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Fractional stochastic partial differential equations in space and time

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FRACTIONAL STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN SPACE AND TIME

Fractional stochastic partial differential equations in space and time

Dissertation

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by

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Contents

Summary

In this thesis, we study deterministic and stochastic abstract evolution equations which are assumed to be of fractional order, either spatially or spatiotemporally. The first chapter contains the preliminary concepts necessary to interpret such equations, and indicates their relations to the subsequent chapters.

In the second chapter, we consider a class of fractional-order linear stochastic partial differential equations of the form

$$(\partial_t + A)^\gamma X(t) = \dot{W}^Q(t), \quad t \in [0,T], \qquad \gamma \in (1/2, \infty),$$

with zero initial data. The linear operator $-A$ generates a strongly continuous semigroup on a separable Hilbert space and the spatiotemporal driving noise $\dot{W}^Q$ is the formal time derivative of a cylindrical Q-Wiener process taking its values in this space. Mild and weak solutions are defined, and these concepts are shown to be equivalent and to lead to well-posed problems. In the case that the Hilbert space is $L^2(\mathcal{D})$ for some spatial domain $\mathcal{D} \subseteq \mathbb{R}^d$, these solutions are interpreted as spatiotemporal Gaussian random fields $(X(t,x))_{(t,x)\in[0,T]\times\mathcal{D}}$. We investigate the temporal and spatial regularity of the solution process X, the former being measured by mean-square or pathwise smoothness and the latter using fractional domain spaces of A. In addition, the covariance of X and its long-time behavior are analyzed. These abstract results are applied to the cases when $A := L^\beta$ and $Q := \widetilde{L}^{-\alpha}$ are fractional powers of symmetric, strongly elliptic second-order differential operators defined on (i) bounded Euclidean domains $\mathcal{D} \subsetneq \mathbb{R}^d$, or (ii) smooth, compact surfaces $\mathcal{M}$. In these cases, the solution processes can be seen as spatiotemporal generalizations of the (Whittle–)Matérn Gaussian random fields widely used in spatial statistics.

In the third chapter, we define a number of higher-order Markov properties for stochastic processes indexed by an interval $\mathbb{T} \subseteq \mathbb{R}$ and taking values in a real separable Hilbert space, and we investigate the relations between them. For the abstract stochastic equation $\mathcal{L}X = \dot{W}$, where $\mathcal{L}$ is a linear operator and W is a cylindrical Wiener process, we identify two sets of additional conditions under which the locality of the precision operator $\mathcal{L}^*\mathcal{L}$ becomes either necessary or sufficient, respectively, for X to possess the weakest Markov property. We apply this theory to the setting of the preceding chapter by taking $\mathcal{L} = (\partial_t + A)^\gamma$ with A and γ as above, and zero initial data if $\inf \mathbb{T} > -\infty$. We prove that the resulting solution process satisfies an Nth order Markov property if $\gamma = N \in \mathbb{N}$ and conversely demonstrate that a necessary condition for the weakest Markov property is, in general, not satisfied if $\gamma \notin \mathbb{N}$. Moreover, complementing the aforementioned link to Whittle–Matérn fields, we again demonstrate the relevance of this class of processes by showing that an infinite-dimensional analog to the fractional Brownian motion with Hurst parameter $H \in (0,1)$ is obtained as the limiting case of $\mathcal{L} = (\partial_t + \varepsilon\,\mathrm{Id})^{H+\frac{1}{2}}$ for $\varepsilon \downarrow 0$.

In the fourth chapter, we turn to the (deterministic) natural Dirichlet problem for $(\partial_t + A)^\gamma$ on an interval (t_0, ∞) with $t_0 \in \mathbb{R}$. That is, instead of considering zero initial data or taking $t_0 = -\infty$ as in the preceding two chapters, in this chapter we impose that the solution equals a given function g on all of $(-\infty, t_0]$:

$$\begin{cases} (\partial_t + A)^\gamma u(t) = 0, & t \in (t_0, \infty), \quad \gamma \in (0, \infty), \\ u(t) = g(t), & t \in (-\infty, t_0], \end{cases}$$

The operator $-A$ now acts on an arbitrary Banach space and generates a semigroup which is strongly measurable and uniformly bounded. Under the additional assumption that the semigroup is exponentially stable, we show that the Dirichlet problem is well-posed in an L^p-sense with $p \in [1, \infty]$. For values of γ and p such that L^p-solutions are continuous, we show that they satisfy a mild solution formula, expressing them in terms of the initial data and the semigroup, and generalizing the well-known variation of constants formula for the first-order abstract Cauchy problem. Although its derivation relies on the exponential stability assumption, the resulting solution formula remains meaningful for uniformly bounded semigroups. Moreover, we include a comparison to analogous solution concepts arising from Riemann–Liouville and Caputo type initial value problems.

Finally, in the fifth chapter, we return to the case of stochastic equations. Namely, we study the convergence as $n \to \infty$ of a sequence of semilinear parabolic stochastic evolution equations of the form

$$\begin{cases} \mathrm{d}X_n(t) = -A_n X_n(t)\,\mathrm{d}t + F_n(t, X_n(t))\,\mathrm{d}t + \mathrm{d}W_n(t), & t \in (0, T], \\ X_n(0) = \xi_n. \end{cases}$$

posed on a sequence of Banach spaces which approximate a limiting space. The abstract 'discrete-to-continuum approximation' setting is encoded by means of projection and lifting operators. These allow us to

- compare the linear operators A_n, semilinear drifts F_n and initial data ξ_n at different indices n,
- define the additive noise $\mathrm{d}W_n$ as a projection of cylindrical Wiener noise on the limiting space, and thus
- formulate conditions on the growth, Lipschitz continuity, and convergence of the coefficients under which we establish convergence of the associated mild solution processes X_n when lifted to a common state space.

Our framework is applied to the case where the limiting problem is a stochastic partial differential equation whose linear part is a generalized Whittle–Matérn operator on a manifold $\mathcal{M}$, discretized by a sequence of graphs constructed from a (random) point cloud. In this setting, we obtain discrete-to-continuum convergence of solutions lifted to the spaces $L^q(\mathcal{M})$ for $q \in [2, \infty]$.

Samenvatting

In dit proefschrift bestuderen we deterministische en stochastische abstracte evolutievergelijkingen van fractionele (tijd)ruimtelijke orde. In het eerste hoofdstuk vatten we concepten samen die nodig zijn om dergelijke vergelijkingen te interpreteren en geven we de verbanden met de daaropvolgende hoofdstukken aan.

In het tweede hoofdstuk behandelen we een klasse van lineaire fractionele stochastische partiële differentiaalvergelijkingen van de vorm

$$(\partial_t + A)^\gamma X(t) = \dot{W}^Q(t), \quad t \in [0,T], \qquad \gamma \in (1/2, \infty),$$

met beginwaarde nul. De lineaire operator $-A$ brengt een sterk continue halfgroep voort op een separabele Hilbertruimte en het tijdruimtelijke ruisproces $\dot{W}^Q$ is de formele tijdsafgeleide van een cilindrisch Q-Wienerproces met waarden in deze ruimte. We definiëren milde en zwakke oplossingen en tonen aan dat deze concepten equivalent zijn en goedgestelde problemen opleveren. Deze oplossingen worden geïnterpreteerd als tijdruimtelijke Gaussische kansvelden $(X(t,x))_{(t,x)\in[0,T]\times\mathcal{D}}$ als we als Hilbertruimte $L^2(\mathcal{D})$ nemen, gegeven een ruimtelijk domein $\mathcal{D} \subseteq \mathbb{R}^d$. We onderzoeken de regulariteit van het oplossingsproces X in tijd en ruimte, waarbij tijdsregulariteit wordt gemeten aan de hand van kwadratisch gemiddelde of padsgewijze gladheid, en ruimtelijke regulariteit via fractionele domeinruimten van A. Daarnaast analyseren we de covariantie van X en het asymptotische gedrag op lange termijn. Deze abstracte resultaten worden toegepast op het geval dat $A := L^\beta$ en $Q := \widetilde{L}^{-\alpha}$ fractionele machten zijn van symmetrische, sterk elliptische differentiaaloperatoren gedefinieerd op (i) begrensde Euclidische domeinen $\mathcal{D} \subsetneq \mathbb{R}^d$ of (ii) gladde, compacte oppervlakken $\mathcal{M}$. In deze gevallen kunnen de oplossingsprocessen worden gezien als tijdruimtelijke generalisaties van de (Whittle–)Matérn Gaussische kansvelden die veelvuldig worden gebruikt in de ruimtelijke statistiek.

In het derde hoofdstuk definiëren we een aantal Markov-eigenschappen van hogere orde voor kansprocessen geïndexeerd door een interval $\mathbb{T} \subseteq \mathbb{R}$ met waarden in een reële separabele Hilbertruimte, en we onderzoeken de onderlinge relaties tussen deze eigenschappen. Voor de vergelijking $\mathcal{L}X = \dot{W}$, waarbij $\mathcal{L}$ een lineaire operator is en W een cilindrisch Wienerproces, stellen we voorwaarden vast die ervoor zorgen dat lokaliteit van de precisie-operator $\mathcal{L}^*\mathcal{L}$ noodzakelijk dan wel voldoende is voor de zwakste Markov-eigenschap van X. We passen deze theorie toe op de situatie van het vorige hoofdstuk door als operator $\mathcal{L} = (\partial_t + A)^\gamma$ te nemen, met A en γ zoals hierboven gedefinieerd en beginvoorwaarde nul als $\inf \mathbb{T} > -\infty$. We bewijzen dat het resulterende oplossingsproces een N^{e}-orde Markov-eigenschap heeft als $\gamma = N \in \mathbb{N}$; omgekeerd laten we zien dat een noodzakelijke voorwaarde voor de zwakste Markov-eigenschap in het algemeen niet vervuld is indien $\gamma \notin \mathbb{N}$.

Ten slotte tonen we, ter aanvulling op het eerder opgemerkte verband met Whittle–Matérnvelden, nogmaals de relevantie van deze klasse processen aan door te bewijzen dat een oneindigdimensionaal analogon van de fractionele Brownse beweging met Hurstparameter $H \in (0,1)$ een limietgeval is van $\mathcal{L} = (\partial_t + \varepsilon \,\mathrm{Id})^{H+\frac{1}{2}}$ als $\varepsilon \downarrow 0$.

In het vierde hoofdstuk richten we ons op het (deterministische) natuurlijke Dirichletprobleem voor $(\partial_t + A)^\gamma$ op een interval (t_0, ∞) met $t_0 \in \mathbb{R}$. In tegenstelling tot de vorige twee hoofdstukken, waarin de beginwaarde nul of $t_0 = -\infty$ was, leggen we nu op dat de oplossing gelijk is aan een gegeven functie g op $(-\infty, t_0]$:

$$\begin{cases} (\partial_t + A)^\gamma u(t) = 0, & t \in (t_0, \infty), \quad \gamma \in (0, \infty), \\ u(t) = g(t), & t \in (-\infty, t_0]. \end{cases}$$

De operator $-A$ werkt nu op een algemene Banachruimte en brengt een sterk meetbare en uniform begrensde halfgroep voort. Onder de verdere aanname dat deze uniform exponentieel stabiel is, tonen we aan dat het Dirichletprobleem goedgesteld is in L^p-zin voor $p \in [1, \infty]$. Voor waarden van γ en p zodanig dat deze L^p-oplossingen continu zijn, tonen we aan dat ze kunnen worden uitgedrukt met een milde oplossingsformule, in termen van de beginwaarden en de halfgroep, die de bekende variatie-van-constantenformule voor het eerste-order abstracte Cauchyprobleem generaliseert. Hoewel de afleiding gebruikmaakt van de exponentiële stabiliteit, blijft de uiteindelijke oplossingsformule welgedefinieerd voor uniform begrensde halfgroepen. Tot slot vergelijken we deze formule met analoge oplossingsconcepten voor beginwaardeproblemen in de zin van Riemann–Liouville of Caputo.

In het vijfde en laatste hoofdstuk keren we terug naar stochastische vergelijkingen. We bestuderen de convergentie als $n \to \infty$ van semilineaire parabolische stochastische evolutievergelijkingen van de vorm

$$\begin{cases} \mathrm{d}X_n(t) = -A_n X_n(t)\,\mathrm{d}t + F_n(t, X_n(t))\,\mathrm{d}t + \mathrm{d}W_n(t), & t \in (0, T], \\ X_n(0) = \xi_n. \end{cases}$$

geformuleerd op een rij Banachruimten die een limietruimte benaderen in een abstracte ‘discreet-naar-continuüm’-zin die formeel wordt beschreven aan de hand van projectie- en liftoperatoren. Deze stellen ons in staat om

- de lineaire operatoren A_n, semilineaire driftoperatoren F_n en beginvoorwaarden ξ_n voor verschillende indices n met elkaar te vergelijken,
- de additieve ruis $\mathrm{d}W_n$ te definiëren als een projectie van cilindrische Wienerruis op de limietruimte, en daarmee
- voorwaarden te formuleren inzake groei, Lipschitzcontinuïteit en convergentie van de coëfficiënten, waarmee we de convergentie van de milde oplossingsprocessen X_n in een gemeenschappelijke toestandsruimte kunnen bewijzen.

We passen onze methode toe op het geval waarin het limietprobleem een stochastische partiële differentiaalvergelijking is met als lineair deel een gegeneraliseerde Whittle–Matérnoperator op een gladde variëteit $\mathcal{M}$, die wordt gediscretiseerd door een rij grafen geconstrueerd op basis van een (willekeurige) puntenwolk. In deze context verkrijgen we discreet-naar-continuümconvergentie van oplossingen, gelift naar de ruimte $L^q(\mathcal{M})$ voor $q \in [2, \infty]$.

1
Introduction

This introductory chapter is devoted to summarizing the common themes which occur in the subsequent main Chapters 2–5 and illustrating the interrelations between them and the rest of this thesis. Since the upcoming chapters are all, at least partially, concerned with (stochastic) partial differential equations of fractional order in space and possibly time, the necessary preliminaries can naturally be divided into the following categories.

In Section 1.1 we describe how (deterministic) partial differential equations can be cast into the abstract functional-analytic setting of evolution equations, i.e., ordinary differential equations taking values in (infinite-dimensional) Banach or Hilbert spaces. An important role is played by the notion of a semigroup generated by a linear operator on such a space, and in Section 1.2 we focus on the concrete situation of elliptic second-order operators defined via sesquilinear (or bilinear) forms. In Section 1.3 we discuss the concept of functional calculus, with the particular aim of defining fractional powers of (differential) operators which enable us to consider fractional-order evolution equations. The final preliminaries are given in Section 1.4, where we consider the probabilistic notions of (Gaussian) random fields and processes, white noise and stochastic integration; these are used for the *stochastic* evolution equations considered throughout the subsequent chapters. Section 1.5 provides an outline of the remaining chapters of this thesis and their relation to the preliminaries from Sections 1.1–1.4.

Throughout this chapter, we assume familiarity with basic concepts from functional analysis, (metric) topology, differential geometry and probability theory, but we will typically emphasize our notational choices at their first occurrence. We make no claim of originality of the results comprising this chapter, and in Section 1.6 we give some further bibliographical notes on the topics treated in this introduction.

1.1. Semigroups and abstract evolution equations

Consider a linear inhomogeneous deterministic partial differential equation of first order in time and arbitrary order in space. That is, given a time interval $J := (t_0, T)$ (where $-\infty \le t_0 < T \le \infty$), an open and connected spatial domain $\emptyset \ne \mathcal{D} \subseteq \mathbb{R}^d$ (where $d \in \mathbb{N} := \{1, 2, \dots\}$), and an inhomogeneity $f : J \times \mathcal{D} \to \mathbb{R}$, we wish to find a

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function $u\colon \overline{J} \times \overline{\mathcal{D}} \to \mathbb{R}$ (where the bar indicates the closure) which solves the initial–boundary value problem

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) + Lu(t,x) = f(t,x), & (t,x) \in J \times \mathcal{D}; \\ u(t_0,x) = u_0(x), & x \in \mathcal{D}; \\ u(t,x) = 0, & (t,x) \in J \times \partial\mathcal{D}. \end{cases} \tag{1.1.1}$$

Here, L represents a linear operator acting on the spatial variable of u (such as the Laplace(–Beltrami) operator, see Sections 1.2.3 and 1.2.4), and $\partial\mathcal{D}$ denotes the (topological) boundary of $\mathcal{D}$. The initial condition $u_0\colon \mathcal{D} \to \mathbb{R}$ and homogeneous boundary data in (1.1.1) are only prescribed if $t_0 > -\infty$ and $\partial\mathcal{D} \neq \emptyset$, respectively.

The aim of this section is to provide a brief introduction to the *abstract evolution equation* viewpoint for problems such as (1.1.1), since we will view all the (fractional-order and possibly stochastic) evolution equations treated in Chapters 2–5 in this way. The key idea is to interpret u and f as functions of time which take their values in a space E of functions from $\mathcal{D}$ to $\mathbb{R}$. If E has sufficient structure, for instance if it is a Banach space, then we can cast (1.1.1) into the form of an E-valued (and typically infinite-dimensional) ordinary differential equation of the form

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in J; \\ u(t_0) = u_0 \in E. \end{cases} \tag{1.1.2}$$

Here, $A\colon \mathsf{D}(A) \subseteq E \to E$ is an (unbounded) linear operator which models the spatial (differential) operator L. Its domain $\mathsf{D}(A)$ can be used to encode boundary conditions, such as the homogeneous Dirichlet condition in (1.1.1). In other words, the evolution equation viewpoint consists in removing the explicit dependence on the spatial variable $x \in \mathcal{D}$ from the equation by encoding it implicitly into E.

Several definitions of a solution to the *abstract Cauchy problem*, as (1.1.2) is called, are given in Section 1.1.3. For this thesis, the most important of these is the notion of a *mild solution*, which is used throughout all of Chapters 2–5. It is formulated in terms of a one-parameter family $(S(t))_{t\ge 0}$ of bounded linear operators on E associated to A, called a *strongly continuous semigroup*, which can intuitively be viewed as an operator-valued exponential function $(e^{-tA})_{t\ge0}$. Owing to their central role in this thesis, the first two subsections of this section are devoted to semigroups. In the final Subsection 1.1.4 we study the sum operator $\partial_t + A$, whose fractional powers give rise to the equations studied in Chapters 2–4.

1.1.1. SEMIGROUPS OF BOUNDED LINEAR OPERATORS

Suppose that $(E, \|\cdot\|_E)$ is a Banach space over the complex scalar field $\mathbb{C}$.[1] Denote by $\mathscr{L}(E)$ the space of bounded linear operators on E equipped with the operator

[1] In many applications, such as those involving stochastic processes, it is more natural to consider real Banach spaces. Although throughout this thesis we will be careful to specify the scalar field over which any given Banach space is considered, we may sometimes apply results to a real Banach space which are only stated for complex spaces. This can be rigorously justified through the use of complexifications (see Section 2.B.2 below).

norm $\|T\|_{\mathscr{L}(E)} := \sup_{\|x\|_E \le 1} \|Tx\|_E$. Before introducing the notion of a semigroup of bounded linear operators on E, let us first comment on the case where A itself is bounded. In this situation, it is immediate from the triangle inequality and submultiplicativity of the operator norm that

$$\left\| \sum_{k=0}^{\infty} \frac{t^k}{k!} (-A)^k \right\|_{\mathscr{L}(E)} \le \sum_{k=0}^{\infty} \frac{t^k}{k!} \|A\|^k_{\mathscr{L}(E)} = e^{t\|A\|_{\mathscr{L}(E)}} < \infty \quad \text{for all } t \ge 0.$$

Hence, we can define the family $(e^{-tA})_{t\ge 0} \subseteq \mathscr{L}(E)$ of operator exponentials by

$$e^{-tA} := \sum_{k=0}^{\infty} \frac{t^k}{k!} (-A)^k, \quad t \ge 0, \tag{1.1.3}$$

since the series on the right-hand side is absolutely convergent in $\mathscr{L}(E)$. If we in fact have $\dim E = n < \infty$, so that we can identify A with an $n \times n$ matrix, then we recover the definition of the matrix exponential. In this finite-dimensional setting, we can view (1.1.2) as a linear system of n scalar-valued ordinary differential equations with initial vector $u_0 \in \mathbb{C}^n$, for which the classical theory yields the unique solution

$$u(t) = e^{-(t-t_0)A} u_0 + \int_{t_0}^{t} e^{-(t-s)A} f(s)\, \mathrm{d}s \quad \text{for all } t \in \overline{J}. \tag{1.1.4}$$

For an unbounded linear operator $A \colon \mathsf{D}(A) \subseteq E \to E$, definition (1.1.3) of e^{-tA} is no longer valid, as the infinite series on the right-hand side does not converge in general. However, to certain classes of unbounded operators we can still associate a one-parameter family $(S(t))_{t\ge 0}$ of bounded linear operators on E which generalizes $(e^{-tA})_{t\ge 0}$ and has a number of analogous properties. We start with the following.

Definition 1.1.1. A one-parameter family $(S(t))_{t\ge 0} \subseteq \mathscr{L}(E)$ of bounded linear operators on $(E, \|\cdot\|_E)$ is said to be an operator semigroup if it satisfies $S(0) = \mathrm{Id}_E$ (the identity operator on E) and

$$S(t+s) = S(t)S(s) \quad \text{for all } t, s \ge 0. \tag{1.1.5}$$

Relation (1.1.5) is called the *semigroup law*. We are primarily interested in operator semigroups $(S(t))_{t\ge 0}$ for which the mapping $t \mapsto S(t)$ has some additional structure such as measurability, continuity or (possibly complex) differentiability.

Definition 1.1.2. An operator semigroup $(S(t))_{t\ge 0} \subseteq \mathscr{L}(E)$ is said to be

- *strongly measurable* if, for all $x \in E$, the *orbit* $t \mapsto S(t)x$ is strongly measurable as a mapping from $(0,\infty)$ to E;
- *strongly continuous on* $J \subseteq [0,\infty)$ if, for all $x \in E$, the orbit $t \mapsto S(t)x$ is continuous as a mapping from J to E;
- *uniformly continuous on* $J \subseteq [0,\infty)$ if $t \mapsto S(t)$ is continuous as a mapping from J to $\mathscr{L}(E)$.

If the interval J is not explicitly specified, then we mean $J = [0,\infty)$.

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We remark that the terms *strongly* and *uniformly* in the above definitions respectively refer to the *strong operator topology* (i.e., the coarsest topology on $\mathscr{L}(E)$ which renders the evaluation mapping $\mathscr{L}(E) \ni T \mapsto Tx \in E$ at x continuous for all $x \in E$), and the *uniform operator topology*, induced by the norm $\|\cdot\|_{\mathscr{L}(E)}$ (and therefore also known as the *(operator) norm topology*).

Every strongly measurable semigroup is strongly continuous on $J = (0,\infty)$ by [109, Theorem 10.2.3]. An operator semigroup which is strongly continuous on all of $[0,\infty)$ is known as a *C_0-semigroup*.[2] Therefore, the only additional condition required for a semigroup to be a C_0-semigroup is that

$$S(t)x \to x \quad \text{in } E \quad \text{as } t \downarrow 0, \quad \text{for all } x \in E.$$

We call an operator semigroup $(S(t))_{t\ge0}$ *locally bounded* if $t \mapsto S(t)$ is bounded in $\mathscr{L}(E)$ on every bounded subinterval of $[0,\infty)$. In this case, there exist constants $M \in [1,\infty)$ and $w \in \mathbb{R}$ such that the following exponential norm bound holds:

$$\|S(t)\|_{\mathscr{L}(E)} \le Me^{-wt} \quad \text{for all } t \ge 0, \tag{1.1.6}$$

cf. [114, Proposition G.2.2]. If $w \ge 0$ (resp. $w > 0$), then we say that $(S(t))_{t\ge0}$ is *uniformly bounded* (resp. *uniformly exponentially stable*), and if in addition $M = 1$, then $(S(t))_{t\ge0}$ is said to be a *semigroup of contractions* or a *contractive semigroup*. If $M = 1$ but $w < 0$, then the semigroup is said to be *quasi-contractive*.

It turns out, see [109, Theorem 9.4.2], that a semigroup $(S(t))_{t\ge0}$ is uniformly continuous if and only if there exists some $G \in \mathscr{L}(E)$ such that $S(t) = e^{tG}$ for all $t \ge 0$; we call G the *(infinitesimal) generator of* $(S(t))_{t\ge0}$.[3]

In the case where $(S(t))_{t\ge0}$ is merely strongly measurable and locally bounded, we can define an unbounded linear operator which plays an analogous role. Its definition requires us to introduce the *resolvent set* $\rho(G)$ consisting of all $\lambda \in \mathbb{C}$ for which $\lambda \operatorname{Id}_E - G$ has a bounded two-sided inverse, in which case $R(\lambda, G) := (\lambda \operatorname{Id}_E - G)^{-1}$ is called a *resolvent operator* of G.

Definition 1.1.3. Let $(S(t))_{t\ge0}$ be a locally bounded and strongly measurable semigroup on a complex Banach space E, satisfying (1.1.6) for some $w \in \mathbb{R}$. The linear operator $G\colon \mathsf{D}(G) \subseteq E \to E$ is said to be the *(infinitesimal) generator* of $(S(t))_{t\ge0}$ if $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -w\} \subseteq \rho(G)$ and, for all $x \in E$ and $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -w$, we have

$$R(\lambda, G)x = \int_0^\infty e^{-\lambda t} S(t)x \,\mathrm{d}t \quad \text{in } E. \tag{1.1.7}$$

[2]This stands for the "*Cesàro summability of order* 0 of $S(t)x$ to x" (for all $x \in E$). This concept, also denoted $(C,0)$, was historically defined alongside weaker notions of "summability of $S(t)x$ near 0", see [109, Section 10.6]: For instance, *Cesàro summability of order* 1, denoted C_1 or $(C,1)$, refers to semigroups for which $\frac{1}{t}\int_0^t S(s)x\,\mathrm{d}s \to x$ as $t \downarrow 0$, and $(S(t))_{t\ge0}$ belongs to the (even larger) class (A) of *Abel-summable* semigroups if $\lambda\int_0^\infty e^{-\lambda t}S(t)x\,\mathrm{d}t \to x$ as $\lambda \to \infty$.

[3]Although it is customary in the literature to denote the generator of a semigroup by $A\colon \mathsf{D}(A) \subseteq E \to E$, we reserve this notation for operators in equations such as (1.1.2), so that in applications we will work with semigroups whose generator is $G = -A$; see also the signs in (1.1.3)–(1.1.4). The reason is the author's preference to let A represent a "non-negative" differential operator in concrete applications in subsequent sections and chapters.

If G is the generator of a locally bounded and strongly measurable semigroup $(S(t))_{t\geq 0}$, with constants $M \in [1,\infty)$ and $w \in \mathbb{R}$ in (1.1.6) and Definition 1.1.3, then (1.1.7) directly implies $\lambda \in \rho(G)$ and

$$\|R(\lambda, G)\|_{\mathscr{L}(E)} \leq \frac{M}{\operatorname{Re}(\lambda) + w} \quad \text{for all } \lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda > -w. \tag{1.1.8}$$

If $(S(t))_{t\geq 0}$ satisfies (1.1.6) for another pair of constants $M' \in [1,\infty)$ and $w' > w$, then we also have $\lambda \in \rho(G)$ and (1.1.7) for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -w'$. Indeed, this follows from the uniqueness of analytic continuations: On the one hand, we know that $\lambda \mapsto R(\lambda, G)x$ is holomorphic on $\rho(G)$ and blows up at the boundary $\partial\rho(G)$ [156, Propositions 10.28 and 10.29]; on the other hand, $\lambda \mapsto \int_0^\infty e^{-\lambda t} S(t)x \, \mathrm{d}t$ turns out to be holomorphic on $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -w'\}$, is bounded on $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \varepsilon - w'\}$ for all $\varepsilon > 0$ (by (1.1.6) with (M', w')), and equal to $R(\lambda, G)x$ on $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -w\}$ (by (1.1.7)). Applying the above with $w' > 0$, we find in particular that the generator G of a uniformly exponentially stable semigroup $(S(t))_{t\geq 0}$ always satisfies $0 \in \rho(G)$.

Generators are closed[4] since, by definition, their resolvent sets are nonempty. If a strongly measurable and locally bounded semigroup $(S(t))_{t\geq 0}$ has a generator, then it is unique and characterizes the semigroup.

If the semigroup $(S(t))_{t\geq 0}$ is *strongly continuous*, then its generator is given by

$$\begin{aligned} \mathsf{D}(G) &= \left\{x \in E : \tfrac{1}{t}(S(t)x - x) \text{ converges in } E \text{ as } t \downarrow 0\right\}; \\ Gx &= \lim_{t\downarrow 0} \tfrac{1}{t}(S(t)x - x) \text{ in } E, \quad x \in \mathsf{D}(G). \end{aligned} \tag{1.1.9}$$

This follows from [73, Chapter II, Theorem 1.10(ii)], which shows that the operator defined by the right-hand sides of (1.1.9) satisfies the conditions of Definition 1.1.3. In other words, the generator of a C_0-semigroup is fully characterized by the derivatives of its orbits evaluated at zero. More generally, we have the following relation between the behavior of orbits of $(S(t))_{t\geq 0}$ at zero and the domain of its generator, see [115, Proposition K.1.5(3)]:

$$\overline{\mathsf{D}(G)} = \{x \in E : S(t)x \to x \text{ in } E \text{ as } t \downarrow 0\}.$$

We conclude that a strongly measurable and locally bounded semigroup is a C_0-semigroup if and only if it admits a densely defined generator G, i.e., $\overline{\mathsf{D}(G)} = E$.

Now we will state our first *semigroup generation theorem*, i.e., a result providing necessary and sufficient conditions for a given linear operator to be a semigroup generator. It is known as the *Hille–Yosida theorem* for C_0-semigroups.

Theorem 1.1.4 (Hille–Yosida [73, Chapter II, Theorem 3.8])**.** *Let $G\colon \mathsf{D}(G) \subseteq E \to E$ be a linear operator on a complex Banach space E, and let the constants $w \in \mathbb{R}$, $M \in [1,\infty)$ be given. The following are equivalent:*

(a) G is the generator of a C_0-semigroup $(S(t))_{t\geq 0}$ satisfying (1.1.6).

[4] An operator $A\colon \mathsf{D}(A) \subseteq E \to E$ is said to be closed if its *graph* $\mathsf{G}(A) := \{(x, Ax) : x \in \mathsf{D}(A)\}$ is closed with respect to the *graph norm* $\|x\|_{\mathsf{G}(A)} := \|x\|_E + \|Ax\|_E$.

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(b) *G is densely defined and we have*

$$\lambda \in \rho(G) \quad \textit{and} \quad \|R(\lambda,G)^k\|_{\mathscr{L}(E)} \le \frac{M}{(\lambda+w)^k} \tag{1.1.10}$$

for every $\lambda \in (-w,\infty)$ *and* $k \in \mathbb{N}$.

(c) *G is densely defined and we have*

$$\lambda \in \rho(G) \quad \textit{and} \quad \|R(\lambda,G)^k\|_{\mathscr{L}(E)} \le \frac{M}{(\mathrm{Re}(\lambda)+w)^k} \tag{1.1.11}$$

for every $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}\,\lambda > -w$ *and* $k \in \mathbb{N}$.

If (1.1.10) or (1.1.11) holds for $M = 1$ and $k = 1$, then the respective condition is in fact satisfied for all $k \in \mathbb{N}$ in view of the submultiplicativity of the $\mathscr{L}(E)$-norm. That is, in the contractive case, Theorem 1.1.4 reduces to the following.

Corollary 1.1.5 (Hille–Yosida—quasi-contraction case [73, Chapter II, Corollary 3.6])**.** *Let* $G\colon \mathsf{D}(G) \subseteq E \to E$ *be a linear operator on a complex Banach space* E*, and let the constant* $w \in \mathbb{R}$ *be given. The following are equivalent:*

(a) *G is the generator of a quasi-contractive* C_0*-semigroup* $(S(t))_{t\ge0}$*, i.e., it satisfies* (1.1.6) *with* $w \in \mathbb{R}$ *and* $M = 1$.

(b) *G is densely defined and we have*

$$\lambda \in \rho(G) \quad \textit{and} \quad \|R(\lambda,G)\|_{\mathscr{L}(E)} \le \frac{1}{\lambda+w} \quad \textit{for all } \lambda \in (-w,\infty).$$

(c) *G is densely defined and we have*

$$\lambda \in \rho(G) \quad \textit{and} \quad \|R(\lambda,G)\|_{\mathscr{L}(E)} \le \frac{1}{\mathrm{Re}(\lambda)+w} \quad \textit{for all } \lambda \in \mathbb{C} \textit{ with } \mathrm{Re}\,\lambda > -w.$$

Thus, we see that the necessary condition (1.1.8) is also sufficient for the generation of a quasi-contractive C_0-semigroup when combined with density of $\mathsf{D}(G)$.

1.1.2. SECTORIAL OPERATORS AND BOUNDED ANALYTIC SEMIGROUPS

Parts of Chapters 2, 3 and 5 are in particular concerned with fractional *parabolic* evolution equations. Such equations are characterized by the fact that the semigroup $(S(t))_{t\ge0}$ generated by $-A$ is *holomorphic*, i.e., complexly differentiable when extended to some open region in the complex plane. Since a function is holomorphic if and only if it is *analytic* (meaning that it equals its Taylor expansion at every point in the region), we use these terms interchangeably. The regions we will consider are the open sectors centered around the positive real line (with opening half-angle $\omega \in (0,\pi)$):

$$\Sigma_\omega := \big\{\lambda \in \mathbb{C}\setminus\{0\} : \arg\lambda \in (-\omega,\omega)\big\}.$$

The corresponding *closed* sector is denoted by

$$\overline{\Sigma}_\omega := \{0\} \cup \big\{\lambda \in \mathbb{C}\setminus\{0\} : \arg\lambda \in [-\omega,\omega]\big\} = \overline{\Sigma_\omega}.$$

Definition 1.1.6. An operator semigroup $(S(t))_{t\geq 0} \subseteq \mathscr{L}(E)$ is said to be

- *analytic on* Σ_η (with $\eta \in (0,\pi)$) if, for every $x \in E$, the orbit $t \mapsto S(t)x$ extends to a holomorphic function from Σ_η to E;
- *bounded analytic on* Σ_η if it is analytic on Σ_η and satisfies

$$\sup_{z\in\Sigma_\eta} \|S(z)\|_{\mathscr{L}(E)} < \infty;$$

- *contractive analytic on* Σ_η if it is bounded analytic on Σ_η with

$$\|S(z)\|_{\mathscr{L}(E)} \leq 1 \quad \text{for all } z \in \Sigma_\eta;$$

- an *analytic* C_0*-semigroup on* Σ_η if it is analytic on Σ_η and, for every $x \in E$,

$$S(z)x \to x \text{ in } E \quad \text{as} \quad \Sigma_\eta \ni z \to 0.$$

It is simply said to be a *(contractive/bounded) analytic* (C_0*-)semigroup* (without explicit reference to a sector) if there exists a sector on which it is a semigroup of the corresponding type.

Some authors impose holomorphy of $t \mapsto S(t)$ with respect to the uniform operator topology, i.e., as a mapping from Σ_η to $\mathscr{L}(E)$; see for instance [146, p. 34], [73, Chapter II, Definition 4.5] and [165, Chapter 2, Definition 5.1]. In view of [113, Corollary B.3.3], this is equivalent to the "strong (or *orbital*) holomorphy" which we require in our definition.

There is a one-to-one relation between bounded analytic semigroups and the class of *sectorial* linear operators, which are defined as follows.

Definition 1.1.7. Let $A\colon \mathsf{D}(A) \subseteq E \to E$ be a linear operator and $\omega \in (0,\pi)$. We say that A is ω*-sectorial* with ω*-sectoriality constant* $M(\omega, A)$, if

$$\sigma(A) \subseteq \overline{\Sigma}_\omega \quad \text{and} \quad M(\omega,A) := \sup\{\|\lambda R(\lambda,A)\|_{\mathscr{L}(E)} : \lambda \in \mathbb{C}\setminus\overline{\Sigma}_\omega\} < \infty. \tag{1.1.12}$$

We denote this by $A \in \operatorname{Sect}(\omega)$. The *angle of sectoriality* of A is defined as

$$\omega(A) := \inf\{\omega \in (0,\pi) : A \in \operatorname{Sect}(\omega)\}. \tag{1.1.13}$$

It is clear from the definition that $\operatorname{Sect}(\omega_1) \subseteq \operatorname{Sect}(\omega_2)$ whenever $\omega_1 \leq \omega_2$.

If $-A$ is the generator of a uniformly bounded and strongly measurable semigroup, then A is sectorial of angle $\omega(A) \leq \frac{1}{2}\pi$, cf. [114, Example 10.1.2]. On the other hand, if $A \in \operatorname{Sect}(\omega)$ for some $\omega < \frac{1}{2}\pi$, which occurs precisely when $\omega(A) \in [0, \frac{1}{2}\pi)$, then for any $\omega' \in (\omega, \frac{1}{2}\pi)$ and $z \in \Sigma_{\frac{1}{2}\pi-\omega'}$ we can define the (suggestively named) operator $S(z) \in \mathscr{L}(E)$ via the Cauchy type integral

$$S(z)x := \frac{1}{2\pi i}\int_{\Gamma_{\varepsilon,\omega'}} e^{-z\lambda} R(\lambda,A)x \,\mathrm{d}\lambda, \quad x \in E. \tag{1.1.14}$$

The contour of integration is given by $\Gamma_{\varepsilon,\omega'} := \partial(B_\varepsilon(0) \cup \Sigma_{\omega'})$, where $B_\varepsilon(0)$ denotes the open ball with radius $\varepsilon > 0$ around the origin in the complex plane. Its shape

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is chosen such that it encloses the spectrum $\sigma(A)$, and to ensure convergence of the (Bochner) integrals. The contour is oriented downwards, so that $\sigma(A)$ is always located to the left of an observer traversing the path. That is, we can decompose $\Gamma_{\varepsilon,\omega'} = \Gamma^{r,+}_{\varepsilon,\omega'} \cup \Gamma^{a}_{\varepsilon,\omega'} \cup \Gamma^{r,-}_{\varepsilon,\omega'}$ into the rays

$$\Gamma^{r,+}_{\varepsilon,\omega'} := \{re^{i\omega'} : r \text{ from } \infty \text{ to } \varepsilon\} \quad \text{and} \quad \Gamma^{r,-}_{\varepsilon,\omega'} := \{re^{-i\omega'} : r \text{ from } \varepsilon \text{ to } \infty\},$$

and the circular arc

$$\Gamma^{a}_{\varepsilon,\omega'} := \{\varepsilon e^{i\vartheta} : \vartheta \text{ from } \omega' \text{ to } -\omega' \text{ counterclockwise}\}.$$

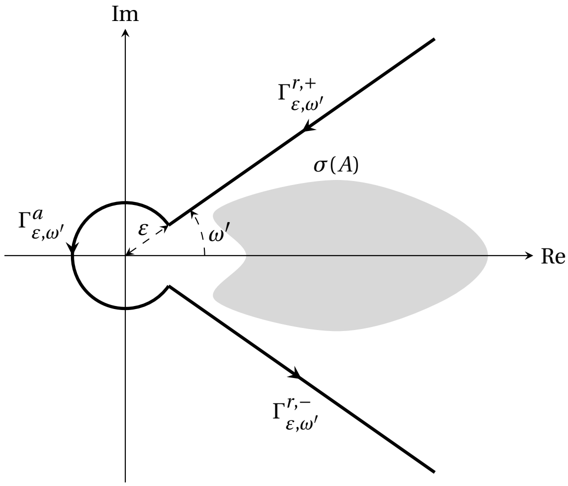

Figure 1.1: Illustration of the rays $\Gamma^{r,\pm}_{\varepsilon,\omega'}$ and circular arc $\Gamma^{a}_{\varepsilon,\omega'}$ which together comprise the integration contour $\Gamma_{\varepsilon,\omega'}$ used in (1.1.14); the arrows signify its orientation. The spectrum $\sigma(A)$, radius ε and opening half-angle ω' are also indicated.

In the situation described above, depicted in Figure 1.1, $\lambda \mapsto e^{-z\lambda}R(\lambda, A)x$ is integrable on $\Gamma^{r,\pm}_{\varepsilon,\omega'}$ owing to (1.1.12) and the choices of ω', z and ε, whereas its integrability on the (bounded) arc $\Gamma^{a}_{\varepsilon,\omega'}$ is due to continuity. In fact, the mapping $\lambda \mapsto R(\lambda, A)x$ is holomorphic on all of $\rho(A)$ by [156, Proposition 10.28]. In other words, the "keyhole" shape of $\Gamma_{\varepsilon,\omega'}$ guarantees integrability by avoiding the origin, which is a singularity of the function $\lambda \mapsto R(\lambda, A)x$ if and only if $0 \in \sigma(A)$. Moreover, it follows from Cauchy's integral theorem that the value of the integral in (1.1.14) does not depend on the particular choices of $\omega' \in (\omega, \frac{1}{2}\pi)$ and $\varepsilon > 0$.

As suggested by the notation, the family $(S(t))_{t\ge 0}$ defined by (1.1.14) (along with $S(0) := \mathrm{Id}_E$) is a bounded and strongly measurable semigroup; its generator is $-A$, see [115, Proposition K.1.11]. In fact, it extends to a bounded analytic semigroup on $\Sigma_{\frac{1}{2}\pi-\omega'}$. Conversely, if $(S(z))_{z\in\Sigma_\eta}$ is a bounded analytic semigroup on some sector Σ_η

with $\eta \in (0, \frac{1}{2}\pi)$ and generator $G = -A$, then one can apply (1.1.8) to the bounded semigroups $(S(te^{i\eta'}))_{t\geq 0}$ for all $\eta' \in (0,\eta)$ in order to obtain (1.1.12) with $\omega = \frac{1}{2}\pi - \eta$.

The equivalence between sectorial operators and bounded analytic semigroups is summarized by the following theorem. Note that the cited source states the result for C_0-semigroups, hence for densely defined generators, but the following more general formulation follows by essentially the same proof:

Theorem 1.1.8 (Cf. [114, Theorem G.5.2])**.** *Let $A\colon \mathsf{D}(A) \subseteq E \to E$ be a linear operator on a complex Banach space E. The following statements are equivalent:*

(a) There exists $\eta \in (0, \frac{1}{2}\pi)$ such that $-A$ is the generator of a bounded analytic semigroup on the sector Σ_η.

(b) There exists $\omega \in (0, \frac{1}{2}\pi)$ such that $A \in \mathrm{Sect}(\omega)$.

Denoting the supremum of all admissible $\eta \in (0, \frac{1}{2}\pi)$ in (a) by $\omega_{\mathrm{res}}(A)$, and recalling the definition of $\omega(A)$ from (1.1.13), we have

$$\omega_{\mathrm{res}}(A) = \tfrac{1}{2}\pi - \omega(A).$$

In either of the equivalent situations (a) or (b), the semigroup $(S(t))_{t\geq 0}$ is given by the formula (1.1.14).

Now we collect some important basic properties of bounded analytic semigroups which are used throughout the sequel of this thesis. Among these is the parabolic smoothing property which essentially distinguishes the class of analytic semigroups $(S(t))_{t\geq 0}$ from more general ones: After any strictly positive instant $t > 0$, the operator $S(t)$ maps E into $\mathsf{D}(G^k)$ for every $k \in \mathbb{N}$, where $G\colon \mathsf{D}(G) \subseteq E \to E$ denotes the generator, and we have a useful estimate for the norm of $G^k S(t)$.

Proposition 1.1.9 ([146, Proposition 2.1.1])**.** *Suppose that $(S(t))_{t\geq 0}$ is a bounded analytic semigroup with generator $G\colon \mathsf{D}(G) \subseteq E \to E$. It has the following properties:*

(a) We have $S(t)x \in \mathsf{D}(G^k)$ for all $t \in (0,\infty)$, $x \in E$ and $k \in \mathbb{N}$. Moreover, $S(t)$ commutes with G^k for every $k \in \mathbb{N}$, meaning that

$$G^k S(t)x = S(t)G^k x \quad \text{if} \quad x \in \mathsf{D}(G^k).$$

(b) For every $k \in \mathbb{N}_0$, there exists a constant $M_k \in [1,\infty)$ such that

$$\|G^k S(t)\|_{\mathscr{L}(E)} \leq M_k t^{-k} \quad \text{for all } t \in (0,\infty).$$

(c) The function $t \mapsto S(t)$ belongs to $C^\infty((0,\infty);\mathscr{L}(E))$; i.e., it has infinitely many classical derivatives. Its kth derivative is given by

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} S(t) = G^k S(t) \quad \text{for all } t \in (0,\infty) \text{ and } k \in \mathbb{N}.$$

The statements in Proposition 1.1.9 can in fact be extended from integers k to arbitrary non-negative real numbers $\alpha \in [0,\infty)$. This necessitates a definition of the fractional power G^α of the operator G, which is the subject of Section 1.3.2 below.

We have already noted after equation (1.1.8) that the generator G of a semigroup $(S(t))_{t\geq 0}$ satisfies $0 \in \rho(G)$ if $(S(t))_{t\geq 0}$ is uniformly exponentially stable. If $(S(t))_{t\geq 0}$ is bounded analytic, then by [115, Proposition K.2.3] the converse is also true.

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1.1.3. THE ABSTRACT CAUCHY PROBLEM

We now return to our goal of defining solution concepts for the (inhomogeneous) abstract Cauchy problem (1.1.2) associated to the operator A and a time interval $J \subseteq \mathbb{R}$. Let a linear operator $A\colon \mathsf{D}(A) \subseteq E \to E$ on a complex Banach space E be given. In this section, we restrict ourselves to the following choices of J:

- a bounded interval $J = (0, T)$, where $T \in (0, \infty)$ is some fixed time horizon;
- the positive half line $J = (0, \infty)$; or
- the entire real line $J = \mathbb{R}$.

The analogous results for arbitrary open intervals J are readily derived from these cases. If $J = (0, T)$ or $J = (0, \infty)$, then we consider the abstract *initial value* problem

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in J; \\ \qquad u(0) = u_0. \end{cases} \tag{1.1.15}$$

Here, $u_0 \in E$ is a given initial value, and we suppose that $f \in L^1_{\mathrm{loc}}(\overline{J}; E)$, meaning that $f\colon \overline{J} \to E$ is a strongly measurable function[5] which is integrable on bounded subsets of J. In particular, we have $L^1_{\mathrm{loc}}([0, T]; E) = L^1(0, T; E)$ for $J = (0, T)$ (where the latter indicates a Bochner space). If $J = \mathbb{R}$, then we do not impose an initial value, and the problem reduces to

$$u'(t) + Au(t) = f(t), \quad t \in \mathbb{R}. \tag{1.1.16}$$

The first goal is to define what it means for a function $u\colon \overline{J} \to E$ to be a solution to (1.1.15) or (1.1.16). This can be done in several ways. We shall collect some definitions which are commonly encountered in the literature, in order to provide some perspectives. However, since in Chapters 2–5 we are mainly concerned with L^p-solutions and mild solutions, we will only briefly touch upon the remaining concepts. More details, e.g. regarding the well-posedness and regularity of such solutions in different settings, can for instance be found in [165, Chapter 4] (for C_0-semigroups) or [146, Chapter 4] (for analytic semigroups).

Inspecting the equations (1.1.15)–(1.1.16), the most obvious (but also restrictive) solution concept is the following.

Definition 1.1.10 (Strict solution)**.** Let $f \in C(\overline{J}; E)$. A function

$$u \in C^1(\overline{J}; E) \cap C(\overline{J}; \mathsf{D}(A)) \tag{1.1.17}$$

is said to be a *strict solution* to the abstract Cauchy problem (1.1.15) or (1.1.16) if

$$u'(t) + Au(t) = f(t) \quad \text{for all } t \in \overline{J}, \tag{1.1.18}$$

(where u' denotes the classical derivative of u), and $u(0) = u_0$ if $\inf J = 0$.

We remark that equations (1.1.17) and (1.1.18) together force $f \in C(\overline{J}; E)$. Moreover, since a strict solution u to the initial value problem (1.1.15) belongs to $C(\overline{J}; E)$

[5] Meaning that it can be approximated a.e. by simple functions, see [113, Definition 1.1.14].

by definition, we have in particular $u_0 = u(0) \in \mathsf{D}(A)$. The difference quotients defining $u'(0)$ also belong to $\mathsf{D}(A)$, and thus $f(0) - Au_0 = u'(0) \in \overline{\mathsf{D}(A)}$.

For $J = (0, T)$, a result towards the converse direction is given by [54, Theorem 8.1]: The authors prove that if $f \in W^{1,p}(0, T; E)$ for some $p \in [1, \infty)$,[6] then the "compatibility conditions" $u_0 \in \mathsf{D}(A)$ and $f(0) - Au_0 \in \overline{\mathsf{D}(A)}$, along with the resolvent estimates (1.1.10) from the Hille–Yosida theorem, are sufficient for the existence of a strict solution to (1.1.15).

A slightly weaker solution concept is that of a classical solution; note the use of J instead of $\overline{J}$, whose main purpose is to respectively exclude or include the value at zero (if applicable). Since $\mathbb{R} = \overline{\mathbb{R}}$, it coincides with the strict solution concept for (1.1.16).

Definition 1.1.11 (Classical solution)**.** Let $f \in C(J; E)$. A function

$$u \in C(\overline{J}; E) \cap C^1(J; E) \cap C(J; \mathsf{D}(A))$$

is said to be a *classical solution* to the abstract Cauchy problem (1.1.15) if it satisfies $u(0) = u_0$ and

$$u'(t) + Au(t) = f(t) \quad \text{for all } t \in J.$$

Since a classical solution u to (1.1.15) satisfies $u(t) \in \mathsf{D}(A)$ for all $t \in J$, as well as $u \in C(\overline{J}; E)$, we find $u_0 \in \overline{\mathsf{D}(A)}$ instead of the more restrictive $u_0 \in \mathsf{D}(A)$.

The following solution concept does not require continuity of f. Throughout this section (and thesis), we use the notational convention $\int_s^t := -\int_t^s$ if $s \geq t$.

Definition 1.1.12 (Strong solution)**.** Let $f \in L^1_{\mathrm{loc}}(\overline{J}; E)$. A strongly measurable function $u\colon \overline{J} \to E$ is said to be a *strong solution* to (1.1.15) or (1.1.16) if it satisfies the following conditions:

(i) We have $u(t) \in \mathsf{D}(A)$ for almost every $t \in J$ and $Au \in L^1_{\mathrm{loc}}(J; E)$.

(ii) The function $u\colon \overline{J} \to E$ solves the integrated version of the corresponding abstract Cauchy problem (1.1.15) or (1.1.16): For almost every $t \in J$, we have

$$u(t) + \int_0^t Au(s)\,\mathrm{d}s = u_0 + \int_0^t f(s)\,\mathrm{d}s \tag{1.1.19}$$

for the given initial value $u_0 \in \overline{\mathsf{D}(A)}$ or, respectively,

$$u(t) + \int_0^t Au(s)\,\mathrm{d}s = u(0) + \int_0^t f(s)\,\mathrm{d}s. \tag{1.1.20}$$

Note that (1.1.20) implies

$$u(t) + \int_s^t Au(r)\,\mathrm{d}r = u(s) + \int_s^t f(r)\,\mathrm{d}r \quad \text{for almost every } s, t \in J$$

by writing $u(t) - u(s) = u(t) - u(0) - (u(s) - u(0))$. Thus, our definition agrees with [115, Definition 17.3.22].

[6] This denotes the Bochner–Sobolev space of functions in $L^p(0, T; E)$ with weak derivatives in $L^p(0, T; E)$.

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As antiderivatives of locally integrable functions are continuous, see [113, Proposition 2.5.9], it follows that any strong solution admits a continuous representative, so that the pointwise evaluation of u in equations (1.1.19) and (1.1.20) is meaningful. In fact, identifying u with this representative, it turns out to be (classically) differentiable for almost all $t \in J$, for which we have $u'(t) + Au(t) = f(t)$, see [115, Equation (17.3)]. This implies $u' = f - Au \in L^1_{\mathrm{loc}}(\overline{J}; E)$, hence u is weakly differentiable and its weak derivative $\partial_t u$ coincides a.e. with u', again by [113, Proposition 2.5.9]. In particular, it follows that our definition of a strong solution (to (1.1.15)) also coincides with [165, Chapter 4, Definition 2.8]. Moreover, combined with the fact that strong solutions take their values in $\mathsf{D}(A)$ almost everywhere, we again derive the requirement $u_0 \in \overline{\mathsf{D}(A)}$.

The next notion of a solution is slightly more restrictive than that of a strong solution. It is central to Section 1.1.4 and parts of Chapter 4.

Definition 1.1.13 (L^p-solution)**.** Let $p \in [1, \infty]$. A strong solution $u \colon \overline{J} \to E$ with corresponding inhomogeneity $f \colon \overline{J} \to E$ is said to be an *L^p-solution* if both f and Au belong to $L^p(J; E)$.

Since $u' = f - Au$ almost everywhere, it immediately follows that $u' \in L^p(J; E)$ as well. Moreover, if $J = (0, T)$, then we can combine (1.1.19) with Hölder's inequality (twice) to obtain $u \in L^p(0, T; E)$ with

$$\|u\|_{L^p(0,T;E)} \le T^{\frac{1}{p}} \|u_0\|_E + T\big(\|Au\|_{L^p(0,T;E)} + \|f\|_{L^p(0,T;E)}\big).$$

However, if J is unbounded, then u itself need not be an $L^p(J; E)$-function. In the case $J = (0, \infty)$, a necessary and sufficient condition for $u \in L^p(0, \infty; E)$ (see, respectively, [115, Corollary 17.2.25 and Proposition 17.2.8]) is that $0 \in \rho(A)$, i.e., A is boundedly invertible.

For the definitions stated up to this point, we have not required that the operator A is associated to a semigroup. In order to define the notion of a *mild solution*, which is the main focus of the rest of this thesis, we do need to assume this.

Definition 1.1.14 (Mild solution)**.** Let $f \in L^1_{\mathrm{loc}}(\overline{J}; E)$. Suppose $-A \colon \mathsf{D}(A) \subseteq E \to E$ is the generator of a strongly measurable and locally bounded semigroup $(S(t))_{t \ge 0}$ on a complex Banach space E.

- Let $u_0 \in \overline{\mathsf{D}(A)}$. The *mild solution to* (1.1.15) is the function $u \colon \overline{J} \to E$ given by the variation of constants formula

$$u(t) := S(t)u_0 + \int_0^t S(t-s) f(s) \, \mathrm{d}s, \quad t \in \overline{J}.$$

- The *mild solution to* (1.1.16) is the function $u \colon \mathbb{R} \to E$ given by

$$u(t) := \int_{-\infty}^t S(t-s) f(s) \, \mathrm{d}s, \quad t \in \mathbb{R}.$$

The mild solutions defined above are bounded and continuous by [115, Proposition K.1.5(3)] and Proposition 4.2.3(b) in Chapter 4.

The relation between strong and mild solutions is given by Proposition 4.2.2 below. In particular, it implies that if $-A$ generates a strongly measurable and locally bounded semigroup $(S(t))_{t\geq 0}$, then strong solutions to (1.1.15) or (1.1.16) are mild. Since mild solutions are defined by an explicit formula, they are trivially unique. Moreover, it can be shown that all the other solution concepts introduced in this section are also mild solutions. Hence, under the additional assumption that $-A$ generates $(S(t))_{t\geq 0}$, we find uniqueness for these types of solutions as well.

1.1.4. THE SUM OPERATOR $\partial_t + A$ AND MAXIMAL L^p-REGULARITY

Since Chapters 2–4 are concerned with evolution equations governed by the space–time fractional (parabolic) operator $(\partial_t + A)^\gamma$ for $\gamma \in (0,\infty)$, we first study the base operator $\partial_t + A$ (i.e., $\gamma = 1$) in some detail here.

Consider the initial value problem (1.1.15) with $J = (0,T)$ and $u_0 = 0$. As observed in the previous subsection, we may assume that for almost every $t \in (0,T)$, a strong solution u in the sense of Definition 1.1.12 is (classically) differentiable and satisfies $u'(t) + Au(t) = f(t)$. It is also weakly differentiable in the Bochner–Sobolev sense, and its weak derivative $\partial_t u$ coincides almost everywhere with u'. Now let us interpret the weak derivative as a linear operator

$$\partial_t \colon W^{1,p}_{0,\{0\}}(0,T;E) \subseteq L^p(0,T;E) \to L^p(0,T;E)$$

(for $p \in [1,\infty]$) with domain

$$W^{1,p}_{0,\{0\}}(0,T;E) := \{u \in W^{1,p}(0,T;E) : u(0) = 0\},$$

and define $\mathcal{A}_{(0,T)} \colon L^p(0,T;\mathsf{D}(A)) \subseteq L^p(0,T;E) \to L^p(0,T;E)$ by

$$[\mathcal{A}_{(0,T)} u](t) := Au(t) \quad \text{for all } u \in L^p(0,T;\mathsf{D}(A)) \text{ and almost every } t \in (0,T).$$

We call $\mathcal{A}_{(0,T)}$ the Bochner space counterpart on $L^p(0,T;E)$ of the operator A. If the meaning is clear from the context, we may simply write $\mathcal{A}$ or A instead of $\mathcal{A}_{(0,T)}$.

If u is an L^p-solution to (1.1.15) with $u_0 = 0$ and $f \in L^p(0,T;E)$, then f belongs to the image of the sum operator

$$\partial_t + \mathcal{A}_{(0,T)} \colon W^{1,p}_{0,\{0\}}(0,T;E) \cap L^p(0,T;\mathsf{D}(A)) \subseteq L^p(0,T;E) \to L^p(0,T;E).$$

In other words, we can reformulate (1.1.15) with $u_0 = 0$ as the operator equation

$$\big(\partial_t + \mathcal{A}_{(0,T)}\big) u = f.$$

We can similarly consider the problems

$$\big(\partial_t + \mathcal{A}_J\big) u = f$$

for $J = (0,\infty)$ or $J = \mathbb{R}$, by defining the operators ∂_t and $\mathcal{A}_J$ on $L^p(J;E)$ analogously. In these cases, we will restrict ourselves to working with operators A whose negatives generate uniformly exponentially stable semigroups, in order to ensure that L^p-solutions belong to $L^p(J;E)$. Note that for $J = \mathbb{R}$, the domain of ∂_t is simply $W^{1,p}(\mathbb{R};E)$ since there is no initial condition. When viewed a linear operator defined according to one of the above definitions, $-\partial_t$ is our first concrete example of a semigroup generator, cf. [115, Propositions 17.3.16 and 17.3.17]:

Example 1.1.15. Let $J = (0,T)$ or $J = (0,\infty)$. For all $p \in [1,\infty)$, the negative $-\partial_t$ of the Bochner–Sobolev weak derivative, equipped with a homogeneous initial condition, generates the uniformly bounded C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ of right-shift operators given by

$$[\mathcal{T}(t)f](s) := \widetilde{f}(s-t) \quad \text{for all } t \geq 0,\ f \in L^p(J;E) \text{ and almost every } s \in J.$$

Here, $\widetilde{f}\colon \mathbb{R} \to E$ denotes the extension by zero to $\mathbb{R}$ of a function $f\colon J \to E$. See Proposition 2.A.5 below for a proof in the case $p = 2$, $J = (0,T)$ and $E = H$.

For $J = \mathbb{R}$, the operator $-\partial_t$ even generates a C_0-*group* $(\mathcal{T}(t))_{t\in\mathbb{R}}$ (note the index set $\mathbb{R}$ instead of $[0,\infty)$), given by

$$[\mathcal{T}(t)f](s) := f(s-t) \quad \text{for all } t \in \mathbb{R},\ f \in L^p(\mathbb{R};E) \text{ and almost every } s \in \mathbb{R},$$

but this fact will not be used in this dissertation.

For later use (see Section 1.3.3 and Example 4.4.6), let us record another important elementary example of a semigroup associated to a differential operator:

Example 1.1.16. Let $p \in [1,\infty)$ and $d \in \mathbb{N}$. The *heat semigroup on* $\mathbb{R}^d$ is the contractive analytic C_0-semigroup $(H(t))_{t\geq 0} \subseteq L^p(\mathbb{R}^d)$ given, for each $t \in (0,\infty)$ and $f \in L^p(\mathbb{R}^d)$, by the convolution $H(t)f := K_t * f$ with the *Gauss–Weierstrass kernel*

$$K_t\colon \mathbb{R}^d \to \mathbb{R} \quad \text{defined by} \quad K_t(x) := (4\pi t)^{-d/2} \exp\Big(-\tfrac{1}{4t}\|x\|^2_{\mathbb{R}^d}\Big), \quad x \in \mathbb{R}^d.$$

Its generator is the Laplace operator on $\mathbb{R}^d$ [156, Section 13.6.c], which is formally given by $\Delta = \sum_{j=1}^d \partial^2_{x_j}$. More precisely, it is the weak L^p-Laplacian (see [156, Section 11.1.e]), an unbounded linear operator on $L^p(\mathbb{R}^d)$ whose domain turns out to coincide with the second-order Sobolev space $W^{2,p}(\mathbb{R}^d)$ for $p \in (1,\infty)$. In Section 1.2.3 we will consider another precise definition of the Laplacian for $p = 2$.

The remainder of this subsection is devoted to analyzing some of the basic properties of $\partial_t + \mathcal{A}_J$. We begin by showing that $\partial_t + \mathcal{A}_J$ is closable if $-A$ is the generator of a locally bounded and strongly measurable semigroup:

Proposition 1.1.17. *Let $p \in [1,\infty]$. Suppose that $-A\colon \mathsf{D}(A) \subseteq E \to E$ generates a locally bounded and strongly measurable semigroup $(S(t))_{t\geq 0}$ on the complex Banach space E, and let the interval $J \subseteq \mathbb{R}$ be as in this subsection. If $J = (0,T)$ or $(S(t))_{t\geq 0}$ is uniformly exponentially stable, then the sum operator $\partial_t + \mathcal{A}_J$ on $L^p(J;E)$ is closable.*

Proof. Suppose that $(u_n)_{n\in\mathbb{N}} \subseteq \mathsf{D}(\partial_t + \mathcal{A}_J)$ and $g \in L^p(J;E)$ are such that $u_n \to 0$ and $(\partial_t + \mathcal{A}_J)u_n \to g$ in $L^p(J;E)$ as $n \to \infty$. Thus, for each $n \in \mathbb{N}$, u_n is trivially an L^p-solution to (1.1.15) with $u_0 = 0$ or (1.1.16) with right-hand side $f_n := (\partial_t + \mathcal{A}_J)u_n$, and thus a strong solution. By Proposition 4.2.2, it follows that u_n is the mild solution, i.e., $u_n = S * f_n$. Since $f \mapsto S * f$ is bounded on $L^p(J;E)$ under the current assumptions (see Proposition 4.2.3(a)), we can apply this operator on both sides of $f_n \to g$ to find $u_n \to S * g$ in $L^p(J;E)$ as $n \to \infty$, and thus $S * g = 0$. Since $v := S * g$ is the mild solution to (1.1.15) with $u_0 = 0$ or (1.1.16) with right-hand side g, and $v = 0 \in L^p(J;\mathsf{D}(A))$, we can once more apply Proposition 4.2.2 to find that v is the strong solution. Thus, applying $\partial_t + \mathcal{A}_J$ to both sides of $S * g = 0$ yields $g = 0$. □

As a consequence of Example 1.1.15 and [114, Example 10.1.2], the operator ∂_t is sectorial of angle at most $\frac{1}{2}\pi$. In what follows, we will also assume that A is sectorial, which implies sectoriality of $\mathcal{A}_J$ (with the same angle). Moreover, ∂_t and $\mathcal{A}_J$ clearly commute. Thus, if ∂_t and A satisfy the *parabolicity condition* $\omega(\partial_t)+\omega(A)<\pi$, then we can apply the following result to obtain that $\partial_t+\mathcal{A}_J$ in fact has a *sectorial extension* (which may coincide with the closure, see item (b)):

Theorem 1.1.18 (Sums of sectorial operators [115, Theorem 16.3.2])**.** *Suppose that $A\colon \mathsf{D}(A)\subseteq E\to E$ and $B\colon \mathsf{D}(B)\subseteq E\to E$ are sectorial linear operators on a complex Banach space E which have commuting resolvents and satisfy $\omega(A)+\omega(B)<\pi$.*

Then there exists a sectorial operator $C\colon \mathsf{D}(C)\subseteq E\to E$ with $\omega(C)\le\max\{\omega(A),\omega(B)\}$ which extends the sum operator $A+B\colon \mathsf{D}(A)\cap\mathsf{D}(B)\subseteq E\to E$. Moreover:

(a) If A or B is injective, then C is injective.

(b) The operator C is densely defined if and only if both A and B are densely defined. In this case, C is the closure of $A+B$.

The proof of Proposition 1.1.17 relied on the fact that $v:=S*g=0$ trivially belonged to $L^p(J;\mathsf{D}(A))$. Alternatively, we could have used that v is (again trivially) weakly differentiable with p-integrable derivative. Had we instead supposed that $u_n\to u$ for an arbitrary $u\in L^p(J;E)$, then $v:=S*g=u$ would not necessarily satisfy either of these properties. Thus, we cannot easily extend the proof of Proposition 1.1.17 to show that the sum operator itself is closed. Under certain additional assumptions (which are typically satisfied in our applications), see Proposition 1.1.20 below, the closedness of the sum operator $\partial_t+\mathcal{A}_J$ on the space $L^p(J;E)$ turns out to be equivalent to the operator A having *maximal L^p-regularity*, defined below.

Although the question whether a given operator A has maximal L^p-regularity has many far-reaching consequences for the study of (nonlinear) evolution equations involving them, we merely focus on its relation to the sum operator. More detailed expositions of the subject can for instance be found in [115, Chapter 17], [63], [130] and [100, Section 9.3].

Definition 1.1.19 (Maximal L^p-regularity)**.** Let $A\colon \mathsf{D}(A)\subseteq E\to E$ be a linear operator on a complex Banach space E and let $p\in[1,\infty]$.

- If $J=(0,T)$ or $J=(0,\infty)$, then A is said to have *maximal L^p-regularity on J* if there exists a constant $C\in[0,\infty)$ such that, for all $f\in L^p(J;E)$, the abstract Cauchy problem (1.1.15) with $u_0=0$ admits a unique L^p-solution $u\colon J\to E$ which satisfies the estimate

$$\|Au\|_{L^p(J;E)}\le C\|f\|_{L^p(J;E)}. \tag{1.1.21}$$

- If $J=\mathbb{R}$, then A is said to have *maximal L^p-regularity on $\mathbb{R}$* if there exists a constant $C\in[0,\infty)$ such that, for all $f\in L^p(\mathbb{R};E)$, the abstract Cauchy problem (1.1.16) admits a unique L^p-solution $u\colon\mathbb{R}\to E$ which satisfies the estimate

$$\|u\|_{L^p(\mathbb{R};E)}+\|Au\|_{L^p(\mathbb{R};E)}\le C\|f\|_{L^p(\mathbb{R};E)}.$$

1

The name "maximal L^p-regularity" refers to the fact that, given a function u satisfying $u' + Au = f$ a.e., one cannot expect u' and Au to belong to a "better" function space than $f \in L^p(J;E)$. Operators with maximal L^p-regularity (on any J) are necessarily closed, see [115, Proposition 17.2.5] (whose proof readily extends to the $J = \mathbb{R}$ case). Thus, we could also add the closedness of A to the definition of maximal L^p-regularity. Under this additional assumption, the existence of a unique L^p-solution for every $f \in L^p(J;E)$ would automatically imply the estimate (1.1.21); this follows from an argument involving the closed graph theorem, see [115, Proposition 17.2.11].

If A has maximal L^p-regularity with $p \in [1,\infty]$ on $J = (0,T)$ or $J = (0,\infty)$, then it also holds for all sub-intervals $(0,T') \subseteq J$. Such a "restriction property" does not hold for maximal L^p-regularity on $\mathbb{R}$.

A result known as Dore's theorem [115, Theorem 17.2.15] states that A having maximal L^p-regularity on $(0,T)$ for $p \in [1,\infty]$ implies that $-A$ generates an analytic semigroup $(S(t))_{t\ge0}$ on E. If A has maximal L^p-regularity on $(0,\infty)$, then $(S(t))_{t\ge0}$ is bounded analytic. In contrast, maximal L^p-regularity on $\mathbb{R}$ does not imply that $-A$ generates an analytic semigroup.

Under the additional assumption that the semigroup generated by $-A$ is strongly continuous (i.e., that A is densely defined), we have the following characterization of maximal regularity in terms of sum operators. The cases (i) and (ii) are proved in [115, Proposition 17.3.14], whereas (iii) can be established analogously.

Proposition 1.1.20 (Maximal L^p-regularity—sum-of-operators characterization)**.** *Let $-A\colon \mathsf{D}(A) \subseteq E \to E$ be the generator of a C_0-semigroup $(S(t))_{t\ge0}$ on the complex Banach space E. Let $p \in [1,\infty]$ and assume that either*

(i) $J = (0,T)$ for some $T \in (0,\infty)$;

(ii) $J = (0,\infty)$ and $(S(t))_{t\ge0}$ is uniformly exponentially stable; or

(iii) $J = \mathbb{R}$ and $(S(t))_{t\ge0}$ is uniformly exponentially stable and analytic.

Then the following statements are equivalent:

(a) The operator A on E has maximal L^p-regularity on J.

(b) There exists a constant $C \in [0,\infty)$ such that the inverse triangle inequality

$$\|u'\|_{L^p(J;E)} + \|Au\|_{L^p(J;E)} \le C\|(\partial_t + \mathcal{A}_J)u\|_{L^p(J;E)} \quad \text{holds for all } u \in \mathsf{D}(\partial_t + \mathcal{A}_J).$$

(c) The operator $\partial_t + \mathcal{A}_J$ on $L^p(J;E)$ is boundedly invertible.

(d) The operator $\partial_t + \mathcal{A}_J$ on $L^p(J;E)$ is closed.

1.2. ELLIPTIC OPERATORS VIA SESQUILINEAR FORMS

In Section 1.1, we introduced the concepts of operator semigroups and their use for abstract evolution equations, the latter serving as abstracted models of concrete partial differential equations. The aim of this section is to go beyond the elementary Examples 1.1.15 and 1.1.16 of differential operators whose negatives generate semigroups: We will rigorously define a more general class of "non-negative" differential operators on a spatial domain, which can be used to concretely define suitable operators A for use in (fractional) evolution equations in subsequent chapters, see Sections 2.5 and 5.3.

To this end, in Subsection 1.2.1 we introduce the notion of a *sesquilinear form* $\mathfrak{a}\colon \mathsf{D}(\mathfrak{a}) \times \mathsf{D}(\mathfrak{a}) \to \mathbb{C}$, i.e., a mapping on some subspace $\mathsf{D}(\mathfrak{a})$ of a complex Hilbert space H which is linear in the first component and conjugate-linear in the second. We will provide a concise overview of the theory of forms, focusing on identifying further conditions on $\mathfrak{a}$ which ensure that the relation

$$(u, Au) \in \mathsf{G}(A) \quad \text{if and only if} \quad \langle Au, v\rangle_H = \mathfrak{a}(u,v) \text{ for all } v \in \mathsf{D}(\mathfrak{a}) \tag{1.2.1}$$

determines a well-defined linear operator $A\colon \mathsf{D}(A) \subseteq H \to H$ whose negative generates an analytic semigroup $(S(t))_{t\ge 0} \subseteq \mathscr{L}(H)$. In Section 1.2.2 we specialize to the case $H := L^2(\mathcal{D})$ and investigate when the semigroup $(S(t))_{t\ge0}$ is contractive or extends to $L^p(\mathcal{D})$. In Section 1.2.3 we consider the form $\mathfrak{a}_V\colon V \times V \to \mathbb{C}$, defined on a domain $\mathsf{D}(a_V) := V \subseteq H^1(\mathcal{D})$ in $H := L^2(\mathcal{D})$ (i.e., square-integrable functions with square-integrable weak derivatives) via the prescription

$$\begin{aligned}\mathfrak{a}_V(u,v) := &\int_{\mathcal{D}} \big(a(x)\nabla u(x) + c(x)u(x)\big)\cdot \overline{\nabla v(x)}\,\mathrm{d}x \\ &+ \int_{\mathcal{D}} \big(b(x)\cdot\nabla u(x)\big)\overline{v(x)}\,\mathrm{d}x + \int_{\mathcal{D}} a_0(x)u(x)\overline{v(x)}\,\mathrm{d}x,\end{aligned} \tag{1.2.2}$$

where the bars indicate complex conjugation, and $a \in L^\infty(\mathcal{D};\mathbb{C}^{d\times d})$, $b, c \in L^\infty(\mathcal{D};\mathbb{C}^d)$ and $a_0 \in L^\infty(\mathcal{D})$. We will motivate why such a form induces a linear operator which, under an additional ellipticity assumption on the of the principal coefficient function a, can be naturally interpreted as a *(uniformly) strongly elliptic second-order differential operator (in divergence form)* on $L^2(\mathcal{D})$.

The results in this section will be formulated for complex Hilbert spaces and, accordingly, complex-valued forms. They remain true when working instead on a real Hilbert space H; in this case, one can simply remove the complex conjugates and real parts from the assumptions and statements.

1.2.1. SESQUILINEAR FORMS AND SEMIGROUPS

Definition 1.2.1 (Sesquilinear form)**.** Let $\mathsf{D}(\mathfrak{a})$ be a given subspace of a complex Hilbert space H. A mapping $\mathfrak{a}\colon \mathsf{D}(\mathfrak{a}) \times \mathsf{D}(\mathfrak{a}) \to \mathbb{C}$ is called a *sesquilinear*[7] *form*, with *domain* $\mathsf{D}(\mathfrak{a})$, if it is linear in the first argument and conjugate-linear in the second:

$$\begin{aligned}\mathfrak{a}(\alpha_1 u_1 + \alpha_2 u_2, v) &= \alpha_1\mathfrak{a}(u_1,v) + \alpha_2\mathfrak{a}(u_2,v) && \text{for all } \alpha_1,\alpha_2 \in \mathbb{C} \text{ and } u_1,u_2,v \in \mathsf{D}(\mathfrak{a});\\ \mathfrak{a}(u, \beta_1 v_1 + \beta_2 v_2) &= \overline{\beta_1}\mathfrak{a}(u,v_1) + \overline{\beta_2}\mathfrak{a}(u,v_2) && \text{for all } \beta_1,\beta_2 \in \mathbb{C} \text{ and } u,v_1,v_2 \in \mathsf{D}(\mathfrak{a}).\end{aligned}$$

If H is instead a real Hilbert space, then we say that $\mathfrak{a}\colon \mathsf{D}(\mathfrak{a}) \times \mathsf{D}(\mathfrak{a}) \to \mathbb{R}$ is a *bilinear form* if it is linear in both arguments. In order for a sesquilinear form to induce an

[7] Derived from the Latin prefix *sesqui-* ("one-and-a-half-"), referring to the fact that we count conjugate linearity as "half-linearity".

operator exhibiting the type of properties that we wish to use the remainder of this thesis, we need to impose more structure. We start with the following.

Definition 1.2.2 (Accretivity and coercivity)**.** A sesquilinear form $\mathfrak{a}\colon \mathsf{D}(\mathfrak{a}) \times \mathsf{D}(\mathfrak{a}) \to \mathbb{C}$ on a complex Hilbert space H is said to be *accretive* if

$$\operatorname{Re} \mathfrak{a}(u,u) \geq 0 \qquad \text{for all } u \in \mathsf{D}(\mathfrak{a}).$$

It is called *coercive* if there exists a constant $\theta \in (0,\infty)$ such that

$$\operatorname{Re} \mathfrak{a}(u,u) \geq \theta \|u\|_H \qquad \text{for all } u \in \mathsf{D}(\mathfrak{a}).$$

For an accretive sesquilinear form $\mathfrak{a}\colon \mathsf{D}(\mathfrak{a}) \times \mathsf{D}(\mathfrak{a}) \to \mathbb{C}$ on H, it holds that

$$\langle u, v\rangle_{\mathfrak{a}} := \tfrac{1}{2}\Big(\mathfrak{a}(u,v) + \overline{\mathfrak{a}(v,u)}\Big) + \langle u, v\rangle_H, \quad u, v \in \mathsf{D}(\mathfrak{a}),$$

defines an inner product on its domain $\mathsf{D}(\mathfrak{a})$, which induces the norm

$$\|u\|_{\mathfrak{a}} := \sqrt{\langle u,u\rangle_{\mathfrak{a}}} = \sqrt{\operatorname{Re}\mathfrak{a}(u,u) + \|u\|_H^2}.$$

If the space $\mathsf{D}(\mathfrak{a})$ is complete with respect to this norm, then we say that $\mathfrak{a}$ is *closed*:

Definition 1.2.3 (Closedness and continuity)**.** An accretive sesquilinear form $\mathfrak{a}$ with domain $\mathsf{D}(\mathfrak{a})$ on a complex Hilbert space H is said to be

- *closed* if $\big(\mathsf{D}(\mathfrak{a}), \langle\cdot,\cdot\rangle_{\mathfrak{a}}\big)$ is a Hilbert space;
- *continuous* if there exists a constant $C \in [0,\infty)$ such that

$$|\mathfrak{a}(u,v)| \leq C\|u\|_{\mathfrak{a}}\|v\|_{\mathfrak{a}} \quad \text{for all } u, v \in \mathsf{D}(\mathfrak{a}). \tag{1.2.3}$$

We are mainly interested in sesquilinear forms which can be used to define linear operators. To this end, we need to assume that the form $\mathfrak{a}$ on H is *densely defined*; just as for linear operators, this means that the domain $\mathsf{D}(\mathfrak{a})$ is dense in H. Our next definition is a slightly more precise version of (1.2.1).

Definition 1.2.4 (Operator associated to form)**.** Let $\mathfrak{a}\colon \mathsf{D}(\mathfrak{a}) \times \mathsf{D}(\mathfrak{a}) \to \mathbb{C}$ be a densely defined sesquilinear form on a complex Banach space H. The *linear operator A associated to* $\mathfrak{a}$ is the operator $A\colon \mathsf{D}(A) \subseteq H \to H$ defined by

$$\begin{aligned} \mathsf{D}(A) &:= \big\{u \in \mathsf{D}(\mathfrak{a}) : \text{there exists } y \in H \text{ such that } \langle y, v\rangle_H = \mathfrak{a}(u,v) \text{ for all } v \in \mathsf{D}(\mathfrak{a})\big\}, \\ Au &:= y \quad \text{(with } y \text{ as above) for all } u \in \mathsf{D}(A). \end{aligned}$$

The assumed density of the domain $\mathsf{D}(\mathfrak{a})$ in H ensures that $y \in H$ in the above definition is unique, and thus that A is well-defined. For a sesquilinear form $\mathfrak{a}$ satisfying all the above properties, we have the following result:

Proposition 1.2.5 ([164, Proposition 1.22])**.** *Suppose that $\mathfrak{a}\colon \mathsf{D}(\mathfrak{a}) \times \mathsf{D}(\mathfrak{a}) \to \mathbb{C}$ is a sesquilinear form on a complex Banach space H which is densely defined, accretive, closed and continuous. Then the linear operator $A\colon \mathsf{D}(A) \subseteq H \to H$ associated to $\mathfrak{a}$ is densely defined and satisfies*

$$-\lambda \in \rho(A) \quad \textit{and} \quad \|(\lambda \operatorname{Id}_H + A)^{-1}\|_{\mathscr{L}(H)} \leq \frac{1}{\lambda} \qquad \textit{for all } \lambda \in (0,\infty).$$

By the contraction case of the Hille–Yosida generation theorem (Corollary 1.1.5 with $w := 0$), it follows that $-A$ is the generator of a C_0-semigroup $(S(t))_{t\geq 0}$ of contractions on H if A is associated to a densely defined, accretive, closed and continuous sesquilinear form. In fact, more can be said:

Theorem 1.2.6 ([164, Theorem 1.52])**.** *Suppose that* $\mathfrak{a}\colon \mathsf{D}(\mathfrak{a})\times\mathsf{D}(\mathfrak{a})\to\mathbb{C}$ *is a sesquilinear form on a complex Banach space H which is densely defined, accretive, closed and continuous. Let* $A\colon \mathsf{D}(A)\subseteq H\to H$ *denote the linear operator associated to* $\mathfrak{a}$.

Then $-A$ is the generator of a contractive C_0-semigroup $(S(t))_{t\geq 0}\subseteq\mathscr{L}(H)$. *Moreover, for all* $\varepsilon\in(0,1]$, *the semigroup* $(e^{-\varepsilon t}S(t))_{t\geq 0}$ *generated by* $-(A+\varepsilon\,\mathrm{Id}_H)$ *extends to a contractive analytic semigroup on* Σ_η, *where* $\eta=\frac{1}{2}\pi-\arctan(C/\varepsilon)$ *and* $C\in[0,\infty)$ *is the continuity constant from* (1.2.3).

In particular, it follows that $(S(t))_{t\geq 0}$ itself is analytic, but not necessarily bounded analytic on the corresponding sector (even though it is contractive on the positive real line). In order to identify a class of sesquilinear forms whose associated C_0-semigroups are in fact contractive analytic, we define the notion of sectorial forms.

Definition 1.2.7 (Sectoriality)**.** Let $\omega\in(0,\frac{1}{2}\pi]$. A sesquilinear form $\mathfrak{a}$ with domain $\mathsf{D}(\mathfrak{a})$ on a complex Hilbert space H is said to be *ω-sectorial* if

$$\mathfrak{a}(u,u)\in\overline{\Sigma}_\omega\quad\text{for all } u\in\mathsf{D}(\mathfrak{a}).$$

It is immediate that sectoriality implies accretivity (since this is in fact the same as $\frac{1}{2}\pi$-sectoriality). If $\omega<\frac{1}{2}\pi$, then by [156, Proposition 13.42], ω-sectorial sesquilinear forms are continuous with continuity constant $1+\tan\omega$. Thus, if such a form is moreover densely defined and closed, then it satisfies the hypotheses of Theorem 1.2.6 and $-A$ generates an analytic semigroup $(S(t))_{t\geq 0}$. The following result states that $(S(t))_{t\geq 0}$ is in fact *contractive analytic*, and that A is then ω-sectorial by Theorem 1.1.8. It is a direct consequence of the Lumer–Phillips generation theorem for analytic contraction C_0-semigroups on Hilbert spaces, see [156, Theorem 13.35].

Theorem 1.2.8 ([156, Theorem 13.40])**.** *Let* $\mathfrak{a}\colon \mathsf{D}(\mathfrak{a})\times\mathsf{D}(\mathfrak{a})\to\mathbb{C}$ *be a sesquilinear form on a complex Banach space H which is densely defined, closed and ω-sectorial for some* $\omega\in(0,\frac{1}{2}\pi)$. *Let* $A\colon\mathsf{D}(A)\subseteq H\to H$ *denote the linear operator associated to* $\mathfrak{a}$. *Then $-A$ is the generator of a C_0-semigroup* $(S(t))_{t\geq 0}\subseteq\mathscr{L}(H)$ *which extends to an analytic C_0-semigroup of contractions on the sector* Σ_η *with* $\eta=\frac{1}{2}\pi-\omega$.

1.2.2. POSITIVITY AND L^p-CONTRACTIVITY OF SEMIGROUPS ON L^2

In this section we specialize to the typical case where $H := L^2(S,\mathscr{A},\nu;\mathbb{C})$ is a Lebesgue space of square-integrable complex-valued functions on a σ-finite measure space $(S,\mathscr{A},\nu)$, typically abbreviated to $L^2(S)$. Let $H_\mathbb{R} := L^2(S,\mathscr{A},\nu;\mathbb{R})$ denote its real-valued counterpart; as before, upon omitting real parts and complex conjugates, all the results in this subsection carry over the real setting, i.e., the case where we take $H := H_\mathbb{R}$ to begin with. Moreover, let $H_\mathbb{R}^+ := \{f\in H_\mathbb{R} : f(x)\geq 0 \text{ for } \nu\text{-a.e. } x\in S\}$ denote the cone of non-negative real functions.

Definition 1.2.9. Let $(S,\mathscr{A},\nu)$ be a σ-finite measure space. A strongly measurable and locally bounded semigroup $(S(t))_{t\geq 0}$ on $H := L^2(S)$ is said to be

- *positive* if $S(t)H_{\mathbb{R}}^+ \subseteq H_{\mathbb{R}}^+$ for all $t \in [0,\infty)$;
- L^∞*-contractive* if, for every $t \in [0,\infty)$ and $u \in L^2(S) \cap L^\infty(S)$, we have

$$\|S(t)u\|_{L^\infty(S)} \leq \|u\|_{L^\infty(S)};$$

- *sub-Markovian* if it is both positive and L^∞-contractive.

The term *sub-Markovian* has the following relation to the (simple) Markov property, mentioned briefly in Section 1.4.3 and defined precisely in Section 3.3.1. As remarked in the latter section, any (real-valued) simple Markov process $(X(t))_{t\geq 0}$ is characterized by a family $(T_{s,t})_{0\leq s\leq t}$ of *transition operators* on the space $B_b(\mathbb{R};\mathbb{R})$ of bounded measurable functions from $\mathbb{R}$ to $\mathbb{R}$. A Markov process is said to be *time-homogeneous* if $T_{s,t} = T_{0,t-s}$ for all $0 \leq s \leq t$; in this case, the *Chapman–Kolmogorov relation* of $(T_{s,t})_{0\leq s\leq t}$, see (TO3) in Section 3.3.1, renders $S(t) := T_{0,t}$ a semigroup on $B_b(\mathbb{R};\mathbb{R})$. Moreover, property (TO1) and equation (3.3.1) of that section imply that $(S(t))_{t\geq 0}$ is positive and L^∞-contractive, respectively. Lastly, property (TO2) asserts that $S(t)\mathbf{1}_{\mathbb{R}} = \mathbf{1}_{\mathbb{R}}$ for all $t \geq 0$ in this case. In general, a *sub*-Markovian semigroup may fail to satisfy the latter property, justifying the prefix.

If a C_0-semigroup $(S(t))_{t\geq 0}$ of contractions on $L^2(S)$ is also L^∞-contractive, then, for $t \in [0,\infty)$ and $p \in [2,\infty]$, one can extend $S(t)$ from $L^2(S) \cap L^p(S)$ to a contraction on $L^p(S)$ by means of the Riesz–Thorin interpolation theorem. By density, it can then be shown that $(S(t))_{t\geq 0}$ can in fact be viewed as a C_0-semigroup of contractions on $L^p(S)$. For *symmetric* bilinear forms on the real Hilbert space $H_{\mathbb{R}} = L^2(S;\mathbb{R})$, meaning that $\mathfrak{a}(u,v) = \mathfrak{a}(v,u)$ for all $u, v \in \mathsf{D}(\mathfrak{a})$, we can characterize the positivity and L^∞-contractivity of its associated semigroup in terms of the *Beurling–Deny criteria*:

Theorem 1.2.10 (Beurling–Deny [164, Corollary 2.18])**.** *Let* $\mathfrak{a}\colon \mathsf{D}(\mathfrak{a}) \times \mathsf{D}(\mathfrak{a}) \to \mathbb{R}$ *be a densely defined,* symmetric *bilinear form on* $L^2(S;\mathbb{R})$ *for which there exists an* $\eta \in \mathbb{R}$ *such that the form* $\mathfrak{a} + \eta$ *defined by*

$$[\mathfrak{a}+\eta](u,v) := \mathfrak{a}(u,v) + \eta\langle u,v\rangle_{L^2(S)}, \quad u,v \in \mathsf{D}(\mathfrak{a}), \tag{1.2.4}$$

is accretive, continuous and closed. Let A and $(S(t))_{t\geq 0}$ respectively denote the linear operator and contraction C_0-semigroup associated to $\mathfrak{a}$. The following are equivalent:

(a) $(S(t))_{t\geq 0}$ *is positive.*

(b) For every $u \in \mathsf{D}(\mathfrak{a})$*, we have* $|u| \in \mathsf{D}(\mathfrak{a})$ *and* $\mathfrak{a}(|u|,|u|) \leq \mathfrak{a}(u,u)$.

Supposing that either (and thus both) of the above conditions are satisfied, the following statements are equivalent as well:

(c) $(S(t))_{t\geq 0}$ *is sub-Markovian.*

(d) For every $u \in \mathsf{D}(\mathfrak{a}) \cap H_{\mathbb{R}}^+$*, we have* $1 \wedge u \in \mathsf{D}(\mathfrak{a})$ *and* $\mathfrak{a}(1 \wedge u, 1 \wedge u) \leq \mathfrak{a}(u,u)$.

1.2.3. ELLIPTIC DIFFERENTIAL OPERATORS ON A EUCLIDEAN DOMAIN

In this section, we show how the theory of sesquilinear forms can be used to define a second-order elliptic differential operator on a Euclidean domain $\mathcal{D} \subseteq \mathbb{R}^d$ which generates a semigroup on $L^2(\mathcal{D})$. We begin by stating the rigorous definition of the sesquilinear form $\mathfrak{a}_V$ defined by (1.2.2) and deriving some basic properties.

GENERAL ELLIPTIC OPERATORS IN DIVERGENCE FORM

Definition 1.2.11 (Uniformly strongly elliptic form)**.** Let $\mathcal{D} \subseteq \mathbb{R}^d$ be an open and connected Euclidean domain, and let the coefficient functions $a \in L^\infty(\mathcal{D};\mathbb{C}^{d\times d})$, $b, c \in L^\infty(\mathcal{D};\mathbb{C}^d)$ and $a_0 \in L^\infty(\mathcal{D};\mathbb{C})$ be given. Suppose that the principal coefficient function $a = (a_{jk})_{j,k=1}^d$ is elliptic, i.e., there exists a constant $\theta \in (0,\infty)$ such that

$$\operatorname*{ess\,inf}_{x\in\mathcal{D}} \operatorname{Re} \sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}(x)\xi_j\overline{\xi_k} \geq \theta\|\xi\|^2_{\mathbb{C}^d} \quad \text{for all } \xi \in \mathbb{C}^d, \tag{1.2.5}$$

where ess inf denotes the essential infimum.

For a closed linear subspace V of the Sobolev space $H^1(\mathcal{D}) := W^{1,2}(\mathcal{D})$ such that $H^1_0(\mathcal{D}) \subseteq V \subseteq H^1(\mathcal{D})$, where $H^1_0(\mathcal{D}) := \overline{C_c^\infty(\mathcal{D})}^{H^1(\mathcal{D})}$, we define the *(uniformly) strongly elliptic* sesquilinear form $\mathfrak{a}_V : V \times V \to \mathbb{C}$ on $L^2(\mathcal{D})$ by (1.2.2) for all $u, v \in V$, where the gradients are interpreted in the weak sense.

We will now motivate why the operator associated to the form $\mathfrak{a}_V$ from Definition 1.2.11, denoted L_V, can rightfully be interpreted as an elliptic second-order differential operator on $L^2(\mathcal{D})$. Consider first the coefficient functions $(a_{jk})_{j,k=1}^d$, $(\widetilde{b}_k)_{k=1}^d$ and a_0, all belonging to $L^\infty(\mathcal{D};\mathbb{C})$. These determine a general second-order differential operator L which acts on functions $u\colon \mathcal{D} \to \mathbb{C}$ as

$$Lu := -\sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}\partial_{x_j}\partial_{x_k}u + \sum_{k=1}^{d}\widetilde{b}_k\partial_{x_k}u + a_0 u, \tag{1.2.6}$$

where $\partial_{x_j} := \frac{\partial}{\partial x_j}$ denotes the (classical or weak) partial derivative in the direction of the jth standard unit vector in $\mathbb{R}^d$. Since the double summation on the right-hand side of (1.2.6) contains the highest-order derivatives, we say that its terms comprise the *principal part* of L. We call L a (uniformly) strongly elliptic operator if its principal part is elliptic, i.e., if (1.2.5) is satisfied.

Equation (1.2.6) can be interpreted in a pointwise sense if u admits classical partial derivatives up to second order (e.g., if $u \in C^2(\mathcal{D})$, which denotes the space of twice continuously differentiable functions on $\mathcal{D}$). If, in addition, the principal coefficient functions $(a_{jk})_{j,k=1}^d$ admit classical first-order partial derivatives, then the product rule implies that the action of L can be written as

$$Lu = -\sum_{j=1}^{d}\sum_{k=1}^{d}\partial_{x_j}(a_{jk}\partial_{x_k}u) + \sum_{k=1}^{d} b_k\partial_{x_k}u + a_0 u,$$

setting $b_k := \widetilde{b}_k + \sum_{j=1}^d \partial_{x_j} a_{jk}$ for all $k \in \{1,\dots,d\}$.

Turning back to the matrix-valued function $a := (a_{jk})_{j,k=1}^d \colon \mathcal{D} \to \mathbb{C}^{d\times d}$ and the vector field $b := (b_k)_{k=1}^d \colon \mathcal{D} \to \mathbb{C}^d$, and using nabla notation for the gradient and divergence, we can express this more concisely as

$$Lu = -\nabla\cdot(a\nabla u) + b\cdot\nabla u + a_0 u.$$

This representation of L is said to be *in divergence form*, since its first (principal) term is the divergence of a vector field. In fact, the most general formal definition of a second-order operator in divergence form is

$$Lu := -\nabla\cdot(a\nabla u + cu) + b\cdot\nabla u + a_0 u, \tag{1.2.7}$$

where $b, c \in L^\infty(\mathcal{D};\mathbb{C}^d)$ are as above.

If the components of a and c are differentiable, then we can again use the product rule to find a transformation of the coefficient functions which allows us to cast the representation (1.2.7) into the shape of (1.2.6) for every $u \in C^2(\mathcal{D})$. In general, however, the divergence form definition (1.2.7) is meaningful for a larger class of functions than $C^2(\mathcal{D})$. In fact, it naturally leads to a weak definition of the operator L, which turns out to be particularly suited for viewing elliptic operators in a functional analytic framework. Indeed, let us multiply both sides of the equation by the complex conjugate of a test function $v \in C_c^1(\mathcal{D})$ (where the subscript c indicates compact support) and integrate over $\mathcal{D}$:

$$\begin{aligned}\int_{\mathcal{D}} Lu(x)\overline{v(x)}\,\mathrm{d}x = -&\int_{\mathcal{D}} \nabla\cdot\big(a(x)\nabla u(x) + c(x)u(x)\big)\overline{v(x)}\,\mathrm{d}x\\ &+\int_{\mathcal{D}}\big(b(x)\cdot\nabla u(x)\big)\overline{v(x)}\,\mathrm{d}x + \int_{\mathcal{D}} a_0(x)u(x)\overline{v(x)}\,\mathrm{d}x.\end{aligned}$$

Integrating by parts now yields

$$-\int_{\mathcal{D}} \nabla\cdot\big(a(x)\nabla u(x) + c(x)u(x)\big)\overline{v(x)}\,\mathrm{d}x = \int_{\mathcal{D}}\big(a(x)\nabla u(x) + c(x)u(x)\big)\cdot\overline{\nabla v(x)}\,\mathrm{d}x,$$

keeping in mind the compact support of v. Combining the previous two displays with (1.2.2) and using $L^2(\mathcal{D})$-inner product notation, we find $\langle Lu, v\rangle_{L^2(\mathcal{D})} = \mathfrak{a}_V(u,v)$ for all $v \in C_c^1(\mathcal{D})$ (without specifying V for the moment), cf. Definition 1.2.4.

Having motivated Definition 1.2.11, let us now study some essential properties of the form $\mathfrak{a}_V \colon V\times V \to \mathbb{C}$ on $L^2(\mathcal{D})$. Firstly, its strong ellipticity guarantees that it satisfies a *Gårding inequality*, i.e., there exists a constant $\eta \in \mathbb{R}$ such that

$$\operatorname{Re}\mathfrak{a}_V(u,u) + \eta\|u\|^2_{L^2(\mathcal{D})} \geq \tfrac{1}{2}\theta\|u\|^2_{H^1(\mathcal{D})} \quad \text{for all } u \in V.$$

In particular, the form $\mathfrak{a}_V + \eta$ defined by (1.2.4) is accretive. Moreover, it is continuous and closed; the estimates verifying these facts can for instance be found on [164, pp. 100–101]. Since $H^1(\mathcal{D})$ is dense in $L^2(\mathcal{D})$, and contained in $V = \mathsf{D}(\mathfrak{a})$ by definition, the form is also densely defined. Thus, if $L_V \colon \mathsf{D}(L_V) \subseteq L^2(\mathcal{D}) \to L^2(\mathcal{D})$ denotes the linear operator associated to $\mathfrak{a}_V$, then by Theorem 1.2.6 $-L_V$ generates a contractive C_0-semigroup $(S_V(t))_{t\geq 0}$ on $L^2(\mathcal{D})$ which extends to an analytic semigroup.

The choice of V serves to impose different types of boundary conditions on the operator L_V. Common choices include the homogeneous *Dirichlet*, *Neumann* and *mixed boundary conditions*, which respectively correspond to

$$\begin{aligned} &V := H^1_0(\mathcal{D}) := \overline{C^\infty_c(\mathcal{D})}^{H^1(\mathcal{D})}, \\ &V := H^1(\mathcal{D}), \quad \text{and} \\ &V := \overline{\{u|_{\mathcal{D}} : u \in C^\infty_c(\mathbb{R}^d \setminus \Gamma)\}}^{H^1(\mathcal{D})} \quad \text{for some closed subset } \Gamma \subseteq \partial\mathcal{D}, \end{aligned} \tag{1.2.8}$$

where $u|_{\mathcal{D}} \colon \mathcal{D} \to \mathbb{C}$ denotes the restriction of $u \colon \mathbb{R}^d \to \mathbb{C}$ to $\mathcal{D}$. Intuitively, the (homogeneous) Dirichlet boundary condition should be interpreted as imposing $u \equiv 0$ on $\partial\mathcal{D}$ as in (1.1.1), whereas the Neumann condition corresponds to

$$\big(a\nabla u + cu\big) \cdot \hat{n} \equiv 0 \quad \text{on } \partial\mathcal{D},$$

where $\hat{n}$ denotes the outward-pointing unit normal vector to $\partial\mathcal{D}$. Mixed boundary conditions correspond to imposing Dirichlet conditions at $\Gamma \subseteq \partial\mathcal{D}$ and Neumann conditions at $\partial\mathcal{D} \setminus \Gamma$. In particular, mixed boundary conditions with $\Gamma = \partial\mathcal{D}$ simply coincide with Dirichlet conditions. On the other hand, taking $\Gamma := \varnothing$ in (1.2.8) yields

$$V := \widetilde{H}^1(\mathcal{D}) := \overline{\{u|_{\mathcal{D}} : u \in C^\infty_c(\mathbb{R}^d)\}}^{H^1(\mathcal{D})},$$

and the corresponding boundary condition is known as *the good Neumann condition*. If the boundary $\partial\mathcal{D}$ is smooth enough, for instance Lipschitz continuous (the minimal assumption used in this thesis), then we have $H^1(\mathcal{D}) = \widetilde{H}^1(\mathcal{D})$, so that the good and regular Neumann conditions coincide.

SYMMETRIC ELLIPTIC OPERATORS WITH REAL-VALUED COEFFICIENTS

In the subsequent chapters of this thesis, we are primarily interested in *symmetric* elliptic differential operators with *real* coefficients, i.e., whose forms satisfy $c = b = 0$, $a \in L^\infty(\mathcal{D}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $a_0 := \kappa^2$ for some $\kappa \in L^\infty(\mathcal{D}; \mathbb{R})$. Thus, the sesquilinear form from Definition 1.2.11 reduces to the symmetric form $\mathfrak{a}_V$ given by

$$\mathfrak{a}_V(u, v) = \int_{\mathcal{D}} a(x)\nabla u(x) \cdot \overline{\nabla v(x)}\,\mathrm{d}x + \int_{\mathcal{D}} \kappa^2(x) u(x)\overline{v(x)}\,\mathrm{d}x \quad \text{for all } u, v \in V. \tag{1.2.9}$$

Hence, the associated operator $L_V \colon \mathsf{D}(L_V) \subseteq L^2(\mathcal{D}) \to L^2(\mathcal{D})$ is formally given by

$$L_V u = -\nabla \cdot (a\nabla u) + \kappa^2 u. \tag{1.2.10}$$

In particular, we mention the important example of the Laplace operator Δ_V, whose negative is obtained by taking $a \equiv \mathrm{Id}_{\mathbb{R}^d}$ and $\kappa \equiv 0$, i.e.,

$$-\Delta_V u = -\nabla \cdot (\nabla u).$$

In the present setting, the natural counterpart of the ellipticity condition (1.2.5) is the existence of $\theta \in (0, \infty)$ such that

$$\operatorname*{ess\,inf}_{x\in\mathcal{D}} \sum_{j=1}^d \sum_{k=1}^d a_{jk}(x)\xi_j\xi_k \ge \theta \|\xi\|^2_{\mathbb{R}^d} \quad \text{for all } \xi \in \mathbb{R}^d;$$

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note that the vectors ξ are in $\mathbb{R}^d$ instead of $\mathbb{C}^d$. In fact, together with the symmetry of the components of a, the above condition implies

$$\operatorname*{ess\,inf}_{x\in\mathcal{D}} \sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}(x)\xi_j\overline{\xi_k} \geq \theta\|\xi\|_{\mathbb{C}^d}^2 \quad \text{for all } \xi\in\mathbb{C}^d,$$

which is stronger than (1.2.5) since it implies that the left-hand side is real, hence the real part does not need to be taken. In particular, it follows that $\mathfrak{a}(u,u)\in[0,\infty)$ for all $u\in V$, hence by Theorem 1.2.8 it holds that $-L_V$ generates a contractive analytic C_0-semigroup $(S_V(t))_{t\geq 0}\subseteq\mathscr{L}(L^2(\mathcal{D}))$ on the sector Σ_η for all $\eta\in(0,\frac{1}{2}\pi)$. Thus, by Theorem 1.1.8, L_V is a sectorial operator of angle $\omega(L_V)=0$.

With homogeneous Dirichlet, Neumann or mixed boundary conditions, (1.2.9) restricts to a symmetric real form on $L^2(\mathcal{D};\mathbb{R})$ whose associated semigroup $(S_V(t))_{t\geq 0}$ is positive and $L^\infty(\mathcal{D})$-contractive by [164, Corollaries 4.3 and 4.10], respectively. It follows that $(S_V(t))_{t\geq 0}$ is bounded analytic on $L^p(\mathcal{D})$ for all $p\in[2,\infty)$ by [164, Proposition 3.12]. We finalize this subsection by recording a result regarding the spectral asymptotics of L_V. We state it in detail for the case of a bounded domain $\mathcal{D}\subsetneq\mathbb{R}^d$ and homogeneous Dirichlet boundary conditions $V=H_0^1(\mathcal{D})$. Under additional smoothness assumptions on the boundary of $\mathcal{D}$, similar asymptotics also hold in the case of Neumann boundary conditions, see for instance the survey [10].

Theorem 1.2.12 (Weyl's law, cf. [61, Theorem 6.3.1])**.** *Let $\mathcal{D}\subsetneq\mathbb{R}^d$ be a bounded region. Consider the form $\mathfrak{a}_V$ from (1.2.9) with domain $V=H_0^1(\mathcal{D})$ and coefficient functions $a\in L^\infty(\mathcal{D};\mathbb{R}^{d\times d}_{\mathrm{sym}})$, $\kappa\in L^\infty(\mathcal{D};\mathbb{R})$. Then the associated operator L_V has empty essential spectrum and compact resolvents, and there exist $c,C\in(0,\infty)$ such that its increasing sequence $(\lambda_j)_{j\in\mathbb{N}}$ of positive eigenvalues satisfies the two-sided estimate*

$$c j^{2/d} \leq \lambda_j \leq C j^{2/d} \quad \textit{for all } j\in\mathbb{N}. \tag{1.2.11}$$

1.2.4. SYMMETRIC ELLIPTIC OPERATORS ON A MANIFOLD

Since certain problems are more naturally formulated on curved spaces, see for instance Sections 2.5.2 and 5.3.1 below, we end this section with a brief summary of how symmetric elliptic differential operators of the form (1.2.10) can be defined on a Riemannian manifold instead of a Euclidean domain. In order to sketch the context, we introduce some notation from (Riemannian) manifold theory. Precise definitions of basic concepts such as manifolds and smooth mappings between them, as well as proofs of the statements below regarding tangent spaces and vector fields, can be found in a textbook such as [98].

Let $\mathcal{M}$ be an m-dimensional connected smooth manifold ($m\in\mathbb{N}$). Its *tangent space* $T_x\mathcal{M}$ at a point $x\in\mathcal{M}$ can be defined as consisting of precisely those linear maps $v_x\colon C^\infty(\mathcal{M};\mathbb{R})\to\mathbb{R}$ which satisfy the following version of the product rule:

$$v_x(fg)=v_x(f)g(x)+f(x)v_x(g) \quad \text{for all } f,g\in C^\infty(\mathcal{M};\mathbb{R}).$$

An element $v_x\in T_x\mathcal{M}$ is called a *tangent vector*, and its action $v_x(f)\in\mathbb{R}$ on a smooth function $f\in C^\infty(\mathcal{M};\mathbb{R})$ can intuitively be interpreted as the directional derivative of

f at the point $x \in \mathcal{M}$ in the direction v_x. The tangent space $T_x\mathcal{M}$ forms a vector space of dimension m.

The disjoint union $T\mathcal{M} := \bigsqcup_{x\in\mathcal{M}} T_x\mathcal{M}$ of all the tangent spaces associated to $\mathcal{M}$ is called the *tangent bundle* of $\mathcal{M}$. Since $T\mathcal{M}$ can be shown to admit a manifold structure as well, we can speak of a smooth mapping $v\colon \mathcal{M} \to T\mathcal{M}$, which we call a *vector field* if it is a *section* of $T\mathcal{M}$, i.e., we have $v(x) \in T_x\mathcal{M}$ for every $x \in \mathcal{M}$.

Since, for $x \in \mathcal{M}$, the tangent space $T_x\mathcal{M}$ is a vector space, we can define the *cotangent space* of $\mathcal{M}$ at x by $T^*_x\mathcal{M} := (T_x\mathcal{M})^* := \mathscr{L}(T_x\mathcal{M};\mathbb{R})$, as well as the *covector bundle* $T^*\mathcal{M} := \bigsqcup_{x\in\mathcal{M}} T^*_x\mathcal{M}$, whose smooth sections are called 1*-forms* (also known as *covector fields*). To any $f \in C^\infty(\mathcal{M};\mathbb{R})$ we associate a 1-form $\mathrm{d}f\colon \mathcal{M} \to T^*\mathcal{M}$ called the *exterior derivative* of f, which is defined by $[\mathrm{d}f(x)](v_x) := v_x(f)$ for all $x \in \mathcal{M}$ and $v_x \in T_x\mathcal{M}$.

Now suppose that $\mathcal{M}$ is a *Riemannian* manifold, i.e., there exists a collection of inner products $(\langle\cdot,\cdot\rangle_{T_x\mathcal{M}})_{x\in\mathcal{M}}$ on the spaces $(T_x\mathcal{M})_{x\in\mathcal{M}}$ which depends smoothly on x in the sense that $x \mapsto \langle v(x), w(x)\rangle_{T_x\mathcal{M}}$ belongs to $C^\infty(\mathcal{M};\mathbb{R})$ for every pair of vector fields $v, w\colon \mathcal{M} \to T\mathcal{M}$. For every $x \in \mathcal{M}$, the inner product $\langle\cdot,\cdot\rangle_{T_x\mathcal{M}}$ renders $T_x\mathcal{M}$ a Hilbert space, so that it can be identified with its dual via the Riesz isomorphism $J_x\colon T_x\mathcal{M} \to (T_x\mathcal{M})^*$. This allows us to define the *gradient* of any $f \in C^\infty(\mathcal{M};\mathbb{R})$ as the vector field $\nabla_{\mathcal{M}} f\colon \mathcal{M} \to T\mathcal{M}$ given by $\nabla_{\mathcal{M}} f(x) := J_x^{-1}\,\mathrm{d}f(x)$ for all $x \in \mathcal{M}$. Unpacking these definitions, we find that the value of the gradient $\nabla_{\mathcal{M}} f$ at $x \in \mathcal{M}$ is characterized by the relation

$$\langle \nabla_{\mathcal{M}} f(x), v_x\rangle_{T_x\mathcal{M}} = v_x(f) \quad \text{for all } v_x \in T_x\mathcal{M}.$$

Moreover, we note that the Riemannian structure of $\mathcal{M}$ enables the construction of the Lebesgue σ-algebra on $\mathcal{M}$, and the resulting measurable space can be equipped with the complete and Radon *Riemann–Lebesgue volume measure* $\mathrm{Vol}_{\mathcal{M}}$ [5, Chapter XII, Section 1].

We have now collected all the ingredients to make sense of an analog to the symmetric sesquilinear form (1.2.9) on a Riemannian manifold $\mathcal{M}$. Namely, given any $\kappa \in C^\infty(\mathcal{M};\mathbb{R})$ and a smooth section $a\colon \mathcal{M} \to \bigsqcup_{x\in\mathcal{M}} \mathscr{L}(T_x\mathcal{M})$, we define

$$\begin{aligned}\mathfrak{a}_{\mathcal{M}}(u,v) := &\int_{\mathcal{M}} \langle a(x)\nabla_{\mathcal{M}} u(x), \nabla_{\mathcal{M}} v(x)\rangle_{T_x\mathcal{M}}\,\mathrm{d}\mathrm{Vol}_{\mathcal{M}}(x)\\ &+ \int_{\mathcal{M}} \kappa^2(x)\langle u(x), v(x)\rangle_{T_x\mathcal{M}}\,\mathrm{d}\mathrm{Vol}_{\mathcal{M}}(x), \quad u, v \in C^\infty(\mathcal{M};\mathbb{R}).\end{aligned}$$

Now suppose that $a(x) \in \mathscr{L}(T_x\mathcal{M})$ is self-adjoint (with respect to the Riemannian inner product) for every $x \in \mathcal{M}$, and that there exists a constant $\theta \in (0,\infty)$ such that

$$\langle a(x)v_x, v_x\rangle_{T_x\mathcal{M}} \ge \theta \|v_x\|^2_{T_x\mathcal{M}} \quad \text{for all } x \in \mathcal{M} \text{ and } v_x \in T_x\mathcal{M}.$$

Then there exists a closed extension to $H^1(\mathcal{M})$ of $\mathfrak{a}_{\mathcal{M}}$ which is moreover sectorial and densely defined, and satisfies the Beurling–Deny criteria, so that the negative of its associated operator $L_{\mathcal{M}}\colon \mathsf{D}(L_{\mathcal{M}}) \subseteq L^2(\mathcal{M}) \to L^2(\mathcal{M})$ generates a positive contractive analytic C_0-semigroup $(S_{\mathcal{M}}(t))_{t\ge 0}$ on $L^p(\mathcal{M})$ for all $p \in [2,\infty)$, cf. [60, Chapter 5]. In particular, this holds for the *Laplace–Beltrami operator* $\Delta_{\mathcal{M}}$ obtained by taking $a(x) := \mathrm{Id}_{T_x\mathcal{M}}$ for all $x \in \mathcal{M}$.

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Finally, if we additionally assume that the manifold $\mathcal{M}$ is compact and that κ^2 is strictly bounded away from zero, i.e., there exists a constant $\kappa_{\min} \in (0,\infty)$ such that $\kappa(x)^2 \geq \kappa_{\min}^2 > 0$ for all $x \in \mathcal{M}$, then we also recover that $L_{\mathcal{M}}$ has a discrete spectrum of positive eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ which satisfy the Weyl asymptotics (1.2.11); see for instance [190, Chapter XII, Theorem 1.3]. The extra assumption on κ^2 is needed to ensure that $0 \notin \sigma(L_{\mathcal{M}})$ since, unlike the Dirichlet Euclidean case, we consider here a domain without boundary, and accordingly there is no boundary condition which excludes nonzero constant functions from $\mathsf{D}(L_{\mathcal{M}})$.

1.3. FUNCTIONAL AND FRACTIONAL CALCULUS

Throughout Chapters 2–5, we are often faced with questions of the form: Given an (unbounded) linear operator $A\colon \mathsf{D}(A) \subseteq E \to E$ on a (complex) Banach space E and a (complex-valued) function f, how can one define a (bounded or unbounded) linear operator $f(A)$ in a consistent way? The most common application (used in each of Chapters 2–5) is to define *fractional powers* $A^\alpha := f_\alpha(A)$, where $f_\alpha(z) := z^\alpha$ for $\alpha \in \mathbb{R}$, but we will also deal with more complicated functions such as incomplete gamma functions in Chapters 3 and 4. Answering this question is the goal of *functional calculus*, which is the subject of this section.

In Section 1.3.1 we present the various definitions that are used in this thesis. This is followed by Section 1.3.2, where we focus on the particularly important example of fractional powers. Finally, in Section 1.3.3 we go one step further and specialize to the case where A is a (first or second order) differential operator, bringing us into the realm of *fractional integration and differentiation* (collectively known as *fractional integro-differentiation* or *fractional calculus*).

1.3.1. ABSTRACT FUNCTIONAL CALCULI

An *abstract functional calculus* consists of three ingredients: A class of admissible linear operators A, a space of $\mathcal{E}$ of scalar-valued functions f (called the *domain of the functional calculus*) and a rule for defining $f(A)$, in such a way that the mapping $f \mapsto f(A)$ respects the operations on the function space $\mathcal{E}$ (e.g., linearity and multiplicativity) for any fixed A.

In the previous section we have already encountered some examples of operators which can be seen as $f(A)$ for certain operators A and functions f:

- If $\lambda \in \rho(A)$, then the resolvent operator $R(\lambda, A) \in \mathscr{L}(E)$ can be viewed as $f_\lambda(A)$ for the function $f_\lambda(z) := (\lambda - z)^{-1}$.
- If $-A$ is the generator of a semigroup $(S(t))_{t\geq 0} \subseteq \mathscr{L}(E)$, then for every $t \in [0,\infty)$ we can interpret $S(t)$ as $f_t(A)$ for $f_t(z) := e^{-tz}$ (or as $f_1(tA)$).

Building upon these observations, we will define the *(extended) Dunford* and *Phillips* functional calculi, respectively, which will be used throughout this thesis: Indeed, the Dunford and H^∞-calculus are used in Chapters 5 and 2, respectively, and the Phillips calculus is used implicitly or explicitly in Chapters 2–4 (mostly to define fractional powers, see also Theorem 1.3.6 below). Afterwards, we also briefly introduce the *spectral* functional calculus which is used to define fractional powers for a class of differential operators on a graph or manifold in Section 5.3.

DUNFORD AND H^∞-CALCULUS FOR SECTORIAL OPERATORS

We start with the *Dunford calculus* for sectorial operators and functions belonging to a *Hardy space* $H^p(\Sigma_\omega)$, which for $p \in [1,\infty]$ and $\omega \in (0,\pi)$ denotes the Banach space consisting of holomorphic functions $f\colon \Sigma_\omega \to \mathbb{C}$ for which the norm

$$\|f\|_{H^p(\Sigma_\omega)} := \sup_{\nu\in(-\omega,\omega)} \left\| t \mapsto f(te^{i\nu}) \right\|_{L^p((0,\infty),t^{-1}\,\mathrm{d}t)}$$

is finite, where $L^p((0,\infty),t^{-1}\,\mathrm{d}t)$ indicates the L^p-norm with respect to the measure $B \mapsto \int_B t^{-1}\,\mathrm{d}t$ on $(0,\infty)$. Note that $H^\infty(\Sigma_\omega)$ reduces to the space of bounded holomorphic functions on Σ_ω equipped with the supremum norm.

Definition 1.3.1 (Dunford calculus)**.** Let $A\colon \mathsf{D}(A) \subseteq E \to E$ be a sectorial linear operator of angle $\omega(A) \in [0,\pi)$, and suppose that $f \in H^1(\Sigma_\omega)$ for some $\omega \in (\omega(A),\pi)$, so that the Cauchy integral formula implies, for $\omega' \in (\omega(A),\omega)$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Sigma_{\omega'}} f(\lambda)(\lambda - z)^{-1}\,\mathrm{d}\lambda \quad \text{for all } z \in \Sigma_{\omega(A)}.$$

Then we can define $f(A) \in \mathscr{L}(E)$ by

$$f(A) := \frac{1}{2\pi i}\int_{\partial\Sigma_{\omega'}} f(\lambda)R(\lambda,A)\,\mathrm{d}\lambda;$$

this operator is well-defined in the sense that the expression on the right-hand side does not depend on the choice of ω'.

By [114, Theorem 10.2.2], the mapping $f \mapsto f(A)$ from $H^1(\Sigma_\omega)$ to $\mathscr{L}(E)$ is linear and multiplicative in the sense that $(fg)(A) = f(A)g(A)$ whenever $f, g \in H^1(\Sigma_\omega)$ are such that $fg \in H^1(\Sigma_\omega)$. We now discuss some circumstances under which we can extend this mapping to a larger class of functions.

Before moving on to this *extended Dunford calculus*, we first briefly record the concept of a bounded H^∞-calculus. It is used as an assumption for parts of Chapter 2 (see Assumption 2.3.1(iii)), and it is related to the concept of maximal L^p-regularity discussed in Section 1.1.4. A sectorial linear operator $A\colon \mathsf{D}(A) \subseteq E \to E$ of angle $\omega(A) \in [0,\pi)$ is said to have a *bounded* $H^\infty(\Sigma_\omega)$*-calculus* for $\omega \in (\omega(A),\pi)$ if there exists $C \in [0,\infty)$ such that

$$\|f(A)\|_{\mathscr{L}(E)} \le C\|f\|_{H^\infty(\Sigma_\omega)} \quad \text{for all } f \in H^1(\Sigma_\omega) \cap H^\infty(\Sigma_\omega), \tag{1.3.1}$$

and we let $\omega_{H^\infty}(A)$ denote the infimum of all $\omega \in (\omega(A),\pi)$ for which A has a bounded $H^\infty(\Sigma_\omega)$-calculus. Whether a given operator has a bounded H^∞-calculus is a highly nontrivial question whose answer has some far-reaching consequences (see [114, Chapter 10]). We only mention here that, by [115, Corollary 17.3.6], any operator A admitting a bounded H^∞-calculus with $\omega_{H^\infty}(A) < \frac{1}{2}\pi$ has maximal L^p-regularity on $(0,\infty)$ for all $p \in (1,\infty)$.

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Now we extend the Dunford calculus into another direction, which does not impose such strong requirements on the operator A. Firstly, for $\omega \in (\omega(A), \pi)$ we introduce the vector space $\mathcal{E}(\Sigma_\omega)$ of holomorphic functions $f\colon \Sigma_\omega \to \mathbb{C}$ of the form

$$f(z) = f_0(z) + a(1+z)^{-1} + b,$$

where $f_0 \in H^1(\Sigma_\omega) \cap H^\infty(\Sigma_\omega)$ and $a, b \in \mathbb{C}$.[8] For $f \in \mathcal{E}(\Sigma_\omega)$, we can naturally define

$$f(A) := f_0(A) + a(\mathrm{Id}_E + A)^{-1} + b\,\mathrm{Id}_E \in \mathscr{L}(E),$$

and the mapping $f \mapsto f(A)$ is linear and multiplicative on $\mathcal{E}(\Sigma_\omega)$ by [115, Proposition 15.1.4]. Examples of such functions include $z \mapsto z^m(1+z)^{-n} \in \mathcal{E}(\Sigma_\omega)$ for all $\omega \in (0,\pi)$ and $n, m \in \mathbb{N}_0$ with $m \geq n$, and $z \mapsto e^{-\zeta z} \in \mathcal{E}(\Sigma_\omega)$ for all $\omega \in (0, \frac{1}{2}\pi)$ if $\zeta \in \Sigma_{\frac{1}{2}\pi - \omega}$. We have $[z \mapsto z^m(1+z)^{-n}](A) = A^m(\mathrm{Id}_E + A)^{-n}$ if A is sectorial [115, Example 15.1.5], and if $\omega(A) < \frac{1}{2}\pi$, the operators $[z \mapsto e^{-\zeta z}](A)$ coincide with the analytic semigroup $S(\zeta)$ given by (1.1.14), see [115, Theorem 15.1.7].

Finally, we can extend the Dunford calculus for a given sectorial linear operator A beyond the class $\mathcal{E}(\Sigma_\omega)$ (where $\omega \in (\omega(A), \pi)$) by means of regularizing functions. We say that $\varrho \in \mathcal{E}(\Sigma_\omega)$ is a *regularizer* for a holomorphic function $f\colon \Sigma_\omega \to \mathbb{C}$ if $\varrho f \in \mathcal{E}(\Sigma_\omega)$ and $\varrho(A) \in \mathscr{L}(E)$ is injective, in which case we define the (possibly unbounded) linear operator $f(A)\colon \mathsf{D}(f(A)) \subseteq E \to E$ by

$$\begin{aligned} \mathsf{D}(f(A)) &:= \{x \in E : [\varrho f](A)x \in \mathsf{R}(\varrho(A))\}, \\ f(A)x &:= \varrho(A)^{-1}[\varrho f](A)x, \quad x \in \mathsf{D}(f(A)), \end{aligned}$$

where we view the inverse of an injective (but not necessarily surjective) operator $T\colon \mathsf{D}(T) \subseteq E \to E$ as an (unbounded) linear operator $T^{-1}\colon \mathsf{D}(T^{-1}) \subseteq E \to E$ whose domain $\mathsf{D}(T^{-1})$ is the range $\mathsf{R}(T)$ of T. By [115, Proposition 15.1.9], the definition of $f(A)$ does not depend on the choice of regularizer. The resulting mapping $f \mapsto f(A)$ is called the *extended Dunford calculus*, and it satisfies analogous properties to linearity and multiplicativity for unbounded linear operators, see [115, Proposition 15.1.12].

PHILLIPS CALCULUS FOR SEMIGROUP GENERATORS

Let $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and $\overline{\mathbb{C}}_+ := \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$ denote the open and closed right-half planes of the complex plane, respectively.

Definition 1.3.2 (Phillips calculus)**.** If $-A$ is the generator of a uniformly bounded and strongly measurable semigroup $(S(t))_{t\geq 0} \subseteq \mathscr{L}(E)$ and $f\colon \overline{\mathbb{C}}_+ \to \mathbb{C}$ is the Laplace transform $\mathcal{L}[\mu]$ of a complex Borel measure μ of bounded variation on $[0,\infty)$, i.e.,

$$f(z) = \int_{[0,\infty)} e^{-zs}\,\mathrm{d}\mu(s) \quad \text{for all } z \in \overline{\mathbb{C}}_+,$$

[8] It is convenient to take $f_0 \in H^1(\Sigma_\omega) \cap H^\infty(\Sigma_\omega)$ instead of the seemingly more natural $f_0 \in H^1(\Sigma_\omega)$, see the discussion below [115, Definition 15.1.3].

then we can define $f(A) \in \mathscr{L}(E)$ by

$$f(A) := \int_{[0,\infty)} S(s)\,\mathrm{d}\mu(s).$$

The operators defined in Definitions 1.3.1 and 1.3.2 coincide whenever $-A$ generates a uniformly bounded and strongly measurable semigroup and $f = \mathcal{L}[\mu]$ extends to a function in $H^1(\Sigma_\omega)$ for some $\omega \in (\frac{1}{2}\pi, \pi)$ by [100, Proposition 3.3.2]. For this reason, we typically use the same notation $f(A)$ for both.

The Dunford and Phillips calculi follow a common recipe for defining operators $f(A)$: Identify a "prototypical" class of functions f for which the definition of $f(A)$ is natural (respectively, the resolvent operators $R(\lambda, A) = [(\lambda - z)^{-1}](A)$ and the semigroup $S(t) = [e^{-tz}](A)$), along with a larger class of scalar-valued functions which can be expressed in terms of those basic functions (e.g., using integral transforms), and define $f(A)$ by substituting the "prototypical" operators into this relation.

SPECTRAL CALCULUS FOR OPERATORS WITH DISCRETE SPECTRA

The definition of the following *spectral* functional calculus is of a different form than the previous two. It applies to linear operators on a separable Hilbert space with an orthonormal eigenbasis whose positive real eigenvalues only accumulate at infinity, such as the symmetric elliptic operators from Section 1.2. More specifically, it is used to define fractional powers of differential operators in Section 5.3.1.

Definition 1.3.3 (Spectral calculus)**.** Let $A\colon \mathsf{D}(A) \subseteq H \to H$ be a linear operator on a separable (real or complex) Hilbert space H. Suppose that there exist $(\psi_j)_{j\in\mathbb{N}} \subseteq H$ and an increasing sequence $(\lambda_j)_{j\in\mathbb{N}} \subseteq (0,\infty)$, accumulating only at infinity, such that $A\psi_j = \lambda_j \psi_j$ for every $j \in \mathbb{N}$. Then for any $f\colon (0,\infty) \to \mathbb{C}$ we can define an (unbounded) linear operator $f(A)$ by

$$\mathsf{D}(f(A)) := \left\{ x \in H : \sum_{j=1}^{\infty} \left| \langle x, \psi_j \rangle_H f(\lambda_j) \right|^2 < \infty \right\},$$

$$f(A)x := \sum_{j=1}^{\infty} \langle x, \psi_j \rangle_H f(\lambda_j) \psi_j, \quad x \in \mathsf{D}(f(A)).$$

This calculus coincides with the previous two whenever applicable, which ultimately follows from the facts that $R(z, A)\psi_j = (z - \lambda_j)^{-1}\psi_j$ (easily verified directly) and $S(t)\psi_j = e^{-\lambda_j t}\psi_j$ (by spectral mapping, cf. [73, Chapter IV, Theorem 3.6]).

1.3.2. FRACTIONAL POWERS OF SECTORIAL OPERATORS

As mentioned earlier, the most common application of functional calculus (appearing in this dissertation) is the definition of fractional powers A^α (where $\alpha \in \mathbb{R}$).[9] In order to define these, let A be a sectorial operator on a complex Banach space E and note that the function $f_\alpha\colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ given by $f_\alpha(z) := z^\alpha := e^{\alpha \log z}$ (taking the

[9] We only use real powers $\alpha \in \mathbb{R}$ in this dissertation, but complex powers can be defined in the same way.

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principal branch of the complex logarithm) belongs to the domain of the extended Dunford calculus, which we use to define $A^\alpha := f_\alpha(A)$. Indeed, f_α is holomorphic on the given domain, as is the family of functions $\varrho_{m,n}\colon \mathbb{C} \to \mathbb{C}$ defined by

$$\varrho_{m,n}(z) := z^m(1+z)^{-m-n}, \quad z \in \mathbb{C},$$

and indexed by $n, m \in \mathbb{N}_0$. Moreover, we have $\varrho_{m,n} f_\alpha \in \mathcal{E}(\Sigma_\omega)$ for any $\omega \in (0,\pi)$ if either $n > \alpha > 0$, or $\alpha = 0$ with $n, m \geq 1$, or $\alpha \leq 0$ with $m > -\alpha$, and the operator $\varrho_{m,n}(A) = A^m(\mathrm{Id}_E + A)^{-n-m}$ is injective if A is injective or $m = 0$. Together, we find:

- $\varrho_{0,n}$ is a regularizer for f_α if $n > \alpha > 0$;
- $\varrho_{n,n}$ is a regularizer for f_α if $\alpha \leq 0$, $n > -\alpha$ and A is injective;

and in these cases we define $A^\alpha := f_\alpha(A)$. The following theorem summarizes some basic properties of fractional powers which will be used throughout this thesis. More details and the proofs of these statements can be found in [115, Proposition 15.2.3, Theorems 15.2.5 and 15.2.7, Corollary 15.2.10].

Theorem 1.3.4. *Let $A\colon \mathsf{D}(A) \subseteq E \to E$ be a sectorial linear operator on a complex Banach space E. Its fractional powers satisfy the following properties:*

(a) *If $\alpha > 0$ and A is densely defined, then A^α is densely defined.*

(b) *For all $n \in \mathbb{N}_0$ (or $n \in \mathbb{Z}$ if A is injective), we have $f_n(A) = A^n$ with equal domains, where the right-hand side indicates A composed with itself n times.*

(c) *If A is injective, then A^α is injective for all $\alpha \in \mathbb{R}$ and $A^{-\alpha} = (A^\alpha)^{-1} = (A^{-1})^\alpha$ with equality of domains.*

(d) *If $\alpha_1 > \alpha_2 > 0$, then $\mathsf{D}(A^{\alpha_1}) \subseteq \mathsf{D}(A^{\alpha_2})$; similarly for $\alpha_2 < \alpha_1 < 0$ if A is injective.*

(e) *If $\alpha_1, \alpha_2 > 0$, then $A^{\alpha_1+\alpha_2} = A^{\alpha_1}A^{\alpha_2}$ with equality of domains; the same holds for $\alpha_1, \alpha_2 < 0$ if A is injective.*

(f) *For any $\alpha \in (0, \pi/\omega(A))$ (with the convention that $\pi/\omega(A) = \infty$ if $\omega(A) = 0$), the operator A^α is sectorial with $\omega(A^\alpha) = \alpha\omega(A)$ and for all $\beta \in (0,\infty)$ we have $(A^\alpha)^\beta = A^{\alpha\beta}$ with equality of domains.*

(g) *If A is injective and A^{-1} is bounded (that is, we have $0 \in \rho(A)$), then $A^{-\alpha}$ is bounded for all $\alpha > 0$.*

Next we present two representation formulae for certain fractional powers, proved in [115, Theorem 15.2.13 and Corollary 15.2.15]. The first of these is the *Balakrishnan representation*, which was historically the first formula in the literature for fractional powers of a general sectorial operator. We only present it for $\alpha \in (0,1)$; higher-order counterparts can be derived by splitting α into integer and fractional parts and using Theorem 1.3.4(b),(e).

Theorem 1.3.5 (Balakrishnan representation)**.** *Let $A\colon \mathsf{D}(A) \subseteq E \to E$ be a sectorial linear operator on a complex Banach space E and let $\omega \in (\omega(A), \pi)$. Then we have*

$$A^\alpha x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{\alpha-1}(t\,\mathrm{Id}_E + A)^{-1}Ax \,\mathrm{d}t \quad \text{for all } \alpha \in (0,1) \text{ and } x \in \mathsf{D}(A).$$

If, moreover, A is injective, then we also have

$$A^{-\alpha}x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha}(t\,\mathrm{Id}_E + A)^{-1} x \,\mathrm{d}t \quad \textit{for all } \alpha \in (0,1) \textit{ and } x \in \mathsf{R}(A). \tag{1.3.2}$$

An application of the Balakrishnan representation—in particular, formula (1.3.2) for negative powers—is the numerical approximation of (stochastic) PDEs of fractional order. Namely, it allows to express the solution operator of a fractional problem as an integral over integer-order problems perturbed by a real parameter. After changing variables to obtain an integral over $\mathbb{R}$, one can truncate and approximate it using a *sinc quadrature* for efficient computations; see [30, 31, 35, 50, 107].

The second representation of fractional powers applies in the case where $-A$ generates a strongly measurable and uniformly bounded semigroup. In this case, one can insert the Laplace transform representation (1.1.7) of the resolvent operators $R(t,-A) = (t\,\mathrm{Id}_E + A)^{-1}$ into Theorem 1.3.5 to derive the following theorem.

Theorem 1.3.6 (Phillips representation)**.** *Let $-A\colon \mathsf{D}(A) \subseteq E \to E$ be the generator of a strongly measurable and uniformly bounded semigroup $(S(t))_{t\ge 0}$ on a complex Banach space E. Then we have, for all $\alpha \in (0,1)$ and $x \in \mathsf{D}(A)$,*

$$A^{\alpha}x = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty t^{-\alpha} S(t)Ax \,\mathrm{d}t = \frac{1}{|\Gamma(-\alpha)|} \int_0^\infty t^{-\alpha-1}\big(x - S(t)x\big)\,\mathrm{d}t. \tag{1.3.3}$$

If, moreover, A is injective, then we also have

$$A^{-\alpha}x = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t)x \,\mathrm{d}t \quad \textit{for all } x \in \mathsf{R}(A). \tag{1.3.4}$$

The name *Phillips representation* is in reference to the analogy between (1.3.4) and the following Laplace transform identity (see [100, Lemma 3.3.4]):

$$z^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-tz} \,\mathrm{d}t \quad \text{for all } z \in \mathbb{C} \text{ with } \operatorname{Re} z \in (0,\infty).$$

Note that (1.3.2) and (1.3.4) hold for all $x \in E$ if A is boundedly invertible. If $(S(t))_{t\ge 0}$ is uniformly exponentially stable, then (1.3.4) holds without the "x" on either side, i.e., it becomes an equation in $\mathscr{L}(E)$, which in fact remains valid for all $\alpha \in (0,\infty)$.

1.3.3. FRACTIONAL INTEGRATION AND DIFFERENTIATION

Among the class of fractional powers A^{α}, the most natural concrete choices of base operators A (as well as the first historical instances) are differential operators. In this section we specialize to such operators, viewed as acting on either temporal or spatial L^p-spaces (in general one could also consider spaces of, for instance, continuous functions), and investigate the resulting formulae.

Spatiotemporal extensions of the fractional time derivatives and integrals defined in the first part of this section are central to Chapters 2–4. We subsequently consider fractional (shifted) Laplacians on $\mathbb{R}^d$, which are related to the Matérn class of Gaussian random fields considered in Example 1.4.5 below. In the last part of this section, we briefly introduce fractional-order Sobolev spaces, which are used in Chapter 2 to measure spatial regularity of solutions to fractional stochastic evolution equations.

FRACTIONAL TIME INTEGRALS AND DERIVATIVES

Consider the (weak) derivative operator ∂_t acting on $L^p(J;E)$ on an interval $J=(0,T)$ or $J=(0,\infty)$, with domain $\mathsf{D}(\partial_t)=W^{1,p}_{0,\{0\}}(J;E)$. Recall from Example 1.1.15 that $-\partial_t$ generates the uniformly bounded C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ of right translations on $L^p(J;E)$; substituting it into (1.3.4) yields

$$\partial_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s \quad \text{for a.e. } t\in J \tag{1.3.5}$$

and any $f\in L^p(J;E)$ and $\alpha\in(0,1)$. The right-hand side is known as the αth-order *Riemann–Liouville fractional integral* of f, see [179, Definition 2.1]. For positive powers $\alpha\in(0,1)$ and $f\in W^{1,p}_{0,\{0\}}(J;E)$, we obtain from (1.3.3):

$$\begin{aligned}\partial_t^{\alpha} f(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\,\partial_s f(s)\,\mathrm{d}s &(1.3.6)\\ &= \frac{1}{|\Gamma(-\alpha)|}\int_0^\infty s^{-\alpha-1}\big(f(t)-\widetilde{f}(t-s)\big)\,\mathrm{d}s; &(1.3.7)\end{aligned}$$

recall that $\widetilde{f}$ denotes the extension of f by zero. Expressions (1.3.6) and (1.3.7) are known as the *Caputo* [121, Section 2.4] and *Marchaud fractional derivatives* [179, Section 5.4], respectively. Since $\partial_t^\alpha=\partial_t\partial_t^{-(1-\alpha)}$ with equality of domains by Theorem 1.3.4(c) and (e), we can apply (1.3.5) with $1-\alpha>0$ to find

$$\partial_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}\partial_t\int_0^t (t-s)^{-\alpha} f(s)\,\mathrm{d}s$$

for all $f\in L^p(J;E)$ such that $\partial_t^{\alpha-1}f\in W^{1,p}_{0,\{0\}}(J;E)$ and a.e. $t\in J$. This is known as the *Riemann–Liouville fractional derivative*, see [179, Definition 2.2]. Note that the Caputo derivative can be derived analogously by instead using $\partial_t^\alpha=\partial_t^{\alpha-1}\partial_t$.

The fractional integrals and derivatives introduced above can be defined for functions on the entire real line by slightly modifying the formulae and the class of admissible functions. For instance, if $f\colon\mathbb{R}\to E$ is such that the integral on the right-hand side converges in some appropriate sense (see [179, Section 5.1]), then we can define its Riemann–Liouville integral as

$$\partial_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s \quad \text{for a.e. } t\in\mathbb{R}.$$

Note that the above examples of fractional (time) integrals and derivatives are all representations of the same operator ∂_t^α (valid for possibly different classes of f), which is unambiguously defined by applying the Dunford calculus to the weak derivative operator on $W^{1,p}_{0,\{0\}}(J;E)$ or $W^{1,p}(\mathbb{R};E)$. The situation is different, however, if we consider ∂_t on the domain of functions in $W^{1,p}(J;E)$ on $J\subseteq(0,\infty)$ which do not necessarily vanish at zero: Its negative no longer generates a semigroup, so the above argument does not generalize, and the commonly used extensions of, for instance, the Caputo and Riemann–Liouville fractional derivatives for functions with nonzero initial values no longer coincide owing to their different boundary terms. This leads to the question of how to naturally impose initial data in fractional (partial) differential equations, which we investigate more thoroughly in Chapter 4.

BESSEL AND RIESZ POTENTIALS; FRACTIONAL LAPLACIANS

Now we turn to multivariate functions $f\colon \mathbb{R}^d \to \mathbb{R}$, where the natural object of study becomes the (shifted) Laplace operator $\kappa^2 - \Delta$ on $L^p(\mathbb{R}^d)$ with $\kappa \in [0,\infty)$, $d \in \{2,3,\dots\}$ and $p \in [1,\infty)$. Its negative $-\kappa^2 + \Delta$ generates the contractive analytic C_0-semigroup $(e^{-\kappa^2 t}H(t))_{t\ge 0}$, recalling the heat semigroup from Example 1.1.16. Substituting its definition into the Phillips representation (1.3.4) and changing the order of integration yields, for all $s \in (0,2)$, $f \in L^p(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$(\kappa^2 - \Delta)^{-s/2} f(x) = \frac{(4\pi)^{-d/2}}{\Gamma(s/2)} \int_{\mathbb{R}^d} \left[\int_0^\infty \tau^{\frac{s-d}{2}-1} e^{-\kappa^2\tau - \frac{1}{4\tau}\|x-y\|^2_{\mathbb{R}^d}} \,\mathrm{d}\tau \right] f(y)\,\mathrm{d}y; \tag{1.3.8}$$

note that we consider $\alpha = s/2$ so that s indicates the order of the operator. The integral inside the brackets is equal to

$$\begin{aligned} 2^{d-s}\Gamma(\tfrac{d-s}{2})\|x-y\|^{s-d}_{\mathbb{R}^d} \quad & \text{if } \kappa = 0 \text{ and } s \in (0,d); \\ 2^{1+\frac{d-s}{2}}\kappa^{\frac{d-s}{2}}\|x-y\|^{(s-d)/2}_{\mathbb{R}^d} K_{(d-s)/2}(\kappa\|x-y\|_{\mathbb{R}^d}) \quad & \text{if } \kappa, s \in (0,\infty); \end{aligned} \tag{1.3.9}$$

where $K_{(d-s)/2}$ denotes a modified Bessel function of the second kind. The first case follows by the change of variables $\tau' := \frac{1}{4}\|x-y\|^2_{\mathbb{R}^d}\tau^{-1}$ and the definition [163, Equation (5.2.1)] of the gamma function; the second case is due to [161, Part I, Equation (5.34)]. Note that both relations hold for larger ranges than $s \in (0,2)$ (if $d > 2$). For $\kappa \in (0,\infty)$, the fact that (1.3.8) holds for all $s \in (0,\infty)$ could already be deduced from the uniform exponential stability of $(e^{-\kappa^2 t}H(t))_{t\ge0}$, and we obtain

$$(\kappa^2 - \Delta)^{-s/2} f(x) = \frac{\kappa^{\frac{d-s}{2}}}{2^{\frac{s}{2}-1}(2\pi)^{d/2}\Gamma(s/2)} \int_{\mathbb{R}^d} \|x-y\|^{(s-d)/2}_{\mathbb{R}^d} K_{(d-s)/2}(\kappa\|x-y\|_{\mathbb{R}^d}) f(y)\,\mathrm{d}y, \tag{1.3.10}$$

which in the case $\kappa = 1$ is known as the *Bessel potential*. For $\kappa = 0$, we can take the formula derived above as a definition of the *Riesz potential* for $s \in (0,d)$:

$$I_s f(x) := \frac{\Gamma(\frac{d-s}{2})}{2^s\pi^{d/2}\Gamma(s/2)} \int_{\mathbb{R}^d} \frac{f(y)}{\|x-y\|^{d-s}_{\mathbb{R}^d}}\,\mathrm{d}y. \tag{1.3.11}$$

This can be viewed as a multi-dimensional analog to the Riemann–Liouville fractional integral. One can derive an expression similar to (1.3.11) for the positive-order fractional Laplacian $(-\Delta)^{s/2}$ by formally substituting the heat semigroup into (1.3.3), interchanging integrals and evaluating the inner integral with analog to (1.3.9). However, in this case the interchange of integrals cannot be rigorously justified, and this is reflected in the fact that the resulting singular integral formula must be interpreted in the *principal value* sense, i.e.,

$$(-\Delta)^{s/2} f(x) := \frac{\Gamma(\frac{d+s}{2})}{2^{-s}\pi^{d/2}|\Gamma(-s/2)|} \lim_{r\downarrow 0} \int_{\mathbb{R}^d\setminus B_r(x)} \frac{f(x)-f(y)}{\|x-y\|^{d+s}_{\mathbb{R}^d}}\,\mathrm{d}y, \quad x \in \mathbb{R}^d.$$

Lastly, we mention that one can define the sth fractional power of $\kappa^2 - \Delta$ on $\mathbb{R}^d$ as a Fourier multiplier operator with the symbol $\xi \mapsto (\kappa^2 + \|\xi\|^2_{\mathbb{R}^d})^{s/2}$. To see this, note that the convolution kernels in (1.3.10) and (1.3.11) are the inverse Fourier transforms of $\xi \mapsto (\kappa^2 + \|\xi\|^2_{\mathbb{R}^d})^{s/2}$ and $\xi \mapsto \|\xi\|^s_{\mathbb{R}^d}$ respectively, see [186, Chapter V, Sections 1 and 3].

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FRACTIONAL SOBOLEV SPACES

Given $p \in [1,\infty)$ and an open set $\mathcal{D} \subseteq \mathbb{R}^d$, recall that $W^{1,p}(\mathcal{D})$ denotes the first-order Sobolev space of functions in $L^p(\mathcal{D})$ whose weak partial derivatives all belong to $L^p(\mathcal{D})$ as well. For $s \in (0,1)$, we define the *Gagliardo seminorm*

$$[u]_{W^{s,p}(\mathcal{D})} := \left(\int_{\mathcal{D}} \int_{\mathcal{D}} \frac{|u(x)-u(y)|^p}{\|x-y\|_{\mathbb{R}^d}^{d+sp}} \,\mathrm{d}x\,\mathrm{d}y \right)^{1/p},$$

along with the *Sobolev–Slobodeckij space* $W^{s,p}(\mathcal{D}) := \{u \in L^p(\mathcal{D}) : [u]_{W^{s,p}(\mathcal{D})} < \infty\}$, which we equip with the norm $\|u\|_{W^{s,p}(\mathcal{D})} := (\|u\|_{L^p(\mathcal{D})}^p + [u]_{W^{s,p}(\mathcal{D})}^p)^{1/p}$ to render it a Banach space. Then for any $s \in (0,1]$ we can define $W^{s+1,p}(\mathcal{D})$ as the space of functions in $W^{1,p}(\mathcal{D})$ whose partial derivatives belong to $W^{s,p}(\mathcal{D})$, and recursively repeat this process to extend the definition of $W^{s,p}(\mathcal{D})$ to any $s \in (0,\infty)$. In the case $p=2$ we use the special notation $H^s(\mathcal{D}) := W^{s,2}(\mathcal{D})$.

If $\mathcal{D} = \mathbb{R}^d$, $p \in (1,\infty)$ and $s \in (0,\infty)$, we can also define the *Bessel potential space* $H^{s,p}(\mathbb{R}^d)$ of functions u for which $\|u\|_{H^{s,p}(\mathbb{R}^d)} := \|(1-\Delta)^{s/2}u\|_{L^p(\mathbb{R}^d)}$ is finite (the latter operator being defined as a Fourier multiplier). The notation is consistent since, for $p=2$, it holds that $H^{s,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d) := W^{s,2}(\mathbb{R}^d)$, see parts (v) and (vii) of the proposition in [178, Section 2.1.2]. This observation raises the question if there is a relation between $H^s(\mathcal{D})$ and the fractional domain spaces $(\mathsf{D}(L^{s/2}), \|L^{s/2}\,\cdot\,\|_{L^2(\mathcal{D})})$ associated to more general elliptic second-order differential operators L on $\mathcal{D} \subseteq \mathbb{R}^d$. The answer depends on the regularity of the coefficient functions and the smoothness of the boundary $\partial\mathcal{D}$; see Lemma 2.5.5 below for some results in this direction.

1.4. STOCHASTIC PROCESSES AND RANDOM FIELDS

The final class of prerequisites for this dissertation belongs to the realm of probability theory. Chapters 2, 3 and 5 are all concerned with *stochastic* evolution equations in space and time, whose solutions can be viewed as spatiotemporal random fields or (function space valued) stochastic processes. In particular, Chapters 2 and 3 deal with linear stochastic partial differential equations (SPDEs) whose solutions are Gaussian random fields (GRFs). Thus, in Sections 1.4.1 and 1.4.2 we describe GRFs and how they arise from linear SPDEs, respectively. In Section 1.4.3 we instead focus on (vector-valued) stochastic *processes*, including important examples such as Wiener processes, and describe how to define stochastic integrals with respect to the latter. This provides a way of defining *mild solutions* to stochastic evolution equations, which is the main solution concept used throughout Chapters 2, 3 and 5. In this last section we also briefly mention the Markov property, which is the subject of Chapter 3 (and explained in more detail there).

Let $(\Omega,\mathcal{F},\mathbb{P})$ denote a complete probability space (meaning that $\mathcal{F}$ contains the collection $\mathcal{N}_{\mathbb{P}}$ of $\mathbb{P}$-null sets), and let $(E,\|\cdot\|_E)$ be a separable Banach space over the real scalar field. A mapping $Z\colon \Omega \to E$ is said to be an *E-valued random variable* if it is (strongly) measurable. Note that since E is separable, the notions of strong measurability (in terms of a.e. approximation by simple functions) and "normal" measurability (in terms of pre-images) coincide, see [113, Corollary 1.1.10]. If $Z \in L^1(\Omega; E)$, then $\mathbb{E}[Z] := \int_\Omega Z(\omega)\,\mathrm{d}\mathbb{P}(\omega)$ denotes its *expectation* (or *expected value*).

A collection $(X(x))_{x\in\mathcal{I}}$ of E-valued random variables indexed by an arbitrary set $\mathcal{I}$ is said to be an E-valued *random field*. We call a random field *spatial* if $\mathcal{I} := \mathcal{D} \subseteq \mathbb{R}^d$ is a (Euclidean or curved) domain, and *spatiotemporal* if $\mathcal{I} := J \times \mathcal{D}$ for some (time) interval $J \subseteq \mathbb{R}$. A collection $(X(t))_{t\in J}$ indexed by J is called a *stochastic process*. The mappings $\mathcal{I} \ni x \mapsto [X(x)](\omega) \in E$ for $\omega \in \Omega$ are called the *trajectories* or *sample paths* of $(X(x))_{x\in\mathcal{I}}$. If the index set $\mathcal{I}$ is a measurable space $(S, \mathscr{A})$, as is the case for J, $\mathcal{D}$ and $J \times \mathcal{D}$, then $(X(x))_{x\in S}$ is said to be *measurable* if $(\omega, x) \mapsto [X(x)](\omega)$ is measurable with respect to the product σ-algebra $\mathcal{F} \otimes \mathscr{A}$ (as usual, E is equipped with its Borel σ-algebra $\mathcal{B}(E)$). Although we will only encounter real-valued random fields in this thesis, we shall frequently use stochastic processes taking values in function spaces such as $E := L^q(\mathcal{D})$ for $q \in [2,\infty)$ or $E := C(\mathcal{D})$ to represent spatiotemporal fields.

1.4.1. GAUSSIAN RANDOM PROCESSES AND FIELDS: TWO VIEWPOINTS

In this section we introduce two different but related definitions of this notion.

Definition 1.4.1. A random field (or process) $(X(x))_{x\in\mathcal{I}}$ indexed by a set $\mathcal{I}$ (most often J, $\mathcal{D}$ or $J\times\mathcal{D}$) is said to be *Gaussian* if its finite subcollections are Gaussian. I.e., for $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathcal{I}$, the $\mathbb{R}^n$-valued random vector $(u(x_i))_{i=1}^n$ is multivariate normal. Its mean vector $\mathbf{m} = (m_j)_{j=1}^n \in \mathbb{R}^n$ and covariance matrix $\mathbf{Q} = (Q_{ij})_{i,j=1}^n$ satisfy $m_j = \mu(x_j)$ and $Q_{ij} = \varrho(x_i, x_j)$, respectively, where $\mu\colon \mathcal{I} \to \mathbb{R}$ and $\varrho\colon \mathcal{I}\times\mathcal{I} \to \mathbb{R}$ are defined by

$$\mu(x) := \mathbb{E}[X(x)] \quad \text{and} \quad \varrho(x,y) := \mathbb{E}[(X(x)-\mu(x))(X(y)-\mu(y))], \quad x, y \in \mathcal{I}. \tag{1.4.1}$$

The *mean function* μ and *covariance function* ϱ fully characterize a Gaussian random field. It is immediate from (1.4.1) that a covariance function ϱ is necessarily symmetric and non-negative definite, i.e., $\varrho(x,y) = \varrho(y,x)$ and $\varrho(x,x) \ge 0$ for all $x, y \in \mathcal{I}$. Conversely, by Kolmogorov's extension theorem [172, Chapter 1, Theorem 3.2], we can associate a random field on indexed by $\mathcal{I}$ with any $\mu\colon \mathcal{I} \to \mathbb{R}$ and symmetric non-negative definite $\varrho\colon \mathcal{I}\times\mathcal{I} \to \mathbb{R}$.

A Gaussian random field with $\mu \equiv 0$ is called *centered* or *mean-zero*. Since any field can be made centered by subtracting its mean, we will often restrict ourselves to mean-zero Gaussian random fields for notational convenience.

Under some further regularity conditions, a Gaussian random field can alternatively be viewed as a Gaussian random variable taking its values in a function space. To illustrate this, consider a measurable centered Gaussian random field $(X(x))_{x\in\mathcal{D}}$ with square-integrable trajectories on a bounded domain $\mathcal{D} \subsetneq \mathbb{R}^d$. In this case, the family $(\langle X, u\rangle_{L^2(\mathcal{D})})_{u\in L^2(\mathcal{D})}$ is also a Gaussian random field, cf. [27, Example 2.3.16]. As such, it is fully characterized by the values of

$$\mathbb{E}\big[\langle X, u\rangle_{L^2(\mathcal{D})} \langle X, v\rangle_{L^2(\mathcal{D})}\big] \quad \text{for all } u, v \in L^2(\mathcal{D}).$$

If, in addition,

$$x \mapsto \varrho(x,x) \quad \text{belongs to } L^1(\mathcal{D}), \tag{1.4.2}$$

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then by Fubini's theorem we have

$$\begin{aligned}\mathbb{E}[\langle X,u\rangle_{L^2(\mathcal{D})}\langle X,v\rangle_{L^2(\mathcal{D})}] &= \mathbb{E}\left[\left(\int_{\mathcal{D}} X(y)u(y)\,\mathrm{d}y\right)\left(\int_{\mathcal{D}} X(x)v(x)\,\mathrm{d}x\right)\right] \\ &= \int_{\mathcal{D}}\left(\int_{\mathcal{D}} \mathbb{E}[X(y)X(x)]u(y)\,\mathrm{d}y\right)v(x)\,\mathrm{d}x = \langle \mathcal{C}u,v\rangle_{L^2(\mathcal{D})},\end{aligned} \tag{1.4.3}$$

where the *covariance operator* $\mathcal{C}\colon L^2(\mathcal{D})\to L^2(\mathcal{D})$ is given by

$$\mathcal{C}u(x) := \int_{\mathcal{D}} \varrho(x,y)u(y)\,\mathrm{d}y, \quad u\in L^2(\mathcal{D}), \text{ a.e. } x\in\mathcal{D}. \tag{1.4.4}$$

In other words, the covariance operator is an integral operator on $L^2(\mathcal{D})$ with integration kernel $\varrho\colon \mathcal{D}\times\mathcal{D}\to\mathbb{R}$. For this reason, ϱ is sometimes referred to as the *covariance kernel* of $(X(x))_{x\in\mathcal{D}}$. We see from (1.4.3) that $\mathcal{C}$ is symmetric (hence bounded by the Hellinger–Toeplitz theorem) and non-negative definite, the latter meaning $\langle \mathcal{C}u,u\rangle_{L^2(\mathcal{D})}\ge 0$ for all $u\in L^2(\mathcal{D})$. By (1.4.2), we in fact have $\operatorname{tr}\mathcal{C}=\int_{\mathcal{D}}\varrho(x,x)\,\mathrm{d}x<\infty$, where the *trace* of an operator $T\in\mathscr{L}(H)$ on a separable Hilbert space H is defined by $\operatorname{tr}T := \sum_{j=1}^{\infty}\langle Te_j,e_j\rangle_H$ for any orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of H. We will see that the random field X, when viewed as an $L^2(\mathcal{D})$-valued random variable, is *Gaussian* in the sense of the following definition.

Definition 1.4.2. A measure μ on a real and separable Hilbert space H (equipped with the Borel σ-algebra $\mathcal{B}(H)$) is said to be *Gaussian* if, for all $h\in H$, the measure $B\mapsto\mu(\langle\cdot,x\rangle_H\in B)$ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ is Gaussian. A random variable $Z\colon\Omega\to H$ is said to be Gaussian if its *law* $\mathbb{P}\circ Z^{-1}$ is a Gaussian measure on H.

Gaussian measures on real separable Hilbert spaces H admit the following characterization in terms of their Fourier transforms, defined for any measure μ by

$$\widehat{\mu}\colon H\to\mathbb{C}, \qquad \widehat{\mu}(h) := \int_H \exp\big(i\langle x,h\rangle_H\big)\,\mathrm{d}\mu(x), \quad h\in H.$$

Theorem 1.4.3 ([27, Theorem 2.3.1])**.** *A measure μ on a real and separable Hilbert space $(H,\mathcal{B}(H))$ is Gaussian if and only if there exist $m\in H$ and a symmetric, non-negative definite operator $Q\in\mathscr{L}(H)$ with finite trace such that*

$$\widehat{\mu}(h) = \exp\big(i\langle m,h\rangle_H - \tfrac{1}{2}\langle Qh,h\rangle_H\big) \quad \textit{for all } h\in H.$$

In this case, we write $X\sim N(m,Q)$ and it holds for all $h,u\in H$ that

$$\langle m,h\rangle_H = \int_H \langle x,h\rangle_H\,\mathrm{d}\mu(x) \quad \textit{and} \quad \langle Qh,u\rangle_H = \int_H \langle x-m,h\rangle_H\langle x-m,u\rangle_H\,\mathrm{d}\mu(x).$$

Before returning to random fields, we note that in the course of the proof of [27, Theorem 2.3.1] it is observed that any Gaussian random variable $Z\colon\Omega\to H$ with mean m and covariance Q can also be expanded as a series. To this end, we note that since Q is a non-negative operator with finite trace, there exists an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ and a sequence of eigenvalues $(\lambda_j)_{j\in\mathbb{N}}\subseteq[0,\infty)$ such that $Qe_j=\lambda_j e_j$ for all $j \in \mathbb{N}$ and $\operatorname{tr} Q = \sum_{j=1}^{\infty} \lambda_j < \infty$. Given a sequence $(\xi_j)_{j\in\mathbb{N}}$ of independent standard normal (real-valued) random variables on Ω, we have the following *Karhunen–Loève expansion* of Z:

$$Z = m + \sum_{j=1}^{\infty} \lambda_j^{1/2} \xi_j e_j; \tag{1.4.5}$$

this series converges $\mathbb{P}$-a.s. and in $L^2(\Omega; H)$.

Now we turn back to the Gaussian random field $(X(x))_{x\in\mathcal{D}}$ viewed as a random variable $X\colon \Omega \to L^2(\mathcal{D})$. The Fourier transform of its law satisfies

$$\widehat{\mathbb{P}\circ X^{-1}}(u) = \mathbb{E}\big[\exp\big(i\langle X, u\rangle_{L^2(\mathcal{D})}\big)\big] = \exp\big(-\tfrac{1}{2}\mathbb{E}\big[\langle X, u\rangle_{L^2(\mathcal{D})}^2\big]\big) = \exp\big(-\tfrac{1}{2}\langle \mathcal{C}u, u\rangle_{L^2(\mathcal{D})}\big)$$

for all $u \in L^2(\mathcal{D})$. Here, we used the change of variables formula for Lebesgue integrals, the Gaussianity of the real-valued random variable $\langle X, u\rangle_{L^2(\mathcal{D})}$, and (1.4.3), respectively. Thus, by Definition 1.4.2 and Theorem 1.4.3, it follows that X is a Gaussian $L^2(\mathcal{D})$-valued random variable with $m = 0$ and $Q = \mathcal{C}$. Conversely, we can associate a measurable Gaussian random field $(X(x))_{x\in S}$ to any centered Gaussian measure on $L^2(S)$ for some measure space $(S, \mathscr{A}, \nu)$, see [27, Example 3.5.12].

1.4.2. GAUSSIAN WHITE NOISE AND LINEAR SPDES

The Gaussian random fields from Chapters 2 and 3 are all given by solutions to linear SPDEs driven by Gaussian noise. In this section, we focus on the relation between such fields and equations. To this end, we first define *Gaussian white noise* $\mathscr{W}$ on domains $(S, \mathscr{A})$ such as $J \subseteq \mathbb{T}$ (temporal), $\mathcal{D} \subseteq \mathbb{R}^d$ (spatial) and $J \times \mathcal{D}$ (spatiotemporal), and subsequently indicate how one can interpret equations of the form $LX = \mathscr{W}$.

Intuitively, Gaussian white noise refers to a completely uncorrelated Gaussian random field. Because a multivariate Gaussian random variable is uncorrelated if its covariance matrix is diagonal (say the identity matrix, in order to normalize the variance), it is tempting given the discussion from the previous section to define Gaussian white noise $\mathscr{W}$ as an $L^2(S)$-valued mean-zero Gaussian random variable with covariance operator $\mathrm{Id}_{L^2(S)}$. However, since the identity operator on an infinite-dimensional Hilbert space has infinite trace, it does not define a proper $L^2(S)$-valued Gaussian random variable for S as above in view of Theorem 1.4.3. Despite this, the left- and right-hand sides of (1.4.3) are still meaningful for $\mathcal{C} = \mathrm{Id}_{L^2(\mathcal{D})}$, leading us to the following definition.

Definition 1.4.4. Let H be a real and separable Hilbert space. A centered real-valued Gaussian random field $(\mathscr{W}(h))_{h\in H}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be an *H-isonormal process* if $\mathbb{E}[\mathscr{W}(u)\mathscr{W}(h)] = \langle u, h\rangle_H$ for all $u, h \in H$.

An $L^2(S)$-isonormal process is said to be *spatial*, *temporal* or *spatiotemporal white noise* if $S = \mathcal{D}$, $S = J$ or $S = J \times \mathcal{D}$, respectively.

Let us briefly comment on the other two informal ways to interpret white noise suggested by the previous section, and the obstacles to making them rigorous. If we wish to view white noise as a Gaussian random field in the sense of Definition 1.4.1, then its kernel would need to be the Dirac delta $\varrho(x, y) := \delta(x - y)$ in order to obtain

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$\mathcal{C} = \mathrm{Id}_{L^2(S)}$ in (1.4.4), which is not a well-defined function. On the other hand, if we want to consider Gaussian white noise as a formal Karhunen–Loève series expansion, then by (1.4.5) we would have $\mathscr{W} = \sum_{j=1}^{\infty} \xi_j e_j$, where $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(S)$ and $(\xi_j)_{j\in\mathbb{N}}$ are independent standard normal random variables. This series does not converge $\mathbb{P}$-a.s. or in $L^2(\Omega; L^2(S))$. However, if $S := \mathcal{D} \subsetneq \mathbb{R}^d$ is bounded with sufficiently smooth boundary, then the series can be shown to converge in a larger space in which $L^2(\mathcal{D})$ has a Hilbert–Schmidt[10] embedding (i.e., a negative-order fractional Sobolev space; see for instance [30, Proposition 2.3]).

Given an H-isonormal process on a real separable Hilbert space H and an (unbounded) linear operator $L\colon \mathsf{D}(L) \subseteq H \to H$ admitting a bounded inverse L^{-1}, we can consider the linear stochastic partial differential equation

$$LX = \mathscr{W}. \tag{1.4.6}$$

For such X, we can formally compute that

$$\langle X, h\rangle_H = \langle L^{-1}LX, h\rangle_H = \langle LX, (L^{-1})^* h\rangle_H = \langle \mathscr{W}, (L^{-1})^* h\rangle_H \quad \text{for all } h \in H,$$

which implies, by the definition of H-isonormal processes,

$$\mathbb{E}[\langle X, u\rangle_H \langle X, h\rangle_H] = \langle (L^{-1})^* u, (L^{-1})^* h\rangle_H = \langle L^{-1}(L^{-1})^* u, h\rangle_H \quad \text{for all } u, h \in H.$$

This motivates us to interpret a centered Gaussian random field X with covariance operator $Q = L^{-1}(L^{-1})^*$ as a generalized solution to (1.4.6). However, in order for this to be meaningful one needs that $L^{-1}(L^{-1})^*$ has finite trace, which is equivalent to L^{-1} being a Hilbert–Schmidt operator. If L^{-1} is merely bounded, then we may still formally consider it as the "covariance operator" of a Gaussian random field in a generalized sense. This is seen in the next example, which forms a crucial part of the background of Chapters 2 and 3 (see the respective introductory sections for more information).

Example 1.4.5 (Matérn Gaussian random fields)**.** Given $\nu, \kappa, \sigma^2 \in (0,\infty)$, let us define $k_{\mathrm{Mat}}\colon [0,\infty) \to (0,\infty)$ by setting $k_{\mathrm{Mat}}(0) = \sigma^2$ and

$$k_{\mathrm{Mat}}(r) := \frac{\sigma^2}{2^{\nu-1}\Gamma(\nu)} (\kappa r)^{\nu} K_{\nu}(\kappa r), \quad r \in (0,\infty),$$

where we recall that K_ν denotes a modified Bessel function of the second kind. Note that the normalizing prefactor is chosen such that $k_{\mathrm{Mat}}(r) \to \sigma^2$ as $r \to 0$. Given a domain $\mathcal{D} \subseteq \mathbb{R}^d$, the symmetric and positive definite function $\varrho_{\mathrm{Mat}}\colon \mathcal{D}\times\mathcal{D} \to (0,\infty)$ defined by $\varrho_{\mathrm{Mat}}(x,y) := k_{\mathrm{Mat}}(\|x-y\|_{\mathbb{R}^d})$ for $x, y \in \mathcal{D}$ is known as a *Matérn covariance function*, giving rise to a *Matérn Gaussian random field* $(X(x))_{x\in\mathcal{D}}$. This three-parameter class of covariance functions, introduced by Matérn in [149], is widely used in spatial statistics [187].

[10] An operator $T \in \mathscr{L}(H;K)$ between separable Hilbert spaces H and K is said to be *Hilbert–Schmidt* if $\sum_{j\in\mathbb{N}} \|Te_j\|_K^2 < \infty$ for any orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of H. The space of Hilbert–Schmidt operators $\mathscr{L}_2(H;K)$ is a Hilbert space with respect to the inner product $\langle T, S\rangle_{\mathscr{L}_2(U;H)} := \sum_{j\in\mathbb{N}} \langle Te_j, Se_j\rangle_H$, which can be shown to be independent of the choice of orthonormal basis.

If $\mathcal{D} \subsetneq \mathbb{R}^d$ is bounded, then by the discussion above we have that X can be seen as a $L^2(\mathcal{D})$-valued Gaussian random variable with covariance operator

$$\mathcal{C}u(x) = \frac{\sigma^2}{2^{\nu-1}\Gamma(\nu)} \int_{\mathcal{D}} (\kappa\|x-y\|_{\mathbb{R}^d})^\nu K_\nu(\kappa\|x-y\|_{\mathbb{R}^d}) u(y)\,\mathrm{d}y. \tag{1.4.7}$$

If $\mathcal{D} = \mathbb{R}^d$, then we can still speak of a Matérn Gaussian random field $(X(x))_{x\in\mathbb{R}^d}$ with covariance kernel ϱ_{Mat}. Substituting $\mathcal{D} = \mathbb{R}^d$ into (1.4.7) would yield the operator

$$\mathcal{C} = \frac{\Gamma(\nu + d/2)}{\Gamma(\nu)} (4\pi)^{d/2} \kappa^{2\nu} \sigma^2 (\kappa^2 - \Delta)^{-(2\nu+d)} \in \mathscr{L}(L^2(\mathbb{R}^d))$$

(by comparison with formula (1.3.10) for the Bessel potentials), which does not have finite trace (for instance because its spectrum is continuous, meaning that it cannot be compact). However, this does show that the unique *stationary* solution (meaning that its covariance kernel $\varrho(x, y)$ depends only on $\|x - y\|_{\mathbb{R}^d}$) to

$$(\kappa^2 - \Delta)^\beta X = \mathscr{W} \quad \text{on } \mathbb{R}^d, \tag{1.4.8}$$

is a Matérn Gaussian random field with smoothness parameter $\nu = 2\beta - d/2$ whenever $\beta > d/4$. The connection between the Matérn class and SPDEs was first observed by Whittle [196]. If we instead consider (1.4.8) with the (e.g. Dirichlet or Neumann) Laplacian on a bounded domain $\mathcal{D} \subsetneq \mathbb{R}^d$, then its solution is called a *Whittle–Matérn field*. The solution to (1.4.8) with $(\kappa^2 - \Delta)^\beta$ replaced by L^β for some general elliptic differential operator L on a Euclidean or curved domain (see for instance Section 1.2) is said to be a *generalized* Whittle–Matérn field.

1.4.3. STOCHASTIC PROCESSES, FILTRATIONS AND INTEGRATION

In this subsection we focus on (E-valued) stochastic *processes* $(X(t))_{t\in J}$, i.e., the case where the index set $J \subseteq \mathbb{R}$ is an interval (interpreted as time), and on the stochastic integration of such processes. These notions are used to define mild solutions to stochastic evolution equations in Chapters 2, 3 and 5 (note that the equations are allowed to be nonlinear in the last chapter).

Suppose that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is *filtered*, meaning that there exists a *filtration* $(\mathcal{F}_t)_{t\in J}$, which is a family of σ-algebras satisfying $\mathcal{F}_t \subseteq \mathcal{F}_s \subseteq \mathcal{F}$ for all $s, t \in J$ such that $t \le s$. Moreover, we will assume that this filtration is *complete* and *right-continuous*, by which we mean that $\mathcal{N}_\mathbb{P} \subseteq \mathcal{F}_t$ and $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for every $t \in J$, respectively. We say that a stochastic process $(X(t))_{t\in J}$ is *adapted to* $(\mathcal{F}_t)_{t\in J}$ if $X(t)$ is $\mathcal{F}_t$-measurable for each $t \in J$; it is said to be *predictable* if it is (strongly) measurable with respect to the *predictable σ-algebra*

$$\mathcal{P}_{J\times\Omega} := \sigma((s, t] \times F_s : s, t \in J \text{ such that } s < t \text{ and } F_s \in \mathcal{F}_s). \tag{1.4.9}$$

Here, $\sigma(\mathscr{B})$ denotes the σ-algebra generated by a family of sets $\mathscr{B}$. Given an indexed family $(\mathscr{B}_j)_{j\in\mathcal{I}}$ we define $\bigvee_{j\in\mathcal{I}} \mathscr{B}_j := \sigma(\bigcup_{j\in\mathcal{I}} \mathscr{B}_j)$. The *filtration generated by a random variable* $Z\colon \Omega \to E$ is given by $\sigma(Z) := \{\{Z \in B\} : B \in \mathcal{B}(E)\}$, and the *filtration generated by a stochastic process* $(X(t))_{t\in J}$ is denoted by $\mathcal{F}_t^X := \sigma(X(t) : t \in J)$. Two

σ-algebras $\mathscr{A}_1$, $\mathscr{A}_2$ are independent (denoted $\mathscr{A}_1 \perp\!\!\!\perp \mathscr{A}_2$) if $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$ whenever $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$. In case $\mathscr{A}_1 = \sigma(Z)$ for a random variable Z, we simply write $Z \perp\!\!\!\perp \mathscr{A}_2$. The processes $(X(t))_{t\in J}$ and $(\widetilde{X}(t))_{t\in J}$ are *modifications* if $\mathbb{P}(X(t) = \widetilde{X}(t)) = 1$ for all $t \in J$, and *indistinguishable* if $\mathbb{P}(X(t) = \widetilde{X}(t)$ for all $t \in J) = 1$.

REAL-VALUED PROCESSES; (FRACTIONAL) BROWNIAN MOTION

One of the most important real-valued stochastic processes is the following:

Definition 1.4.6 (Brownian motion)**.** A real-valued Gaussian process $(B(t))_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a *Brownian motion* if it is centered, its trajectories are continuous and we have and $\mathbb{E}[B(t)B(s)] = s \wedge t$ for all $s, t \geq 0$.

Given a filtration $(\mathcal{F}_t)_{t\geq 0}$, we say that $(B(t))_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-*Brownian motion* if it is $(\mathcal{F}_t)_{t\geq 0}$-adapted and its increments are *independent with respect to* $(\mathcal{F}_t)_{t\geq 0}$, meaning that $B(t) - B(s) \perp\!\!\!\perp \mathcal{F}_s$ for all $t \geq s \geq 0$.

Brownian motion can be constructed from an $L^2(0,\infty)$-isonormal process $\mathscr{W}$ as follows. Firstly, it is easy to check that $B(t) := \mathscr{W}(\mathbf{1}_{[0,t]})$ yields $\mathbb{E}[B(t)B(s)] = s \wedge t$ for all $s, t \geq 0$. From this, it can be derived that $\mathbb{E}[(B(t) - B(s))^2] = |t - s|$, so that $(B(t))_{t\geq 0}$ has a modification whose trajectories are (Hölder) continuous by the Kolmogorov–Chentsov continuity theorem [138, Theorem 2.9].

Another important property of Brownian motion is the fact that it is a Markov process. Intuitively, this means that its "past" and "future" are independent of each other given knowledge of the "present". In the simplest and most common definition of the Markov property (the *simple Markov property*), the past, present and future relative to a time $t \in J$ are represented by the σ-algebras $\mathcal{F}_t$, $\sigma(B(t))$ and $\sigma(B(s) : s > t)$, respectively. However, there also exist more general definitions, some of which are explored in Section 3.3.

The following definition, first introduced by Mandelbrot and Van Ness [147], entails a non-Markovian generalization of Brownian motion.

Definition 1.4.7 (Fractional Brownian motion)**.** A real-valued Gaussian stochastic process $(B_H(t))_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a *fractional Brownian motion* with *Hurst parameter* $H \in (0,1)$ if it is centered, its trajectories are continuous and we have $\mathbb{E}[B(t)B(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$ for all $s, t \geq 0$.

Note that for $H = \frac{1}{2}$, fractional Brownian motion reduces to regular Brownian motion. Next, we claim that a fractional Brownian motion can be expressed in terms of a *stochastic integral* with respect to a (non-fractional) *two-sided Brownian motion*. Such a process $(B(t))_{t\in\mathbb{R}}$ is constructed from two independent Brownian motions $(B_1(t))_{t\geq 0}$ and $(B_2(t))_{t\geq 0}$ by setting $B(t) := B_1(t)$ for $t \in [0,\infty)$ and $B(t) := B_2(-t)$ for $t \in (-\infty, 0)$, and one can integrate functions $f \in L^2(\mathbb{R})$ with respect to it as follows: First define, for elementary functions of the form $f(t) = \sum_{j=1}^n \mathbf{1}_{(a_j,b_j]}(t)x_j$ with $n \in \mathbb{N}$, $x_j \in \mathbb{R}$ and $-\infty < a_j < b_j < \infty$ for all $j \in \{1,\ldots,n\}$,

$$\int_{\mathbb{R}} f(t)\,\mathrm{d}B(t) := \sum_{j=1}^n \big[B(b_j) - B(a_j)\big]x_j.$$

Then $\int_{\mathbb{R}} f(t)\,\mathrm{d}B(t)$ is a well-defined element of $L^2(\Omega)$ and we have the *Itô isometry*

$$\left\| \int_{\mathbb{R}} f(t)\,\mathrm{d}B(t) \right\|_{L^2(\Omega)} = \|f\|_{L^2(\mathbb{R})}, \tag{1.4.10}$$

which allows us to define $\int_{\mathbb{R}} f(t)\,\mathrm{d}B(t) \in L^2(\Omega)$ for *any* $f \in L^2(\mathbb{R})$ by density. In particular, any function $k\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that $k(t,\cdot) \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$ can be used as an integration kernel to define a process

$$X(t) := \int_{\mathbb{R}} k(t,r)\,\mathrm{d}B(r), \quad t \in \mathbb{R}. \tag{1.4.11}$$

Then $(X(t))_{t\in\mathbb{R}}$ is easily seen to be centered and Gaussian. Moreover, combining the Itô isometry (1.4.10) with the polarization identity yields the covariance kernel

$$\mathbb{E}[X(t)X(s)] = \langle X(t), X(s)\rangle_{L^2(\Omega)} = \langle k(t,\cdot), k(s,\cdot)\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} k(t,r)k(s,r)\,\mathrm{d}r$$

for all $s,t \in J$. Therefore, its covariance operator is given by

$$\mathcal{C}f(t) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} k(t,r)k(s,r)\,\mathrm{d}r \right) f(s)\,\mathrm{d}s,$$

for which one easily checks $\mathcal{C} = T_k T_k^*$, where $T_k \in \mathscr{L}(L^2(\mathbb{R}))$ denotes the integral operator $T_k f(t) := \int_{\mathbb{R}} k(t,r)f(r)\,\mathrm{d}r$ associated to k. In particular, if $T_k = L^{-1}$ is the inverse of some linear operator $L\colon \mathsf{D}(L) \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$, then the process defined by (1.4.11) can be seen as a solution to $LX = \mathscr{W}$ on $\mathbb{R}$.

It was already shown in the original work of Mandelbrot and Van Ness [147] that, indeed, there exists a kernel $k := K_H$ such that (1.4.11) yields $X = B_H$; see (3.5.2) and the rest of Section 3.5.1 for more details. An alternative to their definition is the *Riemann–Liouville fractional Brownian motion* $(B_H^{\mathrm{RL}}(t))_{t\ge 0}$, given by

$$B_H^{\mathrm{RL}}(t) := \frac{1}{\Gamma(H+1/2)} \int_0^t (t-r)^{H-1/2}\,\mathrm{d}B(r), \quad H \in (0,1),\ t \in [0,\infty).$$

This is a restriction to $[0,\infty)$ of the process (1.4.11) with $k(t,s) := \frac{1}{\Gamma(H+1/2)}(t-r)^{H-1/2}$ if $t \ge r \ge 0$ and $k(t,s) := 0$ otherwise. In view of the preceding discussion and the results of Section 1.3.3, this process can be seen as a solution to the (time-fractional) SPDE

$$\begin{cases} \partial_t^{H+1/2} X = \mathscr{W} & \text{on } (0,\infty); \\ X(0) = 0. \end{cases} \tag{1.4.12}$$

In particular, we can view a regular Brownian motion B as a solution to $\partial_t X = \mathscr{W}$, i.e., as integrated white noise.

STOCHASTIC INTEGRATION WITH RESPECT TO CYLINDRICAL WIENER PROCESSES

Finally, given a real and separable Hilbert space H, we describe an H-valued analog to Brownian motion and a class of operator-valued functions which can be integrated with respect to it. Namely, for interval $J = (0,T)$ with $T \in (0,\infty]$ we introduce the notion of a *cylindrical Wiener process* $(W(t))_{t\in J}$. If $H = L^2(\mathcal{D})$ for $\mathcal{D} \subseteq \mathbb{R}^d$, then the formal time derivative $(\dot{W}(t))_{t\in J}$ of such a process is interpreted as spatiotemporal white noise. In particular, for any $t \in J$ we wish to interpret $W(t)$ as white noise in space; as discussed in Section 1.4.2, this means that $(W(t))_{t\in J}$ cannot be a proper H-valued stochastic process. One way of circumventing this issue is to first define Q-*Wiener processes* $(W^Q(t))_{t\in J}$ for $Q \in \mathscr{L}(H)$ with finite trace, whose increments $W^Q(t) - W^Q(s)$ are H-valued centered Gaussian random variables with covariance Q, and subsequently defining a cylindrical Wiener process as an $\widetilde{H}$-valued $\widetilde{Q}$-Wiener process for which there exists a Hilbert–Schmidt embedding $\iota\colon H \hookrightarrow \widetilde{H}$ and $\widetilde{Q} := \iota\iota^*$, see [144, Section 2.5].

For the purposes of this introduction we take the alternative, more direct route of identifying a cylindrical Wiener process W with an $L^2(J;H)$-isonormal process $(\mathscr{W}(h))_{h\in L^2(J;H)}$, see [157, Definition 2.2]. Let a real Banach space E be given and consider an elementary operator-valued function $\Phi\colon J \to \mathscr{L}(H;E)$ of the form

$$\Phi(t) = \sum_{j=1}^{n} \mathbf{1}_{(a_j,b_j]}(t) h_j \otimes x_j$$

where $n \in \mathbb{N}$, $-\infty < a_j < b_j < \infty$ and $(h_j, x_j) \in H \times E$ for all $j \in \{1,\dots,n\}$, and the rank-one operator $h_j \otimes x_j \in \mathscr{L}(H;E)$ is given by $[h_j \otimes x_j](u) := \langle h_j, u\rangle_H x_j$ for $u \in H$. For such a process, we set

$$\int_J \Phi(t)\,\mathrm{d}W(t) := \sum_{j=1}^{n} \mathscr{W}\big(\mathbf{1}_{(a_j,b_j]}(\cdot) h_j\big) x_j \in L^2(\Omega;E). \tag{1.4.13}$$

If we moreover assume that the Banach space E has Rademacher type 2 (see [157, Definition 4.1]), then by [157, Proposition 4.2] there exists a constant $C \in (0,\infty)$ such that, for all elementary functions Φ as above,

$$\left\| \int_J \Phi(t)\,\mathrm{d}W(t) \right\|_{L^2(\Omega;E)} \le C \|\Phi\|_{L^2(J;\gamma(H;E))}, \tag{1.4.14}$$

where, $\gamma(H;E)$ denotes the space of γ-*radonifying operators* [157, Definition 3.1], and by density we can define $\int_J \Phi(t)\,\mathrm{d}W(t)$ for any $\Phi \in L^2(J;\gamma(H;E))$. In case that $E := K$ is actually a Hilbert space, then $\gamma(H;K)$ is isometrically isomorphic to the space $\mathscr{L}_2(H;K)$ of Hilbert–Schmidt operators from H to K [114, Proposition 9.1.9], and we recover the following Itô *isometry*

$$\left\| \int_J \Phi(t)\,\mathrm{d}W(t) \right\|_{L^2(\Omega;K)} = \|\Phi\|_{L^2(J;\mathscr{L}_2(H;K))}, \tag{1.4.15}$$

so that accordingly the natural space of integrands is $L^2(J;\mathscr{L}_2(H;K))$. See the survey article [157] for more details, as well as for extensions to more general Banach spaces E and stochastic integrands.

1.5. OUTLINE OF THE DISSERTATION

We will now briefly summarize the contents of Chapters 2–5 and indicate their relation to the concepts discussed in Sections 1.1–1.4.

In Chapter 2 we study an extension to space–time of the generalized Whittle–Matérn fields from Example 1.4.5. To this end, we combine the subjects from Sections 1.1, 1.3 and 1.4 by defining mild solutions to abstract linear fractional stochastic evolution equations of the form $(\partial_t + A)^\gamma X = \dot{W}^Q$ for $\gamma \in (1/2,\infty)$ on $J = (0,T)$ with zero initial data. The operators A and Q can for instance be fractional powers L^β and $\widetilde{L}^{-\alpha}$ (respectively) of elliptic spatial differential operators such as the ones defined in Section 1.2. In this case, we find that the spatiotemporal solutions generalize Whittle–Matérn fields in terms of their smoothness and covariance structure, which depend on the interplay of the fractional parameters α, β and γ. The same class of processes is considered in Chapter 3, both on $J = (0,T)$ with zero initial conditions and on $J = \mathbb{R}$, where the focus is instead on establishing Markov properties of solutions. We define appropriate notions of higher-order or weak Markovianity for processes $(X(t))_{t\in J}$ taking their values in a Hilbert space U, which allows us to separately consider the temporal Markov property of the spatiotemporal fields. We show that such a property is satisfied if $\gamma \in \mathbb{N}$ (and, in general, not satisfied if $\gamma \in (1/2,\infty)\setminus\mathbb{N})$). We also prove that, for $\gamma \in (1/2, 3/2)$, the mild solution processes of the equations $(\partial_t + \varepsilon\,\mathrm{Id}_U)^\gamma X = \dot{W}^Q$ converge to a U-valued counterpart of the fractional Brownian motion from Definition 1.4.7 as $\varepsilon \downarrow 0$ (as suggested by (1.4.12)).

We noted in Section 1.3.3 that, although the various definitions of fractional differentiation on $(0,\infty)$ are unambiguous in that they coincide (on their common domains) whenever they are applied to functions vanishing at zero, the same does not hold for functions which take on a nonzero value there. In this case, the different definitions of fractional differentiation each yield different outcomes, and in particular there is no clear natural candidate for a solution concept of (deterministic) fractional equations such as $\partial_t^s u = f$ on $J = (t_0,\infty)$ with $u(t_0) \neq 0$. Therefore, in Chapter 4 we take a different approach by studying the equation $(\partial_t + A)^s u = f$ for $s \in (0,\infty)$ on J, for which we can instead impose the values $u(t)$ at *all* past times $t \in (-\infty, t_0]$ provided that $-A$ generates a uniformly exponentially stable (and strongly measurable) semigroup. For this problem, we derive a mild solution formula expressing u in terms of the semigroup, the initial data and the forcing function f, and we compare it to analogous solution formulae stemming from Riemann–Liouville and Caputo type derivatives (see the first half of Section 1.3.3). Although its derivation relies on uniformly exponential stability, the resulting mild solution formula remains meaningful for merely uniformly bounded semigroups.

Finally, in Chapter 5 we turn to *semilinear* stochastic evolution equations, posed on a sequence of Banach spaces (indexed by $n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$) which, in an abstract sense, approximate another Banach space in which they are uniformly embedded. Their coefficients can be compared by means of projection and lifting operators between the various Banach spaces, allowing us to formulate sufficient conditions for convergence of the lifted mild solution processes as $n \to \infty$. As an example, we consider a semilinear stochastic partial differential equation whose linear part is a generalized Whittle–Matérn operator on a manifold $\mathcal{M}$ (see Example 1.4.5 and Section 1.2.4), discretized by a sequence of geometric graphs. We demonstrate that this setting fits into the abstract framework described above, from which we obtain convergence of mild solutions when lifted to $L^q(\mathcal{M})$ for $q \in [2,\infty]$.

1

1.6. BIBLIOGRAPHICAL NOTES

SECTION 1.1

References for one-parameter semigroups of bounded linear operators and their applications to abstract Cauchy problems include [9, 73, 109, 146, 165, 200, 201]. Most of these references only deal with the important subclasses of, for instance, C_0-semigroups or (bounded) analytic semigroups. We refer to [115, Appendix K] for a self-contained introduction to the case of measurable semigroups, which we use in Section 1.1 since the results of Chapter 4 are stated in this generality. An overview of the subject of maximal L^p-regularity is given in [130]; see also [115, Chapter 17] and [100, Section 9.3].

SECTION 1.2

The results collected in Sections 1.2.1–1.2.3 regarding sesquilinear forms and their applications to elliptic differential operators on Euclidean domains are taken from the monograph [164] and parts of [156, Chapter 12 and Section 13.4.c]. For operators on manifolds (see Section 1.2.4), we refer to [60, Chapter 5] and [190].

SECTION 1.3

An overview of the subjects of functional calculus and fractional powers for sectorial operators can be found in the monograph [100], as well as in [115, Chapter 15]. In particular, the subject of bounded H^∞-calculus and its implications are treated for instance in [63, 130] and [114, Chapter 10]. We also mention the dedicated treatment of fractional powers given in [148]. Classical references regarding fractional integration and differentiation include [121, 169, 179].

SECTION 1.4

Standard references on (continuous-time) stochastic processes and stochastic differential equations include [97, 119, 138, 162, 172]. A detailed treatment of (Gaussian) random fields, viewed as collections of random variables, is given in [3]. For extensive overviews of the subject of Gaussian measures and random variables on normed spaces, we refer to [27, 139]. We remark that certain Gaussian random fields to which we cannot associate a Gaussian measure on L^2, such as the Matérn field on $\mathbb{R}^d$ from Example 1.4.5, can be related to more general rigorously defined objects such as *cylindrical* measures or processes [173, 193] or *generalized processes* (in the sense of generalized functions) [85, Chapter III]. These approaches are outside of the scope of this introduction since we were only concerned with the intuitive interpretation of the Matérn field as the solution of an SPDE. The standard references for vector-valued stochastic integrals with respect to (cylindrical) Wiener processes and their relation to SPDEs are [56] and [144].

2

Analysis of fractional parabolic stochastic evolution equations

The contents of this chapter are based on the article [125], which is joint work with Kristin Kirchner.

2.1. Introduction to Chapter 2

2.1.1. Background and motivation

Gaussian processes play an important role for modeling in spatial statistics. Typical applications arise in the environmental sciences, where geographically indexed data is collected, including climatology [4, 180], oceanography [22], meteorology [103], and forestry [20, 117, 149]. More generally, hierarchical models based on Gaussian processes have been used in various disciplines, where spatially dependent (or spatiotemporal) data is recorded, such as demography [62, 166], epidemiology [137], finance [76], and neuroimaging [152].

Since a Gaussian process $(X(j))_{j\in\mathcal{I}}$ is fully characterized by its mean and its covariance function, *second-order-based approaches* focus on the construction of appropriate covariance classes. In the case that the index set $\mathcal{I}$ is given by a spatial domain in the Euclidean space $\mathcal{I} = \mathcal{D} \subseteq \mathbb{R}^d$, the *Matérn covariance class* [149] is an important and widely used model. The Matérn covariance function is given by

$$\varrho(x,y) = 2^{1-\nu}\sigma^2[\Gamma(\nu)]^{-1}(\kappa\|x-y\|_{\mathbb{R}^d})^{\nu}K_\nu(\kappa\|x-y\|_{\mathbb{R}^d}), \quad x,y\in\mathcal{D}, \tag{2.1.1}$$

where K_ν denotes the modified Bessel function of the second kind. It is indexed by the three interpretable parameters $\nu,\kappa,\sigma^2 \in (0,\infty)$, which determine *smoothness*, *correlation length* and *variance* of the process. It is this feature that renders the Matérn class particularly suitable for making inference about spatial data [187].

For *spatiotemporal* phenomena, the following two difficulties occur:

1. It is desirable to control the properties of the stochastic process named above (in particular, smoothness and correlation lengths) separately in space and time. For this reason, considering (2.1.1) in $d+1$ dimensions is not expedient and it is a difficult task to construct appropriate spatiotemporal covariance models, see e.g. [51, 79, 92, 170, 171, 188].
2. Second-order-based approaches require the factorization of, in general, dense covariance matrices, causing computational costs which are cubic in the number of observations. The two common assumptions imposed on spatiotemporal covariance models to reduce the computational costs—separability (factorization into merely spatial and temporal covariance functions) and stationarity (invariance under translations)—have proven unrealistic in many situations, see [52, 150, 188]. In particular, Stein [188] criticized the behavior of separable covariance functions with respect to their differentiability.

Owing to these problems, the class of *dynamical models* has gained popularity. The name originates from focusing on the dynamics of the stochastic process which are described either by means of conditional probability distributions or by representing the process as a solution of a stochastic partial differential equation (SPDE). The latter approach was originally proposed in the merely spatial case, motivated by the following observation made by Whittle [196]: A stationary process $(X(x))_{x\in\mathcal{D}}$ indexed by the entire Euclidean space $\mathcal{D}=\mathbb{R}^d$ which solves the SPDE

$$\left(\kappa^2-\Delta\right)^{\beta} X(x)=\mathcal{W}(x), \quad x\in\mathcal{D}, \tag{2.1.2}$$

has a covariance function of Matérn type (2.1.1) with $\nu=2\beta-d/2$. Here, Δ denotes the Laplacian and $\mathcal{W}$ is Gaussian white noise. This relation gave rise to the SPDE approach proposed by Lindgren, Rue, and Lindström [142], where the SPDE (2.1.2) is considered on a bounded domain $\mathcal{D}\subsetneq\mathbb{R}^d$ and augmented with Dirichlet or Neumann boundary conditions. Besides enabling the applicability of efficient numerical methods available for (S)PDEs, such as finite element methods [29–31, 50, 107, 142] or wavelets [32, 104], this approach has the advantage of allowing for

(a) nonstationary or anisotropic generalizations, by replacing the operator $\kappa^2-\Delta$ in (2.1.2) with more general strongly elliptic second-order differential operators such as

$$(Lv)(x)=\kappa^2(x)\,v(x)-\nabla\cdot(a(x)\nabla v(x)), \quad x\in\mathcal{D}, \tag{2.1.3}$$

where $\kappa\colon\mathcal{D}\to\mathbb{R}$ and $a\colon\mathcal{D}\to\mathbb{R}^{d\times d}_{\mathrm{sym}}$ are functions [14, 29–31, 50, 80, 107, 142];

(b) more general domains, such as surfaces [33, 107] or manifolds [104].

In the SPDE (2.1.2) the fractional exponent β defines the (spatial) differentiability of its solution, see e.g. [50]. A realistic description of spatiotemporal phenomena necessitates controllable differentiability in space and time. This motivates to consider the space–time fractional SPDE model

$$\begin{cases} \left(\partial_t+L^{\beta}\right)^{\gamma} X(t,x)=\dot{\mathcal{W}}(t,x), & t\in[0,T], \quad x\in\mathcal{D}, \\ X(0,x)=X_0(x), & x\in\mathcal{D}, \end{cases} \tag{2.1.4}$$

where L in (2.1.3) is augmented with boundary conditions on $\partial\mathcal{D}$, $(X_0(x))_{x\in\mathcal{D}}$ is the initial random field, $\dot{\mathcal{W}}$ denotes space–time Gaussian white noise, and $T \in (0,\infty)$ is the time horizon. Whenever $\beta = \gamma = 1$, the SPDE (2.1.4) simplifies to the stochastic heat equation and this spatiotemporal model had already been mentioned in [142] and it was used for statistical inference in [42, 184]. The novelty and sophistication of the SPDE model (2.1.4) lies in the fractional power $\gamma \in (0,\infty)$ of the parabolic operator. Notably, it is the interplay of the parameters β and γ that will facilitate controlling spatial and temporal smoothness of the solution process. For $\mathcal{D} = \mathbb{R}^d$, this has recently been investigated via Fourier techniques in [140], see also [6, 44, 120].

Besides the aforementioned benefits of the SPDE approach and in contrast to the SPDE $\left(\partial_t^\gamma + L^\beta\right)X = \dot{\mathcal{W}}$, considered for instance in [34, 64], the SPDE model (2.1.4) furthermore exhibits a long-time behavior resembling the spatial model (2.1.2).

2.1.2. CONTRIBUTIONS

We introduce a novel interpretation of (2.1.4) with $X_0 = 0$ as a fractional parabolic stochastic evolution equation, and correspondingly define mild and weak solutions for it. To this end, we first give a meaning to fractional powers of an operator of the form $\partial_t + A$, where $-A$ generates a C_0-semigroup. Generalizing the approach taken for $\gamma = 1$ in [56, Chapter 5], we prove that mild and weak solutions are equivalent under natural assumptions, and we investigate their existence, uniqueness, regularity, and covariance. Our main findings are that the problem (2.1.4) is well-posed, and the properties of its solution X with respect to smoothness and covariance structure generalize those of the spatial Whittle–Matérn SPDE model (2.1.2) and relate to the parameters $\beta, \gamma \in (0,\infty)$ in the desired way. Restricting the analysis to a zero initial field is justified by our primary interest in regularity related to the dynamics of (2.1.4) and the long-time behavior of solutions.

In comparison with [25, 26, 143, 160, 189]—the only previous works on an equation of the form $(\partial_t + L)^\gamma u = f$ known to the authors—the main contributions of this chapter, besides considering a stochastic right-hand side, are the fractional power β in (2.1.4) and the method of proving regularity using semigroups. As opposed to the extension approach in [25, 26, 143, 160, 189], this setting does not require a Euclidean structure in space.

2.1.3. OUTLINE

Preliminary notation and theory will be introduced in Section 2.2. In Section 2.3 we give a meaning to the parabolic operator $\partial_t + A$ and its fractional powers in order to introduce well-defined mild and weak solutions of (2.1.4) with $X_0 = 0$. Subsequently, we analyze these in terms of spatiotemporal regularity. Section 2.4 is concerned with the covariance structure of solutions. Finally, in Section 2.5 we apply our results to the space–time Whittle–Matérn SPDE (2.1.4) considered on a bounded Euclidean domain or on a surface. This chapter is supplemented by two appendices: Appendix 2.A contains several technical auxiliary results used in the proofs of Section 2.3. Appendix 2.B collects some definitions and results from functional calculus.

2.2. PRELIMINARIES FOR CHAPTER 2

In this section we only highlight notation which deviates from the previous chapter or was not used there.

2.2.1. NOTATION

The sets $\mathbb{N} := \{1,2,3,\dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ denote the positive and non-negative integers, respectively. We write $s \wedge t$ (or $s \vee t$) for the minimum (or maximum) of two real numbers $s, t \in \mathbb{R}$. The real and imaginary parts of a complex number $z \in \mathbb{C}$ are denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively; its argument, denoted $\arg z$, takes its values in $(-\pi, \pi]$. We write $\mathbf{1}_D$ for the indicator function of a set D. The restriction of a function $f\colon D \to E$ to a subset $D_0 \subseteq D$ is denoted by $f|_{D_0}\colon D_0 \to E$; the image of D_0 under a linear mapping T is written as TD_0. Given two sets $\mathscr{P}, \mathscr{Q}$ and mappings $\mathscr{F}, \mathscr{G}\colon \mathscr{P} \times \mathscr{Q} \to \mathbb{R}$, we use the expression $\mathscr{F}(p,q) \lesssim_q \mathscr{G}(p,q)$ to indicate that for each $q \in \mathscr{Q}$ there exists a constant $C_q \in (0,\infty)$ such that $\mathscr{F}(p,q) \le C_q \mathscr{G}(p,q)$ for all $p \in \mathscr{P}$. We write $\mathscr{F}(p,q) \eqsim_q \mathscr{G}(p,q)$ if both relations, $\mathscr{F}(p,q) \lesssim_q \mathscr{G}(p,q)$ and $\mathscr{G}(p,q) \lesssim_q \mathscr{F}(p,q)$, hold simultaneously.

2.2.2. BANACH SPACES AND OPERATORS

If not specified otherwise, E or F denote separable Banach spaces. We instead write H or U if we work with separable Hilbert spaces and wish to emphasize this. The scalar field $\mathbb{K}$ is either given by the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. A norm on E will be denoted by $\|\cdot\|_E$ and an inner product on H by $\langle\cdot,\cdot\rangle_H$. We write I for the identity operator. The notation $E \hookrightarrow F$ indicates that E is continuously embedded in F, i.e., there exists a bounded injective map from E to F. The dual space of E is denoted by E^*. We write $\overline{E_0}^E$ for the closure of a subset $E_0 \subseteq E$ with respect to the norm on E; the superscript may be omitted when there is no risk of confusion. The Borel σ-algebra of E is denoted by $\mathcal{B}(E)$.

We write $T^* \in \mathscr{L}(F^*; E^*)$ for the adjoint operator of $T \in \mathscr{L}(E;F)$. In the case that $T \in \mathscr{L}(U;H)$, we identify $U^* = U$ and $H^* = H$ via the Riesz maps, so that T^* belongs to $\mathscr{L}(H;U)$. An operator $T \in \mathscr{L}(H)$ is said to be self-adjoint if $T^* = T$, nonnegative definite if $\langle Tx, x\rangle_H \ge 0$ holds for all $x \in H$, and (strictly) positive definite if there exists a constant $\theta \in (0,\infty)$ such that $\langle Tx, x\rangle_H \ge \theta \|x\|_H^2$ holds for all $x \in H$. The identity operator on a normed space is denoted by I in this chapter.

The range of a linear operator $A\colon \mathsf{D}(A) \subseteq E \to E$ is denoted by $\mathsf{R}(A)$. If $\mathsf{G}(A) \subseteq \mathsf{G}(\widetilde{A})$ for another linear operator $\widetilde{A}$ on E, then $\widetilde{A}$ is called an extension of A and we write $A \subseteq \widetilde{A}$. If $\overline{\mathsf{G}(A)}$ is the graph of a linear operator, then we call this operator the closure of A, denoted $\overline{A}$.

2.2.3. FUNCTION SPACES

Let a measure space $(S, \mathscr{A}, \mu)$ be given. We abbreviate the phrases "almost everywhere" and "almost all" by "a.e." and "a.a.", respectively.

We say that a function $f\colon S \to E$ is strongly measurable if it is the μ-a.e. limit of measurable simple functions. For $p \in [1,\infty]$, the Bochner space of (equivalence classes of) strongly measurable, p-integrable functions is denoted by $L^p(S;E)$. It is equipped with the norm

$$\|f\|_{L^p(S;E)} := \begin{cases} \left(\int_S \|f(t)\|_E^p \,\mathrm{d}\mu(t)\right)^{1/p} & \text{if } p \in [1,\infty), \\ \operatorname{ess\,sup}_{t\in S} \|f(t)\|_E & \text{if } p = \infty, \end{cases}$$

2

where ess sup denotes the essential supremum. The norm on $L^2(S;H)$ is induced by the inner product $\langle f, g\rangle_{L^2(S;H)} := \int_S \langle f(t), g(t)\rangle_H \,\mathrm{d}\mu(t)$.

Now let S be an interval $S := J \subseteq \mathbb{R}$, equipped with the Borel σ-algebra and the Lebesgue measure. The space of continuous functions from J to E will be denoted by $C(J;E)$ or $C^{0,0}(J;E)$ and be endowed with the supremum norm. For $\alpha \in (0,1]$, we consider the space $C^{0,\alpha}(J;E)$ of α-Hölder continuous functions with norm

$$\|f\|_{C^{0,\alpha}(J;E)} := |f|_{C^{0,\alpha}(J;E)} + \|f\|_{C(J;E)},$$

where $|f|_{C^{0,\alpha}(J;E)} := \sup_{t,s\in J,\, t\neq s} \|f(t)-f(s)\|_E / |t-s|^\alpha$ is the α-Hölder seminorm. For $n \in \mathbb{N}_0$ and $0 \leq \alpha \leq 1$, the space $C^{n,\alpha}(J;E)$ consists of functions whose nth derivative exists and belongs to $C^{0,\alpha}(J;E)$. On this space we use the norm

$$\|f\|_{C^{n,\alpha}(J;E)} := \|f^{(n)}\|_{C^{0,\alpha}(J;E)} + \sum_{k=0}^{n-1} \|f^{(k)}\|_{C(J;E)},$$

where $f^{(k)}$ denotes the kth derivative of f. Moreover, we define

$$C^\infty(J;E) := \bigcap_{n\in\mathbb{N}} C^{n,0}(J;E).$$

We say that $f \in C^{n,\alpha}(J;E)$ is compactly supported if the support of f, defined by $\operatorname{supp} f := \overline{\{t \in J : f(t) \neq 0\}}^J$, is compact. The space consisting of such functions is denoted by $C_c^{n,\alpha}(J;E)$. If f vanishes at a point $t \in J$, then we write $f \in C_{0,\{t\}}^{n,\alpha}(J;E)$. The spaces $C_c^\infty(J;E)$ and $C_{0,\{t\}}^\infty(J;E)$ are defined analogously.

For an open interval J, we say that $u \in L^2(J;E)$ belongs to $H^1(J;E)$ if there exists a function $v \in L^2(J;E)$ such that $\int_J v(t)\phi(t)\,\mathrm{d}t = -\int_J u(t)\phi'(t)\,\mathrm{d}t$ for all $\phi \in C_c^\infty(J;\mathbb{R})$. The function $\partial_t u := v$ is called the weak derivative of u and the norm on $H^1(J;E)$ is $\|u\|_{H^1(J;E)} := \left(\|u\|_{L^2(J;E)}^2 + \|\partial_t u\|_{L^2(J;E)}^2\right)^{1/2}$. The completion of $C_c^\infty((0,\infty);E)$ with respect to $\|\cdot\|_{H^1(0,\infty;E)}$ defines the space $H_{0,\{0\}}^1(0,\infty;E)$. Elements of $H_{0,\{0\}}^1(J;E)$ are restrictions of functions in $H_{0,\{0\}}^1(0,\infty;E)$ to $J \subseteq (0,\infty)$.

Whenever the function space contains functions mapping to $E = \mathbb{R}$, we omit the codomain, e.g., we write $L^p(S) := L^p(S;\mathbb{R})$ for the Lebesgue spaces.

2.2.4. VECTOR-VALUED STOCHASTIC PROCESSES

Throughout this chapter, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space equipped with a normal filtration $(\mathcal{F}_t)_{t\geq 0}$. Statements which hold $\mathbb{P}$-almost surely are marked with "$\mathbb{P}$-a.s.". The expectation of $Z \in L^1(\Omega;E)$ is denoted by $\mathbb{E}[Z]$.

An E-valued stochastic process $X = (X(t))_{t\in[0,T]}$ indexed by the interval $[0,T]$, $T \in (0,\infty)$, is called integrable if $(X(t))_{t\in[0,T]} \subseteq L^p(\Omega;E)$ holds for $p=1$, and square-integrable if this inclusion is true for $p=2$. For a self-adjoint strictly positive operator $Q \in \mathscr{L}(H)$, $(W^Q(t))_{t\geq 0}$ denotes a cylindrical Q-Wiener process with respect to $(\mathcal{F}_t)_{t\geq 0}$ which takes its values in H, cf. [144, Proposition 2.5.2]; if $Q = I$, we omit the superscript and call $(W(t))_{t\geq 0}$ a cylindrical Wiener process.

2.3. ANALYSIS OF THE FRACTIONAL STOCHASTIC EVOLUTION EQUATION

The aim of this section is to define and analyze solutions to the following stochastic evolution equation of the general fractional order $\gamma \in (1/2,\infty)$:

$$(\partial_t + A)^\gamma X(t) = \dot{W}^Q(t), \quad t \in [0,T], \qquad X(0) = 0. \tag{2.3.1}$$

We interpret this as an abstraction of (2.1.4) with $X_0 = 0$. As noted in the introduction, we restrict the discussion to a zero initial field, since we are primarily interested in properties resulting from the dynamics of the SPDE (2.1.4), respectively (2.3.1), and the long-time behavior for $0 \ll T < \infty$ of its solution. We also note that imposing non-zero boundary data for fractional problems is, in general, highly non-trivial, see e.g. the recent works [1, 7] on the fractional Laplacian.

In Subsection 2.3.1 we investigate the *parabolic operator* $\mathcal{B}$, which is defined as the closure of the sum operator $\partial_t + A$ on an appropriate domain. In particular, we consider the C_0-semigroup generated by $-\mathcal{B}$, which is used to define fractional powers $\mathcal{B}^\gamma$ for $\gamma \in \mathbb{R}$. Interpreting the expression $(\partial_t + A)^\gamma$ appearing in (2.3.1) as $\mathcal{B}^\gamma$, we use this result to define mild solutions in Subsection 2.3.2. In this part, we furthermore introduce a weak solution concept for (2.3.1), and prove equivalence of the two solution concepts as well as existence and uniqueness of mild and weak solutions. Spatiotemporal regularity of solutions is the subject of Subsection 2.3.3.

2.3.1. THE PARABOLIC OPERATOR AND ITS FRACTIONAL POWERS

In this subsection we define the parabolic operator $\mathcal{B}$ and fractional powers $\mathcal{B}^\gamma$. We start by formulating several assumptions on the linear operator A, to which we shall refer throughout the remainder of this chapter. Recall the semigroup theory from Section 1.1; for a more extensive overview of the theory of C_0-semigroups, we refer the reader to [73] or [165]. The complexification of a normed space or operator is indicated by the subscript $\mathbb{C}$; see Section 2.B.2 in the appendix for details.

Assumption 2.3.1. Let H be a separable Hilbert space over the real scalar field $\mathbb{R}$. We assume that the linear operator $A\colon \mathsf{D}(A) \subseteq H \to H$ satisfies

(i) $-A$ generates a C_0-semigroup $(S(t))_{t\geq 0}$.

Sometimes we additionally require one or more of the following conditions:

(ii) $(S(t))_{t\geq 0}$ is bounded analytic, see Definition 1.1.6;

(iii) $A_\mathbb{C}$ admits a bounded H^∞-calculus with $\omega_{H^\infty}(A_\mathbb{C}) < \frac{\pi}{2}$, see (1.3.1);

(iv) A has a bounded inverse.

Under Assumption 2.3.1(i), Lemma 2.B.3 allows us to use several results from [73, 100, 165] for C_0-semigroups and their generators on complex spaces also for $(S(t))_{t\geq 0}$ and $-A$. For instance, by [73, Theorem II.1.4] and [165, Chapter 1, Theorem 2.2] the operator A is closed and densely defined, and the C_0-semigroup $(S(t))_{t\geq 0}$ satisfies

$$\exists M \in [1,\infty),\ w \in \mathbb{R}: \quad \|S(t)\|_{\mathscr{L}(H)} = \|S_{\mathbb{C}}(t)\|_{\mathscr{L}(H_{\mathbb{C}})} \leq M e^{-wt} \quad \forall t \geq 0. \tag{2.3.2}$$

If the conditions (i), (ii) and (iv) are satisfied, then (2.3.2) holds for some $w \in (0,\infty)$, see e.g. [165, p. 70], i.e., $(S(t))_{t\geq 0}$ is exponentially stable. Moreover, we recall that Assumption 2.3.1(ii) is equivalent to sectoriality of $A_{\mathbb{C}}$ with $\omega(A_{\mathbb{C}}) < \frac{\pi}{2}$ by Theorem 1.1.8, and that consequently condition (iii) implies (ii) since $\omega(A_{\mathbb{C}}) \leq \omega_{H^\infty}(A_{\mathbb{C}})$ by Remark 2.B.2. Whenever the conditions (i) and (ii) are satisfied, we have the following useful estimate (see [100, Proposition 3.4.3]):

$$\forall c \in [0,\infty): \quad \|A^c S(t)\|_{\mathscr{L}(H)} = \|A^c_{\mathbb{C}} S_{\mathbb{C}}(t)\|_{\mathscr{L}(H_{\mathbb{C}})} \lesssim_c t^{-c} \quad \forall t \in (0,\infty). \tag{2.3.3}$$

As a first step towards defining the parabolic operator $\mathcal{B}$, we define the Bochner space counterpart $\mathcal{A}\colon \mathsf{D}(\mathcal{A}) \subseteq L^2(0,T;H) \to L^2(0,T;H)$ of A by

$$\begin{aligned} &[\mathcal{A}v](\vartheta) := Av(\vartheta), \qquad v \in \mathsf{D}(\mathcal{A}),\ \text{a.a. } \vartheta \in (0,T),\\ &\mathsf{D}(\mathcal{A}) = L^2(0,T;\mathsf{D}(A)) := \left\{ v \in L^2(0,T;H) : \|Av(\cdot)\|_{L^2(0,T;H)} < \infty \right\}. \end{aligned} \tag{2.3.4}$$

The C_0-semigroup $(S(t))_{t\geq 0}$ on H, generated by $-A$, can be associated to a family of operators $(\mathcal{S}(t))_{t\geq 0}$ on $L^2(0,T;H)$ in a similar way:

$$[\mathcal{S}(t)v](\vartheta) := S(t)v(\vartheta), \qquad t \geq 0,\ v \in L^2(0,T;H),\ \text{a.a. } \vartheta \in (0,T). \tag{2.3.5}$$

It turns out that $(\mathcal{S}(t))_{t\geq 0} \subseteq \mathscr{L}(L^2(0,T;H))$ is again a C_0-semigroup, with infinitesimal generator $-\mathcal{A}$, see Proposition 2.A.3 in Appendix 2.A.

In addition, recall from Example 1.1.15 the family of zero-padded right-translation operators $(\mathcal{T}(t))_{t\geq 0}$ on $L^2(0,T;H)$, defined by

$$[\mathcal{T}(t)v](\vartheta) := \widetilde{v}(\vartheta - t), \qquad t \geq 0,\ v \in L^2(0,T;H),\ \text{a.a. } \vartheta \in (0,T), \tag{2.3.6}$$

where $\widetilde{v} \in L^2(-\infty,T;H)$ denotes the extension of v by zero to $(-\infty,T)$. As shown in Proposition 2.A.5 in Appendix 2.A, also $(\mathcal{T}(t))_{t\geq 0} \subseteq \mathscr{L}(L^2(0,T;H))$ is a C_0-semigroup and its infinitesimal generator is given by $-\partial_t$, where

$$\partial_t\colon \mathsf{D}(\partial_t) \subseteq L^2(0,T;H) \to L^2(0,T;H), \qquad \mathsf{D}(\partial_t) = H^1_{0,\{0\}}(0,T;H), \tag{2.3.7}$$

denotes the Bochner–Sobolev vector-valued weak derivative. We point out that the domain $\mathsf{D}(\partial_t) = H^1_{0,\{0\}}(0,T;H)$ encodes the zero initial condition of the SPDE (2.3.1). Furthermore, note that it readily follows from the definitions in (2.3.5) and (2.3.6) that, for all $t \geq 0$, every $v \in L^2(0,T;H)$, and a.a. $\vartheta \in (0,T)$,

$$[\mathcal{S}(t)\mathcal{T}(t)v](\vartheta) = [\mathcal{T}(t)\mathcal{S}(t)v](\vartheta) = S(t)\widetilde{v}(\vartheta - t),$$

2

i.e., the semigroups $(\mathcal{S}(t))_{t\geq 0}$ and $(\mathcal{T}(t))_{t\geq 0}$ commute.

We now recall the sum operator $\partial_t + \mathcal{A}\colon \mathsf{D}(\partial_t+\mathcal{A}) \subseteq L^2(0,T;H) \to L^2(0,T;H)$ defined on its natural domain, as introduced in Section 1.1.4. That is,

$$(\partial_t + \mathcal{A})v := \partial_t v + \mathcal{A}v, \quad v \in \mathsf{D}(\partial_t+\mathcal{A}) = H^1_{0,\{0\}}(0,T;H) \cap L^2(0,T;\mathsf{D}(A)), \tag{2.3.8}$$

with $\mathcal{A}$ and ∂_t as given in (2.3.4) and (2.3.7), respectively. The next proposition shows that the closure of $-(\partial_t+\mathcal{A})$ again generates a C_0-semigroup, namely the product semigroup of $(\mathcal{S}(t))_{t\geq 0}$ and $(\mathcal{T}(t))_{t\geq 0}$.

Proposition 2.3.2. *Let Assumption 2.3.1(i) be satisfied. The closure $\mathcal{B} := \overline{\partial_t + \mathcal{A}}$ of the sum operator $\partial_t+\mathcal{A}$ defined in (2.3.8) exists and $-\mathcal{B}$ generates the C_0-semigroup $(\mathcal{S}(t)\mathcal{T}(t))_{t\geq 0}$ on $L^2(0,T;H)$, which satisfies*

$$\|\mathcal{S}(t)\mathcal{T}(t)\|_{\mathscr{L}(L^2(0,T;H))} = \|\mathcal{T}(t)\mathcal{S}(t)\|_{\mathscr{L}(L^2(0,T;H))} = \begin{cases} \|S(t)\|_{\mathscr{L}(H)} & \text{if } 0 \leq t < T, \\ 0 & \text{if } t \geq T, \end{cases}$$

where $(\mathcal{S}(t))_{t\geq 0}$ and $(\mathcal{T}(t))_{t\geq 0}$ are defined as in (2.3.5) and (2.3.6), respectively.

Proof. By the commutativity of the semigroups $(\mathcal{S}(t))_{t\geq 0}$ and $(\mathcal{T}(t))_{t\geq 0}$, we may conclude that $(\mathcal{T}(t)\mathcal{S}(t))_{t\geq 0}$ is a C_0-semigroup whose generator is an extension of the operator $-(\partial_t+\mathcal{A})$, and has a domain containing $H^1_{0,\{0\}}(0,T;H)\cap L^2(0,T;\mathsf{D}(A))$ as a subspace that is dense with respect to the graph norm, see [73, Example II.2.7]. Subsequently, Lemma 2.A.2 shows that the generator is the closure of $-(\partial_t+\mathcal{A})$.

Fix $t\in[0,T)$. The inequality $\|\mathcal{T}(t)\mathcal{S}(t)\|_{\mathscr{L}(L^2(0,T;H))} \leq \|S(t)\|_{\mathscr{L}(H)}$ follows by the contractivity of $\mathcal{T}(t)$ and the operator norm isometry from Lemma 2.A.1(a). Now we turn to the reverse inequality. By definition of the operator norm on $\mathscr{L}(H)$, there exists a normalized sequence $(x_n)_{n\in\mathbb{N}}$ in H such that $\|S(t)x_n\|_H \geq \|S(t)\|_{\mathscr{L}(H)} - \frac{1}{n}$ holds for all $n\in\mathbb{N}$. Correspondingly, let the sequence $(v_n)_{n\in\mathbb{N}}$ in $L^2(0,T;H)$ be defined by $v_n(\vartheta) := (T-t)^{-1/2}\mathbf{1}_{(0,T-t)}(\vartheta)x_n$ for every $\vartheta\in(0,T)$ and all $n\in\mathbb{N}$. Note that $\|v_n\|_{L^2(0,T;H)} = 1$ for every $n\in\mathbb{N}$, and

$$\|\mathcal{T}(t)\mathcal{S}(t)v_n\|_{L^2(0,T;H)} = \|(T-t)^{-1/2}\mathbf{1}_{(t,T)}\|_{L^2(0,T)}\|S(t)x_n\|_H \geq \|S(t)\|_{\mathscr{L}(H)} - \tfrac{1}{n}.$$

As this holds for all $n\in\mathbb{N}$, we conclude that $\|\mathcal{T}(t)\mathcal{S}(t)\|_{\mathscr{L}(L^2(0,T;H))} \geq \|S(t)\|_{\mathscr{L}(H)}$. The final assertion for $t\geq T$ follows from the fact that $\mathcal{T}(t)=0$ for $t\geq T$. □

Remark 2.3.3. The closure $\mathcal{B} = \overline{\partial_t+\mathcal{A}}$ appearing in Proposition 2.3.2 raises the question of when the sum operator itself is closed. As discussed in Section 1.1.4, the answer is intimately related to the subject of maximal L^p-regularity. In the Hilbertian setting, the sum turns out to be closed under Assumptions 2.3.1(i),(ii). Indeed, $[\partial_t]_\mathbb{C}$ has a bounded H^∞-calculus with $\omega_{H^\infty}([\partial_t]_\mathbb{C}) \leq \frac{\pi}{2}$ since $(\mathcal{T}(t))_{t\geq 0}$ and $(\mathcal{T}_\mathbb{C}(t))_{t\geq 0}$ are contractive, see [114, Theorem 10.2.24]. By Assumption 2.3.1(ii) and Theorem 1.1.8, we have $\omega(A_\mathbb{C}) < \frac{\pi}{2}$, and the same follows for $\mathcal{A}_\mathbb{C}$ by applying Lemma 2.A.1(a) to its resolvent operators. Thus, we may conclude with [130, Theorem 12.13] that $[\partial_t+\mathcal{A}]_\mathbb{C}$ is closed, so that the same holds for $\partial_t+\mathcal{A}$.

Under the above assumptions, we can define fractional powers of the parabolic operator using the Phillips representation (see (1.3.4) and the subsequent remarks):

$$\mathcal{B}^{-\gamma} := \frac{1}{\Gamma(\gamma)} \int_0^\infty s^{\gamma-1} \mathcal{S}(s)\mathcal{T}(s)\,\mathrm{d}s = \frac{1}{\Gamma(\gamma)} \int_0^T s^{\gamma-1} \mathcal{S}(s)\mathcal{T}(s)\,\mathrm{d}s, \quad \gamma \in (0,\infty). \tag{2.3.9}$$

Note that this yields a well-defined bounded linear operator on $L^2(0,T;H)$, since the product semigroup $(\mathcal{S}(t)\mathcal{T}(t))_{t\geq 0}$ was seen to be exponentially stable (in fact, eventually zero) in Proposition 2.3.2.

The next result shows that the pointwise evaluation of $\mathcal{B}^{-\gamma} f$ at $t \in [0,T]$ is meaningful, provided that $\gamma > \frac{1}{2}$.

Proposition 2.3.4. *Suppose Assumption 2.3.1(i) and let $p \in (1,\infty), \gamma \in (1/p,\infty)$. Then*

$$f \mapsto \mathscr{B}_{\gamma,p} f, \qquad [\mathscr{B}_{\gamma,p} f](t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} S(t-s) f(s)\,\mathrm{d}s \quad \forall t \in [0,T], \tag{2.3.10}$$

defines a bounded linear operator, mapping $f \in L^p(0,T;H)$ into $C_{0,\{0\}}([0,T];H)$.

In particular, if $\gamma \in (1/2,\infty)$, we have for the negative fractional parabolic operator $\mathcal{B}^{-\gamma}$ defined by (2.3.9) when acting on $f \in L^2(0,T;H)$ the pointwise formula

$$[\mathcal{B}^{-\gamma} f](t) = [\mathscr{B}_{\gamma,2} f](t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} S(t-s) f(s)\,\mathrm{d}s \quad \forall t \in [0,T]. \tag{2.3.11}$$

Proof. By [56, Proposition 5.9], for $p \in (1,\infty)$ and $\gamma \in (1/p,\infty)$, the operator $\mathscr{B}_{\gamma,p}$ defined by (2.3.10) maps continuously from $L^p(0,T;H)$ to $C_{0,\{0\}}([0,T];H)$.

Next, note that for all $f \in L^2(0,T;H)$ and a.a. $t \in [0,T]$, we obtain by (2.3.9)

$$\begin{aligned}
[\mathcal{B}^{-\gamma} f](t) &= \frac{1}{\Gamma(\gamma)} \int_0^\infty s^{\gamma-1} [\mathcal{S}(s)\mathcal{T}(s) f](t)\,\mathrm{d}s = \frac{1}{\Gamma(\gamma)} \int_0^t s^{\gamma-1} S(s) f(t-s)\,\mathrm{d}s \\
&= \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} S(t-s) f(s)\,\mathrm{d}s = [\mathscr{B}_{\gamma,2} f](t).
\end{aligned}$$

Thus, by the first part of this proposition, for every $\gamma \in (1/2,\infty)$, we have that $\mathsf{R}(\mathcal{B}^{-\gamma})$ is contained in $C_{0,\{0\}}([0,T];H)$ and the above identities hold pointwise in $t \in [0,T]$. □

Remark 2.3.5. Propositions 2.3.2 and 2.3.4 require only Assumption 2.3.1(i), i.e., that $-A$ generates the C_0-semigroup $(S(t))_{t\geq 0}$. Exponential stability or uniform boundedness of $(S(t))_{t\geq 0}$ are not needed, since we consider linear operators on $L^2(0,T;H)$ (instead of $L^2(0,\infty;H)$), allowing us to use uniform boundedness of $(S(t))_{t\geq 0}$ on the compact interval $[0,T]$ to derive exponential stability of $(\mathcal{S}(t)\mathcal{T}(t))_{t\geq 0}$.

In what follows, we may also consider the operator $\mathcal{B}^{-\gamma *} := (\mathcal{B}^{-\gamma})^*$. More specifically, we will use it in the next section to define a weak solution to the fractional parabolic SPDE (2.3.1). The following lemma provides useful results for the adjoint $\mathcal{B}^{-\gamma *}$ which are analogous to those for $\mathcal{B}^{-\gamma}$ in Proposition 2.3.4. For ease of presentation, the proof has been moved to Subsection 2.A.3 of Appendix 2.A.

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Lemma 2.3.6. *Suppose Assumption 2.3.1(i) and let $\gamma \in (1/2, \infty)$. The adjoint negative fractional parabolic operator $\mathcal{B}^{-\gamma *}$ maps $g \in L^2(0,T;H)$ into $C_{0,\{T\}}([0,T];H)$, and*

$$[\mathcal{B}^{-\gamma *} g](s) = \frac{1}{\Gamma(\gamma)} \int_s^T (t-s)^{\gamma-1} [S(t-s)]^* g(t) \, \mathrm{d}t \quad \forall s \in [0,T]. \tag{2.3.12}$$

Finally, we note that $\mathcal{B}^{-\gamma *} = (\mathcal{B}^*)^{-\gamma}$. To see that the fractional power on the right-hand side is indeed well-defined, we use [165, Chapter 1, Corollary 10.6] and conclude that $-\mathcal{B}^*$ is the generator of the C_0-semigroup $([\mathcal{S}(t)\mathcal{T}(t)]^*)_{t\geq 0}$, which clearly inherits the exponential stability from $(\mathcal{S}(t)\mathcal{T}(t))_{t\geq 0}$ since their norms are equal. The identity is then obtained as follows,

$$\mathcal{B}^{-\gamma *} = \left(\frac{1}{\Gamma(\gamma)} \int_0^\infty s^{\gamma-1} \mathcal{S}(s)\mathcal{T}(s) \, \mathrm{d}s \right)^* = \frac{1}{\Gamma(\gamma)} \int_0^\infty s^{\gamma-1} [\mathcal{S}(s)\mathcal{T}(s)]^* \, \mathrm{d}s = (\mathcal{B}^*)^{-\gamma},$$

where the first and last identities are due to (2.3.9) and the second is a consequence of the general ability to interchange Bochner integrals and duality pairings.

2.3.2. SOLUTION CONCEPTS, EXISTENCE AND UNIQUENESS

We now turn towards defining solutions to (2.3.1) for fractional powers $\gamma \in (0,\infty)$. Recall from Section 2.2 that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space equipped with a normal filtration $(\mathcal{F}_t)_{t\geq 0}$, and that $(W^Q(t))_{t\geq 0}$ is a cylindrical Q-Wiener process on H with respect to $(\mathcal{F}_t)_{t\geq 0}$, where $Q \in \mathscr{L}(H)$ is self-adjoint and strictly positive.

Having defined and investigated the parabolic operator $\mathcal{B}$, its domain and its fractional powers, we are now in particular able to invert the fractional parabolic operator $\mathcal{B}^\gamma$. Equation (2.3.11) suggests the following definition of a fractional stochastic convolution as a *mild solution* to (2.3.1).

Definition 2.3.7. Let Assumption 2.3.1(i) hold and define, for $\gamma \in (0,\infty)$, the stochastic convolution

$$\widetilde{Z}_\gamma(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} S(t-s) \, \mathrm{d}W^Q(s), \quad t \in [0,T]. \tag{2.3.13}$$

A predictable H-valued stochastic process $Z_\gamma := (Z_\gamma(t))_{t\in[0,T]}$ is called a *mild solution* to (2.3.1) if, for all $t \in [0,T]$, it satisfies $Z_\gamma(t) = \widetilde{Z}_\gamma(t)$, $\mathbb{P}$-a.s.

We first address existence and mean-square continuity of mild solutions. Furthermore, we adapt the Da Prato–Kwapień–Zabczyk factorization method (see [53], [56, Section 5.3]) to establish the existence of a pathwise continuous modification.

Theorem 2.3.8. *Let Assumption 2.3.1(i) be satisfied and let $\gamma \in (0,\infty)$ be such that*

$$\exists \delta \in [0,\gamma): \quad \int_0^T \| t^{\gamma-1-\delta} S(t) Q^{\frac{1}{2}} \|^2_{\mathscr{L}_2(H)} \, \mathrm{d}t < \infty. \tag{2.3.14}$$

The stochastic convolution $\widetilde{Z}_\gamma(t)$ in (2.3.13) belongs to $L^2(\Omega;H)$ for all $t \in [0,T]$ if and only if (2.3.14) holds with $\delta = 0$. In this case, the mapping $t \mapsto \widetilde{Z}_\gamma(t)$ is an element of

$C([0,T];L^p(\Omega;H))$ for all $p \in [1,\infty)$; in particular, there exists a mild solution in the sense of Definition 2.3.7, and it is mean-square continuous.

Whenever (2.3.14) holds for some $\delta \in (0,\gamma)$, then for every $p \in [1,\infty)$ there exists a modification of $\widetilde{Z}_\gamma$ with continuous sample paths belonging to $L^p(\Omega;C([0,T];H))$. In particular, the mild solution has a modification with continuous sample paths.

Proof. We first consider the case $\delta = 0$ in (2.3.14). By the Itô isometry (see e.g. [144, Proposition 2.3.5 and p. 32]), we obtain the identity

$$\sup_{t\in[0,T]} \|\widetilde{Z}_\gamma(t)\|^2_{L^2(\Omega;H)} = \frac{1}{|\Gamma(\gamma)|^2}\int_0^T \|t^{\gamma-1}S(t)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\,\mathrm{d}t.$$

Therefore, $\widetilde{Z}_\gamma(t) \in L^2(\Omega;H)$ holds for all $t\in[0,T]$ if and only if (2.3.14) is satisfied with $\delta = 0$. The fact that $t \mapsto \widetilde{Z}_\gamma(t)$ belongs to $C([0,T];L^p(\Omega;H))$ for all $p \in [1,\infty)$ will be shown in greater generality in Proposition 2.3.18.

Moreover, note that $\widetilde{Z}_\gamma\colon [0,T]\times\Omega \to H$ is measurable and $(\mathcal{F}_t)_{t\in[0,T]}$-adapted, and that mean-square continuity implies continuity in probability, so that we may apply [167, Proposition 3.21] to conclude that there exists a predictable modification Z_γ of $\widetilde{Z}_\gamma$. Then, Z_γ is a mild solution to (2.3.1) in the sense of Definition 2.3.7.

Now suppose that (2.3.14) holds for some $\delta\in(0,\gamma)$ and let $p \in (1/\delta \vee 1,\infty)$. By the above considerations, $\widetilde{Z}_{\gamma-\delta}$ and $\widetilde{Z}_\gamma$ exist as elements of $C([0,T];L^p(\Omega;H))$. In particular, we have

$$\widetilde{Z}_{\gamma-\delta} \in L^p(0,T;L^p(\Omega;H)) \cong L^p(\Omega;L^p(0,T;H)),$$

where the latter identification holds by Fubini's theorem. For this reason, there exists a set $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 0$ such that

$$\forall \omega \in \Omega_0^c = \Omega\setminus\Omega_0: \quad \widetilde{Z}_{\gamma-\delta}(\cdot,\omega) \in L^p(0,T;H).$$

We recall the linear operator

$$\mathscr{B}_{\delta,p}\colon L^p(0,T;H) \to C_{0,\{0\}}([0,T];H)$$

from (2.3.10) and claim that the process $\widehat{Z}_\gamma$ defined for $t\in[0,T]$ and $\omega\in\Omega$ by

$$\widehat{Z}_\gamma(t,\omega) := \begin{cases} [\mathscr{B}_{\delta,p}\widetilde{Z}_{\gamma-\delta}](t,\omega) & \text{if } (t,\omega)\in[0,T]\times\Omega_0^c, \\ 0 & \text{if } (t,\omega)\in[0,T]\times\Omega_0, \end{cases}$$

is the desired continuous modification of $\widetilde{Z}_\gamma$. To this end, firstly note that for every $\omega\in\Omega$ the mapping $t\mapsto\widehat{Z}_\gamma(t,\omega)$ indeed is continuous and $\widehat{Z}_\gamma \in L^p(\Omega;C([0,T];H))$; this follows from Proposition 2.3.4 since $\delta\in(1/p,\infty)$. In order to show that $\widehat{Z}_\gamma$ is a modification of $\widetilde{Z}_\gamma$, we fix $t\in[0,T]$ and employ formulas (2.3.10) and (2.3.13) along with the semigroup property to obtain that

$$\begin{aligned}
\widehat{Z}_\gamma(t) = [\mathscr{B}_{\delta,p}\widetilde{Z}_{\gamma-\delta}](t) &= \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}S(t-s)\widetilde{Z}_{\gamma-\delta}(s)\,\mathrm{d}s \\
&= \frac{1}{\Gamma(\delta)\Gamma(\gamma-\delta)}\int_0^t (t-s)^{\delta-1}S(t-s)\left[\int_0^s (s-r)^{\gamma-\delta-1}S(s-r)\,\mathrm{d}W^Q(r)\right]\mathrm{d}s \\
&= \frac{1}{\Gamma(\delta)\Gamma(\gamma-\delta)}\int_0^t\int_0^s (t-s)^{\delta-1}(s-r)^{\gamma-\delta-1}S(t-r)\,\mathrm{d}W^Q(r)\,\mathrm{d}s, \qquad (2.3.15)
\end{aligned}$$

$\mathbb{P}$-a.s. We set $\widetilde{M}_T := \sup_{t\in[0,T]} \|S(t)\|_{\mathscr{L}(H)}$, $K_T := \int_0^T \|t^{\gamma-1-\delta} S(t) Q^{\frac12}\|^2_{\mathscr{L}_2(H)}\,\mathrm{d}t$ and find

$$\begin{aligned}
&\int_0^t \Big[\int_0^s \|(t-s)^{\delta-1}(s-r)^{\gamma-\delta-1} S(t-r)Q^{\frac12}\|^2_{\mathscr{L}_2(H)}\,\mathrm{d}r\Big]^{1/2}\mathrm{d}s \\
&\quad\le \widetilde{M}_T \int_0^t (t-s)^{\delta-1}\Big[\int_0^s \|(s-r)^{\gamma-\delta-1} S(s-r)Q^{\frac12}\|^2_{\mathscr{L}_2(H)}\,\mathrm{d}r\Big]^{1/2}\mathrm{d}s \\
&\quad= \widetilde{M}_T \int_0^t (t-s)^{\delta-1}\Big[\int_0^s \|r^{\gamma-1-\delta} S(r)Q^{\frac12}\|^2_{\mathscr{L}_2(H)}\,\mathrm{d}r\Big]^{1/2}\mathrm{d}s \le \frac{\widetilde{M}_T T^\delta \sqrt{K_T}}{\delta} < \infty.
\end{aligned}$$

This estimate shows that

$$s \mapsto \mathbf{1}_{(0,t)}(s)\mathbf{1}_{(0,s)}(\cdot)(t-s)^{\delta-1}(s-\cdot)^{\gamma-\delta-1}S(t-\cdot)Q^{\frac12} \in L^1(0,T;L^2(0,T;\mathscr{L}_2(H))),$$

and the stochastic Fubini theorem [167, Theorem 8.14] may be used in (2.3.15), yielding

$$\widehat{Z}_\gamma(t) = \frac{1}{\Gamma(\delta)\Gamma(\gamma-\delta)} \int_0^t \Big[\int_r^t (t-s)^{\delta-1}(s-r)^{\gamma-\delta-1}\,\mathrm{d}s\Big] S(t-r)\,\mathrm{d}W^Q(r), \quad \mathbb{P}\text{-a.s.}$$

Using the change of variables $u(s) := \frac{s-r}{t-r}$ and [163, Formula 5.12.1], we derive that

$$(t-r)^{1-\gamma}\int_r^t (t-s)^{\delta-1}(s-r)^{\gamma-\delta-1}\,\mathrm{d}s = \int_0^1 (1-u)^{\delta-1}u^{\gamma-\delta-1}\,\mathrm{d}u = \frac{\Gamma(\gamma-\delta)\Gamma(\delta)}{\Gamma(\gamma)},$$

which shows that $\widehat{Z}_\gamma(t) = \widetilde{Z}_\gamma(t)$ holds $\mathbb{P}$-a.s. Since $t\in[0,T]$ was arbitrary this implies that $\widehat{Z}_\gamma$ is a modification of $\widetilde{Z}_\gamma$ and completes the proof for $p \in (1/\delta \vee 1, \infty)$. Finally, the case $p \in [1, 1/\delta \vee 1]$ follows from the nestedness of the $L^p(\Omega; C([0,T];H))$ spaces. □

In order to provide a more rigorous justification for the Definition 2.3.7 of a mild solution to (2.3.1), we proceed as follows: We seek a further suitable solution concept of a *weak solution*, which follows "naturally" from (2.3.1) using $L^2(0,T;H)$ inner products, and show that weak and mild solutions are equivalent. For this, we first define the *weak stochastic Itô integral* for $f\colon (0,T)\to\mathscr{L}(H)$ and $g\colon (0,T)\to H$ by

$$\int_0^t \langle f(s)\,\mathrm{d}W^Q(s), g(s)\rangle_H := \int_0^t \widetilde{f}_g(s)\,\mathrm{d}W^Q(s), \quad t\in[0,T],$$

where $\int_0^T \|Q^{\frac12}[f(s)]^* g(s)\|_H^2\,\mathrm{d}s < \infty$ and $\widetilde{f}_g\colon (0,T)\to\mathscr{L}(H;\mathbb{R})$ is defined by

$$\widetilde{f}_g(s)x := \langle f(s)x, g(s)\rangle_H \quad \forall x\in H, \quad \forall s\in(0,T),$$

cf. [144, Lemma 2.4.2].

Definition 2.3.9. Let Assumption 2.3.1(i) hold and let $\gamma\in(0,\infty)$. A predictable H-valued stochastic process $Y_\gamma := (Y_\gamma(t))_{t\in[0,T]}$ is called a *weak solution* to (2.3.1) if it is mean-square continuous and, in addition,

$$\forall \psi \in \mathrm{D}(\mathcal{B}^{\gamma*}): \quad \langle Y_\gamma, \mathcal{B}^{\gamma*}\psi\rangle_{L^2(0,T;H)} = \int_0^T \langle \mathrm{d}W^Q(t), \psi(t)\rangle_H, \quad \mathbb{P}\text{-a.s.} \tag{2.3.16}$$

Remark 2.3.10. For $\gamma = 1$, a natural weak solution concept is the formulation given in [167, Definition 9.11]: A predictable H-valued process $(Y_1(t))_{t\in[0,T]}$ is a weak solution to (2.3.1) if $\sup_{t\in[0,T]} \|Y_1(t)\|_{L^2(\Omega;H)} < \infty$ and, for all $t \in [0,T]$ and $y \in \mathsf{D}(A^*)$,

$$\langle Y_1(t), y\rangle_H = -\int_0^t \langle Y_1(s), A^* y\rangle_H \,\mathrm{d}s + \left\langle W^Q(t), y\right\rangle_H, \quad \mathbb{P}\text{-a.s.}$$

Provided that Assumption 2.3.1(i) and (2.3.14) are satisfied, by [167, Theorem 9.15] an H-valued stochastic process is a weak solution in this sense if and only if it is a mild solution in the sense of Definition 2.3.7 with $\gamma = 1$.

In the next proposition we generalize this result to an arbitrary fractional power γ and show that, under the same conditions, the mild solution in the sense of Definition 2.3.7 is equivalent to the weak solution in the sense of Definition 2.3.9.

Proposition 2.3.11. *Suppose that Assumption 2.3.1(i) holds and let $\gamma \in (0,\infty)$ be such that (2.3.14) is satisfied. Then, a stochastic process is a mild solution in the sense of Definition 2.3.7 if and only if it is a weak solution in the sense of Definition 2.3.9. Moreover, mild and weak solutions are unique up to modification. If one requires continuity of the sample paths, mild and weak solutions are unique up to indistinguishability.*

Proof. First, we show that a mild solution Z_γ is a weak solution. Note that mean-square continuity follows from Theorem 2.3.8. Fix an arbitrary $\psi \in \mathsf{D}(\mathcal{B}^{\gamma*})$. Then,

$$\begin{aligned}\langle Z_\gamma, \mathcal{B}^{\gamma*}\psi\rangle_{L^2(0,T;H)} &= \frac{1}{\Gamma(\gamma)}\int_0^T \left\langle \int_0^t (t-s)^{\gamma-1} S(t-s)\,\mathrm{d}W^Q(s), [\mathcal{B}^{\gamma*}\psi](t)\right\rangle_H \mathrm{d}t \\ &= \frac{1}{\Gamma(\gamma)}\int_0^T\int_0^T \left\langle \mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1} S(t-s)\,\mathrm{d}W^Q(s), [\mathcal{B}^{\gamma*}\psi](t)\right\rangle_H \mathrm{d}t \qquad (2.3.17)\end{aligned}$$

holds $\mathbb{P}$-a.s. Here, we used that $\left\langle \int_0^T f(s)\,\mathrm{d}W^Q(s), x\right\rangle_H = \int_0^T \langle f(s)\,\mathrm{d}W^Q(s), x\rangle_H$ for all $f\colon (0,T)\to\mathscr{L}(H)$ and $x\in H$, which readily is derived from the definition of the weak stochastic integral and the continuity of inner products. We now would like to apply the stochastic Fubini theorem, see e.g. [167, Theorem 8.14], in order to interchange the inner weak stochastic integral and the outer deterministic integral. Again by the definition of the weak stochastic integral we have, for a.a. $t\in(0,T)$,

$$\int_0^T \left\langle \mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1} S(t-s)\,\mathrm{d}W^Q(s), [\mathcal{B}^{\gamma*}\psi](t)\right\rangle_H = \int_0^T \Psi(s,t)\,\mathrm{d}W^Q(s), \quad \mathbb{P}\text{-a.s.},$$

where the integrand $\Psi(s,t)\colon H\to\mathbb{R}$ is deterministic and, for $s,t\in(0,T)$, defined by

$$\Psi(s,t)x := \left\langle \mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1} S(t-s)x, [\mathcal{B}^{\gamma*}\psi](t)\right\rangle_H \quad \forall x\in H. \qquad (2.3.18)$$

Thus, the usage of the stochastic Fubini theorem is justified if $t\mapsto \Psi(\cdot,t)Q^{\frac12}$ is in $L^1(0,T;L^2(0,T;\mathscr{L}_2(H;\mathbb{R})))$. Given an orthonormal basis $(g_j)_{j\in\mathbb{N}}$ for H, we obtain

$$\begin{aligned}\left\|\Psi(s,t)Q^{\frac12}\right\|^2_{\mathscr{L}_2(H;\mathbb{R})} &= \sum_{j=1}^\infty \left|\left\langle \mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1} S(t-s)Q^{\frac12} g_j, [\mathcal{B}^{\gamma*}\psi](t)\right\rangle_H\right|^2 \\ &\le \left\|\mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1} S(t-s)Q^{\frac12}\right\|^2_{\mathscr{L}_2(H)} \left\|[\mathcal{B}^{\gamma*}\psi](t)\right\|^2_H\end{aligned}$$

by the Cauchy–Schwarz inequality on H. From this, it follows that

$$
\begin{aligned}
\big\| t \mapsto \Psi(\cdot, t) Q^{\frac{1}{2}} \big\|_{L^1(0,T;L^2(0,T;\mathscr{L}_2(H;\mathbb{R})))} &= \int_0^T \left(\int_0^T \|\Psi(s,t) Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H;\mathbb{R})} \,\mathrm{d}s \right)^{1/2} \mathrm{d}t \\
&\le \int_0^T \left(\int_0^t \|(t-s)^{\gamma-1} S(t-s) Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)} \|[\mathcal{B}^{\gamma *}\psi](t)\|_H^2 \,\mathrm{d}s \right)^{1/2} \mathrm{d}t \\
&= \int_0^T \left(\int_0^t \|s^{\gamma-1} S(s) Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)} \,\mathrm{d}s \right)^{1/2} \|[\mathcal{B}^{\gamma *}\psi](t)\|_H \,\mathrm{d}t \\
&\le T^{1/2} \|\mathcal{B}^{\gamma *}\psi\|_{L^2(0,T;H)} \left(\int_0^T \|s^{\gamma-1} S(s) Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)} \,\mathrm{d}s \right)^{1/2} < \infty,
\end{aligned}
$$

where we used the Cauchy–Schwarz inequality on $L^2(0,T)$ in the last step. Owing to (2.3.14), the integral in the final expression is finite. Applying the stochastic Fubini theorem to (2.3.17), taking adjoints in (2.3.18) and using the continuity of the inner product $\langle \cdot, \cdot \rangle_H$ gives

$$
\begin{aligned}
\langle Z_\gamma, \mathcal{B}^{\gamma *}\psi \rangle_{L^2(0,T;H)} &= \frac{1}{\Gamma(\gamma)} \int_0^T \int_0^T \Psi(s,t) \,\mathrm{d}t \,\mathrm{d}W^Q(s) \\
&= \int_0^T \left\langle \mathrm{d}W^Q(s), \frac{1}{\Gamma(\gamma)} \int_s^T (t-s)^{\gamma-1} [S(t-s)]^* [\mathcal{B}^{\gamma *}\psi](t) \,\mathrm{d}t \right\rangle_H \\
&= \int_0^T \left\langle \mathrm{d}W^Q(s), [\mathcal{B}^{-\gamma *}\mathcal{B}^{\gamma *}\psi](s) \right\rangle_H = \int_0^T \left\langle \mathrm{d}W^Q(s), \psi(s) \right\rangle_H, \qquad \mathbb{P}\text{-a.s.},
\end{aligned}
$$

where we used (2.3.12) in the third line. Therefore, Z_γ is a weak solution.

Conversely, suppose that Y_γ is a weak solution, let an arbitrary $\phi \in L^2(0,T;H)$ be given and set $\psi := \mathcal{B}^{-\gamma *}\phi \in \mathsf{D}(\mathcal{B}^{\gamma *})$. Substituting this into (2.3.16) gives

$$
\langle Y_\gamma, \phi \rangle_{L^2(0,T;H)} = \int_0^T \left\langle \mathrm{d}W^Q(t), [\mathcal{B}^{-\gamma *}\phi](t) \right\rangle_H, \quad \mathbb{P}\text{-a.s.}
$$

Let $(\widetilde{Z}_\gamma(t))_{t\in[0,T]}$ be the stochastic convolution in (2.3.13). Since the condition for the stochastic Fubini theorem still holds after replacing $\mathcal{B}^{\gamma *}\psi$ by ϕ in (2.3.18), the proof of the previous implication can be read backwards to see that

$$
\forall \phi \in L^2(0,T;H): \quad \mathbb{P}\big(\langle Y_\gamma, \phi \rangle_{L^2(0,T;H)} = \langle \widetilde{Z}_\gamma, \phi \rangle_{L^2(0,T;H)} \big) = 1.
$$

By separability of H, also $\mathbb{P}(Y_\gamma = \widetilde{Z}_\gamma \text{ in } L^2(0,T;H)) = 1$ holds so that by Fubini $Y_\gamma = \widetilde{Z}_\gamma$ in $L^2(0,T;L^2(\Omega;H))$ follows. Since both Y_γ and $\widetilde{Z}_\gamma$ are mean-square continuous, this shows that, for all $t \in [0,T]$, $Y_\gamma(t) = \widetilde{Z}_\gamma(t)$ in $L^2(\Omega;H)$. Therefore, for all $t \in [0,T]$, we have that $Y_\gamma(t) = \widetilde{Z}_\gamma(t)$, $\mathbb{P}$-a.s., i.e., Y_γ is a mild solution.

It thus suffices to prove uniqueness only for mild solutions. By Definition 2.3.7, mild solutions are modifications of the stochastic convolution $\widetilde{Z}_\gamma$ in (2.3.13), hence of each other. If two mild solutions are moreover known to have continuous sample paths, then they are indistinguishable by [167, Proposition 3.17]. □

2.3.3. SPATIOTEMPORAL REGULARITY OF SOLUTIONS

In this section, we will investigate spatiotemporal regularity of the mild solution Z_γ in Definition 2.3.7. We start by stating our main results, Theorem 2.3.12 and Corollary 2.3.13, in the first subsection. In the second subsection we derive a simplified condition for spatiotemporal regularity, which is easier to check in applications and sufficient whenever A satisfies Assumptions 2.3.1(i),(iii),(iv), see Proposition 2.3.14. In addition, we explicitly discuss the setting of a Gelfand triple $V \hookrightarrow H \cong H^* \hookrightarrow V^*$ in the case that the operator A is induced by a (not necessarily symmetric) bilinear form $\mathfrak{a}\colon V \times V \to \mathbb{R}$ which is continuous and satisfies a Gårding inequality. The final subsection is devoted to the proof of Theorem 2.3.12.

MAIN RESULTS

In Theorem 2.3.12 below, temporal regularity is measured by the number of derivatives $n \in \mathbb{N}_0$ as well as the Hölder exponent $\tau \in [0,1)$. Spatial regularity is expressed by means of vector spaces which are defined in terms of fractional powers of A as follows:

$$\dot{H}^\sigma_A := \mathsf{D}\big(A^{\sigma/2}\big), \qquad \langle x, y\rangle_{\dot{H}^\sigma_A} := \big\langle A^{\sigma/2}x, A^{\sigma/2}y\big\rangle_H, \quad \sigma \in [0,\infty).$$

For $\sigma \in (0,\infty)$, $\dot{H}^\sigma_A$ is a Hilbert space provided that Assumptions 2.3.1(i),(ii),(iv) are satisfied. In this case, we have the embeddings $\dot{H}^{\sigma'}_A \hookrightarrow \dot{H}^\sigma_A \hookrightarrow H$ for all $\sigma' \geq \sigma \geq 0$. Note, in particular, that we do not need to assume that A is self-adjoint.

Theorem 2.3.12. *Suppose that Assumptions 2.3.1(i),(ii) are satisfied and let $n \in \mathbb{N}_0$, $\sigma \in [0,\infty)$ and $\gamma \in \big(\frac{\sigma - r}{2} + n, \infty\big)$, where $r \in [0,\sigma]$ is such that $Q^{\frac12} \in \mathscr{L}(H; \dot{H}^r_A)$. In the case that $\sigma \in (0,\infty)$, suppose furthermore that Assumption 2.3.1(iv) is fulfilled. Under the condition*

$$\int_0^T \big\| t^{\gamma-1-n} S(t) Q^{\frac12} \big\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)} \,\mathrm{d}t < \infty, \tag{2.3.19}$$

the mild solution Z_γ (or, equivalently, the weak solution Y_γ) in the sense of Definition 2.3.7 (or 2.3.9) belongs to $C^{n,0}([0,T]; L^p(\Omega; \dot{H}^\sigma_A))$ for every $p \in [1,\infty)$.

If additionally $\gamma \geq n + \tau + \frac12$ and $A^{n+\tau+\frac12-\gamma} Q^{\frac12} \in \mathscr{L}_2(H; \dot{H}^\sigma_A)$ hold for some $\tau \in (0,1)$, then we have $Z_\gamma \in C^{n,\tau}([0,T]; L^p(\Omega; \dot{H}^\sigma_A))$ for every $p \in [1,\infty)$.

An application of the Kolmogorov–Chentsov continuity theorem, see e.g. [49, Theorem 3.9], allows us to (partially) transport the temporal regularity result of Theorem 2.3.12 to the pathwise setting, as seen in the next corollary.

Corollary 2.3.13. *Suppose that Assumptions 2.3.1(i),(ii) are satisfied. Let $\sigma \in [0,\infty)$, $r \in [0,\sigma]$, $\gamma \in \big(\frac{\sigma-r}{2}, \infty\big)$ and $\tau \in (0,1)$ be such that $Q^{\frac12} \in \mathscr{L}(H; \dot{H}^r_A)$ and $\gamma \geq \tau + \frac12$. If $\sigma \in (0,\infty)$, suppose also that Assumption 2.3.1(iv) holds. If the condition*

$$\big\| A^{\tau+\frac12-\gamma} Q^{\frac12} \big\|_{\mathscr{L}_2(H;\dot{H}^\sigma_A)} + \int_0^T \big\| t^{\gamma-1} S(t) Q^{\frac12} \big\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)} \,\mathrm{d}t < \infty$$

is satisfied, then for all $p \in [1,\infty)$ and every $\tau' \in [0,\tau)$ there exists a modification $\widehat{Z}_\gamma$ of the mild solution Z_γ (or, equivalently, of the weak solution Y_γ) in the sense of Definition 2.3.7 (or 2.3.9) such that $\widehat{Z}_\gamma$ has τ'-Hölder continuous sample paths and belongs to $L^p\big(\Omega; C^{0,\tau'}([0,T]; \dot{H}^\sigma_A)\big)$.

Proof. We first invoke Theorem 2.3.12 with $n = 0$ and $\tau \in (0,1)$ to establish that Z_γ belongs to $C^{0,\tau}([0,T]; L^q(\Omega; \dot{H}^\sigma_A))$ for every $q \in [1,\infty)$. The result then follows by choosing $q \geq 1$ sufficiently large, applying the Kolmogorov–Chentsov continuity theorem (see e.g. [49, Theorem 3.9]), and using nestedness of the L^p spaces. □

A SIMPLIFIED CONDITION AND ITS APPLICATION TO THE GÅRDING INEQUALITY CASE

Whenever also Assumption 2.3.1(iii) holds, it is possible to replace the condition in (2.3.19) by one which is simpler to check in practice. In this case, the operator A satisfies *square function estimates* (see Section 2.B.2), one of which is used to prove the next result.

Proposition 2.3.14. *Let Assumptions 2.3.1(i),(iii),(iv) be satisfied. Let $\sigma, \delta \in [0,\infty)$ and $\gamma \in \left(\frac{1}{2}+\delta,\infty\right) \cap \left[\frac{1}{2}+\delta+\frac{\sigma-r}{2},\infty\right)$, where $r \in [0,\sigma]$ is taken such that $Q^{\frac{1}{2}} \in \mathscr{L}(H;\dot{H}^r_A)$. Then it holds that*

$$\int_0^\infty \| t^{\gamma-1-\delta} S(t) Q^{\frac{1}{2}} \|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)} \,\mathrm{d}t \eqsim_{(\gamma,\delta)} \| A^{\delta+\frac{1}{2}-\gamma} Q^{\frac{1}{2}} \|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}.$$

Proof. Applying Lemma 2.B.4, see Appendix 2.B, with $a := \gamma - \delta - \frac{1}{2} \in (0,\infty)$ and $x := A^{\frac{\sigma}{2}+\delta+\frac{1}{2}-\gamma} Q^{\frac{1}{2}} y \in H$ for $y \in H$ shows that

$$\int_0^\infty \| t^{\gamma-1-\delta} A^{\gamma-\delta-\frac{1}{2}} S(t) A^{\delta+\frac{1}{2}-\gamma} Q^{\frac{1}{2}} y \|^2_{\dot{H}^\sigma_A} \,\mathrm{d}t \eqsim_{(\gamma,\delta)} \| A^{\delta+\frac{1}{2}-\gamma} Q^{\frac{1}{2}} y \|^2_{\dot{H}^\sigma_A} \quad \forall y \in H.$$

Summing both sides over an orthonormal basis for H and using the Fubini–Tonelli theorem to interchange integration and summation on the left-hand side yields the desired conclusion. □

Remark 2.3.15. Proposition 2.3.14 shows that under the additional assumption that $A_{\mathbb{C}}$ admits a bounded H^∞-calculus with $\omega_{H^\infty}(A_{\mathbb{C}}) < \frac{\pi}{2}$, which e.g. is satisfied whenever A is self-adjoint and strictly positive, it suffices to check that $\gamma > n + \frac{(\sigma-r)\vee 1}{2}$, $\gamma \geq n + \frac{1+(\sigma-r)\vee(2\tau)}{2}$ and that the Hilbert–Schmidt norm $\| A^{n+\tau+\frac{1}{2}-\gamma} Q^{\frac{1}{2}} \|_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}$ is bounded to conclude the regularity results of Theorem 2.3.12. This condition coincides with the one imposed in [128, Section 4, Theorem 6] to derive regularity in the non-fractional case $\gamma = 1$ for $p = 2$, $\sigma = 0$, $n = 0$ and $\tau \in [0, 1/2]$.

Corollary 2.3.16. *Let $\delta \in [0,\infty)$ and $\gamma \in \left(\frac{1}{2}+\delta,\infty\right)$. Suppose that A satisfies Assumption 2.3.1(i) and that there exists a constant $\eta \in [0,\infty)$ such that $\widehat{A} := A + \eta I$ satisfies Assumptions 2.3.1(i),(iii),(iv) and $\widehat{A}^{\delta+\frac{1}{2}-\gamma} Q^{\frac{1}{2}} \in \mathscr{L}_2(H)$. Then, the mild solution Z_γ in the sense of Definition 2.3.7 exists and belongs to $C([0,T]; L^p(\Omega; H))$ for every $p \in [1,\infty)$. If $\delta > 0$, then for every $p \in [1,\infty)$ there exists a modification of Z_γ in $L^p(\Omega; C([0,T]; H))$ which has continuous sample paths.*

Proof. Note that $S(t) = e^{\eta t} \widehat{S}(t)$ holds for every $t \geq 0$, where $(\widehat{S}(t))_{t\geq 0}$ denotes the C_0-semigroup generated by $-\widehat{A}$. Hence, by Proposition 2.3.14 we find that

$$\begin{aligned}\int_0^T \| t^{\gamma-1-\delta} S(t) Q^{\frac{1}{2}} \|^2_{\mathscr{L}_2(H)} \,\mathrm{d}t &\leq e^{2\eta T} \int_0^T \| t^{\gamma-1-\delta} \widehat{S}(t) Q^{\frac{1}{2}} \|^2_{\mathscr{L}_2(H)} \,\mathrm{d}t \\ &\lesssim_{(\gamma,\delta)} e^{2\eta T} \| \widehat{A}^{\delta+\frac{1}{2}-\gamma} Q^{\frac{1}{2}} \|^2_{\mathscr{L}_2(H)} < \infty.\end{aligned}$$

The claim now follows from Theorem 2.3.8. □

We illustrate the utility of Corollary 2.3.16 in the following example. It is concerned with the case that A is induced by a continuous bilinear form $\mathfrak{a}\colon V \times V \to \mathbb{R}$, where $V \hookrightarrow H$ is dense in H, and $\mathfrak{a}$ is not necessarily coercive on V; see also Section 1.2. We note that this setting applies to a variety of important applications, including symmetric and non-symmetric differential operators of even orders.

Example 2.3.17. Let $(V, \langle\cdot,\cdot\rangle_V)$ be a Hilbert space which is densely and continuously embedded in H. Let $A\colon \mathsf{D}(A) \subseteq H \to H$ be associated to a bilinear form $\mathfrak{a}\colon V \times V \to \mathbb{R}$ which is bounded and satisfies a Gårding inequality, i.e., there exist constants $\alpha_0, \alpha_1 \in (0,\infty)$ and $\eta \in [0,\infty)$ such that

$$|\mathfrak{a}(u,v)| \le \alpha_1 \|u\|_V \|v\|_V \qquad \forall u,v \in V, \tag{2.3.20}$$

$$\mathfrak{a}(u,u) \ge \alpha_0 \|u\|_V^2 - \eta \|u\|_H^2 \qquad \forall u \in V. \tag{2.3.21}$$

The Gårding inequality (2.3.21) can be interpreted as coercivity of the bilinear form $\hat{\mathfrak{a}}(u,v) := \mathfrak{a}(u,v) + \eta\langle u,v\rangle_H$ on V, associated with $\widehat{A} = A + \eta I$, while (2.3.20) implies that $\hat{\mathfrak{a}}$ is bounded. The complexified sesquilinear form $\hat{\mathfrak{a}}_{\mathbb{C}}\colon V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$, which is defined analogously to (2.B.1) and induces the operator $\widehat{A}_{\mathbb{C}}$, inherits the boundedness and coercivity from $\hat{\mathfrak{a}}$. Thus, there exist $\widehat{\alpha}_0, \widehat{\alpha}_1 \in (0,\infty)$ such that

$$|\hat{\mathfrak{a}}_{\mathbb{C}}(u,v)| \le \widehat{\alpha}_1 \|u\|_{V_{\mathbb{C}}} \|v\|_{V_{\mathbb{C}}} \qquad \forall u,v \in V_{\mathbb{C}},$$

$$\operatorname{Re} \hat{\mathfrak{a}}_{\mathbb{C}}(u,u) \ge \widehat{\alpha}_0 \|u\|_{V_{\mathbb{C}}}^2 \qquad \forall u \in V_{\mathbb{C}}.$$

Therefore, $\widehat{\alpha}_0 \|u\|_{V_{\mathbb{C}}}^2 \le \operatorname{Re} \hat{\mathfrak{a}}_{\mathbb{C}}(u,u) \le |\hat{\mathfrak{a}}_{\mathbb{C}}(u,u)| \le \widehat{\alpha}_1 \|u\|_{V_{\mathbb{C}}}^2 \le \frac{\widehat{\alpha}_1}{\widehat{\alpha}_0} \operatorname{Re} \hat{\mathfrak{a}}_{\mathbb{C}}(u,u)$ follows for every $u \in V_{\mathbb{C}}$. If $V_{\mathbb{C}} \neq \{0\}$, these estimates imply that $\widehat{\alpha}_0 \le \widehat{\alpha}_1$ and

$$|\operatorname{Im} \hat{\mathfrak{a}}_{\mathbb{C}}(u,u)| = \sqrt{|\hat{\mathfrak{a}}_{\mathbb{C}}(u,u)|^2 - |\operatorname{Re} \hat{\mathfrak{a}}_{\mathbb{C}}(u,u)|^2} \le \left(\frac{\widehat{\alpha}_1^2}{\widehat{\alpha}_0^2} - 1\right)^{1/2} \operatorname{Re} \hat{\mathfrak{a}}_{\mathbb{C}}(u,u) \quad \forall u \in V_{\mathbb{C}}.$$

This shows that $-\widehat{A}_{\mathbb{C}}$ generates a bounded analytic C_0-semigroup $(\widehat{S}_{\mathbb{C}}(t))_{t\ge0}$ of contractions on $H_{\mathbb{C}}$ by Theorem 1.2.8. Applying [114, Theorems 10.2.24 and 10.4.21] and using that $\omega(\widehat{A}_{\mathbb{C}}) \in [0, \frac{\pi}{2})$, since $(\widehat{S}_{\mathbb{C}}(t))_{t\ge0}$ is bounded analytic (see Theorem 1.1.8), we find that $\widehat{A}_{\mathbb{C}}$ admits a bounded H^∞-calculus of angle $\omega_{H^\infty}(\widehat{A}_{\mathbb{C}}) = \omega(\widehat{A}_{\mathbb{C}}) \in [0, \frac{\pi}{2})$. Thus, we are in the setting of Corollary 2.3.16. In particular, the existence of a mean-square continuous mild solution to (2.3.1) for $\gamma > \frac{1}{2}$ follows if $\|\widehat{A}^{\frac{1}{2}-\gamma} Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)} < \infty$.

THE PROOF OF THEOREM 2.3.12

We split the proof of Theorem 2.3.12 into several intermediate results. Before stating and proving these, we introduce the following function, which generalizes the integrand in (2.3.13) used to define mild solutions. Given $a \in \mathbb{R}$, $b \in [0,\infty)$ and $\sigma \in [0,\infty)$, define $\Phi_{a,b}\colon (0,\infty) \to \mathscr{L}(H; \dot{H}^\sigma_A)$ by

$$\Phi_{a,b}(t) := t^a A^b S(t) Q^{\frac{1}{2}}, \qquad t \in (0,\infty). \tag{2.3.22}$$

Note that a mild solution Z_γ in the sense of Definition 2.3.7 satisfies the relation

$$\forall t \in [0,T]: \quad Z_\gamma(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \Phi_{\gamma-1,0}(t-s)\, \mathrm{d}\widehat{W}(s), \quad \mathbb{P}\text{-a.s.},$$

where $\widehat{W}(t) := Q^{-\frac{1}{2}} W^Q(t)$, $t \geq 0$, is a cylindrical Wiener process.

The first result quantifies spatial regularity of the continuous-in-time stochastic convolution with $\Phi_{a,b}$ in $L^p(\Omega; \dot{H}^\sigma_A)$-sense. Recall from Section 2.2 that $(W(t))_{t\geq 0}$ denotes an (arbitrary) H-valued cylindrical Wiener process with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Proposition 2.3.18. *Let Assumption 2.3.1(i) hold, and let $a \in \mathbb{R}$, $b, \sigma \in [0,\infty)$ and $T \in (0,\infty)$ be given. If $\sigma \neq 0$, then suppose moreover that Assumptions 2.3.1(ii),(iv) are satisfied. If the function $\Phi_{a,b}$ defined in (2.3.22) belongs to $L^2(0,T;\mathscr{L}_2(H;\dot{H}^\sigma_A))$, i.e.,*

$$\int_0^T \|\Phi_{a,b}(t)\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}\, \mathrm{d}t < \infty,$$

then $t \mapsto \int_0^t \Phi_{a,b}(t-s)\, \mathrm{d}W(s)$ belongs to $C([0,T];L^p(\Omega;\dot{H}^\sigma_A))$ for all $p \in [1,\infty)$.

Proof. We first note that the assumption $\Phi_{a,b} \in L^2(0,T;\mathscr{L}_2(H;\dot{H}^\sigma_A))$, combined with the Burkholder–Davis–Gundy inequality (see [144, Theorem 6.1.2]) and the continuous embedding

$$L^2(\Omega;\dot{H}^\sigma_A) \hookrightarrow L^p(\Omega;\dot{H}^\sigma_A), \qquad p \in [1,2),\ \sigma \in [0,\infty), \tag{2.3.23}$$

imply that $\int_0^t \Phi_{a,b}(t-s)\, \mathrm{d}W(s)$ indeed is a well-defined element of $L^p(\Omega;\dot{H}^\sigma_A)$ for all $t \in [0,T]$ and every $p \in [1,\infty)$.

It remains to check the $L^p(\Omega;\dot{H}^\sigma_A)$-continuity of $t \mapsto \int_0^t \Phi_{a,b}(t-s)\, \mathrm{d}W(s)$. For fixed $t \in [0,T)$ and $h \in (0,T-t]$, we split the stochastic integrals as follows:

$$\begin{aligned}
&\int_0^{t+h} \Phi_{a,b}(t+h-s)\, \mathrm{d}W(s) - \int_0^t \Phi_{a,b}(t-s)\, \mathrm{d}W(s) \\
&\quad = \int_t^{t+h} \Phi_{a,b}(t+h-s)\, \mathrm{d}W(s) + \int_0^t [\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)]\, \mathrm{d}W(s).
\end{aligned}$$

For $p \in [2,\infty)$, the Burkholder–Davis–Gundy inequality yields

$$\begin{aligned}
&\left\| \int_t^{t+h} \Phi_{a,b}(t+h-s)\, \mathrm{d}W(s) + \int_0^t [\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)]\, \mathrm{d}W(s) \right\|_{L^p(\Omega;\dot{H}^\sigma_A)} \\
&\lesssim_p \left[\int_t^{t+h} \|\Phi_{a,b}(t+h-s)\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}\, \mathrm{d}s \right]^{1/2} \\
&\qquad + \left[\int_0^t \|\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}\, \mathrm{d}s \right]^{1/2} \\
&= \left[\int_0^h \|\Phi_{a,b}(u)\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}\, \mathrm{d}u \right]^{1/2} + \left[\int_0^t \|\Phi_{a,b}(r+h) - \Phi_{a,b}(r)\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}\, \mathrm{d}r \right]^{1/2},
\end{aligned}$$

where $u := t+h-s$ and $r := t-s$. Since $\Phi_{a,b} \in L^2(0,T;\mathscr{L}_2(H;\dot{H}^\sigma_A))$ the first integral tends to zero as $h \downarrow 0$ by dominated convergence. The second term tends to zero by Lemma 2.A.4, see Appendix 2.A.

For $t \in (0,T]$ and $h \in [-t,0)$, the difference of stochastic integrals can be rewritten using $\int_0^t = \int_0^{t+h} + \int_{t+h}^t$. Thus, we obtain, for every $p \in [2,\infty)$, the bound

$$\begin{aligned}&\left\| \int_0^{t+h} \Phi_{a,b}(t+h-s)\,\mathrm{d}W(s) - \int_0^t \Phi_{a,b}(t-s)\,\mathrm{d}W(s) \right\|_{L^p(\Omega;\dot{H}^\sigma_A)} \\ &\lesssim_p \left[\int_0^{-h} \|\Phi_{a,b}(r)\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}\,\mathrm{d}r \right]^{1/2} + \left[\int_{-h}^t \|\Phi_{a,b}(r+h)-\Phi_{a,b}(r)\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}\,\mathrm{d}r \right]^{1/2},\end{aligned}$$

where we again used the change of variables $r := t-s$. Both terms on the last line tend to zero, again by dominated convergence and Lemma 2.A.4, respectively.

Finally, we note that the result for $p=2$ implies that for $p \in [1,2)$ by (2.3.23). □

Furthermore, we obtain the following result regarding the temporal Hölder continuity of the stochastic convolution with the function $\Phi_{a,b}$ in (2.3.22).

Proposition 2.3.19. *Let Assumptions 2.3.1(i),(ii) hold, let $T \in (0,\infty)$, $a \in \left(-\tfrac{1}{2},\infty\right)$, $b,\sigma \in [0,\infty)$ and $\tau \in \left(0, a+\tfrac{1}{2}\right] \cap (0,1)$. If $\sigma \neq 0$, then suppose that Assumption 2.3.1(iv) holds as well. If $A^{-a-\frac{1}{2}+b+\tau}Q^{\frac{1}{2}} \in \mathscr{L}_2(H;\dot{H}^\sigma_A)$ and $\Phi_{a,b}$ is defined by (2.3.22), then $t \mapsto \int_0^t \Phi_{a,b}(t-s)\,\mathrm{d}W(s)$ belongs to $C^{0,\tau}([0,T];L^p(\Omega;\dot{H}^\sigma_A))$ for all $p \in [1,\infty)$.*

Proof. For $t \in [0,T)$ and $h \in (0,T-t]$, we obtain

$$\begin{aligned}&\left\| \int_0^{t+h} \Phi_{a,b}(t+h-s)\,\mathrm{d}W(s) - \int_0^t \Phi_{a,b}(t-s)\,\mathrm{d}W(s) \right\|_{L^p(\Omega;\dot{H}^\sigma_A)} \\ &\le \left\| \int_0^t \left[\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)\right]\mathrm{d}W(s) \right\|_{L^p(\Omega;\dot{H}^\sigma_A)} \\ &\quad + \left\| \int_t^{t+h} \Phi_{a,b}(t+h-s)\,\mathrm{d}W(s) \right\|_{L^p(\Omega;\dot{H}^\sigma_A)} \lesssim_{(p,a,\tau)} h^\tau \left\| A^{-a-\frac{1}{2}+b+\tau}Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}\end{aligned}$$

by Lemmas 2.A.6 and 2.A.7, see Appendix 2.A. The analogous result for the case that $t \in (0,T]$ and $h \in [-t,0)$ follows upon splitting $\int_0^t = \int_0^{t+h} + \int_{t+h}^t$ and applying the lemmas with $\bar{t} := t+h \in [0,T)$ and $\bar{h} := -h \in (0, T-\bar{t}]$. □

We now investigate temporal mean-square differentiability. To this end, we need the following estimate which is implied by (2.3.3): For all $a \in \mathbb{R}$, $b \in [0,\infty)$, we have

$$\forall c \in [0,\infty): \quad \|\Phi_{a,b}(t)x\|_H \lesssim_c t^{a-c} \left\| A^{b-c}Q^{\frac{1}{2}}x \right\|_H \quad \forall x \in \mathsf{D}\big(A^{b-c}Q^{\frac{1}{2}}\big). \tag{2.3.24}$$

The next lemma records some information about the derivatives of $\Phi_{a,b}$ in (2.3.22).

Lemma 2.3.20. *Let Assumptions 2.3.1(i),(ii) be satisfied, and let $a \in \mathbb{R}$, $b,\sigma \in [0,\infty)$. If $\sigma \in (0,\infty)$, suppose furthermore that Assumption 2.3.1(iv) holds. Then, the function $\Phi_{a,b}$ defined by (2.3.22) belongs to $C^\infty((0,\infty);\mathscr{L}(H;\dot{H}^\sigma_A))$ with kth derivative*

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Phi_{a,b}(t) = \sum_{j=0}^k C_{a,j,k}\, t^{a-(k-j)} A^{b+j} S(t) Q^{\frac{1}{2}} = \sum_{j=0}^k C_{a,j,k} \Phi_{a-(k-j),b+j}(t), \tag{2.3.25}$$

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where $C_{a,j,k} := (-1)^j \binom{k}{j} \prod_{i=1}^{k-j}(a-(k-j)+i)$ *for* $a \in \mathbb{R}$, $j,k \in \mathbb{N}_0$, $j \le k$.

Moreover, if $r \in [0, 2b+\sigma]$ *is such that* $Q^{\frac12} \in \mathscr{L}(H; \dot H^r_A)$ *and the integer* $n \in \mathbb{N}_0$ *satisfies* $n < a - b - \frac{\sigma - r}{2}$, *then* $\Phi_{a,b}$ *has a continuous extension in* $C^n([0,\infty); \mathscr{L}(H; \dot H^\sigma_A))$ *with all* n *derivatives vanishing at zero.*

Proof. Since $(S(t))_{t\ge 0}$ is assumed to be analytic, $S(\cdot)$ is infinitely differentiable from $(0,\infty)$ to $\mathscr{L}(H)$, with jth derivative $(-A)^j S(\cdot)$ and, for $t \in (0,\infty)$, $\varepsilon := \frac{t}{2}$,

$$\begin{aligned}\big[A^{b+\frac{\sigma}{2}} S(\cdot)\big]^{(j)}(t) &= \big[S(\cdot - \varepsilon) A^{b+\frac{\sigma}{2}} S(\varepsilon)\big]^{(j)}(t) \\ &= (-A)^j S(t-\varepsilon) A^{b+\frac{\sigma}{2}} S(\varepsilon) = (-1)^j A^{j+b+\frac{\sigma}{2}} S(t).\end{aligned}$$

Here, the limits for the derivatives are taken in the $\mathscr{L}(H)$ norm. This is equivalent to $[A^b S(\cdot)]^{(j)}(t) = (-1)^j A^{j+b} S(t)$ with respect to the $\mathscr{L}(H; \dot H^\sigma_A)$ norm. The expression for the kth derivative of $\Phi_{a,b}$ thus follows from the Leibniz rule.

Now let $r \in [0, 2b+\sigma]$, $n \in \mathbb{N}_0$ be such that $n < a - b - \frac{\sigma-r}{2}$ and $Q^{\frac12} \in \mathscr{L}(H; \dot H^r_A)$. To prove the second claim, we derive that for all $k \in \{0, 1, \dots, n\}$ and $t \in (0,\infty)$

$$\begin{aligned}\Big\| \frac{\mathrm{d}^k}{\mathrm{d}t^k} \Phi_{a,b}(t) \Big\|_{\mathscr{L}(H;\dot H^\sigma_A)} &= \Big\| \sum_{j=0}^{k} C_{a,j,k} t^{a-(k-j)} A^{b+j+\frac{\sigma-r}{2}} S(t) A^{\frac{r}{2}} Q^{\frac12} \Big\|_{\mathscr{L}(H)} \\ &\lesssim_{(a,b,k,r,\sigma)} t^{a-k-b-\frac{\sigma-r}{2}} \|Q^{\frac12}\|_{\mathscr{L}(H;\dot H^r_A)}\end{aligned}$$

by applying (2.3.24) to each summand with $c := b + j + \frac{\sigma-r}{2} \ge 0$. Furthermore, since $a - k - b - \frac{\sigma-r}{2} \ge a - n - b - \frac{\sigma-r}{2} > 0$, the above quantity tends to zero as $t \downarrow 0$. Hence, extending $t \mapsto \frac{\mathrm{d}^k}{\mathrm{d}t^k}\Phi_{a,b}(t)$ by zero at $t = 0$ gives a function in $C([0,\infty); \mathscr{L}(H; \dot H^\sigma_A))$ for all $k \in \{0, 1, \dots, n\}$. Inductively it follows then that the kth derivative of the zero extension is the zero extension of the original kth derivative. ☐

Proposition 2.3.21. *Let* $\sigma \in [0,\infty)$, *and whenever* $\sigma \in (0,\infty)$ *require additionally Assumptions* 2.3.1(i),(ii),(iv). *Suppose that* $\Psi \in H^1_{0,\{0\}}(0,T; \mathscr{L}_2(H; \dot H^\sigma_A))$ *and let* Ψ' *denote its weak derivative. Then the stochastic convolution* $t \mapsto \int_0^t \Psi(t-s)\,\mathrm{d}W(s)$ *is differentiable from* $[0,T]$ *to* $L^p(\Omega; \dot H^\sigma_A)$ *for all* $p \in [1,\infty)$, *with derivative*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \Psi(t-s)\,\mathrm{d}W(s) = \int_0^t \Psi'(t-s)\,\mathrm{d}W(s) \quad \forall\, t \in [0,T]. \tag{2.3.26}$$

Proof. For $t \in [0,T)$ and $h \in (0, T-t]$, we can write

$$\begin{aligned}&\frac1h \Big[\int_0^{t+h} \Psi(t+h-s)\,\mathrm{d}W(s) - \int_0^t \Psi(t-s)\,\mathrm{d}W(s)\Big] - \int_0^t \Psi'(t-s)\,\mathrm{d}W(s) \\ &= \int_0^t \Big[\frac{\Psi(t+h-s) - \Psi(t-s)}{h} - \Psi'(t-s)\Big]\,\mathrm{d}W(s) + \frac1h \int_t^{t+h} \Psi(t+h-s)\,\mathrm{d}W(s) \\ &=: I_1^{h^+} + I_2^{h^+}.\end{aligned}$$

For $t \in (0,T]$ and $h \in [-t,0)$, we instead have

$$\begin{aligned}
&\frac{1}{h}\Big[\int_0^{t+h} \Psi(t+h-s)\,\mathrm{d}W(s) - \int_0^t \Psi(t-s)\,\mathrm{d}W(s)\Big] - \int_0^t \Psi'(t-s)\,\mathrm{d}W(s) \\
&\quad = \int_0^{t+h}\Big[\frac{\Psi(t+h-s)-\Psi(t-s)}{h} - \Psi'(t-s)\Big]\,\mathrm{d}W(s) \\
&\qquad - \frac{1}{h}\int_{t+h}^t \Psi(t-s)\,\mathrm{d}W(s) - \int_{t+h}^t \Psi'(t-s)\,\mathrm{d}W(s) =: I_1^{h^-} + I_2^{h^-} + I_3^{h^-}.
\end{aligned}$$

We first deal with the terms $I_2^{h^\pm}$. Note that $\Psi \in H^1_{0,\{0\}}(0,T;\mathscr{L}_2(H;\dot{H}^\sigma_A))$ implies that $\Psi(u) = \int_0^u \Psi'(r)\,\mathrm{d}r$ for all $u \in (0,|h|)$, see [74, Section 5.9.2, Theorem 2]. In conjunction with the Burkholder–Davis–Gundy inequality (combined with the embedding (2.3.23) if $p \in [1,2)$) and the Cauchy–Schwarz inequality, this leads to

$$\begin{aligned}
\|I_2^{h^\pm}\|_{L^p(\Omega;\dot{H}^\sigma_A)} &\lesssim_p \frac{1}{|h|}\Big[\int_0^{|h|} \|\Psi(u)\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}\,\mathrm{d}u\Big]^{1/2} \\
&\le \frac{1}{|h|}\Big[\int_0^{|h|}\Big(\int_0^u \|\Psi'(r)\|_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}\,\mathrm{d}r\Big)^2\,\mathrm{d}u\Big]^{1/2} \le \|\Psi'\|_{L^2(0,|h|;\mathscr{L}_2(H;\dot{H}^\sigma_A))}.
\end{aligned}$$

Moreover, we find that

$$\begin{aligned}
\|I_3^{h^-}\|_{L^p(\Omega;\dot{H}^\sigma_A)} &\lesssim_p \Big[\int_{t+h}^t \|\Psi'(t-s)\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}\,\mathrm{d}s\Big]^{1/2} \\
&= \Big[\int_0^{|h|} \|\Psi'(u)\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}\,\mathrm{d}u\Big]^{1/2} = \|\Psi'\|_{L^2(0,|h|;\mathscr{L}_2(H;\dot{H}^\sigma_A))}.
\end{aligned}$$

Since $\Psi' \in L^2(0,T;\mathscr{L}_2(H;\dot{H}^\sigma_A))$, we have that $\|\Psi'\|_{L^2(0,|h|;\mathscr{L}_2(H;\dot{H}^\sigma_A))} \to 0$ as $h \to 0$ by dominated convergence. Thus, it remains to deal with the $I_1^{h^\pm}$ terms. For the case of positive h, we find using the definition of the difference quotient D_h (see Equation (2.A.6) in Subsection 2.A.4) that

$$\begin{aligned}
\|I_1^{h^+}\|_{L^p(\Omega;\dot{H}^\sigma_A)} &\lesssim_p \Big[\int_0^t \Big\|\frac{\Psi(t+h-s)-\Psi(t-s)}{h} - \Psi'(t-s)\Big\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}\,\mathrm{d}s\Big]^{1/2} \\
&= \Big[\int_0^t \Big\|\frac{\Psi(u+h)-\Psi(u)}{h} - \Psi'(u)\Big\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}\,\mathrm{d}u\Big]^{1/2} \\
&= \|D_h\Psi - \Psi'\|_{L^2(0,t;\mathscr{L}_2(H;\dot{H}^\sigma_A))}.
\end{aligned}$$

For the case of negative h, we arrive at

$$\begin{aligned}
\|I_1^{h^-}\|_{L^p(\Omega;\dot{H}^\sigma_A)} &\lesssim_p \Big[\int_0^{t+h} \Big\|\frac{\Psi(t+h-s)-\Psi(t-s)}{h} - \Psi'(t-s)\Big\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}\,\mathrm{d}s\Big]^{1/2} \\
&= \Big[\int_{-h}^t \Big\|\frac{\Psi(u+h)-\Psi(u)}{h} - \Psi'(u)\Big\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}\,\mathrm{d}u\Big]^{1/2} \\
&= \|D_h\Psi - \Psi'\|_{L^2(-h,t;\mathscr{L}_2(H;\dot{H}^\sigma_A))}.
\end{aligned}$$

The convergence $\lim_{h\to 0}\|I_1^{h^\pm}\|_{L^p(\Omega;\dot{H}^\sigma_A)} = 0$ follows then from Proposition 2.A.8. □

We are now ready to prove Theorem 2.3.12.

Proof of Theorem 2.3.12. We first claim that the mild solution, interpreted as a mapping $Z_\gamma\colon [0,T] \to L^p(\Omega; \dot{H}^\sigma_A)$, is n times differentiable and that, for all $k \in \{0,1,\dots,n\}$ and $t \in [0,T]$, its kth derivative satisfies

$$Z^{(k)}_\gamma(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \Phi^{(k)}_{\gamma-1,0}(t-s)\,\mathrm{d}\widehat{W}(s), \quad \mathbb{P}\text{-a.s.,} \tag{2.3.27}$$

where $\Phi^{(k)}_{\gamma-1,0}$ is the kth derivative of $\Phi_{\gamma-1,0}$ given by (2.3.25), and $\widehat{W}$ is the cylindrical Wiener process $\widehat{W}(t) := Q^{-\frac{1}{2}} W^Q(t)$, $t \geq 0$. We prove this by induction with respect to k. For $k=0$, the identity (2.3.27) follows from Definition 2.3.7 and (2.3.22). Now let $k \in \{0,1,\dots,n-1\}$ and suppose that Z_γ is k times differentiable and (2.3.27) holds. Then, the induction hypothesis and Lemma 2.3.20 show that, for all $t \in [0,T]$,

$$\begin{aligned}\frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}} Z_\gamma(t) &= \frac{\mathrm{d}}{\mathrm{d}t} Z^{(k)}_\gamma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\Gamma(\gamma)}\int_0^t \Phi^{(k)}_{\gamma-1,0}(t-s)\,\mathrm{d}\widehat{W}(s)\right] \\ &= \frac{1}{\Gamma(\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \sum_{j=0}^k C_{\gamma-1,j,k}\Phi_{\gamma-1-(k-j),j}(t-s)\,\mathrm{d}\widehat{W}(s), \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

Fixing an arbitrary $j \in \{0,1,\dots,k\}$, it suffices to verify that $\Psi := \Phi_{\gamma-1-(k-j),j}$ satisfies the conditions of Proposition 2.3.21, so that (2.3.26) holds for the cylindrical Wiener process $\widehat{W}$. Indeed, having proved this for an arbitrary j, by linearity

$$\begin{aligned}\frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}} Z_\gamma(t) &= \frac{1}{\Gamma(\gamma)}\int_0^t \sum_{j=0}^k C_{\gamma-1,j,k}\Phi'_{\gamma-1-(k-j),j}(t-s)\,\mathrm{d}\widehat{W}(s) \\ &= \frac{1}{\Gamma(\gamma)}\int_0^t \Phi^{(k+1)}_{\gamma-1,0}(t-s)\,\mathrm{d}\widehat{W}(s), \quad \mathbb{P}\text{-a.s.,}\end{aligned}$$

follows, where the latter identity is an equality of the operator-valued integrands.

Using (2.3.3) with $c := b$, the identity $A^{\frac{\sigma}{2}}\Phi_{a,b}(t) = 2^a (t/2)^a A^b S(t/2) A^{\frac{\sigma}{2}} S(t/2) Q^{\frac{1}{2}}$ and a change of variables $u := t/2$, we observe that

$$\begin{aligned}\|\Phi_{a,b}\|_{L^2(0,T;\mathscr{L}_2(H;\dot{H}^\sigma_A))} &\lesssim_{(a,b)} \left[\int_0^T (t/2)^{2(a-b)} \big\|A^{\frac{\sigma}{2}} S(t/2) Q^{\frac{1}{2}}\big\|^2_{\mathscr{L}_2(H)} \frac{\mathrm{d}t}{2}\right]^{1/2} \\ &= \left[\int_0^T \|\Phi_{a-b,0}(t/2)\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)} \frac{\mathrm{d}t}{2}\right]^{1/2} \leq \|\Phi_{a-b,0}\|_{L^2(0,T;\mathscr{L}_2(H;\dot{H}^\sigma_A))}\end{aligned} \tag{2.3.28}$$

holds for all $a \in \mathbb{R}$ and $b \in [0,\infty)$. For $\Psi = \Phi_{\gamma-1-(k-j),j}$ we use (2.3.28) to obtain

$$\|\Psi\|_{L^2(0,T;\mathscr{L}_2(H;\dot{H}^\sigma_A))} \lesssim_{(\gamma,k,j)} \|\Phi_{\gamma-1-k,0}\|_{L^2(0,T;\mathscr{L}_2(H;\dot{H}^\sigma_A))}.$$

The norm on the right-hand side is finite by (2.3.19), since $k \leq n-1 < n$. Next, noting that $\gamma - 1 - k - \frac{\sigma-r}{2} \geq \gamma - n - \frac{\sigma-r}{2} > 0$, the second assertion of Lemma 2.3.20 implies that $t \mapsto \Psi(t)$ has a continuous extension in $C_{0,\{0\}}([0,T];\mathscr{L}(H;\dot{H}^\sigma_A))$. Furthermore, also by Lemma 2.3.20, Ψ is differentiable from $(0,T)$ to $\mathscr{L}(H;\dot{H}^\sigma_A)$, with derivative

$$\Psi' = (\gamma - 1 - (k-j))\Phi_{\gamma-1-(k-j)-1,j} - \Phi_{\gamma-1-(k-j),j+1}.$$

Applying the triangle inequality and (2.3.28) then shows that

$$\|\Psi'\|_{L^2(0,T;\mathscr{L}_2(H;\dot{H}^\sigma_A))} \lesssim_{(\gamma,k,j)} \|\Phi_{\gamma-1-(k+1),0}\|_{L^2(0,T;\mathscr{L}_2(H;\dot{H}^\sigma_A))},$$

where the norm on the right-hand side is finite by (2.3.19), as $k+1 \le n$. Since it holds that $\mathscr{L}_2(H;\dot{H}^\sigma_A) \hookrightarrow \mathscr{L}(H;\dot{H}^\sigma_A)$, Lemma 2.A.9 implies that $\Psi \in H^1_{0,\{0\}}(0,T;\mathscr{L}_2(H;\dot{H}^\sigma_A))$. Thus, we may indeed use Proposition 2.3.21, and the differentiability follows.

It remains to show that the nth derivative $Z^{(n)}_\gamma$ is (Hölder) continuous, i.e., that we have $Z^{(n)}_\gamma \in C^{0,\tau}([0,T];L^p(\Omega;\dot{H}^\sigma_A))$. To this end, we use (2.3.27) and (2.3.25), and write

$$\forall t \in [0,T]: \quad Z^{(n)}_\gamma(t) = \frac{1}{\Gamma(\gamma)} \sum_{j=0}^{n} C_{\gamma-1,j,n} \int_0^t \Phi_{\gamma-1-(n-j),j}(t-s)\, \mathrm{d}\widehat{W}(s), \quad \mathbb{P}\text{-a.s.}$$

The case $\tau = 0$ (i.e., continuity) follows after applying, for all $j \in \{0,1,\dots,n\}$, Proposition 2.3.18 with $a = \gamma - 1 - (n-j)$ and $b = j$. Note that $\Phi_{\gamma-1-(n-j),j}$ indeed is an element of $L^2(0,T;\mathscr{L}_2(H;\dot{H}^\sigma_A))$ for all $j \in \{0,\dots,n\}$ by (2.3.19) and (2.3.28). For $\tau \in \left(0, \gamma - n - \frac{1}{2}\right] \cap (0,1)$, the Hölder continuity of $Z^{(n)}_\gamma$ follows from Proposition 2.3.19 which we may apply, for all $j \in \{0,1,\dots,n\}$, with $a = \gamma - 1 - (n-j)$ and $b = j$, since $A^{n+\tau+\frac{1}{2}-\gamma} Q^{\frac{1}{2}} \in \mathscr{L}_2(H;\dot{H}^\sigma_A)$ is assumed. □

2.4. COVARIANCE STRUCTURE

In this section, we study the covariance structure of solutions to (2.3.1). More specifically, we consider the mild solution process $(Z_\gamma(t))_{t\in[0,T]}$ from Definition 2.3.7. The covariance structure of Z_γ will be expressed in terms of the family of covariance operators $(Q_{Z_\gamma}(s,t))_{s,t\in[0,T]} \subseteq \mathscr{L}(H)$ which satisfies, for all $s,t \in [0,T]$, that

$$\langle Q_{Z_\gamma}(s,t)x, y\rangle_H = \mathbb{E}[\langle Z_\gamma(s) - \mathbb{E}[Z_\gamma(s)], x\rangle_H \langle Z_\gamma(t) - \mathbb{E}[Z_\gamma(t)], y\rangle_H] \quad \forall x,y \in H. \tag{2.4.1}$$

Note that this family is well-defined whenever Z_γ is square-integrable, e.g., under the assumptions made in Theorem 2.3.8. Note also that $\mathbb{E}[Z_\gamma(t)] = 0$ for all $t \in [0,T]$.

We present three results on the covariance operators of the mild solution Z_γ. The most general result is Proposition 2.4.1, which provides an explicit integral representation of $Q_{Z_\gamma}(s,t)$. Corollary 2.4.2 is concerned with the asymptotic behavior of the covariance operator $Q_{Z_\gamma}(t,t)$ as $t \to \infty$. Subsequently, in Corollary 2.4.3 we consider a situation in which the covariance is separable in time and space, and prove that the temporal part is asymptotically of Matérn type.

Proposition 2.4.1. *Let Assumption 2.3.1(i) be satisfied and let $\gamma \in (0,\infty)$ be such that (2.3.14) holds. The covariance operators $(Q_{Z_\gamma}(s,t))_{s,t\in[0,T]}$ of Z_γ admit the representation*

$$Q_{Z_\gamma}(s,t) = \frac{1}{\Gamma(\gamma)^2} \int_0^{s\wedge t} [(s-r)(t-r)]^{\gamma-1} S(t-r) Q [S(s-r)]^* \,\mathrm{d}r. \tag{2.4.2}$$

Proof. Square-integrability of Z_γ is a consequence of Theorem 2.3.8 and (2.3.14). In order to prove the integral representation (2.4.2), for $s \in [0,T]$, $r \in (0,s)$ and $x \in H$, we define $f(s,r;x) \in \mathscr{L}(H;\mathbb{R})$ by

$$f(s,r;x)z := [\Gamma(\gamma)]^{-1} \langle z, (s-r)^{\gamma-1}[S(s-r)]^* x\rangle_H, \quad z \in H.$$

We proceed similarly as in [123, Lemma 3.10] and obtain (2.4.2) from the Itô isometry combined with the polarization identity:

$$\begin{aligned}\mathbb{E}[\langle Z_\gamma(s),x\rangle_H\langle Z_\gamma(t),y\rangle_H] &= \mathbb{E}\left[\int_0^s f(s,r;x)\,\mathrm{d}W^Q(r)\int_0^t f(t,\tau;y)\,\mathrm{d}W^Q(\tau)\right]\\ &= \int_0^{s\wedge t}\langle f(s,r;x)Q^{\frac12}, f(t,r;y)Q^{\frac12}\rangle_{\mathscr{L}_2(H;\mathbb{R})}\,\mathrm{d}r\\ &= \frac{1}{\Gamma(\gamma)^2}\int_0^{s\wedge t}[(s-r)(t-r)]^{\gamma-1}\langle S(t-r)Q[S(s-r)]^*x,y\rangle_H\,\mathrm{d}r.\end{aligned}$$

Then, (2.4.2) follows from exchanging the order of integration and taking the inner product, which is justified since $(0,s\wedge t)\ni r\mapsto[(s-r)(t-r)]^{\gamma-1}S(t-r)Q[S(s-r)]^*x$ is integrable by (2.3.14). □

By imposing more assumptions on the operator A, one can obtain explicit representations of the asymptotic covariance structure of Z_γ as $t\to\infty$, as the next two corollaries show. Note that, if (2.3.14) holds for $\delta=0$ and $T=\infty$, in Definition 2.3.7 the stochastic convolution $\widetilde{Z}_\gamma$ and the mild solution Z_γ are well-defined on the infinite time interval $[0,\infty)$. It is thus meaningful to consider the asymptotic behavior.

Corollary 2.4.2. *Let Assumptions 2.3.1(i),(ii),(iv) be satisfied and let $\gamma\in(1/2,\infty)$. Suppose that (2.3.14) holds for $\delta=0$ and $T=\infty$. If for every $t\in[0,\infty)$ the operator $S(t)$ is self-adjoint and commutes with the covariance operator Q of W^Q, we have*

$$\lim_{t\to\infty}Q_{Z_\gamma}(t,t)=\Gamma(\gamma-1/2)\big[2\sqrt{\pi}\,\Gamma(\gamma)\big]^{-1}A^{1-2\gamma}Q\quad \textit{in } \mathscr{L}(H).$$

Proof. Starting from the identity (2.4.2) for a fixed $t=s\in[0,\infty)$, we recall the self-adjointness of the operators $(S(t))_{t\ge0}$ and the commutativity with Q to obtain that

$$\begin{aligned}Q_{Z_\gamma}(t,t) &= \frac{1}{\Gamma(\gamma)^2}\int_0^t(t-r)^{2(\gamma-1)}S(t-r)QS(t-r)\,\mathrm{d}r\\ &= \frac{1}{\Gamma(\gamma)^2}\int_0^t(t-r)^{2\gamma-2}QS(2t-2r)\,\mathrm{d}r=\frac{2^{1-2\gamma}}{\Gamma(\gamma)^2}\int_0^{2t}u^{2\gamma-2}QS(u)\,\mathrm{d}u,\end{aligned}$$

where we also used the semigroup property and the change of variables $u:=2(t-r)$. Now we interchange the bounded linear operator Q with the integral, and pass to the limit $t\to\infty$ in $\mathscr{L}(H)$, which by (1.3.4) with $\alpha:=2\gamma-1\in(0,\infty)$ gives

$$\lim_{t\to\infty}Q_{Z_\gamma}(t,t)=2^{1-2\gamma}\Gamma(2\gamma-1)[\Gamma(\gamma)]^{-2}A^{1-2\gamma}Q=\Gamma(\gamma-1/2)\big[2\sqrt{\pi}\,\Gamma(\gamma)\big]^{-1}A^{1-2\gamma}Q.$$

The last equality was obtained by applying the Legendre duplication formula for the gamma function [163, Formula (5.5.5)] to $\Gamma(2\gamma-1)=\Gamma(2[\gamma-1/2])$. □

Corollary 2.4.3. *Suppose the setting of Corollary 2.4.2 and let $A:=\kappa I$ for $\kappa\in(0,\infty)$. Then the covariance function of Z_γ is separable and its temporal part is asymptotically of Matérn type, i.e., there is a function $\varrho_{Z_\gamma}\colon[0,\infty)\times[0,\infty)\to\mathbb{R}$ such that*

$$\begin{aligned}&\forall s,t\in[0,\infty): \quad Q_{Z_\gamma}(s,t)=\varrho_{Z_\gamma}(s,t)\,Q,\\ &\forall h\in\mathbb{R}\setminus\{0\}: \quad \lim_{t\to\infty}\varrho_{Z_\gamma}(t,t+h)=\frac{2^{\frac12-\gamma}\kappa^{1-2\gamma}}{\sqrt{\pi}\,\Gamma(\gamma)}(\kappa|h|)^{\gamma-\frac12}K_{\gamma-\frac12}(\kappa|h|). \qquad (2.4.3)\end{aligned}$$

Remark 2.4.4. On the right-hand side of (2.4.3), one recognizes the Matérn covariance function (2.1.1) with smoothness parameter $\nu = \gamma - 1/2$, correlation length parameter κ and variance $\sigma^2 = \kappa^{1-2\gamma}\Gamma(\gamma - 1/2)\left[2\sqrt{\pi}\Gamma(\gamma)\right]^{-1}$.

Proof of Corollary 2.4.3. For $s, t \geq 0$, the integral representation (2.4.2) yields

$$Q_{Z_\gamma}(s,t) = \frac{1}{\Gamma(\gamma)^2}\int_0^{s\wedge t} [(s-r)(t-r)]^{\gamma-1} e^{-\kappa(s+t-2r)}\,\mathrm{d}r\, Q = \varrho_{Z_\gamma}(s,t)\, Q,$$

where we moved the bounded operator $Q \in \mathscr{L}(H)$ out of the integral. Next, we fix $h \in (0,\infty)$, let $t \in [0,\infty)$ and perform the change of variables $u := h + 2(t-r)$,

$$\varrho_{Z_\gamma}(t, t+h) = \varrho_{Z_\gamma}(t+h, t) = \frac{2^{1-2\gamma}}{\Gamma(\gamma)^2}\int_h^{2t+h} [(u+h)(u-h)]^{\gamma-1} e^{-\kappa u}\,\mathrm{d}u.$$

Thus, by passing to the limit $t \to \infty$, we obtain

$$\begin{aligned}\lim_{t\to\infty} \varrho_{Z_\gamma}(t,t+h) &= \frac{2^{1-2\gamma}}{\Gamma(\gamma)^2}\int_h^{\infty} \left(u^2 - h^2\right)^{\gamma-1} e^{-\kappa u}\,\mathrm{d}u \\ &= \frac{2^{1-2\gamma}}{\Gamma(\gamma)^2}\mathcal{L}\left[u \mapsto \left(u^2-h^2\right)^{\gamma-1}\mathbf{1}_{(h,\infty)}(u)\right](\kappa) = \frac{2^{1-2\gamma}}{\Gamma(\gamma)^2}\frac{(2h)^{\gamma-\frac{1}{2}}\Gamma(\gamma)}{\sqrt{\pi}\kappa^{\gamma-\frac{1}{2}}} K_{\gamma-\frac{1}{2}}(\kappa h),\end{aligned}$$

where $\mathcal{L}[f](\kappa)$ denotes the Laplace transform of the function $f\colon [0,\infty) \to \mathbb{R}$ evaluated at κ, and the last identity follows from [161, Chapter I, Formula (3.13)]. ☐

2.5. SPATIOTEMPORAL WHITTLE–MATÉRN FIELDS

In this section, we demonstrate how the results of the previous Sections 2.3 and 2.4 can be related to the widely used statistical models involving generalized Whittle–Matérn operators (2.1.3) on $H = L^2(\mathcal{X})$, where $\mathcal{X} = \mathcal{D} \subsetneq \mathbb{R}^d$ is a bounded domain in the Euclidean space (see Subsection 2.5.1) or a surface $\mathcal{X} = \mathcal{M}$ (see Subsection 2.5.2).

2.5.1. BOUNDED EUCLIDEAN DOMAINS

Throughout this subsection, let $\emptyset \neq \mathcal{D} \subsetneq \mathbb{R}^d$ be a bounded, connected and open domain. In order to rigorously define the symmetric, strongly elliptic second-order differential operator L, formally given by (2.1.3), as a linear operator on $L^2(\mathcal{D})$, we make the following assumptions on its coefficients $\kappa\colon \mathcal{D} \to \mathbb{R}$ and $a\colon \mathcal{D} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$, as well as on the spatial domain $\mathcal{D} \subsetneq \mathbb{R}^d$.

Assumption 2.5.1 (Euclidean domain—minimal conditions)**.**

(i) $\mathcal{D}$ has a Lipschitz continuous boundary $\partial\mathcal{D}$;

(ii) $a \in L^\infty(\mathcal{D}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ is strongly elliptic, i.e.,

$$\exists \theta > 0: \qquad \operatorname*{ess\,inf}_{x\in\mathcal{D}} \xi^\top a(x)\xi \geq \theta \|\xi\|^2_{\mathbb{R}^d} \qquad \forall \xi \in \mathbb{R}^d;$$

(iii) $\kappa \in L^\infty(\mathcal{D})$.

Under these assumptions, we introduce the bilinear form

$$\mathfrak{a}_L\colon H_0^1(\mathcal{D})\times H_0^1(\mathcal{D})\to\mathbb{R},\qquad \mathfrak{a}_L(u,v):=\langle a\nabla u,\nabla v\rangle_{L^2(\mathcal{D})}+\langle\kappa^2 u,v\rangle_{L^2(\mathcal{D})},$$

which is symmetric, continuous and coercive. We say that $u\in H_0^1(\mathcal{D})$ belongs to the domain $\mathsf{D}(L)$ of the differential operator L if and only if $|\mathfrak{a}_L(u,v)|\lesssim_u\|v\|_{L^2(\mathcal{D})}$ holds for all $v\in H_0^1(\mathcal{D})$. In this case, we define Lu as the unique element of $L^2(\mathcal{D})$ which satisfies the relation $\mathfrak{a}_L(u,v)=\langle Lu,v\rangle_{L^2(\mathcal{D})}$ for all $v\in H_0^1(\mathcal{D})$.

By the Lax–Milgram theorem the inverse $L^{-1}\in\mathscr{L}(L^2(\mathcal{D});H_0^1(\mathcal{D}))$ exists and can be extended to $L^{-1}\in\mathscr{L}(H_0^1(\mathcal{D})^*;H_0^1(\mathcal{D}))$. Moreover, it is a consequence of the Rellich–Kondrachov theorem (see [2, Theorem 6.3]) that L^{-1} is compact on $L^2(\mathcal{D})$. For this reason, the spectral theorem for self-adjoint compact operators is applicable and shows that there exist an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ for $L^2(\mathcal{D})$ and a non-decreasing sequence $(\lambda_j)_{j\in\mathbb{N}}$ of positive real numbers accumulating only at infinity such that $Le_j=\lambda_j e_j$ holds for all $j\in\mathbb{N}$. Furthermore, the eigenvalues of L satisfy the following asymptotic behavior, known as Weyl's law (see Theorem 1.2.12):

$$\lambda_j\eqsim j^{2/d}\quad\forall j\in\mathbb{N}. \tag{2.5.1}$$

In this setting, for two differential operators L and $\widetilde{L}$ on $L^2(\mathcal{D})$ with coefficients a,κ and $\widetilde{a},\widetilde{\kappa}$, respectively, we obtain the following corollary from the regularity results in Section 2.3 for spatiotemporal Whittle–Matérn fields, where $A:=L^\beta$ and $Q:=\widetilde{L}^{-\alpha}$.

Corollary 2.5.2. *Let $\alpha,\beta,\sigma\in[0,\infty)$, set $r:=\frac{\alpha}{\beta}\wedge\sigma$ if $\beta>0$ and $r:=\sigma$ if $\beta=0$, and suppose that $n\in\mathbb{N}_0$, $\tau\in[0,1)$ and $\gamma\in\left(n+\frac{(\sigma-r)\vee 1}{2},\infty\right)$ are such that*

$$\gamma\geq n+\tfrac{1+(\sigma-r)\vee(2\tau)}{2}\quad\text{and}\quad\beta\gamma>\tfrac{d}{4}-\tfrac{\alpha}{2}+\beta\left(n+\tau+\tfrac{1+\sigma}{2}\right). \tag{2.5.2}$$

Let $L\colon\mathsf{D}(L)\subseteq H_0^1(\mathcal{D})\to L^2(\mathcal{D})$ and $\widetilde{L}\colon\mathsf{D}(\widetilde{L})\subseteq H_0^1(\mathcal{D})\to L^2(\mathcal{D})$ be symmetric, strongly elliptic second-order differential operators as defined above, cf. (2.1.3). Suppose that Assumption 2.5.1(i) holds for $\mathcal{D}\subsetneq\mathbb{R}^d$, and that the coefficients a,κ of L and $\widetilde{a},\widetilde{\kappa}$ of $\widetilde{L}$ satisfy Assumptions 2.5.1(ii),(iii). Assume moreover that L and $\widetilde{L}$ diagonalize with respect to the same orthonormal basis $(e_j)_{j\in\mathbb{N}}$ for $L^2(\mathcal{D})$, i.e., there exist non-decreasing sequences $(\lambda_j)_{j\in\mathbb{N}}$, $(\widetilde{\lambda}_j)_{j\in\mathbb{N}}$ of positive real numbers such that $Le_j=\lambda_j e_j$ and $\widetilde{L}e_j=\widetilde{\lambda}_j e_j$ for all $j\in\mathbb{N}$.

Then, setting $A:=L^\beta$ and $Q:=\widetilde{L}^{-\alpha}$, the mild solution Z_γ to (2.3.1) in the sense of Definition 2.3.7, see also (2.1.4), belongs to $C^{n,\tau}([0,T];L^p(\Omega;\dot{H}^\sigma_A))$ for all $p\in[1,\infty)$. If the above conditions hold with $n=0$ and $\tau\in(0,1)$, then for every $p\in[1,\infty)$ and all $\tau'\in[0,\tau)$ the mild solution Z_γ has a modification $\widehat{Z}_\gamma\in L^p\big(\Omega;C^{0,\tau'}([0,T];\dot{H}^\sigma_A)\big)$.

Proof. By the spectral mapping theorem for fractional powers of operators, see e.g. [148, Section 5.3], we obtain that $Ae_j=L^\beta e_j=\lambda_j^\beta e_j$ and $Qe_j=\widetilde{L}^{-\alpha}e_j=\widetilde{\lambda}_j^{-\alpha}e_j$. In particular, A inherits the self-adjointness and strict positive-definiteness from L. This readily implies that $0\in\rho(A)$. It follows from [114, Proposition 10.2.23] that $A_{\mathbb{C}}$ admits a bounded H^∞-calculus of angle $\omega_{H^\infty}(A_{\mathbb{C}})=0$, showing that Assumptions 2.3.1(i)–(iv) are satisfied for A.

Furthermore, we note that, for every $\sigma, s \in [0,\infty)$, we have that $\dot{H}^\sigma_A = \dot{H}^{\sigma\beta}_L$ and the spaces $\dot{H}^s_L$ and $\dot{H}^s_{\widetilde{L}}$ are isomorphic. The latter fact follows from the asymptotic behavior (2.5.1) of the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ and $(\widetilde{\lambda}_j)_{j\in\mathbb{N}}$, since L and $\widetilde{L}$ have the same eigenfunctions. Thus, we obtain that $Q^{\frac{1}{2}} = \widetilde{L}^{-\frac{\alpha}{2}} \in \mathscr{L}(H;\dot{H}^\alpha_L) \subseteq \mathscr{L}(H;\dot{H}^r_A)$.

Since $\gamma \in \left(\frac{1}{2}+n,\infty\right) \cap \left[\frac{1}{2}+n+\frac{\sigma-r}{2},\infty\right)$ is assumed, by Proposition 2.3.14 (see also Remark 2.3.15) the condition (2.3.19) of Theorem 2.3.12 is equivalent to requiring that $A^{n+\frac{1}{2}-\gamma}Q^{\frac{1}{2}} \in \mathscr{L}_2(H;\dot{H}^\sigma_A)$. Since also $\gamma \in \left(\frac{\sigma-r}{2}+n,\infty\right) \cap \left[n+\tau+\frac{1}{2},\infty\right)$, we therefore conclude with Theorem 2.3.12 that it suffices to check that the quantity

$$
\begin{aligned}
\left\| A^{\frac{\sigma}{2}+n+\tau+\frac{1}{2}-\gamma} Q^{\frac{1}{2}} \right\|^2_{\mathscr{L}_2(H)} &= \left\| L^{\beta\left(\frac{\sigma}{2}+n+\tau+\frac{1}{2}-\gamma\right)} \widetilde{L}^{-\frac{\alpha}{2}} \right\|^2_{\mathscr{L}_2(H)} \\
&= \sum_{j=1}^{\infty} \left\| L^{\beta\left(\frac{\sigma}{2}+n+\tau+\frac{1}{2}-\gamma\right)} \widetilde{L}^{-\frac{\alpha}{2}} e_j \right\|^2_H = \sum_{j=1}^{\infty} \lambda_j^{2\beta\left(\frac{\sigma}{2}+n+\tau+\frac{1}{2}-\gamma\right)} \widetilde{\lambda}_j^{-\alpha}
\end{aligned}
\tag{2.5.3}
$$

is finite. Indeed, applying Weyl's law (2.5.1) to both L and $\widetilde{L}$, it follows that

$$
\sum_{j=1}^{\infty} \lambda_j^{2\beta\left(\frac{\sigma}{2}+n+\tau+\frac{1}{2}-\gamma\right)} \widetilde{\lambda}_j^{-\alpha} \eqsim_{(\alpha,\beta,\gamma,\sigma,n,\tau)} \sum_{j=1}^{\infty} j^{\frac{4}{d}\left[\beta\left(n+\tau+\frac{1+\sigma}{2}\right)-\beta\gamma-\frac{\alpha}{2}\right]},
$$

so that (2.5.3) is finite if and only if (2.5.2) holds, as we assume. Then, given any $p \in [1,\infty)$, Theorem 2.3.8, Theorem 2.3.12 and Proposition 2.3.14 yield the existence of a mild solution $Z_\gamma \in C^{n,\tau}([0,T];L^p(\Omega;\dot{H}^\sigma_A))$, which is unique up to modification. The last assertion for $n=0$ and $\tau \in (0,1)$ follows from Corollary 2.3.13. □

The spatial regularity obtained in Corollary 2.5.2 is measured using the spaces $\dot{H}^\sigma_A = \dot{H}^{\beta\sigma}_L$. It would be more practical to express this in terms of fractional-order Sobolev spaces $H^s(\mathcal{D})$, $s \geq 0$. This raises the question of how $\dot{H}^s_L$ and $H^s(\mathcal{D})$ relate. The answer to this question depends on the smoothness of the coefficients a, κ and of the boundary $\partial\mathcal{D}$. We therefore introduce two additional sets of assumptions: Assumption 2.5.3 is only slightly more restrictive than the minimal conditions of Assumption 2.5.1, whereas Assumption 2.5.4 requires a high degree of smoothness.

Assumption 2.5.3 (Euclidean domain—$H^2(\mathcal{D})$-regular setting)**.**

(i) $\mathcal{D}$ is convex.

(ii) $a\colon \overline{\mathcal{D}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is Lipschitz continuous, i.e.,

$$
|a_{ij}(x) - a_{ij}(y)| \lesssim \|x-y\|_{\mathbb{R}^d} \quad \forall x,y \in \overline{\mathcal{D}}, \quad \forall i,j \in \{1,\ldots,d\}.
$$

Assumption 2.5.4 (Euclidean domain—smooth setting)**.**

(i) The boundary $\partial\mathcal{D}$ is of class C^∞;

(ii) $a_{ij} \in C^\infty(\overline{\mathcal{D}})$ holds for all $i,j \in \{1,\ldots,d\}$, i.e., for all entries of a;

(iii) $\kappa \in C^\infty(\overline{\mathcal{D}})$.

The results of the next lemma are taken from [50, Lemma 2] and [28, Lemma 3.4].

Lemma 2.5.5. *Let $L\colon \mathsf{D}(L) \subseteq H_0^1(\mathcal{D}) \to L^2(\mathcal{D})$ be a symmetric second-order differential operator as defined as above, cf.* (2.1.3). *Then, the following assertions hold:*

(a) *If Assumption 2.5.1 is satisfied, then $\dot{H}_L^s \hookrightarrow H^s(\mathcal{D})$ for all $s \in [0,1]$. Moreover, the norms $\|\cdot\|_{\dot{H}_L^s}$ and $\|\cdot\|_{H^s(\mathcal{D})}$ are equivalent on $\dot{H}_L^s$ for $s \in [0,1] \setminus \{1/2\}$;*

(b) *If Assumptions 2.5.1 and 2.5.3 are satisfied, then*

$$\big(\dot{H}_L^s, \|\cdot\|_{\dot{H}_L^s}\big) \cong \big(H^s(\mathcal{D}) \cap H_0^1(\mathcal{D}), \|\cdot\|_{H^s(\mathcal{D})}\big) \quad \forall s \in [1,2];$$

(c) *If Assumptions 2.5.1 and 2.5.4 are satisfied, then $\dot{H}_L^s \hookrightarrow H^s(\mathcal{D})$ for all $s \in [0,\infty)$, and the norms $\|\cdot\|_{\dot{H}_L^s}, \|\cdot\|_{H^s(\mathcal{D})}$ are equivalent on $\dot{H}_L^s$ for all $s \in [0,\infty) \setminus \mathfrak{E}$, where $\mathfrak{E} := \{2k + 1/2 : k \in \mathbb{N}_0\}$ is called the* exclusion set.

Combining Lemma 2.5.5 with the results of Corollary 2.5.2 shows that the mild solution Z_γ is an element of $C^{\eta,\tau}([0,T]; L^p(\Omega; H^{\beta\sigma}(\mathcal{D})))$, provided that $\sigma\beta \in [0,s']$, where $s' \in [1,\infty)$ is prescribed by the smoothness of the coefficients a, κ and the boundary $\partial\mathcal{D}$ via Lemma 2.5.5(a), (b) or (c). Note that we do not have to take the exclusion set $\mathfrak{E}$ into account, as we only need the embedding $\dot{H}_L^s \hookrightarrow H^s(\mathcal{D})$.

Lastly, we consider the covariance structure of the mild solution, as treated in the abstract setting in Section 2.4. The most illustrative results are the asymptotic formulas presented in Corollaries 2.4.2 and 2.4.3, which we translate to the current setting in Corollary 2.5.6. We see that (Whittle–)Matérn operators are recovered as marginal spatial or temporal covariance operators.

Corollary 2.5.6. *Consider the setting of Corollary 2.5.2 with $L = \widetilde{L}$, i.e., $Q := L^{-\alpha}$. Let $\alpha, \beta \in [0,\infty)$ and $\gamma \in (1/2,\infty)$ be such that $\beta\gamma > \frac{1}{2}\big(\frac{d}{2} - \alpha + \beta\big)$, and let Z_γ be the mild solution in the sense of Definition 2.3.7. Then the asymptotic marginal spatial covariance of Z_γ satisfies*

$$\lim_{t\to\infty} Q_{Z_\gamma}(t,t) = \Gamma(\gamma - 1/2)\big[2\sqrt{\pi}\,\Gamma(\gamma)\big]^{-1} L^{\beta(1-2\gamma)-\alpha} \quad \text{in } \mathscr{L}(L^2(\mathcal{D})).$$

For $\beta = 0$, the covariance of Z_γ is separable in the sense that there exists a function $\varrho_{Z_\gamma}\colon [0,\infty) \times [0,\infty) \to \mathbb{R}$ such that

$$Q_{Z_\gamma}(s,t) = \varrho_{Z_\gamma}(s,t)\, L^{-\alpha} \quad \forall s,t \in [0,\infty),$$

and for all $h \in \mathbb{R} \setminus \{0\}$ we have

$$\lim_{t\to\infty} Q_{Z_\gamma}(t,t+h) = 2^{\frac{1}{2}-\gamma}\big[\sqrt{\pi}\,\Gamma(\gamma)\big]^{-1} |h|^{\gamma-\frac{1}{2}} K_{\gamma-\frac{1}{2}}(|h|)\, L^{-\alpha} \quad \text{in } \mathscr{L}(L^2(\mathcal{D})).$$

Proof. Existence and uniqueness of the mild solution Z_γ follow from Corollary 2.5.2 with $L = \widetilde{L}$ and $\eta = \tau = \sigma = 0$. Recall from its proof that the operator A satisfies Assumptions 2.3.1(i)–(iv). Note also that $A = L^\beta$ is self-adjoint and $Q = L^{-\alpha} \in \mathscr{L}(L^2(\mathcal{D}))$ commutes with A, so that it also commutes with $S(t)$ for all $t \in [0,\infty)$, cf. [100, Theorem 1.3.2(a)]. All assertions follow thus from Corollaries 2.4.2 and 2.4.3. □

Remark 2.5.7. The asymptotic results obtained in Corollary 2.5.6 are in accordance with the marginal spatial and temporal covariance functions derived in [140, Section 3, Proposition 1 and Corollary 1] for the case of the operator $L = \gamma_s^2 - \Delta$ acting on functions defined on all of $\mathbb{R}^2$, where $\gamma_s \in (0,\infty)$. Note that, in order to exploit Fourier techniques, in [140] the "time" variable t is an element of the whole real axis, $t \in \mathbb{R}$, instead of only its non-negative part.

Remark 2.5.8. Corollaries 2.5.2 and 2.5.6 explain and justify the roles of the parameters α, β and γ. They control three important properties of spatiotemporal Whittle–Matérn fields. Besides the temporal and spatial smoothness, measured respectively by the quantities $n+\tau$ and σ, we identify a third degree of freedom: The *degree of separability*, expressed by the ratio $\frac{\alpha}{\beta} \in [0,\infty]$. Indeed, if $\frac{\alpha}{\beta} = \infty$, i.e. $\beta = 0$, we observe that the covariance of the field is separable and that its temporal and spatial behavior are exclusively governed by the parameters γ and α, respectively. In contrast, if $\frac{\alpha}{\beta} = 0$, i.e. $\alpha = 0$, the SPDE is driven by spatiotemporal Gaussian white noise and the "coloring" of its solution is fully determined by the fractional parabolic differential operator $\left(\partial_t + L^{\beta}\right)^{\gamma}$.

2.5.2. SURFACES

In this subsection, we provide a brief demonstration of how the above results can be extended to spatiotemporal Whittle–Matérn fields on more general spatial domains. More precisely, we consider a smooth, closed, connected, orientable and compact 2-surface $\mathcal{M}$ immersed in $\mathbb{R}^3$ and endowed with the positive surface measure $\nu_{\mathcal{M}}$ on $\mathcal{B}(\mathcal{M})$, induced by the first fundamental form. An important example of such a surface is given by the 2-sphere, $\mathcal{M} = \mathbb{S}^2$.

On $H := L^2(\mathcal{M})$, we consider the following analog of the symmetric, strongly elliptic second-order differential operator from Subsection 2.5.1, formally given by

$$Lu := -\nabla_{\mathcal{M}} \cdot (a \nabla_{\mathcal{M}} u) + \kappa^2 u, \qquad u \in \mathsf{D}(L) \subseteq L^2(\mathcal{M}),$$

where $\nabla_{\mathcal{M}} \cdot$ and $\nabla_{\mathcal{M}}$ denote the surface divergence and the surface gradient, respectively, see Section 1.2.4. We record the precise conditions on the surface $\mathcal{M}$ and on the coefficients a, κ in Assumption 2.5.9 below; with regard to smoothness, they are analogous to the setting of Assumption 2.5.4 in the case of a bounded Euclidean domain.

Assumption 2.5.9 (Surface—smooth setting)**.**

(i) a is a symmetric tensor field, i.e., $a(x) \colon T_x\mathcal{M} \to T_x\mathcal{M}$ is linear and symmetric for all $x \in \mathcal{M}$, where $T_x\mathcal{M}$ denotes the tangent space of x. Moreover, a is smooth and strongly elliptic in the following sense:

$$\exists \theta > 0 : \quad \forall x \in \mathcal{M}, \forall \xi \in T_x\mathcal{M} : \quad \xi^\top a(x)\xi \geq \theta \|\xi\|_{\mathbb{R}^3}^2.$$

(ii) The coefficient $\kappa \colon \mathcal{M} \to \mathbb{R}$ is smooth and bounded away from zero, i.e., there exists $\kappa_0 \in (0,\infty)$ such that $|\kappa(x)| \geq \kappa_0$ for all $x \in \mathcal{M}$.

The conditions in Assumption 2.5.9 are sufficient to ensure that $L\colon \dot{H}^1_L \to (\dot{H}^1_L)^*$ is boundedly invertible, and has a compact inverse on $L^2(\mathcal{M})$. This allows us to find an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ for $L^2(\mathcal{M})$ and a non-decreasing sequence of positive real eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ of L accumulating only at infinity, as in Subsection 2.5.1. Moreover, fractional powers L^β are well-defined for all $\beta\in\mathbb{R}$, the sequence of eigenvalues still satisfies Weyl's law (2.5.1) (with $d=2$), and a spectral mapping theorem holds, cf. [190, Theorems XII.1.3 and XII.2.1]. These facts are sufficient to repeat the proofs of Corollaries 2.5.2 and 2.5.6 yielding the analogous results, with $d=2$ and other obvious modifications to the conditions. In particular, the analog of Corollary 2.5.2 on the surface $\mathcal{M}$ implies regularity of the solution process in the space $C^{\eta,\tau}([0,T];L^p(\Omega;H^{\beta\sigma}(\mathcal{M})))$.

An important difference from the (smooth) Euclidean setting of Assumption 2.5.4 is that under Assumption 2.5.9, the Sobolev space $H^s(\mathcal{M})$ and $\dot{H}^s_L$ are isomorphic for every $s\in[0,\infty)$, see [190, Example XII.2.1]. In other words, the absence of a boundary $\partial\mathcal{M}$ implies that one does not need to exclude the exception set $\mathfrak{E}$ from the admissible exponents s in the analog of Lemma 2.5.5(c).

APPENDIX TO CHAPTER 2

2.A. AUXILIARY RESULTS

Throughout this section, H denotes a separable Hilbert space which, if not specified otherwise, is considered over the real scalar field $\mathbb{R}$.

2.A.1. BOCHNER COUNTERPARTS

The first auxiliary result records relations between a (possibly unbounded) linear operator $A\colon \mathsf{D}(A)\subseteq H\to H$ and its Bochner space counterpart $\mathcal{A}$ which is defined on a subspace of $L^2(0,T;H)$, where $T\in(0,\infty)$.

Lemma 2.A.1. *Let $T\in(0,\infty)$ and $A\colon \mathsf{D}(A)\subseteq H\to H$ be a linear operator on a real or complex Hilbert space H. Consider the associated operator $\mathcal{A}$ on $L^2(0,T;H)$ as defined in (2.3.4). Then, the following hold:*

(a) *$\mathcal{A}$ is bounded if and only if A is bounded, and in that case we have*

$$\|\mathcal{A}\|_{\mathscr{L}(L^2(0,T;H))} = \|A\|_{\mathscr{L}(H)};$$

(b) *$\mathcal{A}$ is closed if and only if A is.*

Proof. If A is bounded, then the inequality $\|\mathcal{A}\|_{\mathscr{L}(L^2(0,T;H))}\le\|A\|_{\mathscr{L}(H)}$ is easily verified. Now suppose that $\mathcal{A}$ is bounded. Then for all $x\in H$ we have

$$\begin{aligned}\|Ax\|_H &= \|T^{-1/2}\mathbf{1}_{(0,T)}\otimes Ax\|_{L^2(0,T;H)} = \|\mathcal{A}(T^{-1/2}\mathbf{1}_{(0,T)}\otimes x)\|_{L^2(0,T;H)}\\ &\le \|\mathcal{A}\|_{\mathscr{L}(L^2(0,T;H))}\|T^{-1/2}\mathbf{1}_{(0,T)}\otimes x\|_{L^2(0,T;H)} = \|\mathcal{A}\|_{\mathscr{L}(L^2(0,T;H))}\|x\|_H.\end{aligned}$$

Here, given $f\colon(0,T)\to\mathbb{R}$ and $x\in H$, the function $f\otimes x\colon(0,T)\to H$ is defined by $[f\otimes x](t):=f(t)x$ for all $t\in(0,T)$. We thus find that A is bounded with operator norm $\|A\|_{\mathscr{L}(H)}\le\|\mathcal{A}\|_{\mathscr{L}(L^2(0,T;H))}$, which finishes the proof of (a).

To prove part (b), first let A be closed and let the sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathsf{D}(\mathcal{A})$ be such that $v_n \to v$ and $\mathcal{A}v_n \to y$ in $L^2(0,T;H)$. We need to prove that $v \in \mathsf{D}(\mathcal{A})$ and $y = \mathcal{A}v$. Let $(v_{n_k})_{k\in\mathbb{N}}$ be a subsequence such that $v_{n_k} \to v$ and $Av_{n_k} \to y$ in H, a.e. in $(0,T)$, so that by the closedness of A it follows that $v(\vartheta) \in \mathsf{D}(A)$ and $y(\vartheta) = Av(\vartheta)$ for a.a. $\vartheta \in (0,T)$. From the latter we obtain that $y = \mathcal{A}v$, which is meaningful since $v, y \in L^2(0,T;H)$ yields that $v \in \mathsf{D}(\mathcal{A})$.

Now let $\mathcal{A}$ be closed and let $(x_n)_{n\in\mathbb{N}}$ in $\mathsf{D}(A)$ be such that $x_n \to x$ and $Ax_n \to y$ in H. This implies the following convergences in $L^2(0,T;H)$:

$$\begin{aligned} \mathbf{1}_{(0,T)} \otimes x_n &\to \mathbf{1}_{(0,T)} \otimes x, \\ \mathcal{A}(\mathbf{1}_{(0,T)} \otimes x_n) = \mathbf{1}_{(0,T)} \otimes Ax_n &\to \mathbf{1}_{(0,T)} \otimes y. \end{aligned}$$

Since $\mathcal{A}$ is closed, we deduce that $\mathbf{1}_{(0,T)} \otimes x \in \mathsf{D}(\mathcal{A})$ and $\mathbf{1}_{(0,T)} \otimes y = \mathcal{A}(\mathbf{1}_{(0,T)} \otimes x)$, from which we may conclude $x \in \mathsf{D}(A)$ and $y = Ax$. Hence A is closed. □

The following lemma is generally useful for determining the domain of a generator of a given C_0-semigroup, and it will subsequently be used to show that the Bochner space counterpart of a C_0-semigroup is again a C_0-semigroup, see Proposition 2.A.3.

Lemma 2.A.2. *Let $(S(t))_{t\ge0}$ be a C_0-semigroup on H with infinitesimal generator $\widetilde{A}\colon \mathsf{D}(\widetilde{A}) \subseteq H \to H$. If $A\colon \mathsf{D}(A) \subseteq H \to H$ is a linear operator satisfying $A \subseteq \widetilde{A}$ and $\mathsf{D}(A)$ is dense in $\mathsf{D}(\widetilde{A})$ with respect to the graph norm $\|\cdot\|_{\mathsf{D}(\widetilde{A})}$, then $\widetilde{A} = \overline{A}$.*

Proof. Let $(x, \widetilde{A}x) \in \mathsf{G}(\widetilde{A})$ and choose a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathsf{D}(A)$ such that $x_n \to x$ in $\mathsf{D}(\widetilde{A})$. Using $A \subseteq \widetilde{A}$, we have $(x_n, Ax_n) = (x_n, \widetilde{A}x_n) \to (x, \widetilde{A}x)$ with respect to the product norm on $H \times H$, which shows that $(x, \widetilde{A}x) \in \overline{\mathsf{G}(A)}$. Conversely, for any $(x,y) \in \overline{\mathsf{G}(A)}$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathsf{D}(A)$ such that $(x_n, \widetilde{A}x_n) = (x_n, Ax_n) \to (x,y)$ in $H \times H$. Since $\widetilde{A}$ is closed as the generator of a C_0-semigroup, see [73, Theorem II.1.4], we find that $(x,y) \in \mathsf{G}(\widetilde{A})$. This proves $\overline{\mathsf{G}(A)} = \mathsf{G}(\widetilde{A})$. □

Proposition 2.A.3. *Let $T \in (0,\infty)$ and let Assumption 2.3.1(i) be satisfied. The family $(\mathcal{S}(t))_{t\ge0}$ of operators on $L^2(0,T;H)$ given by (2.3.5) is a C_0-semigroup with infinitesimal generator $-\mathcal{A}$, as defined by (2.3.4).*

Proof. First note that the operators $(\mathcal{S}(t))_{t\ge0}$ are well-defined in the sense that they map elements in $L^2(0,T;H)$ to $L^2(0,T;H)$. In fact, Lemma 2.A.1(a) shows that we have $\|\mathcal{S}(t)\|_{\mathscr{L}(L^2(0,T;H))} = \|S(t)\|_{\mathscr{L}(H)}$ for all $t \ge 0$.

We now check that $(\mathcal{S}(t))_{t\ge0}$ is a C_0-semigroup. Clearly, $\mathcal{S}(0) = I$ and the semigroup property holds. Let $M \ge 1$ and $w \in \mathbb{R}$ be as in (2.3.2), so that

$$\forall h \in [0,1]: \quad \|S(h)\|_{\mathscr{L}(H)} \le Me^{-wh} \le Me^{(-w)\vee 0} =: \widetilde{M}.$$

To show strong continuity, let $x \in H$, $h \in (0,1)$ and note that

$$\|S(h)x - x\|_H^2 \le 2\|S(h)x\|_H^2 + 2\|x\|_H^2 \le 2\big(\widetilde{M}^2 + 1\big)\|x\|_H^2.$$

By dominated convergence, $\lim_{h\downarrow0} \|\mathcal{S}(h)v - v\|_{L^2(0,T;H)} = 0$ for $v \in L^2(0,T;H)$.

Next we investigate the infinitesimal generator of $(\mathcal{S}(t))_{t\geq 0}$, which we denote by $-\widetilde{\mathcal{A}}$ for the time being. We wish to show that $\widetilde{\mathcal{A}} = \mathcal{A}$. Let $x \in \mathsf{D}(A)$ and consider

$$\left\| \frac{1}{h}(S(h)x - x) + Ax \right\|_H^2 \leq 2\left\| \frac{1}{h}(S(h)x - x) \right\|_H^2 + 2\|Ax\|_H^2.$$

To bound the first term, we use [165, Chapter 1, Theorem 2.4(d)] and note that, for every $h \in (0,1)$, we obtain

$$\left\| \frac{1}{h}(S(h)x - x) \right\|_H^2 = \left\| \frac{1}{h} \int_0^h S(s)Ax \, \mathrm{d}s \right\|_H^2 \leq \frac{1}{h^2} \left| \int_0^h \|S(s)Ax\|_H \, \mathrm{d}s \right|^2 \leq \widetilde{M}^2 \|Ax\|_H^2.$$

The two previous displays show that, for $v \in L^2(0,T;\mathsf{D}(A))$ and all $h \in (0,1)$,

$$\int_0^T \left\| \frac{1}{h}(S(h)v(\vartheta) - v(\vartheta)) + Av(\vartheta) \right\|_H^2 \mathrm{d}\vartheta \leq 2\big(\widetilde{M}^2 + 1\big)\|\mathcal{A}v\|_{L^2(0,T;H)}^2 < \infty.$$

This justifies the use of the dominated convergence theorem to conclude that

$$-\widetilde{\mathcal{A}}v = \lim_{h\downarrow 0} \frac{1}{h}(\mathcal{S}(h)v - v) = -\mathcal{A}v \qquad \text{in } L^2(0,T;H),$$

i.e., $-\mathcal{A} \subseteq -\widetilde{\mathcal{A}}$ as $v \in \mathsf{D}(\mathcal{A}) = L^2(0,T;\mathsf{D}(A))$ was arbitrary. Since $\mathsf{D}(\mathcal{A})$ is dense in $L^2(0,T;H)$ (by density of $\mathsf{D}(A)$ in H), and $\mathcal{S}(t)$ maps $\mathsf{D}(\mathcal{A})$ to itself for each $t \geq 0$, Proposition II.1.7 of [73] implies that $\mathsf{D}(\mathcal{A})$ is dense in the domain $\mathsf{D}(\widetilde{\mathcal{A}})$ of the generator of $(\mathcal{S}(t))_{t\geq 0}$ with respect to the graph norm $\|\cdot\|_{\mathsf{D}(\widetilde{\mathcal{A}})}$. Applying Lemma 2.A.2 and noting that $\mathcal{A}$ is closed by Lemma 2.A.1(b) completes the proof. □

2.A.2. TRANSLATION OPERATORS

Lemma 2.A.4. *Let U be a real and separable Hilbert space and let $J := (0,T)$ for some $T \in (0,\infty]$. For every $u \in L^2(J;U)$ we have that*

$$\lim_{h\to 0} \|u(\cdot + h) - u\|_{L^2(J_h;U)} = 0.$$

Here, we define for each $h \in \mathbb{R}$ the interval $J_h := ((-h) \vee 0, T \wedge (T-h)) \subseteq J$ and we denote by $u(\cdot + h)\colon J_h \to U$ the function u shifted to the left by an increment h.

Proof. Let $v \in C_c^\infty(J;U)$ and fix an arbitrary $\varepsilon \in (0,\infty)$. Choose a compact interval $[a,b] \subset [0,\infty)$ such that $\operatorname{supp}\big(v(\cdot + h) - v|_{J_h}\big) \subseteq [a,b]$ for all $h \in [-1,1]$. By the uniform continuity of v, there exists a $\delta \in (0,1)$ such that, for all $h \in (-\delta,\delta)$ and every $t \in J_h$, the estimate $\|v(t+h) - v(t)\|_U < \sqrt{\varepsilon/(b-a)}$ holds. Thus,

$$\|v(t+h) - v(t)\|_{L^2(J_h;U)}^2 < \varepsilon \quad \forall h \in (-\delta,\delta).$$

This shows the desired convergence for functions in the space $C_c^\infty(J;U)$, which is dense in $L^2(J;U)$; indeed, since the set of U-valued measurable simple functions is dense in $L^2(J;U)$ [113, Lemma 1.2.19(1)], it suffices to note that the scalar-valued function space $C_c^\infty(J)$ is dense in $L^2(J)$ [2, Corollary 2.30]. Combined with the fact that the translation operator is contractive from $L^2(J;U)$ to $L^2(J_h;U)$ (hence, in particular, bounded uniformly in h), the result extends to $L^2(J;U)$. □

Proposition 2.A.5. *Let $T \in (0,\infty)$. The family $(\mathcal{T}(t))_{t\geq 0} \subseteq \mathscr{L}(L^2(0,T;H))$ defined in (2.3.6) is a C_0-semigroup whose infinitesimal generator is given by $-\partial_t$, where ∂_t is the Bochner–Sobolev vector-valued weak derivative on $\mathsf{D}(\partial_t) = H^1_{0,\{0\}}(0,T;H)$.*

Proof. For each $t \geq 0$, it is clear that $\mathcal{T}(t)$ is a well-defined contractive linear map on $L^2(0,T;H)$. Furthermore, it follows readily from the definition (2.3.6) that $\mathcal{T}(0) = I$ and that the semigroup property is satisfied, since for all $s,t \geq 0$, $v \in L^2(0,T;H)$ and a.a. $\vartheta \in [0,T]$ we have that

$$[\mathcal{T}(t)\mathcal{T}(s)v](\vartheta) = \widetilde{[\mathcal{T}(s)v]}(\vartheta - t) = \widetilde{v}(\vartheta - t - s) = [\mathcal{T}(t+s)v](\vartheta).$$

The strong continuity follows from Lemma 2.A.4 for $h \uparrow 0$.

Next, we turn to the generator of $(\mathcal{T}(t))_{t\geq 0}$. To this end, let $v \in C_c^\infty((0,T];H)$ be arbitrary and note that its extension by zero to $(-\infty,T]$, again denoted by $\widetilde{v}$, is continuously differentiable with classical (and hence weak) derivative $\partial_\vartheta \widetilde{v} = \widetilde{\partial_\vartheta v}$ by the compact support of v in $(0,T]$. Fix an arbitrary $\vartheta \in [0,T]$. The function $t \mapsto \widetilde{v}(\vartheta - t)$ is continuously differentiable on $[0,\infty)$ with derivative $t \mapsto -\widetilde{\partial_\vartheta v}(\vartheta - t)$ by the chain rule. Thus, the fundamental theorem of calculus gives

$$\mathcal{T}(t)v(\vartheta) - v(\vartheta) = \widetilde{v}(\vartheta - t) - \widetilde{v}(\vartheta) = -\int_0^t \widetilde{\partial_\vartheta v}(\vartheta - s)\,\mathrm{d}s = -\int_0^t [\mathcal{T}(s)\partial_\vartheta v](\vartheta)\,\mathrm{d}s$$

for every $t \geq 0$. It follows that

$$\mathcal{T}(t)v - v = -\int_0^t \mathcal{T}(s)\partial_\vartheta v\,\mathrm{d}s.$$

Furthermore, we know from [165, Chapter 1, Theorem 2.4(b)] that if R denotes the generator of $(\mathcal{T}(t))_{t\geq 0}$, then we have

$$\mathcal{T}(t)v - v = R\int_0^t \mathcal{T}(s)v\,\mathrm{d}s,$$

hence, combining the previous two displays yields

$$R\int_0^t \mathcal{T}(s)v\,\mathrm{d}s = -\int_0^t \mathcal{T}(s)\partial_\vartheta v\,\mathrm{d}s. \tag{2.A.1}$$

Set $v_t := \frac{1}{t}\int_0^t \mathcal{T}(s)v\,\mathrm{d}s$ for $t \in (0,\infty)$. It follows that $v_t \to \mathcal{T}(0)v = v$ in $L^2(0,T;H)$ as $t \downarrow 0$, see e.g. [165, Chapter 1, Theorem 2.4(a)]. Dividing both sides of (2.A.1) by $t \in (0,\infty)$ and passing to the limit $t \downarrow 0$, one obtains

$$Rv_t = R\frac{1}{t}\int_0^t \mathcal{T}(s)v\,\mathrm{d}s = -\frac{1}{t}\int_0^t \mathcal{T}(s)\partial_\vartheta v\,\mathrm{d}s \to -\mathcal{T}(0)\partial_\vartheta v = -\partial_\vartheta v.$$

Since R is assumed to be the generator of a C_0-semigroup, it is in particular closed by [73, Proposition II.1.4]. Combined with the convergence $v_t \to v$ and $Rv_t \to -\partial_\vartheta v$ as $t \downarrow 0$, this yields $v \in \mathsf{D}(R)$ and $Rv = -\partial_\vartheta v$, hence $-\partial_\vartheta|_{C_c^\infty((0,T];H)} \subseteq R$.

As $C_c^\infty((0,T];H)$ is dense in $L^2(0,T;H)$ and $\mathcal{T}(t)C_c^\infty((0,T];H) \subseteq C_c^\infty((0,T];H)$ for all $t \geq 0$, we have that $C_c^\infty((0,T];H)$ is dense in $\mathsf{D}(R)$ with respect to the graph norm of R by [73, Proposition II.1.7]. It is evident from the respective definitions that we have $\|\cdot\|_{\mathsf{D}(R)} \eqsim \|\cdot\|_{H^1(0,T;H)}$. These observations together imply

$$\mathsf{D}(R) = \overline{C_c^\infty((0,T];H)}^{\mathsf{D}(R)} = \overline{C_c^\infty((0,T];H)}^{H^1(0,T;H)} = H^1_{0,\{0\}}(0,T;H). \qquad \square$$

2.A.3. THE PROOF OF LEMMA 2.3.6

Proof of Lemma 2.3.6. Analogously to [56, Proposition 5.9] it can be shown that the operator defined by the right-hand side of (2.3.12) maps functions in $L^2(0,T;H)$ to $C_{0,\{T\}}([0,T];H)$. Now we prove the identity in (2.3.12). Let $f,g \in L^2(0,T;H)$ be arbitrary. By (2.3.11) and by continuity of the inner product $\langle\cdot,\cdot\rangle_H$, we find that

$$\begin{aligned}\langle \mathcal{B}^{-\gamma} f, g\rangle_{L^2(0,T;H)} &= \int_0^T \langle [\mathcal{B}^{-\gamma} f](t), g(t)\rangle_H \,\mathrm{d}t \\ &= \int_0^T \left\langle \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} S(t-s) f(s)\,\mathrm{d}s, g(t)\right\rangle_H \mathrm{d}t \\ &= \frac{1}{\Gamma(\gamma)} \int_0^T \int_0^T \langle \mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1} S(t-s) f(s), g(t)\rangle_H \,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{2.A.2}$$

Next, we would like to use Fubini's theorem to exchange the order of integration. By (2.3.2) the semigroup $(S(t))_{t\geq 0}$ is uniformly bounded on the compact interval $[0,T]$,

$$\widetilde{M}_T := \sup_{t\in[0,T]} \|S(t)\|_{\mathscr{L}(H)} \leq M e^{(-wT)\vee 0} < \infty.$$

We then use the Cauchy–Schwarz inequality on H and on $L^2(0,T)$ as well as the fact that $\gamma > \frac{1}{2}$ to check that

$$\begin{aligned} &\int_0^T \int_0^T \left|\langle \mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1} S(t-s) f(s), g(t)\rangle_H\right| \mathrm{d}s\,\mathrm{d}t \\ &\leq \widetilde{M}_T \int_0^T \int_0^t (t-s)^{\gamma-1} \|f(s)\|_H \,\mathrm{d}s\, \|g(t)\|_H \,\mathrm{d}t \\ &\leq \widetilde{M}_T \|f\|_{L^2(0,T;H)} \int_0^T \left(\int_0^t (t-s)^{2\gamma-2}\,\mathrm{d}s\right)^{1/2} \|g(t)\|_H \,\mathrm{d}t \\ &= \frac{\widetilde{M}_T}{\sqrt{2\gamma-1}} \|f\|_{L^2(0,T;H)} \int_0^T t^{\gamma-\frac{1}{2}} \|g(t)\|_H \,\mathrm{d}t \leq \frac{\widetilde{M}_T T^\gamma}{\sqrt{2\gamma(2\gamma-1)}} \|f\|_{L^2(0,T;H)} \|g\|_{L^2(0,T;H)} \end{aligned}$$

is finite. This justifies changing the order of integration in (2.A.2), which gives

$$\begin{aligned} \langle \mathcal{B}^{-\gamma} f, g\rangle_{L^2(0,T;H)} &= \frac{1}{\Gamma(\gamma)} \int_0^T \int_0^T \langle \mathbf{1}_{(s,T)}(t)(t-s)^{\gamma-1} S(t-s) f(s), g(t)\rangle_H \,\mathrm{d}t\,\mathrm{d}s \\ &= \frac{1}{\Gamma(\gamma)} \int_0^T \int_0^T \langle f(s), \mathbf{1}_{(s,T)}(t)(t-s)^{\gamma-1} [S(t-s)]^* g(t)\rangle_H \,\mathrm{d}t\,\mathrm{d}s \\ &= \int_0^T \left\langle f(s), \frac{1}{\Gamma(\gamma)} \int_s^T (t-s)^{\gamma-1} [S(t-s)]^* g(t)\,\mathrm{d}t \right\rangle_H \mathrm{d}s, \end{aligned}$$

where we interchanged integrals and inner products as before in the last step. □

2.A.4. HÖLDER CONTINUITY AND WEAK DERIVATIVES

Recall from Section 2.2 that $(W(t))_{t\geq 0}$ denotes an H-valued cylindrical Wiener process.

Lemma 2.A.6. *Let Assumptions 2.3.1(i),(ii) be satisfied, let $a \in \left(-\frac{1}{2}, \infty\right)$, $b, \sigma \in [0,\infty)$ and $\tau \in \left(0, a+\frac{1}{2}\right] \cap (0,1)$. If $\sigma \neq 0$, then suppose moreover that Assumption 2.3.1(iv) holds. Let $\Phi_{a,b} \colon (0,\infty) \to \mathscr{L}(H; \dot{H}^\sigma_A)$ be defined by (2.3.22) and let $J := (0,T)$ for some $T \in (0,\infty]$. Then, for all $p \in [1,\infty)$, $t \in [0,T)$ and $h \in J$ with $h \le T-t$,*

$$\Big\| \int_0^t [\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)] \,\mathrm{d}W(s) \Big\|_{L^p(\Omega;\dot{H}^\sigma_A)} \lesssim_{(p,a,\tau)} h^\tau \big\| A^{-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \big\|_{\mathscr{L}_2(H;\dot{H}^\sigma_A)}.$$

Proof. We first use the Burkholder–Davis–Gundy inequality (combined with nestedness of the L^p spaces if $p<2$) to bound the quantity of interest $I_\star$,

$$\begin{aligned} I_\star := \Big\| \int_0^t [\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)] \,\mathrm{d}W(s) \Big\|_{L^p(\Omega;\dot{H}^\sigma_A)} \\ \lesssim_p \Big[\int_0^t \|\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)} \,\mathrm{d}s \Big]^{1/2} \\ = \Big[\int_0^t \|\Phi_{a,b}(u+h) - \Phi_{a,b}(u)\|^2_{\mathscr{L}_2(H;\dot{H}^\sigma_A)} \,\mathrm{d}u \Big]^{1/2}, \end{aligned} \tag{2.A.3}$$

where we also applied the change of variables $u := t-s$. For every $u \in (0,t)$, we have by Lemma 2.3.20 that $\Phi_{a,b}(u+\cdot)$ is differentiable as a function from $(0,h)$ to $\mathscr{L}(H;\dot{H}^\sigma_A)$ with derivative $\Phi'_{a,b}(u+\cdot)$ and, moreover, $r \mapsto \|\Phi'_{a,b}(u+r)\|_{\mathscr{L}(H;\dot{H}^\sigma_A)}$ is bounded on $[0,h]$. We conclude that $\Phi_{a,b}(u+\cdot) \in H^1(0,h;\mathscr{L}(H;\dot{H}^\sigma_A))$, so that by [74, Section 5.9.2, Theorem 2] the identity

$$\Phi_{a,b}(u+h) - \Phi_{a,b}(u) = \int_0^h \Phi'_{a,b}(u+r)\,\mathrm{d}r$$

holds as operators in $\mathscr{L}(H;\dot{H}^\sigma_A)$. We now estimate (2.A.3) by exploiting this relation, moving the norm inside the integral, applying formula (2.3.25) for the derivative of $\Phi_{a,b}$ and using the triangle and Minkowski inequalities, which gives

$$\begin{aligned} I_\star \lesssim_p \Big[\int_0^t \Big| \int_0^h \|\Phi'_{a,b}(u+r)\|_{\mathscr{L}_2(H;\dot{H}^\sigma_A)} \,\mathrm{d}r \Big|^2 \,\mathrm{d}u \Big]^{1/2} \le \Big[\int_0^t \big| |a| F(u) + G(u) \big|^2 \,\mathrm{d}u \Big]^{1/2} \\ \le |a| \Big[\int_0^t |F(u)|^2 \,\mathrm{d}u \Big]^{1/2} + \Big[\int_0^t |G(u)|^2 \,\mathrm{d}u \Big]^{1/2}, \end{aligned} \tag{2.A.4}$$

where

$$\begin{aligned} F(u) &:= \int_0^h \|\Phi_{a-1,b}(u+r)\|_{\mathscr{L}_2(H;\dot{H}^\sigma_A)} \,\mathrm{d}r, \\ G(u) &:= \int_0^h \|\Phi_{a,b+1}(u+r)\|_{\mathscr{L}_2(H;\dot{H}^\sigma_A)} \,\mathrm{d}r. \end{aligned}$$

2

Using Minkowski's integral inequality (see e.g. [186, §A.1]), we obtain

$$\begin{aligned}\left[\int_0^t |F(u)|^2\,\mathrm{d}u\right]^{1/2} &= \left[\int_0^t \left|\int_0^h \|\Phi_{a-1,b}(u+r)\|_{\mathscr{L}_2(H;\dot H^\sigma_A)}\,\mathrm{d}r\right|^2 \mathrm{d}u\right]^{1/2}\\ &\le \int_0^h \left[\int_0^t \|\Phi_{a-1,b}(u+r)\|^2_{\mathscr{L}_2(H;\dot H^\sigma_A)}\,\mathrm{d}u\right]^{1/2}\mathrm{d}r\\ &= \int_0^h \left[\int_0^t (u+r)^{2(a-1)}\left\|A^{a+\frac12-\tau}S(u+r)A^{\frac{\sigma}{2}-a-\frac12+b+\tau}Q^{\frac12}\right\|^2_{\mathscr{L}_2(H)}\,\mathrm{d}u\right]^{1/2}\mathrm{d}r\end{aligned}$$

Since the semigroup $(S(t))_{t\ge 0}$ is assumed to be analytic, by (2.3.3) the estimate

$$\begin{aligned}&\left\|A^{a+\frac12-\tau}S(u+r)A^{\frac{\sigma}{2}-a-\frac12+b+\tau}Q^{\frac12}\right\|_{\mathscr{L}_2(H)}\\ &\qquad\lesssim_{(a,\tau)} (u+r)^{-a-\frac12+\tau}\left\|A^{\frac{\sigma}{2}-a-\frac12+b+\tau}Q^{\frac12}\right\|_{\mathscr{L}_2(H)}\end{aligned}\tag{2.A.5}$$

follows, where we also used the assumption that $a+\frac12-\tau\ge 0$. We conclude that

$$\begin{aligned}\left[\int_0^t |F(u)|^2\,\mathrm{d}u\right]^{1/2} &\lesssim_{(a,\tau)} \left\|A^{-a-\frac12+b+\tau}Q^{\frac12}\right\|_{\mathscr{L}_2(H;\dot H^\sigma_A)} \int_0^h \left[\int_0^t (u+r)^{2\tau-3}\,\mathrm{d}u\right]^{1/2}\mathrm{d}r\\ &\le \left\|A^{-a-\frac12+b+\tau}Q^{\frac12}\right\|_{\mathscr{L}_2(H;\dot H^\sigma_A)} \int_0^h \left[\int_r^\infty \bar u^{2\tau-3}\,\mathrm{d}\bar u\right]^{1/2}\mathrm{d}r\\ &= \frac{1}{\sqrt{2-2\tau}}\left\|A^{-a-\frac12+b+\tau}Q^{\frac12}\right\|_{\mathscr{L}_2(H;\dot H^\sigma_A)}\int_0^h r^{\tau-1}\,\mathrm{d}r\\ &= \frac{1}{\tau\sqrt{2-2\tau}}\,h^\tau\left\|A^{-a-\frac12+b+\tau}Q^{\frac12}\right\|_{\mathscr{L}_2(H;\dot H^\sigma_A)}.\end{aligned}$$

The integral $\int_0^t |G(u)|^2\,\mathrm{d}u$ in (2.A.4) can be bounded a similar way. By Minkowski's integral inequality and analogously to (2.A.5), noting that $a+\frac32-\tau > a+\frac12-\tau\ge 0$, we find that

$$\begin{aligned}\left[\int_0^t |G(u)|^2\,\mathrm{d}u\right]^{1/2} &= \left[\int_0^t \left|\int_0^h \|\Phi_{a,b+1}(u+r)\|_{\mathscr{L}_2(H;\dot H^\sigma_A)}\,\mathrm{d}r\right|^2 \mathrm{d}u\right]^{1/2}\\ &\le \int_0^h \left[\int_0^t \|\Phi_{a,b+1}(u+r)\|^2_{\mathscr{L}_2(H;\dot H^\sigma_A)}\,\mathrm{d}u\right]^{1/2}\mathrm{d}r\\ &= \int_0^h \left[\int_0^t (u+r)^{2a}\left\|A^{a+\frac32-\tau}S(u+r)A^{\frac{\sigma}{2}-a-\frac12+b+\tau}Q^{\frac12}\right\|^2_{\mathscr{L}_2(H)}\,\mathrm{d}u\right]^{1/2}\mathrm{d}r\\ &\lesssim_{(a,\tau)} \left\|A^{-a-\frac12+b+\tau}Q^{\frac12}\right\|_{\mathscr{L}_2(H;\dot H^\sigma_A)} \int_0^h \left[\int_0^t (u+r)^{2\tau-3}\,\mathrm{d}u\right]^{1/2}\mathrm{d}r\\ &\le \frac{1}{\tau\sqrt{2-2\tau}}\,h^\tau\left\|A^{-a-\frac12+b+\tau}Q^{\frac12}\right\|_{\mathscr{L}_2(H;\dot H^\sigma_A)},\end{aligned}$$

which completes the proof. □

Lemma 2.A.7. *Let Assumptions 2.3.1(i),(ii) be satisfied, let* $a\in\left(-\frac12,\infty\right)$, $b,\sigma\in[0,\infty)$ *and* $\tau\in\left(0,1\wedge\left(a+\frac12\right)\right]$. *If* $\sigma\ne 0$, *then suppose furthermore that Assumption 2.3.1(iv)* *holds. Let* $J := (0,T)$ *for some* $T \in (0,\infty]$. *Then, for all* $p \in [1,\infty)$, $t \in [0,T)$ *and* $h \in J$ *with* $h \le T - t$, *the function* $\Phi_{a,b} \colon (0,\infty) \to \mathscr{L}(H;\dot H^\sigma_A)$ *in* (2.3.22) *satisfies*

$$\Big\| \int_t^{t+h} \Phi_{a,b}(t+h-s)\,\mathrm{d}W(s) \Big\|_{L^p(\Omega;\dot H^\sigma_A)} \lesssim_{(p,a,\tau)} h^\tau \big\| A^{-a-\frac12+b+\tau} Q^{\frac12} \big\|_{\mathscr{L}_2(H;\dot H^\sigma_A)}.$$

Proof. We apply the Burkholder–Davis–Gundy inequality (combined with nestedness of the L^p spaces if $p<2$), the change of variables $u := t+h-s$, and obtain

$$\begin{aligned}
&\Big\| \int_t^{t+h} \Phi_{a,b}(t+h-s)\,\mathrm{d}W(s) \Big\|^2_{L^p(\Omega;\dot H^\sigma_A)} \\
&\lesssim_p \int_t^{t+h} \|\Phi_{a,b}(t+h-s)\|^2_{\mathscr{L}_2(H;\dot H^\sigma_A)}\,\mathrm{d}s = \int_0^h \|\Phi_{a,b}(u)\|^2_{\mathscr{L}_2(H;\dot H^\sigma_A)}\,\mathrm{d}u \\
&= \int_0^h u^{2a} \big\| A^{a+\frac12-\tau} S(u) A^{\frac{\sigma}{2}-a-\frac12+b+\tau} Q^{\frac12} \big\|^2_{\mathscr{L}_2(H)}\,\mathrm{d}u \\
&\lesssim_{(a,\tau)} \big\| A^{-a-\frac12+b+\tau} Q^{\frac12} \big\|^2_{\mathscr{L}_2(H;\dot H^\sigma_A)} \int_0^h u^{2\tau-1}\,\mathrm{d}u = \frac{h^{2\tau}}{2\tau} \big\| A^{-a-\frac12+b+\tau} Q^{\frac12} \big\|^2_{\mathscr{L}_2(H;\dot H^\sigma_A)},
\end{aligned}$$

where we could proceed as in (2.A.5), since $a + \frac12 - \tau \ge 0$ is assumed. This completes the proof of the assertion. □

Proposition 2.A.8 provides a useful relation between the weak derivative and the difference quotient.

Proposition 2.A.8. *Let* U *be a real and separable Hilbert space and let* $J := (0,T)$ *for some* $T \in (0,\infty]$. *Suppose that* $\Psi \in H^1(J;U)$ *and let* $\Psi' \in L^2(J;U)$ *denote the weak derivative of* Ψ. *For* $h \in \mathbb{R}\setminus\{0\}$, *let* $J_h \subseteq J$ *be as in Proposition* 2.A.4 *and define the* difference quotient $D_h\Psi \colon J_h \to U$ *of* Ψ *by*

$$[D_h\Psi](t) := \frac{\Psi(t+h)-\Psi(t)}{h} \quad \textit{for a.a. } t \in J_h. \tag{2.A.6}$$

Then, we have $\lim_{h\to 0} \|D_h\Psi - \Psi'\|_{L^2(J_h;U)} = 0$.

Proof. Suppose that $\Psi \in E$, where the space E is given by $E := C^\infty([0,T];U)$ if $T<\infty$ and $E := C^\infty_c([0,\infty);U)$ if $T=\infty$, and fix $h \in \mathbb{R}\setminus\{0\}$. Then,

$$[D_h\Psi](t) = \frac1h \int_0^h \Psi'(t+s)\,\mathrm{d}s \quad \forall t \in J_h \tag{2.A.7}$$

holds by the fundamental theorem of calculus, where we use the convention that $\int_0^h = -\int_h^0$ whenever $h \in (-t,0)$. Applying the Cauchy–Schwarz inequality gives

$$\begin{aligned}
\|[D_h\Psi](t)\|^2_U \le \Big| \frac1h \int_0^h \|\Psi'(t+s)\|_U\,\mathrm{d}s \Big|^2 &\le \Big| \frac1h \int_0^h \|\Psi'(t+s)\|^2_U\,\mathrm{d}s \Big| \\
&= \frac1h \int_0^h \|\Psi'(t+s)\|^2_U\,\mathrm{d}s \qquad \forall t \in J_h.
\end{aligned}$$

2

The absolute value can be removed in the last step by the integral sign convention. Integrating this expression over $t \in J_h$ and using Fubini's theorem, we obtain that

$$\begin{aligned}\|D_h\Psi\|^2_{L^2(J_h;U)} &= \int_{J_h} \|[D_h\Psi](t)\|^2_U \,\mathrm{d}t \\ &\le \frac{1}{h}\int_{J_h}\int_0^h \|\Psi'(t+s)\|^2_U \,\mathrm{d}s\,\mathrm{d}t = \frac{1}{h}\int_0^h\int_{J_h} \|\Psi'(t+s)\|^2_U \,\mathrm{d}t\,\mathrm{d}s.\end{aligned} \tag{2.A.8}$$

For all $s \in (0,h)$ (resp., $s \in (h,0)$ if $h < 0$), the change of variables $r := t+s$ gives

$$\int_{J_h} \|\Psi'(t+s)\|^2_U \,\mathrm{d}t = \int_{J_h+s} \|\Psi'(r)\|^2_U \,\mathrm{d}r \le \int_J \|\Psi'(r)\|^2_U \,\mathrm{d}r = \|\Psi'\|^2_{L^2(J;U)}.$$

Hence, we can bound the inner integral in (2.A.8) independently of s, which implies

$$\|D_h\Psi\|^2_{L^2(J_h;U)} \le \|\Psi'\|^2_{L^2(J;U)} \le \|\Psi\|^2_{H^1(J;U)}. \tag{2.A.9}$$

This estimate shows that the linear operator D_h is bounded from $\big(E, \|\cdot\|_{H^1(J;U)}\big)$ to $L^2(J_h;U)$ for all $h \in \mathbb{R}\setminus\{0\}$. By density of E in $H^1(J;U)$ (see [59, XVIII.§1.2, Lemma 1]), the above estimate holds for all $\Psi \in H^1(J;U)$.

Suppose again that $\Psi \in E$. We recall (2.A.7) and find

$$[D_h\Psi](t) - \Psi'(t) = \frac{1}{h}\int_0^h \big(\Psi'(t+s) - \Psi'(t)\big)\,\mathrm{d}s \quad \forall\, t \in J_h. \tag{2.A.10}$$

By the compact support of $D_h\Psi$ and Ψ', there exists a bounded interval $K \subset [0,\infty)$ such that $\operatorname{supp}(D_h\Psi - \Psi'|_{J_h}) \subseteq K$ for all $h \in [-1,1]$. Furthermore, by uniform continuity of $\Psi' \in C^\infty([0,T];U)$ (resp., $\Psi' \in C_c^\infty([0,\infty);U)$), for every $\varepsilon \in (0,\infty)$, there exists some $\delta \in (0,1)$ such that $\|\Psi'(\xi) - \Psi'(\eta)\|_U < \varepsilon$ if $|\xi - \eta| < \delta$. Thus,

$$\|[D_h\Psi](t) - \Psi'(t)\|_U < \varepsilon \quad \forall\, t \in J_h$$

follows for all $h \in (-\delta,\delta)$ by (2.A.10) and, consequently,

$$\|D_h\Psi - \Psi'\|_{L^2(J_h;U)} \lesssim_K \|D_h\Psi - \Psi'\|_{L^\infty(J_h;U)} \to 0 \quad \text{as } h \to 0.$$

This proves the assertion for functions $\Psi \in E$. The general case for $\Psi \in H^1(J;U)$ follows then from density of E and the h-uniform bound (2.A.9): Given $\varepsilon > 0$, we may pick $v \in E$ such that $\|\Psi - v\|_{H^1(J;U)} < \frac{\varepsilon}{3}$, and $h_0 > 0$ such that $\|D_h v - v'\|_{L^2(J_h;U)} < \frac{\varepsilon}{3}$ for all $h \in (-h_0,h_0)$, Thus, we obtain for all $h \in (-h_0,h_0)$

$$\begin{aligned}\|D_h\Psi - \Psi'\|_{L^2(J_h;U)} &\le \|D_h(\Psi - v)\|_{L^2(J_h;U)} + \|D_h v - v'\|_{L^2(J_h;U)} + \|v' - \Psi'\|_{L^2(J_h;U)} \\ &\le 2\|\Psi - v\|_{H^1(J;U)} + \|D_h v - v'\|_{L^2(J_h;U)} < \varepsilon.\end{aligned}$$ □

Lemma 2.A.9. *Let $J := (0,T)$ for some $T \in (0,\infty]$. Let E and F be real separable Banach spaces such that $E \hookrightarrow F$. If $u \in H^1_{0,\{0\}}(J;F)$ and $u, u' \in L^2(J;E)$, where u' denotes the F-valued weak derivative of u, then $u \in H^1_{0,\{0\}}(J;E)$ and its E-valued weak derivative coincides with u' almost everywhere in J.*

Proof. Let $\mathscr{I}_E\colon L^1(J;E)\to E$ and $\mathscr{I}_F\colon L^1(J;F)\to F$ denote, respectively, the E-valued and F-valued Bochner integrals over the interval J. Given an arbitrary $\phi\in C_c^\infty(J)$, the assumption $u\in H^1(J;F)$ implies $\mathscr{I}_F(\phi u')=-\mathscr{I}_F(\phi' u)$, and we wish to show $\mathscr{I}_E(\phi u')=-\mathscr{I}_E(\phi' u)$. To this end, we claim that $\mathscr{I}_E$ and $\mathscr{I}_F$ coincide on the embedded subspace $L^1(J;E)\hookrightarrow L^1(J;F)$ and we apply this fact to $\phi u'$ and $\phi' u$. To verify the claim, fix $f\in L^1(J;E)$. By definition of $\mathscr{I}_E$, there exist E-valued measurable simple functions $(f_n)_{n\in\mathbb{N}}$ satisfying $f_n\to f$ in $L^1(J;E)$ and $\mathscr{I}_E(f_n)\to\mathscr{I}_E(f)$ in E. For all $n\in\mathbb{N}$, it readily follows from the respective definitions and the inclusion $E\subseteq F$ that f_n is an F-valued measurable simple function and $\mathscr{I}_F(f_n)=\mathscr{I}_E(f_n)$. Since $E\hookrightarrow F$, we observe that $f_n\to f$ in $L^1(J;F)$ and $\mathscr{I}_F(f_n)=\mathscr{I}_E(f_n)\to\mathscr{I}_E(f)$ in F, hence $\mathscr{I}_F(f)=\mathscr{I}_E(f)$. We conclude that $u\in H^1(J;E)$ and the E-valued weak derivative coincides with u' a.e. in J. Now it remains to prove that $u\in H^1_{0,\{0\}}(J;E)$. Note that $u\in H^1_{0,\{0\}}(J;F)$ is equivalent to the statement that the unique continuous representative $\tilde u\in C(\bar J;F)$ of u, which exists by virtue of [74, Section 5.9.2, Theorem 2], vanishes at zero, cf. [74, Section 5.5, Theorem 2]. Similarly, from $u\in H^1(J;E)$ we obtain a function $\hat u\in C(\bar J;E)\hookrightarrow C(\bar J;F)$ such that $u=\hat u$ a.e., hence $\hat u=\tilde u$ by uniqueness. In particular, $\hat u(0)=0$ and thus $u\in H_{0,\{0\}}(J;E)$. ☐

2.B. SECTORIAL LINEAR OPERATORS AND FUNCTIONAL CALCULUS

In this appendix, we collect some results regarding sectorial linear operators, semigroups and functional calculus which are relevant to this chapter, complementing those recalled in Sections 1.1 and 1.3.

Throughout this section, $A\colon \mathsf{D}(A)\subseteq H\to H$ denotes a linear operator whose negative $-A$ generates a C_0-semigroup $(S(t))_{t\ge 0}$ on a separable Hilbert space H. The corresponding scalar field is given by the complex numbers $\mathbb{C}$ in Subsection 2.B.1 and the real numbers $\mathbb{R}$ in Subsection 2.B.2.

2.B.1. H^∞-CALCULUS AND MCINTOSH'S THEOREM

Given $\varphi\in(0,\pi)$, we say that a holomorphic function $f\colon\Sigma_\varphi\to\mathbb{C}$ belongs to $H_0^\infty(\Sigma_\varphi)$ if and only if there exist $\alpha\in(0,\infty)$ and $M\in[0,\infty)$ such that

$$|f(z)|\le M\big(|z|^\alpha\wedge|z|^{-\alpha}\big)\quad\text{for all } z\in\Sigma_\varphi.$$

For operators acting on a *complex* Hilbert space H, the admissibility of a bounded H^∞-calculus can be characterized by the following theorem. It is taken from [100, Theorem 7.3.1]; see [114, Theorem 10.4.21] for a generalization to non-injective A.

Theorem 2.B.1. *Let $A\colon \mathsf{D}(A)\subseteq H\to H$ be injective and sectorial. Then*

$$\|x\|_H^2\eqsim_f\int_0^\infty\|f(tA)x\|_H^2\,\frac{\mathrm{d}t}{t}\quad\forall x\in H$$

holds for all $f\in\bigcup_{\varphi\in(\omega(A),\pi)}H_0^\infty(\Sigma_\varphi)\setminus\{0\}$ if and only if A admits a bounded $H^\infty(\Sigma_\varphi)$-calculus for some (or, equivalently, for all) $\varphi\in(\omega(A),\pi)$.

2

Remark 2.B.2. Since $\omega_{H^\infty}(A)$ is defined as an infimum over angles contained in the interval $(\omega(A), \pi)$, any operator admitting a bounded H^∞-calculus always satisfies $\omega_{H^\infty}(A) \geq \omega(A)$. This inequality is also true for operators on a Banach space. Theorem 2.B.1 implies that the reverse inequality holds for operators on a Hilbert space with a bounded $H^\infty(\Sigma_\varphi)$-calculus for some angle $\varphi \in (\omega(A), \pi)$. Indeed, in this case, the same holds for all $\varphi \in (\omega(A), \pi)$, hence we obtain $\omega_{H^\infty}(A) \leq \omega(A)$ upon taking the infimum. We thus have $\omega_{H^\infty}(A) = \omega(A)$.

2.B.2. COMPLEXIFICATIONS, SEMIGROUPS AND FRACTIONAL POWERS

In this section, H denotes a *real* Hilbert space.

COMPLEXIFICATIONS

The complexified Hilbert space $H_\mathbb{C}$ is defined by equipping $H \times H$ with componentwise addition and the respective scalar and inner products

$$
\begin{aligned}
(a+bi)(x,y) &:= (ax-by, bx+ay), && x,y \in H;\ a,b \in \mathbb{R}, \\
\langle (x,y),(u,v)\rangle_{H_\mathbb{C}} &:= \langle x,u\rangle_H + \langle y,v\rangle_H + i[\langle y,u\rangle_H - \langle x,v\rangle_H], && x,y,u,v \in H. \quad (2.B.1)
\end{aligned}
$$

In the sequel, we will write $x + iy := (x,y) \in H_\mathbb{C}$.

A linear operator A on H similarly gives rise to a complexified counterpart $A_\mathbb{C}$ on $H_\mathbb{C}$ by defining $A_\mathbb{C}(x+iy) := Ax + iAy$ on $\mathsf{D}(A_\mathbb{C}) = \{x+iy : x,y \in \mathsf{D}(A)\}$. It follows readily from the above definitions that $T \mapsto T_\mathbb{C} \in \mathscr{L}(\mathscr{L}(H); \mathscr{L}(H_\mathbb{C}))$ is an inverse-preserving and isometric algebra homomorphism. Analogous results hold for unbounded operators, taking natural domains into account. We have the following relation between semigroups and complexifications.

Lemma 2.B.3. *The family $(S(t))_{t\geq 0} \subseteq \mathscr{L}(H)$ is a C_0-semigroup on H if and only if $(S_\mathbb{C}(t))_{t\geq 0} \subseteq \mathscr{L}(H_\mathbb{C})$ is a C_0-semigroup on $H_\mathbb{C}$. In this case, their respective generators $-A\colon \mathsf{D}(A) \subseteq H \to H$ and $-\widehat{A}\colon \mathsf{D}(\widehat{A}) \subseteq H_\mathbb{C} \to H_\mathbb{C}$ satisfy $A_\mathbb{C} = \widehat{A}$.*

Proof. If $(S(t))_{t\geq 0}$ is a C_0-semigroup, then clearly $S_\mathbb{C}(0) = I$ and

$$S_\mathbb{C}(t)S_\mathbb{C}(s) = [S(t)S(s)]_\mathbb{C} = S_\mathbb{C}(t+s) \text{for } s,t \geq 0.$$

Moreover, we have $\|S_\mathbb{C}(t)\widehat{x} - \widehat{x}\|^2_{H_\mathbb{C}} = \|S(t)x - x\|^2_H + \|S(t)y - y\|^2_H \to 0$ as $t \downarrow 0$ for all $\widehat{x} = x + iy \in H_\mathbb{C}$. The reverse implication is readily established by identifying every $x \in H$ with $x + i0 \in H_\mathbb{C}$.

Suppose that $(S(t))_{t\geq 0}$ and $(S_\mathbb{C}(t))_{t\geq 0}$ are C_0-semigroups with respective generators $-A$ and $-\widehat{A}$. Then $\widehat{x} = x + iy \in \mathsf{D}(A_\mathbb{C})$ is equivalent to the existence of the limits $-Ax = \lim_{t\downarrow 0} \frac{1}{t}(S(t)x - x)$ and $-Ay = \lim_{t\downarrow 0} \frac{1}{t}(S(t)y - y)$ in H. Thus,

$$A_\mathbb{C}\,\widehat{x} = Ax + iAy = \lim_{t\downarrow 0}\Big[\frac{1}{t}\big(x - S(t)x\big) + \frac{i}{t}\big(y - S(t)y\big)\Big] = \lim_{t\downarrow 0}\frac{1}{t}\big(\widehat{x} - S_\mathbb{C}(t)\widehat{x}\big) = \widehat{A}\widehat{x},$$

where the limits in the previous display are taken with respect to $\|\cdot\|_{H_\mathbb{C}}$. $\square$

Using Lemma 2.B.3 and interchanging the bounded operator $T \mapsto T_{\mathbb{C}}$ with the Bochner integral, we find

$$[A^{-\alpha}]_{\mathbb{C}} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S_{\mathbb{C}}(t)\,\mathrm{d}t = A_{\mathbb{C}}^{-\alpha} \quad \forall \alpha \in (0,\infty),$$

and the same relation can be derived for arbitrary powers $\alpha \in \mathbb{R}$.

A SQUARE FUNCTION ESTIMATE

The following square function estimate is central to the proof of Proposition 2.3.14.

Lemma 2.B.4. *Let A satisfy Assumptions 2.3.1(i),(iii),(iv). Then, for $a \in (0,\infty)$,*

$$\int_0^\infty \left\| t^{a-\frac{1}{2}} A^a S(t) x \right\|_H^2 \mathrm{d}t \eqsim_a \|x\|_H^2 \quad \forall x \in H.$$

Proof. Given $a \in (0,\infty)$ and $\varphi \in (\omega(A), \pi/2)$, the function $f(z) := z^a e^{-z}$ belongs to $H_0^\infty(\Sigma_\varphi)$ and we have the identity $f(tA_{\mathbb{C}}) = t^a A_{\mathbb{C}}^a S_{\mathbb{C}}(t) = [t^a A^a S(t)]_{\mathbb{C}}$; see the proof of [100, Proposition 3.4.3], which is applicable to our definition of fractional powers as remarked in the previous subsection. By invoking Theorem 2.B.1, we thus find

$$\int_0^\infty \left\| [t^a A^a S(t)]_{\mathbb{C}}\, x \right\|_{H_{\mathbb{C}}}^2 \frac{\mathrm{d}t}{t} = \int_0^\infty \left\| f(tA_{\mathbb{C}}) x \right\|_{H_{\mathbb{C}}}^2 \frac{\mathrm{d}t}{t} \eqsim_a \|x\|_{H_{\mathbb{C}}}^2 \quad \forall x \in H_{\mathbb{C}}.$$

Applying this equivalence to $x + i0$ for all $x \in H$ finishes the proof. □

ACKNOWLEDGMENTS FOR CHAPTER 2

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3

Multiple and weak Markov properties in Hilbert spaces

The contents of this chapter are based on the article [124], which is joint work with Kristin Kirchner.

3.1. Introduction to Chapter 3

3.1.1. Background and motivation

Gaussian Markov random fields play an important role in various applications, such as the analysis of time series or longitudinal data, image processing, as well as in spatial statistics, see for instance [177, Section 1.3]. The latter focuses on the statistical modeling of spatial or spatiotemporal dependence in data collected from phenomena encountered in disciplines such as climatology [4], epidemiology [137] and neuroimaging [152]. The popularity of Gaussian Markov random fields among the larger class of Gaussian random fields is a consequence of their additional conditional independence properties, which entail a sparse precision structure and facilitate efficient computational methods for statistical inference. In particular, hierarchical models based on Gaussian Markov random fields allow for efficient Bayesian inference using Markov chain Monte Carlo methods, see for instance [177, Section 4.1].

Since a Gaussian process is fully characterized by its second-order structure, i.e., by its mean and covariance function, a natural way to specify its distribution is to choose a suitable second-order structure. Alternatively, the dynamics of Gaussian random fields defined on a Euclidean domain $\mathcal{D} \subseteq \mathbb{R}^d$ can be specified by means of stochastic partial differential equations (SPDEs), such as the white noise $(\mathcal{W}(x))_{x\in\mathcal{D}}$ driven equation

$$LX(x) = \mathcal{W}(x), \quad x \in \mathcal{D}. \tag{3.1.1}$$

Here, L is a linear operator acting on real-valued functions defined on $\mathcal{D}$. A spatial Gaussian random field $(X(x))_{x\in\mathcal{D}}$ is said to have the Markov property if the subcollections $(X(x))_{x\in\mathcal{D}_1}$ and $(X(x))_{x\in\mathcal{D}_2}$ corresponding to pairs of disjoint subdomains $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathcal{D}$ are independent conditional on $(X(x))_{x\in\mathcal{D}'}$ for some non-trivial 'splitting' set $\mathcal{D}' \subseteq \mathcal{D}$ separating the two. The precise specification of these sets, which respectively carry the intuitive interpretations of *past*, *future* and *present*, leads to various definitions of the Markov property. By the theory of Rozanov [176], a real-valued Gaussian random field satisfying (3.1.1) has such a Markov property if and only if its precision operator L^*L is local, where L^* denotes the $L^2(\mathcal{D})$-adjoint of L.

An important example in spatial statistics is the choice of a fractional-order differential operator $L := \tau(\kappa^2 - \Delta)^\beta$ in (3.1.1), where Δ is the Laplacian, $\mathcal{W}$ is Gaussian white noise and $\tau, \kappa, \beta \in (0, \infty)$. Whittle [196] observed that the covariance function $\varrho(x, y) := \mathbb{E}[X(x)X(y)]$ of the stationary solution $(X(x))_{x\in\mathcal{D}}$ to (3.1.1) with $\mathcal{D} = \mathbb{R}^d$ then belongs to the widely used *Matérn covariance class* [149]:

$$\varrho(x, y) = C_{\kappa,\tau,\nu,d}(\kappa\|x - y\|_{\mathbb{R}^d})^\nu K_\nu(\kappa\|x - y\|_{\mathbb{R}^d}) \quad \text{for all } x, y \in \mathbb{R}^d, \tag{3.1.2}$$

where $\nu := 2\beta - d/2$, $C_{\kappa,\tau,\nu,d} := \tau^{-2}(4\pi)^{-d/2}2^{1-\nu}[\Gamma(\nu + d/2)]^{-1}\kappa^{-2\nu}$ and K_ν denotes the modified Bessel function of the second kind. This observation motivated the *SPDE approach* for spatial statistical modeling which was proposed by Lindgren, Rue and Lindström [142]. Here, one considers (3.1.1) with $L := \tau(\kappa^2 - \Delta)^\beta$, where Δ is a Laplacian operator on a bounded Euclidean domain $\mathcal{D} \subsetneq \mathbb{R}^d$ augmented with boundary conditions, and approximates the resulting *Whittle–Matérn fields* by means of efficient numerical methods available for (S)PDEs. Owing to its ease of generalization and its computational efficiency as compared to covariance-based techniques, this approach has gained widespread popularity, see e.g. [29–31, 50, 107, 141, 181]. Since in this case the precision operator is given by $L^*L = \tau^2(\kappa^2 - \Delta)^{2\beta}$, we find that Whittle–Matérn fields are Gaussian *Markov* random fields in the sense of Rozanov [176] precisely when $2\beta \in \mathbb{N}$.

Recently, extensions of the SPDE approach incorporating time dependence have been discussed. A class of space–time equations which has been proposed in this context is

$$(\partial_t + L)^\gamma X(t, x) = \dot{\mathcal{W}}^Q(t, x), \quad (t, x) \in \mathbb{T} \times \mathcal{D}, \qquad \gamma \in (1/2, \infty), \tag{3.1.3}$$

where $\mathbb{T} \subseteq \mathbb{R}$ represents a time interval and $\dot{\mathcal{W}}^Q$ is spatiotemporal Gaussian noise, which is spatially colored by an operator Q, see Chapter 2 and [140]. In particular, it has been shown in the previous chapter that (3.1.3) extends the Matérn model in terms of spatial marginal covariance, and that the interplay of its parameters governs smoothness in space and time as well as the degree of separability.

Spatiotemporal random fields can be viewed as U-valued stochastic processes by letting a Hilbert space U encode the spatial variable, so that (3.1.3) corresponds to a stochastic fractional evolution equation of the form

$$(\partial_t + A)^\gamma X(t) = \dot{W}^Q(t), \quad t \in \mathbb{T}. \tag{3.1.4}$$

The (temporal) Markov property of solutions to (3.1.3) is then equivalent to that of the U-valued solution process $(X(t))_{t\in\mathbb{T}}$, where the Markov behavior is considered with respect to the index set $\mathbb{T}$. Moreover, viewing (3.1.4) as a special case of

$$\mathcal{L}X(t) = \dot{W}^Q(t), \quad t \in \mathbb{T}, \tag{3.1.5}$$

where $\mathcal{L}$ is now a linear operator acting on functions from $\mathbb{T}$ to U, the theory of Rozanov [176] suggests that locality of the precision operator $\mathcal{L}^*\mathcal{L}$, also acting on functions $f\colon \mathbb{T} \to U$, can be used to characterize temporal Markov behavior of the solution X.

3.1.2. CONTRIBUTIONS

In this chapter we define *simple*, *multiple* (N*-ple* for $N \in \mathbb{N}$) and *weak* Markov properties for stochastic processes which take values in a Hilbert space U. These definitions generalize those appearing for instance in [108, 172, 176] for real-valued processes to infinite dimensions, see Definitions 3.3.1, 3.3.2 and 3.3.4, respectively. Besides gathering them in once place, we establish their interrelations, see Proposition 3.3.5 and Remark 3.3.6. The main results are Theorems 3.3.7 and 3.3.9, which give necessary and sufficient conditions, in terms of the precision operator $\mathcal{L}^*\mathcal{L}$, for the weakest notion of Markovianity for a U-valued Gaussian process defined via (3.1.5). These results are proved by using a non-trivial extension of the theory by Rozanov [176, Chapters 2 and 3] from the real-valued to the U-valued setting.

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In order to consider processes defined via linear evolution SPDEs such as (3.1.4), we construct a stochastic integral for deterministic operator-valued integrands defined on the whole of $\mathbb{R}$ with respect to a two-sided (cylindrical) Q-Wiener process $(W^Q(t))_{t\in\mathbb{R}}$, see Section 3.2.2. We employ this stochastic integral to define the mild solution process $Z_\gamma = (Z_\gamma(t))_{t\in\mathbb{R}}$ to (3.1.4) on $\mathbb{T} = \mathbb{R}$, see Definition 3.4.4. Our rigorous definition of the fractional space–time operator $(\partial_t + A)^\gamma$ for $\gamma \in \mathbb{R}$, see Definition 3.4.3, extends the Weyl fractional calculus in the sense that one recovers the Weyl fractional derivatives and integrals defined in [121, Section 2.3] upon specializing to $U = \mathbb{R}$ and $A = 0$.

We show that the mild solution Z_γ to (3.1.4) satisfies the N-ple Markov property if $\gamma = N \in \mathbb{N}$, see Theorem 3.4.9. Conversely, we use Theorem 3.3.7 to show that, in general, Z_γ is not weakly Markov for $\gamma \notin \mathbb{N}$. This complements [93, Theorem 2.7], which states that any time-homogeneous U-valued Gaussian simple Markov process is the solution to a first-order stochastic evolution equation.

Finally, we discuss another interesting aspect of the SPDE (3.1.4): A fractional Q-Wiener process $(W^Q_H(t))_{t\in\mathbb{R}}$ with Hurst parameter $H \in (0,1)$, as defined for instance in [68], can be obtained as a limiting case of (3.1.4) with $\gamma = H + 1/2$ and $A = \varepsilon \operatorname{Id}_U$ as $\varepsilon \downarrow 0$, see Proposition 3.5.3. The proof is based on a Mandelbrot–Van Ness [147] type integral representation of W^Q_H, again using the two-sided stochastic integral from Section 3.2.2, see Proposition 3.5.2. The case $H = \frac{1}{2}$ corresponds to a (non-fractional) Q-Wiener process and is thus Markov. Conversely, although the results of Theorems 3.3.7 and 3.3.9 do not apply directly, the above observation provides evidence that W^Q_H does not satisfy a weak Markov property for $H \in (0,1) \setminus \{\frac{1}{2}\}$.

3.1.3. OUTLINE

In Section 3.2 we begin by establishing the necessary notation, see Subsection 3.2.1, followed by the construction of the stochastic integral with respect to a two-sided (cylindrical) Q-Wiener process in Subsection 3.2.2. Section 3.3 is devoted to defining, relating and (for solutions to (3.1.5)) characterizing various notions of Markov behavior for U-valued stochastic processes. The goal of Section 3.4 is to define and analyze the mild solution to (3.1.4) on $\mathbb{T} = \mathbb{R}$. To this end, we first describe the setting and define $(\partial_t + A)^{-\gamma}$ with $\gamma \in (0,\infty)$ in Subsections 3.4.1 and 3.4.2, respectively. We subsequently define the mild solution process in Subsection 3.4.3, and investigate for which values of $\gamma \in (1/2,\infty)$ it exhibits Markov behavior in Subsection 3.4.4. In Section 3.5 we recall the definition from [68] of a Q-fractional Wiener process and prove a Mandelbrot–Van Ness type integral representation, allowing us to exhibit it as a limiting case of (3.1.4).

This chapter is supplemented by two appendices: Appendix 3.A contains auxiliary results relating to specific results from the main text whose statements and proofs were postponed for readability; subjects include conditional independence, filtrations indexed by $\mathbb{R}$ and the mean-square differentiability of stochastic convolutions. Appendix 3.B is a short overview of results regarding fractional powers of linear operators and the interpretation of the fractional parabolic operator $(\partial_t + A)^\gamma$.

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3.2. PRELIMINARIES FOR CHAPTER 3

In this section we mainly highlight notation which deviates from the previous chapters or was not used there. Some of it is listed in Table 3.1, along with some previously established notation which is used throughout this chapter.

3.2.1. NOTATION

Throughout this chapter, we assume that a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given. We write $Z \sim N(m,Q)$ if Z is a U-valued Gaussian random variable with mean $m \in U$ and covariance operator $Q \in \mathscr{L}_1^+(U)$; its existence is guaranteed by [27, Theorem 2.3.1]. Let $\mathcal{G}_1, \mathcal{H}, \mathcal{G}_2 \subseteq \mathcal{F}$ be sub-σ-algebras of $\mathcal{F}$. The expression $\mathbb{E}[Z \mid \mathcal{H}]$ denotes the conditional expectation of a random variable Z given $\mathcal{H}$, and the conditional probability of $A \in \mathcal{F}$ given $\mathcal{H}$ is defined by $\mathbb{P}(A \mid \mathcal{H}) := \mathbb{E}[1_A \mid \mathcal{H}]$, $\mathbb{P}$-a.s. The notation $\mathcal{G}_1 \perp\!\!\!\perp_{\mathcal{H}} \mathcal{G}_2$ indicates that $\mathcal{G}_1$ and $\mathcal{G}_2$ are conditionally independent given $\mathcal{H}$, i.e., for all $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$ we have $\mathbb{P}(G_1 \cap G_2 \mid \mathcal{H}) = \mathbb{P}(G_1 \mid \mathcal{H})\mathbb{P}(G_2 \mid \mathcal{H})$, $\mathbb{P}$-a.s. When conditioning on the σ-algebra $\sigma(Y) := \{\{Y \in B\} : B \in \mathcal{B}(E)\}$ generated by a random variable Y, we write Y instead of $\sigma(Y)$; e.g., $\mathbb{E}[Z \mid Y]$ or $\mathcal{G}_1 \perp\!\!\!\perp_Y \mathcal{G}_2$.

3.2.2. STOCHASTIC INTEGRATION WITH RESPECT TO A TWO-SIDED WIENER PROCESS

Let $(W_1^Q(t))_{t\ge 0}$, $(W_2^Q(t))_{t\ge 0}$ be independent U-valued standard Q-Wiener processes for a given $Q \in \mathscr{L}_1^+(U)$, see for instance [144, Section 2.1], and define $W^Q(t) := W_1^Q(t)$ for $t \in [0,\infty)$ and $W^Q(t) := W_2^Q(-t)$ for $t \in (-\infty, 0)$. Then the *two-sided Q-Wiener process* $W^Q := (W^Q(t))_{t\in\mathbb{R}}$ satisfies the following:

(WP1) $W^Q(t)$ has mean zero and $W^Q(t) - W^Q(s) \sim N(0,(t-s)Q)$ for $t \ge s$;

(WP2) W^Q has continuous sample paths;

(WP3) $W^Q(t_4) - W^Q(t_3) \perp\!\!\!\perp W^Q(t_2) - W^Q(t_1)$ for $t_1 < t_2 \le t_3 < t_4$.

One can define a stochastic integral with respect to such a process using a construction analogous to the one-sided case, as presented for instance in [144, Section 2.3].

Elementary sets and operations

$\mathbb{N}$	positive integers
$\mathbb{N}_0$	non-negative integers
Id_D	identity map on a set D
$\mathbf{1}_{D_0}$	indicator function of a subset $D_0 \subseteq D$
$s \wedge t$	minimum of $s, t \in \mathbb{R}$
$s \vee t$	maximum of $s, t \in \mathbb{R}$

Bounded linear operators

$U, \widetilde{U}$	real and separable Hilbert spaces
$\langle \cdot, \cdot \rangle_U$	inner product of U
E, F	real and separable Banach spaces
$\\| \cdot \\|_E$	norm of E
$\mathscr{L}(E;F)$	bounded linear operators from E to F
$\mathscr{L}(E)$	abbreviation for $\mathscr{L}(E;E)$
T^*	adjoint of $T \in \mathscr{L}(E;F)$
$\mathscr{L}^+(U)$	self-adjoint and positive definite operators on U
$\operatorname{tr} T$	trace of $T \in \mathscr{L}^+(U)$
$\mathscr{L}_1^+(U)$	$T \in \mathscr{L}^+(U)$ with $\operatorname{tr} T < \infty$
$\mathscr{L}_2(U;\widetilde{U})$	Hilbert–Schmidt operators from U to $\widetilde{U}$

Function spaces

J	non-empty (sub)interval of $\mathbb{R}$
$C(J;E)$	continuous functions from J to E
$C_c^\infty(J;E)$	compactly supported infinitely differentiable functions from J to E
$C_c^\infty(J)$	abbreviation for $C_c^\infty(J;\mathbb{R})$
$C_b(E)$	bounded and continuous functions from E to $\mathbb{R}$
$(S, \mathscr{A}, \mu)$	measure space
$B_b(S)$	bounded and measurable functions from S to $\mathbb{R}$
$L^p(S,\mathscr{A},\mu;E)$	Bochner space of p-integrable functions from S to E
$L^p(S;E)$	abbreviation for $L^p(S,\mathscr{A},\mu;E)$
$H^1(J;U)$	functions in $L^2(J;U)$ with weak derivatives in $L^2(J;U)$
$H_0^1(0,\infty;U)$	functions in $H^1(0,\infty;U)$ which vanish at zero

Unbounded linear operators

$\mathsf{D}(A)$	domain of unbounded linear operator $A\colon \mathsf{D}(A) \subseteq E \to E$ on E
$\mathcal{A}_S$	Bochner space counterpart on $L^2(S;E)$ of $A\colon \mathsf{D}(A) \subseteq E \to E$, see (3.4.7)

Table 3.1: Notation used throughout this chapter.

Restricting ourselves to deterministic integrands $\Phi\colon \mathbb{R} \to \mathscr{L}(U;\widetilde{U})$, this procedure yields a square-integrable stochastic integral $\int_{\mathbb{R}} \Phi(t)\,\mathrm{d}W^Q(t)$ belonging to $L^2(\Omega;\widetilde{U})$ which exists if and only if $\Phi(\cdot)Q^{\frac{1}{2}} \in L^2(\mathbb{R};\mathscr{L}_2(U;\widetilde{U}))$; see Table 3.1 for the definitions of these (Bochner) spaces. In this case, it satisfies the following Itô isometry:

$$\left\| \int_{\mathbb{R}} \Phi(t)\,\mathrm{d}W^Q(t) \right\|^2_{L^2(\Omega;\widetilde{U})} = \int_{\mathbb{R}} \|\Phi(t)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(U;\widetilde{U})}\,\mathrm{d}t. \tag{3.2.1}$$

As in the one-sided case, we can extend the definition of the stochastic integral to allow for $Q \in \mathscr{L}^+(U) \setminus \mathscr{L}_1^+(U)$, cf. [144, Section 2.5].

Now we turn to the matter of $\mathbb{R}$-indexed filtrations on $(\Omega, \mathcal{F}, \mathbb{P})$ associated to the process $(W^Q(t))_{t\in\mathbb{R}}$. In the one-sided case, the integral process $\big(\int_0^t \Phi(r)\,\mathrm{d}W_1^Q(r)\big)_{t\geq 0}$ is a martingale with respect to the filtration $\mathcal{F}_t^{W_1^Q} := \sigma\big(W_1^Q(s) : 0 \le s \le t\big) \vee \sigma(\mathcal{N}_{\mathbb{P}})$ whenever $\Phi(\cdot)Q^{\frac{1}{2}} \in L^2(0,t;\mathscr{L}_2(U;\widetilde{U}))$ for all $t \in [0,\infty)$, which is immediate from the definition of the stochastic integral.

In the two-sided case, we instead use the *(completed) filtration* $(\mathcal{F}_t^{\delta W^Q})_{t\in\mathbb{R}}$ *generated by the increments of* W^Q, defined by

$$\mathcal{F}_t^{\delta W^Q} := \sigma(W^Q(u) - W^Q(s) : s < u \le t) \vee \sigma(\mathcal{N}_{\mathbb{P}}), \quad t \in \mathbb{R}. \tag{3.2.2}$$

3

Note that we have $\mathcal{F}_t^{\delta W^Q} \subseteq \mathcal{F}_t^{W^Q}$ for all $t \in \mathbb{R}$ and $\mathcal{F}_t^{W^Q} = \mathcal{F}_t^{\delta W^Q}$ for $t \in [0,\infty)$, where $\mathcal{F}_t^{W^Q}$ is generated by $(W^Q(s))_{s\in(-\infty,t]}$ for each $t \in \mathbb{R}$. We point out that $(\mathcal{F}_t^{\delta W^Q})_{t\in\mathbb{R}}$ is normal, cf. [18, Example 3.6]. By (WP3), the two-sided Wiener process $(W^Q(t))_{t\in\mathbb{R}}$ now satisfies that $W^Q(t) - W^Q(s') \perp\!\!\!\perp \mathcal{F}_s^{\delta W^Q}$ for all $s \le s' < t$, so that, analogously to the one-sided case, $\left(\int_{-\infty}^t \Phi(r)\,\mathrm{d}W^Q(r)\right)_{t\in\mathbb{R}}$ is a martingale with respect to $(\mathcal{F}_t^{\delta W^Q})_{t\in\mathbb{R}}$ for every $\Phi(\cdot)Q^{\frac{1}{2}} \in L^2(\mathbb{R};\mathscr{L}_2(U,\widetilde{U}))$. Unlike $(W_1^Q(t))_{t\ge0}$, however, the two-sided process $(W^Q(t))_{t\in\mathbb{R}}$ itself will not be a martingale with respect to any filtration, see Proposition 3.A.5 in Appendix 3.A. We refer the reader to [18, 19] for more details on the subject of real-valued martingale type processes indexed by $\mathbb{R}$ and stochastic integration with respect to them.

3.3. MARKOV PROPERTIES FOR HILBERT SPACE VALUED STOCHASTIC PROCESSES

Let $X = (X(t))_{t\in\mathbb{T}}$ be a U-valued stochastic process indexed by $\mathbb{T}$, see Section 3.2.1. Intuitively, X is said to be a Markov process if, at any instant, its past and future states are independent conditional on the present. Varying the amount of information from the present gives rise to different Markov properties, which we will list in decreasing order of strength.

3.3.1. SIMPLE MARKOV PROPERTY

The following definition is often just referred to as the Markov property, see also [56, p. 77] or [66, Equation (6.2), p. 81].

Definition 3.3.1. An $(\mathcal{F}_t)_{t\in\mathbb{T}}$-adapted U-valued stochastic process $(X(t))_{t\in\mathbb{T}}$ is said to have the *simple Markov property* if for all $s \le t$ and $B \in \mathcal{B}(U)$, we have

$$\mathbb{P}(X(t) \in B \mid \mathcal{F}_s) = \mathbb{P}(X(t) \in B \mid X(s)), \quad \mathbb{P}\text{-a.s.}$$

The simple Markov property can also be characterized by means of *transition operators*: The process $(X(t))_{t\in\mathbb{T}}$ is simple Markov if and only if there exists a family $(T_{s,t})_{s\le t\in\mathbb{T}}$ of linear operators on $B_b(U)$ satisfying

$$\mathbb{E}[\varphi(X(t)) \mid \mathcal{F}_s] = T_{s,t}\varphi(X(s)), \quad \mathbb{P}\text{-a.s.} \tag{3.3.1}$$

In this case, the *transition operators* $(T_{s,t})_{s\le t\in\mathbb{T}}$ have the following properties:

(TO1) $T_{s,t}\varphi(x) \ge 0$ for all $x \in U$ if $\varphi \in B_b(U)$ is non-negative,

(TO2) $T_{s,t}\mathbf{1}_U = \mathbf{1}_U$,

(TO3) $T_{s,u}\varphi(X(s)) = T_{s,t}T_{t,u}\varphi(X(s))$, $\mathbb{P}$-a.s., for $\varphi \in B_b(U)$ and $s \le t \le u$.

Lastly, we can also characterize the simple Markov property in terms of conditional independence: By Theorem 3.A.1 in Appendix 3.A, the simple Markov property is equivalent to the fact that $\mathcal{F}_s \perp\!\!\!\perp_{X(s)} \sigma(X(t))$ holds for all $s \le t$. In fact, according to [118, Lemma 11.1], this is in turn equivalent to the statement that $\mathcal{F}_s \perp\!\!\!\perp_{X(s)} \sigma(X(t) : t \ge s)$ for all $s \in \mathbb{T}$.

3.3.2. MULTIPLE MARKOV PROPERTY

The following weaker notion of Markovianity dates back to Doob, who introduced it in the context of stationary real-valued Gaussian processes [67, pp. 271–272]. We generalize it to square-integrable U-valued processes with some mean-square differentiability, i.e., $(X(t))_{t\in\mathbb{T}} \subseteq L^2(\Omega;U)$ such that the function $t \mapsto X(t)$ is classically differentiable from $\mathbb{T}$ to $L^2(\Omega;U)$.

Definition 3.3.2. Suppose that $X = (X(t))_{t\in\mathbb{T}} \subseteq L^2(\Omega;U)$ is an $(\mathcal{F}_t)_{t\in\mathbb{T}}$-adapted U-valued stochastic process and let $N \in \mathbb{N}$. Then X has the *N-ple Markov property* if it has $N-1$ mean square derivatives and, for $s \le t$ in $\mathbb{T}$ and $B \in \mathcal{B}(U)$,

$$\mathbb{P}(X(t) \in B \mid \mathcal{F}_s) = \mathbb{P}(X(t) \in B \mid X(s), X'(s), \dots, X^{(N-1)}(s)), \quad \mathbb{P}\text{-a.s.}$$

Setting $\mathbf{X}(t) := (X^{(k)}(t))_{k=0}^{N-1}$ yields a process taking values in the direct product Hilbert space $(U^N, \langle\cdot,\cdot\rangle_{U^N})$, whose inner product

$$\langle \mathbf{x}, \mathbf{y}\rangle_{U^N} := \sum_{j=1}^{N} \langle x_j, y_j\rangle_U, \quad \mathbf{x} = (x_j)_{j=1}^N, \mathbf{y} = (y_j)_{j=1}^N \in U^N$$

induces the product topology on U^N. In particular, the Borel σ-algebra of U^N satisfies $\mathcal{B}(U^N) = \otimes^N \mathcal{B}(U)$ by [118, Lemma 1.2]. Theorem 3.A.1 in Appendix 3.A again yields an equivalent formulation of the N-ple Markov property in terms of conditional independence:

$$\forall s \in \mathbb{T}: \quad \mathcal{F}_s \perp\!\!\!\perp_{\mathbf{X}(s)} \sigma(X(t) : t \ge s). \tag{3.3.2}$$

Note that $\sigma(\mathbf{X}(s)) \vee \mathcal{F}_s = \mathcal{F}_s$ since the mean-square derivatives of X can be replaced by left derivatives, see the proof of Proposition 3.3.5 below. By arguing as in [118, Lemma 11.1], one can show that this is in turn equivalent to the simple Markov property for $\mathbf{X}$. Thus, we can apply the characterization given by (3.3.1) to derive the following corollary.

Corollary 3.3.3. *An $(\mathcal{F}_t)_{t\in\mathbb{T}}$-adapted and square-integrable U-valued stochastic process $X = (X(t))_{t\in\mathbb{T}}$ with $N-1$ mean-square derivatives is N-ple Markov if and only if there exists a family $(T_{s,t})_{s\le t\in\mathbb{T}}$ of linear operators on $B_b(U^N)$ such that*

$$\mathbb{E}[\varphi(\mathbf{X}(t)) \mid \mathcal{F}_s] = T_{s,t}\varphi(\mathbf{X}(s))$$

holds $\mathbb{P}$-a.s. for all $s \le t \le u$ in $\mathbb{T}$ and $\varphi \in B_b(U^N)$. In this case, $(T_{s,t})_{s\le t\in\mathbb{T}}$ satisfies properties (TO1)–(TO3).

3.3.3. WEAK MARKOV PROPERTIES; RELATIONS BETWEEN CONCEPTS

We now define two Markov properties for which the "present" at time $s \in \mathbb{T}$ is represented by information from neighborhoods around s. As we will prove in Proposition 3.3.5 below, these two notions are equivalent. They appear for instance in [176, p. 62] and [108, Equation (5.87), p. 115].

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Definition 3.3.4. An $(\mathcal{F}_t)_{t\in\mathbb{T}}$-adapted U-valued stochastic process $(X(t))_{t\in\mathbb{T}}$ has

(i) the *weak Markov property* if, for every $s\in\mathbb{T}$, there exists $\delta>0$ such that for all $\varepsilon\in(0,\delta)$ it holds that $\mathcal{F}_s \perp\!\!\!\perp_{\mathscr{A}_\varepsilon(s)} \sigma(X(t):t\geq s)$, where we define the σ-algebras $\mathscr{A}_\varepsilon(s):=\sigma(X(u):u\in(s-\varepsilon,s+\varepsilon)\cap\mathbb{T})$;

(ii) the *σ-Markov property* if for all $s\in\mathbb{T}$ we have $\mathcal{F}_s \perp\!\!\!\perp_{\partial\mathscr{A}(s)} \sigma(X(t):t\geq s)$, where $\partial\mathscr{A}(s):=\bigcap_{\varepsilon>0}\mathscr{A}_\varepsilon(s)$.

Proposition 3.3.5. *Let $X=(X(t))_{t\in\mathbb{T}}$ be an $(\mathcal{F}_t)_{t\in\mathbb{T}}$-adapted U-valued stochastic process. We have the following relations between Markov properties:*

$$\textit{simple Markov} \implies \sigma\textit{-Markov} \iff \textit{weak Markov}.$$

If $N,M\in\mathbb{N}$ are such that $N\geq M$ and X has $N-1$ mean-square derivatives, then we moreover have

$$M\textit{-ple Markov} \implies N\textit{-ple Markov} \implies \textit{weak Markov}.$$

Proof. If X has the weak Markov property, then by definition we have the following identity for fixed $s\in\mathbb{T}$, $B_-\in\mathcal{F}_s$ and $B_+\in\sigma(X(t):t\geq s)$:

$$\mathbb{P}(B_-\mid\mathscr{A}_{1/n}(s))\mathbb{P}(B_+\mid\mathscr{A}_{1/n}(s))=\mathbb{P}(B_-\cap B_+\mid\mathscr{A}_{1/n}(s)),\quad \mathbb{P}\text{-a.s.,} \tag{3.3.3}$$

whenever $n\in\mathbb{N}$ is large enough. Now we note that $(\mathscr{G}_n)_{n\in\mathbb{N}}:=(\mathscr{A}_{1/n}(s))_{n\in\mathbb{N}}$ is a non-increasing sequence of sub-σ-algebras of $\mathcal{F}$, i.e., a backward filtration on $(\Omega,\mathcal{F},\mathbb{P})$. Therefore, $(\mathbb{P}(B\mid\mathscr{G}_n))_{n\in\mathbb{N}}$ is a backward martingale with respect to $(\mathscr{G}_n)_{n\in\mathbb{N}}$ for any $B\in\mathcal{F}$. Combined with the fact that $\bigcap_{n\in\mathbb{N}}\mathscr{G}_n=\partial\mathscr{A}(s)$, the backward martingale convergence theorem [97, Section 12.7, Theorem 4] implies that we may take the $\mathbb{P}$-a.s. limit as $n\to\infty$ in (3.3.3) to find that X is σ-Markov.

Now let $N,M\in\mathbb{N}$ with $N\geq M$ be such that X has the M-ple Markov property and $N-1$ mean-square derivatives. When considering $\sigma(X'(s))$ at $s\in\mathbb{T}$, we can restrict ourselves to mean-square left derivatives, i.e., we consider the sequence

$$(\Delta_n(s))_{n\in\mathbb{N}}:=(n[X(s)-X(s-n^{-1})])_{n\in\mathbb{N}} \tag{3.3.4}$$

converging to $X'(s)$ in the $L^2(\Omega;U)$-norm as $n\to\infty$. Consequently, there exists a subsequence $(\Delta_{n_k}(s))_{k\in\mathbb{N}}$ such that $\Delta_{n_k}(s)\to X'(s)$, $\mathbb{P}$-a.s., as $k\to\infty$. Since $\Delta_{n_k}(s)$ is $\mathcal{F}_s$-measurable for each $k\in\mathbb{N}$, we conclude that $X'(s)$ is $\mathcal{F}_s$-measurable and thus $\sigma(X'(s))\subseteq\mathcal{F}_s$. By induction, this extends to

$$\sigma(X(s),X'(s),\dots,X^{(M-1)}(s))\subseteq\sigma(X(s),X'(s),\dots,X^{(N-1)}(s))\subseteq\mathcal{F}_s,$$

so that Lemma 3.A.2(b) yields the N-ple Markov property as formulated in (3.3.2).

It remains to show that the N-ple Markov property for $N\in\mathbb{N}$ and the σ-Markov property imply weak Markovianity of X. Fix $s\in\mathbb{T}$, $\varepsilon>0$, and set

$$\begin{aligned}
\mathcal{H}_1'&:=\sigma(X(u):u\in(s-\varepsilon,s]\cap\mathbb{T})\subseteq\mathcal{F}_s,\\
\mathcal{H}_2'&:=\sigma(X(u):u\in[s,s+\varepsilon)\cap\mathbb{T})\subseteq\sigma(X(t):t\in[s,\infty)\cap\mathbb{T}),\\
\mathcal{H}'&:=\mathcal{H}_1'\vee\mathcal{H}_2'=\mathscr{A}_\varepsilon(s).
\end{aligned}$$

Since $\sigma(X(s)) \subseteq \partial\mathscr{A}(s) \subseteq \mathscr{A}_\varepsilon(s)$, by Lemma 3.A.2(c) the simple (i.e., 1-ple) Markov or σ-Markov property of X would imply

$$\mathcal{F}_s \perp\!\!\!\perp_{\mathscr{A}_\varepsilon(s)} \sigma(X(t) : t \in [s,\infty) \cap \mathbb{T}), \tag{3.3.5}$$

and thus the weak Markov property since $\varepsilon > 0$ was arbitrary. It remains to show that (3.3.5) also holds if X is N-ple Markov. Picking $K \in \mathbb{N}$ so large that $n_k > \varepsilon^{-1}$ for all $k \geq K$, we find that $(\Delta_{n_k}(s))_{k\geq K}$ (see (3.3.4)) is a sequence of $\mathscr{A}_\varepsilon(s)$-measurable random variables converging $\mathbb{P}$-a.s. to $X'(s)$. As before, repeating this argument inductively yields $\mathcal{H} := \sigma(X(s), X'(s), \ldots, X^{(N-1)}(s)) \subseteq \mathscr{A}_\varepsilon(s)$. This justifies the use of Lemma 3.A.2(c) to establish (3.3.5) for the remaining case, and the desired conclusion follows. □

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Remark 3.3.6. An analog to Definition 3.3.4 for *generalized* U-valued stochastic processes $(X(\phi))_{\phi\in C_c^\infty(\mathbb{T})}$ is obtained by replacing $\sigma(X(u) : u \in J)$ with the σ-algebra $\sigma(X(\phi) : \phi \in C_c^\infty(\mathbb{T}), \operatorname{supp}\phi \subseteq J)$ generated by X on an open set $J \subseteq \mathbb{T}$. Since pointwise evaluation is not meaningful for such processes, there is no analog to the simple Markov property. Furthermore, although the proof of the fact that weak Markov implies σ-Markov carries over, its converse now fails: The distributional derivative of white noise is a generalized process which is σ-Markov but not weak Markov, see [176, p. 62].

3.3.4. CHARACTERIZATION OF WEAKLY MARKOV GAUSSIAN PROCESSES

A U-valued stochastic process $X = (X(t))_{t\in\mathbb{T}}$ is said to be Gaussian if the U^n-valued random variable $(X(t_1), X(t_2), \ldots, X(t_n))$ is Gaussian, for any $n \in \mathbb{N}$ and $\{t_i\}_{i=1}^n \subseteq \mathbb{T}$. We will characterize the weak Markov property of Definition 3.3.4 for vector-valued Gaussian processes by extending the theory of Rozanov [176] from real-valued to U-valued processes.

We consider the case of a mean-square continuous Gaussian process X which is the solution of a stochastic evolution equation of the form $\mathcal{L}X = \dot{W}$ for some linear operator $\mathcal{L}\colon \mathsf{D}(\mathcal{L}) \subseteq L^2(\mathbb{T};U) \to L^2(\mathbb{T};U)$; here, $\dot{W}$ denotes spatiotemporal Gaussian white noise, cf. (3.1.1) and (3.1.4). More precisely, we assume that $\mathcal{L}$ has a bounded inverse $\mathcal{L}^{-1}$ which *colors* X, cf. [50, Definition 3], meaning

$$\langle X, \phi\rangle_{L^2(\mathbb{T};U)} \overset{d}{=} \mathscr{W}([\mathcal{L}^{-1}]^*\phi) \quad \forall \phi \in C_c^\infty(\mathbb{T};U), \tag{3.3.6}$$

where $\overset{d}{=}$ indicates equality in distribution. Here, $(\mathscr{W}(f))_{f\in L^2(\mathbb{T};U)}$ is an $L^2(\mathbb{T};U)$-isonormal Gaussian process, see Section 1.4.2. The following theorem then states that the locality of the *precision operator* $\mathcal{L}^*\mathcal{L}$ is necessary for X to be weakly Markov.

Theorem 3.3.7. *Let $\mathcal{L}\colon \mathsf{D}(\mathcal{L}) \subseteq L^2(\mathbb{T};U) \to L^2(\mathbb{T};U)$ be a boundedly invertible linear operator, and suppose that $X = (X(t))_{t\in\mathbb{T}}$ is a mean-square continuous Gaussian U-valued process colored by $\mathcal{L}^{-1}$. Let F be a dense subset of U for which $C_c^\infty(\mathbb{T};F)$ is contained in $\mathsf{D}(\mathcal{L})$. Furthermore, suppose that $C_c^\infty(\mathbb{T};F)$ and its image under $\mathcal{L}$ are dense subsets of $L^2(\mathbb{T};U)$.*

If X has the weak Markov property from Definition 3.3.4 with respect to its natural filtration $(\mathcal{F}_t^X)_{t\in\mathbb{T}}$, then

$$\forall J \in \mathscr{I}: \quad \langle \mathcal{L}\phi, \mathcal{L}\psi\rangle_{L^2(\mathbb{T};U)} = 0 \quad \forall \phi \in C_c^\infty(J;F), \psi \in C_c^\infty(\mathbb{T}\setminus\overline{J};F), \tag{3.3.7}$$

where $\mathscr{I}$ denotes the set of all open intervals $J \subseteq \mathbb{T}$.

Proof. For all $J \in \mathscr{I}$ we define a closed subspace $\mathfrak{H}(J)$ of $L^2(\Omega)$ by

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$$\mathfrak{H}(J) := \overline{\{\langle X, \phi\rangle_{L^2(\mathbb{T};U)} : \phi \in C_c^\infty(J;F)\}}^{L^2(\Omega)}. \tag{3.3.8}$$

Then the family $(\mathfrak{H}(J))_{J\in\mathscr{I}}$ is a Gaussian random field in the sense of [176, Chapter 2, Section 3.1], and we can connect it to the present setting by showing that $\sigma(X(t) : t \in J) = \sigma(\mathfrak{H}(J))$. Indeed, we have $\sigma(\mathfrak{H}(J)) \subseteq \sigma(X(t) : t \in J)$ since $\langle X, \phi\rangle_{L^2(\mathbb{T};U)}$ is measurable with respect to the latter σ-algebra for all $\phi \in C_c^\infty(J;F)$ with $\operatorname{supp}\phi \subseteq J$.

In order to establish the converse inclusion, it suffices to verify the claim that $X(t)$ is $\sigma(\mathfrak{H}(J))$-measurable for each $t \in J$. Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of U and write

$$X(t) = \textstyle\sum_{j=1}^\infty \langle X(t), e_j\rangle_U\, e_j \quad \text{in } L^2(\Omega;U). \tag{3.3.9}$$

Now we will show that $\langle X(t), e_j\rangle_U$ is $\sigma(\mathfrak{H}(J))$-measurable for every $j \in \mathbb{N}$. In fact, by the density of $F \subseteq U$ it suffices to consider $\langle X(t), x\rangle_U$ for $x \in F$. Let $(\phi_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(J)$ be a sequence of bump functions concentrating around t, i.e., we have

$$\lim_{n\to\infty} \int_{\mathbb{T}} f(s)\phi_n(s)\,\mathrm{d}s = f(t) \quad \text{in } E \quad \text{for any } f \in C(\mathbb{T};E),$$

where E is an arbitrary Banach space. It follows from the mean-square continuity of X that $f := \langle X(\cdot), x\rangle_U \in C(\mathbb{T}; L^2(\Omega))$, thus with $E := L^2(\Omega)$ we obtain

$$\langle X(t), x\rangle_U = \lim_{n\to\infty} \int_{\mathbb{T}} \langle X(s), x\rangle_U \phi_n(s)\,\mathrm{d}s = \lim_{n\to\infty} \int_{\mathbb{T}} \langle X(s), \phi_n(s)x\rangle_U\,\mathrm{d}s$$

in $L^2(\Omega)$. Passing to a $\mathbb{P}$-a.s.-convergent subsequence in the rightmost expression, we find that $\langle X(t), x\rangle_U$ is a limit of $\sigma(\mathfrak{H}(J))$-measurable random variables. Thus, each summand in (3.3.9) is $\sigma(\mathfrak{H}(J))$-measurable, and passing to a $\mathbb{P}$-a.s.-convergent subsequence of $(\sum_{j=1}^N \langle X(t), e_j\rangle_U\, e_j)_{N\in\mathbb{N}}$ proves the claim.

Now the theory of [176, Chapter 2, Section 3.1] implies that X has the weak Markov property from Definition 3.3.4 if and only if $(\mathfrak{H}(J))_{J\in\mathscr{I}}$ is Markov in the sense of [176, p. 97]. For a general $B \subseteq \mathbb{T}$, we define

$$\mathfrak{H}_+(B) := \textstyle\bigcap_{\varepsilon>0} \mathfrak{H}(B^\varepsilon), \tag{3.3.10}$$

where $B^\varepsilon := \{t \in \mathbb{T} : \operatorname{dist}(t,B) < \varepsilon\}$ denotes an open ε-neighborhood of B. Using the definition (3.3.10) for $B \in \{\partial J, J, \mathbb{T}\setminus J\}$, the Markov property for $(\mathfrak{H}(J))_{J\in\mathscr{I}}$ implies that [176, Equations (3.14), p. 97] are satisfied for every $J \in \mathscr{I}$:

$$\mathfrak{H}_+(\partial J) = \mathfrak{H}_+(J) \cap \mathfrak{H}_+(\mathbb{T}\setminus J) \quad \text{and} \quad \mathfrak{H}_+(J)^\perp \perp \mathfrak{H}_+(\mathbb{T}\setminus J)^\perp, \tag{3.3.11}$$

where we take $L^2(\Omega)$-orthogonal complements in $\mathfrak{H}(\mathbb{T})$.

Next we define $X^*\colon C_c^\infty(\mathbb{T};F) \to L^2(\Omega)$ by $X^*(\phi) := \mathscr{W}(\mathcal{L}\phi)$ for $\phi \in C_c^\infty(\mathbb{T};F)$, to which we associate the spaces

$$\mathfrak{H}^*(J) := \overline{\{X^*(\phi) : \phi \in C_c^\infty(J;F)\}}^{L^2(\Omega)}, \quad J \in \mathscr{I}. \tag{3.3.12}$$

Then X^* is dual to $(\langle X, \psi\rangle_{L^2(\mathbb{T};U)})_{\psi \in C_c^\infty(\mathbb{T};F)}$ in the sense that

$$\mathbb{E}[\langle X, \phi\rangle_{L^2(\mathbb{T};U)} X^*(\psi)] = \mathbb{E}[\mathscr{W}([\mathcal{L}^{-1}]^*\phi)\mathscr{W}(\mathcal{L}\psi)] = \langle \phi, \psi\rangle_{L^2(\mathbb{T};U)} \tag{3.3.13}$$

for $\phi, \psi \in C_c^\infty(\mathbb{T};F)$. Next we will prove

$$\mathfrak{H}(\mathbb{T}) = \mathfrak{H}^*(\mathbb{T}) \tag{3.3.14}$$

by showing that both of these sets equal $\mathscr{Z} := \overline{\{\mathscr{W}(f) : f \in L^2(\mathbb{T};U)\}}^{L^2(\Omega)}$. First, we note that $\mathfrak{H}(\mathbb{T})$ and $\mathfrak{H}^*(\mathbb{T})$ are clearly contained in $\mathscr{Z}$. Now let $Z \in \mathscr{Z}$ and $\varepsilon > 0$ be arbitrary, and let $f \in L^2(\mathbb{T};U)$ satisfy $\|\mathscr{W}(f) - Z\|_{L^2(\Omega)} < \frac{1}{3}\varepsilon$. Since the image of $C_c^\infty(\mathbb{T};F)$ under $\mathcal{L}$ is assumed to be dense in $L^2(\mathbb{T};U)$, we may furthermore choose $\phi \in C_c^\infty(\mathbb{T};F)$ such that $\|\mathcal{L}\phi - f\|_{L^2(\Omega;U)} < \frac{2}{3}\varepsilon$. Then

$$\begin{aligned}\|Z - X^*(\phi)\|_{L^2(\Omega)} &\le \|Z - \mathscr{W}(f)\|_{L^2(\Omega)} + \|\mathscr{W}(\mathcal{L}\phi - f)\|_{L^2(\Omega)} \\ &= \|Z - \mathscr{W}(f)\|_{L^2(\Omega)} + \|\mathcal{L}\phi - f\|_{L^2(\mathbb{T};U)} < \varepsilon,\end{aligned}$$

which shows $Z \in \mathfrak{H}^*(\mathbb{T})$ since $\varepsilon > 0$ was arbitrary. On the other hand, since $\mathcal{L}$ is densely defined and has a bounded inverse, it is in particular closed, hence $\mathcal{L}^*$ exists and is also densely defined by [156, Proposition 10.22]. It follows that the range of $(\mathcal{L}^{-1})^* = (\mathcal{L}^*)^{-1}$, which equals $\mathsf{D}(\mathcal{L}^*)$, is dense in $L^2(\mathbb{T};U)$, so that there exists a function $g \in L^2(\mathbb{T};U)$ satisfying $\|f - (\mathcal{L}^{-1})^* g\|_{L^2(\mathbb{T};U)} < \frac{1}{3}\varepsilon$. Finally, we choose the function $\psi \in C_c^\infty(\mathbb{T};F)$ such that $\|\psi - g\|_{L^2(\mathbb{T};U)} < \|(\mathcal{L}^{-1})^*\|^{-1}_{\mathscr{L}(L^2(\mathbb{T};U))} \frac{1}{3}\varepsilon$ so that

$$\begin{aligned}\|Z - \langle X, \psi\rangle_{L^2(\mathbb{T};U)}\|_{L^2(\Omega)} &< \tfrac{1}{3}\varepsilon + \|\mathscr{W}(f - [\mathcal{L}^{-1}]^* g)\|_{L^2(\Omega)} + \|\mathscr{W}([\mathcal{L}^{-1}]^*(g - \psi))\|_{L^2(\Omega)} \\ &= \tfrac{1}{3}\varepsilon + \|f - [\mathcal{L}^{-1}]^* g\|_{L^2(\mathbb{T};U)} + \|[\mathcal{L}^{-1}]^*(g - \psi)\|_{L^2(\mathbb{T};U)} < \varepsilon,\end{aligned}$$

hence also $Z \in \mathfrak{H}(\mathbb{T})$. We conclude that (3.3.14) holds.

The necessity of (3.3.7) for the weak Markov property of X will follow from

$$\mathfrak{H}^*(J) \subseteq \mathfrak{H}_+(\mathbb{T} \setminus J)^\perp \quad \forall J \in \mathscr{I}. \tag{3.3.15}$$

Indeed, if X is weakly Markov, then (3.3.15) in combination with (3.3.11) would imply that the random variables defined by

$$\xi := X^*(\phi) \in \mathfrak{H}^*(J) \subseteq \mathfrak{H}_+(\mathbb{T} \setminus J)^\perp \quad \text{and} \quad \eta := X^*(\psi) \in \mathfrak{H}^*(\mathbb{T} \setminus J) \subseteq \mathfrak{H}_+(J)^\perp,$$

are orthogonal, where $\phi \in C_c^\infty(J;F)$ and $\psi \in C_c^\infty(\mathbb{T} \setminus J;F)$. Therefore, we have

$$0 = \langle \xi, \eta\rangle_{L^2(\Omega)} = \mathbb{E}[X^*(\phi)X^*(\psi)] = \langle \mathcal{L}\phi, \mathcal{L}\psi\rangle_{L^2(\mathbb{T};U)},$$

which shows (3.3.7). Note that by definition (3.3.12) and density, the orthogonality extends to all $\xi \in \mathfrak{H}^*(J)$ and $\eta \in \mathfrak{H}^*(\mathbb{T} \setminus J)$.

In order to verify (3.3.15), we again take $\xi = X^*(\phi)$ with ϕ as above. By the compact support of ϕ, there exists $\varepsilon > 0$ such that $\langle \phi, \psi \rangle_{L^2(\mathbb{T};U)} = 0$ for all $\psi \in C_c^\infty((\mathbb{T} \setminus J)^\varepsilon; F)$. Hence, for $\eta = \langle X, \psi \rangle_{L^2(\mathbb{T};U)} \in \mathfrak{H}((\mathbb{T} \setminus J)^\varepsilon)$ we have $\xi \perp \eta$ by (3.3.13), from which we can deduce $\xi \perp \mathfrak{H}_+(\mathbb{T} \setminus J)$, and thus (3.3.15). □

In order to state and prove sufficient conditions for weak Markovianity of X in terms of the locality of its precision operator, we first need to collect some definitions which are based on objects encountered in the proof of Theorem 3.3.7. Namely, we will define spaces $(\mathcal{H}(J))_{J \in \mathscr{I}}$ such that $\mathcal{H}(\mathbb{T})$ is unitarily isomorphic to $\mathfrak{H}(\mathbb{T})$ and there exists a dense injection $\iota \colon C_c^\infty(\mathbb{T}; F) \to \mathcal{H}(\mathbb{T})$.

Associating, to each $\eta \in \mathfrak{H}(\mathbb{T})$, a mapping $I^{-1}\eta \colon C_c^\infty(\mathbb{T}; F) \to \mathbb{R}$ given by

$$I^{-1}\eta(\phi) := \mathbb{E}[\langle X, \phi \rangle_{L^2(\mathbb{T};U)} \eta], \quad \phi \in C_c^\infty(\mathbb{T}; F), \tag{3.3.16}$$

sets up a linear map $I^{-1} \colon \mathfrak{H}(\mathbb{T}) \to \mathcal{H}(\mathbb{T})$, where $\mathcal{H}(\mathbb{T})$ is defined as the range of I^{-1}. It is also injective since $I^{-1}\eta(\phi) = 0$ for all $\phi \in C_c^\infty(\mathbb{T}; F)$ means $\eta \perp C_c^\infty(\mathbb{T}; F)$ in $\mathfrak{H}(\mathbb{T})$, and thus $\eta = 0$ by (3.3.8).

Equipping $\mathcal{H}(\mathbb{T})$ with the inner product

$$\langle v_1, v_2 \rangle_{\mathcal{H}(\mathbb{T})} := \langle I v_1, I v_2 \rangle_{L^2(\Omega)}, \quad v_1, v_2 \in \mathfrak{H}(\mathbb{T}),$$

renders $I \colon \mathcal{H}(\mathbb{T}) \to \mathfrak{H}(\mathbb{T})$ a unitary isomorphism. For $J \in \mathscr{I}$ we can then define

$$\mathcal{H}(J) := \textstyle\bigvee_{\varepsilon > 0} \{ v \in \mathcal{H}(\mathbb{T}) : v(\phi) = 0 \text{ for all } \phi \in C_c^\infty((\mathbb{T} \setminus J)^\varepsilon; F) \}, \tag{3.3.17}$$

where $\bigvee$ denotes the closed linear span.

A dense injection $\iota \colon C_c^\infty(\mathbb{T}; F) \to \mathcal{H}(\mathbb{T})$ is obtained by defining $\iota v \colon C_c^\infty(\mathbb{T}; F) \to \mathbb{R}$ in the following way, for any $v \in C_c^\infty(\mathbb{T}; F)$:

$$\iota v(\phi) := \langle v, \phi \rangle_{L^2(\mathbb{T};U)}, \quad \phi \in C_c^\infty(\mathbb{T}; F).$$

Indeed, we find $\iota v \in \mathcal{H}(\mathbb{T})$ since the duality relations (3.3.13) and (3.3.14) between X and X^* imply that $X^*(v) \in \mathfrak{H}^*(\mathbb{T}) = \mathfrak{H}(\mathbb{T})$ satisfies

$$\iota v(\phi) = \mathbb{E}[\langle X, \phi \rangle_{L^2(\mathbb{T};U)} X^*(v)] = [I^{-1} X^*(v)](\phi) \quad \forall \phi \in C_c^\infty(\mathbb{T}; F).$$

Moreover, the injectivity follows in the same way as for I^{-1}. To show density of the range, fix an arbitrary $v \in \mathcal{H}(\mathbb{T})$. Then $Iv \in \mathfrak{H}(\mathbb{T}) = \mathfrak{H}^*(\mathbb{T})$ and thus there exists a sequence $(\psi_n)_{n \in \mathbb{N}} \subseteq C_c^\infty(\mathbb{T}; F)$ such that $X^*(\psi_n) \to Iv$ in $L^2(\Omega)$. Consequently, we have $\iota \psi_n = I^{-1} X^*(\psi_n) \to v$ in $\mathcal{H}(\mathbb{T})$.

Remark 3.3.8. For centered, real-valued Gaussian random fields $(Z(x))_{x \in \mathcal{X}}$ which are indexed by a compact metric space $(\mathcal{X}, d_{\mathcal{X}})$ and moreover mean-square continuous, a unitary isomorphism can be established between the $L^2(\Omega; \mathbb{R})$-closure of all linear combinations of point evaluations and the dual of the Cameron–Martin space for its associated Gaussian measure on the space $L^2(\mathcal{X}; \mathbb{R})$, see [122, Lemma 4.1(iii)]. We point out its analogy to the unitary isomorphism I between $\mathcal{H}(\mathbb{T})$ and $\mathfrak{H}(\mathbb{T})$ defined above.

Theorem 3.3.9. *Let the linear operator $\mathcal{L}\colon \mathsf{D}(\mathcal{L}) \subseteq L^2(\mathbb{T};U) \to L^2(\mathbb{T};U)$, the U-valued process $X = (X(t))_{t\in\mathbb{T}}$ and the subset $F \subseteq U$ be as in Theorem 3.3.7. Recall that $\mathcal{H}(\mathbb{T})$ is the range of the linear mapping I^{-1} defined by (3.3.16) and $\mathcal{H}(J)$ is given by (3.3.17) for all $J \in \mathscr{I}$. If*

$$\mathcal{H}(J) = \overline{\iota C_c^\infty(J;F)}^{\mathcal{H}(\mathbb{T})} \quad \forall J \in \mathscr{I}, \tag{3.3.18}$$

then (3.3.7) implies that X has the weak Markov property from Definition 3.3.4.

Proof. We will show that, under the additional assumption (3.3.18),

$$\mathfrak{H}^*(J) = \mathfrak{H}_+(\mathbb{T}\setminus J)^\perp \quad \forall J \in \mathscr{I}. \tag{3.3.19}$$

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Note that inclusion (3.3.15) also holds without this assumption, see the proof of Theorem 3.3.7. Identities (3.3.14) and (3.3.19) express that the collections $(\mathfrak{H}(J))_{J\in\mathscr{I}}$ and $(\mathfrak{H}^*(J))_{J\in\mathscr{I}}$ are *dual* in the sense of [176, Chapter 2, Section 3.5]. In this situation, the theorem on [176, p. 100] yields that X is weakly Markov if and only if $(\mathfrak{H}^*(J))_{J\in\mathscr{I}}$ is *orthogonal*, meaning that $\mathfrak{H}^*(J) \perp \mathfrak{H}^*(\mathbb{T}\setminus\overline{J})$ for all $J \in \mathscr{I}$, which is equivalent to (3.3.7) by the respective definitions of X^* and $(\mathfrak{H}^*(J))_{J\in\mathscr{I}}$.

To verify (3.3.19), note that for each $\varepsilon > 0$,

$$\begin{aligned}\mathfrak{H}((\mathbb{T}\setminus J)^\varepsilon)^\perp &= \{\eta \in \mathfrak{H}(\mathbb{T}) : \langle \eta, \xi\rangle_{L^2(\Omega)} = 0 \text{ for all } \xi \in \mathfrak{H}((\mathbb{T}\setminus J)^\varepsilon)\}\\ &= \{\eta \in \mathfrak{H}(\mathbb{T}) : U^{-1}\eta(\phi) = 0 \text{ for all } \phi \in C_c^\infty((\mathbb{T}\setminus J)^\varepsilon; F)\}\\ &\cong \{v \in \mathcal{H}(\mathbb{T}) : v(\phi) = 0 \text{ for all } \phi \in C_c^\infty((\mathbb{T}\setminus J)^\varepsilon; F)\},\end{aligned}$$

and it follows that $\mathcal{H}(J) \cong \mathfrak{H}_+(\mathbb{T}\setminus J)^\perp$. On the other hand, definition (3.3.12) implies $\mathcal{H}(J) \cong \mathfrak{H}^*(J)$, so together we indeed find (3.3.19). □

Remark 3.3.10. In order for locality of the precision operator $\mathcal{L}^*\mathcal{L}$ to imply the weak Markovianity of X, one needs to verify the additional condition (3.3.18). In the real-valued case, two examples of sufficient conditions on $\mathcal{L}$ for (3.3.18) to hold are [176, Lemmas 1 and 2, pp. 108–111], which are expressed in terms of boundedness of multiplication and translation operators, respectively, w.r.t. the norms $\|(\mathcal{L}^{-1})^*\cdot\|_{L^2(\mathbb{T})}$ and/or $\|\mathcal{L}\cdot\|_{L^2(\mathbb{T})}$. In [176, Chapter 3, Section 3.2], these results are applied to differential operators with sufficiently regular coefficients.

Although it is expected that analogous results can be derived in the U-valued setting, this subject is out of scope for the processes considered in the remainder of this chapter, since we establish Markovianity using direct methods instead of Theorem 3.3.9, see Section 3.4.4. However, we do use Theorem 3.3.7 to show when the process lacks Markov behavior in the last subsection of Section 3.4.4.

3.4. FRACTIONAL STOCHASTIC ABSTRACT CAUCHY PROBLEM ON $\mathbb{R}$

The aim of this section is to define a Hilbert space valued process $(Z_\gamma(t))_{t\in\mathbb{R}}$ which can be interpreted as a *mild solution* to the equation

$$(\partial_t + A)^\gamma X(t) = \dot{W}^Q(t), \quad t \in \mathbb{R}, \qquad \gamma \in (1/2, \infty). \tag{3.4.1}$$

This equation differs from (2.3.1) in that it is posed on the whole of $\mathbb{R}$, and consequently does not require initial conditions. In Section 3.4.1 we specify the setting in which equation (3.4.1) will be considered. The fractional parabolic integral operator $(\partial_t + A)^{-\gamma}$ is defined in Section 3.4.2, and the noise term $\dot{W}^Q$ in (3.4.1) is the formal time derivative of the two-sided Q-Wiener process defined in Section 3.2.2. In Section 3.4.3 we combine these two notions to give a rigorous definition of the process $(Z_\gamma(t))_{t\in\mathbb{R}}$, and we indicate its relation to the fractional Q-Wiener process defined in Section 3.5. Lastly, in Section 3.4.4 we prove that $(Z_\gamma(t))_{t\in\mathbb{R}}$ is N-ple Markov if $\gamma = N \in \mathbb{N}$, but in general does not satisfy the weak Markov property when $\gamma \notin \mathbb{N}$.

3

3.4.1. SETTING

The standing assumption throughout this section on the Hilbert space U and the linear operator A is as follows.

Assumption 3.4.1. The linear operator $-A\colon \mathsf{D}(A) \subseteq U \to U$ on the separable real Hilbert space U generates an *exponentially stable* C_0-semigroup $(S(t))_{t\ge 0}$, i.e.,

$$\exists M_0 \in [1,\infty), w \in (0,\infty) : \forall t \in [0,\infty) : \quad \|S(t)\|_{\mathscr{L}(U)} \le M_0 e^{-wt}. \tag{3.4.2}$$

In addition, we may assume one or both of the following conditions on the fractional power γ and the linear operator Q:

Assumption 3.4.2. (i) There exist $\gamma_0 \in (1/2,\infty)$ and $Q \in \mathscr{L}^+(U)$ such that

$$\int_0^\infty \| t^{\gamma_0 - 1} S(t) Q^{\frac{1}{2}} \|^2_{\mathscr{L}_2(U)} \,\mathrm{d}t < \infty.$$

(ii) The C_0-semigroup $(S(t))_{t\ge 0}$ is analytic.

For an introduction to the theory of C_0-semigroups, the reader is referred to Section 1.1 or [73, 165]. Note that the results in the latter works, while stated for complex Hilbert spaces, can be applied to the real setting by employing *complexifications* of (linear operators on) U, see e.g. Section 2.B.2.

We remark that $(1/2,\infty)$ is the maximal range on which Assumption 3.4.2(i) can hold. Moreover, if Assumption 3.4.2(i) holds for some $\gamma_0 \in (1/2,\infty)$ then the same is true for all $\gamma' \in [\gamma_0,\infty)$; see Section 3.A.2 in Appendix 3.A.

Under Assumption 3.4.2(ii), we have $\frac{\mathrm{d}}{\mathrm{d}t} A^j S(t) = -A^{j+1} S(t)$ as the classical derivative from $(0,\infty)$ to $\mathscr{L}(U)$ for all $j \in \mathbb{N}$; moreover,

$$\exists M_j \in [1,\infty) : \ \forall t \in (0,\infty) : \quad \|A^j S(t)\|_{\mathscr{L}(U)} \le M_j t^{-j} e^{-wt} \tag{3.4.3}$$

by [165, Chapter 2, Theorem 6.13(c)].

3.4.2. FRACTIONAL PARABOLIC CALCULUS AND THE DETERMINISTIC PROBLEM

In this section we first consider the following deterministic counterpart to (3.4.1):

$$(\partial_t + A)^\gamma u(t) = f(t), \quad t \in \mathbb{R}, \qquad \gamma \in (0,\infty), \tag{3.4.4}$$

where $f \in L^2(\mathbb{R};U)$. In order to define its mild solution, we introduce the operation of fractional parabolic integration, generalizing the scalar-valued setting with $A = 0$ which is treated in [121, Chapter 2].

Definition 3.4.3. Let Assumption 3.4.1 hold. Given $f : \mathbb{R} \to U$, we define its *fractional parabolic integral* $\mathfrak{I}^\gamma f : \mathbb{R} \to U$ of order $\gamma \in (0,\infty)$ by

$$\mathfrak{I}^\gamma f(t) := \frac{1}{\Gamma(\gamma)} \int_{-\infty}^t (t-s)^{\gamma-1} S(t-s) f(s)\,\mathrm{d}s \tag{3.4.5}$$

if this Bochner integral exists for almost all $t \in \mathbb{R}$.

Note that, when restricted to L^p-functions f vanishing outside of $[0,T]$ for some $p \in [1,\infty]$ and $T \in (0,\infty)$, we obtain the operator $\mathscr{B}_{\gamma,p}$ from Proposition 2.3.4. Viewing it as a linear operator, we have $\mathfrak{I}^\gamma \in \mathscr{L}(L^p(\mathbb{R};U))$ with $\|\mathfrak{I}^\gamma\|_{\mathscr{L}(L^p(\mathbb{R};U))} \le \frac{M_0}{w^\gamma}$, and $(\mathfrak{I}^\gamma)_{\gamma \ge 0}$ is a semigroup of bounded operators on $L^p(\mathbb{R};U)$ if we set $\mathfrak{I}^0 := \mathrm{Id}_U$, see Proposition 4.2.3. The adjoint operator $\mathfrak{I}^{\gamma*}$ of $\mathfrak{I}^\gamma$ satisfies the following formula:

$$\mathfrak{I}^{\gamma*} f(t) = \frac{1}{\Gamma(\gamma)} \int_t^\infty (s-t)^{\gamma-1} [S(s-t)]^* f(s)\,\mathrm{d}s \quad \text{for all } t \in \mathbb{R}; \tag{3.4.6}$$

this can be proven analogously to Lemma 2.3.6.

Given $T : \mathsf{D}(T) \subseteq U \to U$ and a measure space $(S,\mathscr{A},\mu)$, we define the Bochner space counterpart $\mathcal{T}_S : \mathsf{D}(\mathcal{T}_S) \subseteq L^2(S;U) \to L^2(S;U)$ of T by

$$\mathsf{D}(\mathcal{T}_S) = L^2(S;\mathsf{D}(T)) \quad \text{and} \quad [\mathcal{T}_S f](s) := Tf(s), \text{ a.a. } s \in S,\ f \in \mathsf{D}(\mathcal{T}_S), \tag{3.4.7}$$

which is a generalization of (2.3.4). Using (3.4.7) with $T := A$ and $S := \mathbb{R}$, we have

$$(\partial_t + \mathcal{A}_\mathbb{R}) f = \partial_t f + \mathcal{A}_\mathbb{R} f, \qquad f \in \mathsf{D}(\partial_t + \mathcal{A}_\mathbb{R}) = H^1(\mathbb{R};U) \cap L^2(\mathbb{R};\mathsf{D}(A)),$$

where ∂_t denotes the Bochner–Sobolev weak derivative, whose domain is given by $\mathsf{D}(\partial_t) = H^1(\mathbb{R};U) \subset L^2(\mathbb{R};U)$; see Table 3.1. Since $\mathfrak{I}^\gamma$ can be interpreted as a negative fractional power of $\partial_t + \mathcal{A}_\mathbb{R}$, see Appendix 3.B, it is natural to call $\mathfrak{I}^\gamma f$ a *mild solution* to (3.4.4).

3.4.3. MILD SOLUTION PROCESS

Combining the spatiotemporal fractional integration theory from Section 3.4.2 with the stochastic integral defined in Section 3.2.2, we can give a rigorous definition for the mild solution to (3.4.1). Recall the definition of predictability from (1.4.9).

Definition 3.4.4. Let Assumption 3.4.1 be satisfied and let $\gamma \in (1/2,\infty)$ be such that Assumption 3.4.2(i) holds with $\gamma_0 = \gamma$. An $(\mathcal{F}_t^{\delta W^Q})_{t\in\mathbb{R}}$-predictable modification of the process $Z_\gamma = (Z_\gamma(t))_{t\in\mathbb{R}}$ defined by

$$Z_\gamma(t) := \frac{1}{\Gamma(\gamma)} \int_{-\infty}^t (t-s)^{\gamma-1} S(t-s)\,\mathrm{d}W^Q(s), \quad t \in \mathbb{R}, \tag{3.4.8}$$

is said to be a *mild solution* to (3.4.1).

Note that the stochastic integral on the right-hand side of (3.4.8) is convergent, i.e., it is a well-defined element in $L^2(\Omega; H)$, for each $t \in \mathbb{R}$. This is a direct consequence of Assumption 3.4.2(i) and the Itô isometry (3.2.1). Moreover, by Definition 3.4.4, the mild solution process is unique up to modification.

Proposition 3.4.5. *Let Assumption 3.4.1 be satisfied. Suppose that $t_0 \in [-\infty, \infty)$, $\gamma \in (1/2, \infty)$ are given. Define $\mathbb{T} := [t_0, \infty)$ if $t_0 \in \mathbb{R}$ or $\mathbb{T} := \mathbb{R}$ if $t_0 = -\infty$ and let Assumption 3.4.2(i) hold for $\gamma_0 = \gamma$. The process $(Z_\gamma(t \mid t_0))_{t \in \mathbb{T}}$ defined by*

$$Z_\gamma(t \mid t_0) := \frac{1}{\Gamma(\gamma)} \int_{t_0}^{t} (t-s)^{\gamma-1} S(t-s)\, \mathrm{d}W^Q(s), \quad t \in \mathbb{T}, \tag{3.4.9}$$

where $Z_\gamma(\cdot \mid -\infty) := Z_\gamma$, is mean-square continuous on $\mathbb{T}$.

If in addition Assumption 3.4.2(ii) is satisfied and there exists $N \in \mathbb{N}$ such that Assumption 3.4.2(i) holds for $\gamma_0 = \gamma - N$, then $(Z_\gamma(t \mid t_0))_{t \in \mathbb{T}}$ has N mean square derivatives and, for all $t \in [t_0, \infty)$ and $n \in \{0, \ldots, N-1\}$, we have

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right) \tfrac{\mathrm{d}^n}{\mathrm{d}t^n} Z_\gamma(t \mid t_0) = \tfrac{\mathrm{d}^n}{\mathrm{d}t^n} Z_{\gamma-1}(t \mid t_0). \tag{3.4.10}$$

Remark 3.4.6. The first part of Proposition 3.4.5 asserts that Z_γ is mean-square continuous, and thus continuous in probability. Combined with the fact that $(Z_\gamma(t))_{t \in \mathbb{R}}$ is $\big(\mathcal{F}_t^{\delta W^Q}\big)_{t \in \mathbb{R}}$-adapted by definition, we can apply [56, Proposition 3.7(ii)] (the proof of which can be generalized to unbounded index sets) to obtain the existence of an $(\mathcal{F}_t^{\delta W^Q})_{t \in \mathbb{R}}$-predictable modification of $(Z_\gamma(t))_{t \in \mathbb{R}}$. This modification is a mild solution in the sense of Definition 3.4.4.

Proof of Proposition 3.4.5. The mean-square continuity follows by Lemma 3.A.6 in Appendix 3.A, hence we turn to the mean-square differentiability. Define

$$Z_{\beta,j}(t) := \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} A^j S(t-s)\, \mathrm{d}W^Q(s), \quad t \in [t_0, \infty),$$

for $j \in \mathbb{N}_0$ and $\beta \in (1/2, \infty)$ such the right-hand side exists. We claim that, under Assumptions 3.4.2(i)–(ii) with $\gamma_0 = \gamma - N$, the function $t \mapsto t^{\beta-1} A^j S(t) Q^{\frac{1}{2}}$ belongs to $H_0^1(0, \infty; \mathscr{L}_2(U))$ if $\beta - j - \gamma + N \in [1, \infty)$. To this end, we first note that the product rule for the (classical) derivative yields

$$\tfrac{\mathrm{d}}{\mathrm{d}t} t^{\beta-1} A^j S(t) Q^{\frac{1}{2}} = (\beta-1) t^{\beta-2} A^j S(t) Q^{\frac{1}{2}} - t^{\beta-1} A^{j+1} S(t) Q^{\frac{1}{2}}$$

with values in $\mathscr{L}(U)$ for all $t \in (0, \infty)$. Combining (3.4.3) with an argument involving a change of variables and the semigroup property (cf. the proof of Lemma 3.A.4 in Appendix 3.A), one can show that the $L^2(0, \infty; \mathscr{L}_2(U))$-norms of the two functions on the right-hand side can be estimated by that of the function $t \mapsto t^{\beta-j-2} S(t) Q^{\frac{1}{2}}$, which is finite since $\beta - j - 1 \geq \gamma_0$. Again by (3.4.3), we have

$$\| t^{\beta-1} A^j S(t) Q^{\frac{1}{2}} \|_{\mathscr{L}(U)} \leq M_j t^{\beta-j-1} \| Q^{\frac{1}{2}} \|_{\mathscr{L}(U)} \to 0 \quad \text{as } t \downarrow 0$$

since $\beta - j - 1 \geq \gamma_0 > 0$. Noting that $\mathscr{L}_2(U) \hookrightarrow \mathscr{L}(U)$ and using Lemma 2.A.9 then proves the claim.

Thus, we may apply Lemma 3.A.6 from Appendix 3.A, write the result as two separate integrals, and pull the closed operator A out of the stochastic integral defining $Z_{\beta,j+1}$ (cf. [56, Proposition 4.30]) to find

$$Z'_{\beta,j}(t) = Z_{\beta-1,j}(t) - Z_{\beta,j+1}(t). \tag{3.4.11}$$
$$= Z_{\beta-1,j}(t) - AZ_{\beta,j}(t). \tag{3.4.12}$$

Rearranging equation (3.4.12) for $\beta = \gamma$ and $j = 0$ implies (3.4.10) for $n = 0$. Applying (3.4.11) iteratively, we find that $Z_{\beta,j}$ has the nth mean-square derivative

$$Z^{(n)}_{\beta,j}(t) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} Z_{\beta-n+m,j+m}(t), \tag{3.4.13}$$

provided that $\beta - j - \gamma + N \in [n,\infty)$. Now we again let $\beta = \gamma$ and $j = 0$ and apply (3.4.12) with $\beta' = \gamma - n + m$ and $j' = m$ to each term on the right-hand side to derive (3.4.10) for the remaining values of n. □

The next result concerns the covariance structure of the process Z_γ. Analogously to (2.4.1), let us define the covariance operators $(Q_{Z_\gamma}(s,t))_{s,t\in\mathbb{R}} \subseteq \mathscr{L}_1^+(U)$ of Z_γ via the relation

$$\langle Q_{Z_\gamma}(s,t)x, y\rangle_U = \mathbb{E}\big[\big\langle Z_\gamma(s) - \mathbb{E}[Z_\gamma(s)], x\big\rangle_U \big\langle Z_\gamma(t) - \mathbb{E}[Z_\gamma(t)], y\big\rangle_U\big] \tag{3.4.14}$$

for all $s,t \in \mathbb{R}$ and $x,y \in U$; note that $\mathbb{E}[Z_\gamma(\cdot)] \equiv 0$ in this case. The following proposition, which is analogous with Corollary 2.4.3, states that if $A := \kappa\,\mathrm{Id}_U$, then $Q_{Z_\gamma}(s,t)$ is separable, i.e., it can be decomposed into a (scalar) covariance function depending only on the 'time' variables $s,t \in \mathbb{R}$, and a 'spatial' covariance operator acting on U. Moreover, the temporal factor takes the form of a Matérn covariance function (3.1.2). In the language of spatiotemporal statistics, Z_γ has a marginal temporal covariance structure of Matérn type. This motivates the statistical relevance of Z_γ.

Proposition 3.4.7. *Let $\gamma \in (1/2,\infty)$, $A := \kappa\,\mathrm{Id}_U$ with $\kappa \in (0,\infty)$ and suppose that Assumption 3.4.2(i) is satisfied for $\gamma_0 = \gamma$. Then the covariance of Z_γ is separable and its temporal part is of Matérn type, i.e.,*

$$\forall s,t \in \mathbb{R},\, s \neq t: \quad Q_{Z_\gamma}(s,t) = \frac{2^{\frac{1}{2}-\gamma}\kappa^{1-2\gamma}}{\sqrt{\pi}\,\Gamma(\gamma)}(\kappa|t-s|)^{\gamma-\frac{1}{2}} K_{\gamma-\frac{1}{2}}(\kappa|t-s|)\, Q.$$

Proof. For $A = \kappa\,\mathrm{Id}_U$, Assumption 3.4.1 is trivially satisfied and the definition of Z_γ takes on the following form for all $t \in \mathbb{R}$:

$$Z_\gamma(t) = \frac{1}{\Gamma(\gamma)} \int_{-\infty}^{t} (t-r)^{\gamma-1} e^{-\kappa(t-r)}\, \mathrm{d}W^Q(r) = \int_{\mathbb{R}} k_{\gamma,\kappa}(t-r)\, \mathrm{d}W^Q(r),$$

with real-valued convolution kernel $k_{\gamma,\kappa}(t) := \frac{1}{\Gamma(\gamma)} t_+^{\gamma-1} e^{-\kappa t}$, where we set $t_+^{\gamma-1} := t^{\gamma-1}$ if $t \in [0,\infty)$ and $t_+^{\gamma-1} := 0$ otherwise. Define $\widetilde{k}(s,r;x) \in \mathscr{L}(U;\mathbb{R})$ for $s,r \in \mathbb{R}$ and

$x \in U$ by $\widetilde{k}(s,r;x)h := k_{\gamma,\kappa}(s-r)\langle x,h\rangle_U$ for all $h \in U$. Then combining the Itô isometry (3.2.1) and the polarization identity yields

$$\mathbb{E}[\langle Z_\gamma(s),x\rangle_U \langle Z_\gamma(t),y\rangle_U] = \mathbb{E}\Big[\int_{\mathbb{R}} \widetilde{k}(s,r;x)\,\mathrm{d}W^Q(r) \int_{\mathbb{R}} \widetilde{k}(t,r;y)\,\mathrm{d}W^Q(r)\Big]$$
$$= \int_{\mathbb{R}} \langle \widetilde{k}(s,r;x)Q, \widetilde{k}(t,r;y)\rangle_{\mathscr{L}_2(U;\mathbb{R})}\,\mathrm{d}r = \Big\langle \int_{\mathbb{R}} k_{\gamma,\kappa}(s-r)k_{\gamma,\kappa}(t-r)\,\mathrm{d}r\,Qx,y\Big\rangle_U.$$

Since $x,y \in U$ were arbitrary, we find for all $h \in \mathbb{R}\setminus\{0\}$ the covariance operators

$$Q_{Z_\gamma}(t+h,t) = Q_{Z_\gamma}(t,t+h) = \int_{\mathbb{R}} k_{\gamma,\kappa}(t-r)k_{\gamma,\kappa}(t+h-r)\,\mathrm{d}r\,Q.$$

Using the change of variables $u(r) := h + 2(t-r)$ in the integral, we obtain

$$\int_{\mathbb{R}} k_{\gamma,\kappa}(t-r)k_{\gamma,\kappa}(t+h-r)\,\mathrm{d}r = \frac{1}{2}\int_{\mathbb{R}} k_{\gamma,\kappa}(\tfrac{u-h}{2})k_{\gamma,\kappa}(\tfrac{u+h}{2})\,\mathrm{d}u$$
$$= \frac{2^{1-2\gamma}}{[\Gamma(\gamma)]^2}\int_{|h|}^{\infty}(u^2-h^2)^{\gamma-1}e^{-\kappa u}\,\mathrm{d}u = \frac{2^{\frac{1}{2}-\gamma}\kappa^{1-2\gamma}}{\sqrt{\pi}\,\Gamma(\gamma)}(\kappa|h|)^{\gamma-\frac{1}{2}}K_{\gamma-\frac{1}{2}}(\kappa|h|),$$

where the last identity follows by [161, Part I, Equation (3.13)]. □

3.4.4. MARKOV BEHAVIOR

In this section we consider the Markov behavior of the process Z_γ defined in Section 3.4.3. Namely, we will show that Z_N is N-ple Markov for $N \in \mathbb{N}$ (Theorem 3.4.9), whereas in general Z_γ is not weakly Markov if $\gamma \notin \mathbb{N}$ (Example 3.4.15).

INTEGER CASE; MAIN RESULTS

We first introduce the necessary notation and intermediate results leading up to the main theorem asserting the N-ple Markov property of Z_N. The proofs are postponed to the next subsection.

If $\gamma \in (1/2,\infty)$ is such that Assumptions 3.4.1 and 3.4.2(i) hold with $\gamma_0 = \gamma$, then we define for $t_0 \in \mathbb{R}$ the truncated integral process $(\widetilde{Z}_\gamma(t \mid t_0))_{t\in\mathbb{R}}$ by

$$\widetilde{Z}_\gamma(t \mid t_0) := \frac{1}{\Gamma(\gamma)}\int_{-\infty}^{t\wedge t_0}(t-s)^{\gamma-1}S(t-s)\,\mathrm{d}W^Q(s), \tag{3.4.15}$$

so that $\widetilde{Z}_\gamma(\cdot \mid t_0) = Z_\gamma$ on $(-\infty,t_0]$ and $Z_\gamma = \widetilde{Z}_\gamma(\cdot \mid t_0) + Z_\gamma(\cdot \mid t_0)$ on (t_0,∞), where we recall the process $(Z_\gamma(t \mid t_0))_{t\in[t_0,\infty)}$ from (3.4.9). From these two identities, it follows that $t \mapsto \widetilde{Z}_\gamma(t \mid t_0)$ has the same mean-square differentiability at time $t \in \mathbb{R}\setminus\{t_0\}$ as $Z_\gamma(t)$ (and $Z_\gamma(t \mid t_0)$ if $t \in (t_0,\infty)$). In the case $\gamma = N \in \mathbb{N}$, both have $N-1$ mean-square derivatives by Proposition 3.4.5 provided that Assumption 3.4.2(i) is satisfied for $\gamma_0 = \gamma-(N-1) = 1$. The same holds at the critical point $t = t_0$ since the first $N-1$ mean-square (right) derivatives of $Z_\gamma(\cdot \mid t_0)$ vanish there, see (3.4.13) in the proof of Proposition 3.4.5.

Under Assumption 3.4.2(ii), we have $A^j S(t) \in \mathscr{L}(U)$ for all $j \in \mathbb{N}_0$ and $t \in (0,\infty)$, see (3.4.3). Therefore, we can define the function $\overline{\Gamma}(n,(\cdot)A)\colon [0,\infty) \to \mathscr{L}(U)$ by

$$\overline{\Gamma}(n, tA) := \begin{cases} \sum_{j=0}^{n-1} \frac{t^j}{j!} A^j S(t), & t \in (0,\infty); \\ \mathrm{Id}_U, & t = 0, \end{cases} \tag{3.4.16}$$

for $n \in \mathbb{N}$. Note the analogy with integer-order scalar-valued normalized upper incomplete gamma functions [163, Equations (8.4.10) and (8.4.11)]. We will use these functions to derive an expression for $(\widetilde{Z}_N(t \mid t_0))_{t\in[t_0,\infty)}$ in terms of Z_N and its mean-square derivatives at t_0. Recall that the U^N-valued process $\mathbf{Z}_N = (Z_N^{(n)})_{n=0}^{N-1}$ consists of Z_N and its first $N-1$ mean-square derivatives.

Proposition 3.4.8. *Let Assumptions 3.4.1 and 3.4.2(ii) be satisfied and suppose that Assumption 3.4.2(i) holds for $\gamma_0 = 1$. Then for all $N \in \mathbb{N}$, $t_0 \in \mathbb{R}$ and $t \in [t_0,\infty)$,*

$$\widetilde{Z}_N(t \mid t_0) = \zeta_N(t \mid t_0)\mathbf{Z}_N(t_0), \quad \mathbb{P}\text{-a.s.}, \tag{3.4.17}$$

where we define, for any $\boldsymbol{\xi} \in L^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; U^N)$ with $\mathcal{F}_{t_0} := \mathcal{F}_{t_0}^{\delta W^Q}$,

$$\zeta_N(t \mid t_0)\boldsymbol{\xi} := \sum_{k=0}^{N-1} \frac{(t-t_0)^k}{k!} \overline{\Gamma}(N-k, (t-t_0)A)\, \xi_k, \tag{3.4.18}$$

using the incomplete gamma functions defined in (3.4.16).

In particular, adding $Z_N(t \mid t_0)$ on both sides of equation (3.4.17) yields

$$\forall t \in [t_0,\infty): \quad Z_N(t) = Z_N(t \mid t_0, \mathbf{Z}_N(t_0)), \quad \mathbb{P}\text{-a.s.}, \tag{3.4.19}$$

where the process $(Z_N(t \mid t_0, \boldsymbol{\xi}))_{t\in[t_0,\infty)}$ is defined by

$$Z_N(t \mid t_0, \boldsymbol{\xi}) := \zeta_N(t \mid t_0)\boldsymbol{\xi} + Z_N(t \mid t_0), \quad t \in [t_0,\infty). \tag{3.4.20}$$

By (3.4.19), it suffices to show that $(Z_N(t \mid t_0, \boldsymbol{\xi}))_{t\in[t_0,\infty)}$ has the N-ple Markov property in the sense of Definition 3.3.2 for any $t_0 \in \mathbb{R}$ and $\boldsymbol{\xi} \in L^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; U^N)$. In fact, we will show that it is N-ple Markov using Corollary 3.3.3; this is the subject of the following result, which is the main theorem of this section.

Theorem 3.4.9. *Let $N \in \mathbb{N}$, $t_0 \in \mathbb{R}$ and $\boldsymbol{\xi} = (\xi_k)_{k=0}^{N-1} \in L^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; U^{N+1})$ be given. Let Assumptions 3.4.1 and 3.4.2(ii) hold and suppose that Assumption 3.4.2(i) is satisfied for $\gamma_0 = 1$. Then the process $(Z_N(t \mid t_0, \boldsymbol{\xi}))_{t\in[t_0,\infty)}$ from (3.4.20) has the N-ple Markov property in the sense of 3.3.2 with respect to the transition operators $(T_{s,t})_{t_0\le s\le t}$ on $B_b(U^N)$ defined by*

$$T_{s,t}\varphi(\mathbf{x}) := \mathbb{E}[\varphi(\mathbf{Z}_N(t \mid s, \mathbf{x}))], \quad \varphi \in B_b(U^N),\ \mathbf{x} \in U^N,$$

and the increment filtration $(\mathcal{F}_t)_{t\in[t_0,\infty)} := (\mathcal{F}_t^{\delta W^Q})_{t\in[t_0,\infty)}$ from (3.2.2). The process $(Z_N(t))_{t\in\mathbb{R}}$ from (3.4.8) has $N-1$ mean-square derivatives and is N-ple Markov with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}} := (\mathcal{F}_t^{\delta W^Q})_{t\in\mathbb{R}}$, see Definition 3.3.2.

The statements and proofs of Proposition 3.4.8 and Theorem 3.4.9 use the following result regarding the mean-square differentiability of $(\zeta_N(t \mid t_0)\boldsymbol{\xi})_{t\in[t_0,\infty)}$, which is similar to Proposition 3.4.5.

Proposition 3.4.10. *Let $N \in \{2,3,\dots\}$, $t_0 \in \mathbb{R}$ and $\boldsymbol{\xi} \in L^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};U^N)$ be given, where $\xi_k \in L^2(\Omega;\mathsf{D}(A))$ for $k \in \{0,\dots,N-2\}$. Suppose that Assumption 3.4.1 and 3.4.2(ii) hold. Then the process $(\zeta_N(t \mid t_0)\boldsymbol{\xi})_{t\in[t_0,\infty)}$ defined by (3.4.18) is infinitely mean-square differentiable at any $t \in (t_0,\infty)$ and, for $n \in \{0,\dots,N-2\}$,*

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)\tfrac{\mathrm{d}^n}{\mathrm{d}t^n}\zeta_N(t \mid t_0)\boldsymbol{\xi} = \tfrac{\mathrm{d}^n}{\mathrm{d}t^n}\zeta_{N-1}(t \mid t_0)(\xi_{k+1} + A\xi_k)_{k=0}^{N-2}, \quad \mathbb{P}\text{-a.s.} \tag{3.4.21}$$

Moreover, we have $\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)\tfrac{\mathrm{d}^n}{\mathrm{d}t^n}\zeta_1(t \mid t_0)\xi = 0$, $\mathbb{P}$-a.s., for $\xi \in L^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};U)$.

Combining Propositions 3.4.5, 3.4.8 and 3.4.10 yields the following corollary.

Corollary 3.4.11. *Let $N \in \{2,3,\dots\}$, $t_0 \in \mathbb{R}$ and $\boldsymbol{\xi} \in L^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};U^N)$ be given, where $\xi_k \in L^2(\Omega;\mathsf{D}(A))$ for $k \in \{0,\dots,N-2\}$. Let Assumptions 3.4.1 and 3.4.2(ii) hold, and suppose that Assumption 3.4.2(i) is satisfied for $\gamma_0 = 1$. Then the stochastic process $(Z_N(t \mid t_0,\boldsymbol{\xi}))_{t\in[t_0,\infty)}$ from (3.4.20) is $N-1$ times mean-square differentiable at any $t \in (t_0,\infty)$ and satisfies, for $n \in \{0,\dots,N-2\}$,*

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)\tfrac{\mathrm{d}^n}{\mathrm{d}t^n}Z_N(t \mid t_0,\boldsymbol{\xi}) = \tfrac{\mathrm{d}^n}{\mathrm{d}t^n}Z_{N-1}\big(t \mid t_0,(\xi_{k+1} + A\xi_k)_{k=0}^{N-2}\big), \quad \mathbb{P}\text{-a.s.}$$

In particular, it holds for all $t \in \mathbb{R}$ and $n \in \{0,\dots,N-2\}$ that

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)\tfrac{\mathrm{d}^n}{\mathrm{d}t^n}Z_N(t) = \tfrac{\mathrm{d}^n}{\mathrm{d}t^n}Z_{N-1}(t), \quad \mathbb{P}\text{-a.s.}$$

Remark 3.4.12. Corollary 3.4.11 can be interpreted as saying that $(Z_N(t \mid t_0,\boldsymbol{\xi}))_{t\in[t_0,\infty)}$ for $N \in \{2,3,\dots\}$ solves the $L^2(\Omega,\mathcal{F},\mathbb{P};U)$-valued initial value problem

$$\begin{cases} \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)X(t) = Z_{N-1}(t \mid t_0,(\xi_{k+1} + A\xi_k)_{k=0}^{N-2}) & \forall t \in (t_0,\infty), \\ X(t_0) = \xi_0, \end{cases}$$

whenever $\xi_k \in L^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};\mathsf{D}(A))$ for $k \in \{0,\dots,N-2\}$. This observation is the key to the proofs of Propositions 3.4.8 and 3.4.13 below. It is also of interest for computational methods, as it implies that the computation of $Z_N(t \mid t_0,\boldsymbol{\xi})$ amounts to solving a first-order problem $N-1$ times. In fact, inductively applying this result and the fact that $(\tfrac{\mathrm{d}}{\mathrm{d}t} + A)\zeta_1(t \mid t_0)\eta = 0$ for $\eta \in L^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};U)$, we see that for $N \in \mathbb{N}$ we may interpret $(Z_N(t \mid t_0,\boldsymbol{\xi}))_{t\in[t_0,\infty)}$ as the mild solution to the Nth order initial value problem

$$\begin{cases} \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)^N X(t) = \dot{W}^Q(t) & \forall t \in (t_0,\infty), \\ \tfrac{\mathrm{d}^k}{\mathrm{d}t^k}X(t_0) = \xi_k & \forall k \in \{0,\dots,N-1\}. \end{cases}$$

Another key step in the proof of in Theorem 3.4.9 is given by the following result, which essentially amounts to the fact that $(T_{s,t})_{s\le t}$ satisfies (TO3).

Proposition 3.4.13. *Let $N \in \mathbb{N}$, $t_0 \in \mathbb{R}$ and $\boldsymbol{\xi} \in L^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};U^N)$ be given. Let Assumptions 3.4.1 and 3.4.2(ii) hold and suppose that Assumption 3.4.2(i) is satisfied for $\gamma_0 = 1$. The stochastic process $(Z_N(t \mid t_0,\boldsymbol{\xi}))_{t\in[t_0,\infty)}$ from (3.4.20) has the N*-ple Chapman–Kolmogorov property, *i.e., for all $t_0 \le s \le t$ we have*

$$\mathbf{Z}_N(t \mid t_0,\boldsymbol{\xi}) = \mathbf{Z}_N(t \mid s,\mathbf{Z}_N(s \mid t_0,\boldsymbol{\xi})), \quad \mathbb{P}\text{-a.s.} \tag{3.4.22}$$

INTEGER CASE; PROOFS

As indicated in the previous subsection, the statements and proofs of Proposition 3.4.8 and Theorem 3.4.9 rely on Proposition 3.4.10, which we prove first.

Proof of Proposition 3.4.10. We first make some general remarks regarding the operators $\overline{\Gamma}(n,tA)$ from (3.4.16). Under Assumption 3.4.2(ii), estimate (3.4.3) implies that the set $\{t^j A^j S(t) : t \in (0,\infty)\} \subseteq \mathscr{L}(U)$ is uniformly bounded. It follows that $t \mapsto \overline{\Gamma}(n,tA)$ is a strongly continuous function from $[0,\infty)$ to $\mathscr{L}(U)$ for any $n \in \mathbb{N}$, which at $t \in (0,\infty)$ admits a classical derivative satisfying

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)\overline{\Gamma}(n,tA) = \begin{cases} 0, & n = 1; \\ A\overline{\Gamma}(n-1,tA), & n \in \{2,3,\ldots\}. \end{cases}$$

To prove the proposition, we may assume $t_0 = 0$, so fix $t \in (0,\infty)$. For $M \in \mathbb{N}$, $j \in \mathbb{N}_0$ and $\boldsymbol{\eta} \in L^2(\Omega;U^M)$, we define $\zeta_{M,j}(t)\boldsymbol{\eta} := A^j\zeta_M(t \mid 0)\boldsymbol{\eta}$. Combining the product rule in the form $(\frac{\mathrm{d}}{\mathrm{d}t}+A)(uv) = u'v + u[(\frac{\mathrm{d}}{\mathrm{d}t}+A)v]$ with the above recurrence relation yields

$$\begin{aligned}
&(\tfrac{\mathrm{d}}{\mathrm{d}t} + A)\zeta_{M,j}(t)\boldsymbol{\eta} \\
&\quad= \sum_{k=1}^{M-1} \frac{t^{k-1}}{(k-1)!} A^j\overline{\Gamma}(M-k,tA)\eta_k + \sum_{k=0}^{M-2} \frac{t^k}{k!} A^{j+1}\overline{\Gamma}(M-1-k,tA)\eta_k \\
&\quad= \zeta_{(M-1),j}(t)(\eta_{k+1})_{k=0}^{M-1} + \zeta_{(M-1),(j+1)}(t)(\eta_k)_{k=0}^{M-1}.
\end{aligned} \tag{3.4.23}$$

This shows in particular that (3.4.21) holds for integers $N \geq 2$ and $n = 0$, by applying (3.4.23) with $M = N$, $\boldsymbol{\eta} = \boldsymbol{\xi}$ and $j = 0$. Moreover,

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)\zeta_{1,j}(t)\eta = \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + A\right)A^j S(t)\eta = 0.$$

Iteratively applying (3.4.23) and the latter identity then yields that $\zeta_{M,j}\boldsymbol{\eta}$ is $M-1$ times (mean-square) differentiable with an nth derivative of the form

$$\zeta_{M,j}^{(n)}(t)\boldsymbol{\eta} = \sum_{\ell=0}^{n}\sum_{m=0}^{\ell} C_{\ell,m}\zeta_{(M-\ell),(j+n-m)}(t)B_{\ell,m}\boldsymbol{\eta}, \tag{3.4.24}$$

where $C_{\ell,m} \in \mathbb{R}$, $B_{\ell,m} \in \mathscr{L}(U^M;U^{M-\ell})$ and $\zeta_{(M-\ell),(j+n-m)} := 0$ if $M-\ell < 1$. In particular, $\zeta_N(\cdot \mid t_0)\boldsymbol{\xi}$ is $N-1$ times (mean-square) differentiable as claimed. In order to deduce that (3.4.21) also holds for $n \in \{1,\ldots,N-2\}$, we need to justify taking the nth derivative on both sides and commuting it with A. Since A is closed, it suffices to verify that $\zeta_N'(\cdot \mid t_0)\boldsymbol{\xi}$, $A^j\zeta_N(\cdot \mid t_0)\boldsymbol{\xi}$, $\zeta_{N-1}(\cdot \mid t_0)(\xi_{j+1})_{j=0}^{N-1}$ and $A^j\zeta_{N-1}(\cdot \mid t_0)(\xi_j)_{j=0}^{N-1}$ admit nth derivatives for $j \in \{0,1\}$. Indeed, these assertions follow from (3.4.24). □

We can now prove Propositions 3.4.8 and 3.4.13. For the proof of the latter we may use Corollary 3.4.11, since it only combines Propositions 3.4.5, 3.4.8 and 3.4.10.

Proof of Proposition 3.4.8. We use induction on $N \in \mathbb{N}$. For $N = 1$ and $t \in (t_0,\infty)$,

$$\widetilde{Z}_1(t \mid t_0) = S(t-t_0)\int_{-\infty}^{t_0} S(t_0-s)\,\mathrm{d}W^Q(s) = \zeta_1(t \mid t_0)Z_1(t_0), \quad \mathbb{P}\text{-a.s.}$$

3

Now suppose that the statement is true for a given $N \in \mathbb{N}$. By Proposition 3.4.5 and the discussion below (3.4.15), Z_{N+1} and $\widetilde{Z}_{N+1}(\cdot \mid t_0)$ have N mean-square derivatives which satisfy $\left(\frac{d}{dt} + A\right)\widetilde{Z}_{N+1}(t \mid t_0) = \widetilde{Z}_N(t \mid t_0)$ and $\left(\frac{d}{dt} + A\right)Z^{(k)}_{N+1}(t) = Z^{(k)}_N(t)$ for all $k \in \{0,\dots,N-1\}$ and $t \in (t_0,\infty)$. Combined with Proposition 3.4.10 and the induction hypothesis, we find

$$\begin{aligned}\left(\tfrac{d}{dt} + A\right)\zeta_{N+1}(t \mid t_0)\mathbf{Z}_{N+1}(t_0) &= \zeta_N(t \mid t_0)\left[\left(\tfrac{d}{dt} + A\right)Z^{(k)}_{N+1}(t_0)\right]_{k=0}^{N-1}\\ &= \zeta_N(t \mid t_0)\mathbf{Z}_N(t_0) = \widetilde{Z}_N(t \mid t_0) = \left(\frac{d}{dt} + A\right)\widetilde{Z}_{N+1}(t \mid t_0).\end{aligned}$$

Since $\widetilde{Z}_{N+1}(t_0 \mid t_0) = Z_{N+1}(t_0) = \zeta_{N+1}(t_0 \mid t_0)\mathbf{Z}_{N+1}(t_0)$, we find that relation (3.4.17) with $N+1$ holds on $[t_0,\infty)$ by the uniqueness of solutions to $L^2(\Omega,\mathcal{F},\mathbb{P};U)$-valued abstract Cauchy problems, see [165, Chapter 4, Theorem 1.3]. □

Proof of Proposition 3.4.13. Let $t_0 \le s \le t$. We use induction on $N \in \mathbb{N}$. For the base case $N = 1$ we have

$$\begin{aligned}Z_1(t \mid s, Z_1(s \mid t_0,\xi)) &= S(t-s)Z_1(s \mid t_0,\xi) + \int_s^t S(t-r)\,dW^Q(r)\\ &= S(t-s)S(s-t_0)\xi + S(t-s)\int_{t_0}^s S(s-r)\,dW^Q(r) + \int_s^t S(t-r)\,dW^Q(r)\\ &= S(t-t_0)\xi + \int_{t_0}^s S(t-r)\,dW^Q(r) + \int_s^t S(t-r)\,dW^Q(r) = Z_1(t \mid t_0,\xi)\end{aligned}$$

$\mathbb{P}$-a.s. for $\xi \in L^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};\mathsf{D}(A))$. Now suppose that the result holds for $N \in \mathbb{N}$ and let $\boldsymbol{\xi} \in L^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};\mathsf{D}(A)^N)$. Then, for any $t \in (s,\infty)$,

$$\begin{aligned}\left(\tfrac{d}{dt} + A\right)Z_{N+1}(t \mid s, \mathbf{Z}_{N+1}(s \mid t_0,\boldsymbol{\xi})) &= Z_N\left(t \mid s, \left[\left(\tfrac{d}{dt} + A\right)Z^{(k)}_{N+1}(s \mid t_0,\boldsymbol{\xi})\right]_{k=0}^{N-1}\right)\\ &= Z_N\left(t \mid s, \mathbf{Z}_N(s \mid t_0, [\xi_{k+1} + A\xi_k]_{k=0}^{N-1})\right)\\ &= Z_N(t \mid t_0, [\xi_{k+1} + A\xi_k]_{k=0}^{N-1}) = \left(\tfrac{d}{dt} + A\right)Z_{N+1}(t \mid t_0,\boldsymbol{\xi}), \quad \mathbb{P}\text{-a.s.},\end{aligned}$$

where we applied Corollary 3.4.11 in every identity except the third, which uses the induction hypothesis. Moreover, relation $Z_{N+1}(s \mid s, \mathbf{Z}_{N+1}(s \mid t_0,\boldsymbol{\xi})) = Z_{N+1}(s \mid t_0,\boldsymbol{\xi})$ is evident from the definitions. Together, these facts imply that the difference process $Y := Z_{N+1}(\cdot \mid s, \mathbf{Z}_{N+1}(s \mid t_0,\boldsymbol{\xi})) - Z_{N+1}(\cdot \mid t_0,\boldsymbol{\xi})$ solves

$$\begin{cases}\left(\frac{d}{dt} + \mathcal{A}_\Omega\right)Y(t) = 0 & \forall t \in (s,\infty);\\ \qquad Y(s) = 0,\end{cases}$$

where $\mathcal{A}_\Omega \colon L^2(\Omega;\mathsf{D}(A)) \subseteq L^2(\Omega;U) \to L^2(\Omega;U)$ is as in (3.4.7). Since $-\mathcal{A}_\Omega$ is the generator of a C_0-semigroup on $L^2(\Omega;U)$, see Lemma 3.B.1 in Appendix 3.B, the uniqueness result [165, Chapter 4, Theorem 1.3] shows that $Y \equiv 0$ on $[s,\infty)$, meaning that $Z_{N+1}(t \mid t_0,\boldsymbol{\xi}) = Z_{N+1}(t \mid s, \mathbf{Z}_{N+1}(s \mid t_0,\boldsymbol{\xi}))$ holds $\mathbb{P}$-a.s. for all $t \in [s,\infty)$. Taking the nth mean-square derivative for $n \in \{0,\dots,N\}$ on both sides, which is justified by Corollary 3.4.11, we find (3.4.22). In order to establish this identity for general initial data $\boldsymbol{\xi} \in L^2(\Omega,\mathcal{F}_{t_0},\mathbb{P};U^N)$, we use the density of $\mathsf{D}(A)$ in U [165, Chapter 1, Corollary 2.5], which implies the density of $L^2(\Omega;\mathsf{D}(A)^N)$ in $L^2(\Omega;U^N)$, hence it suffices to argue that $\boldsymbol{\eta}\mapsto \mathbf{Z}_{N+1}(t\mid t_0,\boldsymbol{\eta})$ is continuous on $L^2(\Omega;U^N)$ for any fixed $t\in[t_0,\infty)$. Continuity of $\boldsymbol{\eta}\mapsto\zeta_{N+1}(t\mid t_0)\boldsymbol{\eta}$ follows from the fact that $\eta\mapsto\overline{\Gamma}(N+1-k,(t-t_0))\eta$ is bounded on $L^2(\Omega;U)$ for any $k\in\{0,\dots,N\}$. The same holds for the derivatives of $\zeta_{N+1}(\cdot\mid t_0)\boldsymbol{\eta}$ since they are of the same form by Proposition 3.4.10. The conclusion follows. □

With these intermediate results in place, we are ready to prove the main theorem asserting the N-ple Markovianity of Z_N. Its proof is a generalization of [56, Theorem 9.14] and [167, Theorem 9.30], which concern the case $N=1$.

3

Proof of Theorem 3.4.9. **Step 1: Well-definedness of** $(T_{s,t})_{t_0\le s\le t}$. We have to show that $T_{s,t}\varphi=\mathbb{E}[\varphi(\mathbf{Z}_N(t\,|\,s,\cdot))]$ is measurable for $\varphi\in B_b(U^N)$. Let $\mathscr{H}$ be the linear space of bounded $\varphi\colon U^N\to\mathbb{R}$ such that $\mathbb{E}[\varphi(\mathbf{Z}_N(t\mid s,\cdot))]$ is measurable. Arguing as in the proof of Proposition 3.4.13, we find that $[\mathbf{Z}_N(t\mid s,\cdot)](\omega)$ is continuous on U^N for $\mathbb{P}$-a.e. $\omega\in\Omega$. Then, for $\varphi\in\mathscr{C}:=C_b(U^N)$, the dominated convergence theorem implies that $\mathbb{E}[\varphi(\mathbf{Z}_N(t\mid s,\cdot))]$ is also continuous, hence Borel measurable, and thus $\mathscr{C}\subseteq\mathscr{H}$. Moreover, $\mathscr{H}$ contains all constant functions and given $(\varphi_n)_{n\in\mathbb{N}}\subseteq\mathscr{H}$ such that $0\le\varphi_n\uparrow\varphi$ pointwise for some bounded limit function φ, we find $\varphi\in\mathscr{H}$ by monotone convergence. Since $\mathscr{C}$ is closed under pointwise multiplication, we find $B_b(U^N,\sigma(\mathscr{C}))=B_b(U^N)\subseteq\mathscr{H}$ by the monotone class theorem [172, Chapter 0, Theorem 2.2].

Step 2: N**-ple Markovianity**. For $t_0\le s\le t$ and $\varphi\in B_b\left(U^N\right)$, we show

$$\mathbb{E}[\varphi(\mathbf{Z}_N(t\mid t_0,\boldsymbol{\xi}))\mid\mathcal{F}_s]=T_{s,t}\varphi(\mathbf{Z}_N(s\mid t_0,\boldsymbol{\xi})),\quad\mathbb{P}\text{-a.s.},$$

for all $\boldsymbol{\xi}\in L^2(\Omega,\mathcal{F}_s,\mathbb{P};U^N)$. By Proposition 3.4.13, it suffices to verify that the identity $\mathbb{E}[\varphi(\mathbf{Z}_N(t\mid s,\boldsymbol{\xi}))\mid\mathcal{F}_s]=T_{s,t}\varphi(\boldsymbol{\xi})$ holds $\mathbb{P}$-a.s. By a monotone class argument similar to that of Step 1, it suffices to consider $\varphi\in C_b(U^N)$. As in [167, Theorem 9.30], one can first verify it directly for simple $\boldsymbol{\xi}=\sum_{j=1}^n\mathbf{x}_j\mathbf{1}_{A_j}$, with $n\in\mathbb{N}$, $\{\mathbf{x}_1,\dots,\mathbf{x}_n\}\subseteq U^N$ and disjoint events $\{A_1,\dots,A_n\}\subseteq\mathcal{F}_s$ covering Ω, and subsequently extend it to general $\boldsymbol{\xi}\in L^2(\Omega,\mathcal{F}_s,\mathbb{P};U^N)$ by an approximation argument, using the continuity of the functions $\mathbf{Z}_N(t\mid s,\cdot)$ and φ. Finally, the statement regarding $(Z_N(t))_{t\in\mathbb{R}}$ follows from Proposition 3.4.8. □

NON-MARKOVIANITY IN THE FRACTIONAL CASE

We conclude this section by showing how Theorem 3.3.7 can be used to deduce that Z_γ is not weakly Markov (see Definition 3.3.4) if $\gamma\notin\mathbb{N}$. To this end, we determine the coloring operator of Z_γ.

Proposition 3.4.14. *Let Assumption 3.4.1 hold and suppose that* $\gamma\in(1/2,\infty)$ *is such that Assumption 3.4.2(i) holds for* $\gamma_0=\gamma$. *Then the coloring operator of* Z_γ, *see* (3.3.6), *is given by* $\mathcal{L}_\gamma^{-1}=\mathfrak{I}^\gamma Q_{\mathbb{R}}^{\frac12}\in\mathscr{L}(L^2(\mathbb{R};U))$, *where* $\mathfrak{I}^\gamma$ *is as in* (3.4.5).

Proof. Using the stochastic Fubini theorem as in the proof of Proposition 2.3.11, we find that $\langle Z_\gamma,f\rangle_{L^2(\mathbb{R};U)}=\int_{\mathbb{R}}\widetilde{\Phi}_f(s)\,\mathrm{d}W^Q(s)$ holds $\mathbb{P}$-a.s. for every $f\in C_c^\infty(\mathbb{R};U)$, where

$\widetilde{\Phi}_f \colon \mathbb{R} \to \mathscr{L}(U;\mathbb{R})$ is given by

$$\widetilde{\Phi}_f(s)u := \frac{1}{\Gamma(\gamma)} \int_s^\infty \langle (t-s)^{\gamma-1} S(t-s)u, f(t)\rangle_U \,\mathrm{d}t = \langle u, \mathfrak{I}^{\gamma*} f(s)\rangle_U$$

for all $s \in \mathbb{R}$ and $u \in U$, and we recall equation (3.4.6) for the adjoint of $\mathfrak{I}^\gamma$. Thus,

$$\|\langle Z_\gamma, f\rangle_{L^2(\mathbb{R};U)}\|^2_{L^2(\Omega)} = \int_{\mathbb{R}} \|\widetilde{\Phi}_f(t) Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(U;\mathbb{R})} \,\mathrm{d}t = \|\mathcal{Q}_{\mathbb{R}}^{\frac{1}{2}} \mathfrak{I}^{\gamma*} f\|^2_{L^2(\mathbb{R};U)}$$

by (3.2.1), hence (3.3.6) holds with $(\mathcal{L}_\gamma^{-1})^* = \mathcal{Q}_{\mathbb{R}}^{\frac{1}{2}} \mathfrak{I}^{\gamma*}$ by the polarization identity. □

Example 3.4.15. Let Assumptions 3.4.1 and 3.4.2(ii) be satisfied and suppose that $\gamma \in (1/2, \infty)$ is such that Assumption 3.4.2(i) holds for $\gamma_0 = \gamma$. The latter implies that $\partial_t + \mathcal{A}_{\mathbb{R}} = \mathcal{B}$, and we always have $\mathcal{B}^{-\gamma} = \mathfrak{I}^\gamma$, see Section 3.4.2. Thus,

$$\mathcal{L}_\gamma^* \mathcal{L}_\gamma = \left(\mathcal{Q}_{\mathbb{R}}^{-\frac{1}{2}} \mathcal{B}^\gamma\right)^* \mathcal{Q}_{\mathbb{R}}^{-\frac{1}{2}} \mathcal{B}^\gamma = \mathcal{B}^{\gamma*} \mathcal{Q}_{\mathbb{R}}^{-1} \mathcal{B}^\gamma = (\partial_t + \mathcal{A}_{\mathbb{R}})^{\gamma*} \mathcal{Q}_{\mathbb{R}}^{-1} (\partial_t + \mathcal{A}_{\mathbb{R}})^\gamma.$$

Moreover, this assumption implies that $\mathsf{D}(A^n)$ is dense in U for all $n \in \mathbb{N}$ by [165, Chapter 2, Theorem 6.8(c)]; choosing n large enough, we also find that $C_c^\infty(\mathbb{T}; \mathsf{D}(A^n))$ is dense in $\mathsf{D}(\mathcal{B}^\gamma)$, so we can take $F = \mathsf{D}(A^n)$ in Theorem 3.3.7.

Although Q^{-1} may be a nonlocal spatial operator, $\mathcal{Q}_{\mathbb{R}}^{-1}$ is always local in time. Thus for $\gamma \in \mathbb{N}$, the precision operator is local as a composition of three local operators, in accordance with the Markovianity shown in Section 3.4.4.

For $\gamma \notin \mathbb{N}$, we will show that the precision operator is not local in general. Suppose that A has an eigenvector $v \in U$ with corresponding eigenvalue $\lambda \in \mathbb{R}$. Such eigenpairs exists for example if $A = (\kappa^2 - \Delta)^\beta$ with $\kappa, \beta \in (0,\infty)$ and Δ the Dirichlet Laplacian on a bounded Euclidean domain $\mathcal{D} \subsetneq \mathbb{R}^d$. If we moreover assume that $v \in \mathsf{D}(Q^{-\frac{1}{2}})$, then we find $\mathcal{Q}_{\mathbb{R}}^{-\frac{1}{2}} \mathcal{B}^\gamma(\phi \otimes v) = [(\partial_t + \lambda)^\gamma \phi] \otimes Q^{-\frac{1}{2}} v$ for $\phi \in C_c^\infty(\mathbb{R})$ since the spectral mapping theorem implies $S(t)v = e^{-\lambda t} v$ for all $t \in [0,\infty)$. It thus suffices to consider the case $A = \lambda \in \mathbb{R}$, i.e., we wish to find disjointly supported $\phi, \psi \in C_c^\infty(\mathbb{R})$ such that

$$F_\gamma(\phi,\psi) := |\langle (\partial_t + \lambda)^\gamma \phi, (\partial_t + \lambda)^\gamma \psi\rangle_{L^2(\mathbb{R})}| \neq 0. \tag{3.4.25}$$

We will discuss this by means of a numerical experiment for the case $\lambda = 1$, using the following smooth function $\phi \in C_c^\infty(\mathbb{R})$ supported on $[-1,1]$:

$$\phi(t) := \begin{cases} \exp\left(-\frac{1}{1-x^2}\right), & x \in (-1,1), \\ 0, & x \in \mathbb{R} \setminus (-1,1), \end{cases} \tag{3.4.26}$$

and taking $\psi := \phi(\cdot - 2 - \delta)$ for some $\delta \in (0,\infty)$. In Figure 3.1, we see that the parabolic derivatives of ϕ consists of (positive or negative) peaks. For $\gamma \notin \mathbb{N}$, the support of the last of these peaks appears to include the whole of $[1,\infty)$, with its absolute value taking rapidly decaying yet nonzero values there. Therefore, the idea is to take δ small enough, making the right-hand side tail of ϕ overlap with the first peak of ψ to obtain a non-zero $L^2(\mathbb{R})$-inner product. Table 3.2 shows the approximate outcomes of this process for various values of γ and δ, using symbolic differentiation and numerical integration.

Note the contrast with the merely spatial Matérn case, where the self-adjointness of the shifted Laplacian $\kappa^2 - \Delta$ causes $L_\beta^* L_\beta = \tau^2(\kappa^2 - \Delta)^{2\beta}$, thus we find a weak Markov property also for half-integer values of $\beta \in (0, \infty)$.

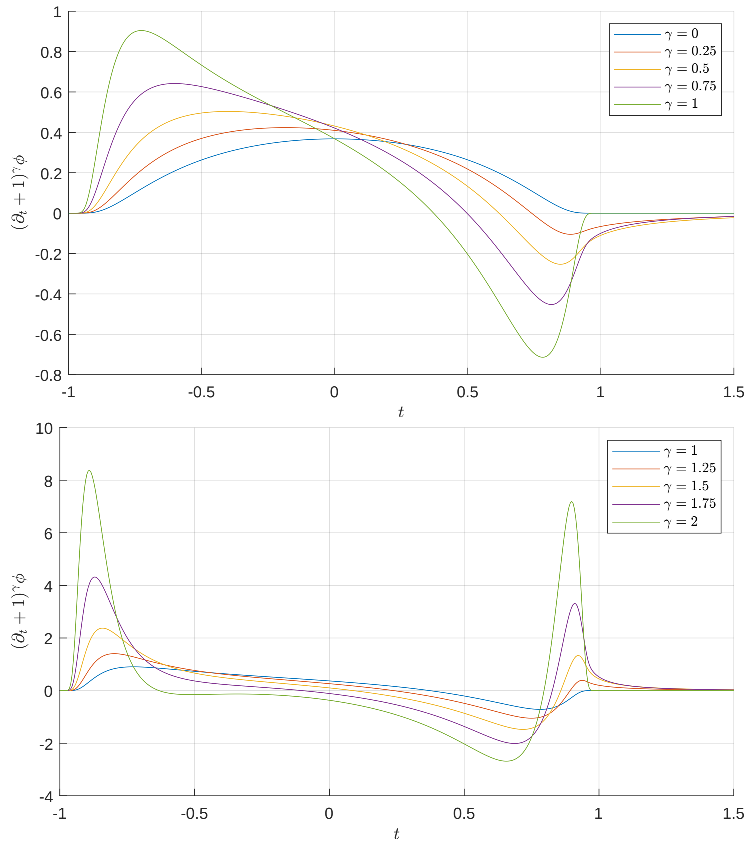

Figure 3.1: Graphs of the fractional parabolic derivative $(\partial_t + 1)^\gamma \phi$ with ϕ from (3.4.26) for certain values of $\gamma \in [0, 2]$. Note the different scales on the y-axes.

3.5. FRACTIONAL Q-WIENER PROCESS

In this final section, we further motivate our interest in solutions to (3.4.1) by relating them to *fractional Q-Wiener processes*, which are U-valued generalizations of the widely studied (real-valued) fractional Brownian motion (see Definition 1.4.7). In Section 3.5.1 we show that, analogously to the real-valued case (see [147, Definition 2.1]), a fractional Q-Wiener process can also be expressed as a Mandelbrot–

δ \ γ	0.25	0.50	0.75	1	1.25	1.50	1.75	2	2.25	2.50	2.75	3
10^{-1}	0.004	0.007	0.007	0	0.016	0.042	0.059	0	0.298	1.078	2.111	0
10^{-2}	0.005	0.009	0.009	0	0.024	0.065	0.098	0	0.622	2.568	5.829	0
10^{-3}	0.005	0.009	0.009	0	0.025	0.068	0.104	0	0.678	2.850	6.601	0

Table 3.2: Numerically approximated values of $F_\gamma(\phi,\psi)$, see (3.4.25), with ϕ from (3.4.26) and $\psi := \phi(\cdot - 2 - \delta)$ for certain values of γ and δ.

Van Ness type stochastic integral over $\mathbb{R}$. Using this representation, we show that fractional Q-Wiener processes are limiting cases of mild solutions to (3.4.1) as introduced in Definition 3.4.4. Finally, in Section 3.5.2 we comment on the Markov behavior of fractional Q-Wiener processes, and we propose possible directions in which to extend or complement the present results of Section 3.3 to establish necessary and sufficient conditions for (weak or N-ple) Markovianity.

The following definition was introduced in [68, Definition 2.1].

Definition 3.5.1. Let $Q \in \mathscr{L}_1^+(U)$. A U-valued Gaussian process $(W_H^Q(t))_{t\in\mathbb{R}}$ is called a *fractional Q-Wiener process with Hurst parameter $H \in (0,1)$* if

(f-WP1) $\mathbb{E}[W_H^Q(t)] = 0$ for all $t \in \mathbb{R}$;

(f-WP2) $Q_H(s,t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})Q$ for all $s,t \in \mathbb{R}$;

(f-WP3) W_H^Q has continuous sample paths.

Here, $(Q_H(s,t))_{s,t\in\mathbb{R}} \subseteq \mathscr{L}_1^+(U)$ are the covariance operators of W_H^Q, cf. (3.4.14).

Note that for $H = \frac{1}{2}$, the above definition reduces to a characterization of a standard (non-fractional) Q-Wiener process when restricted to $[0,\infty)$.

3.5.1. INTEGRAL REPRESENTATION AND RELATION TO Z_γ

Let $Q \in \mathscr{L}_1^+(U)$ and suppose that $(W^Q(t))_{t\in\mathbb{R}}$ is a two-sided Q-Wiener process, see Section 3.2.2. For $H \in (0,1)$, define $(\widehat{W}_H^Q(t))_{t\in\mathbb{R}}$ by

$$\widehat{W}_H^Q(t) := \int_{\mathbb{R}} K_H(t,r)\,\mathrm{d}W^Q(r), \quad t \in \mathbb{R}, \tag{3.5.1}$$

where the Mandelbrot–Van Ness [147] type kernel $K_H\colon \mathbb{R}^2 \to \mathbb{R}$ is given by

$$K_H(t,r) := \frac{1}{C_H}\left[(t-r)_+^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}}\right], \quad (t,r) \in \mathbb{R}^2. \tag{3.5.2}$$

The constant $C_H := \int_{\mathbb{R}} \left|(1-r)_+^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}}\right|^2 \mathrm{d}r = \frac{3-2H}{4H}\mathrm{B}(2-2H, H+\frac{1}{2})$ (where B is the beta function) [168, Theorem B.1], ensures $\widehat{Q}_H(1,1) = Q$, where $(\widehat{Q}_H(t,s))_{t,s\in\mathbb{R}}$ denote the covariance operators of $\widehat{W}_H^Q$. Then $\widehat{W}_H^Q$ has a modification which is a fractional Q-Wiener process:

Proposition 3.5.2. *For all $t \in \mathbb{R}$, (3.5.1) yields a well-defined random variable $\widehat{W}_H^Q(t)$ in $L^2(\Omega, \mathcal{F}_t^{\delta W^Q}, \mathbb{P}; U)$, and there exists a modification of $(\widehat{W}_H^Q(t))_{t\in\mathbb{R}}$ which is a fractional Q-Wiener process in the sense of Definition 3.5.1.*

Proof. To see that $\widehat{W}_H^Q(t) \in L^2(\Omega; U)$, note that the Itô isometry (3.2.1) implies

$$\mathbb{E}\Big[\|\widehat{W}_H^Q(t)\|_U^2\Big] = \int_{\mathbb{R}} \|K_H(t,r)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(U)}\,\mathrm{d}r = \operatorname{tr} Q \int_{\mathbb{R}} |K_H(t,r)|^2\,\mathrm{d}r < \infty.$$

Since K_H is a deterministic kernel integrated with respect to a mean-zero Gaussian process W^Q, it readily follows that $\widehat{W}_H^Q$ is also mean-zero Gaussian. For the covariance operators $(\widehat{Q}_H(t,s))_{t,s\in\mathbb{R}}$ of $\widehat{W}_H^Q$, we can argue as in the proof of Proposition 3.4.7 to find $\widehat{Q}_H(t,s) = \int_{\mathbb{R}} K_H(s,r)K_H(t,r)\,\mathrm{d}r\, Q = \mathbb{E}[B_H(t)B_H(s)]\,Q$ for all $t,s\in\mathbb{R}$, where $B_H = (B_H(t))_{t\in\mathbb{R}}$ denotes (real-valued) fractional Brownian motion. It follows that (f-WP2) holds by the properties of B_H. Lastly, the existence of a (Hölder) continuous modification of (3.5.1) can be established analogously to the real-valued case, by using that $\widehat{W}_H^Q$ is self-similar with stationary increments and applying the Kolmogorov–Chentsov theorem [49, Corollary 3.10]. □

Now we consider the relation between the fractional Q-Wiener process and the process Z_γ considered in Section 3.4. For $\varepsilon \in (0,\infty)$, let Z_γ^ε denote the mild solution to (3.4.1) with $A = \varepsilon\,\mathrm{Id}_U$ and define the process $\big(\overline{Z}_\gamma^\varepsilon(t)\big)_{t\in\mathbb{R}}$ by

$$\overline{Z}_\gamma^\varepsilon(t) := C_{\gamma-1/2}^{-1}\Gamma(\gamma)(Z_\gamma^\varepsilon(t) - Z_\gamma^\varepsilon(0)), \quad t\in\mathbb{R}. \tag{3.5.3}$$

Note that $\widehat{W}_H^Q(t)$ can formally be written as the "convergent difference of divergent integrals" $\frac{1}{C_H}[\int_{-\infty}^t (t-s)^{H-\frac{1}{2}}\,\mathrm{d}W^Q(s) - \int_{-\infty}^0 (-s)^{H-\frac{1}{2}}\,\mathrm{d}W^Q(s)]$, cf. [147, Footnote 3]. This expression would correspond to $\varepsilon = 0$ in (3.5.3), which is ill-defined as Assumption 3.4.2(i) cannot be satisfied. However, the next result shows that a fractional Q-Wiener process can be seen as a limiting case of $\overline{Z}_\gamma^\varepsilon$ as $\varepsilon \downarrow 0$.

Proposition 3.5.3. *Let $Q \in \mathscr{L}_1^+(U)$ and $\gamma \in (1/2, 3/2)$. The family of stochastic processes $(\overline{Z}_\gamma^\varepsilon)_{\varepsilon\in(0,\infty)}$ defined by (3.5.3) converges uniformly on compact subsets of $\mathbb{R}$ in mean-square sense to the fractional Q-Wiener process $\widehat{W}_H^Q$ in (3.5.1) with Hurst parameter $H = \gamma - \frac{1}{2}$ as $\varepsilon\downarrow 0$:*

$$\forall\, T\in(0,\infty): \quad \lim_{\varepsilon\downarrow 0}\ \sup_{t\in[-T,T]} \big\|\widehat{W}^Q_{\gamma-1/2}(t) - \overline{Z}_\gamma^\varepsilon(t)\big\|_{L^2(\Omega;U)} = 0.$$

Proof. For $t\in[0,\infty)$, we can write

$$\begin{aligned}\widehat{W}^Q_{\gamma-1/2}(t) - \overline{Z}_\gamma^\varepsilon(t) &= \frac{1}{C_{\gamma-1/2}}\int_0^t (t-s)^{\gamma-1}\big(1-e^{-\varepsilon(t-s)}\big)\,\mathrm{d}W^Q(s)\\ &\quad + \frac{1}{C_{\gamma-1/2}}\int_{-\infty}^0 \big[(t-s)^{\gamma-1}\big(1-e^{-\varepsilon(t-s)}\big) - (-s)^{\gamma-1}\big(1-e^{\varepsilon s}\big)\big]\,\mathrm{d}W^Q(s).\end{aligned}$$

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Applying the Itô isometry to each of these integrals and using the respective changes of variables $s' := -s$ and $s' := t-s$ yields

$$\|\widehat{W}^Q_{\gamma-1/2}(t) - \overline{Z}^\varepsilon_\gamma(t)\|^2_{L^2(\Omega;U)} = \frac{I_1(t)+I_2(t)}{C_{\gamma-\frac{1}{2}}}(\operatorname{tr} Q)^2, \tag{3.5.4}$$
$$I_1(t) := \int_0^{|t|} s^{2\gamma-2}\big(1-e^{-\varepsilon s}\big)^2 \, \mathrm{d}s,$$
$$I_2(t) := \int_0^\infty \big|(|t|+s)^{\gamma-1}\big(1-e^{-\varepsilon(|t|+s)}\big) - s^{\gamma-1}\big(1-e^{-\varepsilon s}\big)\big|^2 \, \mathrm{d}s.$$

For $t \in (-\infty, 0)$ we find (3.5.4) by instead splitting into integrals over $(-\infty, t)$ and $(t, 0)$ and changing variables $s' := t-s$ and $s' := -s$, respectively. For $t \in [-T, T]$, the elementary inequality $1-e^{-x} \le x$ for $x \in [0,\infty)$ yields the estimate

$$I_1(t) \le \varepsilon^2 \int_0^{|t|} s^{2\gamma}\, \mathrm{d}s = \frac{\varepsilon^2 |t|^{2\gamma+1}}{2\gamma+1} \le \frac{\varepsilon^2 T^{2\gamma+1}}{2\gamma+1}.$$

Applying the fundamental theorem of calculus to $u \mapsto (u+s)^{\gamma-1}(1-e^{-\varepsilon(u+s)})$, followed by Minkowski's integral inequality [186, Section A.1], we find for I_2:

$$\begin{aligned}
I_2(t) &= \int_0^\infty \Big|\int_0^{|t|} [(\gamma-1)(u+s)^{\gamma-2}(1-e^{-\varepsilon(u+s)}) + \varepsilon(u+s)^{\gamma-1}e^{-\varepsilon(u+s)}]\, \mathrm{d}u\Big|^2 \mathrm{d}s \\
&\le \Big|\int_0^T \Big[\int_0^\infty \big|(\gamma-1)(u+s)^{\gamma-2}(1-e^{-\varepsilon(u+s)}) + \varepsilon(u+s)^{\gamma-1}e^{-\varepsilon(u+s)}\big|^2 \mathrm{d}s\Big]^{\frac{1}{2}} \mathrm{d}u\Big|^2 \\
&= \varepsilon^{3-2\gamma}\Big|\int_0^T \Big[\int_{\varepsilon u}^\infty \big|(\gamma-1)v^{\gamma-2}(1-e^{-v}) + v^{\gamma-1}e^{-v}\big|^2 \mathrm{d}v\Big]^{\frac{1}{2}} \mathrm{d}u\Big|^2 \\
&\le \varepsilon^{3-2\gamma}\, T^2 \int_0^\infty \big|(\gamma-1)v^{\gamma-2}(1-e^{-v}) + v^{\gamma-1}e^{-v}\big|^2 \mathrm{d}v,
\end{aligned}$$

where we performed the change of variables $v(s) := \varepsilon(u+s)$ on the third line.

The improper integral on the last line converges: As $v \downarrow 0$, the squares of both terms are of order $\mathcal{O}(v^{2\gamma-2})$, where we again use $1-e^{-v} \le v$ for the first term, and we have $2\gamma-2 \in (-1,1)$; the square of the first term is of order $\mathcal{O}(v^{2\gamma-4})$ as $v \to \infty$, with $2\gamma-4 \in (-3,-1)$, whereas the second term decays exponentially. The convergence thus follows by letting $\varepsilon \downarrow 0$. □

3.5.2. Remarks on Markov behavior

Now we consider the Markov behavior of fractional Q-Wiener processes with Hurst parameter $H \in (0,1)$. Since the case $H = \frac{1}{2}$ corresponds to a standard Q-Wiener process, we find that $W^Q_{1/2}$ is simple Markov, whereas we can expect that W^Q_H is not weakly Markov for $H \ne \frac{1}{2}$.

In the real-valued case, the first published proof of non-Markovianity appears to be [112], which shows that B_H is not simple Markov for $H \ne \frac{1}{2}$ using a characterization in terms of its covariance function. This result can be improved by applying the theory of [108, Chapter V] for Gaussian N-ple Markov processes to the Mandelbrot–Van Ness representation of B_H. Namely, by [108, Theorem 5.1], any real-valued process of the form $(\int_{-\infty}^{t} K(t,s)\,\mathrm{d}B(s))_{t\in\mathbb{R}}$ is N-ple Markov for $N \in \mathbb{N}$ only if there exist functions $(f_j)_{j=1}^N, (g_j)_{j=1}^N$ such that $K(s,t) = \sum_{j=1}^N f_j(s) g_j(t)$ for $s,t \in \mathbb{R}$. The real-valued kernel K_H in (3.5.2) satisfies this condition only if $H = \frac{1}{2}$.

The question arises if one can generalize this characterization of real-valued N-ple Markovianity to the case of Hilbert space valued Gaussian processes, such as W_H^Q and Z_γ (see Definitions 3.5.1 and 3.4.4, respectively). However, it is not evident from the proof in the real-valued case what the analogous condition would be in the Hilbertian setting. For instance, the kernel $K(t,s) = S(t-s)$ of the simple Markov process Z_1 factorizes as $K(t,s) = S(t)S(-s)$ provided that $(S(t))_{t\geq 0}$ extends to a C_0-*group* $(S(t))_{t\in\mathbb{R}}$. Otherwise it is not guaranteed that such a factorization exists, and it is not clear either whether this condition remains necessary for simple Markovianity in the Hilbert space case.

In order to establish that W_H^Q (or B_H) does not have the weak Markov property for $H \neq \frac{1}{2}$, one could also attempt to associate a nonlocal precision operator to the process and apply the necessary condition from Theorem 3.3.7. Formally, its coloring operator $\mathcal{L}_H^{-1}$ acts on $f\colon \mathbb{R} \to U$ as $\mathcal{L}_H^{-1} f(t) = \frac{1}{C_H} \int_{\mathbb{R}} K_H(t,s) f(s)\,\mathrm{d}s$ for all $t \in \mathbb{R}$. For certain ranges of H, see for instance [168, Equation (31)], an explicit formula of its inverse $\mathcal{L}_H$ can also be determined. The operator $\mathcal{L}_H^{-1}$ is bounded on some weighted Hölder space by [168, Theorem 6], but there is no reason to expect that it is bounded on a Hilbert space such as $L^2(\mathbb{R};U)$. Therefore, Theorem 3.3.7 is not directly applicable, as it would need to be extended to the Banach space setting, which is beyond the scope of this chapter.

APPENDIX TO CHAPTER 3

3.A. AUXILIARY RESULTS

This appendix collects auxiliary results which are needed in the main text of the chapter but have been postponed for the sake of readability.

3.A.1. CONDITIONAL INDEPENDENCE

Let $\mathcal{G}_1, \mathcal{H}, \mathcal{G}_2 \subseteq \mathcal{F}$ be σ-algebras on $(\Omega, \mathcal{F}, \mathbb{P})$. We first recall a characterization of conditional independence, see [118, Theorem 8.9], from which we derive a lemma which is useful for establishing relations between various (equivalent formulations of) Markov properties defined in Section 3.3.

Theorem 3.A.1 (Doob's conditional independence property)**.** *We have* $\mathcal{G}_1 \perp\!\!\!\perp_{\mathcal{H}} \mathcal{G}_2$ *if and only if* $\mathbb{P}(G_2 \mid \mathcal{G}_1 \vee \mathcal{H}) = \mathbb{P}(G_2 \mid \mathcal{H})$ *holds* $\mathbb{P}$*-a.s., for all* $G_2 \in \mathcal{G}_2$.

Lemma 3.A.2. *If* $\mathcal{G}_1 \perp\!\!\!\perp_{\mathcal{H}} \mathcal{G}_2$*, then*

(a) $\mathcal{G}_1 \vee \mathcal{H} \perp\!\!\!\perp_{\mathcal{H}} \mathcal{G}_2$;

(b) $\mathcal{G}_1 \perp\!\!\!\perp_{\mathcal{H}'} \mathcal{G}_2$ *for any* σ*-algebra* $\mathcal{H}' \supseteq \mathcal{H}$ *such that* $\mathcal{H}' \subseteq \mathcal{G}_1$;

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(c) $\mathcal{G}_1 \perp\!\!\!\perp_{\mathcal{H}'} \mathcal{G}_2$ *for any* σ*-algebra* $\mathcal{H}' \supseteq \mathcal{H}$ *of the form* $\mathcal{H}' = \mathcal{H}'_1 \vee \mathcal{H}'_2$*, where* $\mathcal{H}'_1, \mathcal{H}'_2$ *are* σ*-algebras satisfying* $\mathcal{H}'_1 \subseteq \mathcal{G}_1 \vee \mathcal{H}$ *and* $\mathcal{H}'_2 \subseteq \mathcal{G}_2 \vee \mathcal{H}$.

Proof. Part (a) is [118, Corollary 8.11(i)]; combining it with the identities $\mathcal{G}_1 = \mathcal{G}_1 \vee \mathcal{H}'$ and $\mathcal{H}' = \mathcal{H} \vee \mathcal{H}'$ yields (b). To prove (c), first note that $\mathcal{G}_1 \vee \mathcal{H} \perp\!\!\!\perp_{\mathcal{H} \vee \mathcal{H}'_1} \mathcal{G}_2 \vee \mathcal{H}$ by parts (a) and (b). Applying (a) again, we find $\mathcal{G}_1 \vee \mathcal{H} \vee \mathcal{H}'_1 \perp\!\!\!\perp_{\mathcal{H} \vee \mathcal{H}'_1} \mathcal{G}_2 \vee \mathcal{H} \vee \mathcal{H}'_1$. Since

$$\mathcal{H} \vee \mathcal{H}'_1 \subseteq \mathcal{H} \vee \mathcal{H}' = \mathcal{H}' = \mathcal{H}'_2 \vee \mathcal{H}'_1 \subseteq \mathcal{G}_2 \vee \mathcal{H} \vee \mathcal{H}'_1,$$

part (b) yields $\mathcal{G}_1 \vee \mathcal{H} \vee \mathcal{H}'_1 \perp\!\!\!\perp_{\mathcal{H}'} \mathcal{G}_2 \vee \mathcal{H} \vee \mathcal{H}'_1$, which proves (c) since $\mathcal{G}_1 \subseteq \mathcal{G}_1 \vee \mathcal{H} \vee \mathcal{H}'_1$ and $\mathcal{G}_2 \subseteq \mathcal{G}_2 \vee \mathcal{H} \vee \mathcal{H}'_1$. □

3.A.2. RESULTS RELATED TO ASSUMPTION 3.4.2(I)

Lemma 3.A.3. *Let Assumption 3.4.1 be satisfied, i.e., suppose that the linear operator* $-A\colon \mathsf{D}(A) \subseteq U \to U$ *on the separable real Hilbert space* U *generates an exponentially stable* C_0*-semigroup* $(S(t))_{t\ge 0}$ *of bounded linear operators on* U*. If* $Q \in \mathscr{L}^+(U)$ *and* $\gamma_0 \in (-\infty, 1/2]$*, then* $\int_0^\infty \|t^{\gamma_0 - 1} S(t) Q^{\frac12}\|^2_{\mathscr{L}_2(U)}\,\mathrm{d}t = \infty$*, that is, Assumption 3.4.2(i) cannot hold for* $\gamma_0 \in (-\infty, 1/2]$.

Proof. Fix any $x \in U$ with $\|x\|_U = 1$. Then we have $\|S(t)Q^{\frac12}\|_{\mathscr{L}_2(U)} \ge \|S(t)Q^{\frac12}x\|_U$ for all $t \in [0,\infty)$. Since $t \mapsto S(t)Q^{\frac12}x$ is continuous at zero and $S(0)Q^{\frac12}x = Q^{\frac12}x$, we can choose $\delta \in (0,\infty)$ so small that $\|S(t)Q^{\frac12}x\|_U \ge \frac12 \|Q^{\frac12}x\|_U$ for all $t \in [0,\delta]$. For $\gamma_0 \in (-\infty, 1/2]$, we then obtain

$$\int_0^\infty \|t^{\gamma_0-1} S(t) Q^{\frac12}\|^2_{\mathscr{L}_2(U)}\,\mathrm{d}t \ge \frac12 \|Q^{\frac12}x\|_U^2 \int_0^\delta t^{2(\gamma_0-1)}\,\mathrm{d}t = \infty. \qquad \square$$

Lemma 3.A.4. *Let Assumption 3.4.1 be satisfied. If Assumption 3.4.2(i) holds for some* $\gamma_0 \in (1/2, \infty)$*, then it also holds for all* $\gamma' \in [\gamma_0, \infty)$.

Proof. The change of variables $\tau := t/2$, the semigroup property and (3.4.2) yield

$$\int_0^\infty \|t^{\gamma'-1} S(t) Q^{\frac12}\|^2_{\mathscr{L}_2(U)}\,\mathrm{d}t \le 2^{2\gamma'-1} M_0^2 \int_0^\infty e^{-2w\tau} \|\tau^{\gamma'-1} S(\tau) Q^{\frac12}\|^2_{\mathscr{L}_2(U)}\,\mathrm{d}\tau.$$

For the latter integral, we split up the domain of integration and estimate each of the resulting integrands to find

$$\begin{aligned}\int_0^\infty e^{-2w\tau} \|\tau^{\gamma'-1} S(\tau) Q^{\frac12}\|^2_{\mathscr{L}_2(U)}\,\mathrm{d}\tau &= \sum_{k=1}^\infty \int_{k-1}^k e^{-2w\tau} \|\tau^{\gamma'-1} S(\tau) Q^{\frac12}\|^2_{\mathscr{L}_2(U)}\,\mathrm{d}\tau \\ &\le \sum_{k=1}^\infty e^{-2w(k-1)} k^{2(\gamma'-\gamma_0)} \int_0^\infty \|\tau^{\gamma_0-1} S(\tau) Q^{\frac12}\|^2_{\mathscr{L}_2(U)}\,\mathrm{d}\tau < \infty,\end{aligned}$$

where the series converges since $|e^{-2wk} k^{2(\gamma'-\gamma_0)}|^{\frac1k} \to e^{-2w} < 1$ as $k \to \infty$. □

3.A.3. FILTRATIONS INDEXED BY THE REAL LINE

Proposition 3.A.5. *A process $(W^Q(t))_{t\in\mathbb{R}}$ satisfying (WP1) cannot be a martingale with respect to any filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$.*

Proof. Suppose that $(W^Q(t))_{t\in\mathbb{R}}$ is a martingale with respect to a filtration $(\mathcal{F})_{t\in\mathbb{R}}$. Then the same holds for the real-valued process $W_h^Q(t) := \langle W^Q(t), h\rangle_U$, choosing $h \in U$ such that $\langle Qh, h\rangle_U^2 > 0$ to ensure that $(W_h^Q(t))_{t\in\mathbb{R}}$ has nontrivial increments. In particular, $(W_h^Q(-n))_{n\in\mathbb{N}}$ is a backward martingale with respect to $(\mathcal{F}_{-n})_{n\in\mathbb{N}}$, implying that it converges $\mathbb{P}$-a.s. and in $L^1(\Omega)$ as $n\to\infty$ by the backward martingale convergence theorem [97, Section 12.7, Theorem 4]. But this contradicts (WP1), since $(W_h^Q(-n))_{n\in\mathbb{N}}$ cannot be a Cauchy sequence in $L^1(\Omega)$ as it has (non-trivial) stationary increments. □

3.A.4. MEAN-SQUARE DIFFERENTIABILITY OF STOCHASTIC CONVOLUTIONS

The following lemma concerning mean-square continuity and differentiation under the integral sign is a straightforward generalization of Propositions 2.3.18 and 2.3.21 to stochastic convolutions with respect to a two-sided Wiener process. Its proof is therefore omitted.

Lemma 3.A.6. *Let $t_0 \in [-\infty,\infty)$ be such that $\Psi(\cdot)Q^{\frac{1}{2}} \in L^2(0,\infty;\mathscr{L}_2(U))$. Define the interval $\mathbb{T} := [t_0,\infty)$ if $t_0 \in \mathbb{R}$ or $\mathbb{T} := \mathbb{R}$ if $t_0 = -\infty$. Then the stochastic convolution $(\int_{t_0}^t \Psi(t-s)\,\mathrm{d}W^Q(s))_{t\in\mathbb{T}}$ is mean-square continuous.*

If $\Psi(\cdot)Q^{\frac{1}{2}} \in H_0^1(0,\infty;\mathscr{L}_2(U))$, then $(\int_{t_0}^t \Psi(t-s)\,\mathrm{d}W^Q(s))_{t\in\mathbb{T}}$ is mean-square differentiable on $\mathbb{T}$ and for all $t\in\mathbb{T}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^t \Psi(t-s)\,\mathrm{d}W^Q(s) = \int_{t_0}^t \partial_t\Psi(t-s)\,\mathrm{d}W^Q(s), \quad \mathbb{P}\text{-a.s.}$$

3.B. FRACTIONAL POWERS OF THE PARABOLIC OPERATOR

Let $A\colon \mathsf{D}(A)\subseteq U\to U$ be a linear operator on a real Hilbert space U. The proof of the following lemma is analogous to that of Proposition 2.A.3 and is therefore omitted.

Lemma 3.B.1. *Let $(S,\mathscr{A},\mu)$ be a measure space such that $L^2(S;\mathbb{R})$ is nontrivial and consider the linear operator $\mathcal{A}_S$ on $L^2(S;U)$ defined by (3.4.7). If $-A$ is the generator of a C_0-semigroup $(T(t))_{t\ge 0}$ on U, then $-\mathcal{A}_S$ generates the C_0-semigroup $(\mathcal{T}_S(t))_{t\ge0}$ on $L^2(S;U)$.*

The next results are respectively analogous to Propositions 2.A.5 and 2.3.2.

Proposition 3.B.2. *For every $t\in\mathbb{R}$, define the shift operator $\mathcal{T}(t)\in\mathscr{L}(L^2(\mathbb{R};U))$ by $\mathcal{T}(t)f := f(\cdot-t)$ for $f\in L^2(\mathbb{R};U)$. The family $(\mathcal{T}(t))_{t\in\mathbb{R}}$ is a C_0-group whose infinitesimal generator is given by $-\partial_t$, where ∂_t is the Bochner–Sobolev weak derivative on $\mathsf{D}(\partial_t) = H^1(\mathbb{R};U)$.*

Proposition 3.B.3. *Suppose that Assumption 3.4.1 holds. The closure $\mathcal{B}$ of the sum operator $\partial_t + \mathcal{A}_{\mathbb{R}}$ exists and $-\mathcal{B}$ is the generator of the product C_0-semigroup given by $(\mathcal{S}_{\mathbb{R}}(t)\mathcal{T}(t))_{t\geq 0}$ on $L^2(\mathbb{R};U)$, where $(\mathcal{S}_{\mathbb{R}}(t))_{t\geq 0}$ and $(\mathcal{T}(t))_{t\geq 0}$ are as in (3.4.7) and Proposition 3.B.2, respectively. The product semigroup satisfies*

$$\|\mathcal{S}_{\mathbb{R}}(t)\mathcal{T}(t)\|_{L^2(\mathbb{R};U)} = \|\mathcal{T}(t)\mathcal{S}_{\mathbb{R}}(t)\|_{L^2(\mathbb{R};U)} = \|S(t)\|_{\mathscr{L}(U)} \quad \textit{for all } t \in \mathbb{R}.$$

It follows that $(\mathcal{S}_{\mathbb{R}}(t)\mathcal{T}(t))_{t\geq 0}$ inherits the exponential stability of $(S(t))_{t\geq 0}$, so that fractional powers of $\mathcal{B}$ can be defined using the Phillips representation, cf. equations (1.3.3) and (1.3.4). Therefore, under Assumption 3.4.1,

$$\mathcal{B}^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^\infty r^{\gamma-1} \mathcal{S}_{\mathbb{R}}(r)\mathcal{T}(r) f(t)\, \mathrm{d}r = \frac{1}{\Gamma(\gamma)} \int_0^\infty r^{\gamma-1} S(r) f(t-r)\, \mathrm{d}r$$

for $\gamma \in (0,\infty)$, $f \in L^2(\mathbb{R};U)$ and almost all $t \in \mathbb{R}$. We conclude that $\mathcal{B}^{-\gamma} = \mathfrak{I}^\gamma$ for all $\gamma \in [0,\infty)$, where the latter is defined by (3.4.5).

4

Abstract nonlocal spatiotemporal Dirichlet problems

The contents of this chapter are based on the article [197].

4.1. Introduction to Chapter 4

4.1.1. Background and motivation

Space–time nonlocal problems involving fractional powers of a parabolic operator arise in physics, biology, probability theory and statistics. The flat parabolic Signorini problem and certain models for semipermeable membranes can be formulated as obstacle problems for the fractional heat operator $(\partial_t - \Delta)^s$, where $s \in (0,1)$ and Δ denotes the Laplacian, acting on functions $u\colon J \times \mathcal{D} \to \mathbb{R}$ for a given time interval $J \subseteq \mathbb{R}$ and a connected non-empty open spatial domain $\mathcal{D} \subseteq \mathbb{R}^d$, see e.g. [13, 189]. In the context of continuous time random walks, equations of the form $(\partial_t - \Delta)^s u = f$ for $f\colon J \times \mathcal{D} \to \mathbb{R}$ are considered examples of *master equations* governing the (non-separable) joint probability distribution of jump lengths and waiting times [39]. The case where f is replaced by spatiotemporal Gaussian noise $\dot{\mathcal{W}}$ has applications to the statistical modeling of spatial and temporal dependence in data: The resulting class of fractional parabolic *stochastic* partial differential equations (SPDEs) has been proposed and analyzed in Chapter 2 and [140] as a spatiotemporal generalization of the *SPDE approach* to spatial statistical modeling, which was initiated by Lindgren, Rue and Lindström [142] and has subsequently gained widespread popularity [141].

After [189] and [160] independently generalized the Caffarelli–Silvestre extension approach from the fractional elliptic to the parabolic setting, there has been a surge of literature on space–time nonlocal problems involving fractional powers of the operator $\partial_t + L$ for more general elliptic operators L acting on functions $u\colon \mathcal{D} \to \mathbb{R}$, see for instance [15, 16, 25, 26, 77, 133, 143]. In particular, in [189, Remark 1.2], the *nat-*

ural Dirichlet problem for the nonlocal space–time operator $(\partial_t + L)^s$, given by

$$\begin{cases} (\partial_t + L)^s u(t,x) = f(t,x), & (t,x) \in J \times \mathcal{D}, \\ u(t,x) = g(t,x), & (t,x) \in \mathbb{R}^{d+1} \setminus (J \times \mathcal{D}), \end{cases} \tag{4.1.1}$$

is introduced, where $g \colon \mathbb{R}^{d+1} \setminus (J \times \mathcal{D}) \to \mathbb{R}$ is a given function prescribing the values of u outside of the spatiotemporal region $J \times \mathcal{D}$. The definition of $(\partial_t + L)^s$, given in Section 4.2.4 below, generalizes that of the Riemann–Liouville fractional time derivative ∂_t^s (i.e., the case where $L = 0$) using the theory of semigroups. Equations involving only a fractional time derivative have been studied widely; see for instance the monographs [81, 121, 169, 179] for an introduction to the subject.

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In the integer-order case, the space–time differential operator is local, so that the analog to (4.1.1) is an initial–boundary value problem. Identifying any function $u \colon J \times \mathcal{D} \to \mathbb{R}$ with $u \colon J \to X$, where $J := (t_0, \infty)$ and X is a Banach space to be thought of as containing functions from $\mathcal{D}$ to $\mathbb{R}$, the corresponding infinite-dimensional initial value problem for $s = 1$ is the *abstract Cauchy problem*

$$\begin{cases} (\partial_t + A) u(t) = f(t), & t \in J, \\ u(t_0) = x \in X. \end{cases} \tag{4.1.2}$$

Here, $A \colon \mathsf{D}(A) \subseteq X \to X$ is a linear operator, whose domain $\mathsf{D}(A)$ can be used to encode (Dirichlet) boundary conditions, and $f \colon J \to X$ is a given forcing function. Although there exist various definitions of solutions to (4.1.2) (e.g., mild, strong and L^p-solutions, see Section 4.2.3), the main focus of this chapter is on mild solutions. The mild solution to (4.1.2) can only be defined under the assumption that $-A$ is the infinitesimal generator of a suitably regular semigroup $(S(t))_{t \ge 0}$ of bounded linear operators on X, see Section 4.2.2. If, moreover, the right-hand side f is sufficiently (Bochner) integrable, then the *mild solution* of (4.1.2) is defined by

$$u(t) := S(t - t_0)x + \int_{t_0}^t S(t-\tau) f(\tau) \, d\tau, \quad t \in J, \tag{4.1.3}$$

which is commonly known as the variation-of-constants formula, by analogy with equation (1.1.4) in the finite-dimensional case.

In this chapter we consider a counterpart of (4.1.1) in the abstract setting of (4.1.2), namely the following Dirichlet problem for $(\partial_t + A)^s$ with $s \in (0,\infty) \setminus \mathbb{N}$:

$$\begin{cases} (\partial_t + A)^s u(t) = 0, & t \in (t_0, \infty), \\ u(t) = g(t), & t \in (-\infty, t_0], \end{cases} \tag{4.1.4}$$

where $g \colon (-\infty, t_0] \to X$. We restrict ourselves to $J = (t_0, \infty)$ since $(\partial_t + A)^s u(t)$ depends only on the values of u to the left of $t \in \mathbb{R}$, see Section 4.2.4 below. Moreover, we only consider $f \equiv 0$ since the problem is linear in u and the mild solution formula for $f \not\equiv 0$ and $g \equiv 0$ (or $J = \mathbb{R}$) is known to be given by a Riemann–Liouville type fractional parabolic integral, cf. [189, Theorem 1.17]. We will define the concept of an

L^p*-solution* to (4.1.4), and show that it can be expressed in terms of g and $(S(t))_{t\geq 0}$ in the following way:

$$\begin{aligned} u(t) := {} & \frac{\sin(\pi\{s\})}{\pi} \int_0^\infty \frac{\tau^{-\{s\}}}{\tau+1} S((t-t_0)(\tau+1)) g(t_0-(t-t_0)\tau)\, \mathrm{d}\tau \\ & + \sum_{k=1}^{\lfloor s \rfloor} \frac{(t-t_0)^{\{s\}+k-1}}{\Gamma(\{s\}+k)} S(t-t_0)[(\partial_t + A)^{\{s\}+k-1} g](t_0), \quad t \in (t_0,\infty), \end{aligned} \tag{4.1.5}$$

where Γ denotes the gamma function and $s = \lfloor s \rfloor + \{s\}$ for $\lfloor s \rfloor \in \mathbb{N}_0$ and $\{s\} \in (0,1)$. This formula generalizes (4.1.3) to fractional orders, and is therefore taken as the definition of the *mild solution* to (4.1.4).

4.1.2. CONTRIBUTIONS

The main contribution of this chapter is the introduction and motivation of (4.1.5) as the definition of the mild solution to (4.1.4) for $s \in (0,\infty) \setminus \mathbb{N}$ and bounded continuous g, rigorously formulated in Definition 4.4.2. This definition is motivated by Theorem 4.4.5, which shows that L^p*-solutions* to (4.1.4) satisfy (4.1.5) under certain natural conditions. Although its proof relies on the uniform exponential stability of $(S(t))_{t\geq 0}$, the resulting formula is well-defined under the more general assumption that $(S(t))_{t\geq 0}$ is uniformly bounded. In particular, this includes the case $A = 0$, meaning that (4.1.5) with $S(\cdot) \equiv \mathrm{Id}_X$ can be viewed as a solution to the Dirichlet problem associated to the fractional time derivative ∂_t^s. Likewise, if $(S(t))_{t\geq 0}$ is uniformly exponentially stable, then the integral in (4.1.5) also converges for $\{s\} = 0$, so that (4.1.5) remains meaningful for integers $s = n \in \mathbb{N}$ and reduces to the integer-order solution formula:

$$u(t) = \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} S(t-t_0)[(\partial_t + A)^k g](t_0), \quad t \in (t_0,\infty).$$

If $(S(t))_{t\geq 0}$ is merely uniformly bounded, then we can still show that the first term of (4.1.5) converges to $S(t-t_0)g(t_0)$ as $\{s\} \uparrow 1$ for all $t \in (t_0,\infty)$, see Proposition 4.4.7. For constant initial data $g \equiv x \in X$, we find that (4.1.5) can be conveniently expressed in terms of an operator-valued incomplete gamma function, see Corollary 4.4.8.

In addition to (4.1.5), we define solution concepts for the Cauchy problems associated to fractional parabolic Riemann–Liouville and Caputo type derivative operators (see Proposition 4.5.1 and Definitions 4.5.2 and 4.5.3) for comparison. The higher-order terms comprising the summation in (4.1.5) turn out to be analogous to the corresponding terms in the Riemann–Liouville solution. The integral term in (4.1.5), however, is continuous at t_0 under mild conditions on $(S(t))_{t\geq 0}$ or g, in contrast to the lowest-order term in the Riemann–Liouville formula, which has a singularity there. As opposed to the Caputo type initial value problem, the solutions to (4.1.4) are in general different for distinct $s_1, s_2 \in (n, n+1)$ for $n \in \mathbb{N}_0$.

To the best of the author's knowledge, the solution formula given by (4.1.5) is new even in the scalar-valued case $X := \mathbb{C}$, $A := a \in \overline{\mathbb{C}}_+$ and $(S(t))_{t\geq 0} = (e^{-at})_{t\geq 0}$, as are the Riemann–Liouville and Caputo type solutions for $a \in \overline{\mathbb{C}}_+ \setminus \{0\}$.

4.1.3. OUTLINE

This chapter is structured as follows. In Section 4.2, we establish some notation and collect preliminary results regarding semigroups, fractional calculus, first-order abstract Cauchy problems and the Phillips functional calculus associated to semigroup generators. These notions are first used in Section 4.3 to investigate problem (4.1.4) for $t_0 = -\infty$, i.e., in the absence of prescribed initial data. Section 4.4 is concerned with the rigorous definition of mild and L^p-solutions to (4.1.4); after establishing the relation between these two concepts, we focus on the mild solution and establish some of its most important properties. The comparison with the solution concepts associated to Riemann–Liouville and Caputo type initial value problems is presented in Section 4.5.

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4.2. PRELIMINARIES FOR CHAPTER 4

4.2.1. NOTATION

In this section we mainly highlight notation which deviates from the previous chapters or was not used there. We write $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ for the floor and ceiling functions; the fractional part of $\alpha \in [0,\infty)$ is defined by $\{\alpha\} := \alpha - \lfloor\alpha\rfloor$. The open and closed right half-planes of the complex plane are denoted by

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \quad \text{and} \quad \overline{\mathbb{C}}_+ := \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\},$$

respectively. The identity map on a set B is denoted by $\mathrm{Id}_B \colon B \to B$ and we write $\mathbf{1}_{B_0} \colon B \to \{0,1\}$ for the indicator function of a subset $B_0 \subseteq B$. The Banach space of bounded continuous functions $u \colon J \to X$, endowed with the supremum norm, is denoted by $(\mathrm{C_b}(J;X), \|\cdot\|_\infty)$.

4.2.2. STRONGLY MEASURABLE SEMIGROUPS

In what follows, we exclusively consider operators A for which $-A$ generates a *locally bounded* strongly measurable semigroups $(S(t))_{t\geq 0}$, which satisfy

$$\exists M_0 \in [1,\infty),\ w \in \mathbb{R} : \quad \|S(t)\|_{\mathscr{L}(X)} \leq M_0 e^{-wt}, \quad \forall t \in [0,\infty). \tag{4.2.1}$$

More precisely, we impose the following assumptions throughout this chapter.

Assumption 4.2.1. Let $-A \colon \mathsf{D}(A) \subseteq X \to X$ be the infinitesimal generator of a locally bounded strongly measurable semigroup $(S(t))_{t\geq 0} \subseteq \mathscr{L}(X)$, which satisfies (4.2.1). More precisely, we suppose that $(S(t))_{t\geq 0}$ is either

(i) *uniformly bounded*, meaning that $w \in [0,\infty)$, or

(ii) *uniformly exponentially stable*, meaning that $w \in (0,\infty)$.

We may sometimes additionally assume that $(S(t))_{t\geq 0}$ is

(iii) *bounded analytic*, i.e., $(0,\infty) \ni t \mapsto S(t) \in \mathscr{L}(X)$ admits a bounded holomorphic extension to $\Sigma_\varphi := \{z \in \mathbb{C} : |\arg z| < \varphi\}$ for some $\varphi \in (0, \frac{1}{2}\pi)$.

4.2.3. FIRST-ORDER ABSTRACT CAUCHY PROBLEMS

Recall the Definitions 1.1.12, 1.1.14 and 1.1.13 of strong, mild and L^p-solutions, respectively, to the first-order abstract Cauchy problem (4.1.2). These are readily generalized to intervals $J = (t_0, \infty)$ for nonzero $t_0 \in \mathbb{R}$. The following is a slight extension of [115, Proposition 17.1.3] for the class of time intervals considered in this chapter.

Proposition 4.2.2. *Suppose that Assumption 4.2.1(ii) holds. Let $J := (t_0, \infty)$ for a given $t_0 \in [-\infty, \infty)$, $f \in L^p(J; X)$ for some $p \in [1, \infty]$ and $x \in \overline{\mathsf{D}(A)}$ if $t_0 \in \mathbb{R}$. Then, for every $u \in \mathrm{C}_{\mathrm{ub}}(\overline{J}; X)$, the following assertions are equivalent:*

(a) u is a strong solution of (4.1.2) in the sense of Definition 1.1.12;

(b) u is the mild solution of (4.1.2) in the sense of Definition 1.1.14 and u is (classically) differentiable almost everywhere with $u' \in L^1_{\mathrm{loc}}(\overline{J}; X)$;

(c) u is the mild solution of (4.1.2) in the sense of Definition 1.1.14, $u(t) \in \mathsf{D}(A)$ for almost all $t \in J$ and $t \mapsto Au(t) \in L^1_{\mathrm{loc}}(\overline{J}; X)$.

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Proof. If $t_0 \in \mathbb{R}$, then it is easy to see which modifications of the proof of [115, Proposition 17.1.3] are necessary. Thus, we only elaborate on the case $t_0 = -\infty$. Let $\lambda \in \mathbb{C}$ be such that $\lambda \mathrm{Id}_X + A$ admits a bounded inverse; since the semigroup $(S(t))_{t \geq 0}$ is uniformly bounded we can take $\lambda = 1$; see (1.1.8).

(a) $\Longrightarrow$ (b): For u as in Definition 1.1.12 and $t \in \mathbb{R}$, we define $v\colon (-\infty, t] \to X$ by

$$v(\tau) := (\lambda \mathrm{Id}_X + A)^{-1} S(t-\tau) u(\tau), \quad \tau \in (-\infty, t].$$

Fixing $t' < t$, and arguing as in the original proof—except for integrating over (t', t) instead of $(0, t)$—we find

$$u(t) = S(t-t')u(t') + \int_{t'}^{t} S(t-\tau) f(\tau)\, \mathrm{d}\tau.$$

As $t' \to -\infty$, the first term vanishes by $u \in \mathrm{C}_{\mathrm{b}}(\mathbb{R}; X)$ and Assumption 4.2.1(ii), and the second term converges to $\int_{-\infty}^{t} S(t-\tau) f(\tau)\, \mathrm{d}\tau$ by dominated convergence.

(b) $\Longrightarrow$ (a) and (c) $\Longrightarrow$ (a): Use the following analog to [115, Equation (17.4)]:

$$\int_0^t A(\lambda \mathrm{Id}_X + A)^{-1} u(\tau)\, \mathrm{d}\tau = -(\lambda \mathrm{Id}_X + A)^{-1} \Big[u(t) - u(0) - \int_0^t f(\tau)\, \mathrm{d}\tau \Big]. \qquad \square$$

4.2.4. FRACTIONAL PARABOLIC CALCULUS

Under the assumption that $-A$ generates a semigroup $(S(t))_{t\geq 0}$ satisfying Assumption 4.2.1(ii), we define the fractional *parabolic* integration and differentiation operators $\mathfrak{J}^s$ and $\mathfrak{D}^s$, which are the rigorous definitions of the expressions $(\partial_t + A)^{-s}$ and $(\partial_t + A)^s$ from Section 4.1, respectively.

The operators $(\mathfrak{J}^s)_{s \in [0,\infty)}$ were defined for Hilbert spaces in Section 3.4.2. The defining formula is the same for Banach spaces, and the resulting operators have the following properties, which will be used throughout this chapter:

Proposition 4.2.3. *Suppose that Assumption 4.2.1(ii) is satisfied. Let $p \in [1, \infty]$ and $s \in [0, \infty)$. The following assertions hold:*

(a) $\mathfrak{J}^s \in \mathscr{L}(L^p(\mathbb{R};X))$ *with* $\|\mathfrak{J}^s\|_{\mathscr{L}(L^p(\mathbb{R};X))} \le \frac{M_0}{w^s}$.

(b) $\mathfrak{J}^s \in \mathscr{L}(L^p(\mathbb{R};X); C_b(\mathbb{R};X))$ *if*

$$\begin{cases} p=1 & \textit{and} \quad s\in[1,\infty), \quad \textit{or}, \\ p\in(1,\infty) & \textit{and} \quad s\in(1/p,\infty). \end{cases} \tag{4.2.2}$$

(c) $\mathfrak{J}^{s_1}\mathfrak{J}^{s_2}u = \mathfrak{J}^{s_1+s_2}u$ *a.e. for all* $s_1, s_2 \in [0,\infty)$ *and* $u \in L^p(\mathbb{R};X)$.

(d) *Given* $x \in X$, *if* $p \in [1, \frac{1}{1-s_1})$ *and* $s_1 \in (0,1)$ *or* $p \in [1,\infty]$ *and* $s_1 \in [1,\infty)$, *then* $k_{s_1} \otimes x \in L^p(\mathbb{R};X)$ *and* $\mathfrak{J}^{s_2}(k_{s_1}\otimes x) = k_{s_1+s_2}\otimes x$ *for all* $s_2 \in [0,\infty)$.

Proof. Estimate (4.2.1) implies $\|k_s\|_{L^1(\mathbb{R};\mathscr{L}(X))} \le M_0 w^{-s}$, so that Minkowski's integral inequality [186, Section A.1] yields (a).

If $p' \in [1,\infty]$ is such that $\frac{1}{p}+\frac{1}{p'}=1$ and $u\in L^p(\mathbb{R};X)$, then $k_s \in L^{p'}(\mathbb{R};\mathscr{L}(X))$ for s as in the statement of (b), and the result follows from Hölder's inequality and the continuity of translations in $L^q(\mathbb{R};X)$ or $L^q(\mathbb{R};\mathscr{L}(X))$ for $q\in[1,\infty)$.

Assertions (c) and (d) follow by combining the semigroup property of $(S(t))_{t\ge0}$, Fubini's theorem and [163, Equation (5.12.1)]. □

For $p \in [1,\infty]$, let $W^{1,p}(\mathbb{R};X)$ denote the Bochner–Sobolev space consisting of functions $u \in L^p(\mathbb{R};X)$ whose weak derivative $\partial_t u$ also belongs to $L^p(\mathbb{R};X)$. Identifying $A\colon \mathsf{D}(A)\subseteq X \to X$ with the operator

$$\mathcal{A}\colon L^p(\mathbb{R};\mathsf{D}(A)) \subseteq L^p(\mathbb{R};X) \to L^p(\mathbb{R};X)$$

defined by $[\mathcal{A}u](\cdot) := Au(\cdot)$, we can view $\partial_t + A$ as an operator on $L^p(\mathbb{R};X)$ with domain $L^p(\mathbb{R};\mathsf{D}(A))\cap W^{1,p}(\mathbb{R};X)$ as in Section 1.1.4. In conjunction with the operators $(\mathfrak{J}^s)_{s\ge0}$, this leads to the definition of the *Riemann–Liouville type fractional parabolic derivative of order* $s\in[0,\infty)$:

$$\begin{aligned} \mathfrak{D}^s u &:= (\partial_t + A)^{\lceil s\rceil}\mathfrak{J}^{\lceil s\rceil - s}u, \\ u \in \mathsf{D}(\mathfrak{D}^s) &:= \{u\in L^p(\mathbb{R};X) : \mathfrak{J}^{\lceil s\rceil - s}u \in \mathsf{D}((\partial_t + A)^{\lceil s\rceil})\}. \end{aligned} \tag{4.2.3}$$

Note that we do not explicitly indicate the dependence of $\mathfrak{D}^s$ and $\mathfrak{J}^s$ on $p\in[1,\infty]$ in the notation, instead leaving it to be inferred from context.

Remark 4.2.4. While the terminology "fractional parabolic" is inspired by the case $A = -\Delta$ acting on a function space such as $X = L^2(\mathcal{D})$, our setting is considerably more general.

4.3. FRACTIONAL-ORDER INHOMOGENEOUS ABSTRACT CAUCHY PROBLEM ON $\mathbb{R}$

In this section, we consider the inhomogeneous abstract Cauchy problem associated to the fractional operator $\mathfrak{D}^s$ on $J := \mathbb{R}$:

$$\mathfrak{D}^s u(t) = f(t), \quad t\in\mathbb{R}, \tag{4.3.1}$$

where $f \in L^p(\mathbb{R}; X)$ for $s \in (0, \infty)$ and $p \in [1, \infty]$. Recall that the Hilbert space case ($X = U$ and $p = 2$) was already considered in the previous chapter, see (3.4.4). The solution concepts for (4.3.1) which we will define are the following fractional-order analogs to the notion of L^p-solutions and mild solutions (given by Definitions 1.1.13 and 1.1.14, respectively).

Definition 4.3.1 (L^p-solution)**.** Suppose that Assumption 4.2.1(ii) is satisfied. Let $s \in (0, \infty)$, $p \in [1, \infty]$ and $f \in L^p(\mathbb{R}; X)$. Then $u \in L^p(\mathbb{R}; X)$ is called an *L^p-solution* to (4.3.1) if $u \in \mathsf{D}(\mathfrak{D}^s)$ and equation (4.3.1) holds almost everywhere on $\mathbb{R}$.

It is a consequence of Proposition 4.3.3(b) below that the L^p-solution to (4.3.1) is unique if it exists. The question of existence of the L^p-solution for all $f \in L^p(\mathbb{R}; X)$ is highly nontrivial: Recall from Section 1.1.4 that in the case $s = 1$ it characterizes the maximal L^p-regularity of $\mathbb{R}$. Since the present chapter is primarily concerned with the concept of mild solutions (which are defined in the same way as in Section 3.4.2) we do not investigate this matter further.

Definition 4.3.2 (Mild solution)**.** Suppose that Assumption 4.2.1(ii) is satisfied. Let $s \in (0, \infty)$ and $p \in [1, \infty]$ satisfy (4.2.2). The *mild solution* to (4.3.1) with $f \in L^p(\mathbb{R}; X)$ is the function $u \in C_b(\mathbb{R}; X)$ defined for all $t \in \mathbb{R}$ by

$$u(t) := \mathfrak{I}^s f(t) = \frac{1}{\Gamma(s)} \int_{-\infty}^{t} (t-\tau)^{s-1} S(t-\tau) f(\tau) \, d\tau.$$

The mild solution exists and is unique by definition, since it is given by an explicit formula. Moreover, in view of Proposition 4.2.3(b), it is indeed continuous under the given assumptions on s and p.

The next proposition shows that the fractional parabolic derivative and integral are inverse to each other whenever the respective left-hand sides are well-defined. In particular, it implies that L^p-solutions are mild solutions whenever the parameters s and p are such that (4.2.2) holds, see Corollary 4.3.5 below.

Proposition 4.3.3. *Suppose that Assumption 4.2.1(ii) holds. Let $s \in [0, \infty)$, $p \in [1, \infty]$ and $u \in L^p(\mathbb{R}; X)$. Then the following assertions hold:*

(a) If $\mathfrak{I}^s u \in \mathsf{D}(\mathfrak{D}^s)$, then $\mathfrak{D}^s \mathfrak{I}^s u = u$ a.e.

(b) If $u \in \mathsf{D}(\mathfrak{D}^s)$, then $\mathfrak{I}^s \mathfrak{D}^s u = u$ a.e.

Proof. (a) For $s = 1$, $v := \mathfrak{I}^1 u$ is the mild solution to (4.1.2) with $f := u \in L^p(J; X)$. Moreover, since $v \in W^{1,p}(J; X) \cap L^p(J; X)$, the conditions of Proposition 4.2.2(b)–(c) are satisfied, so that v is a strong solution, which proves the base case.

Now let $k \in \mathbb{N}$ and suppose that (a) holds for $s = k$. If $\mathfrak{I}^{k+1} u \in \mathsf{D}(\mathfrak{D}^{k+1})$, then by definition we have $\mathfrak{I}^{k+1} u \in \mathsf{D}(\mathfrak{D}^k)$ and $\mathfrak{D}^k \mathfrak{I}^{k+1} u \in \mathsf{D}(\mathfrak{D}^1)$. In view of Proposition 4.2.3(c), this means that $\mathfrak{I}^k \mathfrak{I}^1 u \in \mathsf{D}(\mathfrak{D}^k)$ and $\mathfrak{D}^k \mathfrak{I}^k \mathfrak{I}^1 u \in \mathsf{D}(\mathfrak{D}^1)$. Combining the former expression with the induction hypothesis yields

$$\mathfrak{D}^k \mathfrak{I}^{k+1} u = \mathfrak{D}^k \mathfrak{I}^k \mathfrak{I}^1 u = \mathfrak{I}^1 u, \quad \text{a.e.,} \tag{4.3.2}$$

and thus $\mathfrak{I}^1 u \in \mathsf{D}(\mathfrak{D}^1)$ by the latter. It follows that $\mathfrak{D}^1 \mathfrak{I}^1 u = u$ a.e. by the base case of (a). Using (4.3.2) once more, this becomes $\mathfrak{D}^1 \mathfrak{D}^k \mathfrak{I}^{k+1} u = u$ a.e., whose left-hand side equals $\mathfrak{D}^{k+1}\mathfrak{J}^{k+1}u$ a.e. by the definition of integers powers of $\mathfrak{D}$. This proves (a) for $s = k+1$, and thus for all $s \in \mathbb{N}$ by induction.

For $s \in (0,\infty) \setminus \mathbb{N}$, the assertion follows upon combining the definition (4.2.3) of $\mathfrak{D}^s$ with Proposition 4.2.3(c) and the integer case:

$$\mathfrak{D}^s\mathfrak{J}^s u = (\partial_t + A)^{\lceil s\rceil}\mathfrak{J}^{\lceil s\rceil - s}\mathfrak{J}^s u = (\partial_t + A)^{\lceil s\rceil}\mathfrak{J}^{(\lceil s\rceil - s)+s} u = u \quad \text{a.e.}$$

(b) The case $s = 1$ follows from Proposition 4.2.2 (a) $\implies$ (b) with $f := u' + Au$, and the integer case follows by induction. For fractional s, fix $u \in \mathsf{D}(\mathfrak{D}^s)$ and note

$$\mathfrak{J}^{\lceil s\rceil - s}\mathfrak{J}^s\mathfrak{D}^s u = \mathfrak{J}^{\lceil s\rceil - s}\mathfrak{J}^s(\partial_t + A)^{\lceil s\rceil}\mathfrak{J}^{\lceil s\rceil - s}u = \mathfrak{J}^{\lceil s\rceil}(\partial_t + A)^{\lceil s\rceil}\mathfrak{J}^{\lceil s\rceil - s}u = \mathfrak{J}^{\lceil s\rceil - s}u,$$

holds a.e. Since (a) implies that $\mathfrak{J}^{\lceil s\rceil - s}$ is injective, we conclude $\mathfrak{J}^s\mathfrak{D}^s u = u$ a.e. □

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Combining Propositions 4.2.3(b) and 4.3.3(b) yields the following corollaries:

Corollary 4.3.4. *Suppose that Assumption 4.2.1(ii) holds. If $s \in [0,\infty)$ and $p \in [1,\infty]$ satisfy (4.2.2), then we have $\mathsf{D}(\mathfrak{D}^s) \subseteq C_b(\mathbb{R}; X)$.*

Corollary 4.3.5. *Suppose that Assumption 4.2.1(ii) is satisfied and let u be an L^p-solution to (4.3.1) in the sense of Definition 4.3.1 for some $s \in (0,\infty)$, $p \in [1,\infty]$. If s and p satisfy (4.2.2), then u is the mild solution in the sense of Definition 4.3.2.*

Proof. If u is an L^p-solution, then $u \in \mathsf{D}(\mathfrak{D}^s)$ and $\mathfrak{D}^s u = f$ holds almost everywhere. Thus, by Proposition 4.3.3(b), we can apply $\mathfrak{J}^s$ on both sides to obtain $u = \mathfrak{J}^s f$ a.e., and we have $u \in C_b(\mathbb{R}; X)$ by Corollary 4.3.4 (or by Proposition 4.2.3(b) directly). □

4.4. DIRICHLET PROBLEM ASSOCIATED TO THE FRACTIONAL PARABOLIC DERIVATIVE OPERATOR

In this section we turn to the main subject of the present chapter, namely the natural abstract Dirichlet problem associated to $\mathfrak{D}^s$, which consists in finding a function $u\colon \mathbb{R} \to X$ satisfying

$$\begin{cases} \mathfrak{D}^s u(t) = 0, & t \in (t_0,\infty), \\ u(t) = g(t), & t \in (-\infty, t_0], \end{cases} \tag{4.4.1}$$

for $s \in (0,\infty)$, $t_0 \in \mathbb{R}$ and $g\colon (-\infty, t_0] \to X$. Recall from Section 4.2.4 that $\mathfrak{D}^s$ denotes the Riemann–Liouville type fractional parabolic differentiation operator acting on functions from $\mathbb{R}$ to X, which is our interpretation of the operator $(\partial_t + A)^s$ appearing in (4.1.4).

As in the previous sections, we begin by defining the notion of an L^p-solution to (4.4.1) (cf. Definitions 1.1.13 and 4.3.1), and subsequently define the mild solution (cf. Definitions 1.1.14 and 4.3.2), which is the rigorous formulation of the solution formula formally given by (4.1.5). As before, we note that the existence and uniqueness of the mild solution are immediate from the definition; for the L^p-solution, we have uniqueness but the matter of existence is outside of the scope of this chapter, analogously to the discussion below Definition 4.3.1.

Definition 4.4.1 (L^p-solution)**.** Let Assumption 4.2.1(ii) be satisfied and suppose that $s \in (0,\infty)$, $p \in [1,\infty]$, $t_0 \in \mathbb{R}$ and $g \in L^p(-\infty, t_0; X)$. Then $u \in L^p(\mathbb{R}; X)$ is called an *L^p-solution* to (4.4.1) if $u \in \mathsf{D}(\mathfrak{D}^s)$ and both equations in (4.4.1) hold almost everywhere on their respective sub-intervals of $\mathbb{R}$. In particular, $g \in \mathsf{D}(\mathfrak{D}^s)$ on $(-\infty, t_0]$.

Definition 4.4.2 (Mild solution)**.** Let Assumption 4.2.1(i) be satisfied, suppose that $s \in (0,\infty)$, $p \in [1,\infty]$, $t_0 \in \mathbb{R}$ are given and let $g \in C_b((-\infty, t_0]; X) \cap \mathsf{D}(\mathfrak{D}^{(s-1)\vee 0})$ be such that $\mathfrak{D}^{(s-1)\vee 0} g \in C_b((-\infty, t_0]; X)$. The *mild solution* to (4.4.1) with initial datum g is the function $u \in C_b(\mathbb{R} \setminus \{t_0\}; X)$ defined by

$$u(t) := g(t), \quad t \in (-\infty, t_0],$$

and, for $s \in (0,\infty) \setminus \mathbb{N}$,

$$\begin{aligned} u(t) := {} & \frac{\sin(\pi\{s\})}{\pi} \int_0^\infty \frac{\tau^{-\{s\}}}{\tau+1} S((t-t_0)(\tau+1)) g(t_0 - (t-t_0)\tau) \, d\tau \\ & + \sum_{k=1}^{\lfloor s \rfloor} \frac{(t-t_0)^{\{s\}+k-1}}{\Gamma(\{s\}+k)} S(t-t_0) \mathfrak{D}^{\{s\}+k-1} g(t_0), \quad t \in (t_0, \infty), \end{aligned} \tag{4.4.2}$$

whereas for $s = n \in \mathbb{N}$, we set

$$u(t) := \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} S(t-t_0) \mathfrak{D}^k g(t_0), \quad t \in (t_0, \infty). \tag{4.4.3}$$

The following proposition shows that the mild solution is indeed well-defined in the sense that it possesses the continuity properties asserted in Definition 4.4.2. Its proof is postponed to Section 4.4.1, in which we also state and prove some additional properties of the mild solution.

Proposition 4.4.3. *Suppose Assumption 4.2.1(i) holds. Let $s \in (0,\infty)$, $p \in [1,\infty]$, $t_0 \in \mathbb{R}$ be given and let $g \in L^p(-\infty, t_0; X)$ be as in Definition 4.4.2. Then the mild solution u to (4.4.1) satisfies $u \in C_b(\mathbb{R} \setminus \{t_0\}; X)$ and, for all $t \in (t_0, \infty)$,*

$$\|u(t)\|_X \le M_0 \overline{\Gamma}(s, w(t-t_0)) \max\{\|g\|_\infty, \|\mathfrak{D}^{\{s\}} g(t_0)\|_X, \ldots, \|\mathfrak{D}^{s-1} g(t_0)\|_X\},$$

where $M_0 \in [1,\infty)$ and $w \in [0,\infty)$ are as in (4.2.1).

If moreover $g(t), \mathfrak{D}^{\{s\}+k} g(t_0) \in \overline{\mathsf{D}(A)}$ for all $t \in (-\infty, t_0]$ and $k \in \{0, \ldots, \lfloor s \rfloor\}$, then we in fact have $u \in C_b(\mathbb{R}; X)$.

Remark 4.4.4. Let us emphasize that the solution formula can fail to be continuous at t_0 even in the first-order case $u(t) = S(t-t_0)x$ if $x \notin \overline{\mathsf{D}(A)}$. As an example, we can take $X = C_b(\mathbb{R})$, $A = -\Delta$ and $x(\xi) = \sin(\xi^2)$. Then $-A$ generates the analytic heat semigroup and $\overline{\mathsf{D}(A)} = C_{ub}(\mathbb{R})$ is the space of bounded and uniformly continuous functions on $\mathbb{R}$, cf. [146, Corollary 3.1.9]. In this case, $\|u(t)\|_\infty \le 1$ for all $t \in [t_0, \infty)$, but $S(t-t_0)x$ does not converge uniformly to x as $t \downarrow t_0$.

The motivation for the solution formulae in Definition 4.4.2 is provided by the following theorem, which shows that any L^p-solution to (4.4.1) is a mild solution whenever $s \in (0,\infty)$ and $p \in [1,\infty]$ are such that (4.4.2)–(4.4.3) are meaningful.

Theorem 4.4.5. *Suppose that Assumption 4.2.1(ii) is satisfied and let u be an L^p-solution to (4.4.1) in the sense of Definition 4.4.1 for some $p \in [1,\infty]$, $s \in (0,\infty)$ and $t_0 \in \mathbb{R}$. If s and p satisfy (4.2.2), then u is the mild solution to (4.4.1) in the sense of Definition 4.4.2.*

The proof of Theorem 4.4.5 is presented in Section 4.4.2, where the integer-order and fractional-order cases are treated separately. Before proceeding to the next subsection, we consider the following important example of a situation in which we can write down an explicit mild solution formula for (4.1.2):

Example 4.4.6 (The fractional heat operator $(\partial_t - \Delta)^s$ on $L^2(\mathbb{R}^d)$)**.** Let us consider the function space $X = L^2(\mathbb{R}^d)$ and differential operator $A = -\Delta$ for $d \in \mathbb{N}$, i.e., the negative Laplacian on the full Euclidean space $\mathbb{R}^d$. From Example 1.1.16, we know that $-A = \Delta$ generates the heat semigroup $(S(t))_{t\geq 0}$, which is given by the (spatial) convolution with the Gauss–Weierstrass kernel. Substituting it into equation (4.4.2) for some sufficiently regular initial datum function $g\colon (-\infty, t_0] \times \mathbb{R}^d \to \mathbb{R}$, we obtain an explicit formula for the mild solution $u\colon \mathbb{R}^{d+1} \to \mathbb{R}$ to (4.4.1). For example, if $t_0 = 0$ and $s \in (0,1)$, then we have for all $(t,x) \in (0,\infty) \times \mathbb{R}^d$:

$$\begin{aligned} u(t,x) &= \frac{\sin(\pi s)}{\pi} \int_0^\infty \int_{\mathbb{R}^d} \frac{\tau^{-s}}{\tau+1} K_{t(\tau+1)}(x-y) g(-t\tau, y)\,\mathrm{d}y\,\mathrm{d}\tau \\ &= \frac{\sin(\pi s)}{\pi(\sqrt{4\pi t})^d} \int_0^\infty \int_{\mathbb{R}^d} \tau^{-s}(\tau+1)^{-\frac{d}{2}-1} \exp\Big(-\frac{\|x-y\|^2_{\mathbb{R}^d}}{4t(\tau+1)}\Big) g(-t\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau. \end{aligned}$$

If $s \in (1,2)$ (still with $t_0 = 0$), then we instead find for all $t \in (0,\infty)$ and $x \in \mathbb{R}^d$:

$$\begin{aligned} u(t,x) = {}& \frac{\sin(\pi\{s\})}{\pi(\sqrt{4\pi t})^d} \int_0^\infty \int_{\mathbb{R}^d} \tau^{-\{s\}}(\tau+1)^{-\frac{d}{2}-1} \exp\Big(-\frac{\|x-y\|^2_{\mathbb{R}^d}}{4t(\tau+1)}\Big) g(-t\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau \\ &+ \frac{t^{\{s\}}}{\Gamma(\{s\}+1)} S(t)\mathfrak{D}^{\{s\}} g(0). \end{aligned} \tag{4.4.4}$$

The fractional parabolic derivative $\mathfrak{D}^\alpha f$ of any sufficiently regular $f\colon \mathbb{R} \to X$ (e.g., $f \in \mathsf{D}(\mathfrak{D}^1)$) admits the following Marchaud type representation for all $\alpha \in (0,1)$ and $t \in \mathbb{R}$ (cf. equation (1.3.7), [148, Proposition 3.2.1] or [189, Equation (1.2)]):

$$\mathfrak{D}^\alpha f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \sigma^{-\alpha-1}[S(\sigma)f(t-\sigma) - f(t)]\,\mathrm{d}\sigma.$$

Therefore, supposing that g is sufficiently regular, we have for all $t \in (0,\infty)$,

$$\begin{aligned} &\frac{t^{\{s\}}}{\Gamma(\{s\}+1)} S(t)\mathfrak{D}^{\{s\}} g(0) \\ &\quad = -\frac{t^{\{s\}}\sin(\pi\{s\})}{\pi} \int_0^\infty \sigma^{-\{s\}-1}[S(t+\sigma)g(-\sigma) - S(t)g(0)]\,\mathrm{d}\sigma \\ &\quad = -\frac{\sin(\pi\{s\})}{\pi} \int_0^\infty \tau^{-\{s\}-1}[S(t(\tau+1))g(-\tau t) - S(t)g(0)]\,\mathrm{d}\tau, \end{aligned}$$

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where we used the reflection formula for the gamma function [163, Equation (5.5.3)] to obtain the prefactor on the second line, and the change of variables $\sigma = t\tau$ for the third line. Substituting the heat semigroup once more, we obtain, for all $x \in \mathbb{R}^d$,

$$\begin{aligned}
&\frac{t^{\{s\}}}{\Gamma(\{s\}+1)}[S(t)\mathfrak{D}^{\{s\}}g(0)](x) \\
&\quad= \frac{\sin(\pi\{s\})}{\pi}\int_0^\infty \int_{\mathbb{R}^d} \tau^{-\{s\}-1}[K_t(x-y)g(0,y) - K_{t(1+\tau)}(x-y)g(-\tau t,y)]\,\mathrm{d}y\,\mathrm{d}\tau \\
&\quad= \frac{\sin(\pi\{s\})}{\pi(\sqrt{4\pi t})^d}\int_0^\infty \int_{\mathbb{R}^d} \tau^{-\{s\}-1}\Big[\exp\Big(-\frac{\|x-y\|^2_{\mathbb{R}^d}}{4t}\Big)g(0,y) \\
&\qquad\qquad - (\tau+1)^{-\frac{d}{2}}\exp\Big(-\frac{\|x-y\|^2_{\mathbb{R}^d}}{4t(\tau+1)}\Big)g(-\tau t,y)\Big]\,\mathrm{d}y\,\mathrm{d}\tau.
\end{aligned}$$

Substituting this into (4.4.4), we conclude that the mild solution to the Dirichlet problem associated to the fractional heat operator $(\partial_t - \Delta)^s$, with $s \in (1,2)$ and sufficiently regular initial datum g, admits the explicit expression

$$\begin{aligned}
u(t,x) = &\frac{\sin(\pi\{s\})}{\pi(\sqrt{4\pi t})^d}\int_0^\infty \int_{\mathbb{R}^d} \Big[\tau^{-\{s\}-1}\exp\Big(-\frac{\|x-y\|^2_{\mathbb{R}^d}}{4t}\Big)g(0,y) \\
&+ \Big(\tau^{-\{s\}}(\tau+1)^{-\frac{d}{2}-1} - \tau^{-\{s\}-1}(\tau+1)^{-\frac{d}{2}}\Big)\exp\Big(-\frac{\|x-y\|^2_{\mathbb{R}^d}}{4t(\tau+1)}\Big)g(-t\tau,y)\Big]\,\mathrm{d}y\,\mathrm{d}\tau
\end{aligned}$$

for all $(t,x) \in (0,\infty)\times\mathbb{R}^d$.

4.4.1. PROPERTIES OF THE MILD SOLUTION

In this section, we further investigate the mild solution concept introduced in Definition 4.4.2 by establishing some of its key properties. To this end, we start with the central observation that formula (4.4.2) has close connections to the normalized upper incomplete gamma function, whose principal branch $\overline{\Gamma}(\alpha,\cdot)\colon \mathbb{C}\setminus(-\infty,0) \to \mathbb{C}$ for $\alpha \in \mathbb{C}_+$ is defined by

$$\overline{\Gamma}(\alpha,z) := \frac{1}{\Gamma(\alpha)}\int_z^\infty \zeta^{\alpha-1}e^{-\zeta}\,\mathrm{d}\zeta, \quad z \in \mathbb{C}\setminus(-\infty,0),$$

integrating over any contour from z to ∞ avoiding $(-\infty,0)$, see [163, Chapter 8].

The relation to (4.4.2) follows from the following identities. For every $\alpha \in (0,1)$ and $z \in \mathbb{C}_+ \cup \{0\}$, [163, Equations (5.5.3), (13.4.4) and (13.6.6)] yield

$$\overline{\Gamma}(\alpha,z) = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \frac{\tau^{-\alpha}}{1+\tau}e^{-z(1+\tau)}\,\mathrm{d}\tau. \tag{4.4.5}$$

In particular, for $t \in (0,\infty)$, the change of variables $\sigma := t(1+\tau)$ produces

$$\overline{\Gamma}(\alpha,tz) = \frac{t^\alpha \sin(\pi\alpha)}{\pi}\int_t^\infty (\sigma-t)^{-\alpha}\sigma^{-1}e^{-\sigma z}\,\mathrm{d}\sigma. \tag{4.4.6}$$

Moreover, for $\alpha \in (0,\infty)$ and $n \in \mathbb{N}$, we have the following recurrence relations [163, Equations (8.8.12) and (8.4.10)]:

$$\overline{\Gamma}(\alpha, z) = \overline{\Gamma}(\{\alpha\}, z) + \sum_{k=1}^{\lfloor\alpha\rfloor} \frac{z^{k+\{\alpha\}-1}}{\Gamma(k+\{\alpha\})} e^{-z}, \quad \overline{\Gamma}(n, z) = \sum_{k=0}^{n-1} \frac{z^k}{k!} e^{-z}. \tag{4.4.7}$$

As a first application of these identities, we present the following proof:

Proof of Proposition 4.4.3. The estimate on $\|u(t)\|_X$ follows by applying the triangle inequality, (4.2.1) and identities (4.4.5)–(4.4.7) to (4.4.2). The continuity assertions rely on the strong continuity of $(S(t))_{t\ge 0}$, see the remarks below Assumption 4.2.1. They are immediate for the terms involving $\mathfrak{D}^{\{s\}+k} g(t_0)$. For the integral term, we note that the norm of the integrand is dominated by $\tau \mapsto M_0 \|g\|_\infty \frac{\tau^{-\{s\}}}{\tau+1}$, which is integrable in view of (4.4.5). Combined with the continuity of the function

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$$t \mapsto S((t-t_0)(\tau+1))g(t_0-(t-t_0))$$

on (t_0,∞) and possibly at t_0, the dominated convergence theorem yields the result. □

Next we comment on the precise way in which the fractional-order mild solution formula (4.4.2) of Definition 4.4.2 reduces to formula (4.4.3) for integer orders $s = n \in \mathbb{N}$. Substituting $s = n$ (i.e., $\lfloor s\rfloor = n$ and $\{s\} = 0$) in the higher-order terms of (4.4.2) and shifting the index of summation yields

$$\sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} S(t-t_0)[(\partial_t + A)^k g](t_0), \quad \forall t \in (t_0,\infty), \tag{4.4.8}$$

as desired. Moreover, the first term in (4.4.2) vanishes as required, provided that the integral remains convergent for $\{s\} = 0$. This occurs under Assumption 4.2.1(ii), but may fail in general if only Assumption 4.2.1(i) is satisfied, hence in this case we cannot argue via direct substitution. Instead, we have to consider limits as $s \to n$. Let u_s denote the mild solution from Definition 4.4.2 of order $s \in (0,\infty) \setminus \mathbb{N}$. Then

$$u_{n+\varepsilon}(t) = \frac{\sin(\pi\varepsilon)}{\pi} \int_0^\infty \frac{\tau^{-\varepsilon}}{\tau+1} S((t-t_0)(\tau+1))g(t_0-(t-t_0)\tau)\,d\tau \tag{4.4.9}$$

$$+ \sum_{k=0}^{n-1} \frac{(t-t_0)^{k+\varepsilon}}{\Gamma(k+\varepsilon+1)} S(t-t_0)\mathfrak{D}^{k+\varepsilon} g(t_0), \tag{4.4.10}$$

$$u_{n-\varepsilon}(t) = \frac{\sin(\pi\varepsilon)}{\pi} \int_0^\infty \frac{\tau^{\varepsilon-1}}{\tau+1} S((t-t_0)(\tau+1))g(t_0-(t-t_0)\tau)\,d\tau \tag{4.4.11}$$

$$+ \sum_{k=1}^{n-1} \frac{(t-t_0)^{k-\varepsilon}}{\Gamma(k-\varepsilon+1)} S(t-t_0)\mathfrak{D}^{k-\varepsilon} g(t_0), \tag{4.4.12}$$

for $\varepsilon \in (0,1)$ and $t \in (t_0,\infty)$. Substituting $\varepsilon = 0$ into the summations on lines (4.4.10) and (4.4.12), and comparing the resulting expressions with (4.4.8), we see that the integral terms on lines (4.4.9) and (4.4.11) should converge to zero and $S(t-t_0)g(t_0)$, respectively, as $\varepsilon \to 0$, in order to recover the integer-order case (formally, since we cannot expect the continuity of $\varepsilon \mapsto \mathfrak{D}^{k\pm\varepsilon} g(t_0)$ in general). The following proposition states when these convergences hold:

Proposition 4.4.7. *Let $t_0 \in \mathbb{R}$ and $g \in C_b((-\infty, t_0]; X)$ be given. If Assumption 4.2.1(i) is satisfied, then for all $t \in (t_0, \infty)$ it holds that*

$$\frac{\sin(\pi\varepsilon)}{\pi} \int_0^\infty \frac{\tau^{\varepsilon-1}}{\tau+1} S((t-t_0)(\tau+1)) g(t_0 - (t-t_0)\tau)\, d\tau \to S(t-t_0) g(t_0) \quad \text{as } \varepsilon \to 0.$$

If, in addition, Assumption 4.2.1(ii) is satisfied, then

$$\frac{\sin(\pi\varepsilon)}{\pi} \int_0^\infty \frac{\tau^{-\varepsilon}}{\tau+1} S((t-t_0)(\tau+1)) g(t_0 - (t-t_0)\tau)\, d\tau \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.4.13}$$

Proof. Fix $t_0 \in \mathbb{R}$ and $t \in (t_0, \infty)$. First we define the function $f_{t,t_0} : \mathbb{R} \to X$ by

$$f_{t,t_0}(r) := \begin{cases} S(t-t_0-r) g(t_0+r), & r \in (-\infty, 0]; \\ S(t-t_0) g(t_0), & r \in (0, \infty), \end{cases}$$

which is bounded and continuous at $r = 0$ by the assumptions on $(S(t))_{t\geq 0}$ and g. Next, for any $\varepsilon \in (0,1)$ we define $\psi_{t,t_0,\varepsilon} : \mathbb{R} \to [0,\infty)$ by

$$\psi_{t,t_0,\varepsilon}(r) := \frac{(t-t_0)^{1-\varepsilon} \sin(\pi\varepsilon)}{\pi} (r + (t-t_0))_+^{-1} r_+^{\varepsilon-1}, \quad r \in \mathbb{R}.$$

Shifting the integration variable by $t - t_0$ and applying (4.4.6) with $s = 1-\varepsilon$ and $z = 0$, we find

$$\int_{\mathbb{R}} \psi_{t,t_0,\varepsilon}(r)\, dr = \frac{(t-t_0)^{1-\varepsilon} \sin(\pi\varepsilon)}{\pi} \int_{t-t_0}^\infty r^{-1} (r - (t-t_0))^{\varepsilon-1}\, dr = 1.$$

Moreover, we have for any $\delta > 0$:

$$\begin{aligned} \int_{\{|r|\geq\delta\}} |\psi_{t,t_0,\varepsilon}(r)|\, dr &= \frac{(t-t_0)^{1-\varepsilon} \sin(\pi\varepsilon)}{\pi} \int_\delta^\infty (r+t-t_0)^{-1} r^{\varepsilon-1}\, dr \\ &\leq \frac{(t-t_0)^{1-\varepsilon} \sin(\pi\varepsilon)}{\pi} \int_\delta^\infty r^{\varepsilon-2}\, dr = \frac{(t-t_0)^{1-\varepsilon} \delta^{\varepsilon-1} \sin(\pi\varepsilon)}{\pi(1-\varepsilon)} \longrightarrow \frac{(t-t_0)\cdot 0}{\pi\delta} = 0 \end{aligned}$$

as $\varepsilon \to 0$. Together, these observations show that the family $(\psi_{t,t_0,\varepsilon})_{\varepsilon\in(0,1)}$ forms an approximate identity as $\varepsilon \to 0$ in the sense of [94, Definition 1.2.15]. Since the change of variables $\sigma := (t-t_0)\tau$ yields

$$\frac{\sin(\pi\varepsilon)}{\pi} \int_0^\infty \frac{\tau^{\varepsilon-1}}{\tau+1} S((t-t_0)(\tau+1)) g(t_0 - (t-t_0)\tau)\, d\tau = [\psi_{t,t_0,\varepsilon} * f_{t,t_0}](0),$$

the first assertion now follows by applying the obvious vector-valued generalization of [94, Theorem 1.2.19(2)], which gives

$$[\psi_{t,t_0,\varepsilon} * f_{t,t_0}](0) \longrightarrow f_{t,t_0}(0) = S(t-t_0) g(t_0) \quad \text{as } \varepsilon \to 0.$$

For the second assertion, suppose that Assumption 4.2.1(ii) holds. By Proposition 4.4.3, the left-hand side of (4.4.13) is bounded above by $M_0 \overline{\Gamma}(\varepsilon, w(t-t_0)) \|g\|_\infty$ for all $t \in (t_0, \infty)$. Since $w(t-t_0) > 0$, this expression tends to zero as $\varepsilon \to 0$. □

The final corollary concerns the choice $g \equiv x \in \mathsf{D}(A^{(s-1)\vee 0})$ in Definition 4.4.2, in which case the solution can be expressed in terms of an operator-valued counterpart of the upper incomplete gamma function. Namely, for $\alpha \in (0,1)$ and $t \in (0,\infty)$, we use the Phillips calculus from Definition 1.3.2 to define

$$\overline{\Gamma}(\alpha, tA) := [z \mapsto \overline{\Gamma}(\alpha, tz)](A) = \mathcal{L}\Big[\frac{t^{\alpha}\sigma_+^{-1}(\sigma-t)_+^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}\,\mathrm{d}\sigma\Big](A) \in \mathscr{L}(X),$$

see (4.4.6). For $\alpha \in [1,\infty)$, such a Laplace transform representation is no longer available, so in this case we instead define $\overline{\Gamma}(\alpha, tA)$ by analogy with (4.4.7):

$$\overline{\Gamma}(\alpha, tA)x := \overline{\Gamma}(\{\alpha\}, tA)x + \sum_{k=1}^{\lfloor \alpha \rfloor} \frac{t^{k+\{\alpha\}-1}}{\Gamma(k+\{\alpha\})} A^{k+\{\alpha\}-1}S(t)x, \quad x \in \mathsf{D}(A^{\alpha-1}). \tag{4.4.14}$$

For $t = 0$ we set $\overline{\Gamma}(\alpha, 0A) := \mathrm{Id}_X$. Although $\overline{\Gamma}(\alpha, tA)$ is unbounded in general, we recall that under the additional Assumption 4.2.1(iii) we have, cf. [165, Chapter 2, Theorem 6.13(c)]:

$$\forall \beta \in [0,\infty), \exists M_\beta \in [1,\infty) : \|A^\beta S(t)\|_{\mathscr{L}(X)} \le M_\beta t^{-\beta} e^{-wt}, \quad \forall t \in (0,\infty). \tag{4.4.15}$$

Putting these observations together, we obtain the following formula for the solution with initial datum $g \equiv x$:

Corollary 4.4.8. *Suppose Assumption 4.2.1(i) holds. Let $s \in (0,\infty) \setminus \mathbb{N}$, $p \in [1,\infty]$ and $t_0 \in \mathbb{R}$. If $g \equiv x$ for some given $x \in \mathsf{D}(A^{(s-1)\vee 0})$ and $s \in (0,1)$ or Assumption 4.2.1(ii) holds, then the solution u to (4.4.1) from Definition 4.4.2 becomes*

$$u(t) = \overline{\Gamma}(s, (t-t_0)A)x, \quad \forall t \in (t_0, \infty). \tag{4.4.16}$$

If, in addition, $s \in (0,1)$ or Assumption 4.2.1(iii) is satisfied, then

$$\|u(t)\|_X \le \Big[M_0 \overline{\Gamma}(\{s\}, w(t-t_0)) + e^{-w(t-t_0)} \sum_{k=1}^{\lfloor s \rfloor} \frac{M_{\{s\}+k-1}}{\Gamma(\{s\}+k)}\Big] \|x\|_X, \tag{4.4.17}$$

where $w \in [0,\infty)$ and $M_0, M_{\{s\}}, \dots, M_s \in [1,\infty)$ are as in (4.2.1) and (4.4.15).

Proof. The substitution $g \equiv x$ and the change of variables $\sigma := (t-t_0)(1+\tau)$ in the first term of (4.4.2) produces

$$\frac{\sin(\pi\{s\})}{\pi}(t-t_0)^{\{s\}} \int_{t-t_0}^{\infty} \sigma^{-1}(\sigma-(t-t_0))^{-\{s\}} S(\sigma)\,\mathrm{d}\sigma\, x,$$

which is equal to $\overline{\Gamma}(\{s\}, (t-t_0)A)x$ by (4.4.6).

Now suppose that $s \in (1,\infty)$ and let Assumption 4.2.1(ii) be satisfied. Then for any $\alpha \in [0,\infty)$ we have

$$\mathfrak{I}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \tau^{\alpha-1} S(\tau) x\,\mathrm{d}\tau = A^{-\alpha}x$$

by the Phillips representation of negative fractional powers (cf. (1.3.4)), so that

$$\mathfrak{D}^{\beta} g(t) = (\partial_t + A)^{\lceil \beta \rceil} \mathfrak{J}^{\lceil \beta \rceil - \beta} g(t) = A^{\lceil \beta \rceil} A^{\beta - \lceil \beta \rceil} x = A^{\beta} x$$

for all $\beta \in [0, s-1]$, hence the remaining terms of (4.4.2) become

$$\sum_{k=1}^{\lfloor s \rfloor} \frac{(t-t_0)^{\{s\}+k-1}}{\Gamma(\{s\}+k)} S(t-t_0) A^{\{s\}+k-1} x,$$

proving (4.4.16) in view of (4.4.14). Finally, estimate (4.4.17) follows from the inequalities (4.2.1) (and (4.4.15)). □

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4.4.2. PROOF OF THE RELATION BETWEEN MILD SOLUTIONS AND L^p-SOLUTIONS

The aim of this section is to prove Theorem 4.4.5. We will prove the integer-order and fractional-order cases separately in the following two subsections. Before proceeding to do so, we make a preliminary observation which applies to both cases:

If Assumption 4.2.1(ii) holds, and $u \in \mathsf{D}(\mathfrak{D}^s)$ is an L^p-solution to (4.4.1) for some $t_0 \in \mathbb{R}$, $p \in [1, \infty]$ and $s \in (0, \infty)$ satisfying (4.2.2), then $u \in \mathrm{C_b}(\mathbb{R}; X)$ by Corollary 4.3.4. Moreover, if $s \geq 1$, then we have $\mathfrak{D}^{s-1} u \in \mathsf{D}(\mathfrak{D}^1)$ by the definition (4.2.3), and thus $\mathfrak{D}^{s-1} u \in \mathrm{C_b}(\mathbb{R}; X)$ by applying Corollary 4.3.4 once more. Since $g \equiv u$ on $(-\infty, t_0]$, this shows that the continuity properties of g are as in Definition 4.4.2.

INTEGER-ORDER CASE

Let us consider the case $s = n \in \mathbb{N}$, in which the operator $\mathfrak{D}^n = (\partial_t + A)^n$ is local in time. Note that the first-order case $s = 1$ was already treated in Section 4.2.3. The following proposition is the key ingredient in the proof of Theorem 4.4.5 for $s = n$.

Proposition 4.4.9. *Suppose that Assumption 4.2.1(ii) holds. Let $n \in \mathbb{N}$, $p \in [1, \infty]$, $t_0 \in \mathbb{R}$ and $u \in \mathsf{D}((\partial_t + A)^n)$. For all $t \in (t_0, \infty)$, we have*

$$\mathfrak{J}^n_{t_0} (\partial_t + A)^n u(t) = u(t) - \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} S(t-t_0) [(\partial_t + A)^k u](t_0). \tag{4.4.18}$$

Here, $\mathfrak{J}^s_{t_0} \in \mathscr{L}(L^p(t_0, \infty; X))$ denotes the (Riemann–Liouville type) fractional parabolic integral, defined by

$$\mathfrak{J}^s_{t_0} u(t) := \frac{1}{\Gamma(s)} \int_{t_0}^{t} (t-\tau)^{s-1} S(t-\tau) u(\tau) \,\mathrm{d}\tau, \quad u \in L^p(J; X), \text{ a.e. } t \in (t_0, \infty). \tag{4.4.19}$$

Proof. We use induction on $n \in \mathbb{N}$. For the base case $n = 1$, let us fix an arbitrary $u \in W^{1,p}(J; X) \cap L^p(J; \mathsf{D}(A)) \hookrightarrow \mathrm{C_b}(\overline{J}; X)$. Since $u(t) \in \mathsf{D}(A)$ a.e., we find in particular that $u(t_0) \in \overline{\mathsf{D}(A)}$. Now the result follows by applying Proposition 4.2.2(a) $\Longrightarrow$ (b) with $f := u' + Au \in L^p(J; X)$.

Now suppose that the statement is true for some $n \in \mathbb{N}$. We present the argument for $t_0 = 0$, the other cases being analogous. Fix $u \in \mathsf{D}((\partial_t + A)^{n+1})$ and apply the induction hypothesis to the function $(\partial_t + A)u \in \mathsf{D}((\partial_t + A)^n)$, yielding

$$(\partial_t + A)u(t) = \mathfrak{I}^n_{t_0}(\partial_t + A)^{n+1}u(t) + \sum_{k=0}^{n-1} \frac{t^k}{k!} S(t)[(\partial_t + A)^{k+1}u](0)$$

for all $t \in \overline{J}$. Applying $\mathfrak{I}^1$ to both sides of the above equation, we can use the case $n = 1$ along with Propositions 4.2.3(c)–(d) to find

$$\begin{aligned} u(t) &= \mathfrak{I}^{n+1}_{t_0}(\partial_t + A)^{n+1}u(t) + \sum_{k=0}^{n-1} \frac{t^{k+1}}{(k+1)!} S(t)[(\partial_t + A)^{k+1}u](0) + S(t)u(0) \\ &= \mathfrak{I}^{n+1}_{t_0}(\partial_t + A)^{n+1}u(t) + \sum_{k=0}^{n} \frac{t^k}{k!} S(t)[(\partial_t + A)^k u](0). \end{aligned}$$ □

We can now prove the integer-order case of Theorem 4.4.5:

Proof of Theorem 4.4.5 (for $s = n \in \mathbb{N}$). Let u be an L^p-solution to (4.4.1). By Definition 4.4.2 and the observations in the beginning of Section 4.4.2, it holds that $u \in \mathrm{C_b}(\mathbb{R}; X)$, $u \equiv g$ on $(-\infty, t_0]$ and $(\partial_t + A)^n u = 0$ a.e. on (t_0, ∞). Applying $\mathfrak{I}^s_{t_0}$ to both sides of the latter and using Proposition 4.4.9 on the left-hand side, we find

$$u(t) = \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} S(t-t_0)[(\partial_t + A)^k u](t_0) \quad \text{for all } t \in (t_0, \infty).$$

Note that the operators $(\partial_t + A)^k$ are local in time and that, in fact, we can choose to interpret ∂_t as a left derivative. Thus, since $u \equiv g$ on $(-\infty, t_0]$, we obtain (4.4.3). □

If $u \in \mathsf{D}((\partial_t + A)^k)$ is sufficiently regular, say $u \in C^j(\overline{J}; \mathsf{D}(A^{k-j}))$ (j times continuously differentiable) for $j \in \{0, \dots, k\}$, then we have the pointwise binomial expansion

$$[(\partial_t + A)^k u](t) = \sum_{j=0}^{k} \binom{k}{j} A^{k-j} u^{(j)}(t), \quad \forall t \in \overline{J},$$

where $u^{(j)}$ denotes the jth (classical) derivative of u. Substituting this into (4.4.18), using the definition of binomial coefficients, interchanging the order of summation and shifting the inner summation index yields

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{t^k}{k!} S(t)[(\partial_t + A)^k u](0) &= \sum_{k=0}^{n-1} \sum_{j=0}^{k} \frac{t^k}{j!(k-j)!} S(t) A^{k-j} u^{(j)}(0) \\ &= \sum_{j=0}^{n-1} \sum_{k=j}^{n-1} \frac{t^k}{j!(k-j)!} S(t) A^{k-j} u^{(j)}(0) = \sum_{j=0}^{n-1} \frac{t^j}{j!} \sum_{\ell=0}^{n-j-1} \frac{t^\ell}{\ell!} S(t) A^\ell u^{(j)}(0). \end{aligned}$$

Moreover, note that for $n \in \mathbb{N}$, $x \in \mathsf{D}(A^{n-1})$ and $t \in (0, \infty)$ we have

$$\overline{\Gamma}(n, tA)x = \sum_{k=0}^{n-1} \frac{t^k}{k!} S(t) A^k x,$$

cf. (4.4.7). Together, these observations imply that, in this situation, equation (4.4.18) takes on the following form:

$$\mathfrak{I}^n_{t_0}(\partial_t + A)^n u(t) = u(t) - \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} \overline{\Gamma}(n-k,(t-t_0)A)x_k. \tag{4.4.20}$$

FRACTIONAL-ORDER CASE

Now we turn to the proof of Theorem 4.4.5 in the case that $s \in (0,\infty) \setminus \mathbb{N}$. It relies on the following result, which is the fractional-order analog to Proposition 4.4.9.

Theorem 4.4.10. *Suppose that Assumption 4.2.1(ii) holds. Let $f \in L^p(-\infty, t_0; X)$ for some $p \in [1,\infty]$ and $t_0 \in \mathbb{R}$, and let $\widetilde{f} \in L^p(\mathbb{R}; X)$ denote its extension by zero to the whole of $\mathbb{R}$. Let $s \in (0,\infty)\setminus\mathbb{N}$ be such that (4.2.2) is satisfied. Then the following identity holds for all $t \in (t_0,\infty)$:*

$$\begin{aligned}\mathfrak{I}^s \widetilde{f}(t) = \frac{\sin(\pi\{s\})}{\pi} \int_0^\infty \frac{\tau^{-\{s\}}}{\tau+1} S((t-t_0)(\tau+1))\mathfrak{I}^s f(t_0 - (t-t_0)\tau)\,\mathrm{d}\tau \\ + \sum_{k=1}^{\lfloor s\rfloor} \frac{(t-t_0)^{\{s\}+k-1}}{\Gamma(\{s\}+k)} S(t-t_0)\mathfrak{I}^{\lfloor s\rfloor-k+1} f(t_0).\end{aligned} \tag{4.4.21}$$

Before proving this result, we show how it can indeed be used to finish the proof of Theorem 4.4.5:

Proof of Theorem 4.4.5 (for $s \in (0,\infty) \setminus \mathbb{N}$). Let $u \in \mathsf{D}(\mathfrak{D}^s)$ be an L^p-solution to (4.4.1). Applying $\mathfrak{D}^s$ to the second line of (4.4.1) yields $\mathfrak{D}^s u = \mathfrak{D}^s g$ a.e. on $(-\infty, t_0)$. Combined with the first line, i.e., $\mathfrak{D}^s u = 0$ a.e. on (t_0,∞), we find

$$\mathfrak{D}^s u = \widetilde{f} \quad \text{a.e. on } \mathbb{R},$$

where $f := \mathfrak{D}^s g \in L^p(-\infty, t_0; X)$. Now we apply $\mathfrak{I}^s$ to both sides of this equation, and use Proposition 4.3.3(b) and Theorem 4.4.10 to the left-hand and right-hand sides, respectively. Together, this yields, for all $t \in (t_0,\infty)$,

$$\begin{aligned}u(t) = \frac{\sin(\pi\{s\})}{\pi} \int_0^\infty \frac{\tau^{-\{s\}}}{\tau+1} S((t-t_0)(\tau+1))\mathfrak{I}^s[\mathfrak{D}^s g](t_0 - (t-t_0)\tau)\,\mathrm{d}\tau \\ + \sum_{k=1}^{\lfloor s\rfloor} \frac{(t-t_0)^{\{s\}+k-1}}{\Gamma(\{s\}+k)} S(t-t_0)\mathfrak{I}^{\lfloor s\rfloor-k+1}[\mathfrak{D}^s g](t_0).\end{aligned}$$

In order to conclude that u satisfies equation (4.4.2), it remains to observe that we have $\mathfrak{I}^s\mathfrak{D}^s g = g$ and $\mathfrak{I}^{\lfloor s\rfloor-k+1}\mathfrak{D}^s g = \mathfrak{D}^{s-(\lfloor s\rfloor-k+1)} g = \mathfrak{D}^{\{s\}+k-1} g$ for all $k \in \{1,\dots,\lfloor s\rfloor\}$. Indeed, the former follows from the natural analog of Proposition 4.3.3(b) for functions defined on $(-\infty, t_0]$; for the latter, let $m \in \mathbb{N}_0$ be such that $m \le \lfloor s\rfloor$, for which

$$\begin{aligned}\mathfrak{I}^m\mathfrak{D}^s g = \mathfrak{I}^m\mathfrak{D}^{\lfloor s\rfloor}\mathfrak{I}^{\lfloor s\rfloor-s} g = \mathfrak{I}^m\mathfrak{D}^m\mathfrak{D}^{\lfloor s\rfloor-m}\mathfrak{I}^{\lfloor s\rfloor-s} g = \mathfrak{D}^{\lfloor s\rfloor-m}\mathfrak{I}^{\lfloor s\rfloor-m-(s-m)} g \\ = \mathfrak{D}^{\lfloor s-m\rfloor}\mathfrak{I}^{\lfloor s-m\rfloor-(s-m)} g = \mathfrak{D}^{s-m} g.\end{aligned}$$

Here, we used the definition of $\mathfrak{D}^s$, the additivity of integer powers of $\mathfrak{D}$, and the aforementioned analog to Proposition 4.3.3(b). This completes the proof that u is a mild solution in the sense of Definition 4.4.2. □

The proof of Theorem 4.4.10 involves expressing an integral in terms of fractional binomial coefficients, given by [163, Equations (1.2.6), (5.2.4) and (5.2.5)]:

$$\binom{\alpha}{k} := \frac{1}{k!}\prod_{\ell=0}^{k-1}(\alpha-\ell) = \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)}, \quad \alpha\in(0,\infty),\ k\in\mathbb{N}_0. \tag{4.4.22}$$

Since the author is not aware of a direct reference for the following integral identity, a proof is presented below for the sake of self-containedness.

4

Lemma 4.4.11. *For $\alpha\in(0,1)$, $a,b\in(0,\infty)$ and $n\in\mathbb{N}_0$ we have*

$$\begin{aligned}\frac{\sin(\pi\alpha)}{\pi}\int_0^{a/b}\frac{\tau^{-\alpha}(a-b\tau)^{\alpha+n-1}}{\tau+1}\,\mathrm{d}\tau \\ = (a+b)^{\alpha+n-1} - \sum_{k=1}^{n}\binom{\alpha+n-1}{n-k}a^{n-k}b^{k+\alpha-1}.\end{aligned} \tag{4.4.23}$$

Proof. By the change of variables $\sigma := \frac{b}{a}\tau$ and [163, Equation (5.5.3)], the validity of the identity (4.4.23) is equivalent to that of

$$\begin{aligned}\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^1\frac{\sigma^{-\alpha}(1-\sigma)^{\alpha+n-1}}{\sigma+\frac{b}{a}}\,\mathrm{d}\sigma \\ = a^{1-n}b^{-\alpha}(a+b)^{\alpha+n-1} - \sum_{k=1}^{n}\binom{\alpha+n-1}{n-k}\left(\frac{b}{a}\right)^{k-1}.\end{aligned} \tag{4.4.24}$$

We will verify this identity using induction on $n\in\mathbb{N}_0$. The base case $n=0$ is a consequence of [163, Equations (5.12.4) and (5.12.1)]:

$$\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^1\frac{\sigma^{-\alpha}(1-\sigma)^{\alpha-1}}{\sigma+\frac{b}{a}}\,\mathrm{d}\sigma = \left(1+\frac{b}{a}\right)^{\alpha-1}\left(\frac{b}{a}\right)^{-\alpha} = ab^{-\alpha}(a+b)^{\alpha-1}.$$

Now suppose that (4.4.24) holds for a given $n\in\mathbb{N}_0$. In order to establish the identity for $n+1$, we write $1-\sigma = 1+\frac{b}{a}-(\sigma+\frac{b}{a})$ and apply the induction hypothesis and [163, Equation (5.12.1)], respectively, to the resulting two integrals:

$$\begin{aligned}&\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^1\frac{\sigma^{-\alpha}(1-\sigma)^{\alpha+n}}{\sigma+\frac{b}{a}}\,\mathrm{d}\sigma \\ &= \frac{1+\frac{b}{a}}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^1\frac{\sigma^{-\alpha}(1-\sigma)^{\alpha+n-1}}{\sigma+\frac{b}{a}}\,\mathrm{d}\sigma \\ &\quad - \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^1\sigma^{-\alpha}(1-\sigma)^{\alpha+n-1}\,\mathrm{d}\sigma \\ &= a^{-n}b^{-\alpha}(a+b)^{\alpha+n} - \left(1+\frac{b}{a}\right)\sum_{k=1}^{n}\binom{\alpha+n-1}{n-j}\left(\frac{b}{a}\right)^{j-1} - \frac{\Gamma(\alpha+n)}{n!\,\Gamma(\alpha)}.\end{aligned}$$

For the latter two terms, we have

$$\begin{aligned}
&\left(1+\tfrac{b}{a}\right)\sum_{k=1}^{n}\binom{\alpha+n-1}{n-k}\left(\frac{b}{a}\right)^{k-1}+\frac{\Gamma(\alpha+n)}{n!\Gamma(\alpha)}\\
&=\sum_{k=1}^{n}\binom{\alpha+n-1}{n-k}\left(\frac{b}{a}\right)^{k-1}+\sum_{k=0}^{n}\binom{\alpha+n-1}{n-k}\left(\frac{b}{a}\right)^{k}\\
&=\sum_{k=1}^{n}\left[\binom{\alpha+n-1}{n-k}+\binom{\alpha+n-1}{n-k+1}\right]\left(\frac{b}{a}\right)^{k-1}+\left(\frac{b}{a}\right)^{n}=\sum_{k=1}^{n+1}\binom{\alpha+n}{n+1-k}\left(\frac{b}{a}\right)^{k-1}.
\end{aligned}$$

Indeed, to obtain the second line we note that $\frac{\Gamma(\alpha+n)}{n!\Gamma(\alpha)}=\binom{\alpha+n-1}{n}$ by (4.4.22), which is the term $k=0$ of the second summation on the second line; shifting its index of summation and splitting off the last term yields the first expression on the third line. The final step uses [163, Equation (1.2.7)] and the fact that $(\frac{b}{a})^n$ corresponds to the term $k=n+1$ in the desired formula. Putting the previous two displays together proves the induction step and thereby the lemma. □

Remark 4.4.12. An alternative way to derive (4.4.23) is by noting that the integral can be expressed in terms of a hypergeometric function [163, Equation (15.6.1)] to which one can apply the transformation formula [163, Equation (15.8.2)]. This results in a difference of two hypergeometric functions, whose definitions can be written out to respectively yield an infinite and a finite sum: The former is the fractional binomial expansion of $(a+b)^{\alpha+n-1}$ and the latter consists of its first n terms, and together this gives (4.4.23). In particular, we note that (4.4.23) is formally equal to the tail of a fractional binomial series. The proof of Lemma 4.4.11 is more direct and avoids the need to address the convergence of an infinite series.

Proof of Theorem 4.4.10. Fixing $t\in(t_0,\infty)$, the semigroup law implies

$$S(t-r)=S((t-t_0)(\tau+1))S(t_0-(t-t_0)\tau-r) \tag{4.4.25}$$

for all $\tau\in(0,\infty)$ and $r\in(-\infty,t_0-(t-t_0)\tau)$. This identity, followed by (4.2.1), Hölder's inequality and equation (4.4.5) yields

$$\begin{aligned}
&\frac{1}{\Gamma(s)}\int_0^\infty\int_{-\infty}^{t_0-(t-t_0)\tau}\left\|\frac{\tau^{-\{s\}}(t_0-(t-t_0)\tau-r)^{\gamma-1}}{\tau+1}S(t-r)f(r)\right\|_X\,\mathrm{d}r\,\mathrm{d}\tau\\
&\le M_0\int_0^\infty\frac{\tau^{-\{s\}}}{\tau+1}e^{-w(t-t_0)(\tau+1)}\int_{-\infty}^{t_0-(t-t_0)\tau}\|k_s(t_0-(t-t_0)\tau-r)f(r)\|_X\,\mathrm{d}r\,\mathrm{d}\tau\\
&\le\frac{M_0\pi}{\sin(\pi\{s\})}\overline{\Gamma}(\{s\},w(t-t_0))\|k_s\|_{L^{p'}(0,\infty;\mathscr{L}(X))}\|f\|_{L^p(\mathbb{R};X)}<\infty.
\end{aligned}$$

This justifies the use of Fubini's theorem in the following:

$$
\begin{aligned}
&\int_0^\infty \frac{\tau^{-\{s\}}}{\tau+1} S((t-t_0)(\tau+1))\mathfrak{J}^s f(t_0-(t-t_0)\tau)\,\mathrm{d}\tau \\
&= \frac{1}{\Gamma(s)}\int_0^\infty \int_{-\infty}^{t_0-(t-t_0)\tau} \frac{\tau^{-\{s\}}(t_0-(t-t_0)\tau-r)^{s-1}}{\tau+1} S(t-r)f(r)\,\mathrm{d}r\,\mathrm{d}\tau \\
&= \frac{1}{\Gamma(s)}\int_{-\infty}^{t_0}\left[\int_0^{\frac{t_0-r}{t-t_0}} \frac{\tau^{-\{s\}}(t_0-r-(t-t_0)\tau)^{s-1}}{\tau+1}\,\mathrm{d}\tau\right] S(t-r)f(r)\,\mathrm{d}r,
\end{aligned}
$$

where we used (4.4.25) once more. Lemma 4.4.11 and equation (4.4.22) produce

$$
\begin{aligned}
&\frac{\sin(\pi\{s\})}{\pi}\int_0^{\frac{t_0-r}{t-t_0}} \frac{\tau^{-\{s\}}(t_0-r-(t-t_0)\tau)^{\{s\}+\lfloor s\rfloor-1}}{\tau+1}\,\mathrm{d}\tau \\
&\qquad = (t-r)^{s-1} - \sum_{k=1}^{\lfloor s\rfloor} \frac{\Gamma(s)}{(\lfloor s\rfloor-k)!\,\Gamma(\{s\}+k)}(t_0-r)^{\lfloor s\rfloor-k}(t-t_0)^{k+\{s\}-1}.
\end{aligned}
$$

The previous two displays and the identity $S(t-r) = S(t-t_0)S(t_0-r)$ yield

$$
\begin{aligned}
&\frac{\sin(\pi\{s\})}{\pi}\int_0^\infty \frac{\tau^{-\{s\}}}{\tau+1} S((t-t_0)(\tau+1))\mathfrak{J}^s f(t_0-(t-t_0)\tau)\,\mathrm{d}\tau \\
&= -\sum_{k=1}^{\lfloor s\rfloor} \frac{(t-t_0)^{\{s\}+k-1}S(t-t_0)}{\Gamma(\{s\}+k)}\frac{1}{(\lfloor s\rfloor-k)!}\int_{-\infty}^{t_0}(t_0-r)^{\lfloor s\rfloor-k}S(t_0-r)f(r)\,\mathrm{d}r \\
&\quad + \frac{1}{\Gamma(s)}\int_{-\infty}^{t_0}(t-r)^{s-1}S(t-r)f(r)\,\mathrm{d}r \\
&= -\sum_{k=1}^{\lfloor s\rfloor} \frac{(t-t_0)^{\{s\}+k-1}S(t-t_0)}{\Gamma(\{s\}+k)}\mathfrak{J}^{\lfloor s\rfloor-k+1}f(t_0) + \mathfrak{J}^s\widetilde{f}(t)
\end{aligned}
$$

which is precisely (4.4.21). The final assertion follows from Proposition 4.3.3(b). □

4.5. COMPARISON TO RIEMANN–LIOUVILLE AND CAPUTO CAUCHY PROBLEMS

In this section, we compare the Dirichlet problem (4.4.1) to fractional-order abstract Cauchy problems of the form

$$
(\partial_t + A)^s u(t) = 0, \qquad t \in (t_0,\infty),
$$

augmented with initial conditions which depend on the interpretation of the abstract space–time operator $(\partial_t + A)^s$, acting on functions $u\colon J \to X$ with $J = (t_0,\infty)$ for $t_0 \in \mathbb{R}$ (instead of $J = \mathbb{R}$ as in the previous sections). More precisely, we will interpret $(\partial_t + A)^s$ as a Riemann–Liouville or Caputo type fractional parabolic derivative, respectively, on $L^p(J;X)$ for $p \in [1,\infty]$, and determine the corresponding initial conditions and mild solution formulae, see Definitions 4.5.2 and 4.5.3.

For fractional time derivatives ∂_t^s, i.e., the case $A = 0$ (which we briefly introduced in Section 1.3.3), the resulting solution concepts are well-known and commonly studied, see for instance [121, Chapter 3]. In this case, the fractional integral has less favorable mapping properties, see [121, Section 2.3], and the well-posedness of (4.1.5) is less clear. We will show that, as in the case $A = 0$, the lowest-order term of the solution to the Riemann–Liouville type initial value problem has a singularity at t_0 in general, whereas the Caputo initial value problem yields the same solution for any two $s_1, s_2 \in (n, n+1)$ for $n \in \mathbb{N}_0$. In contrast, the solution from Definition 4.4.2 is continuous at t_0 under mild assumptions on g or $(S(t))_{t\geq 0}$ and changes for all choices of $s \in (0,\infty)$.

Firstly, let us recall the Riemann–Liouville type fractional parabolic integral $\mathfrak{J}^s_{t_0}$ on $L^p(J;X)$ defined by (4.4.19). Then, the Riemann–Liouville and Caputo type fractional parabolic *derivatives* are respectively defined by

$$\mathfrak{D}^s_{\mathrm{RL}} := (\partial_t + A)^{\lceil s\rceil}\mathfrak{J}^{\lceil s\rceil - s}_{t_0} \quad \text{and} \quad \mathfrak{D}^s_{\mathrm{C}} := \mathfrak{J}^{\lceil s\rceil - s}_{t_0}(\partial_t + A)^{\lceil s\rceil}$$

on their maximal domains. In order to derive mild solution type formulae for L^p-solutions to the equations $\mathfrak{D}^s_{\mathrm{RL}} u = 0$ and $\mathfrak{D}^s_{\mathrm{C}} u = 0$, we proceed analogously to [121, Chapter 3] and express $\mathfrak{J}^s_{t_0}\mathfrak{D}^s_{\mathrm{RL}} u$ and $\mathfrak{J}^s_{t_0}\mathfrak{D}^s_{\mathrm{C}} u$ in terms of initial data from u (compare with Proposition 4.4.9 and Theorem 4.4.10), so that applying $\mathfrak{J}^s_{t_0}$ on both sides of the equations motivates the definitions. The integer-order case $s = n \in \mathbb{N}$, where

$$\mathfrak{D}^n = \mathfrak{D}^n_{\mathrm{RL}} = \mathfrak{D}^n_{\mathrm{C}} = (\partial_t + A)^n,$$

was treated in Section 4.4.2. From these results, we derive the following proposition regarding $\mathfrak{D}^s_{\mathrm{RL}}$ and $\mathfrak{D}^s_{\mathrm{C}}$ for fractional $s \in (0,\infty) \setminus \mathbb{N}$:

Proposition 4.5.1. *Let Assumption 4.2.1(i) be satisfied. If $s \in (0,\infty)\setminus\mathbb{N}$, $t_0 \in \mathbb{R}$ and $p \in [1,\infty]$ and $u \in \mathsf{D}(\mathfrak{D}^s_{\mathrm{RL}})$, then for almost all $t \in J := (t_0,\infty)$:*

$$\begin{aligned}\mathfrak{J}^s_{t_0}\mathfrak{D}^s_{\mathrm{RL}} u(t) = u(t) &- \frac{(t-t_0)^{\{s\}-1}}{\Gamma(\{s\})} S(t-t_0)\mathfrak{J}^{1-\{s\}}_{\mathrm{RL}} u(t_0) \\ &- \sum_{k=1}^{\lfloor s\rfloor} \frac{(t-t_0)^{k+\{s\}-1}}{\Gamma(k+\{s\})} S(t-t_0)\mathfrak{D}^{k+\{s\}-1}_{\mathrm{RL}} u(t_0).\end{aligned}$$

If $u \in \mathsf{D}(\mathfrak{D}^s_{\mathrm{C}})$ is such that $u \in C^j(\overline{J};\mathsf{D}(A^{n-1-j}))$ for all $j \in \{0,\dots,n-1\}$, then we have for almost all $t \in J := (t_0,\infty)$:

$$\mathfrak{J}^s_{t_0}\mathfrak{D}^s_{\mathrm{C}} u(t) = u(t) - \sum_{k=0}^{\lfloor s\rfloor} \frac{(t-t_0)^k}{k!}\overline{\Gamma}(\lceil s\rceil - k, (t-t_0)A) u^{(k)}(t_0) \quad a.e.$$

Proof. For the sake of notational convenience we only present the case $t_0 = 0$. The definition of $\mathfrak{D}^s_{\mathrm{RL}}$, along with Propositions 4.2.3(c)–(d) and 4.4.9, yields

$$\begin{aligned}\mathfrak{J}^{\lceil s\rceil - s}_0\mathfrak{J}^s_0\mathfrak{D}^s_{\mathrm{RL}} u &= \mathfrak{J}^{\lceil s\rceil}_0\mathfrak{D}^s_{\mathrm{RL}} u = \mathfrak{J}^{\lceil s\rceil}_0(\partial_t + A)^{\lceil s\rceil}\mathfrak{J}^{\lceil s\rceil - s}_0 u \\ &= \mathfrak{J}^{\lceil s\rceil - s}_0 u - \sum_{k=0}^{n-1}\frac{(\cdot)^k}{k!} S(\cdot)[(\partial_t + A)^k \mathfrak{J}^{\lceil s\rceil - s}_0 u](0) \\ &= \mathfrak{J}^{\lceil s\rceil - s}_0\Big[u - \sum_{k=0}^{n-1}\frac{(\cdot)^{k-\lceil s\rceil + s} S(\cdot)}{\Gamma(k+s-\lceil s\rceil + 1)}[(\partial_t + A)^k\mathfrak{J}^{\lceil s\rceil - s}_0 u](0)\Big]\end{aligned}$$

for any $u \in \mathsf{D}(\mathfrak{D}^s_{\mathrm{RL}})$. The first assertion then follows from Proposition 4.3.3 and the injectivity of $\mathfrak{I}_0^{\lceil s\rceil - s}$.

If $u \in \mathsf{D}(\mathfrak{D}^s_{\mathrm{C}})$, combining the definition with Proposition 4.2.3(c) produces

$$\mathfrak{I}^s_0 \mathfrak{D}^s_{\mathrm{C}} u = \mathfrak{I}^s_0 \mathfrak{I}_0^{\lceil s\rceil - s} (\partial_t + A)^{\lceil s\rceil} u = \mathfrak{I}_0^{\lceil s\rceil} (\partial_t + A)^{\lceil s\rceil} u,$$

so that the result follows from Proposition 4.4.9 and the discussion below it, in particular equation (4.4.20). □

Note that $\mathfrak{I}_0^{1-\{s\}} u$ need not vanish at $t_0 = 0$. Indeed, even if it is continuous, it may not satisfy (4.4.19) pointwise, as evidenced by the example $u := k_{\{s\}} \otimes x$ if we take $p \in [1, \frac{1}{\{s\}-1})$ and $x \in \overline{\mathsf{D}(A)} \setminus \{0\}$, see Proposition 4.2.3(d).

Proposition 4.5.1 motivates the following definition of the Riemann–Liouville fractional abstract Cauchy type problem and its corresponding solution.

4

Definition 4.5.2. Let Assumption 4.2.1(i) be satisfied. For $s \in (0,\infty) \setminus \mathbb{N}$ and $t_0 \in \mathbb{R}$, the mild solution to the Riemann–Liouville abstract Cauchy type problem

$$\begin{cases} \mathfrak{D}^s_{\mathrm{RL}} u(t) = 0, & t \in J := (t_0, \infty), \\ \mathfrak{I}_{t_0}^{1-\{s\}} u(t_0) = x_0 \in X, & \\ \mathfrak{D}_{\mathrm{RL}}^{k+\{s\}-1} u(t_0) = x_k \in \overline{\mathsf{D}(A)}, & k \in \{1, \dots, \lfloor s \rfloor\}, \end{cases} \tag{4.5.1}$$

is the function $u_{\mathrm{RL}} \in C(J; X)$ defined by

$$u_{\mathrm{RL}}(t) := \sum_{k=0}^{\lfloor s \rfloor} \frac{(t-t_0)^{k+\{s\}-1}}{\Gamma(k+\{s\})} S(t-t_0) x_k, \quad t \in J. \tag{4.5.2}$$

Compared with Definition 4.4.2, we first note that the terms $k \in \{1, \dots, \lfloor s \rfloor\}$ in (4.5.2) are almost identical to those of (4.4.2), up to the difference between taking Riemann–Liouville type fractional parabolic derivatives of the function u defined on J and Weyl type derivatives of g defined on $\mathbb{R} \setminus J$. The remaining term, on the other hand, differs significantly. In (4.5.1), we see that x_0 is the prescribed value of $\mathfrak{I}_{t_0}^{1-\{s\}} u$ at t_0 and u_{RL} is continuous at t_0 if and only if $x_0 = 0$, in view of the singularity occurring there for $x_0 \neq 0$. In contrast, the solution to (4.4.1) given by Definition 4.4.2 is bounded by Proposition 4.4.3, does in fact prescribe the value $u(t_0) = g(t_0)$ and is continuous on $\mathbb{R}$ under some further regularity assumptions.

The following definition of a Caputo type initial value problem and corresponding solution can also be derived from Proposition 4.5.1:

Definition 4.5.3. Let Assumption 4.2.1(i) be satisfied. For $s \in (0,\infty) \setminus \mathbb{N}$ and $t_0 \in \mathbb{R}$, the mild solution to the Caputo abstract Cauchy problem

$$\begin{cases} \mathfrak{D}^s_{\mathrm{C}} u(t) = 0, & t \in J := (t_0, \infty), \\ u^{(k)}(t_0) = x_k \in \mathsf{D}(A^{\lfloor s \rfloor - k}), & k \in \{0, \dots, \lfloor s \rfloor\}. \end{cases}$$

is the function $u_{\mathrm{C}} \in C(J; X)$ defined by

$$u_{\mathrm{C}}(t) := \sum_{k=0}^{\lfloor s \rfloor} \frac{(t-t_0)^k}{k!} \overline{\Gamma}(\lceil s \rceil - k, (t-t_0)A) x_k, \quad t \in J.$$

Note that this definition has the same form as the integer-order abstract Cauchy problem from Definition 4.4.2, i.e., formula (4.4.3). Analogously, for sufficiently regular x_k or $(S(t))_{t\geq 0}$, this solution allows for the specification of the value of $u_C(t_0)$. However, in contrast to the solution in the sense of Definition 4.4.2, we observe that the form of u_C only changes "discretely in s," i.e., the solutions for any two $s_1, s_2 \in (n, n+1)$, $n \in \mathbb{N}_0$ are given by the same formula.

ACKNOWLEDGMENTS FOR CHAPTER 4

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5

DISCRETE-TO-CONTINUUM LIMITS OF SPDES

The contents of this chapter are based on the preprint [89], which is joint work with Yves van Gennip and Jonas Latz.

5.1. INTRODUCTION TO CHAPTER 5

5.1.1. BACKGROUND AND MOTIVATION

We establish discrete-to-continuum limits of stochastic evolution equations of the form (5.1.1), i.e., semilinear parabolic stochastic partial diffential equations (SPDEs) driven by Gaussian white noise. Such SPDEs of evolution play an important role in the modeling of physical and other systems, such as fluid dynamics [21, 70, 78, 151], quantum optics [43], phase separation [55], diffusion in random media [105, 127], and population dynamics [199]. Given their significance, there has been a considerable interest in the analysis and numerical analysis of SPDEs; see the introductory textbooks [144] and [145], respectively.

We consider the convergence of a sequence of abstract continuous-time equations, each posed on a different Banach space in order to model the approximation of an evolution SPDE by equations that are continuous in time and discrete in space. This framework covers a typical setting where the spatial domains are finite graphs and the limiting differential operator in space is approximated by the corresponding graphical variants. If the finite graphs approximate an underlying manifold, then the graphical differential operators are related to a finite-difference approximation of the differential operator. As an example, we study a class of semilinear SPDEs whose linear part is given by a Whittle–Matérn differential operator on a manifold, discretized by a sequence of graphs constructed from a (possibly randomly sampled) cloud of points. *Linear* equations of this type have previously been studied in the context of statistics and machine learning [158, 181, 182]. By verifying in detail that the hypotheses of our abstract framework are satisfied in this situation, we establish the discrete-to-continuum convergence of this scheme. Although the main advantage of our general results is their applicability to highly unstructured discretizations, we additionally show that they recover the L^{∞}-convergence of finite-difference discretizations of the fractional stochastic Allen–Cahn equation on the one-dimensional flat torus. To further illustrate the significance of results of this form, we now discuss a few examples of semidiscrete SPDEs as well as their continuum limits.

Semidiscrete models appear frequently in the numerical analysis of (S)PDEs of evolution; since the challenges of spatial, temporal and spatiotemporal discretization are different, these settings are often analyzed separately. This leads to thorough studies of (S)PDEs that are discretized in space but not in time. In the context of SPDEs of evolution, we refer to [36, 99, 129] for examples.

Stochastic PDEs on graphs also appear naturally as models in the physical sciences, e.g., for interacting particle systems [47] or the representation of disordered media [105]. In the former case, the continuum limit represents the large particle limit in the interacting particle system.

In the data science literature, (S)PDEs on graphs have recently gained popularity as semi-supervised learning techniques. In a semi-supervised learning problem we are given a set of labeled features as well as a set of unlabeled features, and the goal is to use the former features to recover the labels of the latter. Features are, for example, images, text, or voice recordings; corresponding labels may be descriptors of the content of the images, the author of the text, or a transcript of the voice recording, respectively. Given an appropriate similarity measure on the space of features, an edge-weighted graph can be constructed in which nodes representing similar features are connected by highly weighted edges. The unknown labels can then be estimated by space-discretized PDEs on this graph, as in [23, 38, 87, 203]. The PDEs often describe gradient flows that minimize a variational functional. Stochastic PDEs appear in this setting if, in addition to finding an estimate for the labels, the uncertainty in the labels is to be quantified as well [24, 181]. The SPDEs of evolution here either form the basis of Markov chain Monte Carlo sampling algorithms [48, 101, 102, 174] or of a randomized global optimization scheme [45, 46] for the solution of the variational problem in the deterministic setting. In this semi-supervised learning setting, discrete-to-continuum limits are of interest because they establish the consistency of the models in the large-data limit. For deterministic PDEs, the literature has grown to encompass pointwise limits of operators, as in [106], Γ-limits of the functionals that underlie the dynamics [83, 86, 136, 191, 195], and more recently, discrete-to-continuum limits for the dynamics themselves [71, 75, 91, 111, 135, 153, 194]. For a more in-depth overview of the literature of discrete-to-continuum limits, we refer to [88].

5.1.2. MAIN RESULTS

We will now summarize the abstract setting and main discrete-to-continuum convergence results from Sections 5.4–5.6, which will already be applied in Section 5.3 to the Whittle–Matérn and stochastic fractional Allen–Cahn equations described in Section 5.1.1.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and some $T \in (0,\infty)$ (called the *time horizon*), we consider a sequence of semilinear parabolic stochastic evolution equations

$$\begin{cases} \mathrm{d}X_n(t) = -A_n X_n(t)\,\mathrm{d}t + F_n(t, X_n(t))\,\mathrm{d}t + \mathrm{d}W_n(t), & t \in (0,T], \\ X_n(0) = \xi_n. \end{cases} \tag{5.1.1}$$

indexed by $n \in \overline{\mathbb{N}} := \{1,2,\dots\} \cup \{\infty\}$. This problem will be rigorously formulated as a stochastic differential equation taking values in a real Banach space E_n (or a smaller embedded space $B_n \hookrightarrow E_n$) called the *state space*. In general, we assume that the terms appearing in (5.1.1) are as follows:

- $-A_n$ is the generator of a bounded analytic semigroup of bounded linear operators on E_n or B_n;
- $u_n \mapsto F_n(\omega, t, u_n)$ is a possibly random and nonlinear drift operator on E_n or B_n for all $(\omega, t) \in \Omega \times [0,T]$;
- $(W_n(t))_{t\ge 0}$ is the projection, onto some appropriate subspace, of a cylindrical Wiener process $(W(t))_{t\ge 0}$ taking values in a real and separable Hilbert space H (more details are specified in Theorem 5.1.1 below);
- ξ_n is a possibly random initial datum with values in E_n or B_n.

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The precise assumptions (in particular, whether the operators and initial data are E_n-valued or B_n-valued) vary throughout Sections 5.3–5.6; an overview is provided in Table 5.1. Depending on the setting, the *mild solutions* to (5.1.1) are either well defined on the whole of $[0,T]$ almost surely, or cease to exist at a time $t < T$ with nonzero probability; such solutions are said to be *global* or *local*, respectively.

The aim of this chapter is to establish conditions on the data of (5.1.1) under which the corresponding solutions $(X_n)_{n\in\mathbb{N}}$ converge to X_∞ as $n \to \infty$. In order to compare processes which take their values in different Banach spaces, we need to assume that each of the families $(E_n)_{n\in\overline{\mathbb{N}}}$, $(H_n)_{n\in\overline{\mathbb{N}}}$ and $(B_n)_{n\in\overline{\mathbb{N}}}$ embeds uniformly into a common space—namely into E_∞, H_∞ and a closed subspace $\widetilde{B} \subseteq B_\infty$, respectively—which they approximate in some appropriate sense as $n \to \infty$. In particular, we shall assume that they share a common sequence of (linear) *lifting operators* $(\Lambda_n)_{n\in\overline{\mathbb{N}}}$ such that each Λ_n maps E_n (resp. H_n, B_n) boundedly into E_∞ (resp. H_∞, $\widetilde{B}$), as well as a sequence of *projection operators* $(\Pi_n)_{n\in\overline{\mathbb{N}}}$ which are left-inverses to the respective lifting operators. That is, each sequence satisfies Assumption 5.2.1 below with the same lifting and projection operators. For example (see Section 5.3), one can take $E_\infty := L^q(\mathcal{D})$ for $q \in [2,\infty)$, $H_\infty := L^2(\mathcal{D})$, $\widetilde{B} := L^\infty(\mathcal{D})$ and $B_\infty := C(\mathcal{D})$ for some spatial domain $\mathcal{D}$, along with $E_n := L^q(\mathcal{D}_n)$, $H_n := L^2(\mathcal{D}_n)$ and $B_n := L^\infty(\mathcal{D}_n)$ for some approximations $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of $\mathcal{D}$.

The projection and lifting operators allow us to compare the (E_n- or B_n-valued) solution processes X_n by instead considering convergence of the lifted processes $\widetilde{X}_n := \Lambda_n X_n$ to X_∞ as $n \to \infty$, which we call *discrete-to-continuum convergence*. Moreover, they allow us to formulate assumptions under which this occurs in terms of conditions imposed on the lifted resolvents $\widetilde{R}_n := \Lambda_n(A_n + \mathrm{Id}_n)^{-1}\Pi_n$ of the linear operators A_n, the lifted drift operators $\widetilde{F}_n(\omega, t, u) := \Lambda_n F_n(t, \omega, \Pi_n u)$, and the lifted initial data $\widetilde{\xi}_n := \Lambda_n \xi_n$. Roughly speaking, we assume that

- $\widetilde{F}_n \to F_\infty$ 'pointwise' (see (F2) in Section 5.5.1 or (F2-B) in Section 5.6.1);

- $\widetilde{R}_n \to R_\infty$ 'pointwise' and there exists a small enough $\beta \in [0, \frac{1}{2})$ such that the fractional powers $\widetilde{R}_n^\beta$ converge to R_∞^β in an appropriate operator norm (see (A3) in Section 5.4 or (A3-B) in Section 5.6.1)
- $\widetilde{\xi}_n \to \xi_\infty$ in $L^p(\Omega; E_\infty)$ or $L^p(\Omega; \widetilde{B})$ for some $p \in [1, \infty)$ (see (IC) in Section 5.5.1 or (IC-B) in Section 5.6.1).

Again we refer to Table 5.1 for an overview of the different settings and types of solutions, with references to the precise formulations of the corresponding assumptions; the setting in the first row (i.e., Section 5.3.2) covers the (fractional) stochastic Allen–Cahn equations announced in Section 5.1.1, see Example 5.3.8 below. The following theorem is a summary of the discrete-to-continuum approximation results for solutions to the abstract equations (5.1.1) in these respective settings.

Theorem 5.1.1 (Discrete-to-continuum convergence—summarized)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $T \in (0, \infty)$ be a terminal time. Consider equations* (5.1.1)*, where the state spaces, linear operators, drift operators and initial data are as in one of the rows of Table 5.1. Let $p \in [1, \infty)$ be the stochastic integrability of the initial data (i.e., the exponent in (IC) or (IC-B)), and let $W_n := \Pi_n W$, where $(W(t))_{t\geq 0}$ is an H-valued cylindrical Wiener process. For all $n \in \overline{\mathbb{N}}$, there exists a unique (local or global, see Table 5.1) mild solution X_n to* (5.1.1)*, and the lifted solution processes $\widetilde{X}_n := \Lambda_n X_n$ satisfy the following:*

(i) If the solutions are global and $p > 1$, then for all $p^- \in [1, p)$ we have

$$\widetilde{X}_n \to X_\infty \quad \text{as } n \to \infty$$

in $L^{p^-}(\Omega; C([0, T]; E_\infty))$ (resp. in $L^{p^-}(\Omega; C([0, T]; \widetilde{B}))$).

In the (semi)linear settings with globally Lipschitz drifts of linear growth, the same in fact holds with $p^- := p$ for any $p \in [1, \infty)$.

(ii) If the solutions are local, with associated explosion times $\sigma_n \colon \Omega \to (0, T]$ (precisely defined in (5.5.9) *below), then we have*

$$\widetilde{X}_n \mathbf{1}_{[0, \sigma_\infty \wedge \sigma_n)} \to X_\infty \mathbf{1}_{[0, \sigma_\infty)} \quad \text{as } n \to \infty$$

in $L^0(\Omega \times [0, T]; E_\infty)$ (resp. in $L^0(\Omega \times [0, T]; \widetilde{B})$), where L^0 indicates convergence in measure.

The full convergence statement for each setting is given in the corresponding part of Sections 5.4–5.6. To be precise, the results comprising Theorem 5.1.1 are: Theorem 5.3.10, Proposition 5.4.5, Theorems 5.5.4 and 5.5.9, Proposition 5.6.1, Theorems 5.6.6 and 5.6.7 and Corollary 5.6.10.

5.1.3. CONTRIBUTIONS

The abstract discrete-to-continuum approximation theorems for stochastic semilinear parabolic evolution equations driven by additive cylindrical Wiener noise—summarized in Theorem 5.1.1 and proved in Sections 5.4–5.6—complement the results from [131, 132], which establish the *continuous dependence on the coefficients*

Section	Description	Assumptions	Sol. type
§5.3.2	graph-based approximation of Whittle–Matérn operators on a manifold	• $A_n := [\mathcal{L}_n^{\tau,\kappa}]^s$ (Whittle–Matérn operators) • $[F_n(t,u)](x) := f_n(t,u(x))$ (Nemytskii drift) • Assumption 5.3.7 (on the functions $(f_n)_{n\in\overline{\mathbb{N}}}$)	global
§5.4	E_n-valued linear	• (A1)–(A3) (linear operators) • $F_n := 0$ and $\xi_n := 0$	global
§5.5.1	E_n-valued semilinear; globally Lipschitz drifts of linear growth	• (A1)–(A3) (linear operators) • (F1)–(F2) (drift operators) • (IC) (initial data)	global
§5.5.2	E_n-valued semilinear; locally Lipschitz and locally bounded drifts	• (A1)–(A3) (linear operators) • (F1′) and (F2) (drift operators) • (IC) (initial data)	local
§5.6.1	B_n-valued semilinear; globally Lipschitz drifts of linear growth	• (A1-B)–(A4-B) with $\theta + 2\beta < 1$ (lin. ops.) • (F1-B)–(F2-B) (drift operators) • (IC-B) (initial data)	global
§5.6.2	B_n-valued semilinear; locally Lipschitz and locally bounded drifts	• (A1-B)–(A4-B) with $\theta + 2\beta < 1$ (lin. ops.) • (F1′-B) and (F2-B) (drift operators) • (IC-B) (initial data)	local
§5.6.3	B_n-valued semilinear; dissipative drifts	• (A1-B)–(A4-B) with $\theta + 2\beta < 1$ (lin. ops.) • (F1″-B) and (F2-B) (drift operators) • (IC-B) (initial data)	global

Table 5.1: Overview of the types of evolution equations considered in the different (sub)sections comprising this chapter. The row in which an assumption appears for the first time also indicates the (sub)section where its definition can be found.

of semilinear equations driven by multiplicative noise in a state space with unconditional martingale differences (UMD), i.e., convergence in the case where $E_n = E$ (and $B_n = B$) for all $n \in \overline{\mathbb{N}}$. Given the motivating applications and our aim to provide a self-contained exposition of the proofs, we make the simplifying assumptions that the UMD spaces $(E_n)_{n\in\overline{\mathbb{N}}}$ have Rademacher type 2 (which was also assumed for E in [132] but not in [131]) and the noise is additive.

Under these conditions, we provide a direct proof of convergence of the E_n-valued stochastic convolutions solving the linear parts of (5.1.1) using a factorization argument in the sense of Da Prato, Kwapień and Zabczyk [53], along with a discrete-to-continuum analog of the Trotter–Kato approximation theorem [116, Theorem 2.1], see Proposition 5.4.5. We extend it to the semilinear E_n-valued settings described in Table 5.1 by adapting the arguments from [131, Sections 3 and 4] and [132, Subsection 3.1] to incorporate the discrete-to-continuum projection and lifting operators, yielding Theorems 5.5.4 and 5.5.9, respectively. In order to state and prove the analogous Theorems 5.6.6 and 5.6.7 for the B_n-valued settings, we impose a uniform ultracontractivity condition on the semigroups which replaces the restriction in [132, Section 3] that the fractional domain spaces $\dot{E}_n^\alpha := \mathsf{D}((\mathrm{Id}_n + A_n)^{\alpha/2})$ also coincide for all $n \in \overline{\mathbb{N}}$.

Theorem 5.3.10, regarding the graph discretization of equations whose linear operators are of generalized Whittle–Matérn type on a manifold $\mathcal{M}$, extends analogous convergence results for linear equations on a spatial domain (cf. [181, Theorem 4.2] and [181, Theorem 7] in $L^2(\mathcal{M})$ and $L^\infty(\mathcal{M})$, respectively) to spatiotemporal and semilinear equations. Like the cited theorems, its proof relies on recent spectral convergence results for graph Laplacians (see [40] and [41] for convergence of eigenfunctions in L^2 and L^∞, respectively), which we use to verify that these SPDEs fit into the abstract framework from Sections 5.4–5.6.

5.1.4. OUTLINE

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The remainder of this chapter is structured as follows. In Section 5.2, we establish some notational conventions and collect preliminaries regarding the (deterministic) discrete-to-continuum Trotter–Kato approximation theorem and stochastic integration in UMD-type-2 Banach spaces. We demonstrate in Section 5.3 how the results summarized by Theorem 5.1.1 can be applied to graph discretizations of stochastic parabolic evolution equations whose linear part is a generalized Whittle–Matérn operator on a manifold. In Section 5.4, we consider the *linear* E_n-valued version of (5.1.1), whose solutions are known as infinite-dimensional Ornstein–Uhlenbeck processes. These results are extended in Section 5.5 to allow for *semilinear* E_n-valued drift operators under (local or global) Lipschitz continuity and boundedness assumptions. In Section 5.6 we first treat the analogous results in the semilinear B_n-*valued* setting, and then establish global well-posedness and convergence for dissipative drifts. Finally, in Section 5.7 we discuss some potential directions for further research. This chapter is complemented by three appendices: Appendix 5.A consists of postponed proofs of some intermediate results from Section 5.3. Appendices 5.B and 5.C are concerned with fractional parabolic integration and (uniformly) sectorial linear operators, respectively. See Figure 5.1 for a schematic overview of the relations between Sections 5.3–5.6 and the appendices.

5.2. PRELIMINARIES FOR CHAPTER 5

5.2.1. NOTATION

In this section we only highlight notation which deviates from the previous chapters or was not used there.

The Cartesian product $\prod_{j\in\mathcal{I}} B_j$ of an indexed family of sets $(B_j)_{j\in\mathcal{I}}$ is comprised of all functions $f : \mathcal{I} \to \bigcup_{j\in\mathcal{I}} B_j$ satisfying $f(j) \in B_j$ for all $j \in \mathcal{I}$. We call $T \in \mathscr{L}(E;F)$ a contraction if $\|T\|_{\mathscr{L}(E;F)} \le 1$; in particular, the inequality need not be strict. The Banach space of (bounded) continuous functions $u : J \to E$, endowed with the supremum norm, is denoted by $C(J;E)$.

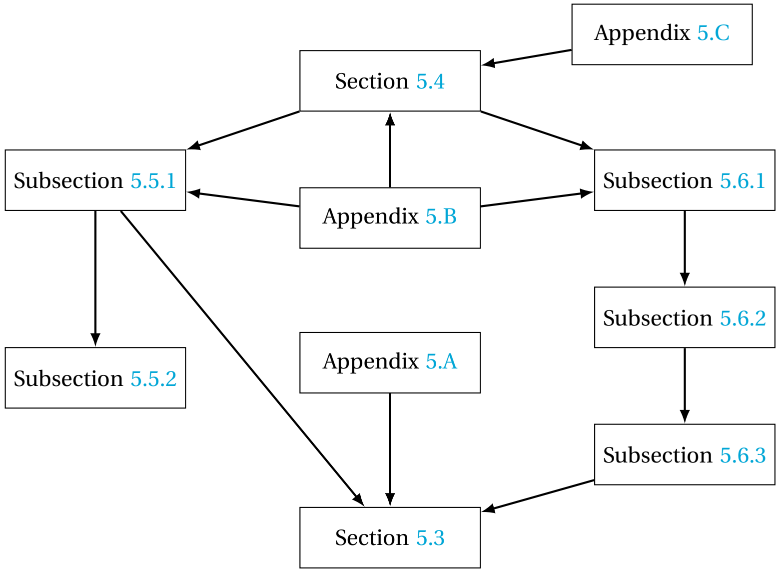

Figure 5.1: Relations between the sections comprising the main part of this chapter and the appendices. Arrows indicate when the results of one section are applied in another.

The meaning of the tensor symbol $\otimes$ will depend on the context: Given a function $\Phi\colon (a,b) \to \mathscr{L}(E;F)$ and $x \in E$, we define $\Phi \otimes x\colon (a,b) \to F$ by $[\Phi \otimes x](t) := \Phi(t)x$. If instead an $h \in H$ is given, we define $h \otimes x \in \mathscr{L}(H;E)$ to be the rank-one operator $[h \otimes x](u) := \langle h, u\rangle_H x$. The space of all (finite) linear combinations of such operators is denoted by $H \otimes E$. We define the convolution $\Psi * f\colon [0,T] \to F$ of the functions $\Psi\colon [0,T] \to \mathscr{L}(E;F)$ and $f\colon [0,T] \to E$ by $[\Psi * f](t) := \int_0^t \Psi(t-s) f(s)\,\mathrm{d}s$.

Table 5.2 (see the next page) lists some notation which is frequently used throughout this chapter. Some of these notional conventions were established in the present section; others will be defined in later sections.

5.2.2. DISCRETE-TO-CONTINUUM TROTTER–KATO APPROXIMATION

We encode the discrete-to-continuum setting in the following way:

Assumption 5.2.1. Let $(E_n, \|\cdot\|_{E_n})_{n\in\overline{\mathbb{N}}}$ and $(\widetilde{E}, \|\cdot\|_{\widetilde{E}})$ be real or complex Banach spaces and suppose that E_∞ is a closed linear subspace of $\widetilde{E}$. We assume that there exist operators $\Pi_n \in \mathscr{L}(\widetilde{E}; E_n)$ and $\Lambda_n \in \mathscr{L}(E_n; \widetilde{E})$ for all $n \in \mathbb{N}$ which satisfy

(i) $M_\Pi := \sup_{n\in\mathbb{N}} \|\Pi_n\|_{\mathscr{L}(\widetilde{E};E_n)} < \infty$ and $M_\Lambda := \sup_{n\in\mathbb{N}} \|\Lambda_n\|_{\mathscr{L}(E_n;\widetilde{E})} < \infty$;

(ii) $\Lambda_n \Pi_n x \to x$ in $\widetilde{E}$ as $n \to \infty$ for all $x \in E_\infty$;

(iii) $\Pi_n \Lambda_n = \mathrm{Id}_{E_n}$ for all $n \in \mathbb{N}$.

In addition, we denote $\Pi_\infty = \Lambda_\infty := \mathrm{Id}_{\widetilde{E}}$ for convenience.

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Elementary sets and operations

$\mathbb{N}$	positive integers
$\mathbb{N}_0$	nonnegative integers
$\overline{\mathbb{N}}$	$\mathbb{N}\cup\{\infty\}$
Id_D	identity map on a set D
$\mathbf{1}_{D_0}$	indicator map on $D_0\subseteq D$
$s\wedge t$	minimum of $s,t\in\mathbb{R}$

Bounded linear operators

H, K	separable Hilbert spaces
E, F	arbitrary Banach spaces
$\langle\cdot,\cdot\rangle_H$	inner product of H
$\lVert\cdot\rVert_E$	norm of E
E^*	dual space of E
$\mathscr{L}(E;F)$	bounded linear operators
$\gamma(H;E)$	γ-radonifying operators
$\mathscr{L}_2(H;K)$	Hilbert–Schmidt operators

Function spaces

$C(K;E)$	continuous functions from a compact space K to E
$C(K)$	abbreviation for $C(K;\mathbb{R})$
$L^p(S;E)$	Bochner space of p-integrable functions from a measure space $(S,\mathscr{A},\nu)$ to E
$L^p(S)$	Lebesgue space $L^p(S;\mathbb{R})$

Graph discretization

$\mathcal{M}$	manifold from Assumption 5.3.1
$\mathbb{T}^m$	m-dimensional flat torus
$d_{\mathcal{M}}$	geodesic metric on $\mathcal{M}$
μ	volume measure on $\mathcal{M}$
$\mathcal{M}_n$	point cloud $(x_n^{(j)})_{j=1}^n\subset\mathcal{M}$
$(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$	probability space of random point cloud from Example 5.3.3
μ_n	empirical measure on $\mathcal{M}_n$
T_n	transport map from $\mathcal{M}$ to $\mathcal{M}_n$
ε_n	$\sup_{x\in\mathcal{M}} d_{\mathcal{M}}(x,T_n(x))$
h_n	graph connectivity length scale, see (5.3.6)
$\mathcal{L}_n^{\tau,\kappa}$	(discretized) Whittle–Matérn operator with coefficient functions $\tau,\kappa\colon\mathcal{M}\to[0,\infty)$
$(\lambda_n^{(j)})_{j=1}^n$	eigenvalues of $\mathcal{L}_n^{\tau,\kappa}$
$(\psi_n^{(j)})_{j=1}^n$	$L^2(\mathcal{M}_n)$-normalized eigenfunctions of $\mathcal{L}_n^{\tau,\kappa}$
$M_{\psi,\infty}$	uniform L^∞-bound of the eigenfunctions, see Assumption 5.3.9
$M_{S,q}$	uniform-ultracontractivity constant, see (5.3.18)

'Discrete-to-continuum' spaces

$(E_n)_{n\in\overline{\mathbb{N}}}$, $\widetilde{E}$	Banach spaces from Assumption 5.2.1 or (A1)
$(H_n)_{n\in\overline{\mathbb{N}}}$, $\widetilde{H}$	Hilbert spaces from (A1)
$(B_n)_{n\in\overline{\mathbb{N}}}$, $\widetilde{B}$	Banach spaces from (A1-B)

Projection and lifting

Π_n	projection operator from E_n (resp. H_n or B_n) to $\widetilde{E}$ (resp. $\widetilde{H}$ or $\widetilde{B}$)
Λ_n	lifting operator from $\widetilde{E}$ (resp. $\widetilde{H}$ or $\widetilde{B}$) to E_n (resp. H_n or B_n)
$\widetilde{T}_n$	lifted version $\Lambda_n T_n \Pi_n$ on $\widetilde{E}$ (resp. $\widetilde{H}$ or $\widetilde{B}$) of operator T_n on E_n (resp. H_n or B_n)
$\widetilde{Y}_n$	lifted version $\Lambda_n Y_n$ on $\widetilde{E}$ (resp. $\widetilde{H}$ or $\widetilde{B}$) of process Y_n on E_n (resp. H_n or B_n)
M_Π	$\sup_{n\in\mathbb{N}}\lVert\Pi_n\rVert_{\mathscr{L}(\widetilde{E};E_n)}$
$\widetilde{M}_\Pi$	$\sup_{n\in\mathbb{N}}\lVert\Pi_n\rVert_{\mathscr{L}(\widetilde{B};B_n)}$
M_Λ	$\sup_{n\in\mathbb{N}}\lVert\Lambda_n\rVert_{\mathscr{L}(E_n;\widetilde{E})}$
$\widetilde{M}_\Lambda$	$\sup_{n\in\mathbb{N}}\lVert\Lambda_n\rVert_{\mathscr{L}(B_n;\widetilde{B})}$

Linear operators in evolution equations

A_n	linear operator on E_n with domain $\mathsf{D}(A_n)$
S_n	semigroup generated by $-A_n$
M_S	uniform-boundedness constant of $(S_n)_{n\in\overline{\mathbb{N}}}$ in $(E_n)_{n\in\overline{\mathbb{N}}}$, see (5.2.1)
$\widetilde{M}_S$	uniform-boundedness constant of $(S_n)_{n\in\overline{\mathbb{N}}}$ in $(B_n)_{n\in\overline{\mathbb{N}}}$, see (5.6.3)
$\rho(A_n)$	resolvent set of A_n
R_n^β	$(A_n+\mathrm{Id}_n)^{-\beta}$
$\mathfrak{I}_{A_n}^s$	fractional parabolic integration operator, see Appendix 5.B

Stochastic evolution equations

Q_n	covariance $\Pi_n\Pi_n^*\in\mathscr{L}(H_n)$
$\mathrm{d}W_n$	H_n-valued Q_n-cylindrical Wiener noise on $(\Omega,\mathcal{F},\mathbb{P})$
W_{A_n}	stochastic convolution, see (5.4.2)
ξ_n	initial datum
F_n	drift operator
L_F, C_F	Lipschitz and growth constants of E_n-valued drifts, see (F1)
$\widetilde{L}_F$, $\widetilde{C}_F$	Lipschitz and growth constants of B_n-valued drifts, see (F1-B)
$L_F^{(r)}$, $C_{F,0}$	local Lipschitz and growth constants of E_n-valued drifts, see (F1′)
f_n	real-valued drift coefficient function
$\widetilde{L}_f$, $\widetilde{C}_f$	Lipschitz and growth constants of $(f_n)_{n\in\overline{\mathbb{N}}}$ see Assumption 5.3.7(i)

Table 5.2: A selection of notation which is used throughout this chapter.

Note that parts (i) and (iii) together imply that the lifting operators are continuous embeddings $\Lambda_n : E_n \hookrightarrow \widetilde{E}$. In applications, they will typically be nested (in the sense that $E_n \hookrightarrow E_{n+1}$ for all $n \in \mathbb{N}$) and finite-dimensional, but neither of these assumptions is strictly necessary in the abstract theory. Moreover, we will often have $\widetilde{E} = E_\infty$, but not always; see Section 5.6.

Now we consider the following sequence of linear operators on $(E_n)_{n\in\overline{\mathbb{N}}}$:

Assumption 5.2.2. For all $n \in \overline{\mathbb{N}}$, let $-A_n : \mathsf{D}(A_n) \subseteq E_n \to E_n$ be a linear operator generating a strongly continuous semigroup $(S_n(t))_{t\geq 0} \subseteq \mathscr{L}(E_n)$. Let $M_S \in [1,\infty)$ and $w \in \mathbb{R}$ be such that

$$\|S_n(t)\|_{\mathscr{L}(E_n)} \leq M_S e^{-wt} \quad \text{for all } n \in \overline{\mathbb{N}} \text{ and } t \in [0,\infty). \tag{5.2.1}$$

Given a sequence $(A_n)_{n\in\overline{\mathbb{N}}}$ of operators generating C_0-semigroups with uniform growth bounds on a single common Banach space $E_n := E$, the Trotter–Kato approximation theorem (see, e.g., [73, Chapter III, Theorem 4.8]) establishes a link between the strong convergence of resolvents and uniform convergence of the semigroups on compact subintervals of $[0,\infty)$. The following discrete-to-continuum analog of this result was proved by Ito and Kappel [116, Theorem 2.1]:

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Theorem 5.2.3 (Discrete-to-continuum Trotter–Kato approximation)**.** *Let Assumptions 5.2.1 and 5.2.2 be satisfied, with $w \in \mathbb{R}$. The following are equivalent:*

(a) There exists a $\lambda \in \bigcap_{n\in\overline{\mathbb{N}}} \rho(A_n)$ such that, for every $x \in E_\infty$,

$$\Lambda_n R(\lambda, A_n)\Pi_n x \to R(\lambda, A_\infty)x \quad \text{in } \widetilde{E} \quad \text{as } n \to \infty.$$

(b) For all $x \in E_\infty$ and $T \in (0,\infty)$ it holds that

$$\Lambda_n S_n \Pi_n \otimes x \to S_\infty \otimes x \quad \text{in } C([0,T];\widetilde{E}) \quad \text{as } n \to \infty.$$

If (a) holds for some $\lambda \in \bigcap_{n\in\overline{\mathbb{N}}} \rho(A_n)$ (or, equivalently, if (b) holds), then (a) holds in fact for every *$\lambda \in \mathbb{C}$ such that* $\operatorname{Re}\lambda < w$.

5.2.3. STOCHASTIC INTEGRATION IN UMD-TYPE-2 BANACH SPACES

Given a real and separable Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$, let $(W(t))_{t\geq 0}$ be an H-valued cylindrical Wiener process on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$ which we fix throughout the chapter; see the last subsection of Section 1.4.3. Let $(E, \|\cdot\|_E)$ be a real Banach space, and let $(\gamma_j)_{j\in\mathbb{N}}$ be a sequence of independent (real-valued) standard normal random variables on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, independent of the spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ used in the rest of this chapter. We define the space $\gamma(H;E)$ of *γ-radonifying operators from H to E* as the completion of the finite-rank operators $H \otimes E$ with respect to the norm $\|\sum_{j=1}^n h_j \otimes x_j\|_{\gamma(H;E)} := \|\sum_{j=1}^n \gamma_j x_j\|_{L^2(\Omega';E)}$, where we assume that the $(h_j)_{j=1}^n$ are H-orthonormal. This norm is well-defined, i.e., it can be checked that the right-hand side is independent of the choice of representation. An important feature of $\gamma(H;E)$ is its *ideal property* (in the algebraic sense) [114, Theorem 9.1.10], which states that for $T \in \gamma(H;E)$, $U \in \mathscr{L}(E;F)$ and $S \in \mathscr{L}(K;H)$, we have

$$UTS \in \gamma(K;F) \quad \text{with} \quad \|UTS\|_{\gamma(K;F)} \le \|U\|_{\mathscr{L}(E;F)} \|T\|_{\gamma(H;E)} \|S\|_{\mathscr{L}(K;H)}. \tag{5.2.2}$$

For any rank-one operator $h \otimes x \in \mathscr{L}(H;E)$, we have $h \otimes x \in \gamma(H;E)$ with

$$\|h \otimes x\|_{\gamma(H;E)} = \|h\|_H \|x\|_E. \tag{5.2.3}$$

The stochastic integral of an elementary integrand $\Phi\colon (0,\infty) \to H \otimes E$ is defined by (1.4.13). In order to extend the definition of the stochastic integral beyond elementary integrands, one needs to impose further geometric assumptions on the Banach space E. In this chapter we work in one of the standard settings, namely that of spaces with *unconditional martingale differences* and *Rademacher type* 2 (abbreviated to *UMD-type-2*). Definitions of these notions can be found in [113, Section 4.2] and [114, Section 7.1], respectively, but we will only use them to ensure existence of stochastic integrals. In this case, one can establish the *Itô inequality*

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$$\left\| \int_0^\infty \Phi(t) \, \mathrm{d}W(t) \right\|_{L^2(\Omega;E)} \lesssim_E \|\Phi\|_{L^2(0,\infty;\gamma(H;E))} \tag{5.2.4}$$

for elementary integrands [157, Proposition 4.2], and use it to extend the definition of the stochastic integral to all $\Phi \in L^2(0,\infty;\gamma(H;E))$. In fact, recall from (1.4.14) that the type 2 assumption suffices since we exclusively deal with deterministic integrands. Despite this, we additionally impose the UMD assumption for the sake of compatibility with some of the literature, and because the concrete examples of Banach spaces in which we are interested (such as the Lebesgue L^q-spaces for $q \in [2,\infty)$) satisfy both properties.

The exponent 2 in L^2 appearing on both sides of (5.2.4) can be replaced by any other $p \in [1,\infty)$ at the cost of a p-dependent constant, see for instance [157, Theorem 4.7]. If E is also a Hilbert space, then $\gamma(H;E)$ is isometrically isomorphic to the space $\mathscr{L}_2(H;E)$, see [114, Proposition 9.1.9], and instead of the inequality (5.2.4) we have the Itô *isometry* between $L^2(0,\infty;\mathscr{L}_2(H;E))$ and $L^2(\Omega;E)$, see (1.4.15).

5.3. GRAPH-DISCRETIZED SEMILINEAR EVOLUTION EQUATIONS WITH WHITTLE–MATÉRN LINEAR PART

Before developing the general discrete-to-continuum convergence results summarized by Theorem 5.1.1 in the upcoming sections, in this section we demonstrate how they can be applied to the particular case of equations whose linear parts are graph discretizations of a generalized Whittle–Matérn operator on a manifold. In the spatial and linear case, such convergence results have been proven in [181, 182]. We also mention the work [158], in which the statistical properties of the spatiotemporal *linear* equation were investigated for fixed $n \in \mathbb{N}$.

5.3.1. GEOMETRIC GRAPHS AND WHITTLE–MATÉRN OPERATORS

Assumption 5.3.1 (Manifold assumption)**.** Given $m, d \in \mathbb{N}$, suppose that $\mathcal{M}$ is an m-dimensional smooth, connected, compact Riemannian manifold without boundary, embedded smoothly and isometrically into $\mathbb{R}^d$. Let μ and $d_{\mathcal{M}}$ denote the normalized volume measure and geodesic metric on $\mathcal{M}$, respectively.

For each $n \in \mathbb{N}$, let a point cloud $\mathcal{M}_n := (x_n^{(j)})_{j=1}^n \subseteq \mathcal{M}$ be given. We suppose that $\mathcal{M}$ can be partitioned into n regions of mass $1/n$, which can be transported to the corresponding n points comprising $\mathcal{M}_n$, in such a way that the maximal geodetic displacement tends to zero as $n \to \infty$. More precisely, we assume that there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of *transport maps* $T_n \colon \mathcal{M} \to \mathcal{M}_n$ such that

$$\mu_n = T_{n\sharp}\mu \qquad \text{for all } n \in \mathbb{N}, \quad \text{and} \tag{5.3.1}$$

$$\varepsilon_n := \sup_{x\in\mathcal{M}} d_{\mathcal{M}}(x, T_n(x)) \to 0 \qquad \text{as } n \to \infty. \tag{5.3.2}$$

Here, $\mu_n := \frac{1}{n}\sum_{j=1}^n \delta_{x_n^{(j)}}$ is the empirical measure on $\mathcal{M}$ associated to $\mathcal{M}_n$, and $T_{n\sharp}\mu$ denotes the pushforward measure $T_{n\sharp}\mu(B) := \mu(\{T_n \in B\})$ on $\mathcal{M}_n$. Two examples in which this assumption is satisfied are presented in Settings 5.3.2 and 5.3.3 below.

Given $u_n \colon \mathcal{M}_n \to \mathbb{R}$ for $n \in \overline{\mathbb{N}}$, these transport maps enable us to define the functions $\Lambda_u u_n \colon \mathcal{M} \to \mathbb{R}$ and $\Pi_n u \colon \mathcal{M}_n \to \mathbb{R}$ by setting

$$\Lambda_n u_n(x) := u_n(T_n(x)) \quad \text{and} \quad \Pi_n u(x_n^{(j)}) := n \int_{V_n^{(j)}} u(x)\,\mathrm{d}\mu(x), \tag{5.3.3}$$

respectively, for all $x \in \mathcal{M}$ and $j \in \{1, \dots, n\}$, where $V_n^{(j)} := \{T_n = x_n^{(j)}\} \subseteq \mathcal{M}$.

It turns out that the operations defined in (5.3.3) satisfy Assumption 5.2.1 with respect to the following function spaces: Given $q \in [1, \infty]$, we set

$$E_n := L^q(\mathcal{M}_n) := L^q(\mathcal{M}, \mu_n), \quad n \in \mathbb{N},$$

as well as $\widetilde{E} := L^q(\mathcal{M})$ and

$$E_\infty := \begin{cases} L^q(\mathcal{M}), & \text{if } q \in [1, \infty); \\ C(\mathcal{M}), & \text{if } q = \infty. \end{cases}$$

Later on, we will need $q \in [2, \infty)$, so that E_n is a UMD-type-2 space for use in stochastic integration, but the statements here hold for all $q \in [1, \infty]$.

For these spaces, Assumption 5.2.1(i) is satisfied with $M_\Lambda = 1$ and $M_\Pi \le 1$. Indeed, the fact that Λ_n is an isometry follows from (5.3.1) if $q \in [1, \infty)$, whereas for $q = \infty$ we see directly from the definition that

$$\|\Lambda_n u_n\|_{L^\infty(\mathcal{M})} = \sup_{x\in\mathcal{M}} |u_n(T_n(x))| = \max_{j=1}^{n} |u_n(x_n^{(j)})| = \|u_n\|_{L^\infty(\mathcal{M}_n)}.$$

To show that Π_n is a contraction, we first apply Hölder's inequality in (5.3.3) with $\frac{1}{q} + \frac{1}{q'} = 1$ to find $|\Pi_n u(x_n^{(j)})| \le n\|u\|_{L^q(V_n^{(j)})}\mu(V_n^{(j)})^{\frac{1}{q'}} = n^{\frac{1}{q}}\|u\|_{L^q(V_n^{(j)})}$, so that

$$\|\Pi_n u\|_{L^q(\mathcal{M}_n)}^q = \frac{1}{n}\sum_{j=1}^n |\Pi_n u(x_n^{(j)})|^q \le \sum_{j=1}^n \|u\|_{L^q(V_n^{(j)},\mu)}^q = \|u\|_{L^q(\mathcal{M})}^q.$$

Assumption 5.2.1(ii) is a consequence of (5.3.2), which implies that $\Lambda_n \Pi_n u \to u$ in $L^q(\mathcal{M})$ for any $u \in C(\mathcal{M})$. For $q = \infty$, this is what we wanted to show; if instead $q \in [1, \infty)$, then $C(\mathcal{M})$ is dense in $L^q(\mathcal{M})$, and the fact that $(\Lambda_n \Pi_n)_{n\in\mathbb{N}}$ is uniformly bounded in $\mathscr{L}(L^q(\mathcal{M}))$ by Assumption 5.2.1(i) which we have just proven to hold, yields $\Lambda_n \Pi_n u \to u$ for all $u \in L^q(\mathcal{M})$ as desired. Assumption 5.2.1(iii) can be verified via direct computation using the definitions. Finally, we have

$$\begin{aligned}\int_{\mathcal{M}} \Lambda_n u_n(x) v(x)\,\mathrm{d}\mu(x) &= \int_{\mathcal{M}} u_n(T_n(x)) v(x)\,\mathrm{d}\mu(x) = \sum_{j=1}^{n} \int_{V_n^{(j)}} u_n(x_n^{(j)}) v(x)\,\mathrm{d}\mu(x) \\ &= \frac{1}{n}\sum_{j=1}^{n} u_n(x_n^{(j)}) \Pi_n v(x_n^{(j)}) = \int_{\mathcal{M}_n} u_n(x) \Pi_n v(x)\,\mathrm{d}\mu_n(x). \end{aligned} \tag{5.3.4}$$

which shows that the adjoint of $\Pi_n \in \mathscr{L}(L^q(\mathcal{M}); L^q(\mathcal{M}_n))$, where $q \in [1,\infty)$, is given by $\Pi_n^* = \Lambda_n \in \mathscr{L}(L^{q'}(\mathcal{M}_n); L^{q'}(\mathcal{M}))$.

The concrete choices of $\mathcal{M}$ and their discretizations which we will consider in this section are the following two:

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Setting 5.3.2 (Square grid on $\mathbb{T}^m$)**.** Let $\mathcal{M} := \mathbb{T}^m$ be the m-dimensional flat torus, which we view as the cube $[0,1]^m$ endowed with periodic boundary conditions. For notational convenience, we will index our sequence of discretizations of $\mathbb{T}^m$ only by the natural numbers n such that $n^{1/m} \in \mathbb{N}$, for which we define the following square equidistant grid with mesh size $h_n := n^{-1/m}$:

$$\mathcal{M}_n := \{\tfrac{1}{2} n^{-1/m}, \tfrac{3}{2} n^{-1/m}, \dots, 1 - \tfrac{1}{2} n^{-1/m}\}^m.$$

Then the grid points of $\mathcal{M}_n$ can be written as $\mathbf{x}_n^{(\mathbf{j})} = n^{-1/m}(\mathbf{j} - \frac{1}{2}\mathbf{1})$ for some m-tuple $\mathbf{j} \in \{1, \dots, n^{1/m}\}^m$, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^m$. To each of these points $\mathbf{x}_n^{(\mathbf{j})} \in \mathcal{M}_n$, we associate the half-open cube $U_n^{(\mathbf{j})} := \prod_{k=1}^m \big[n^{-1/m}(j_k - 1), n^{-1/m} j_k\big)$. Since these cubes form a partition of $\mathcal{M}$ (recalling that opposite sides are identified), we can define the transport map $T_n \colon \mathcal{M} \to \mathcal{M}_n$ by $T_n(x) := \mathbf{x}_n^{(\mathbf{j})}$ whenever $x \in U_n^{(\mathbf{j})}$. It readily follows that (5.3.1) holds, as does (5.3.2), since $\varepsilon_n = \frac{1}{2}\sqrt{m}\, n^{-1/m}$ for all $n \in \mathbb{N}$.

Setting 5.3.3 (Randomly sampled point cloud)**.** Let $\mathcal{M}$ be any manifold satisfying Assumption 5.3.1, and let $(x^{(n)})_{n\in\mathbb{N}} \subseteq \mathcal{M}$ be a sequence of points independently sampled from μ. This sequence can be viewed as a sample from the probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}) := \prod_{n\in\mathbb{N}} (\mathcal{M}, \mathcal{B}(\mathcal{M}), \mu)$, where $\mathcal{B}(\mathcal{M})$ is the Borel σ-algebra on $\mathcal{M}$. If we set $\mathcal{M}_n := (x^{(j)})_{j=1}^n$ for all $n \in \mathbb{N}$, then [181, Proposition 4.1] states that, $\widetilde{\mathbb{P}}$-a.s., there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of transport maps $T_n \colon \mathcal{M} \to \mathcal{M}_n$ for which (5.3.1) holds and we have

$$\varepsilon_n \lesssim_{\mathcal{M}} (\log n)^{c_m} n^{-1/m} \to 0 \quad \text{as } n \to \infty,$$

where $c_m = \frac{3}{4}$ if $m = 2$ and $c_m = \frac{1}{m}$ otherwise.

We will now introduce the linear operators which we consider on the domains $\mathcal{M}$ and $(\mathcal{M}_n)_{n\in\mathbb{N}}$ as in Settings 5.3.2 and 5.3.3. Given the coefficient functions

$$\tau \colon \mathcal{M} \to [0,\infty) \quad \text{and} \quad \kappa \colon \mathcal{M} \to [0,\infty),$$

respectively assumed to be Lipschitz and continuously differentiable, we consider the nonnegative and symmetric second-order linear differential operator $\mathcal{L}_\infty^{\tau,\kappa}$ formally defined by

$$\mathcal{L}_\infty^{\tau,\kappa} u := \tau u - \nabla \cdot (\kappa \nabla u), \tag{5.3.5}$$

for u belonging to some appropriate domain $\mathsf{D}(\mathcal{L}_\infty^{\tau,\kappa}) \subseteq L^2(\mathcal{M})$, see also Section 1.2.4.

For each $n \in \mathbb{N}$, we endow $\mathcal{M}_n$ with a (weighted, undirected) graph structure by viewing its points as vertices and defining the weight matrix $\mathbf{W}_n \in \mathbb{R}^{n\times n}$ by

$$(\mathbf{W}_n)_{ij} := \frac{2(m+2)}{\nu_m} \frac{1}{n h_n^{m+2}} \mathbf{1}_{[0,h_n]}\big(\|x_n^{(i)} - x_n^{(j)}\|_{\mathbb{R}^d}\big), \tag{5.3.6}$$

where ν_m denotes the volume of the unit sphere in $\mathbb{R}^m$ and $h_n \in (0,\infty)$ is a given graph connectivity length scale. With these weights, the resulting graph is an example of a *geometric graph* (or in fact a *random geometric graph* if the nodes are sampled randomly as in Setting 5.3.3). The results in this section are likely to remain valid if the indicator function $\mathbf{1}_{[0,h_n]}$ in (5.3.6) is replaced by a more general (e.g., Gaussian) cut-off kernel (such as in [41]), but we only consider $\eta = \mathbf{1}_{[0,h_n]}$ in order to also cite sources which are not formulated in this generality.

The graph-discretized counterpart $\mathcal{L}_n^{\tau,\kappa}$ of (5.3.5) is then the operator which acts on a given function $u\colon \mathcal{M}_n \to \mathbb{R}$ as

$$\mathcal{L}_n^{\tau,\kappa} u(x_n^{(i)}) := \tau(x_n^{(i)}) u(x_n^{(i)}) + \sum_{j=1}^n (\mathbf{W}_n)_{ij} \sqrt{\kappa(x_n^{(i)})\kappa(x_n^{(j)})}\big(u(x_n^{(i)}) - u(x_n^{(j)})\big). \tag{5.3.7}$$

This can be seen as a generalized version of the (unnormalized) graph Laplacian $\Delta_{\mathcal{M}_n}$, and in fact reduces to it if $\tau \equiv 0$ and $\kappa \equiv 1$.

Assumption 5.3.4 (Coefficients of $\mathcal{L}_n^{\tau,\kappa}$)**.** Let $\tau\colon \mathcal{M} \to [0,\infty)$ and $\kappa\colon \mathcal{M} \to [0,\infty)$ be the coefficient functions used in the definitions of the base operators $(\mathcal{L}_n^{\tau,\kappa})_{n\in\overline{\mathbb{N}}}$, see equations (5.3.5) and (5.3.7). We shall suppose that

(i) τ is Lipschitz, whereas κ is continuously differentiable and bounded below away from zero.

For some results, we specialize to the case that

(ii) $\tau \equiv 0$ and $\kappa \equiv 1$, i.e., $\mathcal{L}_n^{\tau,\kappa} = \Delta_{\mathcal{M}_n}$ and $\mathcal{L}_\infty^{\tau,\kappa}$ and reduces to the Laplace–Beltrami operator on $\mathcal{M}$.

Assumption 5.3.5 (Connectivity length scale of random graph)**.** Let the manifold $\mathcal{M}$ and the random point clouds $(\mathcal{M}_n)_{n\in\mathbb{N}}$ on the probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ be as in Setting 5.3.3. Let $(h_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ determine the connectivity length scales of the graphs associated to $(\mathcal{M}_n)_{n\in\mathbb{N}}$ via the weights (5.3.6), and suppose that $s \in (0,\infty)$. We will assume one of the following:

(i) There exists a $\beta > \frac{m}{4s}$ such that $(\log n)^{c_m} n^{-\frac{1}{m}} \ll h_n \ll n^{-\frac{1}{4s\beta}}$.

(ii) There exists a $\delta > 0$, so small that $\frac{m}{1-\delta} < m+4+\delta$, and a $\beta > \frac{m+4+\delta}{2s}$ such that $n^{-\frac{1}{m+4+\delta}} \lesssim h_n \ll n^{-\frac{1}{2s\beta}}$.

Given two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ of positive real numbers, the notation $a_n \ll b_n$ means $a_n/b_n \to 0$ as $n\to\infty$.

Since $\mathrm{Id}_n + \mathcal{L}_n^{\tau,\kappa}$ is self-adjoint, positive definite and has a compact inverse for all $n\in\overline{\mathbb{N}}$ (cf. [190, Chapter XII] in the case $n=\infty$), there exists an orthonormal basis $(\psi_n^{(j)})_{j=1}^n$ of $L^2(\mathcal{M}_n)$ and a non-decreasing sequence $(\lambda_n^{(j)})_{j=1}^n \subseteq [0,\infty)$, accumulating only at infinity for $n=\infty$, such that $\mathcal{L}_n^{\tau,\kappa}\psi_n^{(j)} = \lambda_n^{(j)}\psi_n^{(j)}$ for all $j\in\{1,\dots,n\}$. We summarize this state of affairs by saying that $(\psi_n^{(j)},\lambda_n^{(j)})_{j=1}^n$ is an orthonormal eigenbasis of $L^2(\mathcal{M}_n)$ associated to $\mathcal{L}_n^{\tau,\kappa}$. Recall from Section 1.2.4 that the asymptotic behavior of the eigenvalues $(\lambda_\infty^{(j)})_{j\in\mathbb{N}}$ is described by *Weyl's law*, cf. [190, Theorem XII.2.1]:

$$\lambda_\infty^{(j)} \eqsim_{(\mathcal{M},\tau,\kappa)} j^{2/m} \quad \text{for all } j\in\mathbb{N}. \tag{5.3.8}$$

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Given any of the above settings and $n\in\overline{\mathbb{N}}$, we define the generalized Whittle–Matérn operator A_n on $L^2(\mathcal{M}_n)$ as a fractional power of the symmetric elliptic operator $\mathcal{L}_n^{\tau,\kappa}$ given by (5.3.5) and (5.3.7). That is, we set $A_n := (\mathcal{L}_n^{\tau,\kappa})^s$ for some $s\in[0,\infty)$, where we use the spectral definition of fractional powers (see Definition 1.3.3):

$$A_n u = (\mathcal{L}_n^{\tau,\kappa})^s u := \sum_{j=1}^n [\lambda_n^{(j)}]^s \langle u,\psi_n^{(j)}\rangle_{L^2(\mathcal{M}_n)}\psi_n^{(j)}, \quad u\in \mathsf{D}(A_n)\subseteq L^2(\mathcal{M}_n). \tag{5.3.9}$$

These will be used as the linear operators $(A_n)_{n\in\overline{\mathbb{N}}}$ in the stochastic partial differential equations in the next subsection.

Since A_n is a nonnegative definite and self-adjoint operator on $L^2(\mathcal{M}_n)$ for any $n\in\overline{\mathbb{N}}$, the Lumer–Phillips theorem [156, Theorem 13.35] implies that $-A_n$ generates a contractive analytic C_0-semigroup $(S_n(z))_{z\in\Sigma_\eta}\subseteq\mathscr{L}(L^2(\mathcal{M}_n))$ on the sector

$$\Sigma_\eta := \{\lambda\in\mathbb{C}\setminus\{0\} : \arg\lambda\in(-\eta,\eta)\} \tag{5.3.10}$$

for every $\eta\in(0,\frac{1}{2}\pi)$. Thus, the operators $(A_n)_{n\in\overline{\mathbb{N}}}$ on $(L^2(\mathcal{M}_n))_{n\in\overline{\mathbb{N}}}$ are uniformly sectorial of angle 0, see Appendix 5.C.

The following assumption(s) on the $L^\infty(\mathcal{M}_n)$-boundedness of the semigroups will be needed for some of the results in Section 5.3.2:

Assumption 5.3.6 (Uniform L^∞-boundedness of semigroups)**.** Suppose that

(i) there exists a constant $M_{S,\infty}\in[1,\infty)$ such that

$$\|S_n(t)\|_{\mathscr{L}(L^\infty(\mathcal{M}_n))} \le M_{S,\infty} \quad \text{for all } n\in\overline{\mathbb{N}} \text{ and } t\ge 0.$$

We may sometimes additionally assume that

(ii) $(S_n(t))_{t\ge0}$ is $L^\infty(\mathcal{M}_n)$-contractive for all $n\in\overline{\mathbb{N}}$, i.e., $M_{S,\infty}=1$ in (i).

Under this assumption, it follows from [164, Proposition 3.12] that $(S_n(z))_{z\in\Sigma_{\eta_q}}$ is bounded analytic on $L^q(\mathcal{M}_n)$ with $\eta_q=\frac{2}{q}\eta$ for all $n\in\overline{\mathbb{N}}$ and $q\in(2,\infty)$, and its uniform norm bound on the sector Σ_{η_q} only depends on q and $M_{S,\infty}$. Therefore, the sequence of operators $(A_n)_{n\in\overline{\mathbb{N}}}$ on $(L^q(\mathcal{M}_n))_{n\in\overline{\mathbb{N}}}$ is uniformly sectorial of angle at most $(\frac{1}{2}-\frac{1}{q})\pi$.

5.3.2. CONVERGENCE OF GRAPH-DISCRETIZED SEMILINEAR SPDES

Let $(W(t))_{t\geq 0}$ be an $L^2(\mathcal{M})$-valued cylindrical Wiener process with respect to a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. The spaces $\mathcal{M}$ and $(\mathcal{M}_n)_{n\in\mathbb{N}}$ are as in Setting 5.3.2 or 5.3.3 above; in the latter case, note that the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ associated to the Wiener noise is independent of the probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ describing the randomness of the point cloud. For every $n \in \overline{\mathbb{N}}$, we set $W_n := \Pi_n W$ and consider the following semilinear stochastic partial differential equation (SPDE):

$$\begin{cases} \mathrm{d}u_n(t,x) + [\mathcal{L}_n^{\tau,\kappa}]^s u_n(t,x)\,\mathrm{d}t = f_n(t,u_n(t,x))\,\mathrm{d}t + \mathrm{d}W_n(t,x), \\ \qquad\qquad u_n(0,x) = \xi_n(x), \qquad\qquad (t,x) \in (0,T] \times \mathcal{M}_n, \end{cases} \tag{5.3.11}$$

where $s \in (0,\infty)$, $T \in (0,\infty)$ is a finite time horizon, $f_n \colon \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is the nonlinearity, and $\xi_n \colon \Omega \times \mathcal{M}_n \to \mathbb{R}$ is the initial datum. Note that $(W_n(t))_{t\geq 0}$ is a Q_n-cylindrical Wiener process with $Q_n = \Pi_n^* \Pi_n = \Lambda_n \Pi_n = \mathrm{Id}_n$, where we recall (5.3.4) for the second identity. Therefore, $(W_n(t))_{t\geq 0}$ is a cylindrical Wiener process on $L^2(\mathcal{M}_n)$ for all $n \in \overline{\mathbb{N}}$, and its formal time derivative $\mathrm{d}W_n$ represents spatiotemporal Gaussian white noise on $[0,T] \times \mathcal{M}_n$.

Solutions to (5.3.11)—and all the other (semi)linear SPDEs that we consider in this chapter—are always interpreted in the mild sense. This notion of solutions is defined using the semigroup $(S_n(t))_{t\geq 0}$ generated by $-[\mathcal{L}_n^{\tau,\kappa}]^s$. We say that u_n is a *global mild solution* to (5.3.11) if it satisfies the following relation for all $t \in [0,T]$:

$$u_n(t) = S_n(t)\xi_n + \int_0^t S_n(t-s) F_n(s, u_n(s))\,\mathrm{d}s + \int_0^t S_n(t-s)\,\mathrm{d}W_n(s), \quad \mathbb{P}\text{-a.s.}$$

Here, we interpret $u_n = (u_n(t))_{t\in[0,T]}$ as a process taking its values in an infinite-dimensional Banach space of functions on $\mathcal{M}_n$ (such as $L^q(\mathcal{M}_n)$ or $C(\mathcal{M}_n)$), and we define for every $(\omega, t) \in \Omega \times [0,T]$ the *Nemytskii operator* $u_n \mapsto F_n(\omega, t, u_n)$ on this function space by setting $[F_n(\omega,t,u_n)](\xi) := f_n(\omega, t, u_n(\xi))$ for all $\xi \in \mathcal{M}_n$.

This notion of solution is called "global" because it exists on the whole of $[0,T]$, in contrast with "local" solutions, which may blow up before time T. However, we note that global solutions generally grow unbounded as $T \to \infty$. We will not consider local solutions in this section, but we do work with them in Section 5.5.2.

In this section, the real-valued functions f_n are supposed to satisfy the following:

Assumption 5.3.7 (Nonlinearities)**.** We will assume one of the following conditions:

(i) The nonlinearities $(f_n)_{n\in\overline{\mathbb{N}}}$ are globally Lipschitz continuous and grow linearly, both uniformly in n. I.e., there exist $\widetilde{L}_f, \widetilde{C}_f \in [0,\infty)$ such that, for all $n \in \overline{\mathbb{N}}$ and $x, y \in \mathbb{R}$,

$$|f_n(\omega,t,x) - f_n(\omega,t,y)| \leq \widetilde{L}_f |x-y| \quad \text{and} \quad |f_n(\omega,t,x)| \leq \widetilde{C}_f(1+|x|).$$

(ii) The nonlinearities $(f_n)_{n\in\overline{\mathbb{N}}}$ are of the polynomial form

$$f_n(\omega,t,x) := -a_{2k+1,n}(\omega,t)x^{2k+1} + \sum_{j=0}^{2k} a_{j,n}(\omega,t)x^j, \tag{5.3.12}$$

where $k \in \mathbb{N}_0$ and $a_{j,n} \colon \Omega \times [0,T] \to \mathbb{R}$ for each $j \in \{0,\dots,2k+1\}$, and there exist constants $c, C \in (0,\infty)$ such that

$$c \le a_{2k+1,n}(\omega,t) \le C \quad \text{and} \quad |a_{j,n}(\omega,t)| \le C \tag{5.3.13}$$

for all $j \in \{0,\dots,2k\}$, $n \in \overline{\mathbb{N}}$ and $(\omega,t) \in \Omega \times [0,T]$.

In either case, we suppose moreover that $f_n \to f$ uniformly on compact intervals; i.e., for all $r \in [0,\infty)$ and $(\omega,t) \in \Omega \times [0,T]$,

$$\sup_{x \in [-r,r]} |f_n(\omega,t,x) - f_\infty(\omega,t,x)| \to 0 \quad \text{as } n \to \infty. \tag{5.3.14}$$

Example 5.3.8. The cubic polynomial $f_n(\omega,t,x) := -x^3 + x$, which turns (5.3.11) into the (fractional) stochastic Allen–Cahn equation in case $\tau \equiv 0$, $\kappa \equiv 1$ and $s \le 1$, is of the form asserted in Assumption 5.3.7(ii). Note that this is also an example of the important situation where (5.3.14) is trivially satisfied by taking the same function $f_n := f$ for all $n \in \overline{\mathbb{N}}$.

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The final technical assumption that we record before moving on to the main theorem of this section is the following:

Assumption 5.3.9 (Uniform L^∞-boundedness of eigenfunctions)**.** There exists some $M_{\psi,\infty} \in (0,\infty)$ such that

$$\|\psi_n^{(j)}\|_{L^\infty(\mathcal{M}_n)} \le M_{\psi,\infty} \quad \text{for all } n \in \overline{\mathbb{N}} \text{ and } j \in \{1,\dots,n\}.$$

The interplay of the various choices of spatial domains $\mathcal{M}_n$, linear operators A_n and nonlinearity functions f_n determines the class of SPDEs to which (5.3.11) belongs. Rigorous definitions of the corresponding mild solution concepts, as well as well-posedness and discrete-to-continuum convergence results can be found in Sections 5.4–5.6, respectively. Applying these results in their respective regimes of applicability yields the following discrete-to-continuum convergence theorem for the solutions to (5.3.11); note that the setting of part (c) covers the stochastic (fractional) Allen–Cahn equation on the one-dimensional torus, see Example 5.3.8.

Theorem 5.3.10. *Let $\mathcal{M}$ and $(\mathcal{M}_n)_{n\in\overline{\mathbb{N}}}$ be as in Setting 5.3.2 or 5.3.3.*

(a) Consider Setting 5.3.3. Let $s > \frac{1}{2}m$ and suppose that Assumption 5.3.5(i) holds with $\beta \in (\frac{m}{4s}, \frac{1}{2})$. If Assumptions 5.3.4(i) and 5.3.7(i) are satisfied, and $p \in [1,\infty)$ is such that $\Lambda_n \xi_n \to \xi_\infty$ in $L^p(\Omega,\mathcal{F}_0,\mathbb{P};L^2(\mathcal{M}))$, then there exists a unique global mild solution u_n in $L^p(\Omega;C([0,T];L^2(\mathcal{M}_n)))$ to (5.3.11) for every $n \in \overline{\mathbb{N}}$, and as $n \to \infty$ we have

$$\Lambda_n u_n \to u_\infty \quad \widetilde{\mathbb{P}}\text{-a.s. in } L^p(\Omega;C([0,T];L^2(\mathcal{M}))).$$

(b) In Setting 5.3.3, let $\delta > 0$ be such that $\frac{m}{1-\delta} < m+4+\delta$, suppose $s > m+4+\delta$ and Assumption 5.3.5(ii) holds with $\beta \in (\frac{m+4+\delta}{2s}, \frac{1}{2})$. Let Assumptions 5.3.4(ii), 5.3.6(i), 5.3.7(i) and 5.3.9 be satisfied.

If $p \in [1,\infty)$ *is such that* $\Lambda_n \xi_n \to \xi_\infty$ *in* $L^p(\Omega, \mathcal{F}_0, \mathbb{P}; L^\infty(\mathcal{M}))$, *then there exists a unique global mild solution* u_n *in* $L^p(\Omega; C([0,T]; L^\infty(\mathcal{M}_n)))$ *to* (5.3.11) *for every* $n \in \mathbb{N}$, *as well as* u_∞ *in* $L^p(\Omega; C([0,T]; C(\mathcal{M})))$ *for* $n = \infty$, *and as* $n \to \infty$ *we have*

$$\Lambda_n u_n \to u_\infty \quad \text{in } L^0(\widetilde{\Omega}, L^p(\Omega; C([0,T]; L^\infty(\mathcal{M})))).$$

(c) *Consider Setting* 5.3.2 *with* $\mathcal{M} := \mathbb{T}$. *Let* $s \in (\frac{1}{2}, 1]$ *and suppose that Assumptions* 5.3.4(ii) *and* 5.3.7(ii) *are satisfied.*

If $p \in (1,\infty)$ *is such that* $\Lambda_n \xi_n \to \xi_\infty$ *in* $L^p(\Omega, \mathcal{F}_0, \mathbb{P}; L^\infty(\mathbb{T}))$, *then there exists a unique global mild solution* u_n *in* $L^p(\Omega; C([0,T]; L^\infty(\mathcal{M}_n)))$ *to* (5.3.11) *for every* $n \in \mathbb{N}$, *as well as* u_∞ *in* $L^p(\Omega; C([0,T]; C(\mathbb{T})))$ *for* $n = \infty$ *and for all* $p^- \in [1,p)$ *we have, as* $n \to \infty$,

$$\Lambda_n u_n \to u_\infty \quad \text{in } L^{p^-}(\Omega; C([0,T]; L^\infty(\mathbb{T}))).$$

The proof is presented in Section 5.3.4. In the next section, we list the intermediate results on which it relies. The motivations behind the various assumptions listed above, as well as their role in Theorem 5.3.10, are discussed in Section 5.3.5.

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5.3.3. INTERMEDIATE RESULTS

In this subsection, we collect a number of intermediate results which imply that the conditions imposed in Theorem 5.3.10 are sufficient to fit into the setting of the various convergence theorems in Sections 5.4–5.6. More precisely, depending on the setting, we wish to verify a subset of the following: Conditions (A1)–(A3) from Section 5.4 on the linear operators, conditions (F1)–(F2) and (IC) from Section 5.5 on the nonlinearities and initial conditions, respectively, as well as their extended counterparts (A1-B)–(A4-B), (IC-B), (F1-B)–(F2-B) and (F1″-B) from Section 5.6. The proofs of the results in this section are deferred to Appendix 5.A for ease of exposition.

The necessary convergence of the linear operators, given by (A3) and (A3-B), will ultimately be derived from the spectral convergence of $(\mathcal{L}_n^{\tau,\kappa})_{n\in\mathbb{N}}$ to $\mathcal{L}_\infty^{\tau,\kappa}$, i.e., the convergence of the respective eigenvalues and (lifted) eigenfunctions. In the square grid Setting 5.3.2, we can argue directly using closed-form expressions of all the eigenvalues and eigenfunctions involved, see Lemma 5.3.11 below. A subtlety arising in the random graph Setting 5.3.3 is that, for any $n \in \mathbb{N}$, we cannot in general control the errors $|\lambda_n^{(j)} - \lambda_\infty^{(j)}|$ and $\|\psi_n^{(j)} \circ T_n - \psi_\infty^{(j)}\|_{L^q(\mathcal{M})}$ for all $j \in \{1,\dots,n\}$, but only for indices j up to a sufficiently small integer k_n. We present the precise statements below: Theorems 5.3.12 and 5.3.13(a), which cover eigenvalue convergence and $L^2(\mathcal{M})$-convergence of eigenfunctions, are respectively taken from [181, Theorems 4.6 and 4.7]. Theorem 5.3.13(b), concerning the $L^\infty(\mathcal{M})$-convergence of Laplacian eigenvalues, is a consequence of the main results from [41], as shown in [182, Lemma 15] and the discussion preceding it.

Lemma 5.3.11 (Spectral convergence—square grid). *Let* $\mathcal{M} = \mathbb{T}^m$ *be discretized by the sequence of square grids described in Setting* 5.3.2. *If* $\tau \equiv 0$ *and* $\kappa \equiv 1$, *then for all* $n \in \mathbb{N}$ *such that* $n^{1/m} \in \mathbb{N}$, *the eigenfunction–eigenvalue pairs* $(\psi_n^{(j)}, \lambda_n^{(j)})_{j=1}^n$ *and*

$(\psi_\infty^{(j)}, \lambda_\infty^{(j)})_{j\in\mathbb{N}}$ *of the graph Laplacian* $\mathcal{L}_n^{\tau,\kappa} = \Delta_n$ *and the Laplace–Beltrami operator* $\mathcal{L}_\infty^{\tau,\kappa} = -\Delta_{\mathcal{M}}$*, respectively, satisfy*

$$0 \le \lambda_\infty^{(j)} - \lambda_n^{(j)} \le \tfrac{1}{12} j^4 \pi^4 n^{-\frac{2}{m}} \quad \text{for all } j \in \{1,\dots,n\}; \tag{5.3.15}$$

$$\|\psi_\infty^{(j)} - \psi_n^{(j)} \circ T_n\|_{L^\infty(\mathcal{M})} \le \tfrac{1}{2}\sqrt{2} j\pi n^{-\frac{1}{m}} \quad \text{for all } j \in \{1,\dots,n-1\}. \tag{5.3.16}$$

Theorem 5.3.12 (Eigenvalue convergence—random graphs)**.** *Let the manifold* $\mathcal{M}$ *and the random point clouds* $(\mathcal{M}_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ *on the probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ *be as in Setting 5.3.3. Suppose that* $\tau \colon \mathcal{M} \to [0,\infty)$ *is Lipschitz, and that* $\kappa \colon \mathcal{M} \to [0,\infty)$ *is continuously differentiable and bounded below away from zero.*

If the graph connectivity length scales $(h_n)_{n\in\mathbb{N}}$ *(see (5.3.6)) are chosen in such a way that there exist positive integers* $(k_n)_{n\in\mathbb{N}}$ *satisfying*

$$\varepsilon_n \ll h_n \ll [\lambda_\infty^{(k_n)}]^{-\frac{1}{2}}, \tag{5.3.17}$$

then there exists a constant $C_{(\mathcal{M},\tau,\kappa)} > 0$ *such that*

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$$\widetilde{\mathbb{P}}\left(\frac{|\lambda_n^{(j)} - \lambda_\infty^{(j)}|}{\lambda_\infty^{(j)} + 1} \le C_{(\mathcal{M},\tau,\kappa)}\, \varepsilon_n h_n^{-1} + h_n [\lambda_\infty^{(j)}]^{\frac{1}{2}} \text{ for all } n \in \mathbb{N},\ j \in \{1,\dots,k_n\} \right) = 1.$$

Theorem 5.3.13 (Eigenfunction convergence—random graphs)**.** *Let the manifold* $\mathcal{M}$ *and the random point clouds* $(\mathcal{M}_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ *on the probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ *be as in Setting 5.3.3. Let* $(h_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ *be the connectivity length scales of the graphs associated to* $(\mathcal{M}_n)_{n\in\mathbb{N}}$ *via the weights (5.3.6), and consider the (graph-discretized) differential operators* $\mathcal{L}_n^{\tau,\kappa}$ *with coefficients* $\tau \colon \mathcal{M} \to [0,\infty)$ *and* $\kappa \colon \mathcal{M} \to [0,\infty)$.

(a) If Assumption 5.3.4(i) holds, and there exist integers $(k_n)_{n\in\mathbb{N}}$ *such that (5.3.17) is satisfied, then there exists a constant* $C_{(\mathcal{M},\tau,\kappa)} > 0$ *such that, for all* $n \in \mathbb{N}$*,*

$$\widetilde{\mathbb{P}}\Big(\|\psi_n^{(j)} \circ T_n - \psi_\infty^{(j)}\|_{L^2(\mathcal{M})} \le C_{(\mathcal{M},\tau,\kappa)}\, j^{\frac{3}{2}} \big(\varepsilon_n h_n^{-1} + h_n [\lambda_\infty^{(j)}]^{\frac{1}{2}}\big)^{\frac{1}{2}} \quad \text{for all } j \in \{1,\dots,k_n\} \Big) = 1.$$

(b) Let Assumption 5.3.4(ii) be satisfied. If there exist $(k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ *and* $\delta > 0$ *such that*

$$n^{-\frac{1}{m+4+\delta}} \lesssim_{\mathcal{M}} h_n \lesssim_{\mathcal{M}} [\lambda_\infty^{(k_n)}]^{-1} \quad \text{and} \quad \lambda_\infty^{(k_n)} \lesssim_{\mathcal{M}} n^{\frac{1-\delta}{m}},$$

then there exists a constant $C_{\mathcal{M}} > 0$ *such that, as* $n \to \infty$*,*

$$\widetilde{\mathbb{P}}\Big(\|\psi_n^{(j)} \circ T_n - \psi_\infty^{(j)}\|_{L^\infty(\mathcal{M})} \le C_{\mathcal{M}} [\lambda_\infty^{(j)}]^{m+1} j^{\frac{3}{2}} \big(\varepsilon_n h_n^{-1} + h_n [\lambda_\infty^{(j)}]^{\frac{1}{2}}\big)^{\frac{1}{2}} \quad \text{for all } j \in \{1,\dots,k_n\} \Big) \to 1.$$

From the above results, we can derive the following convergence of the sequence $(A_n)_{n\in\overline{\mathbb{N}}}$. Its proof is analogous to that of [181, Theorem 4.2], see Appendix 5.A.

Theorem 5.3.14. *Given* $\tau \colon \mathcal{M} \to [0,\infty)$, $\kappa \colon \mathcal{M} \to [0,\infty)$ *and* $s \in [0,\infty)$*, consider the generalized Whittle–Matérn operators* $A_n := (\mathcal{L}_n^{\kappa,\tau})^s$ *defined by (5.3.9), and define the operator* $\widetilde{R}_n^\alpha := \Lambda_n(\mathrm{Id}_n + A_n)^{-\alpha}\Pi_n$*, for all* $\alpha \in [0,\infty)$ *and* $n \in \overline{\mathbb{N}}$.

(a) *Suppose that $\mathcal{M}$ and its discretizations $(\mathcal{M}_n)_{n\in\mathbb{N}}$ are as in Setting 5.3.3, Assumption 5.3.5(i) holds with $\beta \in (\frac{m}{4s},\infty)$ and Assumption 5.3.4(i) is satisfied. Then we have for all $\beta' \in [\beta,\infty)$:*

$$\widetilde{R}_n^{\beta'} \to R_\infty^{\beta'} \quad \widetilde{\mathbb{P}}\text{-a.s. in } \mathscr{L}_2(L^2(\mathcal{M})) \quad \text{as } n\to\infty.$$

(b) *Suppose that $\mathcal{M}$ and its discretizations $(\mathcal{M}_n)_{n\in\mathbb{N}}$ are as in Setting 5.3.3, Assumption 5.3.5(ii) holds with $\beta \in (\frac{m+4+\delta}{2s},\infty)$, and Assumptions 5.3.4(ii) and 5.3.9 are satisfied. Then we have for all $\beta' \in [\beta,\infty)$:*

$$\widetilde{R}_n^{\beta'} \to R_\infty^{\beta'} \quad \text{in } L^0(\widetilde{\Omega};\mathscr{L}(L^2(\mathcal{M});L^\infty(\mathcal{M}))) \quad \text{as } n\to\infty.$$

Here, $L^0(\widetilde{\Omega})$ denotes convergence in probability with respect to $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$.

(c) *Suppose that $\mathcal{M} := \mathbb{T}^m$ is discretized using the square grids $(\mathcal{M}_n)_{n\in\mathbb{N}}$ from Setting 5.3.2, and that Assumption 5.3.4(ii) holds. For all $\beta \in (\frac{m}{4s},\infty)$,*

$$\widetilde{R}_n^{\beta} \to R_\infty^{\beta} \quad \text{in } \mathscr{L}(L^2(\mathcal{M});L^\infty(\mathcal{M})) \quad \text{as } n\to\infty.$$

The following property, which we call the *uniform ultracontractivity* of the semigroups $(S_n)_{n\in\overline{\mathbb{N}}}$, will be needed in order to obtain the $L^\infty(\mathcal{M})$-convergence in Theorem 5.3.10(b) and (c). Its proof relies on the Riesz–Thorin interpolation theorem, Assumption 5.3.6, and some arguments from Theorem 5.3.14.

Lemma 5.3.15 (Uniform ultracontractivity)**.** *Let $s \in (0,\infty)$ and consider the generalized Whittle–Matérn operators $A_n := (\mathcal{L}_n^{\kappa,\tau})^s$ defined by (5.3.9) for all $n \in \overline{\mathbb{N}}$. Assume either of the following:*

(a) *In Setting 5.3.3, Assumption 5.3.5(i) or (ii) holds with corresponding β, as well as Assumptions 5.3.4(i), 5.3.6(i) and 5.3.9.*

(b) *In Setting 5.3.2, $\beta \in (\frac{m}{4s},\infty)$ is arbitrary, and Assumptions 5.3.4(ii) and 5.3.6(i) hold.*

Then, for every $q \in [2,\infty]$, there exists $M_{S,q} \in [1,\infty)$ such that

$$\|S_n(t)\|_{\mathscr{L}(L^q(\mathcal{M}_n);L^\infty(\mathcal{M}_n))} \le M_{S,q}\, t^{-\frac{2}{q}\beta} \quad \text{for all } n \in \overline{\mathbb{N}} \text{ and } t > 0. \tag{5.3.18}$$

In case of (a), (5.3.18) holds $\widetilde{\mathbb{P}}$-a.s.

5.3.4. Proof of convergence

Using the intermediate results from Section 5.3.3, we can now prove Theorem 5.3.10:

Proof of Theorem 5.3.10. In order to prove parts (a)–(c), we will apply Theorems 5.5.4 and 5.6.6 as well as Corollary 5.6.10, which are the rigorous counterparts of Theorem 5.1.1 in the respective settings.

The argument preceding Setting 5.3.2 shows that (A1) and (A1-B) hold in any of the given situations, with $H_n := L^2(\mathcal{M}_n)$ and $E_n := L^q(\mathcal{M}_n)$ for $n \in \overline{\mathbb{N}}$ and $q \in [2,\infty)$, as well as $B_n := L^\infty(\mathcal{M}_n)$ for all $n \in \mathbb{N}$, $B_\infty := C(\mathcal{M})$ and $\widetilde{B} := L^\infty(\mathcal{M})$. Moreover, note that (IC) (or (IC-B)) is explicitly assumed in each case.

(a) Here, we take $q = 2$, i.e., $E_n = H_n = L^2(\mathcal{M}_n)$ for all $n \in \overline{\mathbb{N}}$. As discussed at the end of Subsection 5.3.1, the operators $(A_n)_{n\in\overline{\mathbb{N}}} := ([\mathcal{L}_n^{\tau,\kappa}]^s)_{n\in\overline{\mathbb{N}}}$ are uniformly sectorial of angle 0 on $(L^2(\mathcal{M}_n))_{n\in\overline{\mathbb{N}}}$. Letting $\beta \in (\frac{m}{4s}, \frac{1}{2})$ be as in Assumption 5.3.5(i), it follows from Theorem 5.3.14(a) that $\widetilde{R}_n^{\beta'} \to R_\infty^{\beta'}$, $\widetilde{\mathbb{P}}$-a.s., in $\mathscr{L}_2(L^2(\mathcal{M}))$ as $n \to \infty$, for all $\beta' \geq \beta$. Applying this with $\beta' := \beta \in (0, \frac{1}{2})$ and $\beta' := 1$ yields (A2) and (A3). Setting, for all $(\omega, t) \in \Omega \times [0,T]$, $u \in L^2(\mathcal{M}_n)$ and $x \in \mathcal{M}_n$,

$$[F_n(\omega,t,u)](x) := f_n(\omega,t,u(x)), \tag{5.3.19}$$

it is immediate from Assumption 5.3.7(i) that (F1) is satisfied. Moreover, combining the definition of $\widetilde{F}_n$ from (5.5.3) with (5.3.19) yields

$$[\widetilde{F}_n(\omega,t,u)](x) = [\Lambda_n F_n(\omega,t,\Pi_n u)](x) = f_n(t,\omega,\Lambda_n\Pi_n u(x)),$$

so that

$$\begin{aligned}\|\widetilde{F}_n(\omega,t,u) - F_\infty(\omega,t,u)\|_{L^2(\mathcal{M})} &= \|f_n(\omega,t,\Lambda_n\Pi_n u(\cdot)) - f_\infty(\omega,t,u(\cdot))\|_{L^2(\mathcal{M})} \\ &\leq \widetilde{L}_f\|\Lambda_n\Pi_n u - u\|_{L^2(\mathcal{M})} + \|f_n(\omega,t,u(\cdot)) - f_\infty(\omega,t,u(\cdot))\|_{L^2(\mathcal{M})}.\end{aligned}$$

As $n \to \infty$, the first term vanishes by Assumption (A1), and the second term by dominated convergence using (5.3.14) and the uniform linear growth condition in Assumption 5.3.7(i). Therefore, condition (F2) is also satisfied.

(b) Now we need Assumption 5.3.6(i) in order for $([\mathcal{L}_n^{\tau,\kappa}]^s)_{n\in\overline{\mathbb{N}}}$ to be uniformly sectorial of angle less than $\frac{1}{2}\pi$ on $(L^q(\mathcal{M}_n))_{n\in\overline{\mathbb{N}}}$ for all $q \in [2,\infty)$. Letting $\delta > 0$ and $\beta \in (\frac{m+4+\delta}{2s}, \frac{1}{2})$ be as in Assumption 5.3.5(ii), it follows from Theorem 5.3.14(b) that $\widetilde{R}_n^{\beta'} \to R_\infty^{\beta'}$ in $L^0(\widetilde{\Omega};\mathscr{L}(L^2(\mathcal{M});L^\infty(\mathcal{M})))$ as $n \to \infty$, for all $\beta' \geq \beta$, under Assumptions 5.3.4(ii) and 5.3.9. In particular, we have $\widetilde{R}_n^{\beta} \to R_\infty^{\beta}$ in $L^0(\widetilde{\Omega};\gamma(L^2(\mathcal{M});L^q(\mathcal{M})))$ for all $q \in [1,\infty)$ by [114, Corollary 9.3.3], and $\widetilde{R}_n \to R_\infty$ in $L^0(\widetilde{\Omega};\mathscr{L}(L^\infty(\mathcal{M})))$. This shows (A2-B) and (A3-B). By Lemma 5.3.15(a), we have (A4-B) with $\theta = \frac{4}{q}\beta$. Thus, choosing $q > \frac{4\beta}{1-2\beta}$ yields $\theta + 2\beta < 1$. Conditions (F1-B) and (F2-B) follow similarly to part (a).

(c) As in part (b), we need to verify conditions (A1-B)–(A4-B), now with contractive semigroups $(S_n(t))_{t\geq 0}$, i.e., Assumption 5.3.6(ii). For $s = 1$, $(S_\infty(t))_{t\geq 0}$ is $L^\infty(\mathbb{T})$-contractive since the $L^1(\mathbb{T})$-norm of its heat kernel coincides with the $L^1(\mathbb{R})$-norm of the Gauss–Weierstrass kernel, which is equal to 1. For finite n, the $L^\infty(\mathcal{M}_n)$-contractivity of $S_n(t) = e^{-tA_n}$ is equivalent to A_n being diagonally dominant with positive diagonal by [154, Lemma 6.1], which holds for Laplacian matrices. Since these assertions can be extended to all $s \in (0,1]$ by subordination, see e.g. [115, Theorem 15.2.17], we indeed find that Assumption 5.3.6(ii) holds. Thus, we can proceed to argue as in (b), using Theorem 5.3.14(c) and Lemma 5.3.15(b) for an arbitrary $\beta \in (\frac{1}{4s}, \frac{1}{2})$, to obtain (A1-B)–(A4-B) with $\widetilde{M}_S = 1$ and $\theta + 2\beta < 1$ for $q \in [2,\infty)$ large enough.

It remains to establish that the nonlinearities from Assumption 5.3.7(ii) are such that (F1″-B) holds. This is done in Example 5.6.8, noting that the space $L^\infty(\mathcal{M}_n)$ coincides with $C(\mathcal{M}_n)$ if we equip $\mathcal{M}_n$ with the discrete topology. □

5.3.5. DISCUSSION OF THE ASSUMPTIONS

Here, we comment on the various assumptions made in Theorem 5.3.10, the extent to which they are necessary, and how one might check them in practice.

The distinction between parts (i) and (ii) of Assumption 5.3.4, i.e., whether to allow for spatially varying coefficient functions τ and κ in the second-order symmetric base operators $(\mathcal{L}_n^{\tau,\kappa})_{n\in\overline{\mathbb{N}}}$ instead of merely considering Laplacians, is mainly due to the availability of spectral convergence theorems in the respective situations. Most of the literature on eigenfunction convergence of graph-discretized second-order operators is focused on the Laplacian, see for instance [40, 82] for L^2-convergence and [41, 69, 198] for L^∞-convergence. However, the authors of [181] show how the L^2-convergence results can be extended to coefficient functions satisfying Assumption 5.3.4(i). We expect that most spectral convergence results for graph Laplacians can be extended to allow for varying coefficients, but doing so requires significant effort, hence we sometimes make Assumption 5.3.4(ii) for the sake of convenience.

Similarly, the difference between the bounds on the graph connectivity length scales in the two parts of Assumption 5.3.5 is a result of the current availability of spectral convergence literature. Eigenfunction convergence of $(\mathcal{L}_n^{\tau,\kappa})_{n\in\overline{\mathbb{N}}}$ in L^2 has for instance been proved in [181] under Assumption 5.3.5(i), but for graphs and manifolds as in our setting, the optimal available L^∞-convergence results (for graph Laplacians) seem to be those of [41], which require Assumption 5.3.5(ii). However, according to [41, Remark 2.7], it is plausible that the $L^\infty(\mathcal{M})$-convergence of Laplacian eigenfunctions can be established under the same assumptions as the $L^2(\mathcal{M})$-convergence, with the same rate. Some recent results in this direction can be found in [12], where the authors show L^∞-convergence of Laplacian eigenvectors with optimal rates and loose lower bounds on the connectivity lengths, using homogenization theory, for point clouds on less general spatial domains.

Assumption 5.3.6 is natural in the sense that the results regarding L^∞-convergence in space (for instance Theorem 5.3.10(b) and (c)) rely on uniform L^∞-convergence of semigroup orbits on compact time intervals. The latter necessitates that Assumption 5.3.6(i) is satisfied, at least for $t \in [0,T]$ with arbitrarily large $T \in (0,\infty)$.

Moreover, for typical choices of differential operators, one can often check that Assumption 5.3.6(ii) holds, meaning that the semigroups are in fact L^∞-contractive. One such example is outlined in the proof of Theorem 5.3.10(c): Matrix exponentials $(e^{-tL_n})_{t\ge 0}$ are L^∞-contractive if and only if $L_n \in \mathbb{R}^{n\times n}$ is diagonally dominant with nonnegative diagonal entries [154, Lemma 6.1]. Sufficient conditions for the L^∞-contractivity of the semigroup $(S_\infty(t))_{t\ge 0}$ generated by the negative of a uniformly elliptic second-order differential operator on a Euclidean domain $\mathcal{D} \subsetneq \mathbb{R}^d$, subject to appropriate boundary conditions, can be found in [164, Section 4.3]. Likewise, the heat semigroup associated to the Laplace–Beltrami operator on a compact Riemannian manifold $\mathcal{M}$ is L^∞-contractive, cf. [60, p. 148].

As mentioned in the proof of Theorem 5.3.10(c), all of the above L^∞-contractivity results for second-order differential operators can be extended to fractional powers $s \in (0,1)$ by using a subordination formula such as [115, Theorem 15.2.17], noting that the definition of fractional power operators in this reference coincides with ours (more details are given in the first half of the proof of Lemma 5.C.1 below). The

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semigroups generated by higher-order differential operators, however, are in general not contractive on L^∞ (or any L^q for $q \neq 2$, see for instance [134]); this is closely related to their lack of positivity preservation. As an example, the fractional heat kernel associated to $(-\Delta)^s$ on $\mathbb{R}^d$ with $s \in (0,\infty)$ at time $t \in (0,\infty)$ is given by the inverse Fourier transform of $\xi \mapsto \exp(-t\|\xi\|^{2s}_{\mathbb{R}^d})$, which is positive for $s \le 1$ but fails to be sign-definite if $s > 1$, see [72, p. 626 and pp. 632–633], respectively.

Thus, for operators $([\mathcal{L}_n^{\tau,\kappa}]^s)_{n\in\overline{\mathbb{N}}}$ with $s > 1$, we have to content ourselves with uniform L^∞-boundedness of the semigroups $(S_n(t))_{t\ge 0}$ in n and t, i.e., Assumption 5.3.6(i). In the absence of positivity preservation, one route to verifying such uniformity is through Gaussian upper bounds on the integral kernels corresponding to the semigroups, cf. [164, Proposition 7.1]. Such bounds have been established for higher-order differential operators on Euclidean domains, as well as Laplacian operators on more general domains such as manifolds, graphs and fractals (see [164, pp. 194–196] and the references therein). While it may be possible to unify these results in the setting of graph-discretized higher-order differential operators on manifolds, and thus obtain the uniform L^∞-bounds required by Assumption 5.3.6(i), this appears to be highly nontrivial and outside the scope of this chapter.

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We also remark that certain higher-order operators have been shown to exhibit *(local) eventual positivity*, meaning that for every nonnegative initial datum $u_0 \ge 0$ and subset $\mathcal{D}^*$ of the spatial domain $\mathcal{D}$, there exists $t^* > 0$ such that $S(t)u_0 \ge 0$ on $\mathcal{D}^*$ for all $t \ge t^*$. For instance, in [84], this was shown for the bi-Laplacian Δ^2 on $\mathcal{D} = \mathbb{R}^d$. In [95], the authors apply the theory of [57, 58] to treat the squared graph Laplacian Δ_n^2, deduce that $(e^{-\Delta_n^2 t})_{t\ge 0}$ is eventually L^∞-contractive [95, Proposition 6.7], and note that this implies L^∞-boundedness uniformly in $t \ge 0$ [95, Remark 6.8]. However, as $n \to \infty$, their upper bound $\|e^{-\Delta_n^2 t}\|_\infty \le \exp(\|\Delta_n\|^2_\infty t^*)$ blows up in our setting. Hence, these results do not appear to be directly useful for our purpose of verifying Assumption 5.3.6(i).

If we restrict ourselves to nonlinearities of the form $[F_n(\omega,t,u)](x) := f_n(\omega,t,u(x))$ (see (5.5.1) and (5.6.1)), then the conditions in Assumption 5.3.7 are the natural ones to ensure the global (in time) convergence results formulated in Theorem 5.3.10. Convergence results for more general nonlinearities, possibly formulated only in terms of local-in-time convergence and with respect to weaker norms, can be found in Sections 5.5 and 5.6.

Assumption 5.3.9 was used to establish the L^∞-convergence asserted in Theorem 5.3.14 and the uniform L^2–L^∞-ultracontractivity in Lemma 5.3.15. The L^∞-norms of the L^2-normalized eigenfunctions of the Laplace–Beltrami operator on a general compact Riemannian manifold $\mathcal{M}$ of dimension m satisfy the upper bound $\|\psi^{(j)}\|_{L^\infty(\mathcal{M})} \lesssim [\lambda^{(j)}]^{\frac{m-1}{4}}$ due to Hörmander [110]. This bound is sharp in the sense that it is attained by the symmetric spherical harmonics on the sphere. On the other hand, the L^∞-norms are uniformly bounded (i.e., satisfy Assumption 5.3.9) if $m = 1$ or if $\mathcal{M} = \mathbb{T}^m$ is a flat torus. Some results relating the L^∞-growth rate of these eigenfunctions to the geometry of the manifold can be found in [65, 185, 192].

These observations indicate that Assumption 5.3.9 poses strong restrictions on the curvature of the manifold, which raises the question whether this assumption could be removed. Its central role in the proofs of Theorem 5.3.14(b), (c) and Lemma 5.3.15

is due to the L^2–L^∞-norm bounds (5.A.2) of operators which are defined in terms of eigenvalue expansions, such as the fractional powers defined by (5.3.9). This suggests that disposing of Assumption 5.3.9 would involve techniques which are not based on spectral representations and spectral convergence of the operators involved. For Lemma 5.3.15 in particular, one indication that this should be possible is the fact that, like L^∞-boundedness, the L^2–L^∞-ultracontractivity of $(S_\infty(t))_{t\geq 0}$ follows from certain upper bounds on its heat kernel [96, Theorem 3.2]. For the Laplace–Beltrami operator on a compact Riemannian manifold, we indeed have such bounds by [60, Proposition 5.5.1 and Theorem 5.5.2], which imply (5.3.18) with $\beta = \frac{m}{4}$ (for $n = \infty$ and $s = 1$).

5.4. INFINITE-DIMENSIONAL ORNSTEIN–UHLENBECK PROCESS

In this section and the subsequent Sections 5.5 and 5.6, we state and prove the abstract discrete-to-continuum approximation results which were shown to be applicable to the Whittle–Matérn graph discretization setting in Section 5.3. Thus, from this point onwards we no longer necessarily work with graphs or Whittle–Matérn operators. Instead, we have the following abstract setting.

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Let the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be given. For any $n \in \overline{\mathbb{N}}$, we consider the following *linear* stochastic evolution equation, whose state space is a real and separable UMD-type-2 Banach space $(E_n, \|\cdot\|_{E_n})$:

$$\begin{cases} \mathrm{d}X_n(t) = -A_n X_n(t)\,\mathrm{d}t + \mathrm{d}W_n(t), & t \in (0,T], \\ X_n(0) = 0. \end{cases} \tag{5.4.1}$$

Here, $A_n \colon \mathsf{D}(A_n) \subseteq E_n \to E_n$ is a linear operator and $T \in (0,\infty)$ is a time horizon. Moreover, we take $W_n := \Pi_n W_\infty$, where $(W_\infty(t))_{t\geq 0}$ denotes a cylindrical Wiener process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ taking values in a separable Hilbert space $(H_\infty, \langle\cdot,\cdot\rangle_{H_\infty})$ and the operator $\Pi_n \in \mathscr{L}(H_\infty; H_n)$ is as in assumption (A1) below. Thus, the formal time derivative $\dot{W}_\infty$ of W_∞ represents space–time Gaussian white noise and $(W_n)_{t\geq 0}$ is an H_n-valued Q_n-cylindrical Wiener process colored in space by the covariance operator $Q_n := \Pi_n \Pi_n^* \in \mathscr{L}(H_n)$.

We impose the following uniformity assumptions on the spaces $(E_n)_{n\in\overline{\mathbb{N}}}$, $(H_n)_{n\in\overline{\mathbb{N}}}$ and the operators $(A_n)_{n\in\overline{\mathbb{N}}}$:

(A1) Assumption 5.2.1 holds for the UMD-type-2 Banach spaces $(E_n)_{n\in\overline{\mathbb{N}}}$ and the Hilbert spaces $(H_n)_{n\in\overline{\mathbb{N}}}$, with $\widetilde{E} := E_\infty$ and $\widetilde{H} := H_\infty$, both with the same sequence $(\Lambda_n, \Pi_n)_{n\in\overline{\mathbb{N}}}$ of lifting and projection operators.

(A2) The operators $(A_n)_{n\in\overline{\mathbb{N}}}$ on $(E_n)_{n\in\overline{\mathbb{N}}}$ are uniformly sectorial of angle less than $\frac{1}{2}\pi$ (see Appendix 5.C for the definition of this concept). In particular, their negatives generate bounded analytic C_0-semigroups $(S_n(t))_{t\geq 0} \subseteq \mathscr{L}(E_n)$ which satisfy Assumption 5.2.2 with $w = 0$. Moreover, there exists a $\beta \in [0, \frac{1}{2})$ such that $R_n^\beta := (\mathrm{Id}_n + A_n)^{-\beta} \in \gamma(H_n; E_n)$ for every $n \in \overline{\mathbb{N}}$.

In general, the fractional powers of the sectorial operators A_n appearing in (A2) can be defined using any of the equivalent definitions in [100, Chapter 3]. If, as in Section 5.3, the operator A_n given as the restriction of an operator whose eigenvalues form an orthonormal basis on some Hilbert space containing E_n, then one can use the spectral definition (5.3.9) of fractional powers of A_n.

The solution concept which we consider for all of the equations in this chapter is that of a *mild solution*, which was also the main object of study in the previous chapters. Recall that for the linear equation (5.4.1), it is given by a stochastic convolution. Namely, we have the following result.

Proposition 5.4.1. *Let $n \in \overline{\mathbb{N}}$ and $T \in (0,\infty)$. Under (A1)–(A2), the stochastic convolution*

$$W_{A_n}(t) := \int_0^t S_n(t-s)\,\mathrm{d}W_n(s), \quad t \in [0,T], \tag{5.4.2}$$

is a well-defined process in $C([0,T];L^p(\Omega;E_n))$ for every $p \in [1,\infty)$.

Proof. For every $p \in [1,\infty)$ and $t \in [0,T]$, we have by the Itô inequality (5.2.4):

$$\|W_{A_n}(t)\|^2_{L^p(\Omega;E_n)} \lesssim_{(p,E_n)} \int_0^t \|S_n(t-s)\|^2_{\gamma(H_n;E_n)}\,\mathrm{d}s \le \|S_n\|^2_{L^2(0,T;\gamma(H_n;E_n))}$$

To thow that the right-hand side is bounded, we use the ideal property (5.2.2) of $\gamma(H_n;E_n)$ and the estimate (5.C.2) for analytic semigroups (in conjuction with Assumptions (A1) and (A2)) to see that

$$\begin{aligned}
\|S_n\|^2_{L^2(0,T;\gamma(H_n;E_n))} &= \int_0^T \|(\mathrm{Id}_n + A_n)^\beta S_n(t) R_n^\beta\|^2_{\gamma(H_n;E_n)}\,\mathrm{d}t \\
&\le \|R_n^\beta\|^2_{\gamma(H_n;E_n)} \int_0^T \|(\mathrm{Id}_n + A_n)^\beta S_n(t)\|^2_{\mathscr{L}(E_n)}\,\mathrm{d}t \\
&\lesssim_\beta \|R_n^\beta\|^2_{\gamma(H_n;E_n)} \int_0^T t^{-2\beta}\,\mathrm{d}t = \|R_n^\beta\|^2_{\gamma(H_n;E_n)} \frac{T^{1-2\beta}}{1-2\beta} < \infty.
\end{aligned}$$

Note that $\|R_n^\beta\|^2_{\gamma(H_n;E_n)}$ is finite by (A2). Next, applying the Itô inequality (5.2.4) to the difference $W_{A_n}(t+h) - W_{A_n}(t)$ for small enough $h \in \mathbb{R}$ yields

$$\|W_{A_n}(t+h) - W_{A_n}(t)\|_{L^p(\Omega;E_n)} \lesssim_{(p,E_n)} \|S_n(\cdot+h) - S_n\|_{L^2(0,T;\gamma(H_n;E_n))} \to 0,$$

as $h \to 0$, by the strong continuity of translation operators on the Bochner space $L^2(0,T;\gamma(H_n;E_n))$. This shows $W_{A_n} \in L^p(\Omega;C([0,T];E_n))$. □

Definition 5.4.2. An E_n-valued stochastic process $X_n = (X_n(t))_{t\in[0,T]}$ belonging to $C([0,T];L^p(\Omega;E_n))$ for some $p \in [1,\infty)$ is said to be a *mild solution* to (5.4.1) if it is a modification of the process W_{A_n} defined in (5.4.2).

The existence and uniqueness (up to modification) of the mild solution to (5.4.1) in $C([0,T];L^p(\Omega;E_n))$ is then immediate from Definition 5.4.2.

As remarked in Section 5.2.2, we mainly have applications in mind where the problem corresponding to $n = \infty$ is interpreted as a spatiotemporal stochastic partial differential equation. In the present linear setting, its solution is also known as an *infinite-dimensional Ornstein–Uhlenbeck process*, and solutions to (5.4.1) for $n \in \mathbb{N}$ are spatially discretized approximations. Therefore, it is natural to ask whether we can identify the right mode of convergence of the operators $(A_n)_{n\in\mathbb{N}}$ to A_∞ as $n \to \infty$ to ensure the convergence of the processes $(W_{A_n})_{n\in\mathbb{N}}$ to W_{A_∞}.

The answer is provided by Proposition 5.4.4 below, which is a stochastic counterpart of the discrete-to-continuum Trotter–Kato approximation theorem for strongly continuous semigroups recalled in Theorem 5.2.3. In fact, with an eye towards the proof of Proposition 5.4.5 below, we consider a more general class of auxiliary processes, see equation (5.4.3).

Before stating any discrete-to-continuum results, let us introduce some convenient notation for this goal. Using the operators $(\Lambda_n)_{n\in\mathbb{N}}$ and $(\Pi_n)_{n\in\mathbb{N}}$, we can take a mapping which has E_n as its domain or state space, and turn it into an analogous mapping from or to E_∞. E.g., we define the E_∞-valued processes $\widetilde{W}_{A_n} := \Lambda_n W_{A_n}$, as well as the operators $\widetilde{R}_n^\alpha := \Lambda_n R_n^\alpha \Pi_n$ and $\widetilde{S}_n(t) := \Lambda_n S_n(t)\Pi_n$ in $\mathscr{L}(E_\infty)$ for all $\alpha, t \in [0,\infty)$. Now we can formulate our notion of the convergence $A_n \to A_\infty$ as $n \to \infty$ as follows:

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(A3) For every $x \in E_\infty$, we have $\widetilde{R}_n^1 x \to R_\infty^1 x$ in E_∞ as $n \to \infty$. Moreover, for $\beta \in [0, \frac{1}{2})$ as in (A2), we have $\widetilde{R}_n^\beta \to R_\infty^\beta$ in $\gamma(H_\infty; E_\infty)$.

We will see in Proposition 5.4.4 below that the convergence assumption (A3) of the linear operators is sufficient to ensure convergence of the solutions to the linear stochastic evolution equation (5.4.1). Its proof uses Lemma 5.C.1 in Appendix 5.C, as well as the following general approximation lemma for square-integrable functions with values in the space of γ-radonifying operators. It is a simplified analog to [131, Lemma 2.6], which was only necessary to allow for stochastic equations in UMD Banach spaces without type 2.

Lemma 5.4.3. *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces, and let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space. Given $a, b \in \mathbb{R}$ with $a < b$, let $M_n \colon (a,b) \to \mathscr{L}(E;F)$ for all $n \in \overline{\mathbb{N}}$ and suppose that*

(i) $M_n \otimes x \to M \otimes x$ uniformly on compact subsets of (a,b) for all $x \in E$, and

(ii) $\sup_{n\in\overline{\mathbb{N}}} \sup_{t\in(a,b)} \|M_n(t)\|_{\mathscr{L}(E;F)} < \infty$.

For all $R \in L^2(a,b;\gamma(H;E))$ and $n \in \overline{\mathbb{N}}$, we have $M_n \otimes R \in L^2(a,b;\gamma(H;F))$ and

$$M_n R \to MR \quad \text{in } L^2(a,b;\gamma(H;F)) \quad \text{as } n \to \infty.$$

Proof. Arguing as in the proof of Proposition 5.B.3 and using assumption (ii), it follows that we only need to prove the claim for all R belonging to some dense subset D of $L^2(a,b;\gamma(H;E))$. Note that every $R \in L^2(a,b;\gamma(H;E))$ can be approximated by a step function $\sum_{j=1}^N \mathbf{1}_{(a_j,b_j')} \otimes T_j$ with $a < a_j' < b_j' < b$ and $T_j \in \gamma(H;E)$, and by definition of $\gamma(H;E)$ the latter can be approximated by finite-rank operators. By linearity, it thus suffices to prove the statement for R of the form

$$R(t) = \mathbf{1}_{(a',b')}(t)\, h \otimes x, \quad \text{where } a < a' < b' < b \text{ and } (h,x) \in H \times E.$$

Substituting this representation, using (5.2.3) and (i), we find as $n \to \infty$:

$$\begin{aligned}\|M_nR - MR\|^2_{L^2(a,b;\gamma(H;F))} &= \int_{a'}^{b'} \|h\otimes[M_n(t)x - M(t)x]\|^2_{\gamma(H;F)}\,\mathrm{d}t\\ &= \|h\|^2_H \int_{a'}^{b'} \|M_n(t)x - M(t)x\|^2_F\,\mathrm{d}t\\ &\le \|h\|^2_H(b'-a') \sup_{t\in(a',b')} \|M_n(t)x - M(t)x\|^2_F \to 0. \quad \square\end{aligned}$$

Proposition 5.4.4. *Suppose that Assumptions (A1) and (A2) hold. Let us define the auxiliary processes*

$$W^\delta_{A_n}(t) := \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1} S_n(t-s)\,\mathrm{d}W_n(s), \quad \delta\in(1/2,\infty),\ t\in[0,\infty), \tag{5.4.3}$$

where Γ denotes the Gamma function [163, Section 5.2]. Then, for every $\beta'\in(\beta,\infty)$, $T\in(0,\infty)$ and $p\in[1,\infty)$, we have $W^{\beta'+\frac12}_{A_n}\in C([0,T];L^p(\Omega;E_n))$ for all $n\in\overline{\mathbb{N}}$. If we suppose in addition that Assumption (A3) is satisfied, then

$$\widetilde{W}^{\beta'+\frac12}_{A_n} \to W^{\beta'+\frac12}_{A_\infty} \quad \text{in } C([0,T];L^p(\Omega;E_\infty)) \quad \text{as } n\to\infty.$$

Proof. The fact that $W^{\beta'+\frac12}_{A_n}\in C([0,T];L^p(\Omega;E_n))$ for all $n\in\overline{\mathbb{N}}$ can be established by arguing as in Proposition 5.4.1, thus using Assumptions (A1) and (A2). For $t\in[0,T]$, the Itô inequality (5.2.4) yields

$$\begin{aligned}&\Big\|\widetilde{W}^{\beta'+\frac12}_{A_n}(t) - W^{\beta'+\frac12}_{A_\infty}(t)\Big\|_{L^p(\Omega;E_\infty)}\\ &\quad\lesssim_{(p,E_\infty)} \frac{1}{\Gamma(\beta'+\frac12)}\Big(\int_0^t (t-s)^{2\beta'-1}\|\widetilde{S}_n(t-s) - S_\infty(t-s)\|^2_{\gamma(H_\infty;E_\infty)}\,\mathrm{d}s\Big)^{\frac12}\\ &\quad\le \frac{1}{\Gamma(\beta'+\frac12)}\Big(\int_0^T s^{2\beta'-1}\|\widetilde{S}_n(s) - S_\infty(s)\|^2_{\gamma(H_\infty;E_\infty)}\,\mathrm{d}s\Big)^{\frac12}.\end{aligned}$$

Since semigroups commute with fractional powers of their infinitesimal generators, we can write the difference between the semigroups as follows:

$$\begin{aligned}\widetilde{S}_n(s) - S_\infty(s) &= \Lambda_n(\mathrm{Id}_n + A_n)^\beta S_n(s)\Pi_n\widetilde{R}^\beta_n - (\mathrm{Id}_\infty + A_\infty)^\beta S_\infty(s)R^\beta_\infty\\ &= \Lambda_n(\mathrm{Id}_n + A_n)^\beta S_n(s)\Pi_n(\widetilde{R}^\beta_n - R^\beta_\infty)\\ &\quad + (\Lambda_n(\mathrm{Id}_n + A_n)^\beta S_n(s)\Pi_n - (\mathrm{Id}_\infty + A_\infty)^\beta S_\infty)R^\beta_\infty.\end{aligned}$$

Thus, by the triangle inequality, it suffices to show that

$$\begin{aligned}\text{(I)} &:= \int_0^T s^{2\beta'-1}\|\Lambda_n(\mathrm{Id}_n + A_n)^\beta S_n(s)\Pi_n(\widetilde{R}^\beta_n - R^\beta_\infty)\|^2_{\gamma(H_\infty;E_\infty)}\,\mathrm{d}s \quad\text{and}\\ \text{(II)} &:= \int_0^T s^{2\beta'-1}\|[\Lambda_n(\mathrm{Id}_n + A_n)^\beta S_n(s)\Pi_n - (\mathrm{Id}_\infty + R_\infty)^\beta S_\infty]R^\beta_\infty\|^2_{\gamma(H_\infty;E_\infty)}\,\mathrm{d}s\end{aligned}$$

tend to zero as $n\to\infty$. Applying the ideal property (5.2.2) of $\gamma(H_\infty;E_\infty)$, followed by the analytic semigroup estimate (5.C.2) in conjunction with Assumptions (A1) and (A2), we find

$$\begin{aligned}(\mathrm{I}) &\le \|\widetilde{R}_n^\beta - R_\infty^\beta\|^2_{\gamma(H_\infty;E_\infty)}\int_0^T s^{2\beta'-1}\|\Lambda_n(\mathrm{Id}_n+A_n)^\beta S_n(s)\Pi_n\|^2_{\mathscr{L}(E_\infty)}\,\mathrm{d}s\\ &\lesssim_{(\beta,M_\Lambda,M_\Pi)} \|\widetilde{R}_n^\beta - R_\infty^\beta\|^2_{\gamma(H_\infty;E_\infty)}\int_0^T s^{2(\beta'-\beta)-1}\,\mathrm{d}s\\ &= \frac{T^{2(\beta'-\beta)}}{2(\beta'-\beta)}\|\widetilde{R}_n^\beta - R_\infty^\beta\|^2_{\gamma(H_\infty;E_\infty)}\to 0,\end{aligned}\tag{5.4.4}$$

where the convergence on the last line follows from the second part of (A3). The convergence (II) $\to 0$ follows by applying Lemma 5.4.3 with

$$M_n(t):=t^\beta\Lambda_n(\mathrm{Id}_n+A_n)^\beta S_n(t)\Pi_n\quad\text{and}\quad R(t):=t^{\beta'-\beta-\frac12}R_\infty^\beta,$$

Indeed, this is justified since $R\in L^2(0,T;\gamma(H_\infty;E_\infty))$ with

$$\|R\|^2_{L^2(0,T;\gamma(H_\infty;E_\infty))}=\frac{T^{2(\beta'-\beta)}}{2(\beta'-\beta)}\|R_\infty^\beta\|^2_{\gamma(H_\infty;E_\infty)}<\infty,$$

condition (ii) is verified by applying (5.C.2) to $\|M_n(t)\|_{\mathscr{L}(E_\infty)}$ combined with Assumptions (A1) and (A2) as in (5.4.4), and hypothesis (i) holds by Lemma 5.C.1. □

We will show that there exist modifications of W_{A_∞} and $(\widetilde{W}_{A_n})_{n\in\mathbb{N}}$ which, for all $p\in[1,\infty)$ and $T\in(0,\infty)$, belong to $L^p(\Omega;C([0,T];E_\infty))$ and converge in this norm. In particular, as $n\to\infty$, their trajectories converge uniformly on bounded time intervals, $\mathbb{P}$-a.s.

The proof is based on the Da Prato–Kwapień–Zabczyk factorization method, first formulated in [53] for Hilbert spaces (see also [56, Section 5.3]), and later extended to UMD-type-2 Banach spaces in [37, Theorem 3.2]. This method was also used in Chapter 2. Recall that the general idea is to express the process W_{A_n} as the 'product' $\mathfrak{I}_{A_n}^{\frac12-\beta'}W_{A_n}^{\frac12+\beta'}$ of a fractional parabolic integral operator $\mathfrak{I}_{A_n}^{\frac12-\beta'}$ (whose definition and necessary properties are recalled in Appendix 5.B) and auxiliary process $W_{A_n}^{\frac12+\beta'}$ from (5.4.3), and using the smoothing properties of the former.

Proposition 5.4.5. *Let $p\in[1,\infty)$ and $T\in(0,\infty)$. If Assumptions (A1)–(A2) hold, then for every $n\in\overline{\mathbb{N}}$ there exists a modification of W_{A_n} which belongs to $L^p(\Omega;C([0,T];E_n)$, which we will identify with W_{A_n} itself.*

If, in addition, Assumption (A3) holds, then the sequence $(\widetilde{W}_{A_n})_{n\in\overline{\mathbb{N}}}$ satisfies

$$\widetilde{W}_{A_n}\to W_{A_\infty}\quad \text{in } L^p(\Omega;C([0,T];E_\infty))\quad \text{as } n\to\infty.$$

Proof. Let $\beta'\in(\beta,\frac12)$, where $\beta\in[0,\frac12)$ is as in (A2). Since $L^p(\Omega)$-spaces with higher exponents are embedded in those with lower ones contractively (which follows from $\mathbb{P}(\Omega)=1$), we assume without loss of generality that $p\in((\frac12-\beta)^{-1},\infty)$. By the first part of Proposition 5.4.4 (thus using Assumptions (A1) and (A2)) and the Fubini theorem, we have for all $n \in \overline{\mathbb{N}}$:

$$W_{A_n}^{\frac{1}{2}+\beta'} \in C([0,T];L^p(\Omega;E_n)) \hookrightarrow L^p(0,T;L^p(\Omega;E_n)) \cong L^p(\Omega;L^p(0,T;E_n)),$$

where the constants for the first embedding depend only on p and T. In particular, there exists $\Omega_1 \subseteq \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that $W_{A_n}^{\frac{1}{2}+\beta'}(\omega,\cdot)$ belongs to $L^p(0,T;E_n)$ for all $\omega \in \Omega_1$. It then follows from Proposition 5.B.1(b) that $\mathfrak{I}_{A_n}^{\frac{1}{2}-\beta'} W_{A_n}^{\frac{1}{2}+\beta'}(\omega,\cdot)$ belongs to $C([0,T];E_n)$, where $(\mathfrak{I}_{A_n}^s)_{s\in[0,\infty)}$ are the fractional parabolic integral operators defined by (5.B.2) in Appendix 5.B. In this case, the process $\mathfrak{I}_{A_n}^{\frac{1}{2}-\beta'} W_{A_n}^{\frac{1}{2}+\beta'}$ (set to zero outside of Ω_1) belongs to $L^p(\Omega;C([0,T];E_n))$, and by the factorization theorem [37, Theorem 3.2] it is a modification of W_{A_n}.

For the lifted processes, the properties of the embeddings and projections from assumption (A1) imply that $\widetilde{\mathfrak{I}}_{A_n}^{\frac{1}{2}-\beta'} \widetilde{W}_{A_n}^{\frac{1}{2}+\beta'} \in L^p(\Omega;C([0,T];E_\infty))$ is a continuous modification of $\widetilde{W}_{A_n}$, where $\widetilde{\mathfrak{I}}_{A_n}^s := \Lambda_n \mathfrak{I}_{A_n}^s \Pi_n$. Identifying $\widetilde{W}_{A_n}$ with its factorized continuous modification for every $n \in \overline{\mathbb{N}}$, we can estimate as follows:

$$\begin{aligned}\|\widetilde{W}_{A_n} - W_{A_\infty}\|_{L^p(\Omega;C([0,T];E_\infty))} &= \|\widetilde{\mathfrak{I}}_{A_n}^{\frac{1}{2}-\beta'} \widetilde{W}_{A_n}^{\frac{1}{2}+\beta'} - \mathfrak{I}_{A_\infty}^{\frac{1}{2}-\beta'} W_{A_\infty}^{\frac{1}{2}+\beta'}\|_{L^p(\Omega;C([0,T];E_\infty))} \\ &\le \|\widetilde{\mathfrak{I}}_{A_n}^{\frac{1}{2}-\beta'}(\widetilde{W}_{A_n}^{\frac{1}{2}+\beta'} - W_{A_\infty}^{\frac{1}{2}+\beta'})\|_{L^p(\Omega;C([0,T];E_\infty))} \\ &\quad + \|(\widetilde{\mathfrak{I}}_{A_n}^{\frac{1}{2}-\beta'} - \mathfrak{I}_{A_\infty}^{\frac{1}{2}-\beta'}) W_{A_\infty}^{\frac{1}{2}+\beta'}\|_{L^p(\Omega;C([0,T];E_\infty))}.\end{aligned}$$

Since $\frac{1}{2}-\beta' > \frac{1}{p}$, we can apply Corollary 5.B.2(b) to find that $\widetilde{\mathfrak{I}}_{A_n}^{\frac{1}{2}-\beta'}$ is a bounded linear operator from $L^p(0,T;E_\infty)$ to $C([0,T];E_\infty)$ whose norm can be bounded independently of n. Thus, by the above discussion and the second part of Proposition 5.4.4 (which uses Assumption (A3)), we find

$$\begin{aligned}&\|\widetilde{\mathfrak{I}}_{A_n}^{\frac{1}{2}-\beta'}(\widetilde{W}_{A_n}^{\frac{1}{2}+\beta'} - W_{A_\infty}^{\frac{1}{2}+\beta'})\|_{L^p(\Omega;C([0,T];E_\infty))} \\ &\lesssim \|\widetilde{W}_{A_n}^{\frac{1}{2}+\beta'} - W_{A_\infty}^{\frac{1}{2}+\beta'}\|_{L^p(\Omega\times(0,T);E_\infty)} \le T^{\frac{1}{p}} \|\widetilde{W}_{A_n}^{\frac{1}{2}+\beta'} - W_{A_\infty}^{\frac{1}{2}+\beta'}\|_{C([0,T];L^p(\Omega;E_\infty))} \to 0.\end{aligned}$$

Now we note that, for all $\omega \in \Omega$, Proposition 5.B.3(a) implies that

$$\|\widetilde{\mathfrak{I}}_{A_n}^{\frac{1}{2}-\beta'} W_{A_\infty}^{\frac{1}{2}+\beta'}(\omega,\cdot) - \mathfrak{I}_{A_\infty}^{\frac{1}{2}-\kappa} W_{A_\infty}^{\frac{1}{2}+\beta'}(\omega,\cdot)\|_{C([0,T];E_\infty)} \to 0.$$

Again by Corollary 5.B.2(b), we moreover have

$$\|\widetilde{\mathfrak{I}}_{A_n}^{\frac{1}{2}-\beta'} W_{A_\infty}^{\frac{1}{2}+\beta'}(\omega,\cdot) - \mathfrak{I}_{A_\infty}^{\frac{1}{2}-\beta'} W_{A_\infty}^{\frac{1}{2}+\beta'}(\omega,\cdot)\|_{C([0,T];E_\infty)} \lesssim 2\|W_{A_\infty}^{\frac{1}{2}+\beta'}\|_{L^p(0,T;E_\infty)}$$

with constant independent of $n \in \mathbb{N}$, and since $W_{A_\infty}^{\frac{1}{2}+\beta'} \in L^p((0,T)\times\Omega;E_\infty)$, the dominated convergence theorem yields

$$\|(\widetilde{\mathfrak{I}}_{A_n}^{\frac{1}{2}-\beta'} - \mathfrak{I}_{A_\infty}^{\frac{1}{2}-\beta'}) W_{A_\infty}^{\frac{1}{2}+\beta'}\|_{L^p(\Omega;C([0,T];E_\infty))} \to 0. \qquad \square$$

5.5. APPROXIMATION OF SEMILINEAR EQUATIONS WITH ADDITIVE CYLINDRICAL WIENER NOISE

In this section, we shall extend the results from Section 5.4 regarding the *linear* E_n-valued equation (5.4.1) to the *semilinear case*. As before, let the spaces $(E_n)_{n\in\overline{\mathbb{N}}}$, $(H_n)_{n\in\overline{\mathbb{N}}}$ and the operators $(A_n)_{n\in\overline{\mathbb{N}}}$ satisfy assumptions (A1) and (A2), respectively, and suppose that $W_n := \Pi_n W_\infty$ is an H_n-valued Q_n-cylindrical Wiener process (with $Q_n := \Pi_n \Pi_n^*$), supported on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. Let $T \in (0,\infty)$ be a finite time horizon. In this section, we suppose moreover that we are given a drift coefficient function $F_n \colon \Omega \times [0,T] \times E_n \to E_n$ and initial datum $\xi_n \colon \Omega \to E_n$. We will consider the following semilinear stochastic evolution equation:

$$\begin{cases} \mathrm{d}X_n(t) = -A_n X_n(t)\,\mathrm{d}t + F_n(t, X_n(t))\,\mathrm{d}t + \mathrm{d}W_n(t), & t \in (0,T], \\ X_n(0) = \xi_n. \end{cases} \tag{5.5.1}$$

Note that $F_n(\omega, t, \cdot)$ is a (nonlinear) operator on E_n for all $(\omega, t) \in \Omega \times [0,T]$; in Section 5.3, we considered the specific case $[F_n(\omega, t, u_n)](x) := f_n(\omega, t, u_n(x))$ for some real-valued nonlinearity f_n.

In what follows, we shall impose more precise conditions on the F_n and ξ_n to ensure the well-posedness of (5.5.1) for every fixed $n \in \overline{\mathbb{N}}$ and to obtain discrete-to-continuum convergence of the respective solutions as $n \to \infty$.

In Section 5.5.1 we will assume in particular that the drifts $(F_n)_{n\in\overline{\mathbb{N}}}$ are uniformly globally Lipschitz and of linear growth to obtain the existence of unique global solutions $(X_n(t))_{t\in[0,T]}$, whose lifted counterparts $\widetilde{X}_n$ converge to X_∞ with respect to the $L^p(\Omega; C([0,T]; E_\infty))$-norm as $n \to \infty$, where $p \in [1,\infty)$ is the stochastic integrability of the initial data. These assumptions are relaxed in Section 5.5.2, where we suppose that the drifts are uniformly locally Lipschitz and uniformly bounded near zero. In general, this comes at the cost of obtaining merely local solutions, converging in a weaker norm. However, if one can show independently that the solutions are global and the $L^p(\Omega; C([0,T]; E_n))$-norms of X_n are uniformly bounded in $n \in \mathbb{N}$, then we recover the stronger sense of convergence.

5.5.1. GLOBALLY LIPSCHITZ DRIFTS OF LINEAR GROWTH

In this section we suppose that the drift coefficients $F_n \colon \Omega \times [0,T] \times E_n \to E_n$ in (5.5.1) for $n \in \overline{\mathbb{N}}$ are uniformly globally Lipschitz continuous and of linear growth. More precisely:

(F1) There exist constants $L_F, C_F \in (0,\infty)$ such that, for every $t \in [0,T]$, $\omega \in \Omega$, $n \in \overline{\mathbb{N}}$ and $x_n, y_n \in E_n$,

$$\|F_n(\omega, t, x_n) - F_n(\omega, t, y_n)\|_{E_n} \le L_F \|x_n - y_n\|_{E_n};$$
$$\|F_n(\omega, t, x_n)\|_{E_n} \le C_F(1 + \|x_n\|_{E_n}).$$

Moreover, the process $(\omega, t) \mapsto F_n(\omega, t, x_n)$ is strongly measurable and adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$.

Now let us comment on the existence and uniqueness of solutions to (5.5.1) for fixed $n \in \overline{\mathbb{N}}$. We will use the following concept of global mild solutions, see [155,

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pp. 969–970]. In Subsection 5.5.2, we also introduce the concept of local solutions, which may blow up in finite time. In particular, a local solution which exists $\mathbb{P}$-a.s. on the whole of $[0,T]$ is in fact global.

Recall that $(S_n(t))_{t\geq 0}$ denotes the C_0-semigroup on E_n generated by $-A_n$.

Definition 5.5.1. An E_n-valued stochastic process $X_n = (X_n(t))_{t\in[0,T]}$ is a global mild solution to (5.5.1) with coefficients (A_n, F_n, ξ_n) if

(i) $X_n\colon \Omega\times[0,T]\to E_n$ is strongly measurable and $(\mathcal{F}_t)_{t\in[0,T]}$-adapted;

(ii) $s\mapsto S_n(t-s)F_n(s,X_n(s))\in L^0(\Omega;L^1(0,t;E_n))$ for every $t\in[0,T]$;

(iii) $s\mapsto S_n(t-s)\in L^2(0,t;\gamma(H_n;E_n))$ for every $t\in[0,T]$;

(iv) for all $t\in[0,T]$, we have

$$X_n(t) = S_n(t)\xi_n + \int_0^t S_n(t-s)F_n(s,X_n(s))\,\mathrm{d}s + W_{A_n}(t), \quad \mathbb{P}\text{-a.s.}$$

In the present framework, existence and uniqueness can be proven by showing that the operator $\Phi_{n,T}$ given by

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$$[\Phi_{n,T}(u_n)](t) := S_n(t)\xi_n + \int_0^t S_n(t-s)F_n(s,u_n(s))\,\mathrm{d}s + W_{A_n}(t) \tag{5.5.2}$$

is a well-defined and Lipschitz-continuous mapping on $L^p(\Omega;C([0,T];E_n))$, whose Lipschitz constant tends to zero as $T\downarrow 0$ (see [155, Proposition 6.1] or [131, Theorem 3.7] for more general results):

Proposition 5.5.2. *Suppose that Assumptions (A1), (A2) and (F1) hold. Let $n\in\overline{\mathbb{N}}$, $p\in[1,\infty)$, $\xi_n\in L^p(\Omega,\mathcal{F}_0,\mathbb{P};E_n)$ and $T\in(0,\infty)$. The operator $\Phi_{n,T}$ given by (5.5.2) is well defined and Lipschitz continuous on $L^p(\Omega;C([0,T];E_n))$. Its Lipschitz constant is independent of ξ_n, depends on A_n and F_n only through M_S and L_F, and tends to zero as $T\downarrow 0$.*

Proof. The fact that $S_n\otimes\widetilde{\xi}_n\in L^p(\Omega;C([0,T];E_n))$ is immediate from (A1)–(A2) and $\xi_n\in L^p(\Omega,\mathcal{F}_0,\mathbb{P};E_n)$. We also have $W_{A_n}\in L^p(\Omega;C([0,T];E_n))$ by the first part of Proposition 5.4.5. Given $u_n\in L^p(\Omega;C([0,T];E_n))$, it follows from (F1) that

$$\|s\mapsto F_n(s,u_n(s))\|_{L^\infty(0,T;E_n)} \leq C_F(1+\|u_n\|_{C([0,T];E_n)}),$$

so that $S_n * F_n(\cdot,u_n)$ belongs to $L^p(\Omega;C([0,T];E_n))$ with

$$\|S_n * F_n(\cdot,u_n)\|_{L^p(\Omega;C([0,T];E_n))} \leq C_F(1+\|u_n\|_{L^p(\Omega;C([0,T];E_n))})$$

by Proposition 5.B.1(b) with $E=F=E_n$, $\alpha=0$ and $s=1$ (noting that $S_n * f = \mathfrak{I}^1_{A_n} f$, see Appendix 5.B). This shows that $\Phi_{n,T}$ is well-defined.

Now let $u_n, v_n\in L^p(\Omega;C([0,T];E_n))$ and observe that

$$\Phi_{n,T}(u_n)-\Phi_{n,T}(v_n) = \int_0^t S_n(t-s)[F_n(s,u_n(s))-F_n(s,v_n(s))]\,\mathrm{d}s.$$

A straightforward estimate involving Assumptions (A1), (A2) and (F1) then yields

$$\|\Phi_{n,T}(u_n)-\Phi_{n,T}(v_n)\|_{L^p(\Omega;C([0,T];E_n))} \leq M_S L_F T\,\|u_n-v_n\|_{L^p(\Omega;C([0,T];E_n))}. \qquad \square$$

Under the conditions of Proposition 5.5.2, it follows from the Banach fixed-point theorem that (5.5.1) has a unique solution on a small enough time interval $[0, T_0]$, which can be extended to a unique global mild solution on any $[0, T]$ by "patching together" solutions on small time intervals:

Proposition 5.5.3. *Suppose that Assumptions (A1), (A2) and (F1) are satisfied, and let $n \in \overline{\mathbb{N}}$, $p \in [1,\infty)$, $\xi_n \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; E_n)$ and $T \in (0,\infty)$. Then (5.5.1) has a unique global mild solution $X_n \in L^p(\Omega; C([0, T]; E_n))$.*

Proof. By Proposition 5.5.2, there exists $T_0 \in (0,\infty)$ such that Φ_{n,T_0} is a strict contraction on $L^p(\Omega; C([0, T_0]; E_n))$, and thus has a unique fixed point X_n. Since the bound on the Lipschitz constant of $\Phi_{n,T}$ only depended on M_S, L_F and T, we can repeat the previous argument to obtain a unique solution $Y \in L^p(\Omega; C([0, T_0]; E_n))$ for (5.5.1) with initial datum $\eta_n := X_n(\frac{1}{2}T_0)$, drift $G_n(\,\cdot\,, u_n) := F_n(\,\cdot\, + \frac{1}{2}T_0, u_n)$ and noise $\widehat{W}_n := W_n(\,\cdot\, + \frac{1}{2}T_0)$. It can then be argued directly using Definition 5.5.1 that the concatenation of the processes X_n and $Y_n(\,\cdot\, + \frac{1}{2}T_0)$ is the unique mild solution to (5.5.1) with the original data on $[0, \frac{3}{2}T_0]$. Proceeding inductively, we find the same conclusion for all intervals $[0, (k + \frac{1}{2})T_0]$ with $k \in \mathbb{N}$ and thus for $[0, T]$. ☐

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For $n \in \mathbb{N}$, we analogously define the lifted initial datum $\widetilde{\xi}_n : \Omega \to E_\infty$ by $\widetilde{\xi}_n := \Lambda_n \xi_n$ and the lifted drift coefficient $\widetilde{F}_n : \Omega \times [0, T] \times E_\infty \to E_\infty$ by

$$\widetilde{F}_n(\omega, t, x) := \Lambda_n F_n(\omega, t, \Pi_n x), \quad (\omega, t, x) \in \Omega \times [0, T] \times E_\infty. \tag{5.5.3}$$

We will assume that the initial data and drift coefficients are approximated in the following way:

(IC) There exists $p \in [1,\infty)$ such that $(\xi_n)_{n\in\overline{\mathbb{N}}} \in \prod_{n\in\overline{\mathbb{N}}} L^p(\Omega, \mathcal{F}_0, \mathbb{P}; E_n)$ and

$$\widetilde{\xi}_n \to \xi_\infty \quad \text{in } L^p(\Omega; E_\infty) \text{ as } n \to \infty.$$

(F2) For a.e. $(\omega, t) \in \Omega \times [0, T]$ and every $x \in E_\infty$, we have

$$\widetilde{F}_n(\omega, t, x) \to F_\infty(\omega, t, x) \quad \text{in } E_\infty \text{ as } n \to \infty.$$

Under these assumptions, we obtain the main result of this section, namely the following discrete-to-continuum convergence theorem in the context of uniformly globally Lipschitz nonlinearities of linear growth. It is analogous to [131, Theorem 4.3].

Theorem 5.5.4. *Suppose that Assumptions (A1)–(A3), (F1)–(F2) and (IC) are satisfied, with $p \in [1,\infty)$. For all $n \in \overline{\mathbb{N}}$ and $T \in (0,\infty)$, let $X_n = (X_n(t))_{t\in[0,T]}$ denote the unique global solution to (5.5.1), and let $\widetilde{X}_n := \Lambda_n X_n$. Then we have*

$$\widetilde{X}_n \to X_\infty \quad \textit{in } L^p(\Omega; C([0, T]; E_\infty)) \quad \textit{as } n \to \infty.$$

Its proof involves the lifted counterparts of $\Phi_{n,T}$, defined by

$$\widetilde{\Phi}_{n,T} := \Lambda_n \Phi_{n,T} \circ \Pi_n : L^p(\Omega; C([0, T]; E_\infty)) \to L^p(\Omega; C([0, T]; E_\infty)),$$

i.e., for all $u \in L^p(\Omega; C([0,T]; E_\infty))$ and $t \in [0,T]$, we have

$$\begin{aligned}[\widetilde{\Phi}_{n,T}(u)](t) &:= \Lambda_n S_n(t)\xi_n + \int_0^t \Lambda_n S_n(t-s) F_n(s, \Pi_n u(s))\,\mathrm{d}s + \Lambda_n W_{A_n}(t) \\ &= \widetilde{S}_n(t)\widetilde{\xi}_n + \int_0^t \widetilde{S}_n(t-s)\widetilde{F}_n(s,u(s))\,\mathrm{d}s + \widetilde{W}_{A_n}(t), \quad \mathbb{P}\text{-a.s.},\end{aligned}$$

where the second identity is due to Assumption 5.2.1(iii). Using the tensor and convolution notations from Section 5.2.1, it can be expressed even more concisely as

$$\widetilde{\Phi}_{n,T}(u) = \widetilde{S}_n \otimes \xi_n + \widetilde{S}_n * \widetilde{F}_n(\cdot, u) + \widetilde{W}_{A_n} \tag{5.5.4}$$

In particular, we will show that all three terms of (5.5.4) converge to their "continuum" counterparts; they are addressed by Lemmas 5.5.5–5.5.6 below (which are analogous to [131, Lemma 4.4, 4.5(1) and 4.5(3)]), as well as Proposition 5.4.5.

Lemma 5.5.5. *If (A1)–(A3) and (IC) hold with $p \in [1,\infty)$, then we have*

$$\widetilde{S}_n \otimes \xi_n \to S_\infty \otimes \xi_\infty \quad \text{in } L^p(\Omega; C([0,T]; E_\infty)) \quad \text{as } n \to \infty.$$

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Proof. As in the beginning of the proof of Proposition 5.5.2, it follows from (A1)–(A2) and (IC) that $S_n \otimes \xi_n \in L^p(\Omega; C([0,T]; E_n))$ for all $n \in \overline{\mathbb{N}}$. Applying the projection and lifting operators from (A1), we thus find $\widetilde{S}_n \otimes \widetilde{\xi}_n \in L^p(\Omega; C([0,T]; E_\infty))$.

The triangle inequality implies

$$\begin{aligned}&\|\widetilde{S}_n \otimes \widetilde{\xi}_n - S_\infty \otimes \xi_\infty\|_{L^p(\Omega;C([0,T];E_\infty))} \\ &\quad \le \|\widetilde{S}_n \otimes (\widetilde{\xi}_n - \xi_\infty)\|_{L^p(\Omega;C([0,T];E_\infty))} + \|(\widetilde{S}_n - S_\infty) \otimes \xi_\infty\|_{L^p(\Omega;C([0,T];E_\infty))}.\end{aligned}$$

By (A1)–(A2) and (IC), for the first term we have, as $n \to \infty$:

$$\|\widetilde{S}_n \otimes (\widetilde{\xi}_n - \xi_\infty)\|_{L^p(\Omega;C([0,T];E_\infty))} \le M_\Lambda M_S M_\Pi \|\widetilde{\xi}_n - \xi_\infty\|_{L^p(\Omega;E_\infty)} \to 0$$

For the second term, first note that $\widetilde{S}_n \otimes \xi_\infty(\omega) \to S_\infty \otimes \xi_\infty(\omega)$ in $C([0,T]; E_\infty)$, $\mathbb{P}$-a.s., by Theorem 5.2.3, where we now also use (A3). Since we moreover have

$$\|\widetilde{S}_n \otimes \xi_\infty(\omega) - S_\infty \otimes \xi_\infty(\omega)\|_{C([0,T];E_\infty)} \le M_S(M_\Lambda M_\Pi + 1)\|\xi_\infty(\omega)\|_{E_\infty},$$

and the right-hand side belongs to $L^p(\Omega)$ by assumption, we deduce that also

$$\|(\widetilde{S}_n - S_\infty) \otimes \xi_\infty\|_{L^p(\Omega;C([0,T];E_\infty))} \to 0 \quad \text{as } n \to \infty. \qquad \square$$

Lemma 5.5.6. *Suppose that Assumptions (A1), (A2), (F1) and (F2) are satisfied. Let $p \in [1,\infty)$ and $u \in L^p(\Omega; C([0,T]; E_\infty))$ be given. Then we have*

$$\widetilde{S}_n * \widetilde{F}_n(\cdot, u) \to S_\infty * F_\infty(\cdot, u) \quad \text{in } L^p(\Omega; C([0,T]; E_\infty)) \text{ as } n \to \infty.$$

Proof. Similarly to the proof of Proposition 5.5.2, it follows from (A1)–(A2) and (F1) that $\widetilde{S}_n * \widetilde{F}_n(\cdot, u) \in L^p(\Omega; C([0,T]; E_\infty))$ for all $n \in \overline{\mathbb{N}}$. By the triangle inequality, we can split up the statement into the following two assertions:

(i) $\widetilde{S}_n * \widetilde{F}_n(\cdot, u) - \widetilde{S}_n * F_\infty(\cdot, u) \to 0$ in $L^p(\Omega; C([0,T]; E_\infty))$ as $n \to \infty$;

(ii) $\widetilde{S}_n * F_\infty(\cdot, u) \to S_\infty * F_\infty(\cdot, u)$ in $L^p(\Omega; C([0,T]; E_\infty))$ as $n \to \infty$.

For almost every $(\omega, t) \in \Omega \times [0,T]$, we have by (F1) and (A1):

$$\begin{aligned} &\|\widetilde{F}_n(\omega, t, u(\omega,t)) - F_\infty(\omega, t, u(\omega,t))\|_{E_\infty} \\ &\quad \le C_F(M_\Lambda + 1 + (M_\Pi M_\Lambda + 1)\|u(\omega,t)\|_{E_\infty}). \end{aligned} \tag{5.5.5}$$

It follows that

$$\begin{aligned} \|\widetilde{F}_n(\cdot, u) - F_\infty(\cdot, u)\|_{L^p(\Omega\times(0,T);E_\infty)} &\lesssim_{(C_F, M_\Lambda, M_\Pi)} \|u\|_{L^p(\Omega\times(0,T);E_\infty)} \\ &\lesssim_{(p,T)} \|u\|_{L^p(\Omega;C([0,T];E_\infty))} < \infty. \end{aligned} \tag{5.5.6}$$

Since $\widetilde{S}_n * f = \widetilde{\mathfrak{I}}^1_{A_n} f$ for all $f \in L^p(0,T;E_\infty)$, we can apply Proposition 5.B.1(b) with $E = F = E_\infty$, $\alpha = 0$ and $s = 1$ to find that

$$\begin{aligned} &\|\widetilde{S}_n * (\widetilde{F}_n(\cdot, u) - F_\infty(\cdot, u))\|_{C([0,T];E_\infty)} \\ &\quad \lesssim_{(s,r,T,M_S)} \|\widetilde{F}_n(\cdot, u) - F_\infty(\cdot, u)\|_{L^p(\Omega\times(0,T);E_\infty)}. \end{aligned}$$

The latter tends to zero as $n \to \infty$ by the dominated convergence theorem, which applies in view of (F2) and (5.5.5)–(5.5.6). This shows (i).

For (ii), we derive in the same way that, for almost every $\omega \in \Omega$,

$$t \mapsto F_\infty(\omega, t, u(\omega,t)) \in L^p(0,T;E_\infty),$$

which implies, cf. Proposition 5.B.3(a) with $\widetilde{E} := E_\infty$, that

$$\widetilde{S}_n * F_\infty(\omega, \cdot, u(\omega, \cdot)) \to S_\infty * F_\infty(\omega, \cdot, u(\omega, \cdot)) \quad \text{in } C([0,T];E_\infty) \quad \text{as } n \to \infty.$$

The conclusion follows by using the uniform boundedness of the operators $(\widetilde{\mathfrak{I}}^1_{A_n})_{n\in\overline{\mathbb{N}}}$ in $\mathscr{L}(L^p(0,T;E_\infty); C([0,T];E_\infty))$, asserted in Corollary 5.B.2(b) (with $\widetilde{E} := E_\infty$ once more), and finishing the dominated convergence argument as in part (i). ☐

Proof of Theorem 5.5.4. By Proposition 5.5.2 and (A1), for small enough $T_0 \in (0,\infty)$ there exists a constant $c \in [0,1)$, depending only on L_F, M_S, M_Λ and M_Π, such that, for all $u, v \in L^p(\Omega; C([0,T_0]; E_\infty))$,

$$\sup_{n\in\mathbb{N}} \|\widetilde{\Phi}_{n,T_0}(u) - \widetilde{\Phi}_{n,T_0}(v)\|_{L^p(\Omega;C([0,T_0];E_\infty))} \le c\|u - v\|_{L^p(\Omega;C([0,T_0];E_\infty))}.$$

Moreover, by Proposition 5.5.3, for every $n \in \overline{\mathbb{N}}$, there exists a unique global solution $X_n \in L^p(\Omega; C([0,T]; E_n))$ to (5.5.1), which in particular satisfies $X_n = \Phi_{n,T_0}(X_n)$ when restricted to $[0,T_0]$. By (A1), this implies $\widetilde{X}_n = \widetilde{\Phi}_{n,T_0}(\widetilde{X}_n)$. Hence,

$$\begin{aligned} \|X_\infty - \widetilde{X}_n\|_{L^p(\Omega;C([0,T_0];E_\infty))} &= \|\Phi_{\infty,T_0}(X_\infty) - \widetilde{\Phi}_{n,T_0}(\widetilde{X}_n)\|_{L^p(\Omega;C([0,T_0];E_\infty))} \\ &\le \|\Phi_{\infty,T_0}(X_\infty) - \widetilde{\Phi}_{n,T_0}(X_\infty)\|_{L^p(\Omega;C([0,T_0];E_\infty))} \\ &\quad + \|\widetilde{\Phi}_{n,T_0}(X_\infty) - \widetilde{\Phi}_{n,T_0}(\widetilde{X}_n)\|_{L^p(\Omega;C([0,T_0];E_\infty))} \\ &\le \|\Phi_{\infty,T_0}(X_\infty) - \widetilde{\Phi}_{n,T_0}(X_\infty)\|_{L^p(\Omega;C([0,T_0];E_\infty))} \\ &\quad + c\|X_\infty - \widetilde{X}_n\|_{L^p(\Omega;C([0,T_0];E_\infty))}, \end{aligned}$$

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so that Lemmas 5.5.5–5.5.6 and Proposition 5.4.5 yield (using all of the Assumptions (A1)–(A3), (F1)–(F2) and (IC)):

$$\begin{aligned}&\|X_\infty - \widetilde{X}_n\|_{L^p(\Omega;C([0,T_0];E_\infty))}\\ &\quad\le \frac{1}{1-c}\|\Phi_{\infty,T_0}(X_\infty) - \widetilde{\Phi}_{n,T_0}(X_\infty)\|_{L^p(\Omega;C([0,T_0];E_\infty))} \to 0\end{aligned} \tag{5.5.7}$$

as $n\to\infty$. In order to extend the convergence to arbitrary time horizons, we write

$$\begin{aligned}\|X_\infty - \widetilde{X}_n\|_{L^p(\Omega;C([0,\frac{3}{2}T_0];E_\infty))} &\le \|X_\infty - \widetilde{X}_n\|_{L^p(\Omega;C([0,\frac{1}{2}T_0];E_\infty))}\\ &\quad + \|X_\infty - \widetilde{X}_n\|_{L^p(\Omega;C([\frac{1}{2}T_0,\frac{3}{2}T_0];E_\infty))}.\end{aligned}$$

The first term tends to zero as $n\to\infty$ by (5.5.7). As for the second term, we note that $X_n|_{[\frac{1}{2}T_0,\frac{3}{2}T_0]}$ are the respective solutions to (5.5.1) with shifted drift functions $(F_n(\cdot+\frac{1}{2}T_0,\cdot))_{n\in\overline{\mathbb{N}}}$ and initial values $(X_n(\frac{1}{2}T_0))_{n\in\overline{\mathbb{N}}}$. Since the Lipschitz constants of the fixed point operators defined above did not depend on the initial datum and only depended on F through its (time-independent) Lipschitz constant L_F, we can repeat the same argument to find that the second term tends to zero. Proceeding by induction, we obtain the convergence $\widetilde{X}_n \to X_\infty$ in $L^p(\Omega;C([0,(1+\frac{k}{2})T_0];E_\infty))$ for any $k\in\mathbb{N}$, and thus in $L^p(\Omega;C([0,T];E_\infty))$ for any $T\in(0,\infty)$. □

5.5.2. LOCALLY LIPSCHITZ NONLINEARITIES

In this section, we work under the weaker assumption that the drift coefficients $(F_n)_{n\in\overline{\mathbb{N}}}$ are *locally* Lipschitz continuous, with local-Lipschitz constants uniformly bounded in $t\in[0,T]$, $\omega\in\Omega$ and $n\in\overline{\mathbb{N}}$. Moreover, we replace the uniform linear growth condition from (F1) by the assumption that $F_n(t,\omega,0)$ is bounded, again uniformly in t, ω and n; we will also call this notion of boundedness local since it only involves $u=0$. Thus, we assume that the $(F_n)_{n\in\overline{\mathbb{N}}}$ are locally uniformly Lipschitz and locally uniformly bounded:

(F1′) For all $r\in(0,\infty)$ there exists a constant $L_F^{(r)}\in(0,\infty)$ such that for almost every $(\omega,t)\in\Omega\times[0,T]$, all $n\in\overline{\mathbb{N}}$ and every $x_n,y_n\in E_n$ such that $\|x_n\|_{E_n},\|y_n\|_{E_n}\le r$, we have

$$\|F_n(\omega,t,x_n)-F_n(\omega,t,y_n)\|_{E_n}\le L_F^{(r)}\|x_n-y_n\|_{E_n}.$$

Moreover, for every $x_n\in E_n$, $n\in\overline{\mathbb{N}}$ the process $(\omega,t)\mapsto F_n(\omega,t,x_n)$ is strongly measurable and adapted, and there exists a constant $C_{F,0}$ such that

$$\|F_n(\omega,t,0)\|_{E_n}\le C_{F,0}\quad\text{for all } n\in\overline{\mathbb{N}}.$$

Under these conditions, we can in general not expect to obtain global solutions of (5.5.1) in the sense of Definition 5.5.1. Instead, we need to work with locally defined E_n-valued stochastic processes, i.e., with mappings of the form

$$Y\colon\{(\omega,t)\in\Omega\times[0,T]: t\in[0,\tau(\omega))\}\to E_n \tag{5.5.8}$$

for some stopping time $\tau\colon\Omega\to[0,T]$. We denote such a process by $Y=(Y(t))_{t\in[0,\tau)}$. If the half-open interval $[0,\tau(\omega))$ in (5.5.8) is replaced by $[0,\tau(\omega)]$, then instead we write $Y=(Y(t))_{t\in[0,\tau]}$. We say that $(Y(t))_{t\in[0,\tau)}$ is *admissible* if

- for all $t \in [0,T]$, the mapping $\{\omega \in \Omega : t < \tau(\omega)\} \ni \omega \mapsto Y(\omega,t) \in E_n$ is measurable with respect to $\mathcal{F}_t$;
- the mapping $[0,\tau(\omega)) \ni t \mapsto Y(\omega,t) \in E_n$ is continuous, $\mathbb{P}$-a.s.

Let $V^{\mathrm{loc}}([0,\tau)\times\Omega;E_n)$ be the space of admissible E_n-valued processes $(Y(t))_{t\in[0,\tau)}$ for which there exists a sequence $(\tau_m)_{m\in\mathbb{N}}$ of stopping times such that, for $\mathbb{P}$-a.e. $\omega\in\Omega$, we have $\tau_m(\omega)\uparrow\tau(\omega)$ as $m\to\infty$ and $\|Y\|_{C([0,\tau_m(\omega)];E_n)}<\infty$ for all $m\in\mathbb{N}$. As in [155, Section 8], we define local solutions to (5.5.1) as follows:

Definition 5.5.7. An admissible E_n-valued stochastic process $X_n = (X_n(t))_{t\in[0,\tau)}$ is said to be a local solution to (5.5.1) with coefficients (A_n, F_n, ξ_n) if there exists a sequence $(\tau_m)_{m\in\mathbb{N}}$ of stopping times such that $\tau_m\uparrow\tau$ as $m\to\infty$, $\mathbb{P}$-a.s., and for all $m\in\mathbb{N}$ we have

(i) for every $t\in[0,T]$, the process $(\omega,s)\mapsto S_n(t-s)F_n(\omega,s,X_n(\omega,s))\mathbf{1}_{[0,\tau_m]}(s)$ belongs to $L^0(\Omega;L^1(0,t;E_n))$;

(ii) for every $t\in[0,T]$, $s\mapsto S_n(t-s)\mathbf{1}_{[0,\tau_m]}(s)\in L^2(0,t;\gamma(H_n;E_n))$;

(iii) it holds $\mathbb{P}$-a.s. that for all $t\in[0,\tau_m]$, we have

$$X_n(t) = S_n(t)\xi_n + \int_0^t S_n(t-s)F_n(s,X_n(s))\mathbf{1}_{[0,\tau_m]}(s)\,\mathrm{d}s + \int_0^t S_n(t-s)\mathbf{1}_{[0,\tau_m]}(s)\,\mathrm{d}W_n(s).$$

We say that a local solution $(X_n(t))_{t\in[0,\tau)}$ to (5.5.1) is *maximal* if for any other local solution $(\overline{X}_n(t))_{t\in[0,\overline{\tau})}$ it holds $\mathbb{P}$-a.s. that $\overline{\tau}\le\tau$ and $X_n|_{[0,\overline{\tau})}\equiv\overline{X}_n$. It is called *global* if $\tau = T$ holds $\mathbb{P}$-a.s. and there exists a solution $(\widehat{X}_n(t))_{t\in[0,T]}$ to (5.5.1) in the sense of Definition 5.5.1 such that $\widehat{X}_n|_{[0,\tau)}\equiv X_n$, $\mathbb{P}$-a.s. The stopping time τ is called an *explosion time* if

$$\limsup_{t\uparrow\tau(\omega)}\|X_n(\omega,t)\|_{E_n}=\infty \quad \text{for a.e. } \omega\in\Omega \text{ such that } \tau(\omega)<T. \tag{5.5.9}$$

The following local well-posedness result then follows from [155, Theorem 8.1]:

Theorem 5.5.8 ([155, Theorem 8.1])**.** *Suppose that Assumptions (A1), (A2) and (F1′) are satisfied, and let $n\in\overline{\mathbb{N}}$, $p\in[1,\infty)$, $\xi_n\in L^p(\Omega,\mathcal{F}_0,\mathbb{P};E_n)$ and $T\in(0,\infty)$ be given. Then (5.5.1) has a unique maximal local mild solution $(X_n(t))_{t\in[0,\sigma_n)}$ in the space $V^{\mathrm{loc}}([0,\sigma_n)\times\Omega;E_n)$, where $\sigma_n\colon\Omega\to[0,T]$ is an explosion time.*

Combined with the convergence assumptions (IC) and (F2), we can argue analogously to [132, Theorem 3.3 and Corollary 3.4] to derive the following extension of Theorem 5.5.4 to the present setting.

Theorem 5.5.9. *Suppose that Assumptions (A1), (A2), (IC), (F1′) and (F2) are satisfied. For $n\in\overline{\mathbb{N}}$, let $(X_n(t))_{t\in[0,\sigma_n)}$ be the maximal local solution to (5.5.1) with explosion time $\sigma_n\colon\Omega\to[0,T]$, and set $\widetilde{X}_n := \Lambda_n X_n$. Then the following is true:*

(i) We have $\widetilde{X}_n\mathbf{1}_{[0,\sigma_\infty\wedge\sigma_n)}\to X_\infty\mathbf{1}_{[0,\sigma_\infty)}$ in $L^0(\Omega\times[0,T];E_\infty)$ as $n\to\infty$.

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If, moreover, $\sigma_n = T$ holds $\mathbb{P}$-a.s. for all $n \in \mathbb{N}$ and $p \in [1,\infty)$ is such that

$$\sup_{n\in\mathbb{N}} \|X_n\|_{L^p(\Omega;C([0,T];E_n))} < \infty, \tag{5.5.10}$$

then the following assertions also hold:

(ii) We have $\sigma_\infty = T$, $\mathbb{P}$-a.s.

(iii) If $p \in (1,\infty)$, then for all $p^- \in [1,p)$ we have

$$\widetilde{X}_n \to X_\infty \quad \text{in } L^{p^-}(\Omega; C([0,T];E_\infty)) \text{ as } n \to \infty.$$

Similarly to [132, Theorem 3.3 and Corollary 3.4], the proof of Theorem 5.5.9 relies on the following general approximation results for locally defined processes:

Theorem 5.5.10 ([132, Theorem 2.1 and Corollary 2.5])**.** *Let $(E, \|\cdot\|_E)$ be a real and separable Banach space and $T \in (0,\infty)$. For every $n \in \overline{\mathbb{N}}$, suppose that $(Y_n(t))_{t\in[0,\sigma_n)}$ is a continuous and adapted E-valued locally defined process with explosion time $\sigma_n : \Omega \to (0,T]$, and define the stopping times $\rho_n^{(r)} : \Omega \to [0,T]$ by*

$$\rho_n^{(r)} := \inf\{t \in (0,\sigma_n) : \|Y_n(t)\|_E > r\}, \quad r \in (0,\infty), \tag{5.5.11}$$

with the convention that $\inf\varnothing := T$. Moreover, suppose that for each $r \in (0,\infty)$ there exists a (globally defined) continuous and adapted E-valued process $(Y_n^{(r)}(t))_{t\in[0,T]}$ which satisfies the following two conditions:

(a) *For all $n \in \overline{\mathbb{N}}$ and $r \in (0,\infty)$, it holds $\mathbb{P}$-a.s. that*

$$Y_n^{(r)} \mathbf{1}_{[0,\rho_n^{(r)}]} \equiv Y_n \mathbf{1}_{[0,\rho_n^{(r)}]} \quad \text{on } [0,T];$$

(b) *For all $r \in (0,\infty)$ we have*

$$Y_n^{(r)} \to Y_\infty^{(r)} \quad \text{in } L^0(\Omega; C([0,T];E)) \text{ as } n \to \infty.$$

Then the following assertions hold:

(i) For all $r \in (0,\infty)$ and $\varepsilon > 0$ it holds $\mathbb{P}$-a.s. that

$$\liminf_{n\to\infty} \rho_n^{(r)} \le \rho_\infty^{(r)} \le \limsup_{n\to\infty} \rho_n^{(r+\varepsilon)}.$$

(ii) For all $r \in (0,\infty)$ and $\varepsilon > 0$, we have

$$Y_n \mathbf{1}_{[0,\rho_\infty^{(r)} \wedge \rho_n^{(r+\varepsilon)})} \to Y_\infty \mathbf{1}_{[0,\rho_\infty^{(r)})} \quad \text{in } L^0(\Omega; B_{\mathrm{b}}([0,T];E)) \text{ as } n \to \infty,$$

where $B_{\mathrm{b}}([0,T];E)$ denotes the space of bounded and strongly measurable functions from $[0,T]$ to E.

(iii) We have

$$Y_n \mathbf{1}_{[0,\sigma_\infty\wedge\sigma_n)} \to Y_\infty \mathbf{1}_{[0,\sigma_\infty)} \quad \text{in } L^0(\Omega \times [0,T]; E) \text{ as } n \to \infty.$$

If, in addition, we have $\mathbb{P}(\sigma_n = T) = 1$ *for all* $n \in \mathbb{N}$*, and* $p \in [1,\infty)$ *is such that*

$$\sup_{n\in\mathbb{N}} \|Y_n\|_{L^p(\Omega;C([0,T];E))} < \infty,$$

then the following assertions also hold:

(iv) We have $\mathbb{P}(\sigma_\infty = T) = 1$ *and* $X_\infty \in L^q(\Omega; C([0,T];E))$.

(v) If $p \in (1,\infty)$*, then for all* $p^- \in [1,p)$ *we have*

$$Y_n \to Y_\infty \quad \text{in } L^{p^-}(\Omega; C([0,T];E)) \text{ as } n\to\infty.$$

Proof of Theorem 5.5.9. For every $n \in \overline{\mathbb{N}}$ and $r \in (0,\infty)$, let us define the mapping $F_n^{(r)}\colon \Omega\times[0,T]\times E_n \to E_n$ by

$$F_n^{(r)}(\omega,t,x_n) := \begin{cases} F_n(\omega,t,x_n), & \text{if } \|\Lambda_n x_n\|_{E_\infty} \le r,\\ F_n\Big(\omega,t,\frac{r x_n}{\|\Lambda_n x_n\|_{E_\infty}}\Big), & \text{otherwise.}\end{cases} \tag{5.5.12}$$

For any fixed $r>0$, the sequence $(F_n^{(r)})_{n\in\overline{\mathbb{N}}}$ satisfies conditions (F1) and (F2). Indeed, to establish the former, we first note that $F_n^{(r)}$ can be written as

$$F_n^{(r)}(\omega,t,x_n) = F_n(\omega,t,\Pi_n R_r(\Lambda_n x_n)), \tag{5.5.13}$$

where $R_r\colon E_\infty \to E_\infty$ denotes the canonical retraction of E_∞ onto the closed ball around $0\in E_\infty$ with radius r:

$$R_r(x) := \begin{cases} x, & \text{if } \|x\|_{E_\infty} \le r;\\ \frac{rx}{\|x\|_{E_\infty}}, & \text{otherwise.}\end{cases}$$

An elementary estimate shows that R_r is Lipschitz with constant 2. It follows that $(F_n^{(r)})_{n\in\overline{\mathbb{N}}}$ is uniformly globally Lipschitz with constant $L_{F^{(r)}} \le L_F^{(rM_\Pi)} M_\Pi M_\Lambda$, and thus of linear growth with uniform constant $C_{F^{(r)}} \le \max\{L_{F^{(r)}}, C_{F,0}\}$, so that it satisfies (F1).

In order to show that $(F_n^{(r)})_{n\in\overline{\mathbb{N}}}$ satisfies (F2), first note that for every sequence $(y_n)_{n\in\mathbb{N}} \subseteq E_\infty$ converging to some y in E_∞, we have $\widetilde{F}_n(\omega,t,y_n) \to F_\infty(\omega,t,y)$. Indeed, by the triangle inequality it suffices to note that

$$\|\widetilde{F}_n(\omega,t,y_n) - \widetilde{F}_n(\omega,t,y)\|_{E_\infty} \to 0 \quad \text{and} \quad \|\widetilde{F}_n(\omega,t,y) - F_\infty(\omega,t,y)\|_{E_\infty} \to 0$$

as $n\to\infty$, respectively because $(\widetilde{F}_n)_{n\in\overline{\mathbb{N}}}$ is uniformly locally Lipschitz (with constants $L_{\widetilde{F}}^{(r)} \le M_\Pi M_\Lambda L_F^{(r)}$) and since (F2) was assumed for $(F_n)_{n\in\mathbb{N}}$. Writing

$$\widetilde{F}_n^{(r)}(\omega,t,x) = \widetilde{F}_n(\omega,t,R_r(\Lambda_n\Pi_n x)),$$

see (5.5.13), we can apply the above observation to the sequence $y_n := R_r(\Lambda_n\Pi_n x)$, which converges to $y := R_r(x)$ in E_∞ as $n\to\infty$ in view of Assumption 5.2.1(ii) and the (Lipschitz) continuity of R_r. Therefore, we find $\widetilde{F}_n^{(r)}(\omega,t,x) \to F_\infty^{(r)}(\omega,t,x)$, thus proving the claim that (F2) holds for $(F_n^{(r)})_{n\in\overline{\mathbb{N}}}$ as well.

For each $r > 0$, condition (F1) for $(F_n^{(r)})_{n\in\overline{\mathbb{N}}}$ yields the existence of a unique global solution $(X_n^{(r)}(t))_{t\in[0,T]}$ to (5.5.1) with coefficients $(A_n, F_n^{(r)}, \xi_n)$. In order to establish statement (i) of the present theorem, we will apply the corresponding parts (i)–(iii) of Theorem 5.5.10 to the processes $Y_n := \widetilde{X}_n$; hence we need to verify its conditions (a) and (b) for $(\widetilde{X}_n)_{n\in\overline{\mathbb{N}}}$. First note that we have $\rho_n^{(r)} \le \sigma_n$, where $\rho_n^{(r)}$ is defined by (5.5.11), and that the restrictions of $X_n^{(r)}$ and X_n to $[0,\rho_n^{(r)})$ are local solutions to (5.5.1) with coefficients $(A_n, F_n^{(r)}, \xi_n)$ and (A_n, F_n, ξ_n), respectively. Since it holds $\mathbb{P}$-a.s. that $\|\Lambda_n X_n(t)\|_{E_\infty} \le r$ for $t \in [0,\rho_n^{(r)})$, we find

$$F_n(\cdot, X_n) \equiv F_n^{(r)}(\cdot, X_n) \quad \text{on } [0,\rho_n^{(r)}), \quad \mathbb{P}\text{-a.s.},$$

hence $(X_n(t))_{t\in[0,\rho_n^{(r)})}$ is in fact also a local solution of the equation with coefficients $(A_n, F_n^{(r)}, \xi_n)$. Therefore, the local uniqueness of (5.5.1) (cf. [155, Lemma 8.2]) implies that $X_n^{(r)} \equiv X_n$ on $[0,\rho_n^{(r)}]$ holds $\mathbb{P}$-a.s., and applying Λ_n on both sides verifies (a). Condition (b) follows by applying Theorem 5.5.4 to $(F_n^{(r)})_{n\in\overline{\mathbb{N}}}$, proving (i).

Finally, since $(\Lambda_n)_{n\in\mathbb{N}}$ is uniformly bounded in view of Assumption 5.2.1(i), we see that (5.5.10) implies that the conditions of Theorem 5.5.10(iv) and (v) are satisfied, which directly yields the remaining assertions (ii) and (iii). □

5.6. REACTION–DIFFUSION TYPE EQUATIONS

In this section, we introduce another family of Banach spaces $(B_n)_{n\overline{\mathbb{N}}}$ such that each B_n embeds into E_n and $\widetilde{B} \supseteq B_\infty$ (along with other assumptions, given in Subsection 5.6.1), and consider the B_n-valued counterparts of (5.5.1). The main purpose of this setting is to eventually specialize to the class of stochastic reaction–diffusion type equations which are formally given, for any $n \in \overline{\mathbb{N}}$, by

$$\begin{cases} dX_n(t,x) = -A_n X_n(t,x)\,dt + f_n(t, X_n(t,x))\,dt + dW_n(t,x), \\ X_n(0,x) = \xi_n(x), \end{cases} \tag{5.6.1}$$

where $t \in (0,T]$, $T \in (0,\infty)$, $x \in \mathcal{D}_n \subseteq \mathbb{R}^d$ and $f_n\colon \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz (real-valued) function. This problem amounts to letting the drift F_n in (5.5.1) be a Nemytskii operator (also called a *superposition operator*), i.e., defining it by

$$[F_n(\omega,t,u)](x) := f_n(\omega,t,u(x)), \quad (\omega,t,x) \in \Omega \times [0,T] \times \mathcal{D}_n, \tag{5.6.2}$$

for a given function $u\colon \mathcal{D}_n \to \mathbb{R}$. However, in order for a Nemytskii operator F_n to inherit the local Lipschitz continuity from f_n, we cannot view it as acting on the UMD-type-2 Banach space $E_n = L^q(\mathcal{D}_n)$ with $q \in [1,\infty)$. In fact, by [8, Theorem 3.9], the operator $u \mapsto F_n(\omega,t,u)$ defined by (5.6.2) is weakly continuous on $L^q(\mathcal{D}_n)$ (meaning that it maps weakly convergent sequences to weakly convergent sequences) if and only if it is affine in u, i.e., there exist coefficients $a_n(\omega,t), b_n(\omega,t) \in \mathbb{R}$ such that $[F_n(\omega,t,u)](x) = a_n(\omega,t) + b_n(\omega,t)u(x)$ for all $x \in \mathcal{D}_n$. In particular, if $(F_n)_{n\in\overline{\mathbb{N}}}$ is a family of Nemytskii operators which is uniformly locally Lipschitz in the sense of (F1′) on the spaces $E_n = L^q(\mathcal{D}_n)$, then it is in fact globally Lipschitz and of linear growth in the sense of (F1).

Thus, we will instead view F_n as an operator on $B_n := C(\overline{\mathcal{D}_n})$ (which coincides with $B_n = L^\infty(\mathcal{D}_n)$ if $\mathcal{D}_n$ is discrete). This, in turn, poses a difficulty for stochastic evolution equations, since there is no theory for the stochastic integration of integrands taking their values in a space of continuous functions. This is due to the poor geometric properties of $(C(\overline{\mathcal{D}_n}), \|\cdot\|_\infty)$ as a Banach space: The most general notion of stochastic integration in Banach spaces (see [157]) requires at least the UMD property; such spaces are, in particular, reflexive [113, Theorem 4.3.3], which $C(\overline{\mathcal{D}_n})$ fails to be.

One way to circumvent this issue is to proceed as in [132, Section 3.2]; namely, defining the fractional domain spaces

$$\dot{E}_n^\alpha := \mathsf{D}(A_n^{\alpha/2}), \quad \|x_n\|_{\dot{E}_n^\alpha} := \|(\mathrm{Id}_n + A_n)^{\alpha/2} x_n\|_{E_n},$$

and supposing that $\dot{E}_n^\theta \hookrightarrow C(\overline{\mathcal{D}_n}) \hookrightarrow L^q(\mathcal{D}_n)$ continuously and densely for $\theta \in [0,1)$, one can carry out the stochastic integration in the space $\dot{E}_n^\theta$, while working with $C(\overline{\mathcal{D}_n})$-valued processes for the fixed-point arguments.

In applications, we typically assume that A_n is an (unbounded) linear differential operator on $L^q(\mathcal{D}_n)$, where $q \in [2,\infty)$, augmented with some boundary conditions ("b.c.") such that $\dot{E}_n^\alpha$ is the fractional Sobolev space $W^{\alpha,q}_{\text{b.c.}}(\mathcal{D}_n)$ of order α. We then suppose that θ is chosen large enough in relation to the dimension d that we have the continuous and dense Sobolev embedding $W^{\alpha,q}_{\text{b.c.}}(\mathcal{D}_n) \hookrightarrow C_{\text{b.c.}}(\overline{\mathcal{D}_n})$.

In Section 5.6.1 we will specify the abstract formulation of the setting outlined above, as well as some additional uniformity conditions with respect to $n \in \overline{\mathbb{N}}$. These will be used to, as a first step, derive B_n-valued analogs to the E_n-valued discrete-to-continuum approximation results for globally Lipschitz drifts of linear growth from Subsection 5.5.1; in Subsection 5.6.2 we do the same for the B_n-valued setting with locally Lipschitz and locally bounded drifts. In the latter case, the solution are local in general, but in Section 5.6.3 we state an extra dissipativity assumption on F_n under which the existence of global solutions to (5.6.1) has been proven in [132, Section 4]. These processes then also converge in an improved sense, and we can apply this to Section 5.3.

5.6.1. Setting and convergence for globally Lipschitz drifts

We start by specifying the abstract setting for the treatment of reaction–diffusion type equations which was outlined at the beginning of this section. That is, we complement the Hilbert spaces $(H_n)_{n\in\overline{\mathbb{N}}}$ and UMD-type-2 Banach spaces $(E_n)_{n\in\overline{\mathbb{N}}}$ from Section 5.5 with a sequence of real separable Banach spaces $(B_n, \|\cdot\|_{B_n})_{n\in\overline{\mathbb{N}}}$, embedded continuously and densely into E_n for each $n \in \overline{\mathbb{N}}$. Moreover, we introduce the real Banach space $(\widetilde{B}, \|\cdot\|_{\widetilde{B}})$, containing B_∞ as a closed subspace, and we suppose that all the B_n are embedded by into $\widetilde{B}$ by the lifting operators $(\Lambda_n)_{n\in\mathbb{N}}$ from (A1). The spaces $B_\infty \subseteq \widetilde{B}$ should respectively be thought of as $C(\overline{\mathcal{D}}) \subseteq L^\infty(\mathcal{D})$. More precisely, we will work with the following extensions of assumptions (A1)–(A3):

(A1-B) Assumption (A1) holds, and Assumption 5.2.1 is satisfied for $(B_n, \|\cdot\|_{B_n})_{n\in\overline{\mathbb{N}}}$ and $(\widetilde{B}, \|\cdot\|_{B_\infty})$, with the same projection and lifting operators $(\Pi_n)_{n\in\overline{\mathbb{N}}}$,

$(\Lambda_n)_{n\in\overline{\mathbb{N}}}$ from (A1), for which we set

$$\widetilde{M}_\Pi := \sup_{n\in\mathbb{N}} \|\Pi_n\|_{\mathscr{L}(\widetilde{B};B_n)} \quad \text{and} \quad \widetilde{M}_\Lambda := \sup_{n\in\mathbb{N}} \|\Lambda_n\|_{\mathscr{L}(B_n;\widetilde{B})}.$$

(A2-B) Assumption (A2) holds, the semigroup $(S_n(t))_{t\geq 0} \subseteq \mathscr{L}(E_n)$ leaves B_n invariant for all $n \in \overline{\mathbb{N}}$, and its restriction $(S_n(t)|_{B_n})_{t\geq 0}$ to B_n is a strongly continuous semigroup in $\mathscr{L}(B_n)$. Moreover, there exists $\widetilde{M}_S \in [1,\infty)$ such that

$$\|S_n(t)\|_{\mathscr{L}(B_n)} \leq \widetilde{M}_S \quad \text{for all } n \in \overline{\mathbb{N}} \text{ and } t \in [0,\infty). \tag{5.6.3}$$

(A3-B) Assumption (A3) holds, and $\widetilde{R}_n x \to R_\infty x$ in $\widetilde{B}$ as $n \to \infty$ for all $x \in B_\infty$.

By [73, Chapter II, Proposition 2.3], assumptions (A1-B) and (A2-B) imply that the generator of $(S_n(t)|_{B_n})_{t\geq 0}$ is the operator $-A_n|_{B_n} : \mathsf{D}(A_n|_{B_n}) \subseteq B_n \to B_n$ defined by

$$-A_n|_{B_n} x_n := -A_n x_n \quad \text{on } \mathsf{D}(A_n|_{B_n}) := \{x_n \in B_n \cap \mathsf{D}(A_n) : A_n x_n \in B_n\},$$

which is known as *the part of* $-A_n$ *in* B_n. Therefore, by Theorem 5.2.3, assumption (A3-B) implies $\widetilde{S}_n|_{B_n} \otimes x \to S|_{B_\infty} \otimes x$ in $C([0,T];\widetilde{B})$, as $n \to \infty$, for all $x \in B_\infty$ and $T \in (0,\infty)$.

The following *uniform ultracontractivity* assumption is necessary in order to circumvent the aforementioned problem regarding stochastic integration in arbitrary separable Banach spaces B_n. It replaces assumption (A3) in [132], which forces all the spaces $(\dot{E}_n^\alpha)_{n\in\overline{\mathbb{N}}}$ to essentially be the same, which is not satisfied in applications such as discrete-to-continuum approximation, where each $\dot{E}_n^\alpha$ consists of functions defined on a different domain.

(A4-B) There exist constants $\theta \in [0,1)$ and $M_\theta \in [0,\infty)$ such that, for all $n \in \overline{\mathbb{N}}$, we have $\dot{E}_n^\theta \hookrightarrow B_n \hookrightarrow E_n$ continuously and densely, with

$$\|S_n(t)x_n\|_{B_n} \leq M_\theta t^{-\theta/2} \|x_n\|_{E_n} \quad \text{for all } x_n \in E_n \text{ and } t \in [0,\infty).$$

The uniformity in n of the constant M_θ enables us to prove the following discrete-to-continuum approximation result for the stochastic convolutions $(W_{A_n})_{n\in\overline{\mathbb{N}}}$ as B_n-valued processes (i.e., a B_n-valued counterpart to Proposition 5.4.5):

Proposition 5.6.1. *Let $p \in [1,\infty)$ and $T \in (0,\infty)$ be given, and suppose that Assumptions (A1-B), (A2-B) and (A4-B) hold, with $\theta + 2\beta < 1$. For every $n \in \overline{\mathbb{N}}$, there exists a modification of W_{A_n} which belongs to $L^p(\Omega; C([0,T]; B_n))$, and we identify these modifications with the processes $(W_{A_n})_{n\in\overline{\mathbb{N}}}$ themselves.*

Under the additional Assumption (A3-B), we have

$$\widetilde{W}_{A_n} \to W_{A_\infty} \quad \text{in } L^p(\Omega; C([0,T];\widetilde{B})) \text{ as } n \to \infty.$$

Proof. Fix a $\beta' \in (\beta, \frac{1}{2})$ such that $\theta + 2\beta' < 1$, where $\beta \in [0, \frac{1}{2})$ is as in (A2). Without loss of generality, suppose that $p \in (2,\infty)$ is so large that, in fact, $\theta + 2\beta' < 1 - \frac{2}{p}$. As in the proof of Proposition 5.4.5, we see that $W_{A_n}^{\frac{1}{2}+\beta'}(\omega,\cdot)$, where $W_{A_n}^{\frac{1}{2}+\beta'}$ is the auxiliary process defined by (5.4.3), belongs to $L^p(0,T;E_n)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, hence $\mathfrak{I}_{A_n}^{\frac{1}{2}-\beta'} W_{A_n}^{\frac{1}{2}+\beta'}(\omega,\cdot) \in C([0,T];B_n)$ by applying Proposition 5.B.1(b) with the spaces $E := E_n$, $F := B_n$ and the constant $\alpha := \theta$ from assumption (A4-B). Moreover, one finds that the process $\mathfrak{I}_{A_n}^{\frac{1}{2}-\beta'} W_{A_n}^{\frac{1}{2}+\beta'}$ is a continuous modification of W_{A_n}, belonging to $L^p(\Omega;C([0,T];B_n))$. It now suffices to establish the following, as $n\to\infty$:

$$\|\widetilde{\mathfrak{I}}_{A_n}^{\frac{1}{2}-\kappa}(\widetilde{W}_{A_n}^{\frac{1}{2}+\kappa} - W_{A_\infty}^{\frac{1}{2}+\kappa})\|_{L^p(\Omega;C([0,T];\widetilde{B}))} \to 0,$$
$$\|\widetilde{\mathfrak{I}}_{A_n}^{\frac{1}{2}-\kappa} W_{A_\infty}^{\frac{1}{2}+\kappa} - \mathfrak{I}_{A_\infty}^{\frac{1}{2}-\kappa} W_{A_\infty}^{\frac{1}{2}+\kappa}\|_{L^p(\Omega;C([0,T];\widetilde{B}))} \to 0.$$

These convergences follow by arguing as in the proof of Proposition 5.4.5, where we now need Corollary 5.B.2(c) and Proposition 5.B.3(b) instead of Corollary 5.B.2(b) and Proposition 5.B.3(a), respectively. □

For the initial data, we consider the following analog to (IC):

(IC-B) There exists $p \in [1,\infty)$ such that $(\xi_n)_{n\in\overline{\mathbb{N}}} \in \prod_{n\in\overline{\mathbb{N}}} L^p(\Omega;B_n)$ and

$$\widetilde{\xi}_n \to \xi_\infty \quad \text{in } L^p(\Omega;\widetilde{B}) \text{ as } n\to\infty.$$

Finally, regarding the drift coefficients $(F_n)_{n\in\overline{\mathbb{N}}}$, we now suppose that

(F1-B) Assumption (F1) holds with $(B_n)_{n\in\overline{\mathbb{N}}}$ in place of $(E_n)_{n\in\overline{\mathbb{N}}}$, and B_n-valued Lipschitz and growth constants respectively denoted by $\widetilde{L}_F$ and $\widetilde{C}_F$.

(F2-B) For almost every $(\omega,t) \in \Omega\times[0,T]$ and every $x \in B_\infty$ we have

$$\widetilde{F}_n(\omega,t,x) \to F_\infty(\omega,t,x) \quad \text{in } \widetilde{B} \text{ as } n\to\infty.$$

Note that in the approximation assumptions (A3-B) and (F2-B), we only impose convergence for $x \in B_\infty \subseteq \widetilde{B}$, and similarly we only consider $\xi_\infty \in L^p(\Omega;B_\infty)$ in (IC-B). Recall that this is sufficient since Theorem 5.2.3, on which the approximation results ultimately rely, is formulated in this setting.

Under (A1-B), (A2-B), (A4-B) (with $\theta+2\beta<1$), (IC-B) and (F1-B), we can derive well-posedness of B_n-*valued* global solutions to (5.5.1); these are defined by simply replacing the space E_n by B_n in Definition 5.5.1. To this end, we again investigate the fixed-point operators $\Phi_{n,T}$ defined by (5.5.2), viewing them now as acting on $L^p(\Omega;C([0,T];B_n))$. We have the following analog to Proposition 5.5.2:

Proposition 5.6.2. *Suppose that (A1-B), (A2-B), (A4-B) are satisfied with $\theta+2\beta<1$, and (F1-B) is satisfied. Let $n\in\overline{\mathbb{N}}$, $p\in[1,\infty)$, $\xi_n \in L^p(\Omega,\mathcal{F}_0,\mathbb{P};B_n)$ and $T\in(0,\infty)$ be given. The operator $\Phi_{n,T}$ given by* (5.5.2) *is a well-defined and Lipschitz continuous mapping on $L^p(\Omega;C([0,T];B_n))$. Its Lipschitz constant is independent of ξ_n, depends on A_n and F_n only through $\widetilde{M}_S$ and $\widetilde{L}_F$, and tends to zero as $T\downarrow 0$.*

Proof. The fact that $S_n\otimes\xi_n \in L^p(\Omega;C([0,T];B_n))$ is an immediate consequence of Assumptions (A1-B), (A2-B) and $\xi_n \in L^p(\Omega,\mathcal{F}_0,\mathbb{P};E_n)$. By the first part of Proposition 5.6.1 (which also uses (A4-B)), we find $W_{A_n} \in L^p(\Omega;C([0,T];B_n))$. By (F1-B) and

Proposition 5.B.1(b) (with $E = F = B_n$, $\alpha = 0$ and $s = 1$), we find that $S_n * F_n(\cdot, u_n)$ belongs to $L^p(\Omega; C([0,T]; B_n))$ for all $u_n \in L^p(\Omega; C([0,T]; B_n))$. This shows that $\Phi_{n,T}$ is well-defined. A straightforward estimate involving Assumptions (A1-B), (A2-B) and (F1-B) yields, for all $u_n, v_n \in L^p(\Omega; C([0,T]; B_n))$,

$$\|\Phi_{n,T}(u_n) - \Phi_{n,T}(v_n)\|_{L^p(\Omega;C([0,T];B_n))} \le \widetilde{M}_S \widetilde{L}_F T \|u_n - v_n\|_{L^p(\Omega;C([0,T];B_n))}. \qquad \square$$

Consequently, the proof of the following global well-posedness result is entirely analogous to that of Proposition 5.5.3:

Proposition 5.6.3. *Suppose (A1-B), (A2-B), (A4-B) hold with $\theta + 2\beta < 1$, and let assumption (F1-B) be satisfied. Suppose that $n \in \overline{\mathbb{N}}$, $p \in [1,\infty)$, $\xi_n \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; E_n)$ and $T \in (0,\infty)$. Then (5.5.1) has a unique global mild solution X_n in $L^p(\Omega; C([0,T]; B_n))$.*

Under the additional convergence assumptions (A3-B), (IC-B) and (F2-B), we can again set out to prove discrete-to-continuum convergence of B_n-valued global mild solutions to (5.5.1) by showing that all the expressions appearing in the fixed point maps $\widetilde{\Phi}_{n,T}$ from (5.5.2) are continuous as mappings on $L^p(\Omega; C([0,T]; \widetilde{B}))$. For the first term, the following can be proven exactly in the same way as Lemma 5.5.5:

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Lemma 5.6.4. *If Assumptions (A1-B)–(A3-B) and (IC-B) are satisfied, then we have $\widetilde{S}_n \otimes \widetilde{\xi}_n \to S_\infty \otimes \xi_\infty$ in $L^p(\Omega; C([0,T]; \widetilde{B}))$ as $n \to \infty$.*

The following is an analog to Lemma 5.5.6:

Lemma 5.6.5. *Suppose that Assumptions (A1-B)–(A3-B), (F1-B) and (F2-B) are satisfied. Let $p \in [1,\infty)$ and $u \in L^p(\Omega; C([0,T]; B_\infty))$ be given. Then we have*

$$\widetilde{S}_n * \widetilde{F}_n(\cdot, u) \to S_\infty * F_\infty(\cdot, u) \quad \text{in } L^p(\Omega; C([0,T]; \widetilde{B})) \text{ as } n \to \infty.$$

Proof. As in Lemma 5.5.6, we split up the statement into the following two assertions:

(i) $\widetilde{S}_n * \widetilde{F}_n(\cdot, u) - \widetilde{S}_n * F_\infty(\cdot, u) \to 0$ in $L^p(\Omega; C([0,T]; \widetilde{B}))$ as $n \to \infty$;

(ii) $\widetilde{S}_n * F_\infty(\cdot, u) \to S_\infty * F_\infty(\cdot, u)$ in $L^p(\Omega; C([0,T]; \widetilde{B}))$ as $n \to \infty$.

Part (i) is shown exactly as Lemma 5.5.6(i), up to replacing E_∞ by B_∞ (or $\widetilde{B}$).

For (ii), we instead note that (F1-B) implies, for almost every $\omega \in \Omega$,

$$t \mapsto F_\infty(\omega, t, u(\omega, t)) \in L^p(0,T; B_\infty).$$

Hence, in order to argue as in Lemma 5.5.6(ii), we apply Proposition 5.B.3(a) and Corollary 5.B.2(b) with $E_n := B_n$ for all $n \in \overline{\mathbb{N}}$ and $\widetilde{E} := \widetilde{B}$. $\square$

With these auxiliary results in place, we can now prove the first main discrete-to-continuum approximation result for solutions to (5.6.1) with globally Lipschitz drift coefficients of linear growth, analogously to Theorem 5.5.4:

Theorem 5.6.6. *Let Assumptions (A1-B)–(A4-B), (F1-B), (F2-B) and (IC-B) be satisfied, with $\theta + 2\beta < 1$ and $p \in [1,\infty)$. Denoting by X_n the unique B_n-valued global mild solution to (5.5.1), we have*

$$\widetilde{X}_n \to X_\infty \quad \text{in } L^p(\Omega; C([0,T]; \widetilde{B})) \quad \text{as } n \to \infty.$$

Proof. By Proposition 5.6.2 and (A1-B), for small enough $T_0 \in (0,\infty)$ there exists a constant $c \in [0,1)$, depending only on $\widetilde{L}_F$, $\widetilde{M}_S$, $\widetilde{M}_\Lambda$ and $\widetilde{M}_\Pi$, such that, for all u, v in $L^p(\Omega; C([0,T_0]; B_\infty))$,

$$\sup_{n\in\mathbb{N}} \|\widetilde{\Phi}_{n,T_0}(u) - \widetilde{\Phi}_{n,T_0}(v)\|_{L^p(\Omega;C([0,T_0];\widetilde{B}))} \le c\|u-v\|_{L^p(\Omega;C([0,T_0];\widetilde{B}))}.$$

By Proposition 5.6.3, there exists a unique global solution $X_n \in L^p(\Omega; C([0,T]; B_n))$ to (5.5.1) for every $n \in \overline{\mathbb{N}}$. In particular, note that X_∞ takes its values in $B_\infty \subseteq \widetilde{B}$. Thus, in order to finish the argument analogously to the proof of Theorem 5.5.4, it suffices to establish that $\widetilde{\Phi}_{n,T}(\phi) \to \Phi_{\infty,T}(\phi)$ in $L^p(\Omega; C([0,T]; \widetilde{B}))$ as $n \to \infty$ for all $\phi \in L^p(\Omega; C([0,T]; B_\infty))$. This is precisely the combined content of (the second part of) Proposition 5.6.1, along with Lemmas 5.6.4 and 5.6.5. □

5.6.2. LOCALLY LIPSCHITZ DRIFTS

As in Section 5.5.2, we can extend Theorem 5.6.6 to the locally Lipschitz setting. Namely, we assume that

(F1′-B) Assumption (F1′) holds with $(B_n)_{n\in\overline{\mathbb{N}}}$ in place of $(E_n)_{n\in\overline{\mathbb{N}}}$. For every $r > 0$, the B_n-valued local Lipschitz and local boundedness constants are respectively denoted by $\widetilde{L}_F^{(r)}$ and $\widetilde{C}_{F,0}^{(r)}$.

Then, arguing in the same way as Theorem 5.5.9, we obtain the following result.

Theorem 5.6.7. *Suppose that Assumptions (A1-B)–(A4-B), (F1′-B), (F2-B) and (IC-B) are satisfied, with $\theta+2\beta<1$. For $n \in \overline{\mathbb{N}}$, let $(X_n(t))_{t\in[0,\sigma_n)}$ be the maximal local solution to (5.6.1) with explosion time $\sigma_n \colon \Omega \to [0,T]$. Then we have*

(i) $\widetilde{X}_n \mathbf{1}_{[0,\sigma_\infty\wedge\sigma_n)} \to X_\infty \mathbf{1}_{[0,\sigma_\infty)}$ in $L^0(\Omega\times[0,T];\widetilde{B})$ as $n\to\infty$.

If, moreover, $\sigma_n = T$ holds $\mathbb{P}$-a.s. for all $n \in \mathbb{N}$ and $p \in [1,\infty)$ is such that

$$\sup_{n\in\mathbb{N}} \|X_n\|_{L^p(\Omega;C([0,T];B_n))} < \infty, \tag{5.6.4}$$

then the following assertions also hold:

(ii) We have $\sigma_\infty = T$, $\mathbb{P}$-a.s.

(iii) If $p \in (1,\infty)$, then for all $p^- \in [1,p)$ we have

$$\widetilde{X}_n \to X_\infty \quad \text{in } L^{p^-}(\Omega; C([0,T];\widetilde{B})) \text{ as } n\to\infty.$$

5.6.3. GLOBAL WELL-POSEDNESS AND CONVERGENCE FOR DISSIPATIVE DRIFTS

In this section, we consider a class of equations whose drift coefficients satisfy not only (F1′-B) (which would only guarantee *local* well-posedness), but additionally a type of dissipativity condition, also used in [132], which allows us to deduce global existence.

Let the subdifferential $\partial\|x_n\|_{B_n}$ of the norm $\|\cdot\|_{B_n}$ at $x_n \in B_n$ be given by

$$\partial\|x_n\|_{B_n} = \{x_n^* \in B_n^* : \|x_n^*\|_{B_n^*} \le 1 \text{ and } {}_{B_n}\langle x_n, x_n^*\rangle_{B_n^*} = \|x_n\|_{B_n}\}.$$

The assumptions on $(F_n)_{n\in\overline{\mathbb{N}}}$ under which we can derive global well-posedness, see Lemma 5.6.9 below (which is a simplified version of [132, Theorem 4.3] for equations driven by additive noise), are as follows:

(F1″-B) Let the conditions of (F1′-B) hold. Suppose that there exist $a', b' \in [0,\infty)$ and $N \in \mathbb{N}$ such that

$$_{B_n}\langle -A_n x_n + F_n(\omega, t, x_n + y_n), x_n^*\rangle_{B_n^*} \le a'(1 + \|y_n\|_{B_n})^N + b'\|x_n\|_{B_n}$$

for all $n \in \overline{\mathbb{N}}$, $(\omega, t) \in \Omega \times [0, T]$, $x_n \in \mathsf{D}(A_n|_{B_n})$, $x_n^* \in \partial\|x_n\|_{B_n}$ and $y_n \in B_n$.

If the semigroups $(S_n(t)|_{B_n})_{t\ge 0}$ are contractive on B_n, i.e., if $\widetilde{M}_S = 1$ in (A2-B), then we know that $A_n|_{B_n}$ is *accretive*, i.e.,

$$_{B_n}\langle A_n x_n, x_n^*\rangle_{B_n^*} \ge 0 \quad \text{for all } x_n \in \mathsf{D}(A_n|_{B_n}),\ x_n^* \in \partial\|x\|_{B_n}.$$

Thus, in this case, it suffices to check that

$$_{B_n}\langle F_n(\omega, t, x_n + y_n), x_n^*\rangle_{B_n^*} \le a'(1 + \|y_n\|_{B_n})^N, \tag{5.6.5}$$

5

in order to establish that (F1″-B) holds for $b' = 0$. The next example shows how the relation (5.6.5) can be verified in our situation of main interest. It is an elaborated version of [132, Example 4.2].

Example 5.6.8. Given $n \in \overline{\mathbb{N}}$, let $B_n := C(\mathcal{M}_n)$ be the space of continuous real-valued functions on a compact Hausdorff space $\mathcal{M}_n$ equipped with the supremum norm $\|u_n\|_{B_n} := \sup_{\xi\in\mathcal{M}_n} |u_n(\xi)|$. In this case, for all $u_n \in B_n$, the subdifferential $\partial\|u_n\|_{B_n}$ is the weak*-closed convex hull of

$$\{r\delta_{\hat{\xi}} : r \in \operatorname{sgn} u_n(\hat{\xi}) \text{ for } \hat{\xi} \in \mathcal{M}_n \text{ such that } \|u_n\|_{B_n} = |u_n(\hat{\xi})|\}, \tag{5.6.6}$$

where $\delta_\xi \in C(\mathcal{M}_n)^*$ denotes the evaluation functional at $\xi \in \mathcal{M}_n$ and, for $y \in \mathbb{R}$,

$$\operatorname{sgn} y := \begin{cases} \{-1\}, & \text{if } y < 0; \\ \{-1, 1\}, & \text{if } y = 0; \\ \{1\}, & \text{if } y > 0. \end{cases}$$

Indeed, since subdifferential sets are convex by definition and $\partial\|u_n\|_{B_n}$, being contained in the closed unit ball in B_n^*, is weak* compact by the Banach–Alaoglu theorem, the Krein–Milman theorem implies that it suffices to argue that the extreme points of $\partial\|u_n\|_{B_n}$ are precisely given by (5.6.6). This, in turn, follows from a characterization of the extreme points of the closed unit ball in $C(\mathcal{M}_n)^*$ due to Arens and Kelley [11].

Moreover, suppose that $(F_n)_{n\in\overline{\mathbb{N}}}$ is a family of Nemytskii operators on $(B_n)_{n\in\overline{\mathbb{N}}}$ (see equation (5.6.2)), generated by a family $(f_n)_{n\in\overline{\mathbb{N}}}$ of functions satisfying the polynomial form introduced in (5.3.12)–(5.3.13). Fixing $n \in \overline{\mathbb{N}}$, $(\omega, t) \in \Omega \times [0, T]$ and $u_n, v_n \in B_n$, inequality (5.6.5) becomes

$$_{B_n}\langle F_n(\omega, t, u_n + v_n), x_n^*\rangle_{B_n^*} \le a'(1 + \|v_n\|_{B_n})^N \quad \text{for all } x_n^* \in \partial\|u_n\|_{B_n}. \tag{5.6.7}$$

Since the inequality is preserved under convex combinations and weak* limits of x_n^*, the above characterization of $\partial\|u_n\|_{B_n}$ shows that it only needs to be checked for $x_n^* = r\delta_{\hat{\xi}}$, where $r \in \operatorname{sgn} u_n(\hat{\xi})$ for $\hat{\xi} \in \mathcal{M}_n$ such that $\|u_n\|_{B_n} = |u_n(\hat{\xi})|$. I.e., it suffices to verify that

$$r f_n(\omega, t, u_n(\hat{\xi}) + v_n(\hat{\xi})) \le a'(1 + \|v_n\|_{B_n})^N.$$

Indeed, for $(f_n)_{n\in\overline{\mathbb{N}}}$ satisfying (5.3.12)–(5.3.13), we can establish the estimate

$$r f_n(\omega, t, y+z) \lesssim_{(c,C,k)} (1+|z|)^{2k+1}$$

for all $(\omega, t) \in \Omega\times[0,T]$, $y, z \in \mathbb{R}$ and $r \in \operatorname{sgn} y$. This implies the existence of a constant $a' \in [0,\infty)$, depending only on $c, C \in (0,\infty)$ from (5.3.13) and $k \in \mathbb{N}_0$ from (5.3.12), such that (5.6.7) holds with $N = 2k+1$.

Lemma 5.6.9. *Let (A1-B)–(A4-B) and (F1″-B) hold with $\widetilde{M}_S = 1$, let $n \in \overline{\mathbb{N}}$ and suppose that $\theta + 2\beta < 1$. If $\xi_n \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; B_n)$ for some $p \in [1,\infty)$, then the maximal solution $(X_n(t))_{t\in[0,\sigma_n)}$ to (5.6.1) is global (i.e., it holds $\mathbb{P}$-a.s. that $\sigma_n = T$), and we have*

$$\|X_n\|_{L^p(\Omega;C([0,T];B_n))} \lesssim_{(a',b',T,N)} 1 + \|\xi_n\|_{L^p(\Omega;B_n)} + \|W_{A_n}\|^N_{L^{Np}(\Omega;C([0,T];B_n))}.$$

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Proof. Fix $n \in \overline{\mathbb{N}}$. For each $m \in \mathbb{N}$, let $F_{n,m}$ denote the globally Lipschitz retraction of F_n onto the closed ball of radius m around $0 \in B_n$, cf. (5.5.12) (replacing E_n by B_n). Then $F_{n,m}$ satisfies the global Lipschitz and (global) linear growth estimates in (F1-B), hence by Proposition 5.6.3 there exists a unique global B_n-valued mild solution $X_{n,m} \in L^p(\Omega; C([0,T]; B_n))$ to (5.5.1) with drift coefficient operator $F_{n,m}$. By the triangle inequality,

$$\begin{aligned}&\|X_{n,m}\|_{L^p(\Omega;C([0,T];B_n))}\\ &\quad\le \|S_n \otimes \xi_n + S_n * F_{n,m}(\cdot, X_{n,m})\|_{L^p(\Omega;C([0,T];B_n))} + \|W_{A_n}\|_{L^p(\Omega;C([0,T];B_n))}.\end{aligned}$$

As shown in the proof of [132, Theorem 4.3], $F_{n,m}$ inherits the dissipativity estimate (F1″-B), with the same constants a', b' and N, from F_n. Thus, from [132, Lemma 4.4] it follows that

$$\begin{aligned}&\|S_n \otimes \xi_n + S_n * F_{n,m}(\cdot, X_m)\|_{C([0,T];B_n)}\\ &\quad\le e^{b'T}\Big(\|\xi\|_{B_n} + a'\int_0^T (1+\|W_{A_n}(s)\|_{B_n})^N\,ds\Big)\\ &\quad\le e^{b'T}\|\xi\|_{B_n} + a'T2^{N-1}e^{b'T}(1+\|W_{A_n}\|^N_{C([0,T];B_n)}), \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

Hence,

$$\begin{aligned}&\|S_n \otimes \xi_n + S_n * F_{n,m}(\cdot, X_m)\|_{L^p(\Omega;C([0,T];B_n))}\\ &\quad\lesssim_{(a',b',T,N)} 1 + \|\xi_n\|_{L^p(\Omega;B_n)} + \|W_{A_n}\|^N_{L^{Np}(\Omega;C([0,T];B_n))}.\end{aligned}$$

Note that $W_{A_n} \in L^{Np}(\Omega; C([0,T]; B_n))$ by Proposition 5.6.1 (with Np taking the role of p). Combining these estimates, we find

$$\sup_{m\in\mathbb{N}} \|X_{n,m}\|_{L^p(\Omega;C([0,T];B_n))} \lesssim_{(a',b',T,N)} 1 + \|\xi_n\|_{L^p(\Omega;B_n)} + \|W_{A_n}\|^N_{L^{Np}(\Omega;C([0,T];B_n))},$$

so the result follows by Theorem 5.5.10(iv)–(v) applied to $Y_m := X_{n,m}$. □

Combined with Theorem 5.6.7(ii)–(iii), whose boundedness hypothesis (5.6.4) is verified by the combination of Lemma 5.6.9 and Proposition 5.6.1 under the assumption in the following corollary, we derive:

Corollary 5.6.10. *Suppose that (A1-B)–(A4-B), (F1′′-B), (F2-B) and (IC-B) are satisfied with $\widetilde{M}_S = 1$, $p \in (1,\infty)$ and $\theta + 2\beta < 1$. Then for any $p^- \in [1,p)$, the sequence $((X_n)_{t\in[0,T]})_{n\in\overline{\mathbb{N}}}$ of B_n-valued global solutions to* (5.6.1) *satisfies*

$$\widetilde{X}_n \to X_\infty \quad \text{in } L^{p^-}(\Omega; C([0,T];\widetilde{B})) \text{ as } n \to \infty.$$

5.7. OUTLOOK

In this section we suggest some possible extensions of the results in this chapter, both the convergence of graph-discretized Whittle–Matérn SPDEs shown in Section 5.3 as well as the underlying abstract results from Sections 5.4–5.6.

As discussed in Subsection 5.3.5, the approximation results from Theorem 5.3.10 for (5.3.11) might be extended to broader classes of domains $\mathcal{M}$, connectivity length regimes $(h_n)_{n\in\mathbb{N}}$, coefficient functions $\tau,\kappa\colon \mathcal{M} \to [0,\infty)$ and powers $s \in (0,\infty)$. Possible advancements to this end include establishing more general L^∞-convergence results for graph Laplacian eigenfunctions (or convergence of Whittle–Matérn operators without using spectral convergence), as well as uniform L^∞-boundedness of the semigroups (for instance via heat kernel estimates). Under more restrictive assumptions on the connectivity parameter regime, rates of convergence for the case of purely spatial (i.e., stationary) graph-discretized linear SPDEs were established in [181]. The same might be possible in the linear spatiotemporal setting since the discrete-to-continuum Trotter–Kato theorem can be extended to yield error estimates, see [116, Section 2.2]. The semilinear cases, however, appear to be out of reach for the methods used in this chapter.

The proofs of the abstract discrete-to-continuum approximation results from Sections 5.4–5.6 largely rely on incorporating projection and lifting operators into arguments from [131, 132] in an appropriate way. By adapting other proofs from these sources along similar lines, it is likely that our results can be extended further, in particular enabling us to relax the simplifying assumptions that the UMD Banach spaces $(E_n)_{n\in\overline{\mathbb{N}}}$ have type 2 and that the driving noise is additive. In fact, we expect more generally that many results asserting continuous dependence of stochastic evolution problems on their coefficients can be extended to discrete-to-continuum approximation theorems via this procedure.

One particular type of problem for which this would be interesting is the class of stochastic evolution *inclusions*, whose drift operators are allowed to be multivalued; this occurs, for instance, in the Langevin setting if $F_n = \partial\varphi_n$ is the subdifferential of a convex and lower-semicontinuous but non-differentiable functional φ_n on the state space, taking values in $(-\infty,\infty]$. Continuous dependence results for stochastic inclusion problems have been established in two different settings in [90, 183]; however, neither of these covers the class of *semilinear* inclusions driven by *cylindrical* (i.e., *white*) noise. Not much theory appears to be available for such problems, with even the question of well-posedness (for fixed $n \in \overline{\mathbb{N}}$) being highly nontrivial. In fact, to the best of our knowledge, the only results in this direction concern the (important) subclass of *stochastic reflection problems* (also known as *Skorokhod problems* in the scalar-valued case), given by

$$\begin{cases} \mathrm{d}X(t) \in -AX(t)\,\mathrm{d}t - \partial I_\Gamma(X(t))\,\mathrm{d}t + \mathrm{d}W(t), & t \in (0,T], \\ X(0) = \xi, \end{cases} \tag{5.7.1}$$

where Γ is a convex subset of a (Hilbertian) state space H and the *indicator functional* $I_\Gamma \colon H \to (-\infty,\infty]$ is defined to vanish on Γ and equal ∞ outside of it. In the first work on this problem, Nualart and Pardoux [159] used a direct approach to show existence and uniqueness in a setting which corresponds to $H := L^2(0,1)$, $A := -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with homogeneous Dirichlet or Neumann boundary conditions, and $\Gamma := K_0$, where

$$K_\alpha := \{u \in L^2(0,1) : u(x) \geq -\alpha \text{ for a.e. } x \in (0,1)\}, \quad \alpha \in [0,\infty).$$

In [175], the authors first use the theory of Dirichlet forms to establish that (5.7.1) is well posed in the case that Γ is a “regular” convex subset of a general Hilbert space H, a condition which includes $\Gamma := K_\alpha$ for $\alpha > 0$ but not K_0, which is treated separately using different techniques. Lastly, the work [17] describes a variational approach to study (5.7.1) in a similar setting under the assumption that 0 belongs to the interior of Γ, which excludes the choice $\Gamma := K_0$. We point out that the argument used on [17, p. 362] to extract a weak*-convergent sequence from the set $(u_\varepsilon)_{\varepsilon>0}$ in the dual space of $L^\infty(0,T;H)$ appears to be flawed, as it seems to imply that the closed unit ball of the dual of this (non-separable) space is *sequentially* compact, which is not the case. Hence, the argument would need to be finished using a *generalized* subsequence (also known as a sub*net*) converging in the weak* sense to some u^*. For this reason, and since the theory for convergence of Dirichlet forms and their associated processes is well established—see for instance [126] for general results and [202] for an application to Markov chain Monte Carlo scaling—the setting of [175] is perhaps the most promising for an attempt at establishing discrete-to-continuum convergence results for (5.7.1).

5

Appendix to Chapter 5

5.A. Proofs of intermediate results in Section 5.3.3

We start with the proof of Lemma 5.3.11 which establishes spectral convergence rates for the Laplace–Beltrami operator on the torus, discretized by a uniform grid.

Proof of Lemma 5.3.11. First suppose $m = 1$. In this case, the continuum Laplace–Beltrami operator reduces to the second derivative $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with periodic boundary conditions. Its eigenvalues and $L^2(\mathcal{M})$-normalized eigenfunctions are respectively given, for all $j \in \mathbb{N}$ and $x \in [0,1]$, by

$$\lambda_\infty^{(j)} = \begin{cases} j^2\pi^2 & \text{if } j \text{ is even,} \\ (j-1)^2\pi^2 & \text{if } j \text{ is odd;} \end{cases} \qquad \psi_\infty^{(j)}(x) = \begin{cases} 1 & \text{if } j = 1, \\ \sqrt{2}\sin(j\pi x) & \text{if } j \text{ is even,} \\ \sqrt{2}\cos((j-1)\pi x) & \text{if } j \text{ is odd.} \end{cases}$$

I.e., 0 is an eigenvalue corresponding to the constant 1 eigenfunction, and $(2k)^2\pi^2$ is an eigenvalue with eigenfunctions $x \mapsto \sin(2k\pi x), \cos(2k\pi x)$ for all $k \in \mathbb{N}$.

The eigenvalues $(\lambda_n^{(j)})_{j=1}^n$ of the corresponding graph Laplacian, which in this case reduces to the finite difference approximation of the second derivative on the grid with $n \in \mathbb{N}$ points, are given by

$$\lambda_n^{(j)} = \begin{cases} 4n^2 \sin^2\left(\frac{\pi j}{2n}\right) & \text{if } j \text{ is even,} \\ 4n^2 \sin^2\left(\frac{\pi (j-1)}{2n}\right) & \text{if } j \text{ is odd.} \end{cases}$$

The corresponding $L^2(\mathcal{M}_n)$-normalized eigenfunctions are

$$\psi_n^{(j)}(x_n^{(i)}) = \begin{cases} 1 & \text{if } j = 1, \\ (-1)^i & \text{if } j = n \text{ is even,} \\ \sqrt{2}\sin\big(j\pi x_n^{(i)}\big) & \text{if } j \neq n \text{ and } j \text{ is even,} \\ \sqrt{2}\cos\big((j-1)\pi x_n^{(i)}\big) & \text{if } j \text{ is odd.} \end{cases}$$

Let $j \in \{1,\ldots,n\}$. Supposing that j is even (the odd case being analogous), we write

$$\lambda_\infty^{(j)} - \lambda_n^{(j)} = j^2\pi^2\left(1 - \frac{4n^2}{j^2\pi^2}\sin^2\left(\frac{\pi j}{2n}\right)\right) = j^2\pi^2\left(1 - \left[\frac{2n}{j\pi}\sin\left(\frac{\pi j}{2n}\right)\right]^2\right),$$

so that the estimate in (5.3.15) for $m = 1$ follows from the elementary inequality

$$0 \le 1 - (\sin(x)/x)^2 \le \tfrac{1}{3}x^2 \quad \text{for all } x \in \mathbb{R}.$$

Estimate (5.3.16) is a consequence of the fact that the sine and cosine functions are 1-Lipschitz; note that we only consider $j \in \{1,\ldots,n-1\}$ to avoid the case where $j = n$ for even n.

The result for higher dimensions $m \in \mathbb{N}$ can be derived from the $m = 1$ case. Indeed, by separation of variables in the continuum case, or by writing the discretized operator as a Kronecker sum of m one-dimensional discretizations, one finds that the eigenvalues and eigenvectors of the m-dimensional operators are given by sums and products, respectively, of their 1-dimensional counterparts. From this, one can deduce the desired result. □

Next we prove Theorem 5.3.14 regarding the convergence of the fractional resolvent operators $\widetilde{R}_n^{\beta'}$ in various settings and norms.

Proof of Theorem 5.3.14. Assertions (a)–(c) can all be shown using analogous arguments. Thus, we only provide a detailed proof for part (b), being the most involved case, and subsequently summarize the changes needed for (a) and (c).

(b) **Step 1** (Setup and notation). The operator $\widetilde{R}_n^{\beta'}$ acts on functions $f \in L^2(\mathcal{M})$ in the following way:

$$\begin{aligned}\widetilde{R}_n^{\beta'} f &= \sum_{j=1}^{n} \left(1 + [\lambda_n^{(j)}]^s\right)^{-\beta'} \langle \Pi_n f, \psi_n^{(j)} \rangle_{L^2(\mathcal{M}_n)} \Lambda_n \psi_n^{(j)} \\ &= \sum_{j=1}^{n} \left(1 + [\lambda_n^{(j)}]^s\right)^{-\beta'} \langle f, \Lambda_n \psi_n^{(j)} \rangle_{L^2(\mathcal{M})} \Lambda_n \psi_n^{(j)}.\end{aligned}$$

Here, we used the fact that $\Pi_n^* = \Lambda_n$ (see (5.3.4)) on the second line. Using the tensor notation from Section 5.2.1 and denoting $\widetilde{\psi}_n^{(j)} := \Lambda_n \psi_n^{(j)}$, we can write these operators more concisely as follows:

$$U_n := \widetilde{R}_n^{\beta'} = \sum_{j=1}^{n} \left(1 + [\lambda_n^{(j)}]^s\right)^{-\beta'} \widetilde{\psi}_n^{(j)} \otimes \widetilde{\psi}_n^{(j)}.$$

By Assumption 5.3.5(ii), there exists a sequence of natural numbers $(k_n)_{n\in\mathbb{N}}$ which satisfies the conditions of Theorem 5.3.13(b), as well as the relation

$$k_n \gg n^{\frac{m}{4s\beta}} \gtrsim n^{\frac{m}{4s\beta'}}. \tag{5.A.1}$$

We use the sequence $(k_n)_{n\in\mathbb{N}}$ to define the following approximations for $n \in \mathbb{N}$:

$$\begin{aligned}U_n^1 &:= \sum_{j=1}^{k_n} \left(1 + [\lambda_\infty^{(j)}]^s\right)^{-\beta'} \psi_\infty^{(j)} \otimes \psi_\infty^{(j)}, \\ U_n^2 &:= \sum_{j=1}^{k_n} \left(1 + [\lambda_\infty^{(j)}]^s\right)^{-\beta'} \widetilde{\psi}_n^{(j)} \otimes \widetilde{\psi}_n^{(j)}, \\ U_n^3 &:= \sum_{j=1}^{k_n} \left(1 + [\lambda_n^{(j)}]^s\right)^{-\beta'} \widetilde{\psi}_n^{(j)} \otimes \widetilde{\psi}_n^{(j)}.\end{aligned}$$

For notational convenience, we abbreviate the $\mathscr{L}(L^2(\mathcal{M}); L^\infty(\mathcal{M}))$-norm by $\|\cdot\|_{2\to\infty}$ throughout this proof. We will make repeated use of the following estimate: Given an operator of the form $U = \sum_j \alpha_j e_j \otimes f_j$ for some scalars $(\alpha_j)_j \subseteq \mathbb{R}$, an orthonormal system $(e_j)_j \subseteq L^2(\mathcal{M})$ and some functions $(f_j)_j \subseteq L^\infty(\mathcal{M})$, we have by the Cauchy–Schwarz inequality:

$$\|U\|_{2\to\infty} \le \sup_j \|f_j\|_{L^\infty(\mathcal{M})} \Big(\sum_j |\alpha_j|^2\Big)^{\frac{1}{2}}. \tag{5.A.2}$$

Moreover, it is immediate from the definition of the $\|\cdot\|_{2\to\infty}$-norm that

$$\|h \otimes f\|_{2\to\infty} = \|h\|_{L^2(\mathcal{M})} \|f\|_{L^\infty(\mathcal{M})} \quad \text{for all } h \in L^2(\mathcal{M}) \text{ and } f \in L^\infty(\mathcal{M}). \tag{5.A.3}$$

Step 2 ($U_\infty - U_n^1$ and $U_n^3 - U_n$). For the difference between U_∞ and U_n^1, we find using (5.A.2) and Assumption 5.3.9:

$$\|U_\infty - U_n^1\|_{2\to\infty}^2 \le M_{\psi,\infty}^2 \sum_{j=k_n+1}^{\infty} \left(1 + [\lambda_\infty^{(j)}]^s\right)^{-2\beta'}. \tag{5.A.4}$$

Recalling Weyl's law (5.3.8), which implies that $\big(1+[\lambda_\infty^{(j)}]^s\big)^{-2\beta'} \eqsim_{\mathcal{M}} j^{-4s\beta'/m}$, we observe that the series on the right-hand side is convergent precisely when $\beta' > \frac{m}{4s}$. Since $k_n \to \infty$ as $n \to \infty$, this implies $U_\infty \to U_n^1$ in $\mathscr{L}(L^2(\mathcal{M}); L^\infty(\mathcal{M}))$.

Similarly, for the difference between U_n^3 and U_n, we have

$$\|U_n^3 - U_n\|_{2\to\infty}^2 \le M_{\psi,\infty}^2 \sum_{j=k_n+1}^{n} \big(1+[\lambda_n^{(j)}]^s\big)^{-2\beta'} \le M_{\psi,\infty}^2 (n-k_n)\big(1+[\lambda_n^{(k_n)}]^s\big)^{-2\beta'},$$

where the second inequality is due to the non-decreasing order of $(\lambda_n^{(j)})_{j=1}^n$. Moreover, as a consequence of Theorem 5.3.12, there exists a constant $C' > 0$ such that, $\widetilde{\mathbb{P}}$-a.s., for all $n \in \mathbb{N}$ and $j \in \{1, \ldots, k_n\}$, we have $\lambda_n^{(j)} \ge C' \lambda_\infty^{(j)}$. In particular, $\lambda_n^{(k_n)} \gtrsim \lambda_\infty^{(k_n)}$. Together with Weyl's law, we find $1 + [\lambda_n^{(k_n)}]^s \gtrsim_{\mathcal{M}} k_n^{2s/m}$, so that the convergence of this difference is due to (5.A.1):

$$\|U_n^3 - U_n\|_{2\to\infty}^2 \lesssim_{\mathcal{M}} M_{\psi,\infty}^2 n k_n^{-4s\beta'/m} \to 0 \quad \text{as } n \to \infty. \tag{5.A.5}$$

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Step 3 $(U_n^1 - U_n^2)$**.** In order to show that

$$U_n^1 - U_n^2 \to 0 \quad \text{in } L^0(\widetilde{\Omega}; \mathscr{L}(L^2(\mathcal{M}); L^\infty(\mathcal{M}))) \quad \text{as } n \to \infty,$$

we first fix an arbitrary $\varepsilon > 0$. Then, for all $\ell, n \in \mathbb{N}$ such that $k_n > \ell$, we split off the first ℓ terms and use the triangle inequality to obtain

$$\begin{aligned}\|U_n^1 - U_n^2\|_{2\to\infty} &\le \sum_{j=1}^{\ell} \big(1+[\lambda_\infty^{(j)}]^s\big)^{-\beta'} \|\psi_\infty^{(j)} \otimes \psi_\infty^{(j)} - \widetilde{\psi}_n^{(j)} \otimes \widetilde{\psi}_n^{(j)}\|_{2\to\infty} \\ &+ \Big\| \sum_{j=\ell+1}^{k_n} \big(1+[\lambda_\infty^{(j)}]^s\big)^{-\beta'} \psi_\infty^{(j)} \otimes \psi_\infty^{(j)} \Big\|_{2\to\infty} + \Big\| \sum_{j=\ell+1}^{k_n} \big(1+[\lambda_\infty^{(j)}]^s\big)^{-\beta'} \widetilde{\psi}_n^{(j)} \otimes \widetilde{\psi}_n^{(j)} \Big\|_{2\to\infty}.\end{aligned}$$

Using the triangle inequality once more, followed by (5.A.3) and Assumption 5.3.9, the norms in the summation over $j \in \{1, \ldots, \ell\}$ can be bounded by

$$\begin{aligned}&\|\psi_\infty^{(j)} \otimes \psi_\infty^{(j)} - \widetilde{\psi}_n^{(j)} \otimes \widetilde{\psi}_n^{(j)}\|_{2\to\infty} \\ &\quad\le \|\psi_\infty^{(j)} \otimes (\psi_\infty^{(j)} - \widetilde{\psi}_n^{(j)})\|_{2\to\infty} + \|(\psi_\infty^{(j)} - \widetilde{\psi}_n^{(j)}) \otimes \widetilde{\psi}_n^{(j)}\|_{2\to\infty} \\ &\quad= \|\psi_\infty^{(j)} - \widetilde{\psi}_n^{(j)}\|_{L^\infty(\mathcal{M})} + \|\psi_\infty^{(j)} - \widetilde{\psi}_n^{(j)}\|_{L^2(\mathcal{M})} \|\widetilde{\psi}_n^{(j)}\|_{L^\infty(\mathcal{M})} \\ &\quad\le (1 + M_{\psi,\infty}) \|\psi_\infty^{(j)} - \widetilde{\psi}_n^{(j)}\|_{L^\infty(\mathcal{M})},\end{aligned}$$

whereas the remaining two summations can be treated by arguing as for (5.A.4). Together, this yields

$$\begin{aligned}\|U_n^1 - U_n^2\|_{2\to\infty} \le (1 + M_{\psi,\infty}) &\sum_{j=1}^{\ell} \big(1+[\lambda_\infty^{(j)}]^s\big)^{-\beta'} \|\psi_\infty^{(j)} - \widetilde{\psi}_n^{(j)}\|_{L^\infty(\mathcal{M})} \\ &+ 2\Big(\sum_{j=\ell+1}^{\infty} \big(1+[\lambda_\infty^{(j)}]^s\big)^{-2\beta'}\Big)^{\frac{1}{2}}.\end{aligned}$$

Since we have already seen in Step 2 that the latter series converges, we can fix $\ell \in \mathbb{N}$ so large that the second sum on the right-hand side is less than $\frac{1}{2}\varepsilon$. Moreover, it follows from Theorem 5.3.13(b) that $\|\psi_\infty^{(j)} - \widetilde{\psi}_n^{(j)}\|_{L^\infty(\mathcal{M})} \to 0$ in $L^0(\widetilde{\Omega}, \widetilde{\mathbb{P}})$ as $n \to \infty$ for every $j \in \{1, \ldots, \ell\}$. In particular, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, the first sum on the right-hand side is less than $\frac{1}{2}\varepsilon$, and thus the whole right-hand side is less than ε, with probability $\widetilde{\mathbb{P}} \geq 1 - \varepsilon$. This shows $\|U_n^1 - U_n^2\|_{2\to\infty} \to 0$ in $\widetilde{\mathbb{P}}$, as desired.

Step 4 ($U_n^2 - U_n^3$)**.** Finally, the difference $U_n^2 - U_n^3$ can be treated in the same manner as Step 3, namely by writing, for all $\ell, n \in \mathbb{N}$ such that $k_n > \ell$,

$$\begin{aligned}\|U_n^2 - U_n^3\|_{2\to\infty}^2 &\lesssim_{M_{\psi,\infty}} \sum_{j=1}^{k_n} \left|\left(1 + [\lambda_n^{(j)}]^s\right)^{-\beta'} - \left(1 + [\lambda_\infty^{(j)}]^s\right)^{-\beta'}\right|^2 \\ &\leq \sum_{j=1}^{\ell} \left|\left(1 + [\lambda_n^{(j)}]^s\right)^{-\beta'} - \left(1 + [\lambda_\infty^{(j)}]^s\right)^{-\beta'}\right|^2 \\ &\quad + 2 \sum_{j=\ell+1}^{k_n} \left(1 + [\lambda_n^{(j)}]^s\right)^{-2\beta'} + 2 \sum_{j=\ell+1}^{k_n} \left(1 + [\lambda_\infty^{(j)}]^s\right)^{-2\beta'}.\end{aligned}$$

Using the fact that, $\widetilde{\mathbb{P}}$-a.s., we have $\lambda_n^{(j)} \gtrsim \lambda_\infty^{(j)}$ for all $j \in \{1, \ldots, k_n\}$ (see Step 2), the latter two summations can be bounded, up to a multiplicative constant, by the convergent series $\sum_{j=\ell+1}^{\infty} \left(1 + [\lambda_\infty^{(j)}]^s\right)^{-2\beta'}$. Combined with the eigenvalue convergence asserted by Theorem 5.3.12, which can be applied to the remaining summation, we obtain $\|U_n^2 - U_n^3\|_{2\to\infty} \to 0$ as $n \to \infty$, $\widetilde{\mathbb{P}}$-a.s., by arguing as in Step 3. Thus, we have shown part (b).

(a) Replace (5.A.2) by the identity $\|U\|_{\mathscr{L}_2(L^2(\mathcal{M}))}^2 = \sum_j |\alpha_j|^2 \|f_j\|_{L^2(\mathcal{M})}^2$, which follows directly from the definition of the Hilbert–Schmidt norm. Assumption 5.3.9 is not required since all the eigenfunctions are L^2-normalized. The sufficiency of Assumption 5.3.5(i) and the $\widetilde{\mathbb{P}}$-a.s. convergence in the conclusion are due to the use of Theorem 5.3.13(a) instead of Theorem 5.3.13(b).

(c) Recall from Setting 5.3.2 that $h_n := n^{-\frac{1}{m}}$ by definition. In view of Lemma 5.3.11, we can take $k_n := n - 1$, hence neither of the bounds on h_n from Assumption 5.3.5 is needed. Indeed, in the proof of (b), the lower bound (5.A.1) on k_n was only used in (5.A.5), which becomes $\|U_n^3 - U_n\|_{2\to\infty}^2 \lesssim_{\mathcal{M}} M_{\psi,\infty}^2 k_n^{-4s\beta'/m}$ in the current situation, and this tends to zero since, trivially, $k_n \to \infty$ as $n \to \infty$. □

Lastly we prove Lemma 5.3.15 regarding uniform ultracontractivity of the semigroups $(S_n(t))_{t\geq 0}$ associated to the (discretized) generalized Whittle–Matérn operators $-(\mathcal{L}_n^{\kappa,\tau})^s$.

Proof of Lemma 5.3.15. (a) For $p = \infty$, the statement holds by Assumption 5.3.6(i). For $p = 2$, we note that, for all $n \in \overline{\mathbb{N}}$, $t > 0$, and $f \in L^2(\mathcal{M}_n)$,

$$\|S_n(t) f\|_{L^\infty(\mathcal{M}_n)} = \|R_n^\beta (\mathrm{Id}_n + A_n)^\beta S_n(t) f\|_{L^\infty(\mathcal{M}_n)} \leq t^{-\beta} \|R_n^\beta\|_{2\to\infty} \|f\|_{L^2(\mathcal{M}_n)},$$

where we used (5.C.2), as $(S_n(t))_{t\geq 0}$ is a contractive analytic semigroup on $L^2(\mathcal{M}_n)$. In the proof of Theorem 5.3.14(b), we found $\|R_n^\beta\|_{2\to\infty} \leq M_{\psi,\infty} \sum_{j=1}^n (1 + [\lambda_n^{(j)}]^s)^{-2\beta}$

under Assumption 5.3.9. Since Assumption 5.3.5(ii) implies 5.3.5(i) with the same β, and since the estimate of $\|R_n^\beta\|_{2\to\infty}$ only involves the eigenvalues (and not the eigenfunctions), we can in both cases argue as in Theorem 5.3.14(a), under Assumption 5.3.4(i), to deduce that, $\widetilde{\mathbb{P}}$-a.s., the right-hand side can be bounded independently of n. This proves the statement for $p = 2$, hence by the Riesz–Thorin interpolation theorem [94, Theorem 1.3.4], the lemma holds for all $p \in [2,\infty]$, with $M_{S,p} \le M_{S,2}^{\frac{2}{p}} M_{S,\infty}^{1-\frac{2}{p}}$.

(b) The differences with the proof of part (a) are the use of Theorem 5.3.14(c) instead of 5.3.14(b), and the fact that Assumption 5.3.9 is automatically satisfied. □

5.B. FRACTIONAL PARABOLIC INTEGRATION

Let $(E, \|\cdot\|_E)$ be a Banach space. Suppose that $-A\colon \mathsf{D}(A) \subseteq E \to E$ generates a strongly continuous semigroup $(S(t))_{t\ge 0}$, and let the constants $M \in [1,\infty)$, $w \in \mathbb{R}$ be such that

$$\|S(t)\|_{\mathscr{L}(E)} \le Me^{wt} \quad \text{for all } t \in [0,\infty). \tag{5.B.1}$$

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We define the fractional parabolic integral of $f \in L^p(0,T;E)$ of order $s \in (0,\infty)$ by

$$\mathfrak{I}_A^s f(t) := \frac{1}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} S(t-\tau) f(\tau)\,\mathrm{d}\tau, \quad \text{a.e. } t \in (0,T); \tag{5.B.2}$$

for $s = 0$ we set $\mathfrak{I}_A^0 := \mathrm{Id}_E$. The following properties of $\mathfrak{I}_A^s$ (for a single operator A) are well known—see also [56, Proposition 5.9] and Proposition 4.2.3(a),(b)–and used throughout the main text of the chapter. We will state them here for the sake of self-containedness.

Proposition 5.B.1. *Suppose that $-A\colon \mathsf{D}(A) \subseteq E \to E$ generates a strongly continuous semigroup $(S(t))_{t\ge 0} \subseteq \mathscr{L}(E)$ satisfying (5.B.1). Then, for every $s \in [0,\infty)$, $p \in [1,\infty]$ and $T \in (0,\infty)$, we have:*

(a) $\mathfrak{I}_A^s$ is bounded from $L^p(0,T;E)$ to itself, with an operator norm depending only on s, p, T, w and M.

If $(F, \|\cdot\|_F)$ is a Banach space for which there exist $M' \in [1,\infty)$ and $\alpha \in [0,\infty)$ such that

$$S(t) \in \mathscr{L}(E;F) \quad \textit{with} \quad \|S(t)\|_{\mathscr{L}(E;F)} \le M' t^{-\alpha/2} \quad \textit{for all } t \in [0,\infty),$$

and in addition we have either $p = 1$, $s \ge 1 + \frac{\alpha}{2}$ or $p > 1$, $s > \frac{1}{p} + \frac{\alpha}{2}$, then

(b) $\mathfrak{I}_A^s$ is bounded from $L^p(0,T;E)$ to $C([0,T];F)$, with an operator norm depending only on s, p, T and M'.

For a sequence of operators $(A_n)_{n\in\overline{\mathbb{N}}}$ on Banach spaces which satisfy the appropriate discrete-to-continuum assumptions from the main text of the chapter, the proposition above implies the following corollary regarding uniform boundedness of the sequence $(\widetilde{\mathfrak{I}}_{A_n}^s)_{n\in\overline{\mathbb{N}}}$, where $\widetilde{\mathfrak{I}}_{A_n}^s := \Lambda_n \mathfrak{I}_{A_n}^s \Pi_n$ for all $n \in \overline{\mathbb{N}}$. From this, in turn, one can derive Proposition 5.B.3 below asserting the strong convergence of these operators.

Corollary 5.B.2. *Let the Banach spaces $(E_n, \|\cdot\|_{E_n})_{n\in\overline{\mathbb{N}}}$, $(\widetilde{E}, \|\cdot\|_{\widetilde{E}})$ and the linear operators $(A_n)_{n\in\overline{\mathbb{N}}}$ satisfy the assumptions of Theorem 5.2.3, and suppose that $p \in [1,\infty]$ and $s \in [0,\infty)$. The following assertions hold:*

(a) *The sequence $(\widetilde{\mathfrak{I}}^s_{A_n})_{n\in\overline{\mathbb{N}}}$ is uniformly bounded in $\mathscr{L}(L^p(0,T;\widetilde{E}))$.*

(b) *The sequence $(\widetilde{\mathfrak{I}}^s_{A_n})_{n\in\overline{\mathbb{N}}}$ is uniformly bounded in $\mathscr{L}(L^p(0,T;\widetilde{E});C([0,T];\widetilde{E}))$ if, either, $p=1$ and $s\geq 1$, or $p>1$ and $s>\frac{1}{p}$.*

If the spaces $(E_n)_{n\in\overline{\mathbb{N}}}$, $(B_n)_{n\in\overline{\mathbb{N}}}$ and $\widetilde{B}$ are as in Assumptions (A1-B), (A2-B) and (A4-B), and we have $s > \frac{1}{p} + \frac{\theta}{2}$, then

(c) *the sequence $(\widetilde{\mathfrak{I}}^s_{A_n})_{n\in\overline{\mathbb{N}}}$ is uniformly bounded in $\mathscr{L}(L^p(0,T;E_\infty);C([0,T];\widetilde{B}))$.*

Proposition 5.B.3. *Let the Banach spaces $(E_n, \|\cdot\|_{E_n})_{n\in\overline{\mathbb{N}}}$, $(\widetilde{E}, \|\cdot\|_{\widetilde{E}})$ and the linear operators $(A_n)_{n\in\overline{\mathbb{N}}}$ satisfy the assumptions of Theorem 5.2.3. Suppose that $p \in [1,\infty]$ and $s \in [0,\infty)$. The following assertions hold:*

(a) *If either $p=1$ and $s\geq 1$, or $p>1$ and $s>\frac{1}{p}$, then we have $\widetilde{\mathfrak{I}}^s_{A_n} f \to \mathfrak{I}^s_{A_\infty} f$ in $C([0,T];\widetilde{E})$, as $n\to\infty$, for every $f\in L^p(0,T;E_\infty)$.*

Moreover, let Assumptions (A1-B), (A2-B), (A3-B) and (A4-B) hold.

(b) *If $s > \frac{1}{p}+\frac{\theta}{2}$, then $\widetilde{\mathfrak{I}}^s_{A_n} f \to \mathfrak{I}^s_{A_\infty} f$ in $C([0,T];\widetilde{B})$ as $n\to\infty$ for all $f \in L^p(0,T;E_\infty)$.*

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Proof. We only present the details of the argument for part (b), the proof of (a) being similar.

Let $p\in[1,\infty)$, $s\in(\frac{1}{p}+\frac{\theta}{2},\infty)$, $f\in L^p(0,T;E_\infty)$ and fix an arbitrary $\varepsilon>0$. By the density of B_∞ in E_∞ (see (A4-B)), and that of B_∞-valued simple functions in $L^p(0,T;B_\infty)$, there exists a function $g\colon [0,T]\to B_\infty$ of the form

$$g = \sum_{j=1}^{K} \mathbf{1}_{(a_j,b_j)} \otimes x_j, \quad K\in\mathbb{N};\, 0\leq a_j<b_j\leq T,\ x_j\in B_\infty \text{ for all } j\in\{1,\dots,K\}$$

such that

$$\|f-g\|_{L^p(0,T;E_\infty)} < \frac{\varepsilon}{4}\Big(\sup\nolimits_{n\in\overline{\mathbb{N}}} \|\widetilde{\mathfrak{I}}^s_{A_n}\|_{\mathscr{L}(L^p(0,T;E_\infty);C([0,T];\widetilde{B}))}\Big)^{-1}.$$

Note that the expression between the parentheses is finite by Corollary 5.B.2(c) and can be assumed to be nonzero without loss of generality, as otherwise $\widetilde{\mathfrak{I}}^s_{A_n}=0$ for all $n\in\overline{\mathbb{N}}$ and the asserted convergence would be trivial. Thus, for every $n\in\mathbb{N}$,

$$\begin{aligned}
&\|\widetilde{\mathfrak{I}}^s_{A_n} f - \mathfrak{I}^s_A f\|_{C([0,T];\widetilde{B})} \\
&\quad\leq \|\widetilde{\mathfrak{I}}^s_{A_n}(f-g)\|_{C([0,T];\widetilde{B})} + \|\widetilde{\mathfrak{I}}^s_{A_n} g - \mathfrak{I}^s_A g\|_{C([0,T];\widetilde{B})} + \|\mathfrak{I}^s_A(g-f)\|_{C([0,T];\widetilde{B})} \\
&\quad< \tfrac{1}{2}\varepsilon + \|\widetilde{\mathfrak{I}}^s_{A_n} g - \mathfrak{I}^s_A g\|_{C([0,T];\widetilde{B})}.
\end{aligned}$$

For any $j\in\{1,\dots,K\}$, by (A3-B) and the discrete-to-continuum Trotter–Kato approximation theorem, we can choose $N_j\in\mathbb{N}$ so large that

$$\|\widetilde{S}_n\otimes x_j - S\otimes x_j\|_{C([0,T];\widetilde{B})} < \frac{s\Gamma(s)}{2T^sK}\varepsilon \quad \text{for all } n\geq N_j.$$

Thus, setting $N := \max_{j=1}^K N_j$, we find for all $n \geq N$ and $t \in [0,T]$:

$$\begin{aligned}\|\widetilde{\mathfrak{I}}^s_{A_n} g(t) - \mathfrak{I}^s_A g(t)\|_{\widetilde{B}} &\leq \frac{1}{\Gamma(s)} \sum_{j=1}^K \int_{a_j}^{b_j} (t-r)^{s-1} \|\widetilde{S}_n(t-r)x_j - S(t-r)x_j\|_{\widetilde{B}}\, \mathrm{d}r \\ &\leq \frac{1}{\Gamma(s)} \sum_{j=1}^K \int_0^T r^{s-1} \|\widetilde{S}_n(r)x_j - S(r)x_j\|_{\widetilde{B}}\, \mathrm{d}r \\ &\leq \frac{T^s}{s\Gamma(s)} \sum_{j=1}^K \|\widetilde{S}_n \otimes x_j - S \otimes x_j\|_{C([0,T];\widetilde{B})} < \frac{\varepsilon}{2}.\end{aligned}$$

Since $t \in [0,T]$ was arbitrary, we conclude that $\|\widetilde{\mathfrak{I}}^s_{A_n} g - \mathfrak{I}^s_A g\|_{C([0,T];\widetilde{B})} < \frac{1}{2}\varepsilon$, and therefore $\|\widetilde{\mathfrak{I}}^s_{A_n} f - \mathfrak{I}^s_A f\|_{C([0,T];\widetilde{B})} < \varepsilon$ for all $n \geq N$. □

5.C. UNIFORMLY SECTORIAL SEQUENCES OF OPERATORS

5

We first recall the concept of sectorial operators from Definition 1.1.7. A linear operator $A\colon \mathsf{D}(A) \subseteq E \to E$ on a (real or complex) Banach space E, with spectrum $\sigma(A)$, is said to be ω-*sectorial* (with $\omega \in (0,\pi)$) if

$$\sigma(A) \subseteq \overline{\Sigma}_\omega \quad \text{and} \quad M(\omega, A) := \sup\{\|\lambda R(\lambda, A)\|_{\mathscr{L}(E)} : \lambda \in \mathbb{C} \setminus \overline{\Sigma}_\omega\} < \infty, \tag{5.C.1}$$

where Σ_ω is as in (5.3.10) and $M(\omega, A)$ is called the ω-*sectoriality constant*. Its *angle of sectoriality* $\omega(A) \in [0,\pi)$ is defined as the infimum of all ω for which (5.C.1) holds.

If A is closed and densely defined, then by Theorem 1.1.8, A is ω-sectorial for some $\omega \in (0, \frac{1}{2}\pi)$ if and only if there exists $\eta \in (0, \frac{1}{2}\pi)$ such that $-A$ generates a bounded analytic semigroup $(S(t))_{t\geq 0}$ on Σ_η. Inspecting the proof of the cited theorem reveals that, whenever these equivalent conditions hold, we have

$$\sup_{z\in\Sigma_\eta} \|S(z)\|_{\mathscr{L}(E)} \eqsim_{(\omega,\eta)} M(\omega, A).$$

Moreover, recall that the supremum of the set of $\eta \in (0, \frac{1}{2}\pi)$ for which $(S(t))_{t\geq 0}$ extends to a bounded analytic semigroup on Σ_η equals $\frac{1}{2}\pi - \omega(A)$.

By [100, Propositions 3.4.1 and 3.4.3] we have

$$\|A^\alpha S(t)\|_{\mathscr{L}(E)} \lesssim_{(\omega,\alpha)} M(\omega, A)\, t^{-\alpha}, \tag{5.C.2}$$

for all $\omega \in (\omega(A), \frac{1}{2}\pi)$ and $t \in (0,\infty)$, where the implicit constant is non-decreasing in α for any fixed ω.

We say that a sequence $(A_n)_{n\in\mathbb{N}}$ of linear operators $A_n\colon \mathsf{D}(A_n) \subseteq E_n \to E_n$ is *uniformly sectorial of angle* $\omega \in [0,\pi)$ if A_n is sectorial of angle ω for all $n \in \mathbb{N}$ and

$$M_{\mathrm{Unif}}(\omega', A) := \sup_{n\in\mathbb{N}} M(\omega', A_n) < \infty \quad \text{for all } \omega' \in (\omega, \pi). \tag{5.C.3}$$

Lemma 5.C.1 complements Theorem 5.2.3 in the situation where the semigroup generators are uniformly sectorial of angle less than $\frac{1}{2}\pi$, in which case we obtain the uniform convergence $\Lambda_n A_n^\alpha S(\cdot)\Pi_n x \to A_\infty^\alpha S_\infty(\cdot)x$ on compact subsets of $(0,\infty)$. It is an analog to [131, Lemma 4.1(2)] in the discrete-to-continuum setting and for general $\alpha \in (0,\infty)$ (instead of $\alpha = 1$).

Lemma 5.C.1. *Let the linear operators $A_n\colon \mathsf{D}(A_n) \subseteq E_n \to E_n$ on the Banach spaces $(E_n, \|\cdot\|_{E_n})_{n\in\overline{\mathbb{N}}}$ be uniformly sectorial of angle $\omega \in [0, \frac{1}{2}\pi)$, and denote by $(S_n(t))_{t\geq 0}$ the bounded analytic C_0-semigroups generated by $-A_n$. Let Assumptions 5.2.1 and 5.2.2 be satisfied with $w = 0$, and suppose that the equivalent statements (a) and (b) in Theorem 5.2.3 hold. Then we have, for all $\alpha \in (0,\infty)$, $x \in E_\infty$ and $0 < a < b < \infty$,*

$$\sup_{t\in[a,b]} \|\Lambda_n A_n^\alpha S_n(t)\Pi_n x - A_\infty^\alpha S_\infty(t)x\|_{\widetilde{E}} \to 0 \quad \text{as } n \to \infty.$$

Proof. Fix $\omega' \in (\omega, \frac{1}{2}\pi)$, $n \in \overline{\mathbb{N}}$ and $t \in (0,\infty)$. We begin by sketching the functional calculus argument (see Section 1.3) which shows that we have the following Cauchy integral representation:

$$A_n^\alpha S_n(t) = \frac{1}{2\pi i}\int_{\partial\Sigma_{\omega'}} z^\alpha e^{-tz} R(z, A_n)\, \mathrm{d}z \tag{5.C.4}$$

To see this, define the functions $f_\alpha, g_t\colon \Sigma_{\omega'} \to \mathbb{C}$ by $f_\alpha(z) := z^\alpha$ and $g_t(z) := e^{-tz}$ for $z \in \Sigma_{\omega'}$ and $t \in (0,\infty)$. Denote by $f_\alpha(A_n)$ and $g_t(A_n)$ the operators obtained via the extended Dunford calculus. Then $f_\alpha(A_n)$ is the fractional power A_n^α, which satisfies $f_\alpha(A_n)x = f_\alpha(\lambda)x = \lambda^\alpha x$ if $A_n x = \lambda x$, hence this (more general) definition agrees with the spectrally defined fractional powers in the setting of (5.3.9). Moreover, we have $g_t(A_n) = S_n(t)$ by [115, Theorem 15.1.7], and $(f_\alpha g_t)(A_n) = f_\alpha(A_n)g_t(A_n)$ by [115, Proposition 15.1.12] since $S(t)$ is bounded. The function $(f_\alpha g_t)(z) = z^\alpha e^{-z}$ is holomorphic and has (super)polynomial decay at 0 and ∞, and thus belongs to the domain of the primary Dunford calculus [115, Definition 15.1.1], which implies that $(f_\alpha g_t)(A_n) = \frac{1}{2\pi i}\int_{\partial\Sigma_{\omega'}} z^\alpha e^{-tz}R(z,A_n)\,\mathrm{d}z$. Putting all these observations together yields (5.C.4). Applying the projection and lifting operators and parametrizing the complex integral yields, for all $x \in E_\infty$,

$$\begin{aligned}\Lambda_n A_n^\alpha S_n(t)\Pi_n x = &-\frac{e^{i(\alpha+1)\omega'}}{2\pi i}\int_0^\infty r^\alpha \exp(-te^{i\omega'}r)\widetilde{R}(e^{i\omega'}r, A_n)x\,\mathrm{d}r \\ &+\frac{e^{-i(\alpha+1)\omega'}}{2\pi i}\int_0^\infty r^\alpha \exp(-te^{-i\omega'}r)\widetilde{R}(e^{-i\omega'}r, A_n)x\,\mathrm{d}r,\end{aligned}$$

where we recall that $\Pi_\infty = \Lambda_\infty = \mathrm{Id}_{\widetilde{E}}$ for $n = \infty$. It follows that the above estimate implies the following uniform bound on the interval $[a,b]$:

$$\begin{aligned}&\sup_{t\in[a,b]} \|\Lambda_n A_n^\alpha S_n(t)\Pi_n x - A_\infty^\alpha S_\infty(t)x\|_{\widetilde{E}} \\ &\leq \frac{1}{2\pi}\int_0^\infty r^\alpha e^{-a\cos(\omega')r}\Big[\|\widetilde{R}(re^{i\omega'}, A_n)x - R(re^{i\omega'}, A_\infty)x\|_{\widetilde{E}} \\ &\qquad\qquad + \|\widetilde{R}(re^{-i\omega'}, A_n)x - R(re^{-i\omega'}, A_\infty)x\|_{\widetilde{E}}\Big]\,\mathrm{d}r.\end{aligned} \tag{5.C.5}$$

Setting $\eta := \frac{1}{2}\pi - \vartheta$ for some $\vartheta \in (\omega, \omega')$, the uniform sectoriality of $(A_n)_{n\in\overline{\mathbb{N}}}$ implies that the operators $(-A_n e^{\pm i\eta})_{n\in\overline{\mathbb{N}}}$ generate C_0-semigroups $(S_n(te^{\pm i\eta}))_{t\geq 0}$ which are uniformly bounded in t and n. Therefore, we can apply Theorem 5.2.3 to these two sequences of semigroups to find that $\widetilde{R}(\lambda, A_n)x \to R(\lambda, A_\infty)x$ for all $|\arg\lambda| > \vartheta$ if this convergence holds for one such λ. We have in fact $\widetilde{R}(\lambda, A_n)x \to R(\lambda, A_\infty)x$ for *all* λ such that $|\arg\lambda| > \frac{1}{2}\pi > \vartheta$ by our hypothesis that the operators $(A_n)_{n\in\overline{\mathbb{N}}}$ satisfy statements (a) and (b) in Theorem 5.2.3, so we conclude

$$\widetilde{R}(re^{\pm i\omega'}, A_n)x \to R(re^{\pm i\omega'}, A_\infty)x$$

for all $r \in (0,\infty)$. On the other hand, by (5.C.3) and Assumption 5.2.1 we have

$$\|\widetilde{R}(re^{\pm i\omega'}, A_n)x\|_{\widetilde{E}} \le M_\Pi M_\Lambda M_{\mathrm{Unif}}(\vartheta, A)\|x\|_{\widetilde{E}} r^{-1}$$

for all $n \in \overline{\mathbb{N}}$. Hence, we can bound the integrand in (5.C.5), up to n-independent constants, by the integrable function $r \mapsto r^{\alpha-1}\exp(-a\cos(\omega')r)$, so that the integrals tend to zero by the dominated convergence theorem. □

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Curriculum Vitae

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List of publications

1. K. Kirchner and J. Willems. "Regularity theory for a new class of fractional parabolic stochastic evolution equations". In: *Stoch. Partial Differ. Equ. Anal. Comput.* 12.3 (2024), pp. 1805–1854
2. K. Kirchner and J. Willems. "Multiple and weak Markov properties in Hilbert spaces with applications to fractional stochastic evolution equations". In: *Stoch. Process. Appl.* 186 (2025), Paper No. 104639
3. J. Willems. "Dirichlet problems associated to abstract nonlocal space-time differential operators". In: *J. Evol. Equ.* 25.1 (2025), Paper No. 19, 30
4. Y. van Gennip, J. Latz, and J. Willems. *Discrete-to-continuum limits of semilinear stochastic evolution equations in Banach spaces.* Preprint, arXiv:2504.05142. 2025