 1m$, the moment of inertia of the columns and girder $= I_{cln} = I_{bm} = 1,33 \cdot 10^{-1} m^4$ (rectangular beam of $0,02 \times 0,02$) and the elasticity modulus of the columns and girder equals $E = 2,1 \cdot 10^{11} N/m^2$ (steel).

For both situations the left column will be considered. If the effective length of the left column is determined the critical normal load on the column can be determined. By using statics it follows for both situations that if $\xi = 0$ and $q = 0$, the normal load in the columns equals F.

2

Single column buckling

The first step is looking at the buckling behavior of a single column under compression. This leads to effective buckling lengths for different sets of boundary conditions. For each set of boundary conditions a displacement, curvature and moment line have been established to give more insight in the buckling behavior.

2.1. Governing equation

The figure below shows a single beam and a segment of the beam

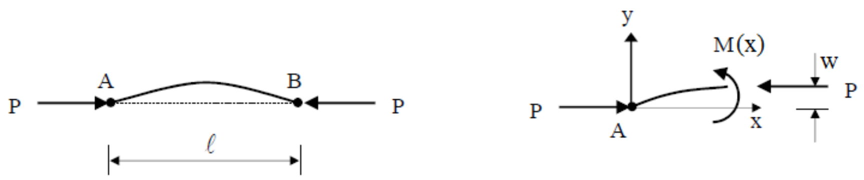


Figure 2.1: Single pinned pinned column and small segment

the governing equation relating the bending moment to the deflection w of the column along its axis is given by [**?**]

$$\frac{d^4w}{dx^4} + \frac{P}{EI}\frac{d^2w}{dx^2} = \frac{q}{EI} \tag{2.1}$$

Considering a column with axial load only, such that the lateral load $q(x)$ vanishes, and substitution of $\lambda^2 = P/EI$ gives

$$\frac{d^4w}{dx^4} + \lambda^2\frac{d^2w}{dx^2} = 0 \tag{2.2}$$

which is a homogeneous fourth- order differential equation.

2.2. Solution

The general solution of the governing equation is defined as

$$w(x) = A\sin(\lambda x) + B\cos(\lambda x) + Cx + D \tag{2.3}$$

The four constant A, B, C and D can be determined by satisfying the boundary conditions, where there are four types:

- Pinned end: (rotation free and translation fixed) $w = 0$ and $M = \frac{d^2w}{dx^2} = 0$

- Fixed end (rotation fixed and translation fixed): $w = 0$ and $\frac{dw}{dx} = 0$
- Sliding end (rotation fixed and translation free): $\frac{dw}{dx} = 0$
- Free end (rotation free and translation free): $M = \frac{d^2w}{dx^2} = 0$ and $V = \frac{d^3w}{dx^3} + \lambda^2 \frac{dw}{dx} = 0$

where the derivatives of $w(x)$ are given by

$$\begin{aligned} dw/dx &= A\lambda \cos(\lambda x) - B\lambda \sin(\lambda x) + C \\ d^2w/dx^2 &= -A\lambda^2 \sin(\lambda x) - B\lambda^2 \cos(\lambda x) \\ d^3w/dx^3 &= -A\lambda^3 \cos(\lambda x) + B\lambda^3 \sin(\lambda x) \end{aligned} \tag{2.4}$$

Below the different combinations of boundary conditions are solved. The constants are solved numerically so only the results are shown. The solutions are also plotted with displacement, curvature and moment lines.

1. **pinned - pinned** Because $w(0) = 0 \rightarrow A = 0$ and $w(l) = 0 \rightarrow B\sin(\lambda l) = 0$. So, $\sin(\lambda l)$ must be zero and thus $\lambda_n l = n\pi$ for $n = 0, 1, 2, ...$ With $\lambda^2 = \frac{P}{EI}$ it follows that the critical load corresponding to the nth buckling mode is given by

$$P_n = \frac{n^2\pi^2 EI}{l^2} \tag{2.5}$$

From this we define the critical Euler buckling load for the first buckling mode by

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} \tag{2.6}$$

with effective buckling length $L_e = K \cdot L$ and K the effective buckling length factor. For the pinned - pinned case it follows that $K = 1$ because the first buckling mode ($n = 1$) will be used for each case. Below the resulting displacement, curvature and moment lines for the buckling mode are shown

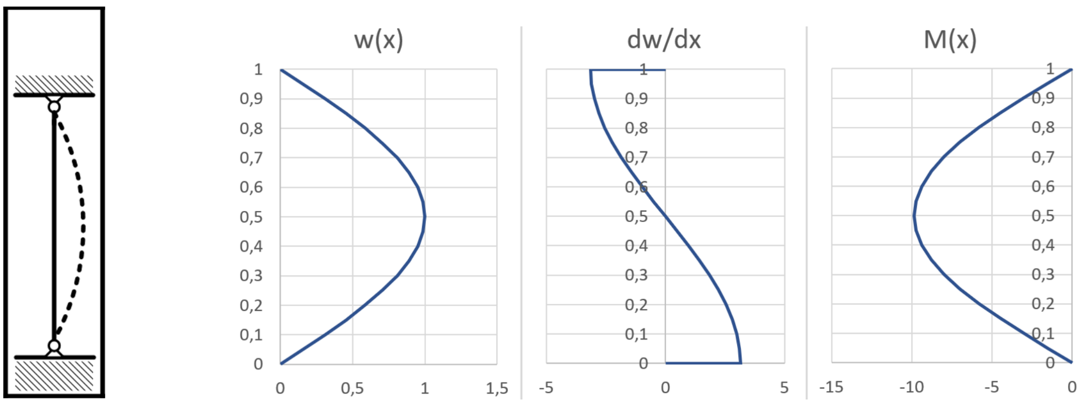


Figure 2.2: Displacement, curvature and moment lines for boundary condition set 1

2. **fixed - fixed** Because $w(0) = 0$, $\left.\frac{dw}{dx}\right|_{x=0} = 0$ and $w(L) = 0$, $\left.\frac{dw}{dx}\right|_{x=L} = 0$ it follows by numerical evaluation that $A = 0, C = 0, B = -D$ and for the effective buckling length $K = 0.5$. Again, the corresponding displacement, curvature and moment lines for the buckling mode are shown below

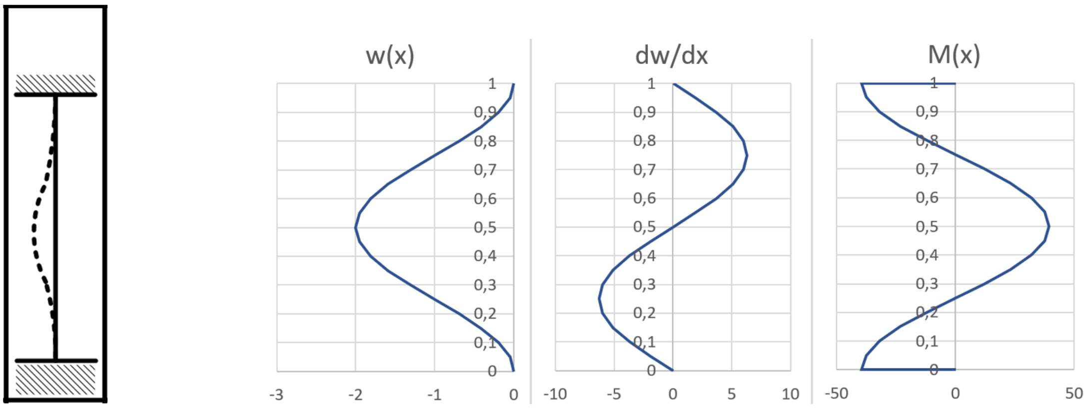


Figure 2.3: Displacement, curvature and moment lines for boundary condition set 2

3. **fixed - rotation free / translation fixed** Because $w(0) = 0$, $\frac{dw}{dx}\Big|_{x=0} = 0$ and $w(L) = 0$ it follows that $A \approx -0.7c, B \approx 0.305c, C \approx \pi c$ and $D \approx -3.05c$ with c a parameter determining the perturbation in the buckling mode. For the effective buckling length it follows that $K = 0.7$

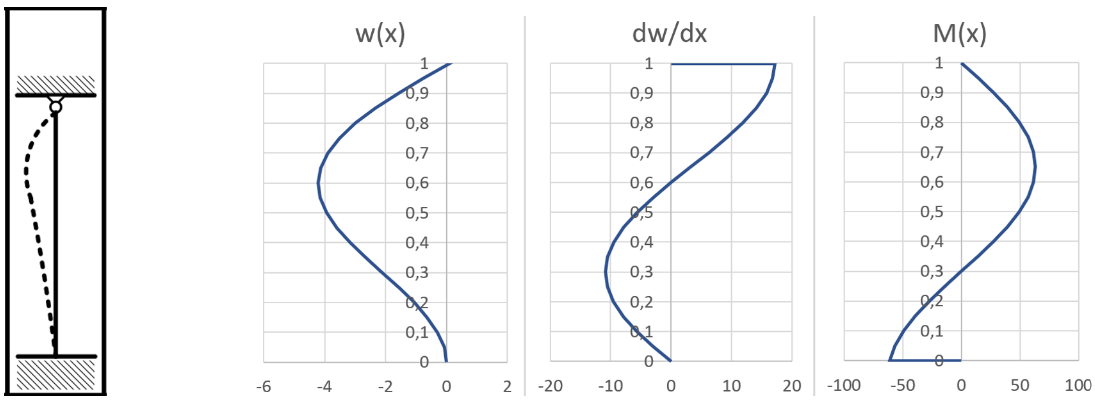


Figure 2.4: Displacement, curvature and moment lines for boundary condition set 3

4. **fixed - rotation fixed / translation free** Because $w(0) = 0$, $\frac{dw}{dx}\Big|_{x=0} = 0$ and $\frac{dw}{dx}\Big|_{x=L} = 0$ it follows that $A = 0, B = -D, C = 0$ and for the effective buckling length $K = 1.0$

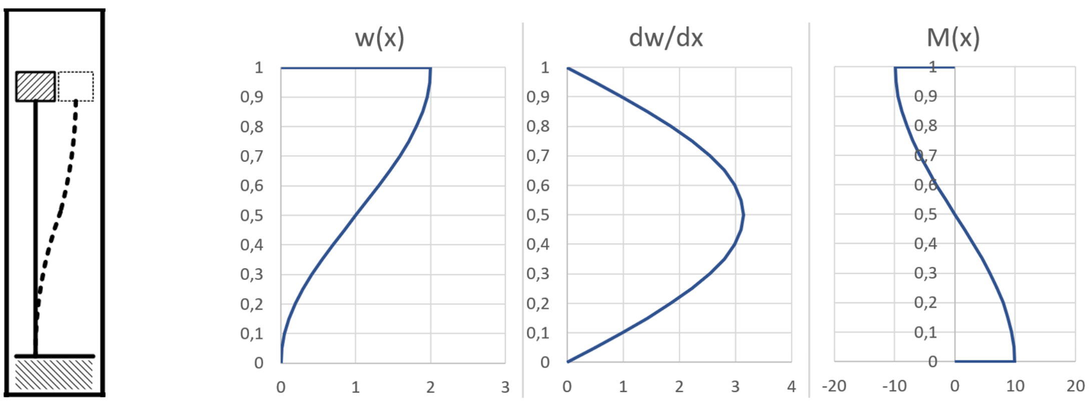


Figure 2.5: Displacement, curvature and moment lines for boundary condition set 4

5. **fixed - free** Because $w(0) = 0$, $\left.\frac{dw}{dx}\right|_{x=0} = 0$ and $\left.\frac{d^2w}{dx^2}\right|_{x=L} = 0$, $\left(\frac{d^3w}{dx^3} + \lambda^2 \frac{dw}{dx}\right)_{x=L} = 0$ it follows that $A = 0, B = -D, C = 0$ for the effective buckling length $K = 2.0$

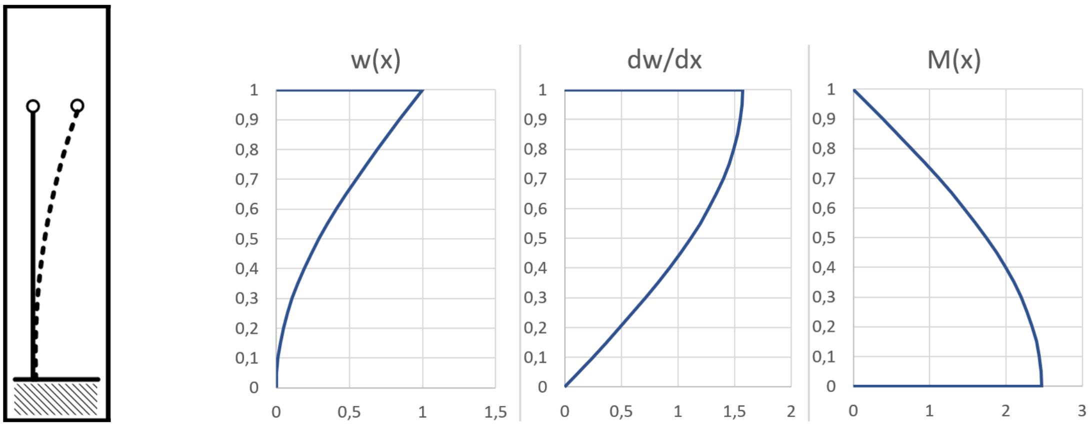


Figure 2.6: Displacement, curvature and moment lines for boundary condition set 5

6. **pined / rotation fixed translation free** Because $w(0) = 0$, $\left.\frac{d^2w}{dx^2}\right|_{x=0} = 0$ and $\left.\frac{dw}{dx}\right|_{x=L} = 0$ it follows that $B = C = D = 0$ and for the effective buckling length $K = 2.0$

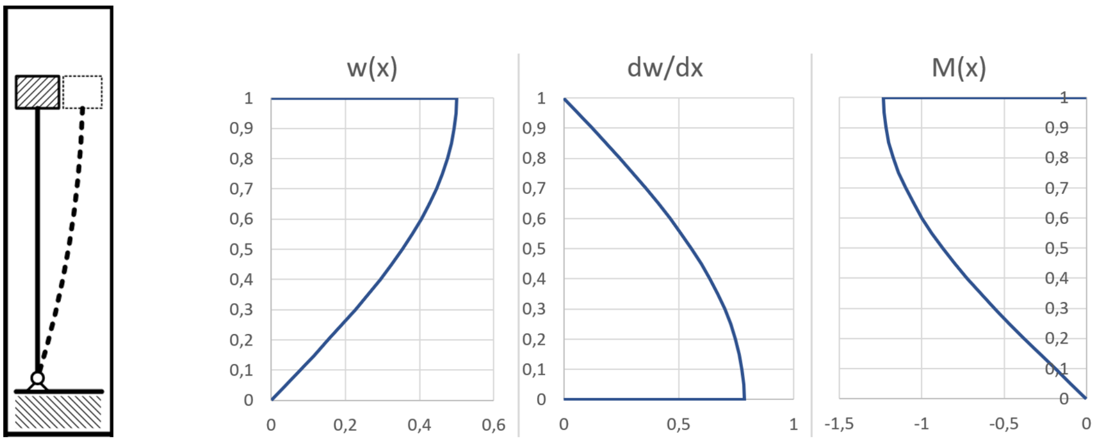
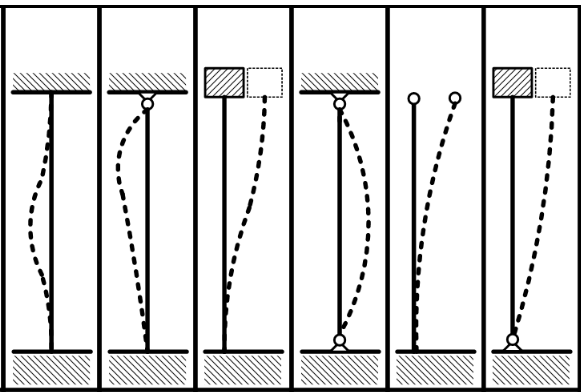
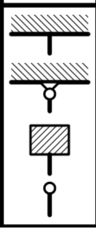


Figure 2.7: Displacement, curvature and moment lines for boundary condition set 6

Effective lengths in summary In the figure below the effective length factors K of the different situations are summarized



Buckled shape of column shown by dashed line	(a)	(b)	(c)	(d)	(e)	(f)
Theoretical K value	0.5	0.7	1.0	1.0	2.0	2.0
Recommended design value K	0.65	0.80	1.2	1.0	2.10	2.0



End condition key:
- Rotation fixed and translation fixed
- Rotation free and translation fixed
- Rotation fixed and translation free
- Rotation free and translation free

Figure 2.8: Effective buckling length factors

3

Eurocode 3

This chapter discusses the methods and rules used in standards for frame buckling. First a general buckling stability criteria will be defined based on Eurocode 3. For this criteria the effective buckling lengths are needed, which will be treated in section 3.2 for the sway frame and section 3.3 for the non sway frame. Finally the method from the AISC will be shortly discussed.

3.1. General buckling stability criteria

In Eurocode 3 a criteria for a column under compression is defined which must be satisfied in order to be stable against buckling. For this criteria first the effective buckling length of the column under consideration must be determined, which is treated in the next sections.

Looking at a single member in a frame there are 4 classes of cross sections according to Eurocode (see table 5.2 [**?**]). The different classes are defined as

- Class 1 cross sections: are cross sections where a plastic pin can occur where the rotation capacity is enough for a plastic calculation without resistance loss
- Class 2 cross sections: are cross sections where the plasticity can be reached with limited rotation capacity by local buckling
- Class 3 cross sections: are cross sections where elasticity can be reached, and even local buckling doesn't cause plasticity
- Class 4 cross sections: are cross sections where local buckling will occur before the yield criteria will be reached in one of more parts of the cross section.

Considering a prismatic beam under an axial compression load, the buckling stability needs to satisfy

$$\frac{N_{Ed}}{N_{b,Rd}} \leq 1 \tag{3.1}$$

where N_{Ed} is the normal compression force and $N_{b,Rd}$ is the buckling resistance. For bars with a asymmetric cross section of class 4, an additional moment ΔM_{Ed} needs to be included. The buckling resistance can be determined by

- $N_{b,Rd} = \frac{\chi A f_y}{\gamma_{M1}}$ for cross sections of class 1,2 and 3
- $N_{b,Rd} = \frac{\chi A_{eff} f_y}{\gamma_{M1}}$ for cross sections of class 4

where f_y is the yield point, γ_{M1} is the partial factor for the resistance of elements corresponding to instability (usually $\gamma_{M1} = 1$) and χ is the reduction factor for the corresponding buckling shape

$$\chi = \frac{1}{\Phi + \sqrt{\Phi^2 - \bar{\lambda}^2}} \quad \text{but } \chi \leq 1.0 \tag{3.2}$$

where

$$\Phi = 0.5\left(1 + \alpha\left(\bar{\lambda} - 0.2\right) + \bar{\lambda}^2\right) \tag{3.3}$$

here α is the imperfection factor and N_{cr} is the critical buckling load. The imperfection factor α can be determined depending on the corresponding buckling shape for the table or graph below

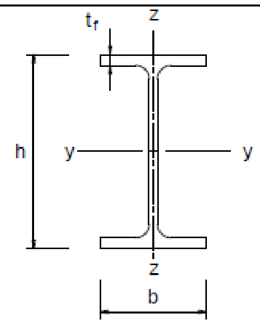
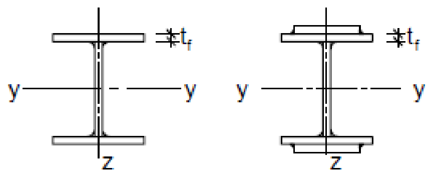
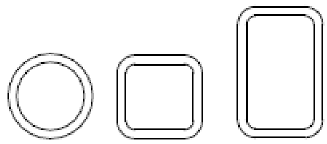
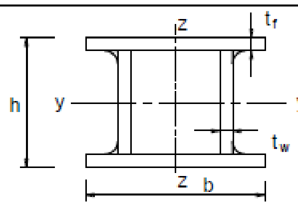
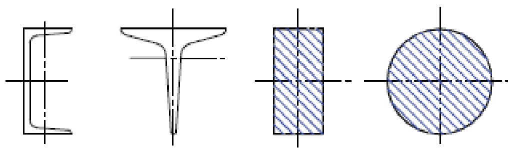
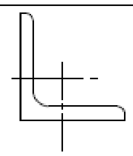
Doorsnede		Begrenzingen		Knik om de as	Knikkromme S 235, S 275, S 355, S 420	Knikkromme S 460
Gewalste profielen		h/b > 1,2	$t_f \le 40$ mm	y – y	a	a_0
				z – z	b	a_0
			40 mm < $t_f \le 100$ mm	y – y	b	a
				z – z	c	a
		h/b ≤ 1,2	$t_f \le 100$ mm	y – y	b	a
				z – z	c	a
			$t_f > 100$ mm	y – y	d	c
				z – z	d	c
Gelaste profielen		$t_f \le 40$ mm		y – y	b	b
				z – z	c	c
		$t_f > 40$ mm		y – y	c	c
				z – z	d	d
Buisprofielen		Warmvervaardigd		Elke as	a	a_0
		Koudgevormd en gelast		Elke as	c	c
Gelaste kokerprofielen		Algemeen (behalve in het hieronder gegeven geval)		Elke as	b	b
		Dikke lassen: $a > 0{,}5t_f$; $b/t_f < 30$; $h/t_w < 30$		Elke as	c	c
U-, T- en massieve profielen				Elke as	c	c
L-profielen				Elke as	b	b

Figure 3.1: Buckling shapes for cross sections

or we can use the graph below if the buckling shape is known

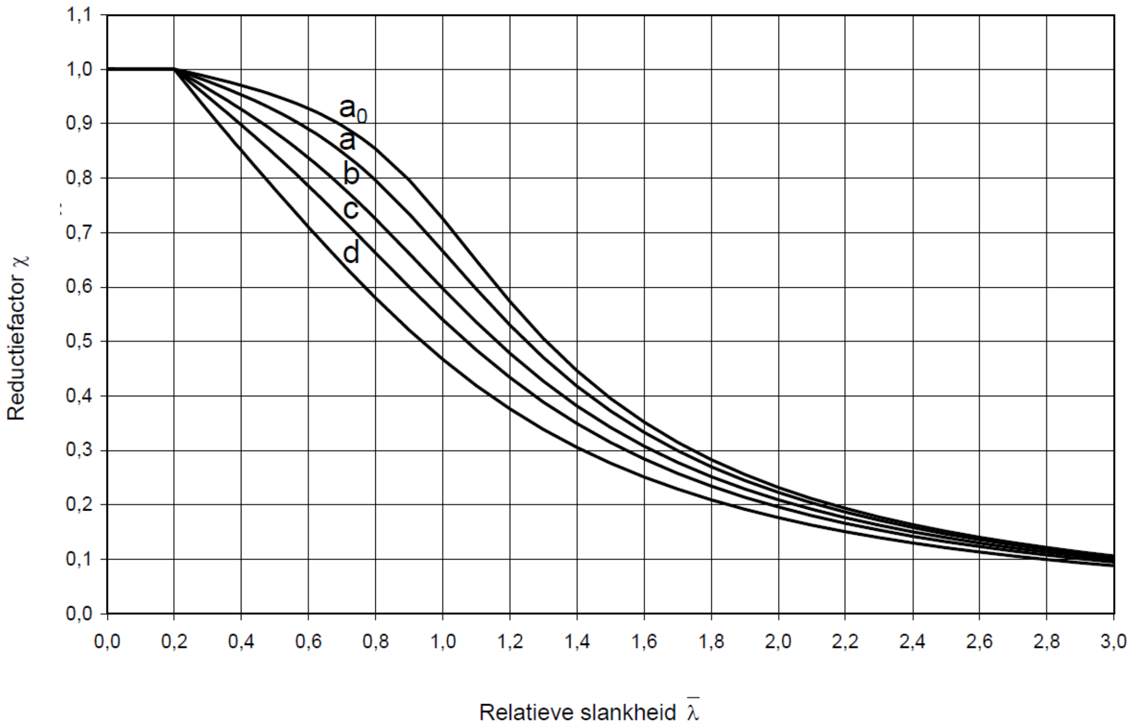


Figure 3.2: Buckling shapes

If the buckling shape is known the imperfection factor α can be determined from the table below

Table 3.1: Imperfection factor for the buckling shapes

Buckling shape	a_0	a	b	c	d
Imperfectionfactor α	0.13	0.21	0.34	0.49	0.76

For the relative slenderness $\bar{\lambda}$

- *Buckling by bending*: The relative slenderness $\bar{\lambda}$ is defined by
  - $\bar{\lambda} = \sqrt{\frac{Af_y}{N_{cr}}} = \frac{L_{cr}}{i}\frac{1}{\lambda_1}$ for cross sections of class 1,2, and 3
  - $\bar{\lambda} = \sqrt{\frac{A_{eff}f_y}{N_{cr}}} = \frac{L_{cr}}{i}\frac{\sqrt{\frac{A_{eff}}{A}}}{\lambda_1}$ for cross sections of class 4

  where L_{cr} is the buckling length of the considered buckling direction and i is the inertia radius for the considered axis, $\lambda_1 = \pi\sqrt{\frac{E}{f_y}} = 93.3\varepsilon$ where $\varepsilon = \sqrt{\frac{235}{f_y}}$ with f_y in N/mm^2.
- *Buckling by torsion*: The relative slenderness $\bar{\lambda}_T$ for torsion stability and torsional buckling stability is defined by
  - $\bar{\lambda}_T = \sqrt{\frac{Af_y}{N_{cr}}} = \frac{L_{cr}}{i}\frac{1}{\lambda_1}$ for cross sections of class 1,2, and 3
  - $\bar{\lambda}_T = \sqrt{\frac{A_{eff}f_y}{N_{cr}}}$ for cross sections of class 4

  where $N_{cr} = N_{cr,TF}$ but $N_{cr} < N_{cr,T}$ with $N_{cr,TF}$ the critical elastic force for torsional buckling stability and $N_{cr,T}$ the critical elastic force for torsion stability

For a relative slenderness $\bar{\lambda} \leq 0.2$ or $\frac{N_{Ed}}{N_{cr}} \leq 0.04$ the buckling effects can be neglected.

3.2. Sway frames

Support moment For beams and their connections to columns in sway frames, the first order force distribution (no buckling) must be adjusted by adding an extra moment when the structure buckles (also called the support moment) per column end on the connected beam [**?**]. The support moment can be calculated by

$$M_{r,Ed} = \left(\frac{1}{\alpha_{cr} - 1} \times M_{1,Ed}\right) + \left(\frac{\alpha_{cr}}{\alpha_{cr} - 1} \times F_{t,Ed} \times e_0\right) \quad (3.4)$$

with

$$\alpha_{cr} = \frac{N_{cr}}{F_{t,ed}} \quad \text{and} \quad N_{cr} = \frac{\pi^2 \times E \times I}{L_{cr}^2} \quad (3.5)$$

where $M_{1,Ed}$ is the first order bending moment in the column, E is the elasticity modulus, I is the bending moment, L_{cr} is the effective buckling length, $F_{t,ed}$ is the total vertical load which causes the column instability and e_0 is the initial bending (imperfection).

The equations can also be rewritten to

$$M_{r,Ed} = M_{2,Ed} - M_{1,Ed} \quad \text{with} \quad M_{2,Ed} = \frac{\alpha_{cr}}{\alpha_{cr} - 1} \times (M_{1,Ed} + F_{t,Ed} \times e_0) \quad (3.6)$$

where $M_{2,Ed}$ is the second order bending moment.

Buckling mode The first order elastic moment lines and the moment lines of the buckling mode of the sway frame considered in this literature assignment can be seen below [**?**]

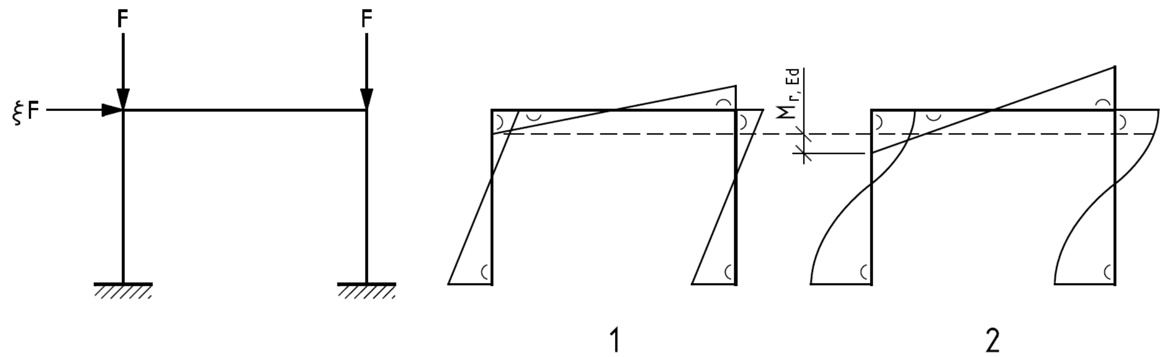


Figure 3.3: Sway frame: 1. first order elastic moment lines and 2. moment lines at failure

It can be seen that the second order effects change the force distribution in the buckling mode. The figure also illustrates where the support moment is added when the structure buckles. The effect of the horizontal load ξF mostly effects the first order elastic moment lines, but its effect is negligible (if ξ is small enough) when the structure buckles. In the following the effect of ξ will be neglected.

Effective buckling length determination Looking at the moment lines of the buckling mode of the situation 1 sway frame above and the single column buckling moment lines it can be observed that the effective length factor of the columns must be between $1 < K < 2$. Indeed, according to Eurocode for general columns in sway frames, $L_{cr} > L$ so $K > 1$. For rigidly connected beams Eurocode states that one of the following situations can be used

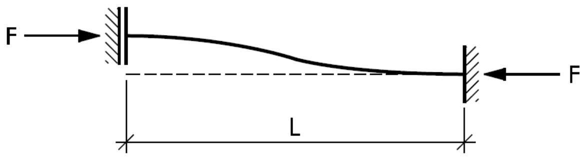
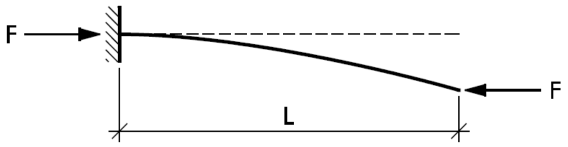
a		$L_{cr} = L$
b		$L_{cr} = 2 \times L$

Figure 3.4: Effective buckling lengths with fixed columns in a sway frame

which satisfies the observation that the effective length observed from the buckling mode must be between 1 and 2.

As a more exact solution for sway frames Eurocode states that the relation between effective length en the length of the column is given by

$$\frac{L_{cr}}{L} = \frac{\pi}{\lambda} \tag{3.7}$$

where λ for $0 \leq \lambda \leq \pi$, can be determined by solving λ from

$$C_A \times C_B \times \lambda^2 \times \sin\lambda = ((C_A + C_B) \times \lambda \times \cos\lambda) + \sin\lambda \tag{3.8}$$

or by drawing a straight line between C_A and C_B and take L_{cr}/L from the figure below

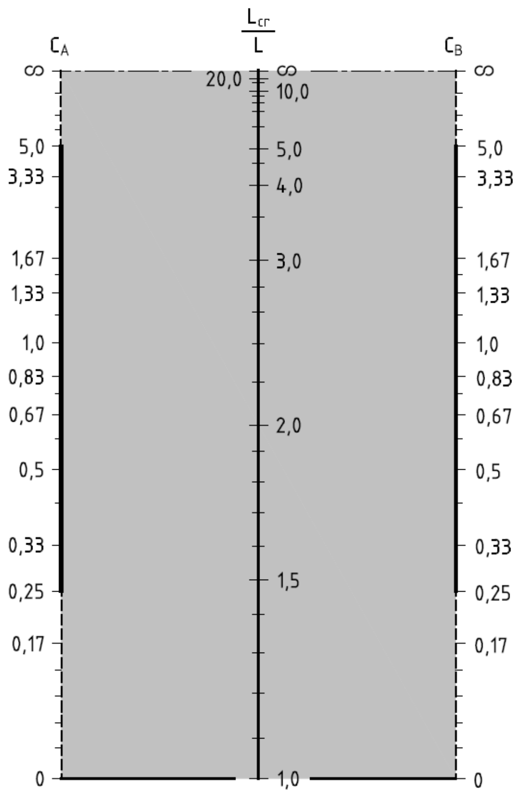


Figure 3.5: Elastic effective buckling length or elastic fixed columns in sway frames

Here C is the flexibility parameter, where the subscribt A and B denote the ends of the considered column

$$C = \frac{\sum I_{cln}/L_{cln}}{\sum \mu \times (I_{bm}/L_{bm})} \tag{3.9}$$

where the sum is over all beams which are rigid connected to the considered column end. I_{cln} is the moment if inertia of the column axis, L_{cln} is the column length, I_{bm} is the moment of inertia of the beam and L_{bm} is the length of the beam. μ is a correction factor in order to implement a constraint

- $\mu = 6$ if the other end is rigid connected to one or more columns
- $\mu = 4$ if the other end is fixed
- $\mu = 3$ if the other end is pinned

For a column with a pinned end, the theoretical value is $C = \infty$, but $C = 5$ must be used. For a column end which is fixed the theoretical value is $C = 0$, but $C = 0.25$ must be used.

The flexibility parameter C of a column end can also be calculated by

$$C = \frac{E \times (I_{cln}/L_{cln})}{k_\varphi} \tag{3.10}$$

where k_φ is the rotation elastic stiffness at the column end.

For column ends that are elastic fixed to beams which are not totally stiff connected to the column, L_{cr} can be determined in the same way where we replace the beam length L_{bm} by the equivalent beam length L_{equ}

$$L_{equ} = L_{bm} + \frac{\mu \times E \times I_{bm}}{S_i} \tag{3.11}$$

where S_i is the rotational stiffness of the flexible connection.

Situation 1 elaboration Looking at the left column of the first situation is follows that for the lower end which is fixed $C_A = 0$ and thus $C_A = 0.25$.

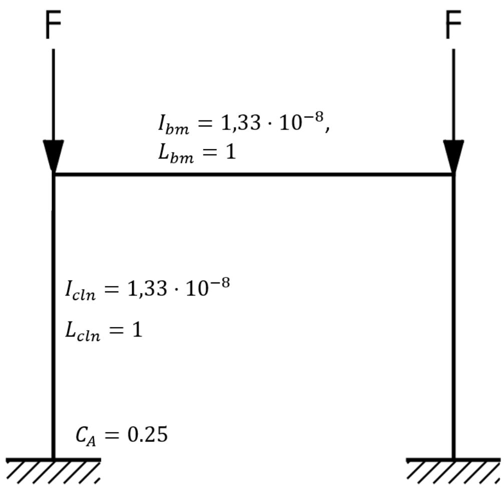


Figure 3.6: Situation 1

For the upper end we assume that the girder and column have the same length and cross section, and so we can use $I_{cln} = 1, L_{cln} = 1, I_{bm} = 1, L_{bm} = 1$. Here 1 is used for simplicity but any other value gives the same solution as long as $I_{cln} = I_{bm}$ and $L_{cln} = I_{bm}$ holds (and so the assumtion holds). Because there is a fixed connection of the column and the girder $\mu = 4$, such that

$$C_B = \frac{1/1}{4 \times 1/1} = \frac{1}{4} \tag{3.12}$$

Solving formula 3.8 gives

$$\frac{L_{cr}}{L} = \frac{\pi}{\lambda} \approx 1.45 \tag{3.13}$$

This also satisfies the observation that the effective length factor must be between 1 and 2. Using the critical euler buckling formula

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} \tag{3.14}$$

the critical buckling load follows as $13143,79N$.

3.3. Non Sway frames

Support moment Also for the non sway frame a support moment has to be added to the first order linear elastic moment lines when the frame buckles. The support moment $M_{r,Ed}$ at the end of the column with the largest fixings parameter ρ_1 is the minimum of the following three values

$$\begin{aligned} M_{r,1,Ed} &= \left(k_M \times R_\rho \times (\lambda_L - \lambda_{Lcr}) \times \tfrac{(N_{cr}-N_{b,Rd})}{(N_{cr}-N_{ed})} \times \tfrac{N_{Ed}}{N_{b,Rd}}\right) \times M_{c,Rd} \\ M_{r,1,ED} &= (-0.015 + C \times \lambda_L) \times M_{c,Rd} \\ M_{r,1,Ed} &= M_{N,Rd} \end{aligned} \tag{3.15}$$

The support moment $M_{r,2,Ed}$ at the end of the column with the smallest fixing parameter ρ_2 is given by

$$M_{r,2,Ed} = \left(2 \times M_{r,1,Ed}\right) - M_{r,1,Ed} \tag{3.16}$$

with

$$\begin{aligned} k_M &= 10^{-6} \times (-439 + 129\lambda_L - 0.451\lambda_L^2) \\ R_\rho &= \tfrac{2\times\rho_1}{\rho_2+\rho_1} \\ \lambda_L &= \tfrac{L}{i} \\ \lambda_{Lcr} &= \tfrac{L_{cr}}{i} \\ C &= 10^{-6} \times \left(3300 + 834,4\log\left(\tfrac{\rho_1}{\rho_2}\right) - 124\left(\log\left(\tfrac{\rho_1}{\rho_2}\right)\right)^2\right) \\ \rho_i &= \tfrac{k_{\phi j}\times L}{E\times I} \end{aligned} \tag{3.17}$$

If $\rho_1 \to \infty$ than take $\frac{\rho_1}{\rho_2} = 10000$ and if $\rho_2 \to \infty$ than take $\frac{\rho_1}{\rho_2} = 10000$. Here R_ρ is a ratio expressed in relative stiffness from the column ends, $M_{c,Rd}$ is the moment resistance from the column cross section, $M_{N,Rd}$ is the moment resistance from the column cross section taking into account the normal force, N_{cr} is the critical elastic force of the column assuming the buckling length, $N_{b,Rd}$ is the buckling resistance from the axial loaded bar, N_{ed} is the normal force in the column, L is the length of the column, λ_L is slenderness of the column, bases on length L, λ_{Lcr} is the slenderness of the column based on buckling length L_{cr}, I is the inertia moment radius, ρ_i is the relative stiffness of the column end $i = 1, 2$ and $k_{\varphi j}$ is the rotation stiffness at the column end i by the beams.

Buckling mode The moment lines of the non sway frame used in this assignment and the moment lines of its buckling mode can be seen in the figure below

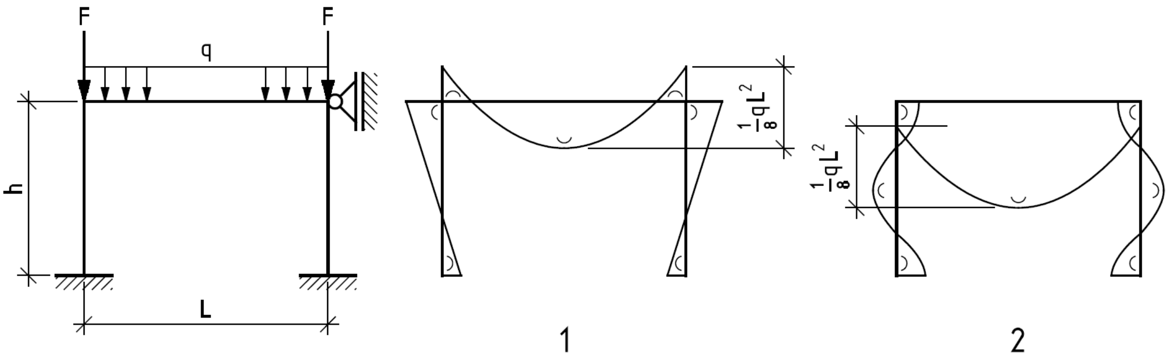


Figure 3.7: Non Sway frame: 1. first order elastic moment lines and 2. moment lines when the structure buckles

As can be seen, the moment lines again change when the structure buckles due to the second order effects. The change in height of the beam moment line can be incorporated by $M_{r,Ed}$.

In the figure it can be seen that the distributed load q has an influence on the moment lines and thus the buckling. It enlarges the moments at the rigid connections of the columns and thus influence the moment lines of the columns. Because the found methods do not include this type of pre applied loads which causes an extra moment on the connection it is assumed that $q = 0$ for the next sections.

Effective buckling length From the moment lines of the figure above and by using the moment lines of the single column buckling moment lines, we can observe that the effective length of the columns must be between $0.5 < K < 1$. Indeed according to Eurocode, in general $L_{cr} \leq L$ and so $K < 1$ for columns in a non sway frame. For the effective buckling length of pinned or rigid connections in a non sway frame one of the following situations can be used according to Eurocode

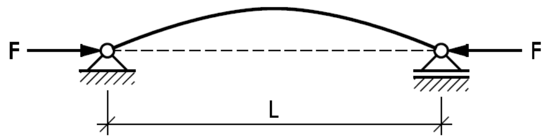
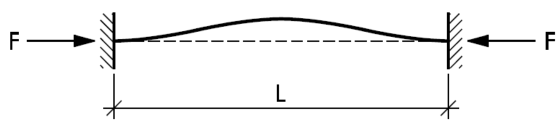
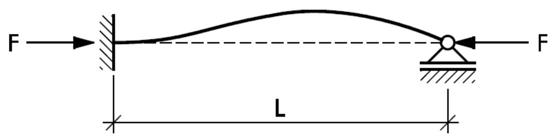
a		$L_{cr} = L$
b		$L_{cr} = \frac{1}{2} L$
c		$L_{cr} = \frac{L}{\sqrt{2}} \approx 0{,}71 \times L$

Figure 3.8: Effective buckling lengths with pinned or fixed columns in a non sway frame

For columns where the ends are connected to other beams by their stiffness, L_{cr} can be determined with

$$\frac{L_{cr}}{L} = \frac{\pi}{\lambda} \tag{3.18}$$

where λ for $\pi \leq \lambda \leq 2\pi$ can be determined by solving for λ from

$$C_A \times C_B \times \lambda^2 \times \sin\lambda = ((C_A \times C_B) \times \lambda \times \cos\lambda) + ((1 - C_A - C_B) \times \sin\lambda) - \left(2 \times \frac{1 - \cos\lambda}{\lambda}\right) \tag{3.19}$$

of by drawing a straight line between C_A and C_B and read L_{cr}/L from the figure below

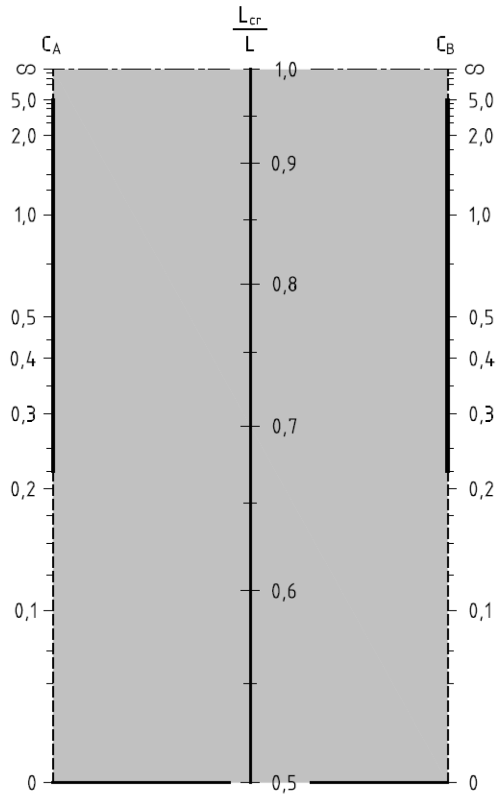


Figure 3.9: Elastic effective buckling length or elastic fixed columns in non sway frames

Here, the flexibility parameters C, where subscripts A and B denote the ends of the considered column, need to be deteremined first. These parameters can be determined by

$$C = \frac{\sum I_{cln}/L_{cln}}{\sum \mu \times (I_{bm}/L_{bm})} \tag{3.20}$$

where the sum is over all beams which are rigid connected to the considered column end. I_{cln} is the moment if inertia of the column axis, L_{cln} is the column length, I_{bm} is the moment of inertia of the beam and L_{bm} is the length of the beam. μ is a correction factor in order to implement a constraint

- $\mu = 2$ if the other end is rigid connected to one or more columns
- $\mu = 3$ if the other end has a pinned connection
- $\mu = 4$ if the other end is fixed

For an end which is pinned, the value of C is in theory ∞, but an ideal pin is not possible and therefore $C = 5$ is used. For a column fixed at one end, $C = 0.25$ for that end, because an ideal fixed end is not possible. For columns where the ends have an elastic (spring) connection by beams which are not totally stiff connected to the column, L_{cr} can be determined by using the same method, where L_{bm} is replaced by the equivalent beam length L_{equ}

$$L_{equ} = L_{bm} + \frac{\mu \times E \times I_{bm}}{S_j} \tag{3.21}$$

where S_j is the rotational stiffness in a flexible connection.

Situation 2 elaboration Looking at the left column of the second situation it follows that for the lower end which is fixed $C_A = 0$ and so $C_A = 0.25$.

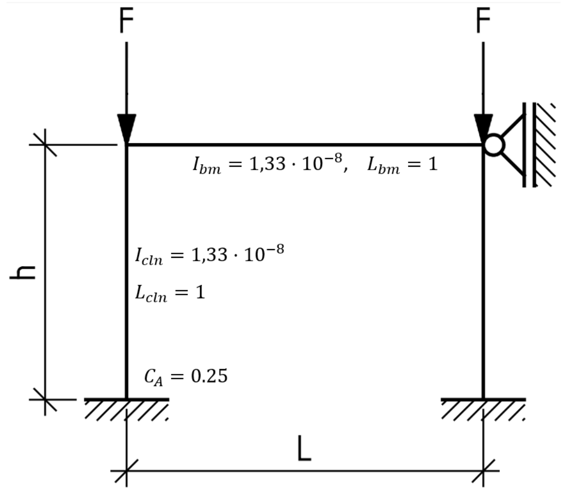


Figure 3.10: Situation 2

For the upper end we again make the following assumptions and thus $I_{cln} = 1, L_{cln} = 1, I_{bm} = 1, L_{bm} = 1$ and because there is a fixed connection with the girder $\mu = 3$, such that

$$C_B = \frac{1/1}{4 \times 1/1} = \frac{1}{4} \tag{3.22}$$

Solving formula 3.18 gives

$$\frac{L_{cr}}{L} = \frac{\pi}{\lambda} \approx 0.66 \tag{3.23}$$

This satisfies the observation that the effective length factor must be between 1 and 2. Using the critical euler buckling formula the critical buckling load follows as $63440, 82N$.

3.4. AISC method

A method which looks like the Eurocode 3 method is the method used in the AISC manual [**?**], where in case of a sway frame and a non sway frame respectively, the effective length factor K must be solved from

$$\begin{aligned} &\frac{G_A G_B}{4}(\pi/K)^2 + \left(\frac{G_A+G_B}{2}\right)\left(1 - \frac{\pi/K}{\tan(\pi/K)}\right) + \frac{2\tan(\pi/2K)}{\pi/K} - 1 = 0 \\ \text{or} \quad &\frac{G_A G_B(\pi/K)^2 - 36}{6(G_A+G_B)} - \frac{\pi/K}{\tan(\pi/K)} = 0 \end{aligned} \tag{3.24}$$

The relative stiffness factors are given by

$$G_A = \frac{\sum_A (EI/L)_c}{\sum_A (EI/L)_g} \quad \text{and} \quad G_B = \frac{\sum_B (EI/L)_c}{\sum_B (EI/L)_g} \tag{3.25}$$

where the subscripts A and B indicate the joints for the column, and the subscripts c and g indicate the terms for the column and girder. Because this methods looks like the Eurocode method it will not be used to calculate the effective lengths for the situations in this assignment.

4

Calculation methods for effective length

In this chapter first it is investigated which methods the different FEM programs use that handle buckling for structures. Also another modified Eurocode 3 method will be discussed and demonstrated.

4.1. Methods used in FEM programs

The table below gives an overview of the different methods used in FEM programs

Table 4.1: Buckling methods used in FEM programs

Program	Method
staad	nonlinear buckling analysis and linear buckling analysis [**?**]
etabs	nonlinear buckling analysis and linear buckling analysis [**?**]
revit	/
SAP2000	nonlinear buckling analysis and linear buckling analysis [**?**]
SACS	/
strand7	Linear buckling analysis [**?**]
nastran	Linear buckling analysis [**?**]
nauticus	/
genie	/
dnv genie	/
sesam	/
maestro	/
rstab	nonlinear buckling analysis and linear buckling analysis [**?**] *With Eurocode 3 extension*: Effective length factors (can be obtained by eigenvalue analysis) and Eurocode 3 [**?**]
rfem	nonlinear buckling analysis and linear buckling analysis [**?**] *With Eurocode 3 extension*: Effective length factors (can be obtained by eigenvalue analysis) and Eurocode 3 [**?**]
civilfem	/
robot	
csi	/
midas	Linear buckling analysis [**?**]
scia	/
tekla	Linear buckling analysis and Eurocode 3 [**?**]
krasta	/
sdc verifier	Eurocode 3 method [**?**]
FEMAP	nonlinear buckling analysis and linear buckling analysis

4.1.1. Linear buckling analysis

The linear buckling analysis is the most used method for FEM programs. For the linear buckling analysis, first a linear static analysis has to be executed. From this linear static analysis a material K^{mat} and an geometrical K^{geo} stiffness matrix can be made. The linear buckling analysis is then performed by solving the following eigenvalue problem [**?**]

$$\left(\mathrm{K}^{mat} + \lambda_i \mathrm{K}^{geo}\right)_i = 0 \tag{4.1}$$

where the eigenvalues λ_i are the load parameters and the eigenvectors $_i$ is the buckling mode corresponding to load parameter λ_i. The load parameter multiplied by the applied load for the analysis defines the critical buckling load where the structure buckles in the corresponding buckling mode.

This method can be applied to all finite element types beams, shells, volume elements, ect.

Situation 1 With the FEM program FEMAP a linear buckling analysis has been performed. For the dimensions and geometry the properties defined in section one are used: length of columns and girder $= 1m$, the moment of inertia of the columns and girder $= I_{cln} = I_{bm} = 1,33 \cdot 10^{-1} m^4$ (rectangular beam of $0,02 \times 0,02$) and the elasticity modulus of the columns and girder equals $E = 2,1 \cdot 10^{11} N/m^2$ (steel). For the model beam elements are used. In the figure below the fixed constraint is given by 123456. In the top corners there is an translation in the z direction (3). The applied load F is defines as 1 such that the resulting eigenvalue defines the critical load in the structure.

From the analysis it follows that the critical buckling load is given by $20635,59N$. This result, and the corresponding buckling mode can be seen in the figure below

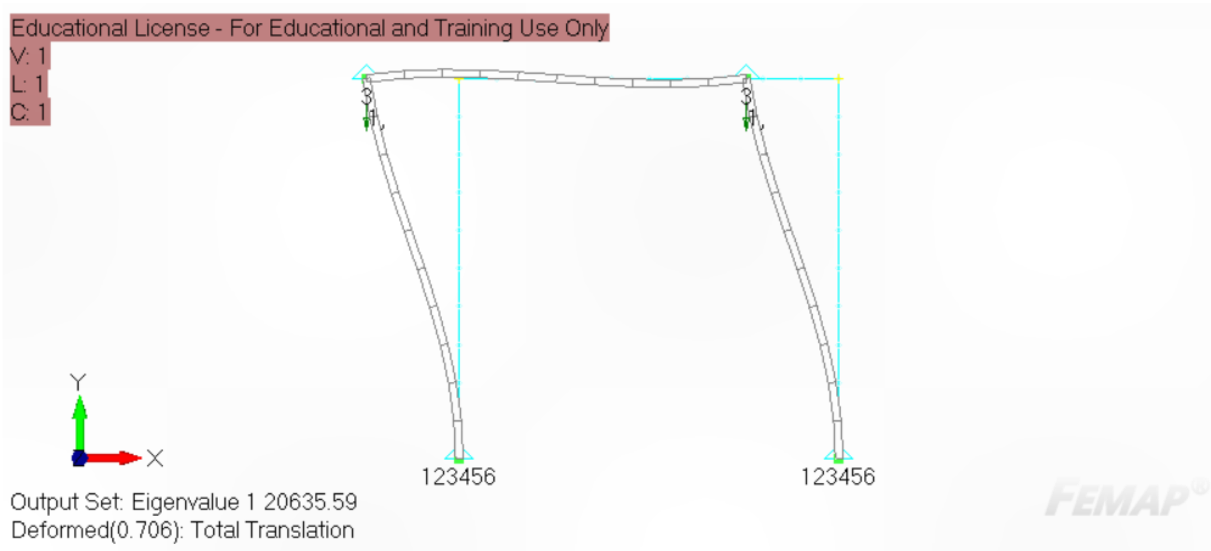


Figure 4.1: Linear buckling analysis result for situation 1

Using statics it follows that the normal force P_{cr} in the column is equal to F. In this analysis the force F equals 20635.59. For the corresponding effective length the effective buckling length formula can be reformulated as

$$K = \frac{1}{L}\sqrt{\frac{\pi^2 EI}{P_{cr}}} \tag{4.2}$$

Using this formula it follows that the effective length is given by $K = 1,16$.

Situation 2 Again the same dimensions and geometry is used, however an extra constraint is added to prevent the sway mode. Also, at the middle of the columns an translation constraint has been added in the z direction. This prevents that the first buckling mode is in the z direction. The critical buckling load now follows as 70329.13. The result and the corresponding buckling mode can be seen in the figure below

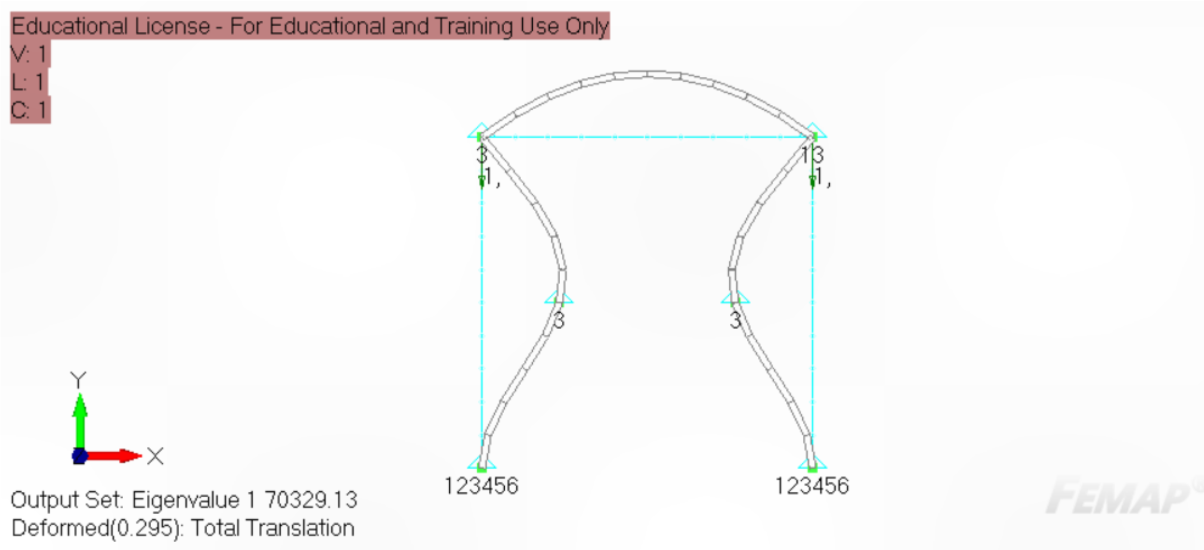


Figure 4.2: Linear buckling analysis result for situation 2

Again, the same formula will be used to determine the effective length factor, which results in $K = 0,627$

4.1.2. Nonlinear buckling analysis

The nonlinear buckling analysis is more expensive then the linear buckling analysis in the sense that a lot more calculation steps are needed [**?**]. In a general nonlinear structural analysis the incremental iterative method is a many used method. With this method the total applied load on the structure will be applied in multiple load steps, where for each step the displacement will be calculated. With a nonlinear buckling analysis also such an incremental iterative method will be used and after each load step the system will be checked for bifurcation. This will be done by an eigenvalue problem with the so called tangential stiffness matrix, where a negative eigenvalue indicates that in the so far applied load a bifurcation has occurred. In this analysis this bifurcation indicates loss of stability and thus the structure has buckled.

This method can also be applied to all finite element types beams, shells, volume elements, ect.

4.1.3. Effective length factors

The programs rstab and rfem also have the possibility to select the effective length factors for all the beam elements in a structure, as can be seen in the figure below

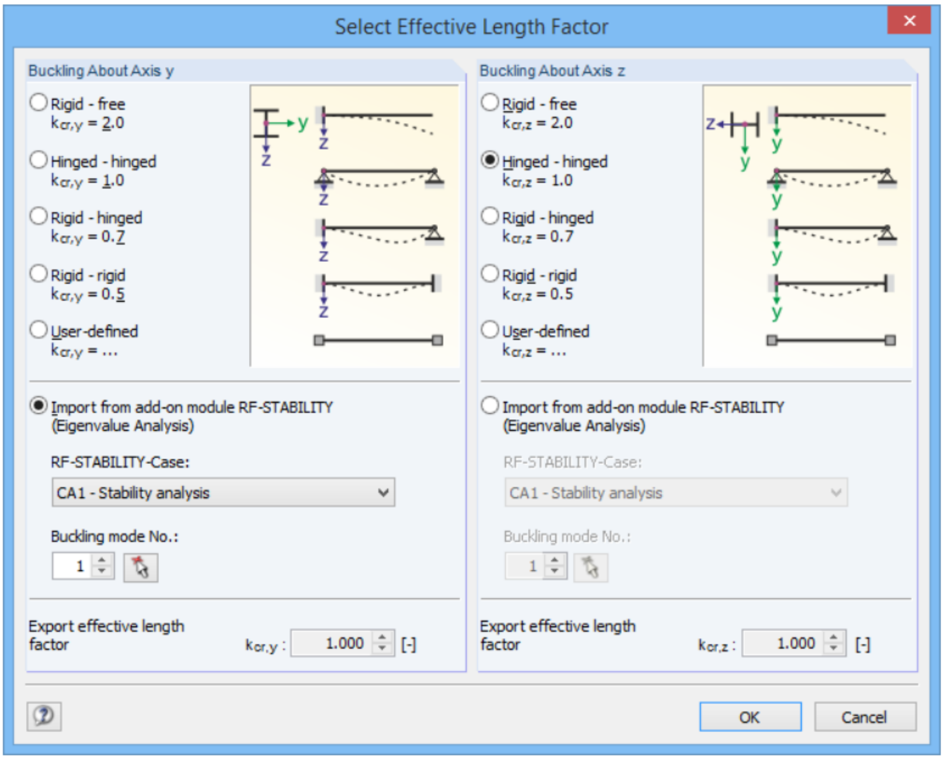


Figure 4.3: Effective length factors

In rstab and rfem there is also the possibility to import the effective lengths from the linear buckling analysis of a buckling mode.

If the effective length factors are known for each member of the frame, the program can execute a linear static analysis and check if the normal force in each member does not exceed the critical compression load corresponding to the effective length.

4.2. StruSoft calculation method

An other calculation method based on the Eurocode 3 method is given by StruSoft [**?**]. In these method there again are two different theoretical planer models, namely for non- sway and sway frames

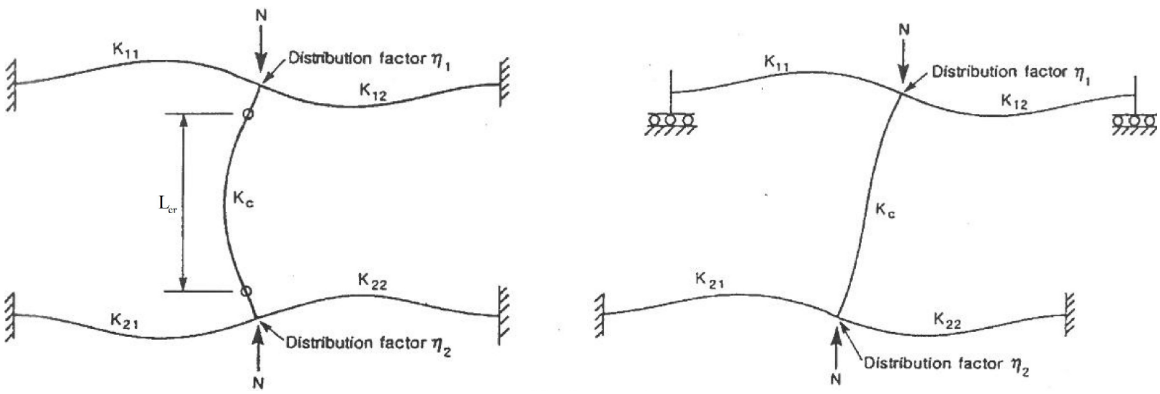


Figure 4.4: Theoretical model for non-sway and sway mode

The distribution factors (representing the sum of the rotational stiffness at the end) at the end of the considered column are defined as

$$\eta_1 = \frac{K_c}{K_c + K_{11} + K_{12}} = \frac{K_c}{K_c + \sum_{j=1}^{n} K_{1j}} \quad \text{and} \quad \eta_2 = \frac{K_c}{K_c + K_{21} + K_{22}} = \frac{K_c}{K_c + \sum_{j=1}^{n} K_{2j}} \tag{4.3}$$

where the K values are the rotational stiffness coefficients. $\eta_i = 0$ if the column is fixed against rotation and $\eta_i = 1$ if the end is hinged. The K_c column stiffness coefficient of the considered column is

$$K_c = 4\frac{EI}{L} \tag{4.4}$$

The effective stiffness coefficients of the connecting beams at the ends of the considered column are K_{ij} with $i = 1, 2$ representing the point of the considered column.

If the considered column has a parallel sequel at one end then the distribution is given by

$$\eta_1 = \frac{K_c + K_1}{K_c + K_1 + K_{11} + K_{12}} = \frac{K_c + K_1}{K_c + K_1 + \sum_{j=1}^{n} K_{1j}} \quad \text{and} \quad \eta_2 = \frac{K_c + K_2}{K_c + K_2 + K_{21} + K_{22}} = \frac{K_c + K_2}{K_c + K_2 + \sum_{j=1}^{n} K_{2j}} \tag{4.5}$$

where K_1 and K_2 are the effective coefficients of the continuous (sequel part) column at the end points.

The effective stiffness coefficients of the connecting beams K_{ij} depends on

- The far end support condition of the beam (e.g. fixed, hinged, elastic support, ect.)
- The connecting elements to the beam at far ends (e.g. vertical columns, ect.)
- The previous property depends on whether the column is non-sway or sway
- The end release condition of the connecting beam at both ends (e.g. hinges)

In general case if the connecting beam has a transnational K_T and rotational K_R point support at the far end the effective rotational stiffness parameter in the principal plain of the straight beam is given by

$$K_{ij} = 4\frac{EI_B\left(K_T L_B^2 \left(K_R L_B + 3EI_B\right) + 3K_R EI_B\right)}{K_T L_B^3 \left(K_R L_B + 4EI_B\right) + 12EI_B \left(K_R L_B + EI_B\right)} \tag{4.6}$$

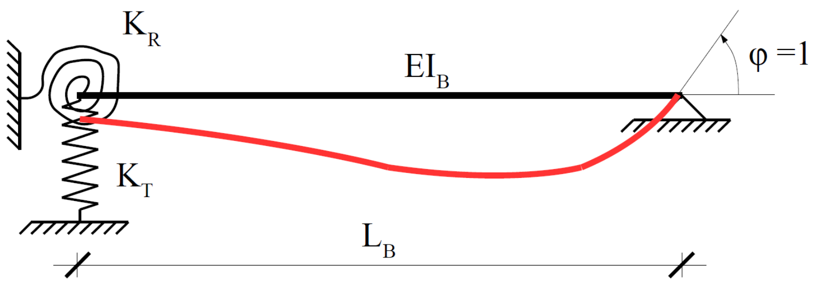


Figure 4.5: The effective rotational parameter case with general support condition at far end

The following far end support conditions of the connecting beam

Table 4.2: Far end support conditions and effective stiffness

Conditions			Effective stiffness
$K_T \to \infty$	$K_R \to \infty$	Fixed end	$K_{ij} = 4\frac{EI_B}{L_B}$
$K_T \to \infty$	$K_R \to 0$	Hinged end	$K_{ij} = 3\frac{EI_B}{L_B}$
$K_T \to 0$	$K_R \to 0$	Free end	$K_{ij} = 0$
$K_T \to 0$	$K_R \to \infty$		$K_{ij} = \frac{EI_B}{L_B}$

- *Non- Sway frame*: in a sway frame if the connecting beam has a connection with an adjacent column then the rotational stiffness parameter comes from the case when the rotation equal and opposite to that at the near end. The stiffness parameter in this case

$$K_{ij} = 2\frac{EI_B}{L_B} \tag{4.7}$$

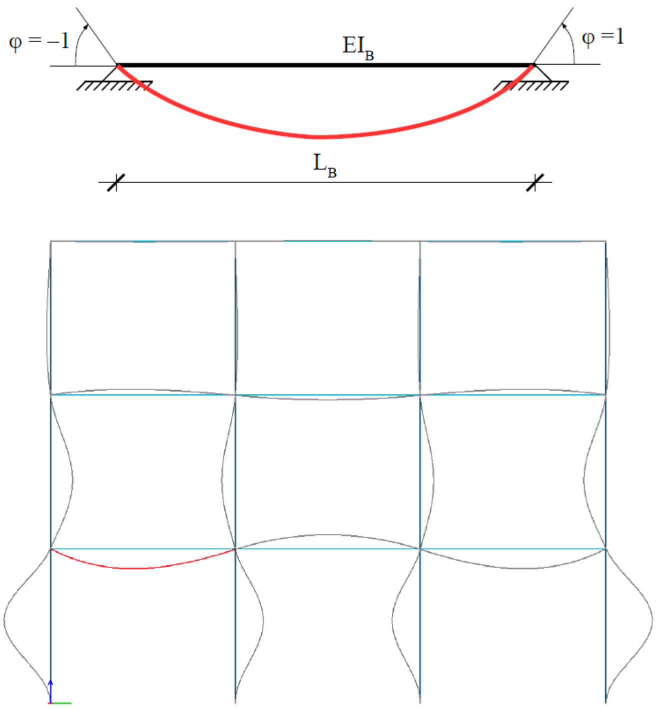


Figure 4.6: Beam in non- sway frame

- *Sway frame*: in a sway frame if the connecting beam has a connection with an adjacent column then the rotational stiffness parameter comes from the case when the rotation equal to that at the near end. The stiffness parameter in this case

$$K_{ij} = 6\frac{EI_B}{L_B} \tag{4.8}$$

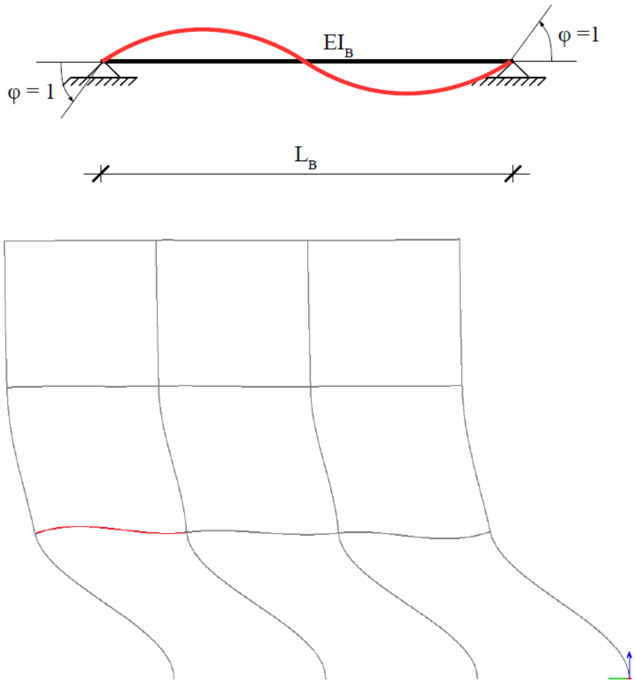


Figure 4.7: Beam in sway frame

Finally, the ratio between the flexural buckling length and the geometric length of the considered column (thus the effective length) is given by

$$\beta = L_{cr}/L \tag{4.9}$$

where

- Non- sway frame: $\beta = \frac{1+0.145(\eta_1+\eta_2)-0.265\eta_1\eta_2}{2-0.364(\eta_1+\eta_2)-0.247\eta_1\eta_2}$ where the factor is between 0.5 and 1
- Sway frame: $\beta = \sqrt{\frac{1-0.2(\eta_1+\eta_2)-0.12\eta_1\eta_2}{1-0.8(\eta_1+\eta_2)+0.6\eta_1\eta_2}}$ where the factor is between 1.0 and $+\infty$.

Situation 1 Again the same geometry, dimensions and columns and girder will be used, such that

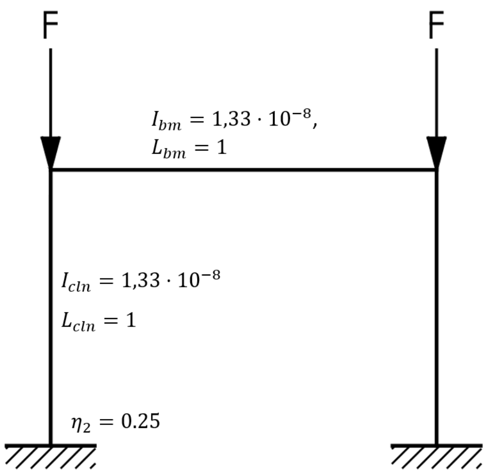


Figure 4.8: Situation 1

Considering the left column it follows that $\eta_2 = 0$. For η_1

$$\eta_1 = \frac{K_c}{K_c + K_{11} + K_{12}} = \frac{K_c}{K_c + 0 + K_{12}} \tag{4.10}$$

where $K_{12} = 6\frac{EI_{bm}}{L_{bm}}$ because there is a fixed connection and $K_c = 4\frac{EI_{cln}}{L_{cln}}$. It follows that $\eta_1 = 0,4$. Solving β gives $\beta = 1,16216$. By using the critical buckling load formula

$$P_{cr} = \frac{\pi^2 EI}{(K \cdot L)^2} = 20425.74N \tag{4.11}$$

Situation 2 Also for situation 2 the same geometry, dimensions, columns and girder will be used

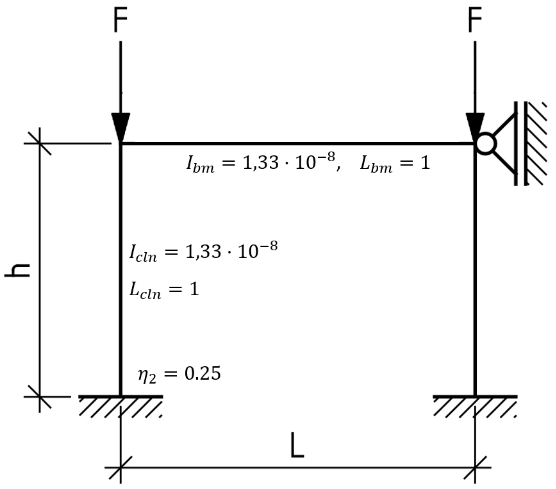


Figure 4.9: Situation 2

Considering the left column it again follows that $\eta_2 = 0$. For η_1

$$\eta_1 = \frac{K_c}{K_c + K_{11} + K_{12}} = \frac{K_c}{K_c + 0 + K_{12}} \tag{4.12}$$

where $K_{12} = 2\frac{EI_{bm}}{L_{bm}}$ because there is a fixed connection and $K_c = 4\frac{EI_{cln}}{L_{cln}}$. It follows that $\eta_1 = 0,667$. Solving β gives $\beta = 0,624052$. By using the critical buckling load formula

$$P_{cr} = \frac{\pi^2 EI}{(K \cdot L)^2} = 70960,34N \tag{4.13}$$

5

Result and conclusions

5.1. Results

The results of the found methods are summarized in the table below

Table 5.1: Results of effective length calculation methods

	Sway frame		Non Sway frame	
Method	Critical load	Effective length	Critical load	Effective length
Eurocode 3	$13143,79N$	$1,45$	$63440,82N$	$0,66$
FEMAP	$20635,59N$	$1,16$	$70329,13N$	$0,63$
StruSoft	$20425,74N$	$1,16$	$70960,34N$	$0,62$

It can be seen that, especially with the sway frame, there is a significant error between the Eurocode 3 method and the FEMAP simulation. However, the improved Eurocode 3 method from StruSoft has an negligible error compared to the FEMAP simulation. However, with the methods used for only two situations no conclusion can be made about what is the more accurate method.

In this case it is however recommenced to use the Eurocode 3 results. This because the eurocode 3 method gives the largest effective length results and thus the lowest critical load. Buckling is a failure mode where the exact value of the critical load can not be determined. So, it is more useful to underestimate the strength of the frame against buckling. It is therefore recommended design the frame such that is can resits the critical loads of all methods. In this case a frame which satisfies eurocode 3 also satisfies the FEMAP and StruSoft outcome, and not vice versa.

The above does not apply to all methods. The method used in this assignment are well tested and many used and thus the results are not significantly inaccurate. There are many methods and not all methods show good results and thus not all methods need to be satisfied.

5.2. Conclusion

First of all, there are two types of frames, a sway and a non sway frame. Which type of frame determines the type of buckling mode and the necessary calculation method. There are different ways of determining which type of calculation method has to be used

- Classical method (chapter 1) to determine if the frame is a sway frame
- Use both methods (sway and non sway) and use the worst case scenario

- Preform a buckling analysis with a finite element analysis program and observe if the first buckling mode is a sway or non sway mode (or just use the results of the linear buckling analysis of a FEM program).
- Use own insight if the frame has the possibility to sway when it buckles. If not, the non sway frame methods need to be used

In chapter 2, the buckling of single columns has been discussed. The buckling mode and the critical buckling load are depended on the type of boundary conditions. Each combination of boundary conditions has its own effective length factor and characteristic displacement, curvature and moment lines. This moment lines can be used to determine the effective length of the columns in sway and non sway frame looking at the moment lines of the buckling mode of that frame.

Eurocode 3 uses a similar simplified method, which uses the single column with boundary condition cases of chapter 2 in order to determine the effective lengths of columns in a frame. This is discussed in chapter 3. An more advanced method from Eurocode 3 is also discussed in chapter 3, and tested on the frames of the two situations.

In chapter 4 the methods used in FEM programs are discussed, where it can be seen that the most used methods are the linear and nonlinear buckling analysis. This methods however do not give an effective length for each column but only the critical buckling load and the corresponding buckling mode. The only found programs which are able to determine effective lengths out of a buckling mode are rfem and rstab. In chapter 4 also a modified version of the Eurocode 3 method is discussed. Both the modified Eurocode method and the linear buckling analysis of a fem program (FEMAP) are used to determine the effective lengths of both situations.

So looking at the research question, *How can the effective length of a column under compression, as part of a frame, be determined?* there are multiple methods to calculate effective lengths. However, not all methods can be trusted and one should determine the plausibility of the results. In the case of this assignment, all the three considered methods give plausible results to determine the effective length en thus all the methods should be satisfied.

5.3. Recommendation

If we want to tackle a buckling problem where we want to determine of a structure remains stable against buckling to a corresponding load there are multiple ways. If we look at the situations used in this assignment it can observed that for almost the same frame there are significantly different results depended on if it is a sway or a non sway frame. A first recommendation is thus to determine if the considered frame is a sway or a non sway frame, and if this is not known the worst case scenario is recommended.

Based on the results on this assignment not a single method can be recommended on which is more accurate. Therefore it is always better to use multiple methods an let the structure satisfy all applicable methods. In this assignment three methods are tested and they all have plausible results but there is a difference in the results. The three used methods in this assignment are well used and trusted methods and thus it is recommended to let the structure satisfy all methods.

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